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Abstract In this work, we study the minimal time to steer a crowd to a desired configuration. The control is a vector field, representing a perturbation of the crowd displacement, localized on a fixed control set.

We give a characterization of the minimal time both for discrete and continuous crowds. We show that the minimal time to steer one initial configuration to another is related to the condition of having enough mass in the control region to feed the desired final configuration.

The construction of the control is explicit, providing a numerical algorithm for computing it. We then provide some examples of numerical simulations.

Keywords Minimal time · Controllability · Transport equation

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1 Introduction and main results

In recent years, the study of systems describing a crowd of interacting autonomous agents has drawn a great interest from the control community. A better understanding of such interaction phenomena can have a strong impact in several key applications, such as road traffic and egress problems for pedestrians. For a few reviews about this topic, see *e.g.* [4,5,8,17,23,24,30,33].

Beside the description of interactions, it is now relevant to study problems of control of crowds, *i.e.* of controlling such systems by acting on few agents, or on a small subset of the configuration space. Roughly speaking, basic problems for such models are controllability (i.e. reachability of a desired configuration) and optimal control (i.e. the minimization of a given functional). We already have addressed the controllability problem in [19], thus identifying reachable configurations for crowd models. The present article deals with the subsequent step, that is the study of a classical optimal control problem: the minimal time to reach a desired configuration.

The nature of the control problem relies on the model used to describe the crowd. Two main classes are widely used. In microscopic models, the position of each agent is clearly identified; the crowd dynamics is described by a large dimensional ordinary differential equation, in which couplings of terms represent interactions. For control of such models, a large literature is available, see *e.g.* reviews [7,27,28], as well as applications, both to pedestrian crowds [20,29] and to road traffic [36,22].

In macroscopic models, instead, the idea is to represent the crowd by the spatial density of agents; in this setting, the evolution of the density solves a partial differential equation, usually of transport type. Nonlocal terms (such as convolutions) model the interactions between the agents. To our knowledge, there exist few studies of control of this family of equations. In [32], the authors provide approximate alignment of a crowd described by the macroscopic Cucker-Smale model [18]. The control is the acceleration, and it is localized in a control region ω which moves in time. In a similar situation, a stabilization strategy has been established in [9,10], by generalizing the Jurdjevic-Quinn method to partial differential equations. Finally, a different approach is given by mean-field type control, *i.e.* control of mean-field equations and of mean-field games modelling crowds, see *e.g.* [1,2,11,21]. In this case, problems are often of optimization nature, *i.e.* the goal is find a control minimizing a given cost, with no final constraint. In this article, we are interesting in the minimal time to reach a specific configuration, for which mean-field type control approaches seem not adapted.

This article deals with the problem of steering one initial to a final configuration in minimal time. We recently discussed in [19] the problem of controllability for the systems described here, which main results are recalled in Section 2. We proved that one can approximately steer an initial to a final configuration if they satisfy the Geometric condition 1 recalled below. Roughly speaking, it requires that the whole initial configuration crosses the control set

ω forward in time, and the final one crosses it backward in time. From now on, we will always assume that this condition is satisfied, so to ensure that the final configuration is approximately reachable.

When the controllability condition is satisfied, it is then interesting to study minimal time problems. Indeed, from the theoretical point of view, it is the first problem in which optimality conditions can be naturally defined. More related to applications described above, minimal time problems play a crucial role: egress problems can be described in this setting, while traffic control is often described in terms of minimization of (maximal or average) total travel time.

For microscopic models, the dynamics can be written in terms of finite-dimensional control systems. For this reason, minimal time problems can sometimes be addressed with classical (linear or non-linear) control theory, see *e.g.* [3,6,25,34]. Instead, very few results are known for macroscopic models, that can be recasted in terms of control of the transport equation. The linear case is classical, see *e.g.* [16]. Instead, more recent developments in the non-linear case (based on generalization of differential inclusions) have been recently described by [12,14,13].

The originality of our research lies in the constraint given on the control: it is localized in a given region ω of the space. Such constraint is highly non-trivial, since the control problem is clearly non-linear even though the uncontrolled dynamics is. At the best of our knowledge, minimal time problems with this constraints have not been studied, neither for microscopic nor for macroscopic models.

Here, we first study a microscopic model, where the crowd is represented by a vector with nd components ($n, d \in \mathbb{N}^*$) representing the positions of n agents in $\mathbb{R}^d$. The natural (uncontrolled) vector field is denoted by $v : \mathbb{R}^d \to \mathbb{R}^d$, assumed Lipschitz and uniformly bounded. We act on the vector field in a fixed subdomain ω of the space, which will be a nonempty open convex subset of $\mathbb{R}^d$. The admissible controls are thus functions of the form $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$. We then consider the following ordinary differential equation (microscopic model)

$$\begin{cases} \dot{x}_i(t) = v(x_i(t)) + \mathbb{1}_\omega(x_i(t))u(x_i(t),t) \text{ for a.e. } t \geqslant 0, \\ x_i(0) = x_i^0 \end{cases} \tag{1}$$

eq ODE

for $i \in \{1, ..., n\}$, where $X^0 := \{x_1^0, ..., x_n^0\}$ is the initial configuration of the crowd.

We also study a macroscopic model, where the crowd is represented by its density, that is a time-evolving measure $\mu(t)$ defined on the space $\mathbb{R}^d$. We consider the same natural vector field v, control region ω and admissible controls $\mathbb{1}_\omega u$. We then consider the following linear transport equation (macroscopic model)

$$\begin{cases} \partial_t \mu + \nabla \cdot ((v + \mathbb{1}_\omega u)\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases} \tag{2}$$

eq:transport

where μ^0 is the initial density of the crowd. The function $v + \mathbb{1}_\omega u$ represents the vector field acting on μ.

To a discrete configuration $\{x_1, ..., x_n\}$, we can associate the empirical measure

$$\mu := \sum_{i=1}^{n} \frac{1}{n}\delta_{x_i}.$$

With this notation, System (1) is a particular case of System (2). This identification will be used several times in the following, namely to approximate continuous crowds with discrete ones.

Systems (1) and (2) are first approximations for crowd modeling, since the uncontrolled vector field v is given, and it does not describe interactions between agents. Nevertheless, it is necessary to understand control properties for such simple equations as a first step, before dealing with vector fields depending on the crowd itself. Thus, in a future work, we will study control problems for crowd models with a non-local term $v[\mu]$, based on the results for linear problems presented here.

Wenow recall the notion of approximate and exact controllability for Systems (1) and (2). We say that they are *approximately controllable* from the initial configuration from μ^0 to the final one μ^1 at time T if we can steer the solution from μ^0 at time 0 to a configuration at time T as close to the final configuration as we want with an appropriate control $\mathbb{1}_\omega u$. Similarly, *exact controllability* means that we can steer the solution from μ^0 at time 0 exactly to μ^1 at time T. In Definition 5 below, we give a formal definition of the notion of approximate controllability in terms of Wasserstein distance.

In all this paper, we assume that the following geometrical condition is satisfied:

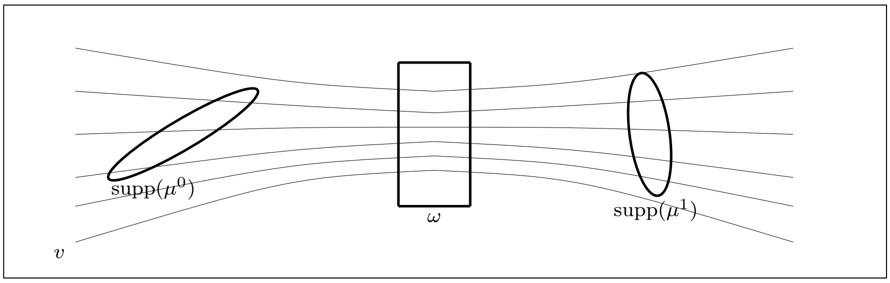
⟨cond1⟩ **Condition 1 (Geometrical condition)** *Let μ^0, μ^1 be two probability measures on $\mathbb{R}^d$ satisfying:*

(i) For each $x^0 \in \mathrm{supp}(\mu^0)$, there exists $t^0 > 0$ such that $\Phi^v_{t^0}(x^0) \in \omega$, where Φ^v_t is the flow associated to v (see Definition 3 below).
(ii) For each $x^1 \in \mathrm{supp}(\mu^1)$, there exists $t^1 > 0$ such that $\Phi^v_{-t^1}(x^1) \in \omega$.

Condition 1 means that particle crosses the control region. This geometrical aspect is illustrated in Figure 1. It is the minimal condition that we can expect to steer any initial condition to any target. Indeed, if Item (i) of Condition 1 is not satisfied, then there exists a whole subpopulation of the measure μ^0 that never intersects the control region, thus we cannot act on it.

We have proved in [19] that if we consider μ^0, μ^1 two probability measures on $\mathbb{R}^d$ compactly supported, absolutely continuous with respect to the Lebesgue measure and satisfying Condition 1, then there exists T such that System (2) is **approximately controllable** at time T from μ^0 to μ^1 with a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time.

For arbitrary continuous measures, one can expect approximate controllability only, since for general measures there exists no homeomorphism sending



⟨fig:cond geo⟩

Fig. 1 Geometrical condition.

one to another. Indeed, if we impose the classical Carathéodory condition of $\mathbb{1}_\omega u$ being Lipschitz in space, measurable in time and uniformly bounded, then the flow $\Phi_t^{v+\mathbb{1}_\omega u}$ is an homeomorphism (see [6, Th. 2.1.1]). Similarly, in the discrete case, such control vector field u cannot separate points, due to uniqueness of the solution of (1).

In the microscopic model, we assume that the initial configuration X^0 and the final one X^1 are disjoint, in the following sense:

$$\begin{cases} x_i^0 \neq x_j^0 \text{ for all } i \neq j, \\ x_i^1 \neq x_j^1 \text{ for all } i \neq j. \end{cases} \tag{3}$$ def:distinct

In other words, two agents can not be at the same point of the space. We will try to preserve this property at each time in the control strategy.

In this article, we aim to study the minimal time problem. We denote by T_a the minimal time to approximately steer the initial configuration μ^0 to a final one μ^1 in the following sense: it is the infimum of times for which there exists a control with a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time steering μ^0 arbitrarily close to μ^1. We similarly define the minimal time T_e to exactly steer the initial configuration μ^0 to a final one μ^1. A precise definition will be given following the situation. Since the minimal time is not always reached, we will speak about *infimum time*.

In the sequel, we will use the following notation for all $x \in \mathbb{R}^d$:

$$\begin{cases} \bar{t}^0(x) := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x) \in \overline{\omega}\}, \\ \bar{t}^1(x) := \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x) \in \overline{\omega}\}, \\ t^0(x) := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x) \in \omega\}, \\ t^1(x) := \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x) \in \omega\}. \end{cases}$$

The quantity $\bar{t}^0(x)$ is the infimum time at which the particle localized at point x with the velocity v belongs to $\overline{\omega}$. Idem for the other quantities.

1.1 Infimum time for discrete crowds

We denote by

$$\bar{t}_i^0 := \bar{t}^0(x_i^0),\ \bar{t}_i^1 := t^1(x_i^1),\ t_i^0 := t^0(x_i^0) \text{ and } t_i^1 := t^1(x_i^1), \tag{4}$$ def:t^1_i

for $i \in \{1, ..., n\}$. We now state our main result on the infimum time to approximately control System (1).

Theorem 1 (Main result - inf. time for exact. contr. discrete crowd) *Let $X^0 := \{x_1^0, ..., x_n^0\}$ and $X^1 := \{x_1^1, ..., x_n^1\}$ be two distinct configurations* th:discret exact⟩ *(see (3)), satisfying Condition 1. Arrange the sequences $\{t_i^0\}_i$ and $\{t_j^1\}_j$ to be increasingly and decreasingly ordered, respectively. Define $M_e^* := \max\{t_i^0, t_i^1 : i = 1, ..., n\}$. Then*

$$M_e(X^0, X^1) := \max_{i \in \{1,\dots,n\}} |t_i^0 + t_i^1| \tag{5}$$ OT disc CE

is the infimum time $T_e(X^0, X^1)$ for exact control of System (1) in the following sense:

(i) For each $T > M_e$, System (1) is exactly controllable from X^0 to X^1 at time T, i.e. there exists a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time steering X^0 exactly to X^1. Moreover, at each time $t \in [0, T]$, the configuration is distinct, i.e. $x_i(t) \neq x_j(t)$ for all $i \neq j$.
(ii) For each $T \in (M_e^*, M_e]$, System (1) is not exactly controllable from X^0 to X^1.
(iii) There exists (at most) a finite number of time $T \in [0, M_e^*]$ for which System (1) is exactly controllable from X^0 to X^1.

We give a proof of Theorem 1 in Section 3. The value M_e^* means that we wait a time long enough such that we can act on each particles. It can exists some time $T \leqslant M_e^*$ at which it is possible to steer μ^0 to μ^1, but it will be not entirely thanks to the control. We give an example of this situation in Remark 4.

We now give a characterisation of the minimal to approximatively steer a configuration to another. The distance in which this notion of approximate controllability is understood will be specified in Sectin 2.

Theorem 2 (Main result - inf. time for approx. contr. discr. crowd) *Let $X^0 := \{x_1^0, ..., x_n^0\}$ and $X^1 := \{x_1^1, ..., x_n^1\}$ be two distinct configurations* h:discret approx⟩ *(see (3)) satisfying Condition 1. Arrange the sequences $\{t_i^0\}_i$ and $\{\bar{t}_j^1\}_j$ to be increasingly and decreasingly ordered, respectively. Define $M_a^* := \max\{t_i^0, \bar{t}_i^1 : i = 1, ..., n\}$. Then*

$$M_a(X^0, X^1) := \max_{i \in \{1,\dots,n\}} |t_i^0 + \bar{t}_i^1|$$

is the infimum time $T_a(X^0, X^1)$ to approximately control System (1) in the following sense:

(i) For each $T > M_a$, System (1) *is approximatively controllable from X^0 to X^1 at time T, i.e. there exists a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time steering μ^0 arbitrarily close to μ^1.*
(ii) For each $T \in (M_a^*, M_a]$, System (1) *is not approximatively controllable from X^0 to X^1.*
(iii) There exists (at most) a finite number of time $T \in [0, M_a^*]$ for which System (1) *is approximately controllable from X^0 to X^1.*

We give a proof of Theorem 2 in Section 3. As for the exact controllability, when the approximate controllability holds for a time $T \leqslant M_a^*$, it will be not entirely due to the control and happens only in some pathological situations. We refer also to Remark 4 for an example.

It is well know that the notions of approximate and exact controllability are equivalent for finite dimensional linear systems when the control acts linearly. We remark that it is not the case for System (1) which highlights the fact that we are dealing with a non linear control problem. An example is given in Remark 5.

1.2 Infimum time for continuous crowds

Introduce first the maps $\mathcal{F}_0$ and $\mathcal{F}_1$ defined for all $t \geqslant 0$ by

$$\begin{cases} \mathcal{F}_0(t) := \mu^0(\{x \in \mathrm{supp}(\mu^0) : t^0(x) \leqslant t\}), \\ \mathcal{F}_1(t) := \mu^1(\{x \in \mathrm{supp}(\mu^1) : t^1(x) \leqslant t\}). \end{cases}$$

The function $\mathcal{F}_0$ (resp. $\mathcal{F}_1$) gives the quantity of mass coming from μ^0 (resp. the quantity of mass coming from μ^1 backward in time) which has entered in ω at time t. Observe that we do not decrease $\mathcal{F}_0$ when the mass eventually leaves ω, and similarly for $\mathcal{F}_1$. Define the generalised inverse functions $\mathcal{F}_0^{-1}$ and $\mathcal{F}_1^{-1}$ of $\mathcal{F}_0$ and $\mathcal{F}_1$ given for all $m \in [0,1]$ by

$$\begin{cases} \mathcal{F}_0^{-1}(m) := \inf\{t \geq 0 : \mathcal{F}_0(t) \geqslant m\}, \\ \mathcal{F}_1^{-1}(m) := \inf\{t \geq 0 : \mathcal{F}_1(t) \geqslant m\}. \end{cases} \qquad (6)$$ def F0 F1 -1

The function $\mathcal{F}_0^{-1}$ is increasing, lower semi-continuous and gives the time at which a mass m has entered in ω, and similarly for $\mathcal{F}_1^{-1}$. We then have the following main result about infimum time in the continuous case:

Theorem 3 (Main result - infimum time for approx. cont. crowd)
⟨th opt⟩ *Let μ^0 and μ^1 be two probability measures, with compact support, absolutely continuous with respect to the Lebesgue measure and satisfying Condition 1. Then*

$$S(\mu^0, \mu^1) := \sup_{m \in [0,1]} \{\mathcal{F}_0^{-1}(m) + \mathcal{F}_1^{-1}(1-m)\} \qquad (7)$$ def T0

is the infimum time $T_a(\mu^0, \mu^1)$ to approximately steer μ^0 to μ^1 in the following sense:

(i) For all $T > S(\mu^0, \mu^1)$, System (2) is approximatively controllable from μ^0 to μ^1 at time T with a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time.

(ii) For all $T \in (S^*, S(\mu^0, \mu^1)]$, System (2) is not approximatively controllable from μ^0 to μ^1,

where $S^* := \sup\{\bar{t}^l(x) : x \in \mathrm{supp}(\mu^l)$ and $l \in \{0,1\}\}$.

We give a proof of Theorem 3 in Section 4. We observe that $S(\mu^0, \mu^1)$ in (7) is the continuous equivalent of M_e in (5). Moreover, we will see in the proof of Theorem 3 that we can replace t^0 and t^1 by $\bar{t}^0$ and $\bar{t}^1$ in the definition the definition of $\mathcal{F}^0$ and $\mathcal{F}^1$, since the measure by μ^0 of the set composed of the points satisfying $t^0(x) = \bar{t}^0(x)$ is equal to zero (idem for μ^1). Contrarily to the discrete case, System (2) can be approximately controllable at each time $T \in (0, S^*)$. Since such time are sparse in the discrete case and the proof of Theorem 3 is a quid a passage to the limit, we can think that there are also sparse in the continuous case. System (2) can also be never approximately controllable from μ^0 to μ^1 on the whole time interval $(0, S^*)$. We give some examples in Remark 6.

This paper is organised as follow. In Section 2, we recall basic properties of the Wasserstein distance, ordinary differential equations and continuity equations. We prove our main results Theorems 1 and 2 in Section 3 and Theorem 3 in Section 4. We finally introduce an algorithm to compute the infimum time and give some numerical examples in Section 5.

2 Models and controllability

⟨section 2⟩ In this section, we recall some properties of the microscopic and macroscopic models (1) and (2). We highlight their connection with the Wasserstein distance, that is the natural distance associated to these dynamics. We denote by $\mathcal{P}_c(\mathbb{R}^d)$ the space of probability measures in $\mathbb{R}^d$ with compact support.

Definition 1 For $\mu,\ \nu \in \mathcal{P}_c(\mathbb{R}^d)$, we denote by $\Pi(\mu, \nu)$ the set of *transference plans* from μ to ν, *i.e.* the probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal μ and second marginal ν. Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$. Define

$$W_p(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left\{ \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p d\pi \right)^{1/p} \right\}, \tag{8}$$
def:Wp

$$W_\infty(\mu,\nu) := \inf\{\pi - \mathrm{esssup}|x-y| : \pi \in \Pi(\mu,\nu)\}. \tag{9}$$
def:Winf

For $p \in [1, \infty]$, this quantity is called the **Wasserstein distance**.

This is the idea of *optimal transportation*, consisting in finding the optimal way to transport mass from a given measure to another. For a thorough introduction, see *e.g.* [35].

Between two discrete configuration, we will use the following distance

Definition 2 For all $p \in [1, \infty)$ and all configurations $X^0 := \{x_1^0, ..., x_n^0\}$ and $X^1 := \{x_1^1, ..., x_n^1\}$ of $\mathbb{R}^d$, we define the distance between X^0 and X^1 as

$$d_p(X^0, X^1) := \inf_{\sigma \in S_n} \left(\sum_{i=1}^{n} \frac{1}{n} \|x_i^0 - x_{\sigma(i)}^1\|^p \right)^{1/p},$$

where S_n is the set of permutations on $\{1, ..., n\}$.

⟨rmq:dp Wp⟩ *Remark 1* If we denote by $\mu^0 := \frac{1}{n}\sum_i \delta_{x_i^0}$ and $\mu^1 := \frac{1}{n}\sum_i \delta_{x_i^1}$, for all $p \in [1, \infty)$, it holds $d_p(X^0, X^1) = W_p(\mu^0, \mu^1)$.

The Wasserstein distance satisfies some useful properties.

⟨prop Wp⟩ **Property 1 (see [35, Chap. 7] and [15])** *For all $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$, the infimum in* (8) *or* (9) *are achieved by at least one minimizer $\pi \in \Pi(\mu, \nu)$.*

For $p \in [1, \infty]$, the quantity W_p is a distance on $\mathcal{P}_c(\mathbb{R}^d)$. Moreover, for $p \in [1, \infty)$, the topology induced by the Wasserstein distance W_p on $\mathcal{P}_c(\mathbb{R}^d)$ coincides with the weak topology.

The Wasserstein distance can be extended to all pairs of measures μ, ν compactly supported with the same total mass $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d) \neq 0$, by the formula

$$W_p(\mu, \nu) = |\mu|^{1/p} W_p\left(\frac{\mu}{|\mu|}, \frac{\nu}{|\nu|}\right).$$

In the rest of the paper, the following properties of the Wasserstein distance will be helpful.

⟨prop:ine wass⟩ **Property 2 (see [31,35])** *Let μ, ρ, ν, η be four positive measures compactly supported satisfying $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$ and $\rho(\mathbb{R}^d) = \eta(\mathbb{R}^d)$.*

(i) For each $p \in [1, \infty]$, it holds

$$W_p^p(\mu + \rho, \nu + \eta) \leqslant W_p^p(\mu, \nu) + W_p^p(\rho, \eta).$$

(ii) For each p_1, $p_2 \in [1, \infty]$ with $p_1 \leqslant p_2$, it holds

$$\begin{cases} W_{p_1}(\mu, \nu) \leqslant W_{p_2}(\mu, \nu), \\ W_{p_2}(\mu, \nu) \leqslant \operatorname{diam}(X)^{1 - p_1/p_2} W_{p_1}^{p_1/p_2}(\mu, \nu), \end{cases} \tag{10}$$

ine wasser 4

where X contains the supports of μ and ν.

Consider the following system

$$\begin{cases} \partial_t \mu + \nabla \cdot (w\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases} \tag{11}$$

eq:transport sec 2

where $w : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$. This equation is called the **continuity equation**. We now introduce the flow associated to System (11).

⟨def:flow⟩ **Definition 3** We define the **flow** associated to a vector field $w : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, measurable in time and Lipschitz in space as the application $(x^0, t) \mapsto \Phi_t^w(x^0)$ such that, for all $x^0 \in \mathbb{R}^d$, $t \mapsto \Phi_t^w(x^0)$ is the solution to

$$\begin{cases} \dot{x}(t) = w(x(t), t) \text{ for a.e. } t \geqslant 0, \\ x(0) = x^0. \end{cases} \tag{12}$$

eq charac

(For such velocity, System (12) is well-posed. See for instance [6].)

We denote by Γ the set of the Borel maps $\gamma : \mathbb{R}^d \to \mathbb{R}^d$. We recall the definition of the *push-forward* of a measure.

Definition 4 For a $\gamma \in \Gamma$, we define the *push-forward* $\gamma \# \mu$ of a measure μ of $\mathbb{R}^d$ as follows:

$$(\gamma \# \mu)(E) := \mu(\gamma^{-1}(E)),$$

for every subset E such that $\gamma^{-1}(E)$ is μ-measurable.

We denote by "AC measures" the measures which are absolutely continuous with respect to the Lebesgue measure and by $\mathcal{P}_c^{ac}(\mathbb{R}^d)$ the subset of $\mathcal{P}_c(\mathbb{R}^d)$ of AC measures. We now recall a standard result linking (12) and (11), known as the method of characteristics.

Theorem 4 (see [35, Th. 5.34]) *Let $T > 0$, $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$ and w a vector field uniformly bounded, Lipschitz in space and measurable in time. Then, System (11) admits a unique solution μ in $\mathcal{C}^0([0,T]; \mathcal{P}_c(\mathbb{R}^d))$, where $\mathcal{P}_c(\mathbb{R}^d)$ is equipped with the weak topology. Moreover:*

(i) it holds $\mu(\cdot, t) = \Phi_t^w \# \mu^0$;
(ii) if $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$, then $\mu(\cdot, t) \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$.

We also recall the following property which will be useful in the rest of the paper.

Property 3 *Let μ, $\nu \in \mathcal{P}_c(\mathbb{R}^d)$ and $w : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be a vector field uniformly bounded, Lipschitz in space and measurable in time with a Lipschitz constant equal to L. For each $t \in \mathbb{R}$ and $p \in [1, \infty)$, it holds*

$$W_p(\Phi_t^w \# \mu, \Phi_t^w \# \nu) \leqslant e^{L|t|} W_p(\mu, \nu). \tag{13}$$

ine wasser 2

Proof Consider π a minimiser of the Wasserstein distance (8) between μ and ν. Then $(\Phi_t^v, \Phi_t^v)\#\pi \in \Pi(\Phi_t^v \# \mu, \Phi_t^v \# \nu)$ and it holds by Gronwall's Lemma

$$W_p^p(\Phi_t^v \# \mu, \Phi_t^v \# \nu) \leqslant \int_{\mathbb{R}^d} |\Phi_t^v(x) - \Phi_t^v(y)|^p d\pi(x,y) \leqslant e^{pL|t|} \int_{\mathbb{R}^d} |x - y|^p d\pi(x,y).$$

□

We now give the precise notions of approximate controllability for System (1) and System (2) in terms of the Wasserstein distance.

⟨def:approx⟩ **Definition 5** We say that

- System (1) is **approximately controllable** from X^0 to X^1 at time T $(0,T)$ if for each $\varepsilon > 0$ there exists a control $\mathbb{1}_\omega u$ uniformly bounded, Lipschitz in space and measurable in time such that the corresponding solution X to System (1) satisfies

$$d_p(X^1, X(T)) \leqslant \varepsilon. \tag{14}$$ estim Wp approx bis

- System (2) is **approximately controllable** from μ^0 to μ^1 at time T if for each $\varepsilon > 0$ there exists a control $\mathbb{1}_\omega u$ uniformly bounded, Lipschitz in space and measurable in time such that the corresponding solution μ to System (2) satisfies

$$W_p(\mu^1, \mu(T)) \leqslant \varepsilon. \tag{15}$$ estim Wp approx ter

Remark 2 Properties (10) imply that all the Wasserstein distances W_p are equivalent for measures compactly supported and $p \in [1, \infty)$, see [35]. Thus, combining with Remark 1, we can replace (14) and (15) by

$$W_1(\mu^1, \mu(T)) \leqslant \varepsilon,$$

where in the discrete case $\mu^1 := \sum_i \frac{1}{n}\delta_{x_i^1}$ and μ is the solution to System (2) associated to $\mu^0 := \sum_i \frac{1}{n}\delta_{x_i^0}$.

Thus, in this work, we study approximate controllability by considering the distance W_1 only. We will use the distances W_2 and W_∞, in some other specific cases.

We now recall the results obtained in [19] concerning the approximate and exact controllability of System (2) :

Theorem 5 (see [19]) *Let μ^0, μ^1 be two probability measures on $\mathbb{R}^d$ compactly supported, absolutely continuous with respect to the Lebesgue measure and satisfying Condition 1. Then there exists T such that System (2) is **approximately controllable** on the time interval $[0, T]$ from μ^0 to μ^1 with a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time.*

Theorem 6 (see [19]) *Let μ^0, μ^1 be two probability measures on $\mathbb{R}^d$ compactly supported and satisfying Condition 1. Then, there exists $T > 0$ such that System (2) is **exactly controllable** on the time interval $[0, T]$ from μ^0 to μ^1 in the following sense: there exists a couple $(\mathbb{1}_\omega u, \mu)$ composed of a Borel vector field $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ and a time-evolving measure μ being weak solution to System (2) and satisfying $\mu(T) = \mu_1$.*

3 Infimum time in the discrete case

time finite dim⟩ In this section, we prove Theorem 1 and 2, *i.e.* the infimum time for the approximate and exact controllability in the discrete case. We first obtain the following result:

⟨prop: dim finie⟩ **Proposition 1** *Let $X^0 := \{x_1^0, ..., x_n^0\} \subset \mathbb{R}^d$ and $X^1 := \{x_1^1, ..., x_n^1\} \subset \mathbb{R}^d$ two distinct configurations (see* (3)*) satisfying Condition 1. Consider the sequences $\{t_i^0\}_i$ and $\{t_i^1\}_i$ given in* (4)*. Then*

$$\widetilde{M}_e(X^0, X^1) = \min_{\sigma \in S_n} \max_{i \in \{1,\ldots,n\}} |t_i^0 + t_{\sigma(i)}^1| \tag{16}$$ minimal time

is the infimum times $T_e(X^0, X^1)$ to exactly control System (1) *in the following sense:*

(i) For each $T > \widetilde{M}_e$, System (1) *is exactly controllable from X^0 to X^1 at time T, i.e. there exists a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time steering X^0 exactly to X^1. Moreover, at each time $t \in [0, T]$, the configuration is distinct, i.e. $x_i(t) \neq x_j(t)$ for all $i \neq j$.*
(ii) For each $T \in (M_e^*, \widetilde{M}_e]$, System (1) *is not exactly controllable from X^0 to X^1.*
(iii) There exists (at most) a finite number of time $T \in [0, M_e^*]$ for which System (1) *is exactly controllable from X^0 to X^1.*

Proof Let $T := \widetilde{M}_e(X^0, X^1) + \delta$ with $\delta > 0$. Using Condition 1, for all $i \in \{1, ..., n\}$, there exist $s_i^0 \in (t_i^0, t_i^0 + \delta/3)$ and $s_i^1 \in (t_i^1, t_i^1 + \delta/3)$ such that

$$y_i^0 := \Phi_{s_i^0}^v(x_i^0) \in \omega \text{ and } y_i^1 := \Phi_{-s_i^1}^v(x_i^1) \in \omega.$$

The proof is divided into two steps:

(i) In Step 1, we build a permutation σ and a flow on ω sending y_i^0 to $y_{\sigma(i)}^1$ for all $i \in \{1, ..., n\}$ with no intersection of these trajectories.
(ii) In Step 2, we deduce a control sending x_i^0 to $x_{\sigma(i)}^1$ for all $i \in \{1, ..., n\}$.

Step 1: Consider a permutation σ minimizing (16). The goal is to build a flow without intersection of the characteristic. For all $i, j \in \{1, ..., n\}$, we define the cost

$$K_{ij} := \begin{cases} \|(y_i^0, s_i^0) - (y_j^1, T - s_j^1)\|_{\mathbb{R}^{d+1}} & \text{if } s_i^0 < T - s_j^1, \\ \infty & \text{otherwise.} \end{cases}$$

Consider the minimization problem:

$$\inf_{\pi \in \mathcal{B}_n} \frac{1}{n} \sum_{i,j=1}^{n} K_{ij} \pi_{ij}, \tag{17}$$ eq:inf

where $\mathcal{B}_n$ is the set of the bistochastic $n \times n$ matrices, *i.e.* the matrices $\pi := (\pi_{ij})_{1 \leqslant i,j \leqslant n}$ satisfying, for all $i, j \in \{1, ..., n\}$, $\sum_{i=1}^n \pi_{ij} = 1$, $\sum_{j=1}^n \pi_{ij} = 1$, $\pi_{ij} \geqslant 0$. Using the definition of $\widetilde{M}_e(X^0, X^1)$, the infimum in (17) is attained. It is a linear minimization problem on the closed convex set $\mathcal{B}_n$. Hence, as a consequence of Krein-Milman's Theorem (see [26]), the functional (17) admits a minimum at a extremal point, *i.e.* a permutation matrix. Let σ be a permutation, for which the associated matrix minimizes (17). Consider the linear applications y_i equal to y_i^0 at time s_i^0 and to $y^1_{\sigma(i)}$ at time $T - s^1_{\sigma(i)}$ defined by

$$y_i(t) := \frac{T - s^1_{\sigma(i)} - t}{T - s^1_{\sigma(i)} - s_i^0} y_i^0 + \frac{t - s_i^0}{T - s^1_{\sigma(i)} - s_i^0} y^1_{\sigma(i)}.$$

We now prove by contradiction that these trajectories have no intersection: Assume that there are i and j such that the associated trajectories $y_i(t)$ and $y_j(t)$ intersect. If we associate y_i^0 and y_j^0 to $y^0_{\sigma(j)}$ and $y^0_{\sigma(i)}$ respectively, *i.e.* we consider the permutation $\sigma \circ \mathcal{T}_{i,j}$, where $\mathcal{T}_{i,j}$ is the transposition between the i^{th} and the j^{th} elements, then the associated cost (17) is strictly smaller than the cost associated to σ. Indeed, using some geometrical considerations (see Figure 2), we obtain

$$\begin{cases} \|(y_i^0, s_i^0) - (y^1_{\sigma(j)}, T - s^1_{\sigma(j)})\| < \|(y_i^0, s_i^0) - (y^1_{\sigma(i)}, T - s^1_{\sigma(i)})\|, \\ \|(y_j^0, s_j^0) - (y^1_{\sigma(i)}, T - s^1_{\sigma(i)})\| < \|(y_j^0, s_j^0) - (y^1_{\sigma(j)}, T - s^1_{\sigma(j)})\|. \end{cases}$$

This is in contradiction with the fact that σ minimizes (17).

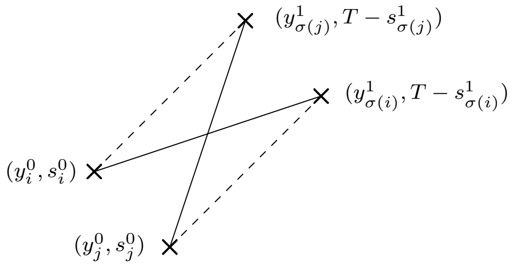


Fig. 2 Optimal permutation

⟨fig:geo⟩

Step 2: Consider a flow z_i satisfying:

$$z_i(t) := \begin{cases} \Phi_t^v(x_i^0) & \text{for all } t \in (0, s_i^0), \\ y_i(t) & \text{for all } t \in (s_i^0, T - s^1_{\sigma(i)}), \\ \Phi^v_{t-T}(x_i^1) & \text{for all } t \in (T - s^1_{\sigma(i)}, T). \end{cases}$$

The applications z_i have no intersection. Since ω is convex, then, using the definition of the application y_i, the points $y_i(t)$ belong to ω for all $t \in (s_i^0, T - s^1_{\sigma(i)})$. For all $i \in \{1, ..., n\}$, consider $R_i > r_i > 0$ such that for all $t \in (s_i^0, T - s^1_{\sigma(i)})$

$$B_{r_i}(z_i(t)) \subset B_{R_i}(z_i(t)) \subset \omega$$

and, for all $t \in (0, T)$ and $i, j \in \{1, ..., n\}$,

$$B_{R_i}(z_i(t)) \cap B_{R_j}(z_j(t)) = \varnothing.$$

The corresponding control can be chosen as a $\mathcal{C}^\infty$ function satisfying

$$u(x,t) := \begin{cases} \dfrac{y^1_{\sigma(i)} - y^0_i}{T - s^1_{\sigma(i)} - s^0_i} & \text{if } t \in (s^0_i, T - s^1_{\sigma(i)}) \text{ and } x \in B_{r_i}(z_i(t)), \\ 0 & \text{if } t \in (s^0_i, T - s^1_{\sigma(i)}) \text{ and } x \notin B_{R_i}(z_i(t)). \end{cases}$$

Assume now that System (1) is exactly controllable at a time $T \geqslant M^*_e$. Consider σ the corresponding permutation satisfying $x_i(T) = x^1_{\sigma(i)}$. Since $T \geqslant M^*_e$, then t^0_i and t^1_i are finite and

$$T > |t^0_i + t^1_{\sigma(i)}|$$

for all $i \in \{1, ..., n\}$. Thus, using the definition of $\widetilde{M}_e(X^0, X^1)$, it holds $T > \widetilde{M}_e(X^0, X^1)$.

We now prove Item (iii). Consider a sequence $\{T_k\}_{k\in\mathbb{N}} \subset (0, T)$ of distinct times at which System (1) is exactly controllable. There exists $l \in \{0, 1\}$ and $m \in \{1, ..., n\}$ such that $t^l(x^l_m) = M^*_e$. Assume for the moment that $l = 0$. Since X^1 is distinct, for a $R > 0$, it holds

$$B_R(x^1_i) \cap B_R(x^1_j) = \varnothing, \tag{18}$$ B_R

for each $i \neq j$. Since System (1) is exactly controllable at time T_k, there exists $m_k \in \{1, ..., n\}$ and a control u_k for which

$$\Phi^{v+\mathbb{1}_\omega u_k}_{T_k}(x^0_m) = x^1_{m_k}.$$

Since $T_k < T < M^*_e = t^0_m$, then $x^1_{m_k} \notin \omega$. Thus, it holds

$$\Phi^{v+\mathbb{1}_\omega u_k}_{t}(x^0_m) \in B_R(x^1_{m_k}),$$

for all $t \in (T_k - R/\sup|v|, T_k + R/\sup|v|)$. Equality (18) implies that System (1) is not exactly controllable for all $T \in (T_k - R/\sup|v|, T_k + R/\sup|v|)$. We obtain a contradiction with the fact that the interval $(0, M^*_e)$ is bounded (see Condition 1). The proof is similar in the case $l = 1$, since the equation is reversible in time. □

Remark 3 Proposition 1 can be interpreted as follows: Each particle at point x^0_i needs to be sent on a target point $x^1_{\sigma(i)}$. $\widetilde{M}_e(X^0, X^1)$ is larger than the infimum time for the particle at x^0_i to enter in ω and then go from ω to $x^1_{\sigma(i)}$. We are thus assuming that the particle travels with a quasi infinite velocity in ω.

⟨rmq:T2*⟩ *Remark 4* We illustrate Item (iii) of Theorem 1 by giving an example in which System (1) is never exactly controllable on $(0, M^*_e)$ and another where System (1) is exactly controllable at a time $T \in (0, M^*_e)$:

- Consider $\omega := (-1,1) \times (-1.5, 1.5)$, $v := (1,0)$, $X^0 := (-2,0)$ and $X^1 := (2,0)$ (see Figure 3 (left)). The time $M_e^*(X^0, X^1)$ at which we can act on the particles and the minimal time $M_e(X^0, X^1)$ are respectively equal to 1 and 2. We observe that System (1) is never exactly controllable or approximately controllable on the interval $(0, M_e^*)$.
- Consider $\omega := (-1,1) \times (-1.5, 1.5)$, v be a vector field equal to $(-y, x)$ at each point (x, y) of the circle centred at $(1,0)$ of radius 1 and X^0, X^1 given by

$$\begin{cases} X^0 := \{(1+\sqrt{2}/2, -\sqrt{2}/2)\}, \\ X^1 := \{(1+\sqrt{2}/2, -\sqrt{2}/2)\}. \end{cases}$$

  See Figure 3 (right). The time $M_e^*(X^0, X^1)$ at which we can act on the particles and the minimal time $M_e(X^0, X^1)$ are respectively equal to $3\pi/4$ and π. We remark that for $T := \pi/2 \in (0, M_e^*(X^0, X^1))$, System (1) is exactly controllable.

These two examples illustrate also Item (iii) of Theorem 2 concerning the approximate controllability of System (1).

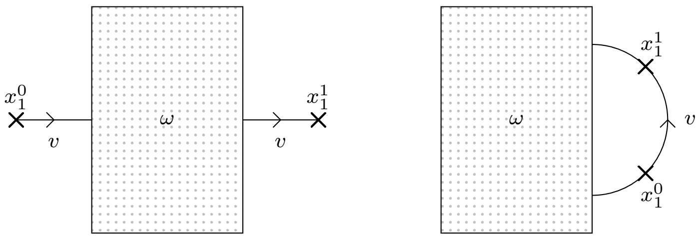


Fig. 3 Examples in the case $T \in (0, M_e^*)$.
⟨fig:ex (0,T*)⟩

Formula (16) leads to the proof of Theorem 1.

Proof (of Theorem 1) Consider $\widetilde{M}_e(X^0, X^1)$ given in (16). We assume that the sequence $\{t_i^0\}_{i\in\{1,...,n\}}$ is increasingly ordered. Let σ_0 be a minimizing permutation in (16). We build recursively a sequence of permutations $\{\sigma_1, ..., \sigma_n\}$ as follows:

- Let k_1 be such that $t^1_{\sigma_0(k_1)}$ is one of the maximisers of $\{t^1_{\sigma_0(1)}, ..., t^1_{\sigma_0(n)}\}$. We denote by $\sigma_1 := \sigma_0 \circ \mathcal{T}_{1,k_1}$, where, for all $i, j \in \{1, ..., n\}$, $\mathcal{T}_{i,j}$ is the transposition between the i^{th} and the j^{th} elements. As illustrated in Figure 4, we clearly have

$$\begin{cases} t^0_{k_1} + t^1_{\sigma_0(k_1)} \geqslant t^0_1 + t^1_{\sigma_0(1)}, \\ t^0_{k_1} + t^1_{\sigma_0(k_1)} \geqslant t^0_1 + t^1_{\sigma_1(1)}, \\ t^0_{k_1} + t^1_{\sigma_0(k_1)} \geqslant t^0_{k_1} + t^1_{\sigma_1(k_1)}. \end{cases}$$

Thus σ_1 minimizes (16) too:

$$\max_{i\in\{1,\ldots,n\}}\{t_i^0+t^1_{\sigma_0(i)}\}\geqslant\max_{i\in\{1,\ldots,n\}}\{t_i^0+t^1_{\sigma_1(i)}\}.$$

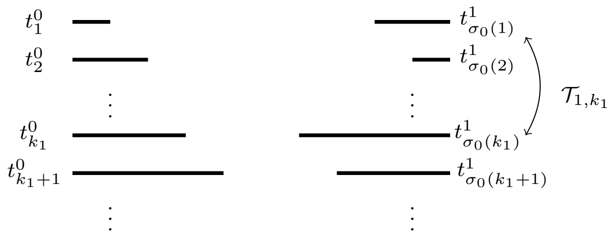


Fig. 4 Computation of the infimum time

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- Assume that σ_j is built. Let k_{j+1} be such that $t^1_{\sigma_j(k_{j+1})}$ is a maximizer of $\{t^1_{\sigma_j(j+1)},\ldots,t^1_{\sigma_j(n)}\}$. Again, we clearly have

$$\begin{cases} t^0_{k_{j+1}}+t^1_{\sigma_j(k_{j+1})}\geqslant t^0_{j+1}+t^1_{\sigma_j(j+1)},\\ t^0_{k_{j+1}}+t^1_{\sigma_j(k_{j+1})}\geqslant t^0_{j+1}+t^1_{\sigma_{j+1}(j+1)},\\ t^0_{k_{j+1}}+t^1_{\sigma_j(k_{j+1})}\geqslant t^0_{k_{j+1}}+t^1_{\sigma_{j+1}(k_{j+1})}.\end{cases}$$

Thus $\sigma_{j+1}:=\sigma_j\circ\mathcal{T}_{j+1,k_{j+1}}$ minimizes (16) too:

$$\max_{i\in\{1,\ldots,n\}}\{t_i^0+t^1_{\sigma_j(i)}\}\geqslant\max_{i\in\{1,\ldots,n\}}\{t_i^0+t^1_{\sigma_{j+1}(i)}\}.$$

The sequence $\{t^1_{\sigma_n(1)},\ldots,t^1_{\sigma_n(n)}\}$ is then decreasing and σ_n is a minimizing permutation in (16). We deduce that $\widetilde{M}_e(X^0,X^1)=M_e(X^0,X^1)$. $\square$

With Theorem 1, we give an explicit and simple expression of the infimum time. This result is useful in numerical simulations of Section 5 (in particular, see Algorithm 1).

Let μ^0 and μ^1 be two probability measures given by

$$\mu^0:=\sum_{i=1}^n\frac{1}{n}\delta_{x_i^0}\text{ and }\mu^1:=\sum_{i=1}^n\frac{1}{n}\delta_{x_i^1},\tag{19}$$ def mu dim finie

where the points x_i^0 (resp. x_i^1) are disjoint. We now deduce Theorem 3 in the discrete case, *i.e.* for μ^0, μ^1 given in (19).

⟨cor:disc Fi⟩ **Corollary 1** *Let $X^0:=\{x_1^0,\ldots,x_n^0\}\subset\mathbb{R}^d$ and $X^1:=\{x_1^1,\ldots,x_n^1\}\subset\mathbb{R}^d$. Assume Condition 1 is satisfied and the points x_i^0 (resp. x_i^1) are disjoint. Consider μ^0 and μ^1 the corresponding measures given in* (19). *Then the infimum time $T_e(X^0,X^1)$ is equal to $S(\mu^0,\mu^1)$ given in Theorem 3.*

Proof Remark that if the sequences $\{t_i^0\}_{i\in\{1,...n\}}$ and $\{t_i^1\}_{i\in\{1,...n\}}$ are increasingly and decreasingly ordered respectively, then for all $m \in \left(\frac{i-1}{n}, \frac{i}{n}\right)$ it holds

$$\begin{cases} \mathcal{F}_0^{-1}(m) = t_i^0, \\ \mathcal{F}_1^{-1}(1-m) = t_i^1. \end{cases}$$

□

We now prove Theorem 2 which characterize the infimum time to approximatively control System (1).

Proof (of Theorem 2) Consider $X^0 := \{x_1^0, ..., x_n^0\}$ and $X^1 := \{x_1^1, ..., x_n^1\}$. We first prove that the infimum time $T_a(X^0, X^1)$ is equal to

$$\widetilde{M}_a(X^0, X^1) = \min_{\sigma\in S_n} \max_{i\in\{1,...,n\}} |t_i^0 + \bar{t}_{\sigma(i)}^1|.$$

Assume first that $T > \widetilde{M}_a$. Let $\varepsilon > 0$. By continuity of the flow, for all $i \in \{1, ..., n\}$, there exists $y_i^1 \in \mathbb{R}^d$ such that it holds

$$\|y_i^1 - x_i^1\|_1 \leqslant \varepsilon \text{ and } y_i := \Phi^v_{-\bar{t}_i^1}(y_i^1) \in \omega.$$

We denote by $Y^1 := \{y_1^1, ..., y_n^1\}$. For all $i \in \{1, ..., n\}$, since $y_i \in \omega$, then $t^1(y_i^1) \leqslant \bar{t}_i^1$ and

$$\widetilde{M}_e(X^0, Y^1) \leqslant \widetilde{M}_a(X^0, X^1) < T.$$

Theorem 1 implies that we can exactly steer X^0 to Y^1 at time T with a control u Lipschitz in space, measurable in time and uniformly bounded. Consider X the solution to System (1) for the initial condition X^0 and the control u. It holds

$$d_1(X^1, X(T)) = d_1(X^1, Y^1) \leqslant \sum_{i=1}^n \frac{1}{n} \|\Phi^v_{\bar{t}_i^1}(y_i^1) - x_i^1\| \leqslant \varepsilon.$$

We deduce that we can approximately steer X^0 to X^1 at time T.

Consider now a time $T > M_a^* := \max\{t_i^0, \bar{t}_i^1 : i = 1, ..., n\}$ at which System (1) is approximately controllable. For all $k \in \mathbb{N}^*$, there exists a control u_k Lipschitz in space, measurable in time and uniformly bounded such that the corresponding solution X_k to System (2) satisfies

$$d_1(X^1, X_k(T)) \leqslant 1/k.$$

We denote by $Y_k^1 := \{y_{k,1}^1, ..., y_{k,n}^1\}$, where $y_{k,i}^1 := x_{k,i}(T)$. Notice that we can exactly steer X^0 to Y_k^1. To apply Theorem 1, we need to prove that

$$T > M_e^*(X^0, Y_k^1). \tag{20}$$

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As $T > M_a^* := \max\{t_i^0, \bar{t}_i^1 : i = 1, ..., n\}$ it only remains to prove that $T > \max\{t_i^1(y_{k,i}) : i = 1, ..., n\}$. By contradiction, assume that there is $j \in \{1, \ldots, n\}$ such that $t^1(y_{k,j}^1) \geq T$. We distinguish two cases:

- If $t^1(y^1_{k,j}) > T$, then for any $t \in [0,T]$, $\Phi^v_{-t}(y^1_{k,j}) \notin \omega$. Thus, for each $t \in [0,T]$,
$$\Phi^v_{-t}(y^1_{k,j}) = \Phi^{v+\mathbb{1}_\omega u_k}_{-t}(y^1_{k,j}).$$
Noticing that $\Phi^{v+\mathbb{1}_\omega u_k}_{-T}(y^1_{k,j}) = x^0_j$ and $t^0(x^0_j) < T$ this leads to a contradiction.
- Similarly, $t^1(y^1_{k,j}) = T$ implies that $x^0_j \in \omega$ but $\Phi^v_t(x^0_j) \notin \omega$ for $t \in (0,T]$. As ω is an open set and $\Phi^v_\cdot(x^0_j)$ is continuous this is also a contradiction.

Thus (20) holds and Theorem 1 implies that
$$T > \widetilde{M}_e(X^0, Y^1_k). \tag{21}$$
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Denote by σ_k the permutation corresponding to u_k. Up to extract a subsequence, for all k large enough, σ_k is equal to a permutation σ. We deduce that for all $i \in \{1, ..., n\}$
$$y^1_{k,i} \underset{k\to\infty}{\longrightarrow} x^1_{\sigma(i)}. \tag{22}$$
y_k,i

Since $t^1(y^1_{k,i}) \leqslant \widetilde{M}_e(X^0, Y^1_k) < T$, up to a subsequence, for a $s_i \geqslant 0$, it holds
$$t^1(y^1_{k,i}) \underset{k\to\infty}{\longrightarrow} s_i. \tag{23}$$
t_k,i

Using (22), (23) and the continuity of the flow, it holds
$$\begin{aligned}&|\Phi^v_{-t^1(y^1_{k,i})}(y^1_{k,i}) - \Phi^v_{-s_i}(x^1_{\sigma(i)})|\\ &\quad\leqslant |\Phi^v_{-t^1(y^1_{k,i})}(y^1_{k,i}) - \Phi^v_{-s_i}(y^1_{k,i})| + |\Phi^v_{-s_i}(y^1_{k,i}) - \Phi^v_{-s_i}(x^1_{\sigma(i)})| \underset{k\to\infty}{\longrightarrow} 0.\end{aligned}$$
The fact that $\Phi^v_{-t^1(y^1_{k,i})}(y^1_{k,i}) \in \overline{\omega}$ leads to
$$\Phi^v_{-s_i}(x^1_{\sigma(i)}) \in \overline{\omega}.$$
Thus
$$\bar{t}^1(x^1_{\sigma(i)}) \leqslant \lim_{k\to\infty} t^1(y^1_{k,\sigma(i)}).$$
Combining with (21), we obtain
$$T > \widetilde{M}_a(X^0, X^1).$$
Applying the permutation method used in the proof of Theorem 1, it holds
$$\widetilde{M}_a(X^0, X^1) = M_a(X^0, X^1).$$

Finally let us prove the third item of Theorem 2. Let $T \in (0, M^*_a)$ be such that System (1) is approximately controllable.

Assume that there exists $j \in \{1, \ldots, n\}$ such that $t^0(x^0_j) = M^*_a > T$. For any $\varepsilon > 0$, there exists u_ε such that the associated trajectory to System (1) satisfies
$$d_1(X(T), X^1) < \varepsilon.$$

Thus there exists $k \in \{1, \dots, n\}$ such that

$$\|x_j(T) - x_k^1\| < \varepsilon. \tag{24}$$ proche_cible

As $t^0(x_j^0) > T$, it comes that $x_j(T) = \Phi_T^v(x_j^0)$ is independent of ε. Considering only

$$\varepsilon \in \left(0, \frac{1}{2} \min \left\{ \|x_p^1 - x_q^1\| : p, q \in \{1, \dots, n\} \right\} \right)$$

implies that k is also independent of ε. Thus for ε sufficiently small, the estimate (24) gives $x_j(T) = x_k^1$. Using the fact that $x_j(t) = \Phi_t^v(x_j^0)$ for all $t \in [0, M_a^*)$ together with Condition 1, we obtain that the velocity field v does not vanish on the trajectory $\{x_j(t) : t \in [0, M_a^*)\}$. This implies that

$$\{t \in [0, M_a^*) : x_j(t) = x_k\}$$

is finite.

The other situation is dealt with in the exact same way. Indeed if there exists $j \in \{1, \dots, n\}$ such that $M_a^* = \bar{t}^1(x_j^1)$ then $t^1(x_j^1) \geq \bar{t}^1(x_j^1) > T$. □

⟨rmq: ex CA CE⟩ *Remark 5* (Example of difference approximate control / exact control) Consider $X^0 := \{(-2, -2)\}$, $X^1 := \{(2, -2)\} \subset \mathbb{R}^2$, $\omega := (-1, 1) \times (3, 3)$ and v a vector field satisfying

$$v(x, y) = \begin{cases} (y + 2, -x - 1) & \text{if } (x + 2)^2 + (y + 1)^2 = 1, \\ (-y - 2, x - 1) & \text{if } (x + 2)^2 + (y - 1)^2 = 1, \\ (1, 0) & \text{if } (x, y) \in (-2, 2) \times \{2\}, \\ (-y + 2, x - 1) & \text{if } (x - 2)^2 + (y - 1)^2 = 1, \\ (y - 2, -x - 1) & \text{if } (x - 2)^2 + (y + 1)^2 = 1. \end{cases}$$

This situation is illustrated in Figure 5. The two quantities $M_e(X^0, X^1)$ and $M_a(X^0, X^1)$, given in Theorem 1 and 2, are respectively equal to $2 + 4\pi$ and $1 + 9\pi/4$. Thus, in this example, the notion of exact and approximate controllability are not equivalent.

4 Infimum time for absolutely continuous measures

ptimal time cont⟩ This section is devoted to the proof of main Theorem 3 about infimum time for AC measures. We first introduce the notion of infimum time up to small mass in Section 4.1. We then give some comparisons between the discrete and continuous case in Section 4.2 and we finally use the obtained results to prove Theorem 3 in Section 4.3.

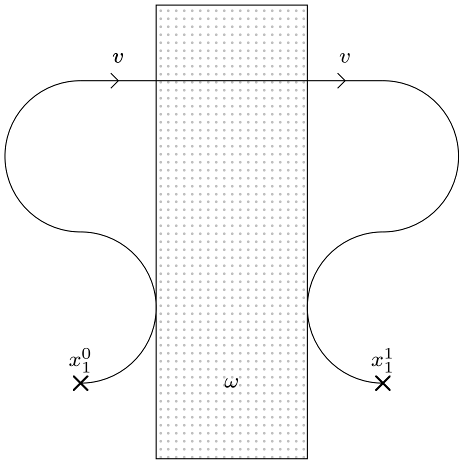


Fig. 5 Example in which the exact and approximate controllability are not equivalent.
⟨fig:ex CE CA⟩

4.1 Infimum time in the discrete setting up to small masses

⟨sec:up mass⟩ In this section, we introduce the notion of infimum time up to small mass and prove some results concerning this notion in the discrete case. Let $M > 0$ be a positive mass, not necessarily 1, and μ^0, μ^1 be two measures given by

$$\mu^0 := \sum_{i=1}^{n} \frac{M}{n}\delta_{x_i^0} \text{ and } \mu^1 := \sum_{i=1}^{n} \frac{M}{n}\delta_{x_i^1} \tag{25}$$
def mu dim finie m

and satisfying (3). It is possible to compute the infimum time to steer μ^0 to μ^1 up to a small mass.

Definition 6 (Infimum time up to small mass) Let $X^0 := \{x_1^0, ..., x_n^0\} \subset \mathbb{R}^d$ and $X^1 := \{x_1^1, ..., x_n^1\} \subset \mathbb{R}^d$ be two distinct configurations (see (3)) satisfying Condition 1. Let $M > 0$ and the corresponding measures μ^0 and μ^1 defined in (25). Fix $R \in \{1, ..., n\}$ and $\varepsilon := MR/n$. We define the infimum time $T_{e,\varepsilon}$ to exactly steer μ^0 to μ^1 (or X^0 to X^1) up to a mass ε (or R particles) as the infimum of time $T \geqslant M_e^*$ for which there exists a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time and $\sigma_0, \sigma_1 \in S_n$ such that for all $i \in \{1, ..., n-R\}$ it holds $x_{\sigma_0(i)}(T) = x^1_{\sigma_1(i)}$. We similarly define the minimal time to approximately steer μ^0 to μ^1 (or X^0 to X^1) up to a mass ε (or R particles).

We have the following result:

to mass discret⟩ **Proposition 2** *Let $X^0 := \{x_1^0, ..., x_n^0\} \subset \mathbb{R}^d$ and $X^1 := \{x_1^1, ..., x_n^1\} \subset \mathbb{R}^d$. Assume Condition 1 is satisfied and the points x_i^0 (resp. x_i^1) are disjoint. Let $R \in \{1, ..., n\}$ and $\varepsilon := MR/n$. Consider μ^0 and μ^1 the corresponding measures* *given in* (25)*. Assume that the sequences* $\{t_i^0\}_{i\in\{1,...n\}}$ *and* $\{t_i^1\}_{i\in\{1,...n\}}$ *are increasingly and decreasingly ordered respectively. The infimum time* $T_{e,\varepsilon}(\mu^0,\mu^1)$ *to exactly steer* μ^0 *to* μ^1 *up to a mass* ε *(or up to R particles) is equal to*

$$M_{e,\varepsilon}(X^0,X^1) := \max_{1\leq i\leq n-R}\{t^0(x_i^0)+t^1(x_{i+R}^1)\}.$$

Proof We can adapt the proof of Proposition 1 as follows. We first replace the bistochastic matrices in (17) by the matrices satisfying

$$\sum_i \pi_{ij} \leqslant 1, \quad \sum_j \pi_{ij} \leqslant 1, \quad \sum_{ij} \pi_{ij} = n-R \text{ and } \pi_{ij} \geqslant 0.$$

The set of such matrices is clearly closed and convex. The problem of minimizing $\sum_{i,j=1}^n K_{ij}\pi_{ij}$ is still linear, hence, again, as a consequence of Krein-Milman's Theorem (see [26]), some minimisers of this functional are extremal points, that are matrices composed of a permutation sub-matrix for some rows and columns, and zeros for other rows and columns. We then define

$$\widetilde{M}_{e,\varepsilon}(X^0,X^1) := \min_{\sigma_0,\sigma_1\in S_n}\max_{1\leq i\leq n-R}|t^0_{\sigma_0(i)}+t^1_{\sigma_1(i)}|.$$

By applying the permutation method of the proof of Theorem 1, we prove that $M_{e,\varepsilon}(X^0,X^1)$ is equal to $\widetilde{M}_{e,\varepsilon}(X^0,X^1)$. Indeed, if the sequences $\{t_i^0\}_{i\in\{1,...n\}}$ and $\{t_i^1\}_{i\in\{1,...n\}}$ are increasingly and decreasingly ordered respectively, then the optimality of (17) is reached by taking $\tilde{X}^0 = \{x_1^0,...,x_{n-R}^0\}$ and $\tilde{X}^1 = \{x_{R+1}^1,...,x_n^1\}$. □

Proposition 2 can be adapted to the case of the approximate controllability:

Proposition 3 *Let* $X^0 := \{x_1^0,...,x_n^0\} \subset \mathbb{R}^d$ *and* $X^1 := \{x_1^1,...,x_n^1\} \subset \mathbb{R}^d$*. Assume Condition 1 is satisfied and the points* x_i^0 *(resp.* x_i^1*) are disjoint. Let* $R\in\{1,...,n\}$ *and* $\varepsilon := MR/n$*. Consider* μ^0 *and* μ^1 *the corresponding measures given in* (25)*. Assume that the sequences* $\{t_i^0\}_{i\in\{1,...n\}}$ *and* $\{t_i^1\}_{i\in\{1,...n\}}$ *are increasingly and decreasingly ordered respectively. The infimum time* $T_{a,\varepsilon}(\mu^0,\mu^1)$ *to approximately steer* X^0 *to* X^1 *up to a mass* ε *(or up to R particles) is equal to*

$$M_{a,\varepsilon}(X^0,X^1) := \max_{1\leq i\leq n-R}\{t^0(x_i^0)+\bar{t}^1(x_{i+R}^1)\}.$$

As in the proof of Corollary 1, we use the definition of $\mathcal{F}_0$, $\mathcal{F}_1$, t_i^0 and t_i^1, together with applying Theorem 1 to suitable subsets of X^0, X^1, to have the following result.

⟨corol T0eps⟩ **Corollary 2** *Let* $X^0 := \{x_1^0,...,x_n^0\} \subset \mathbb{R}^d$ *and* $X^1 := \{x_1^1,...,x_n^1\} \subset \mathbb{R}^d$ *satisfying Condition 1 and the points* x_i^0 *(resp.* x_i^1*) are disjoint. Let* μ^0 *and* μ^1 *be the measures given in* (25)*. Fix* $\varepsilon := MR/n$ *with* $R\in\{1,...,n\}$*. The infimum time* $T_{e,\varepsilon}$ *to exactly steer* μ^0 *to* μ^1 *up to a mass* ε *is equal to*

$$S_\varepsilon(\mu^0,\mu^1) := \sup_{m\in[0,M-\varepsilon]}\{\mathcal{F}_0^{-1}(m)+\mathcal{F}_1^{-1}(M-\varepsilon-m)\}, \tag{26}$$ def T0eps

where $\mathcal{F}_0^{-1}$ *and* $\mathcal{F}_1^{-1}$ *are given in* (6)*.*

4.2 Comparison of the continuous and discrete case

⟨sec: comp⟩ We compare here the discrete and continuous situation in Proposition 4. The results of the previous section and this one will be used in the proof of Theorem 3 in Section 4.3. Since we will not always consider the same control region, it will be specified in the different notations.

⟨prop cont T0⟩ **Proposition 4** *Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two real positive sequences converging to zero and for all $n \in \mathbb{N}$ let ω_n be defined by*

$$\omega_n := \{x \in \mathbb{R}^d : d(x, \omega^c) > f_n\}. \tag{27}$$ omega n

Consider $\mu^0, \mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ satisfying Condition 1 with respect to ω, some sequences $\{\mu_n^0\}_{n\in\mathbb{N}}$, $\{\mu_n^1\}_{n\in\mathbb{N}}$ of measures compactly supported satisfying Condition 1 with respect to ω and two sequences of sets $\{R_n^0\}_{n\in\mathbb{N}}$, $\{R_n^1\}_{n\in\mathbb{N}}$ of $\mathbb{R}^d$ such that

$$\begin{cases} r_n := \mu^0(R_n^0) = \mu^1(R_n^1) \underset{n\to\infty}{\longrightarrow} 0, \\ \mu_n^0(\mathbb{R}^d) = \mu_n^1(\mathbb{R}^d) = 1 - r_n, \\ W_\infty(\mu^0_{|(R_n^0)^c}, \mu_n^0) < g_n \underset{n\to\infty}{\longrightarrow} 0, \\ W_\infty(\mu^1_{|(R_n^1)^c}, \mu_n^1) < g_n \underset{n\to\infty}{\longrightarrow} 0. \end{cases}$$

Consider the quantity S_ε given in (26). Then for all $\varepsilon, \delta > 0$, there exists $N \in \mathbb{N}^*$ such that for all $n \geq N$, it holds

(i) $S_{2\varepsilon}(\mu_n^0, \mu_n^1, \omega_n) \leqslant S_\varepsilon(\mu^0, \mu^1, \omega) + \delta$.
(ii) $S_{2\varepsilon}(\mu^0, \mu^1, \omega) \leqslant S_\varepsilon(\mu_n^0, \mu_n^1, \omega_n) + \delta$.
(iii) There exist some Borel set $E_{0,n}, E_{1,n} \subset \mathbb{R}^d$ such that

$$\begin{cases} \mu_n^0(E_{0,n}), \mu_n^1(E_{1,n}) \underset{n\to\infty}{\longrightarrow} 1, \\ M_e^*(\mu^0_{n|E_{0,n}}, \mu^1_{n|E_{1,n}}, \omega_n) \leqslant S^*(\mu^0, \mu^1, \omega) + \delta. \end{cases}$$

Proof We first prove Item (i). Consider $h_n := \max\{f_n, g_n\}$ for all $n \in \mathbb{N}$. We define

$$S_n^0 := \{x^0 \in \operatorname{supp}(\mu^0) : \Phi_t^v(B_{h_n}(x^0)) \subset\subset \omega_n \text{ for a } t \leqslant t^0(x^0, \omega) + \delta/2\}.$$

We define $r_n^0 := \mu^0((S_n^0)^c)$ and prove that it holds

$$\lim_{n\to\infty} r_n^0 = 0. \tag{28}$$ pr T ep0

We prove it by contradiction. Assume that there exists a subsequence $\{k_n\}_{n\in\mathbb{N}}$ such that it holds $\mu^0((S_{k_n}^0)^c) > C$ for all $n \in \mathbb{N}^*$. Then there exists $x^0 \in \operatorname{supp}(\mu^0)$ such that

$$\Phi_t^v(B_{h_{k_n}}(x^0)) \not\subset \omega_{k_n}, \tag{29}$$ pr T ep1

for all $t \leqslant t^0(x^0, \omega) + \delta/2$ and $n \in \mathbb{N}^*$. But, since ω is open, for a $t^*(x^0) \in (t^0(x^0, \omega), t^0(x^0, \omega) + \delta/2)$ and a $r(x^0) > 0$, it holds

$$B_{r(x^0)}(\Phi^v_{t^*(x^0)}(x^0)) \subset\subset \omega.$$

By continuity of Φ_t^v, there exists $r(x^0) > 0$ such that

$$\Phi^v_{t^*(x^0)}(B_{r(x^0)}(x^0)) \subset\subset \omega. \qquad (30) \quad \boxed{\texttt{pr T ep2}}$$

Since (29) and (30) are in contradiction, for n large enough, we conclude that (28) holds. We deduce that, for all $x^0 \in S_n^0$,

$$\xi^0 \in \overline{B_{h_n}(x^0)} \Rightarrow t^0(\xi^0, \omega_n) \leqslant t^0(x^0, \omega) + \delta/2. \qquad (31) \quad \boxed{\texttt{lemme gamma1}}$$

For each $n \in \mathbb{N}^*$, consider an optimal transference plan π_n associated to the Wasserstein distance d_n (see Property 1). We remark that it holds

$$|x^0 - \xi^0| \leqslant g_n \leqslant h_n \qquad (32) \quad \boxed{\texttt{lemme gamma2}}$$

for π_n-almost every $x^0 \in \mathrm{supp}(\mu^0) \cap (R_n^0)^c \cap S_n^0$ and $\xi^0 \in \mathrm{supp}(\mu_n^0)$. Thus, combining (31) and (32), it holds

$$t^0(\xi^0, \omega_n) \leqslant t^0(x^0, \omega) + \delta/2 \qquad (33) \quad \boxed{\texttt{ine t0}}$$

for π_n-almost every $x^0 \in \mathrm{supp}(\mu^0) \cap (R_n^0)^c \cap S_n^0$ and $\xi^0 \in \mathrm{supp}(\mu_n^0)$. We consider

$$\begin{cases} \mathcal{F}_0(t) := \mu^0(\{x^0 \in \mathrm{supp}(\mu^0) : t^0(x^0, \omega) \leqslant t\}), \\ \mathcal{F}_1(t) := \mu^1(\{x^1 \in \mathrm{supp}(\mu^1) : t^1(x^1, \omega) \leqslant t\}), \\ \mathcal{F}_{0,n}(t) := \mu_n^0(\{x^0 \in \mathrm{supp}(\mu_n^0) : t^0(x^0, \omega_n) \leqslant t\}), \\ \mathcal{F}_{1,n}(t) := \mu_n^1(\{x^1 \in \mathrm{supp}(\mu_n^1) : t^1(x^1, \omega_n) \leqslant t\}), \end{cases}$$

for all $t \in \mathbb{R}^+$. Using (33), we obtain

$$\begin{aligned} \mathcal{F}_0(t) &\leqslant \mu^0(\{x^0 \in \mathrm{supp}(\mu^0) \cap (R_n^0)^c \cap S_n^0 : t^0(x^0, \omega) \leqslant t\}) + r_n + r_n^0 \\ &= \pi_n(\{x^0 \in \mathrm{supp}(\mu^0) \cap (R_n^0)^c \cap S_n^0, \xi^0 \in \mathrm{supp}(\mu_n^0) : \\ &\qquad\qquad t^0(x^0, \omega) \leqslant t\}) + r_n + r_n^0 \\ &\leqslant \pi_n(\{x^0 \in \mathrm{supp}(\mu^0) \cap (R_n^0)^c \cap S_n^0, \xi^0 \in \mathrm{supp}(\mu_n^0) : \\ &\qquad\qquad t^0(\xi^0, \omega_n) \leqslant t + \delta/2\}) + r_n + r_n^0 \\ &\leqslant \mu_n^0(\{\xi^0 \in \mathrm{supp}(\mu_n^0) : t^0(\xi^0, \omega_n) \leqslant t + \delta/2\}) + r_n + r_n^0 \\ &= \mathcal{F}_{0,n}(t + \delta/2) + r_n + r_n^0, \end{aligned}$$

for all $t \in \mathbb{R}^+$. Similarly, we have

$$\mathcal{F}_1(t) \leqslant \mathcal{F}_{1,n}(t + \delta/2) + r_n + r_n^1,$$

for all $t \in \mathbb{R}^+$, where r_n^1 is similarly defined. We deduce that the generalized inverse satisfies

$$\begin{aligned} \mathcal{F}_{0,n}^{-1}(m) &:= \inf\{t \geqslant 0 : \mathcal{F}_{0,n}(t) \geqslant m\} \\ &\leqslant \inf\{t \geqslant \delta/2 : \mathcal{F}_0(t - \delta/2) - r_n^0 - r_n \geqslant m\} \\ &= \inf\{s \geqslant 0 : \mathcal{F}_0(s) \geqslant m + r_n^0 + r_n\} + \delta/2 \\ &= \mathcal{F}_0^{-1}(m + r_n^0 + r_n) + \delta/2, \end{aligned}$$

for all $m \in (0, 1 - r_n^0 - r_n)$. Similarly, we obtain

$$\mathcal{F}_{1,n}^{-1}(1 - r_n - 2\varepsilon - m) \leqslant \mathcal{F}_1^{-1}(1 + r_n^1 - 2\varepsilon - m) + \delta/2,$$

for all $m \in (r_n^1 - 2\varepsilon, 1 - r_n - 2\varepsilon)$. For n large enough, we have

$$\begin{aligned}
S_{2\varepsilon}(\mu_n^0, \mu_n^1) &:= \sup_{m \in [0, 1 - r_n - 2\varepsilon]} \{\mathcal{F}_{0,n}^{-1}(m) + \mathcal{F}_{1,n}^{-1}(1 - r_n - 2\varepsilon - m)\} \\
&\leqslant \sup_{m \in [0, 1 - r_n - 2\varepsilon]} \{\mathcal{F}_0^{-1}(m + r_n^0 + r_n) + \\
&\qquad\qquad \mathcal{F}_1^{-1}(1 + r_n^1 - 2\varepsilon - m)\} + \delta \\
&\leqslant \sup_{m \in [r_n^0 + r_n, 1 + r_n^0 - 2\varepsilon]} \{\mathcal{F}_0^{-1}(m) \\
&\qquad\qquad + \mathcal{F}_1^{-1}(1 + r_n^1 + r_n^0 + r_n - 2\varepsilon - m)\} + \delta.
\end{aligned}$$

Thus, taking n large enough such that $r_n^1 + r_n^0 + r_n \leqslant \varepsilon$, we conclude by using Corollary 2 and the fact that $\mathcal{F}_1^{-1}$ is increasing.

We now prove Item (ii). Define

$$\widetilde{S}_n^0 := \{x^0 \in \mathrm{supp}(\mu_n^0) : \Phi_t^v(B_{h_n}(x^0)) \subset\subset \omega \quad \text{for a} \quad t \leqslant t^0(x^0, \omega_n) + \delta/2\}$$

and $\widetilde{r}_n^0 := \mu^0((\widetilde{S}_n^0)^c)$. Using the same argument as previously, we can prove that

$$t^0(\xi^0, \omega) \leqslant t^0(x^0, \omega_n) + \delta/2 \tag{34}$$

ine t0 (ii)

for π_n-almost every $x^0 \in \mathrm{supp}(\mu_n^0) \cap \widetilde{S}_n^0$ and $\xi^0 \in \mathrm{supp}(\mu^0) \cap (R_n^0)^c$. Inequality (34) implies

$$\begin{aligned}
\mathcal{F}_{0,n}(t) &\leqslant \mu_n^0(\{x^0 \in \mathrm{supp}(\mu_n^0) \cap \widetilde{S}_n^0 : t^0(x^0, \omega_n) \leqslant t\}) + \widetilde{r}_n^0 \\
&= \pi_n(\{x^0 \in \mathrm{supp}(\mu_n^0) \cap \widetilde{S}_n^0, \xi^0 \in \mathrm{supp}(\mu^0) \cap (R_n^0)^c : \\
&\qquad\qquad t^0(x^0, \omega_n) \leqslant t\}) + \widetilde{r}_n^0 \\
&\leqslant \pi_n(\{x^0 \in \mathrm{supp}(\mu_n^0) \cap \widetilde{S}_n^0, \xi^0 \in \mathrm{supp}(\mu^0) \cap (R_n^0)^c : \\
&\qquad\qquad t^0(\xi^0, \omega) \leqslant t + \delta/2\}) + \widetilde{r}_n^0 \\
&= \mu^0(\{\xi^0 \in \mathrm{supp}(\mu^0) \cap (R_n^0)^c : t^0(\xi^0, \omega) \leqslant t + \delta/2\}) + \widetilde{r}_n^0 \\
&= \mathcal{F}_0(t + \delta/2) + r_n + \widetilde{r}_n^0,
\end{aligned}$$

for all $t \in \mathbb{R}^+$. We also have

$$\mathcal{F}_{1,n}(t) \leqslant \mathcal{F}_1(t + \delta/2) + r_n + \widetilde{r}_n^1,$$

for all $t \in \mathbb{R}^+$, where $\widetilde{r}_n^1$ is similarly defined. We conclude as before.

Item (iii) holds for $E_{0,n} := S_n^0$ and $E_{1,n} := S_n^1$. $\square$

4.3 Proof of Theorem 3

⟨sec:proof th 3⟩ In this section, we prove Theorem 3 using the results obtained in Section 4.1 and 4.2.

Proof (of Theorem 3) We first prove Item (i). Fix $\varepsilon, s > 0$. We prove that we can steer μ^0 to a W_1-neighbourhood of μ^1 of size ε at time

$$T := S(\mu^0, \mu^1, \omega) + s.$$

We assume that $d := 2$, but the reader will see that the proof can be clearly adapted to any space dimension. The proof is divided into four steps.

Step 1: We first discretize uniformly in space the supports of μ^0 and μ^1. To simplify the presentation, assume $\text{supp}(\mu^0) \subset (0,1)^2$ and $\text{supp}(\mu^1) \subset (0,1)^2$. Consider the sequence of uniform meshes $\mathcal{T}_n := \bigcup\limits_{k\in\{0,...,2^n-1\}^2} S_{n,k}$ with

$$S_{n,k} := \left[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}\right) \times \left[\frac{k_2}{2^n}, \frac{k_2+1}{2^n}\right),$$

where $k := (k_1, k_2)$.

Step 2: To send a measure to another, these measures need to have the same total mass. Thus, for each $n \in \mathbb{N}^*$ and $k \in \{0, ..., 2^n - 1\}^2$ such that $\mu^0(S_{n,k}) > 1/n^4$, we discretize measures $\mu^0_{|S_{n,k}}$ and $\mu^1_{|S_{n,k}}$ with some measures with the same total mass $1/n^4$. As illustrated in Figure 6, we partition $S_{n,k}$ into some subsets $\{A^0_{ki}\}_i$ with $A^0_{ki} = [a^0_i, a^0_{i+1}) \times (0, 1)$ such that $\mu^0_{|S_{n,k}}(A^0_{ki}) = 1/n^2$ (if $\mu^0_{|S_{n,k}\cap S^0_n}(A^0_{ki}) < 1/n^2$, then $A^0_{k0} := S_{n,k}$) and, for each i, we partition A^0_{ki} into some subsets $\{A^0_{kij}\}_j$ with $A^0_{kij} = [a^0_i, a^0_{i+1}) \times [a^0_{ij}, a^0_{i(j+1)})$ such that

$$\mu^0_{|S_{n,k}}(A^0_{kij}) = 1/n^4.$$

If $\mu^0(S_{n,k})$ is not a multiple of $1/n^4$, it remains a global small mass (smaller than $1/n^2$) that we do not control (see (35)). For more details on such discretization, we refer to [19, Prop. 3.1]. We discretize similarly the measure μ^1 on some sets A^1_{kij}. As in Figure 7, we then build $B^0_{kij} := [b^0_i, b^0_{i+1}) \times [b^0_{ij}, b^0_{i(j+1)}) \subset\subset A^0_{kij}$ and $B^1_{kij} := [b^1_i, b^1_{i+1}) \times [b^1_{ij}, b^1_{i(j+1)}) \subset\subset A^1_{kij}$ such that

$$\mu^0(B^0_{kij}) = \mu^1(B^1_{kij}) = (n^2 - 2)^2/n^8.$$

For more details of such construction, we also refer to [19, Prop. 3.1].

Step 3: In this step, we send the mass of μ^0 from each B^0_{kij} to some $B^1_{k'i'j'}$, while we do not control the rest of the mass outside B^0_{kij}.

We first explain why this rest is negligible. Consider

$$I^0_n := \{(k, i, j) : \mu^0(B^0_{kij}) > 1/n^4\}.$$

We define similarly I^1_n. Without loss of generality, we can assume that $|I^0_n| = |I^1_n|$. Indeed, for example in the case $n_0 := |I^0_n| - |I^1_n| > 0$, we remove the n_0 last

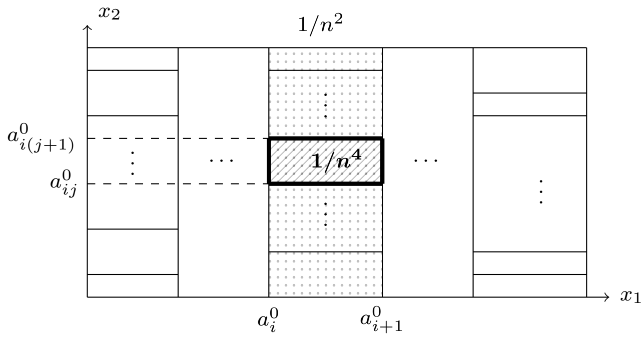


Fig. 6 Example of a partition of $S_{n,k}$ with a cell A^0_{kij} (hashed).
⟨fig: mesh⟩

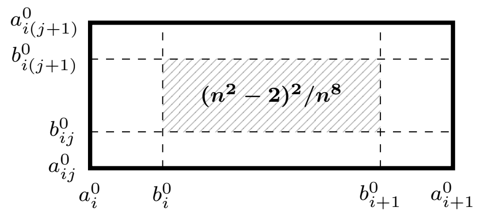


Fig. 7 Example of cells B^0_{ij} (hashed).
⟨fig:cell⟩

cells in the set of indices I^0_n, the total corresponding removed mass is smaller that $1/n^2$, then negligible when $n \to \infty$ (see (35)). We define for $l = 0, 1$ the sets

$$R^l_n := \mathbb{R}^d \setminus \bigcup_{kij \in I^l_n} B^l_{kij}.$$

We remark that, for $l = 0, 1$, it holds

$$\mu^l(R^l_n) \leqslant 1 - \tfrac{(n^2-2)^2}{n^4} + \tfrac{2}{n^2} = \tfrac{6n^2-4}{n^4} \underset{n\to\infty}{\longrightarrow} 0. \qquad (35)$$ eq:R_n^l

The term $2/n^2$ correspond to the cases where the masses $\mu^0(S_{n,k})$ are not a multiple of $1/n^4$ and $|I^0_n|$ is not equal to $|I^1_n|$.

We now approximate the measures μ^l restricted to $(R^l_n)^c$ ($l = 0, 1$) by a sum of Dirac masses μ^l_n defined by

$$\mu^l_n := \sum_{kij \in I^l_n} \frac{(n^2-2)^2}{n^8} \delta_{x^l_{kij}},$$

where the points x^l_{kij} are the center of the cell B^l_{kij}. We denote by $t^l_{kij} := t^l(x^l_{kij})$. Using the definition of $S_{n,K}$, it holds

$$W_\infty(\mu^l_{|(R^l_n)^c}, \mu^l_n) \leqslant g_n := \sqrt{2}/n \underset{n\to\infty}{\longrightarrow} 0. \qquad (36)$$ eq:wass inf

For all $n \in \mathbb{N}^*$, we define $f_n := 1/2^{n-1}$ and consider the sets ω_n given in (27). We remark that the measures μ^0, μ^1 and the sequences $\{\mu_n^0\}_{n\in\mathbb{N}^*}$, $\{\mu_n^1\}_{n\in\mathbb{N}^*}$, $\{R_n^0\}_{n\in\mathbb{N}^*}$ and $\{R_n^0\}_{n\in\mathbb{N}^*}$ satisfy the hypotheses of Proposition 4. Since

$$S_{\varepsilon/4}(\mu^0,\mu^1,\omega) \leqslant S(\mu^0,\mu^1,\omega),$$

applying Proposition 4 for $\delta := s/2$, it holds

$$S_{\varepsilon/2}(\mu_n^0,\mu_n^1,\omega_n) + \frac{s}{2} \leqslant S_{\varepsilon/4}(\mu^0,\mu^1,\omega) + s \leqslant S(\mu^0,\mu^1,\omega) + s = T.$$

Using Corollary 2,

$$T_{e,\varepsilon/2}(\mu_n^0,\mu_n^1,\omega_n) = S_{\varepsilon/2}(\mu_n^0,\mu_n^1,\omega_n).$$

Then, there exists a control u_n^d satisfying $\operatorname{supp}(u_n^d) \subset\subset \omega_n$ such that, denoting by μ_n^d the solution to System (2) associated to u_n^d and the initial data μ_n^0, it holds

$$W_1(\mu_n^1,\mu_n^d(T)) \leqslant \varepsilon/2 \tag{37}$$

ine 0

and denoting by σ_0 and σ_1 the associated permutations, we have

$$\Phi_T^{v+\mathbb{1}_\omega u_n^d}(x^0_{\sigma_0(kij)}) = x^1_{\sigma_1(kij)},$$

for all $kij \in \{1,...,|I_n^0| - M_n^0\}$ (indices kij are assumed ordered). Since we have no intersection of the trajectories $\Phi_\cdot^{v+\mathbb{1}_\omega u_n^d}(x^0_{kij})$ (see argument given in the proof of Proposition 1), there exist $r, R > 0$ such that $r < R$ and, for all $t \in (t^0_{n,k}, T - t^1_{n,k})$, it holds

$$\Phi_t^{v+\mathbb{1}_\omega u_n^d}(B_r(x^0_{kij})) \subset\subset \Phi_t^{v+\mathbb{1}_\omega u_n^d}(B_R(x^0_{kij})) \subset\subset \omega$$

and, for all $t \in (0,T)$, it holds

$$\Phi_t^{v+\mathbb{1}_\omega u_n^d}(B_R(x^0_{kij})) \cap \Phi_t^{v+\mathbb{1}_\omega u_n^d}(B_R(x^0_{k'i'j'})) = \varnothing$$

for all $kij, k'i'j' \in I_n^0$. Denote by $\tau_{kij} := (T - t^0_{kij} - t^1_{kij})/2$. There exists $s^0_{kij} \in (t^0_{kij}, t^0_{kij} + \tau^0_{kij})$ such that

$$\Phi_{t^0_{kij}}^{v+\mathbb{1}_\omega u_n^d}(B_r(x^0_{kij})) \subset\subset \Phi_t^{v+\mathbb{1}_\omega u_n^d}(B_R(x^0_{kij}))$$

for all $t \in (t^0_{kij}, s^0_{kij})$. Using the same argument developed in [19, Prop. 3.3], there exists a control u_n^0 on the time interval $(0, s^0_{kij})$ measurable in time, Lipschitz in space and uniformly bounded, such that $\operatorname{supp}(u_n^0) \subset\subset \omega$,

$$\operatorname{supp}(\Phi_t^{v+\mathbb{1}_\omega u_n^0} \# \mu^0_{|B^0_{kij}}) \subset\subset \operatorname{supp}(\Phi_t^v \# \mu^0_{|A^0_{kij}}))$$

for all $t \in (t^0_{kij}, s^0_{kij})$ and

$$\operatorname{supp}(\Phi_{s^0_{kij}}^{v+\mathbb{1}_\omega u_n^0} \# \mu^0_{|B^0_{kij}}) \subset\subset \Phi_{t^0_{kij}}^{v+\mathbb{1}_\omega u_n^d}(B_r(x^0_{kij})) \subset\subset \Phi_{s^0_{kij}}^{v+\mathbb{1}_\omega u_n^d}(B_R(x^0_{kij})).$$

We similarly build a control u_n^1 on some intervals (s_{kij}^1, t_{kij}^1). We define a control u_n such that

$$u_n := \begin{cases} u_n^0 \text{ in } \cup_{t\in(t_{kij}^0, s_{kij}^0)} \operatorname{supp}(\Phi_t^{v+\mathbb{1}_\omega u_n^0} \# \mu^0_{|B_{kij}^0}), \\ u_n^1 \text{ in } \cup_{t\in(s_{kij}^1, t_{kij}^1)} \operatorname{supp}(\Phi_{T-t}^{v+\mathbb{1}_\omega u_n^1} \# \mu^1_{|B_{kij}^1}), \\ u_n^d \text{ otherwise.} \end{cases}$$

Step 4: We now estimate the Wasserstein distance between $\Phi_T^{v+\mathbb{1}_\omega u_n} \# \mu^0$ and μ^1. Using Property 2, it holds

$$\begin{aligned} W_1(\Phi_T^{v+\mathbb{1}_\omega u_n} \# \mu^0, \mu^1) \leqslant\ & W_1(\Phi_T^{v+\mathbb{1}_\omega u_n} \# \mu^0_{|(R_n^0)^c}, \mu^1_{|(R_n^1)^c}) \\ & + W_1(\Phi_T^{v+\mathbb{1}_\omega u_n} \# \mu^0_{|R_n^0}, \mu^1_{|R_n^1}). \end{aligned} \qquad (38)\ \boxed{\texttt{ine1 ter}}$$

By triangular inequality,

$$\begin{aligned} & W_1(\Phi_T^{v+\mathbb{1}_\omega u_n} \# \mu^0_{|(R_n^0)^c}, \mu^1_{|(R_n^1)^c})) \\ & \quad \leqslant W_1(\Phi_T^{v+\mathbb{1}_\omega u_n} \# \mu^0_{|(R_n^0)^c}, \Phi_T^{v+\mathbb{1}_\omega u_n} \# \mu_n^0) \\ & \qquad + W_1(\Phi_T^{v+\mathbb{1}_\omega u_n} \# \mu_n^0, \mu_n^1) + W_1(\mu_n^1, \mu^1_{|(R_n^1)^c}). \end{aligned} \qquad (39)\ \boxed{\texttt{ine2 ter}}$$

We now estimate each term in the right-hand side in (39). Using inequalities (13) and (36), it holds

$$\begin{aligned} W_1(\Phi_T^{v+\mathbb{1}_\omega u_n} \# \mu^0_{|(R_n^0)^c}, \Phi_T^{v+\mathbb{1}_\omega u_n} \# \mu_n^0) & \leqslant e^{2LT} W_1(\mu^0_{|(R_n^0)^c}, \mu_n^0) \\ & \leqslant e^{2LT} W_\infty(\mu^0_{|(R_n^0)^c}, \mu_n^0) \\ & \leqslant e^{2LT}\sqrt{2}/n \end{aligned} \qquad (40)\ \boxed{\texttt{ine3 ter}}$$

and

$$W_1(\mu_n^1, \mu^1_{|(R_n^0)^c}) \leqslant W_\infty(\mu_n^1, \mu^1_{|(R_n^0)^c}) \leqslant \sqrt{2}/n. \qquad (41)\ \boxed{\texttt{ine4 ter}}$$

Combining (37), (39), (40) and (41), it holds

$$W_1(\Phi_T^{v+\mathbb{1}_\omega u_n} \# \mu^0_{|(R_n^0)^c}, \mu^1_{|(R_n^1)^c})) \leqslant \varepsilon/2 + (1+e^{LT})\sqrt{2}/n. \qquad (42)\ \boxed{\texttt{ine5 ter}}$$

Using Property 1, there exists $\pi_n \in \Pi(\Phi_T^{v+\mathbb{1}_\omega u_n} \# \mu^0_{|R_n^0}, \mu^1_{|R_n^1})$ such that

$$\begin{aligned} W_1(\Phi_T^{v+\mathbb{1}_\omega u_n} \# \mu^0_{|R_n^0}, \mu^1_{|R_n^1}) & = \int_{(-T\sup|v|, 1+T\sup|v|)^2\times(0,1)^2} |x-y| d\pi_n(x,y) \\ & \leqslant \sqrt{2}(1+2T\sup|v|) \times \tfrac{6n^2-4}{n^4}. \end{aligned} \qquad (43)\ \boxed{\texttt{ine6 ter}}$$

Combining (38), (42), (43), we obtain

$$W_1(\Phi_T^{v+\mathbb{1}_\omega u_n} \# \mu^0, \mu^1) \leqslant \varepsilon/2 + (1+e^{LT})\sqrt{2}/n + \sqrt{2}(1+2T\sup|v|)\tfrac{6n^2-4}{n^4},$$

which leads to the conclusion when $n \to \infty$.

We now prove Item (ii) of Theorem 3. Consider

$$T \in (S^*, S(\mu^0, \mu^1, \omega)].$$

Since $\mathcal{F}_0$ and $\mathcal{F}_1$ are continuous and increasing, then

$$m \mapsto \mathcal{F}_0^{-1}(m) + \mathcal{F}_1^{-1}(1-m)$$

is lower semi-continuous. Thus there exists a sequence $\{m_k\}_{k\in\mathbb{N}^*} \subset (0,1)$ such that

$$\mathcal{F}_0^{-1}(m_k) + \mathcal{F}_1^{-1}(1-m_k) \longrightarrow S(\mu^0, \mu^1, \omega).$$

For K large enough, denoting $\varepsilon := \min\{m_K, 1-m_K\}$, it holds

$$\begin{aligned} T &< \mathcal{F}_0^{-1}(m_K) + \mathcal{F}_1^{-1}(1-m_K), \\ &\leqslant \sup_{m\in(\varepsilon,1-\varepsilon)} \{\mathcal{F}_0^{-1}(m) + \mathcal{F}_1^{-1}(1-m)\} \\ &\leqslant \sup_{m\in(\varepsilon,1-\varepsilon)} \{\mathcal{F}_0^{-1}(m+\varepsilon/2) + \mathcal{F}_1^{-1}(1-m)\} \\ &\leqslant \sup_{m\in(3\varepsilon/2,1-\varepsilon/2)} \{\mathcal{F}_0^{-1}(m) + \mathcal{F}_1^{-1}(1-\varepsilon/2-m)\} \\ &\leqslant S_{\varepsilon/2}(\mu^0, \mu^1, \omega). \end{aligned}$$

Consider μ_n^0 and μ_n^1 build in the proof of Item (i). Applying successively Item (ii) and Item (iii) of Proposition 4 with $\delta = \delta_1 := (S_\varepsilon(\mu^0, \mu^1, \omega) - T)/2$ and $\delta = \delta_2 := (T - S^*(\mu^0, \mu^1, \omega))/2$, there exists sets E_n^0, $E_n^1 \subset \mathbb{R}^d$ and $N \in \mathbb{N}^*$ such that $e_n := \mu_n^0(E_n^0) = \mu_n^0(E_n^1) \underset{n\to\infty}{\longrightarrow} 1$ and for all $n \geqslant N$, it holds

$$T < S_{\varepsilon/2}(\mu^0, \mu^1, \omega) - \delta_1 \leqslant S_{\varepsilon/4}(\mu_n^0, \mu_n^1, \omega_n)$$

and

$$T > S^*(\mu^0, \mu^1, \omega) + \delta_2 \geqslant M_e^*(\mu^0_{n|E_n^0}, \mu^1_{n|E_n^1}, \omega_n).$$

Since for n large enough

$$S_{\varepsilon/4}(\mu_n^0, \mu_n^1, \omega_n) \leqslant S_{\varepsilon/8}(\mu^0_{n|E_n^0}, \mu^1_{n|E_n^1}, \omega_n),$$

Corollary 2 implies that

$$\begin{cases} \text{there exists no control} \\ \text{steering at time } T\ \mu^0_{n|E_n^0} \text{ to } \mu^1_{n|E_n^1} \text{ up to a mass } \varepsilon/8. \end{cases} \tag{44}$$ non contr

We now prove by contradiction that System (2) is not approximately controllable at time T. Assume that System (2) is approximately controllable at time T. Then there exists a control $\mathbb{1}_\omega u$ Lipschitz in space, measurable in time and uniformly bounded for which

$$W_1(\Phi_T^{v+\mathbb{1}_\omega u}\#\mu^0, \mu^1) \leqslant \varepsilon/16. \tag{45}$$ eq: eps rec

Fix $n \geqslant N$. By triangular inequality,

$$\begin{aligned} W_1(\Phi_T^{v+\mathbb{1}_\omega u}\#\mu^0_{n|E_n^0}, \mu^1_{n|E_n^1}) &\leqslant W_1(\Phi_T^{v+\mathbb{1}_\omega u}\#\mu^0_{n|E_n^0}, \Phi_T^{v+\mathbb{1}_\omega u}\#\mu^0_{|(R_n^0)^c\cap E_n^0}) \\ &\quad + W_1(\Phi_T^{v+\mathbb{1}_\omega u}\#\mu^0_{|(R_n^0)^c\cap E_n^0}, \mu^1_{|(R_n^1)^c\cap E_n^1}) + W_1(\mu^1_{|(R_n^1)^c\cap E_n^1}, \mu^1_{n|E_n^1}). \end{aligned} \tag{46}$$ ine1

Item (ii) of Property 2 implies

$$W_1(\Phi_T^{v+\mathbb{1}_\omega u}\#\mu^0_{|(R_n^0)^c\cap E_n^0},\mu^1_{|(R_n^1)^c\cap E_n^1}) \leqslant W_1(\Phi_T^{v+\mathbb{1}_\omega u}\#\mu^0_{|R_n^0\cup(E_n^0)^c},\mu^1_{|R_n^1\cup(E_n^0)^c}) \\ +W_1(\Phi_T^{v+\mathbb{1}_\omega u}\#\mu^0,\mu^1). \quad (47)$$

ine2

Using Property 1, there exists $\pi\in\Pi(\Phi_T^{v+\mathbb{1}_\omega u}\#\mu^0_{|R_n^0\cup(E_n^0)^c},\mu^1_{|R_n^1\cup(E_n^1)^c})$ such that

$$\begin{array}{l} W_1(\Phi_T^{v+\mathbb{1}_\omega u}\#\mu^0_{|R_n^0\cup(E_n^0)^c},\mu^1_{|R_n^1\cup(E_n^1)^c}) \\ \quad =\int_{(-T\sup|v|,1+T\sup|v|)^2\times(0,1)^2}|x-y|d\pi(x,y) \\ \quad \leqslant\sqrt{2}(1+2T\sup|v|)\times(\frac{6n^2-4}{n^4}+e_n). \end{array} \quad (48)$$

ine6 ter2

Inequality (47), (48) and (45) leads to

$$\begin{array}{l} W_1(\Phi_T^{v+\mathbb{1}_\omega u}\#\mu^0_{|(R_n^0)^c\cap E_n^0},\mu^1_{|(R_n^1)^c\cap E_n^1}) \\ \quad \leqslant\sqrt{2}(1+2T\sup|v|)(\frac{6n^2-4}{n^4}+e_n)+\varepsilon/16. \end{array} \quad (49)$$

ine3

Using inequalities (13) and (36), it holds

$$W_1(\Phi_T^{v+\mathbb{1}_\omega u}\#\mu^0_{n|E_n^0},\Phi_T^{v+\mathbb{1}_\omega u}\#\mu^0_{|(R_n^0)^c\cap E_n^0})\leqslant e^{2LT}\sqrt{2}/n \quad (50)$$

ine3 ter2

and

$$W_1(\mu^1_{|(R_n^1)^c\cap E_n^1},\mu^1_{n|E_n^1})\leqslant\sqrt{2}/n. \quad (51)$$

ine3 ter3

Combining (46), (49), (50) and (51), it holds

$$W_1(\Phi_T^{v+\mathbb{1}_\omega u}\#\mu^0_{n|E_n^0},\mu^1_{n|E_n^1})\leqslant\sqrt{2}(1+2T\sup|v|)(\frac{6n^2-4}{n^4}+e_n) \\ +\varepsilon/16+(1+e^{LT})\sqrt{2}/n.$$

For n large enough, we obtain a contradiction with (44). Thus System (2) is not approximately controllable at time T.

□

⟨rmq:T2* cont⟩ *Remark 6* We given bellow an example in which System (1) is never exactly controllable on $(0,S^*)$ and another where System (1) is exactly controllable at each time $T\in(0,S^*)$:

- Consider $\omega:=(-1,1)\times(-1.5,1.5)$, $v:=(1,0)$ and μ^0, μ^1 given by

$$\begin{cases}\mu^0:=\mathbb{1}_{(-2.5,-2)\times(-1,1)}dx,\\ \mu^1:=\mathbb{1}_{(2,2.5)\times(-1,1)}dx.\end{cases}$$

  See Figure 3 (left). The time $S^*(\mu^0,\mu^1)$ at which we can act on the particles and the minimal time $S(\mu^0,\mu^1)$ are respectively equal to 1.5 and 2.5. We observe that for each time $T\in[0,S^*]$ System (2) is not approximately controllable.

- Consider $\omega := (-1,1)\times(-1.5,1.5)$, v of class $\mathcal{C}^\infty$ satisfying

$$v(x,y) = \sqrt{x^2+y^2}\begin{pmatrix} y \\ -x \end{pmatrix}$$

  for all $(x,y) \in B_1(1,0)\backslash B_{0.5}(1,0)$ and μ^0, μ^1 given by

$$\mu^0 = \mu^1 := \mathbb{1}_{B_1(1,0)\backslash B_{0.5}(1,0)}dx.$$

  See Figure 8 (right). In this case, both quantities $S^*(\mu^0,\mu^1)$ and $S(\mu^0,\mu^1)$ are equal to π. Since $\Phi_t^v\#\mu^0 = \mu^1$ for all $t \geqslant 0$, we remark that System (1) is exactly controllable for all $T \in (0, S^*(\mu^0,\mu^1))$.

There exists at most a finite number of time $t \in [0, M_e^*]$ at which the microscopic model (1) is exactly controllable and it represents some very pathological cases. Moreover the proof of Theorem 3 is a kind a passage to the limits from the microscopic model to the macroscopic one. So, even if System (2) can be controllable at some time $T \in [0, S^*]$, then we can think that these situations are also pathological and sparse in the macroscopic model.

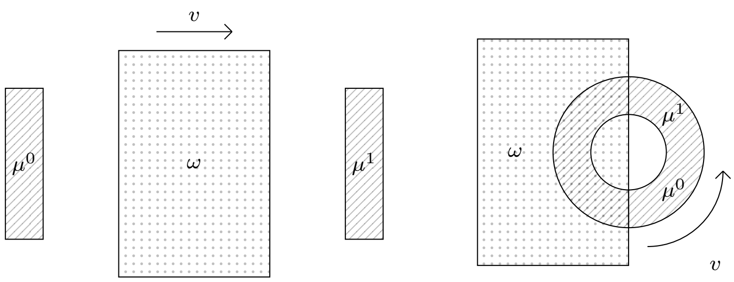


Fig. 8 Examples in the case $T \in (0, S^*)$.

g:ex (0,T*) cont⟩

We can adapt the proof of Theorem 3 to obtain the minimal time to approximately steer μ^0 to μ^1 up to mass ε:

Corollary 3 *Let $\mu^0,\mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ satisfying Condition 1. Then the infimum time $T_{a,\varepsilon}$ to approximately steer μ^0 to μ^1 up tu a mass ε is equal to $S_\varepsilon(\mu^0,\mu^1)$ given in* (26)*.*

5 Numerical simulations

⟨sec:num sim⟩ In this section, we give some numerical illustrations of the algorithm developed in the proof of Theorem 3 to compute the infimum time and the solution associated to the minimal time problem to approximately steer a AC measure to another. We use a Lagrangian scheme for these simulations. We first recall the algorithm:

Algorithm 1

⟨algo 1⟩ Consider two AC measures μ^0 and $\mu 1$ satisfying Condition 1

Step 1: Construction of the uniform mesh (see step 1 in the proof of Theorem 3)

Step 2: Construction of the cell A^0_{kij} and A^1_{kij} following the mass (see Step 2 in the proof of Theorem 3)

Step 3: Consider $X^0 := \{x^0_{kij}\}$ and $X^1 := \{x^1_{kij}\}$ composed with the centres of the A^0_{kij} and A^1_{kij}

Step 4: For $t^0_{kij} := t^0(x^0_{kij})$ and $t^1_{kij} := t^1(x^1_{kij})$, compute the minimal time to steer X^0 to X^1

$$M_e(X^0, X^1) := \max_{kij}\{t^0_{kij} + t^1_{kij}\},$$

with $\{t^0_{kij}\}_{kij}$ and $\{t^1_{kij}\}_{kij}$ increasingly and decreasingly ordered

Step 5: Compute of the optimal permutations σ minimizing (17) to steer X^0 to X^1

Step 6: Concentration of the mass contained in the cells near $\delta^0_{x^k ij}$ in order to obtain no intersection of the cells when they follow the trajectories of $\delta^0_{x^k ij}$

Step 7: Final computation

5.1 Example in the one dimensional case

Consider the initial data μ^0 and the target μ^1 defined by

$$\mu^0 := 0.5 \times \mathbb{1}_{(0,2)}(x)dx$$

and

$$\mu^1 := 0.5 \times \mathbb{1}_{(7,8)\cup(10,11)}(x)dx.$$

We fix the velocity field $v := 1$ and the control region $\omega := (5, 6)$. We remark that the infimum time to steer μ^0 to μ^1 is equal to 8. We fix $\delta := 1$, then we want to approximately steer μ^0 to μ^1 at time 9. Following Algorithm 1, we obtain the solution presented in Figure 9. It is interesting to observe a concentration of the mass in the control region ω, which can be dramatic in some practical cases.

5.2 Example in the two dimensional case

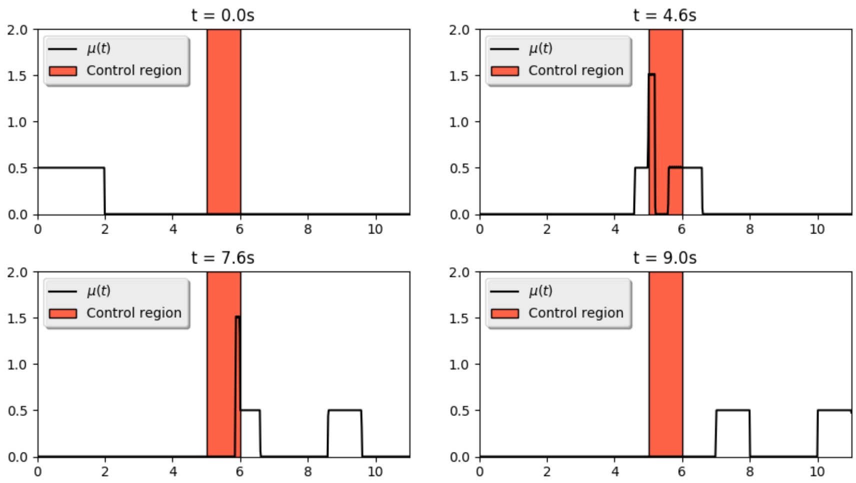
We now give an example in the two dimension case. Consider the initial data μ^0 and the target μ^1 defined by

$$\mu^0 := \begin{cases} 1/8 & \text{if } (x, y) \in (0, 4) \times (1, 3), \\ 0 & \text{otherwise} \end{cases}$$

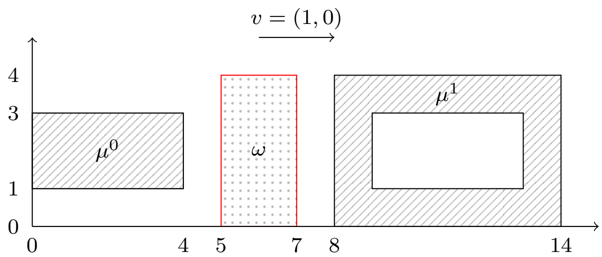
and

$$\mu^1 := \begin{cases} 1/16 & \text{if } (x, y) \in (8, 14) \times (0, 4)\backslash(9, 13) \times (1, 3), \\ 0 & \text{otherwise.} \end{cases}$$

We fix the velocity field $v := (1, 0)$ and the control region $\omega := (5, 7) \times (0, 4)$. This situation is illustrated in Figure 10. The infimum time to steer μ^0 to μ^1



⟨fig:simu1D⟩ **Fig. 9** Solution at time $t = 0$, $t = 4.6$, $t = 7.6$ and $t = T = 9.0$.



⟨fig:situation2D⟩ **Fig. 10** Configuration of the simulation in the two dimensional case.

is equal to 8. We fix $\delta := 0.8$, then we want to approximately steer μ^0 to μ^1 at time 8.8. Following algorithm 1, we present the computed solution in Figure 11. The maximum of the final solution is equal to 3.07. It is due to the fact that we want no cross of the cells when they follows the trajectories of Dirac. Again, it can be dramatic in some piratical situations. So, in can be interesting to evaluate this concentration in a future work.

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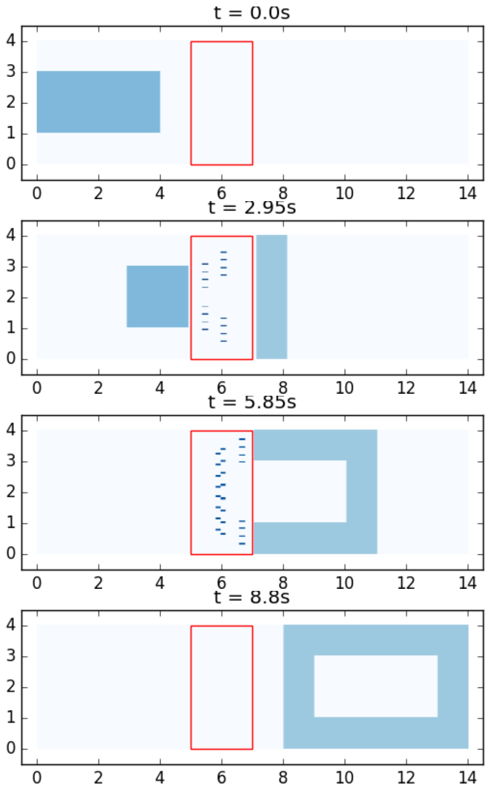

Fig. 11 Solution in the two dimensional case at time $t = 0$, $t = 2.95$, $t = 5.85$ and $t = T = 8.8$.
⟨fig:simu2D⟩

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