Delay Margin in Controlling a Furuta Pendulum

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José-Enrique Hernández-Díez[a], Silviu-Iulian Niculescu[b], César-Fernando Méndez-Barrios[a], Emilio-Jorge González-Galván[a], Ambrocio Loredo-Flores[c] & Juan-Antonio Escareno[d]

***Abstract*— This paper focuses on the design of an LQR based control scheme for the stabilization of the Furuta Pendulum in its unstable equilibrium point at the upright position. More precisely, we are interested in characterizing the corresponding delay margin under the assumption that the feedback loop includes time-delay. The paper provides an explicit tool to compute the critical delay value in the state feedback loop and a delicate tuning to reach larger delay values. In order to illustrate the effectiveness of the proposed control scheme, some numerical results are presented.**

I. Introduction

In mechanical systems, one of the classical problems of automatic control is the stabilization of the inverted pendulum on its unstable equilibrium point at the upright position. In this work we study this task in the case of the so-called Furuta pendulum (see Fig. 1), also known as the rotatory inverted pendulum.

In order to achieve this task, we propose the use of some standard Linear Quadratic Regulation (LQR) controller. Some insights concerning this control law for three underactuated systems (inverted pendulum on a cart, inverted wedge and ball and beam system) can be found in [1]. Furthermore, we consider a time-delay in the state feedback loop which can be inherent to the system due to data processing, or even designed for performance requirements.

It is well known that adverse effects as oscillations, instability and bandwidth sensitivity, among others, are the consequence of the presence of delay in the control loop (see, for instance, [2], [3]). However, it is worth mentioning that there exist some situations when the delay may improve the system stability as explained in the classical example [4], [5], where an oscillator is controlled by one gain-delay "block", with positive gains and small delay values (a detailed analysis of such an approach can be found in [6]). Finally, it is worth noticing that even in the simple case of the inverted pendulum, the presence of delays in the input may induce some unexpected properties as, for instance a triple root at the origin (see, for instance [7] and the references therein).

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In order to consider this scenario in the control scheme design, we use the results shown in [8] for the inverted pendulum and cart system. This shows a simple method on how to compute the critical delay value in the state feedback loop in which the closed-loop system loses stability. Moreover, we explore the behavior of the stability conditions by considering an auxiliary pair of gains for the position regulation of both angles of the Furuta pendulum. Numerical simulations were conducted in Matlab-Simulink to illustrate how sensitive the tuning of these parameters can be. The main contribution of the paper is to construct some appropriate tuning rules able to enlarge the delay margins. Illustrative examples confirm such an approach.

The remaining part of this paper is organized as follows. Section II concerns to the Furuta pendulum modelling and the LQR control design for the stabilization in the delay-free scenario. Section III-A shows a simple method easy to implement for computing the critical time-delay value in the state feedback loop at which the closed-loop system loses stability. In Section III-B the stability boundaries for two auxiliary gains and the time-delay are characterized. Section IV shows some numerical results obtained on the software "Matlab". Finally, Sections V and VI discuss some concluding remarks and future work, respectively.

Furthermore, we invite the reader to visit the website refereed in Section IV, where illustrative support material for the understanding of the results discussed in this paper is proposed.

II. Prerequisites on Control Scheme Design

This section includes the basics of the design of an LQR based control scheme in the delay-free scenario for the Furuta pendulum stabilization problem. It covers the Furuta pendulum modelling and the LQR controller gain tuning.

A. Furuta Pendulum Modeling

As discussed in the sequel, we introduce the Furuta pendulum nonlinear model and a linear representation valid uniquely around an operating point of interest.

Figure 1 depicts the representation of the Furuta pendulum, also known as the rotatory inverted pendulum. This mechanical system has two degrees of freedom and two rotatory joints. It consists in three essential components: a motor and two bars known as *arm* and *pendulum*. The motor's shaft is fixed at one end of the arm inducing a rotatory movement of this bar. The pendulum is placed at the opposite end to the motor's shaft with a rotatory joint which provides a free rotatory movement in a normal plane to the arm. As shown in the schematic representation illustrated in Fig. 1, θ_0 and θ_1 are the arm and pendulum angular positions, respectively. θ_0 is measured with respect of the X-axis and θ_1 with respect to the upright position. $\mathscr{T}$ concerns to the torque applied to the arm and it is provided by the motor. I_0 and J_1 stands for the motor-arm and pendulum inertia values and L_0 and l_1 represent the arm length and the pendulum's center of mass location, respectively. Finally, m_1 represents the mass of the pendulum, while g denotes the gravitational acceleration.

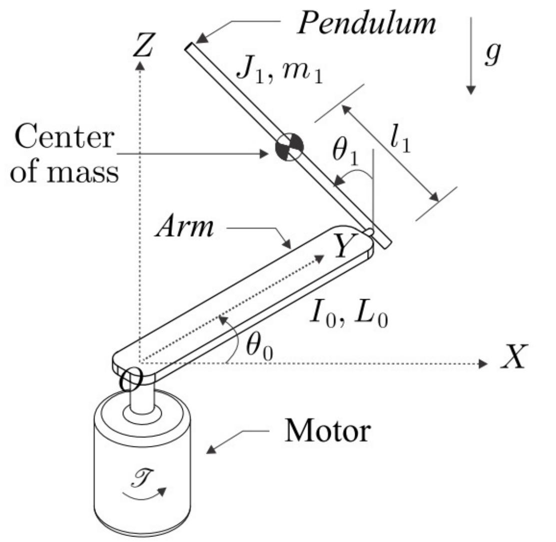

Fig. 1. Furuta Pendulum Diagram ([9]).

As detailed in [9] (see also, [10] and [11]), the Lagrangian formulation of the Furuta pendulum consists in the following:

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q)=F, \tag{1}$$

where:

$$M(q):=\begin{bmatrix} I_0+m_1(L_0^2+l_1^2\sin^2\theta_1) & m_1l_1L_0\cos\theta_1 \\ m_1l_1L_0\cos\theta_1 & J_1+m_1l_1^2 \end{bmatrix},$$

$$C(q,\dot{q}):=\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & 0 \end{bmatrix}, \qquad g(q):=\begin{bmatrix} 0 \\ -m_1l_1g\sin\theta_1 \end{bmatrix},$$

$$\begin{aligned} c_{11} &:= \frac{1}{2}m_1l_1^2\dot{q}_2\sin(2q_2), \\ c_{12} &:= -m_1l_1L_0\dot{q}_2\sin q_2+\frac{1}{2}m_1l_1^2\dot{q}_1\sin(2q_2), \\ c_{21} &:= -\frac{1}{2}m_1l_1^2\dot{q}_1\sin(2q_2), \end{aligned}$$

$$F:=\begin{bmatrix} \mathscr{T} \\ 0 \end{bmatrix}, \qquad q:=\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}=\begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}.$$

In this work we focus on the control problem of stabilization and regulation of the solution pair $(\theta_0(t),\theta_1(t))$ around an operating point. Inspired by [9], we consider the corresponding system's linearization:

$$\dot{x}=Ax+Bu, \tag{2}$$

where the state of the system x and the control variable u are defined as:

$$x:=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} \theta_0 \\ \dot{\theta}_0 \\ \theta_1 \\ \dot{\theta}_1 \end{bmatrix}, \qquad u:=\mathscr{T}. \tag{3}$$

The constant matrices A and B are given by:

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \beta & 0 \end{bmatrix}, \qquad B=\begin{bmatrix} 0 \\ \gamma \\ 0 \\ \varepsilon \end{bmatrix}, \tag{4}$$

where:

$$\begin{aligned} \alpha &:= \frac{-gm_1^2l_1^2L_0}{I_0(J_1+m_1l_1^2)+J_1m_1L_0^2}, \\ \beta &:= \frac{(I_0+m_1L_0^2)m_1l_1g}{I_0(J_1+m_1l_1^2)+J_1m_1L_0^2}, \\ \gamma &:= \frac{J_1+m_1l_1^2}{I_0(J_1+m_1l_1^2)+J_1m_1L_0^2}, \\ \varepsilon &:= \frac{-m_1l_1L_0}{I_0(J_1+m_1l_1^2)+J_1m_1L_0^2}. \end{aligned}$$

Notice that this system is valid for any operation close to the unstable equilibrium point:

$$\begin{bmatrix} \theta_0^* \\ \dot{\theta}_0^* \\ \theta_1^* \\ \dot{\theta}_1^* \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{5}$$

(that is, the pendulum is located at the upright position).

B. State Feedback Gain Design

As mentioned above, in order to stabilize the linear system (2) through a state feedback control law we use a similar approach to the one presented in [8]. This is known as the standard LQR control problem. The technique consists in computing the optimal solution for the linear quadratic cost functional:

$$J=\inf_{u(t)\in L_2(0,\infty)}\int_0^\infty\left[x(t)^TQx(t)+ru(t)^2\right]dt. \tag{6}$$

In this expression, the weights $Q\geq 0$ and $r\geq 0$ are chosen with the purpose of reducing the states x and the cost of the control u. Qualitatively, if Q is a diagonal matrix, the position of the greater value of this matrix represents the most important state to be reduced. In the same manner, as r is chosen with a greater value, in such a way that the energy provided by the control law u must be lower.

The solution to the functional (6) is the state feedback control law:

$$u(t)=-K^*x(t)=r^{-1}B^TPx(t), \tag{7}$$

where:

$$K^*=[k_a,k_b,k_c,k_d]=-r^{-1}B^TP, \tag{8}$$

and P is the unique symmetric positive-defined solution to the Riccati equation:

$$A^TP+PA-PBr^{-1}B^TP+Q=0. \tag{9}$$

III. Computing Delay Margin and Retarded Gains Tuning

This section contains two stability analysis by considering a time-delay in the state feedback loop. First, we compute the critical delay value in the feedback loop for a proper choice of K^*. Second, we propose two auxiliary gains which will give us two degrees of freedom (2-DOF), allowing to improve the system's response.

A. Delay Margin and Robustness Issues

It is well known that in a closed-loop system, if the control law is implemented by means of a digital platform, then, there always be present a time-delay due to the computational data processing. In this regard, the delay is a consequence of sensor with built-in data processing. In this section we aim to characterize this behavior by considering a time-delay in the control law.

Having designed the vector gain K^* as shown in the previous section, we propose the following:

$$u(t) = -K^* x(t-\tau), \tag{10}$$

where $\tau > 0$ is a fixed delay value. Furthermore, one may notice that τ can be defined as $\tau := \tau_p + \tau_d$ where the time-delay values τ_p and τ_d refers to the data processing and, to a control design parameter, respectively.

Remark 1: It is well known that the stability of the closed-loop system is directly related to the location of the roots of the characteristic equation (see, [3], for further details). More precisely, the closed-loop system is stable if and only if all the roots of the characteristic equation are located in the LHP (Left-Half Plane) of the complex plane.

Since for $\tau = 0$ the closed-loop system is locally asymptotically stable around the origin, therefore, all of the roots of the characteristic equation given by:

$$\Delta_0(s) := \det\{sI - (A - BK^*)\} = 0, \tag{11}$$

have negative real parts. In other words, all of its roots remain in the LHP of the complex plane for a proper choice of K^*. Now, by taking into account the control law (10), the characteristic equation of the closed-loop system can be expressed as:

$$\Delta_\tau(s) = \det\left\{sI - (A - BK^* e^{-\tau s})\right\} = 0, \tag{12}$$

or more compactly as:

$$\Delta_\tau(s) = P(s) + Q(s)e^{-\tau s}, \tag{13}$$

where:

$$\begin{aligned} P(s) &= s^4 - \beta s^2, \\ Q(s) &= (k_b\gamma + k_d\varepsilon)s^3 + (k_a\gamma + k_c\varepsilon)s^2 \\ &+ k_b(\alpha\varepsilon - \beta\gamma)s + k_a(\alpha\varepsilon - \beta\gamma). \end{aligned}$$

Remark 2: As mentioned by [3], this type of function ($\Delta_\tau(s)$) is known as a quasi-polynomial, one of its main differences with respect to a common polynomial, is that it has an infinite number of roots. Furthermore, the roots of $\Delta_\tau(s)$ move continuously with respect to variations of its parameters (coefficients, delay) and there is always a finite number of roots at the right side of any vertical line of the complex plane.

The appropriate computation of the critical delay value at which the closed-loop system loses stability is given below:

Proposition 1: The closed-loop system is asymptotically stable for any delay value $\tau \in [0, \tau_c)$, where:

$$\tau_c = \min\left\{\tau^* \in \mathbb{R} \,\middle|\, \tau^*(\omega^*) > 0, \omega^* \in \Omega_p\right\}. \tag{14}$$

where:

$$\tau^*(\omega^*) = \frac{1}{\omega^*}\left[\arg\left\{\frac{Q(i\omega^*)}{P(i\omega^*)}\right\} + 2n\pi\right], \qquad n \in \mathbb{Z}, \tag{15}$$

and where the set Ω_p is defined as the set of all real roots of the following equation:

$$|Q(i\omega^*)|^2 - |P(i\omega^*)|^2 = 0. \tag{16}$$

Proof: By taking into account Remark 2, and the fact that the closed-loop system is stable for $\tau = 0$ implies that for $\tau > 0$ sufficiently small all the roots of (13) will remain on the LHP of the complex plane. Moreover, there is a critical value τ such that (13) has at least one root on the imaginary axis and hence, such a value induces to the closed-loop system to lose stability.

As can be seen in [12], there exists a value τ such that the quasi-polynomial $\Delta_\tau(s)$ has a root on the imaginary axis in $s = i\omega^*$, if and only if, the following condition:

$$\left|\frac{Q(i\omega^*)}{P(i\omega^*)}\right| = 1, \tag{17}$$

holds for some value $\omega^* \in \mathbb{R}_+$. Moreover, the correspondent time-delay value can be computed by (15). Furthermore, notice that the condition (17) can be rewritten easily as (16), which is a polynomial, implying that it has a finite number of solutions. Finally, by defining Ω_p as the set of all real roots of (16), the critical delay value can be computed as in (14). ■

B. Extended Controller (2-DOF)

In the previous section, we show a method for computing the margin delay in order to maintain stability in the closed-loop system. In this section, we propose a 2-DOF controller, which will be shown to be useful when the inherent delay in the system is larger than the critical delay computed above.

Let K^* be a stabilizing gain for the delay-free scenario, computed using the results shown in section II-B. We consider as our new state feedback gain:

$$K = K^* + [k_1, 0, k_2, 0], \tag{18}$$

where $k_1, k_2 \in \mathbb{R}$ are compensating gains in both positions (θ_0, θ_1) feedback loop. Considering this extended controller, the characteristic function of the closed-loop system is given by:

$$\Delta_\tau^*(s) = P(s) + \left(Q(s) + (\alpha k_1 + \varepsilon k_2)s^2 + k_1(\alpha\varepsilon - \beta\gamma)\right)e^{-\tau s}.$$

Remark 3: It is worth noticing that the proposed state feedback gain (18) has a particular structure which provides two degrees of freedom in the positions regulation problem.

We are interested in such structure, since the appropriate regulation of position implies the regulation of velocity to the origin.

Now, we introduce some notation: let $\rho(\omega) := \alpha\varepsilon - \gamma(\omega^2 + \beta)$ and, $R(\omega)$ and $I(\omega)$ be the real and imaginary part of $Q(i\omega)$, respectively. Furthermore, it is worth noticing that $P(i\omega) \in \mathbb{R}$, for all real ω. Having explained this approach, the following result characterize the triplet (k_1, k_2, τ) at which the system has at least one root on the imaginary axis.

Proposition 2: Let K^* be a stabilizing gain of the delay-free scenario and let ρ, I and R be as defined previously. Then, the characteristic function $\Delta^*_\tau(s)$ has at least two roots on the imaginary axis at $s = \pm i\omega$, if and only if:

$$\tau_\delta = \frac{1}{\omega}\sin^{-1}\left\{-\frac{I(\omega)}{P(i\omega)}\right\}, \qquad \forall \omega \in \Omega_d, \tag{19}$$

where:

$$\Omega_d := \left\{\omega \in \mathbb{R} \left| \left|\frac{I(\omega)}{P(i\omega)}\right| \leq 1 \right.\right\}, \tag{20}$$

and the gains k_1 and k_2 belong to the family of lines:

$$k_2 = \frac{1}{\varepsilon\omega^2}\left\{\rho(\omega)k_1 + R(\omega) - I(\omega)\cot(\tau\omega)\right\}, \tag{21}$$

for any $\omega \in \Omega_d$. Furthermore, it has a single root at the origin, if and only if:

$$k_1 = -k_a. \tag{22}$$

Proof: Consider the characteristic function $\Delta^*_\tau(s)$, by setting $s = i\omega$ the following equations system is obtained:

$$\Re\{\Delta^*_\tau(s)\} = 0, \qquad \Im\{\Delta^*_\tau(s)\} = 0, \tag{23}$$

by trying to solve this system for k_1 and k_2, the following is computed:

$$\begin{bmatrix} \rho(\omega)\cos(\tau\omega) & -\varepsilon\omega^2\cos(\tau\omega) \\ -\rho(\omega)\sin(\tau\omega) & \varepsilon\omega^2\sin(\tau\omega) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = r(\omega), \tag{24}$$

where $r(\omega)$ is a vector-valued function which can be easily deduced and for the sake of brevity is omitted. It is clear to see that the determinant of the matrix related to equation (24) is equal to zero and, therefore, does not have a unique solution.

However, we can rewrite (23) as:

$$\ell(\omega)\cos(\tau\omega) + I(\omega)\sin(\tau\omega) + P(i\omega) = 0, \tag{25}$$
$$\ell(\omega)\sin(\tau\omega) + I(\omega)\cos(\tau\omega) = 0, \tag{26}$$

where:

$$\ell(\omega) = \rho(\omega)k_1 - \varepsilon\omega^2 k_2 + R(\omega). \tag{27}$$

By solving the system of equations formed by (25) and (26) for $\ell(\omega)$, and consequently comparing the obtained expressions the following condition must be fulfilled:

$$P(i\omega)\sin(\tau\omega) + I(\omega) = 0. \tag{28}$$

On one hand, any pair $(\tau, \hat{\omega})$ satisfying condition (28) also induces to equations (25) and (26) to be equivalent. On the other hand, for every $\hat{\omega} \in \mathbb{R}$ there exist an infinite set of pairs (k_1, k_2) along the line (26), which also solves (25).

The proof ends by solving τ and k_2 from (28) and (26) and obtaining conditions (19) and (21), respectively. Furthermore, condition (22) can be verified simply by solving k_1 from $\Delta^*_\tau(0) = 0$ and the set Ω_d is defined to consider only real solutions of (19). ■

IV. Illustrative Results

For further details on the examples proposed in the sequel, we refer to the website [1]. Such material is composed by a variety of animations of the Furuta Pendulum system and system response signals behavior. The support material is listed below:

A.1 Free Motion Behavior of the Furuta Pendulum with initial conditions:

$$\left[\theta_0(0), \dot{\theta}_0(0), \theta_1(0), \dot{\theta}_1(0)\right]^T = \left[\frac{\pi}{4}, 0, \frac{\pi}{4}, 0\right]^T. \tag{29}$$

A.2 Controlled Motion with a delay value $\tau = 0$.

A.3 Controlled Motion with a delay value $\tau = 0.5\tau_c$.

A.4 Controlled Motion with a delay value $\tau = 0.9\tau_c$.

A.5 Unstable Response of the Furuta Pendulum with a delay value $\tau = \tau_c$.

A.6 Smooth Time Delay Variation of the System Transient Response from $\tau = 0$ to $\tau = \tau_c$.

The parameters used in these simulations are taken from the experimental test bench studied in [9] and are summarized in Tab. I. The initial conditions settled for the following numerical results are chosen near the origin as:

$$\left[\theta_0(0), \dot{\theta}_0(0), \theta_1(0), \dot{\theta}_1(0)\right]^T = \left[\frac{\pi}{10}, 0, \frac{\pi}{9}, 0\right]^T.$$

TABLE I

Parameters of the System

Symbol	Value	Unit
g	9.81	$\frac{m}{s^2}$
l_1	129×10^{-3}	m
L_0	155×10^{-3}	m
m_1	22.18×10^{-3}	Kg
J_1	184.50×10^{-6}	$Kg.m^2$
I_0	238.49×10^{6}	$Kf.m^2$

As stated in section II-B, in order to compute the state feedback gain K^* we need to chose the weights Q and r. We set $r = 1$ and:

$$Q = 1 \times 10^{-4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.01 \end{bmatrix}, \tag{30}$$

with the purpose of giving more importance to the convergence of the states θ_0 and θ_1 than to the angular velocities or the control effort. More precisely, we set the position correspondent values in Q one hundred times larger than the ones set for the angular velocities. Given this parameters, the state feedback gain is computed as:

$$K^* = -\left[0.0100, 0.0049, 0.1755, 0.0161\right]^T. \tag{31}$$

[1] https://furutablind.wixsite.com/furuta

Now, as mentioned in section III-A, the proposed strategy is meant to find the critical delay in the state feedback loop such that the closed-loop system becomes unstable. Having calculated the gain value K^* and according to Proposition 1 we use equation (16) to compute the set Ω_p. This consists in one element $\omega^* = 30.88$. Second, we calculate the critical delay value $\tau_c = 0.0344s$ from expression (14).

We show the results obtained under these considerations. Fig. 2 exhibits the closed-loop system response under different time-delay values below the critical condition: $\tau = 0, 0.5\tau_c, 0.9\tau_c$. The results are illustrated from the nonlinear and linear model point of view. As can be seen in Fig. 2, as the time-delay value tends to approximate to the critical value τ_c, the system response tends to have a more oscillatory behavior in both models. This can be explained from the linear model perspective since as $\tau \to \tau_c$, the rightmost root of the characteristic equation (13) approaches the imaginary axis. Illustrative animations of the Furuta Pendulum for this particular cases can be found in A.2, A.3 and A.4. Furthermore, A.6 corresponds to the continuous change in the transient response as $\tau \to \tau_c$.

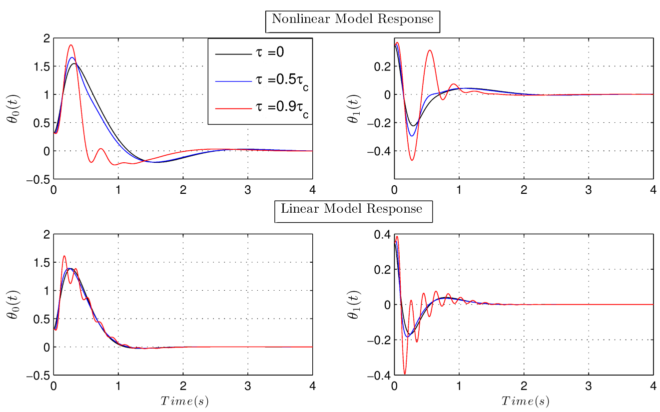

Fig. 2. Closed-Loop System Response Under Different Delay Values.

Moreover, in Fig. 3, we present an unstable response of the closed-loop system by setting $\tau = \tau_c$. At this value, the characteristic equation of the closed-loop system has at least one pair of roots on the imaginary axis. One of the main features of this test is that the nonlinear model clearly loses stability against the linear model which behaves more similar to a marginally stable system. The behavior of the Furuta Pendulum in this conditions is illustrated in A-5.

Finally, we use the result shown in Proposition 2 to compute the stability boundaries for the auxiliary gains k_1 and k_2. The results are depicted in Fig. 4, notice that since K^* is a stabilizing gain for the delay-free scenario then, the origin of this figure illustrated by A is a stable point. Moreover, by remark 2 any variation of the parameters (k_1, k_2, τ) around the origin without crossing any stability boundary is stable. Therefore, the region around the origin delimited by the stability boundaries is a stable region. The point B depicts the parameters setting $(0, 0, \tau_c)$ which corresponds to the boundary of the previous analysis. As expected, this point lays on the stability boundaries. Furthermore, the line from

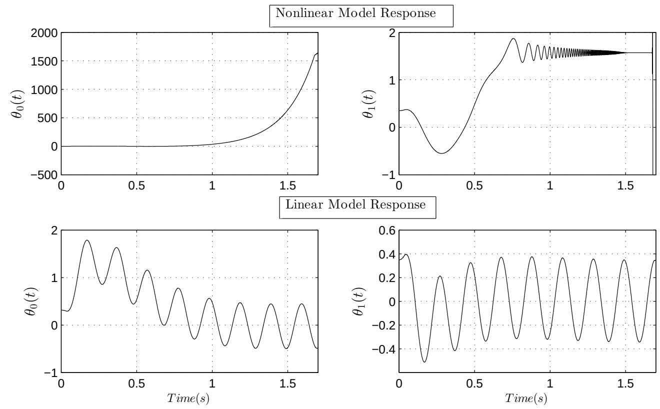

Fig. 3. Closed-Loop System Unstable Response Under a Critical Delay.

A to B corresponds to the test illustrated in Figs. 2 and 3.

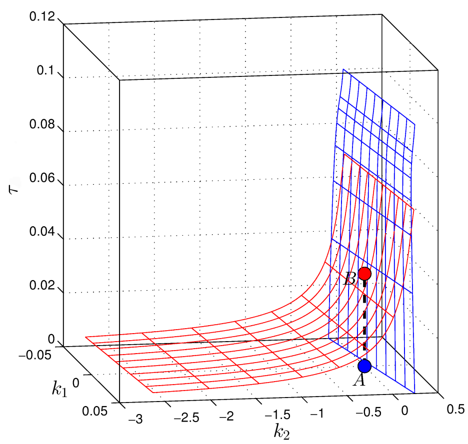

Fig. 4. Stability Boundaries for (k_1, k_2, τ).

In order to verify this result, in Fig. 5 we explore the region in which $k_1 = 0.005$ illustrating the critical delay τ_c. Furthermore, considering a larger delay $\tau = 0.04$, we test this regions by setting the parameters p_1, p_2 and p_3 in the closed-loop system with initial conditions:

$$\left[\theta_0(0), \dot{\theta}_0(0), \theta_1(0), \dot{\theta}_1(0)\right]^T = \left[\frac{\pi}{20}, 0, \frac{\pi}{20}, 0\right]^T.$$

The results are shown in Fig. 6, where is clear to see that the system has a stable response when p_1 and p_2 are chosen and an unstable response for p_3.

V. Concluding Remarks

A methodology for the design of an LQR based control scheme considering a time-delay value in the state feedback loop for the stabilization of the Furuta pendulum is addressed. As stated, the design methodology can be applied and developed straightforwardly, showing that the presented results are easy to implement. Furthermore, support didactic material in form of animations of this control scheme is also addressed.

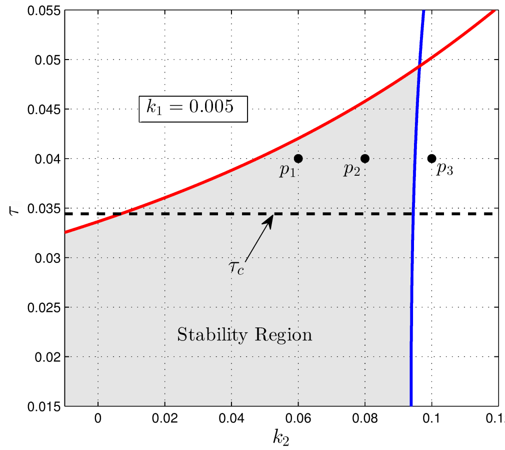

Fig. 5. Stability Boundaries for $(0.005, k_2, \tau)$.

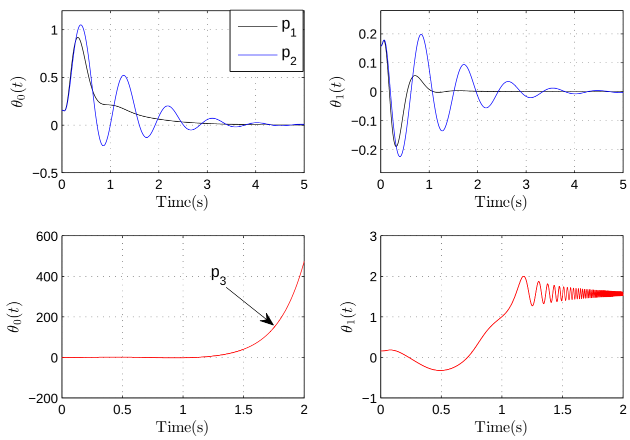

Fig. 6. Closed-Loop System Response Setting the Tunning Parameters p_1, p_2 and p_3.

VI. Future Work

We are interested in study two different design features for the Furuta pendulum stabilization problem, these are discussed in the sequel:

First, we aim to solve the problem of designing a control law such that the system can be driven from its stable equilibrium point in rest, to at least near the unstable point at the upright position. Moreover, considering a time-delayed nature in such a control law.

Second, in order to use the least requirements, we are interested in designing a $P-\delta$-controller type as in [13], which does not require the full state. Moreover, we aim to attack this problem from a different point of view as made in past works in [14], [15] considering also the controllers fragility.

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