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Addendum to Pontryagin’s maximum principle for
dynamic systems on time scales

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Abstract

This note is an addendum to [1, 2], pointing out the differences between these papers and raising open questions.

Keywords: time scale; optimal control; Pontryagin maximum principle; Ekeland variational principle; packages of needle-like variations.

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The main differences. In view of establishing a time scale version of the Pontryagin Maximum Principle (PMP), the authors of [1, Theorem 1] have developed in 2013 a strategy of proof based on the Ekeland variational principle. This strategy was originally considered for the classical continuous case by Ivar Ekeland in his seminal paper [3].

The authors of [2, Theorem 2.11] developed in 2017 a different approach, with packages of needle-like variations and necessary conditions for an extreme in a cone. Note that the authors of [2] prove moreover in [2, Theorem 2.13] that the necessary conditions derived in the PMP are also sufficient in the linear-convex case.

In the sequel of this paragraph, we focus on the major pros and cons of each approach:

1. In [1]:
   (a) The set Ω of control constraints is assumed to be closed. This is in order to apply the Ekeland variational principle on a complete metric space.
   (b) There is no assumption on the time scale T.

2. In [2]:
   (a) The set Ω of control constraints is assumed to be convex, but need not to be closed.

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(b) The time scale $\mathcal{T}$ is assumed to satisfy density conditions (see [2, Definition 2.4]) of the kind
\[
\lim_{\beta \to 0^+} \frac{\mu(s + \beta)}{\beta} = 0,
\]
for every right-dense points $s$, in order to guarantee that
\[
\lim_{\beta \to 0^+} \frac{1}{\beta} \int_{[s,s+\beta)} x(\tau) \Delta \tau = x(s),
\]
for $\Delta$-integrable function $x$ and for right-dense $\Delta$-Lebesgue points $s$, even for $\beta > 0$ such that $s + \beta \notin \mathcal{T}$. Note that a discussion about this issue was provided in [1, Section 3.1].

Hence, the method developed in [2] allows to remove the closedness assumption done on $\Omega$ in [1], but this is at the price of an additional assumption on the time scale $\mathcal{T}$.

In [1, Section 3.1], the authors explained why other approaches (other than the Ekeland variational principle), based for instance on implicit function arguments, or on Brouwer fixed point arguments, or on separation (Hahn-Banach) arguments, fail for general time scales.

As a conclusion, a time scale version of the PMP without closedness assumption on $\Omega$ and without any assumption on the time scale $\mathcal{T}$ still remains an open challenge.

**Additional comments on the terminal constraints.** In [1] the authors considered constraints on the initial/final state of the kind $g(x(t_0), x(t_1)) \in S$, where $S$ is a nonempty closed convex set and $g$ is a general smooth function.

In [2] the authors considered constraints on the initial/final state of the kind $\Phi_i(x(t_0), x(t_1)) = 0$ for $i = 1, \ldots, k$, and $\Phi_i(x(t_0), x(t_1)) \leq 0$ for $i = k+1, \ldots, n$, where $\Phi_i$ are general smooth functions.

Contrarily to what is claimed in [2], the terminal constraints considered in [2] are only a particular case of the ones considered in [1]. Indeed, it suffices to take
\[
g = (\Phi_1, \ldots, \Phi_k, \Phi_{k+1}, \ldots, \Phi_n)
\]
and
\[
S = \{0\} \times \ldots \times \{0\} \times \mathbb{R}^- \times \ldots \times \mathbb{R}^-.
\]

Moreover, note that the necessary condition $-\Psi \in \mathcal{O}_S(g(x(t_0), x(t_1)))$ obtained in [1, Theorem 1] encompasses both the sign condition (1) and the complementary slackness (2) obtained in [2, Theorem 2.11]. For the sign condition, it is sufficient to recall that the orthogonal of $\mathbb{R}^-$ at a point $x \in \mathbb{R}^-$ is included in $\mathbb{R}^+$. For the complementary slackness, it is sufficient to recall that the orthogonal of $S$ at $g(x(t_0), x(t_1))$ is reduced to $\{0\}$ when $g(x(t_0), x(t_1))$ belongs to the interior of $S$.

**Additional comments on the convexity of $\Omega$.** The set $\Omega$ is assumed to be convex in [2], while it is not in [1]. As explained in [1, Section 3.1], in order to apply necessary conditions of an extreme in a cone, the authors of [2] require that the parameters of perturbations live in intervals. As a consequence, in order to remove the convexity assumption on $\Omega$, one would need (local-directional) convexity of the set $\Omega$ for perturbations at right-scattered points, which is a concept that differs from the stable $\Omega$-dense directions used in [1]. Hence, in spite of the claim done in [2], the convexity assumption on $\Omega$ does not seem to be easily removable.
On the universal Lagrange multipliers. This paragraph is devoted to providing more details on the existence of universal Lagrange multipliers claimed in [2, page 25]. In the sequel, we use the notations of [2], and we denote by \( S \) the unit sphere of \( \mathbb{R}^{n+1} \).

A package \( P \) consists of:

- \( N \in \mathbb{N} \) and \( \nu \in \mathbb{N} \);
- \( \tau = (\tau_1, \ldots, \tau_N) \) where \( \tau_i \) are right-dense points of \( T \);
- \( \nu = (\nu_1, \ldots, \nu_N) \) where \( \nu_i \in U \);
- \( \tau = (r_1, \ldots, r_\nu) \) where \( r_i \) are right-scattered points of \( T \).
- \( \tau = (z_1, \ldots, z_\nu) \) where \( z_i \in U \).

Let \( (P_i)_{i \in I} \) denotes the set of all possible packages.

Following the proof of [2, Theorem 2.11], for every \( i \in I \), there exists a nonzero vector \( \lambda = (\lambda_0, \ldots, \lambda_n) \) (that we renormalize in \( S \)) of Lagrange multipliers such that:

(i) (1) and (2) in [2, Theorem 2.11] are satisfied;
(ii) the adjoint vector \( \Psi \) solution of (2.9), with the final condition (3.65) which depends on \( \lambda \), satisfies the initial condition \( \Psi(t_0) = Lx_0 \);
(iii) (4a) and (4b) in [2, Theorem 2.11] are satisfied, but only at the points contained in \( \tau \) and \( \tau \) respectively.

For every \( i \in I \), the above vector \( \lambda \) is not necessarily unique. Then, for every \( i \in I \), we denote by \( K_i \) the set of all nonzero and renormalized Lagrange multiplier vectors associated with \( P_i \) satisfying the above properties.

By continuity of the adjoint vector \( \Psi \) with respect to the Lagrange multipliers (dependence from its final condition), we infer that \( K_i \) is a nonempty closed subset contained in the compact \( S \). This is true for every \( i \in I \).

Now, let us prove that the family \( (K_i)_{i \in I} \) satisfies the finite intersection property. Let \( J \subset I \) be a finite subset and let us prove that \( \cap_{i \in J} K_i \neq \emptyset \). Note that we can construct a package \( P \) corresponding to the union of all packages \( P_i \) with \( i \in J \). It follows that \( P \in (P_i)_{i \in I} \), and thus there exists a nonzero and renormalized Lagrange multiplier vector \( \lambda \) associated with \( P \) satisfying the above properties. Since \( \lambda \in K_i \) for every \( i \in J \), we conclude that \( \cap_{i \in J} K_i \neq \emptyset \).

It follows from the lemma of a centered system in a compact set that \( \cap_{i \in I} K_i \neq \emptyset \), and we deduce the existence of a universal Lagrange multiplier vector.

On the density conditions and the Cantor set. Contrarily to what is claimed in [2, Example 2.5], the classical Cantor set does not satisfy the density conditions. However, generalized versions of the Cantor set (see, e.g., [4]) that satisfy density conditions can be constructed as follows.

Let \( (\alpha_k)_{k \in \mathbb{N}} \) be a real sequence such that \( 0 < \alpha_k < \frac{1}{2} \) for all \( k \in \mathbb{N} \), and such that \( \lim_{k \to +\infty} \alpha_k = \frac{1}{2} \). Let \( (A_k)_{k \in \mathbb{N}} \) be the sequence of compact subsets defined by the induction

\[
A_0 = [0, 1], \quad A_{k+1} = \mathcal{J}_k(A_k) \quad \forall k \in \mathbb{N},
\]
where $T_k$ denotes the operator removing the open $(\alpha_k, 1 - \alpha_k)$-central part of all intervals. Note that the classical Cantor set corresponds to the case where $\alpha_k = \frac{1}{3}$ for every $k \in \mathbb{N}$.

In our situation, we obtain

$$A_1 = [0, \alpha_0] \cup [1 - \alpha_0, 1],$$

$$A_2 = ([0, \alpha_1] \cup [(1 - \alpha_1)\alpha_0, \alpha_0]) \cup ([1 - \alpha_0, 1 - (1 - \alpha_1)\alpha_0]) \cup [1 - \alpha_1\alpha_0, 1],$$

etc. We define the generalized Cantor set $T = \cap_{k \in \mathbb{N}} A_k$. In order to prove that the time scale $T$ satisfies the density conditions, from the fractal properties of $T$, it suffices to prove that the density condition is satisfied at the right-dense point $0 \in T$. More precisely, it is sufficient to prove that

$$\lim_{\beta \to 0^+} \frac{\mu(\beta)}{\beta} = 0.$$ 

Since $\mu(\beta) = 0$ for every right-dense point $\beta$, we only have to consider the case where $\beta$ is a right-scattered point of $T$. In that case, one can easily see that $\frac{\mu(\beta)}{\beta} \leq \frac{1 - 2\alpha_k}{\alpha_k}$ for some $k \in \mathbb{N}$ and that $k$ tends to $+\infty$ when $\beta$ tends to 0. The conclusion follows from the fact that $\lim_{k \to +\infty} \alpha_k = \frac{1}{2}$.

References


