Direct and inverse problem in multiple-scattering of small obstacles in homogeneous media.

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Journée "Jeunes Chercheur-e-s" sur les problèmes d'ondes harmoniques de grande taille .

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Motivation

Theme

Simulating propagation of acoustic waves through ***(highly) heterogeneous media*** .

Goal

Create effective numerical tools for

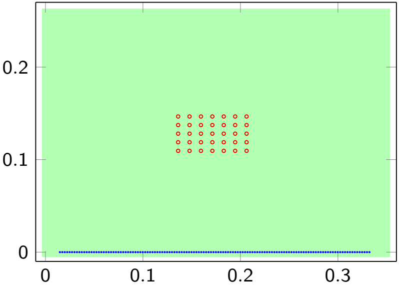
- direct simulation
- and inverse problems

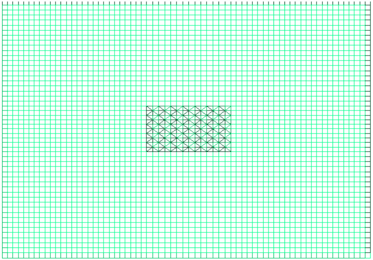
Collaboration with the acoustic lab I2M (`i2m.u-bordeaux.fr`)

Motivation (cont)

Heterogeneities produced by obstacles

- Domains of size ≥ 100 incidence wavelength λ
- Obstacles of radius $\leq 0.3\lambda$.





Volume-discretization based methods lose their robustness in these settings :

large linear systems, numerical pollution caused by dispersion, etc.

- I2M uses COMSOL (commercial software, finite-element based).
- Highly-optimized Software in Magique3D : MONTJOIE, HOU10NI.

Overview

1. Introduction of method
2. Comparison with Montjoie
3. Solver's robustness comparison
    - Closely spaced obstacles
    - Far away obstacles
4. Discussion of the inverse problems
    - An example of an localization problem and data
    - Discussion of reconstruction method
5. Numerical inversion experiments
    - Periodic configuration of 6 obstacles with 30dB
    - Periodic configuration of 12 obstacles with 25dB
    - Random configuration of 12 obstacles with 30 dB

Plan

1. Introduction of method

Multiple obstacle scattering as Exterior Boundary Value problems

Propagation of acoustic waves of freq. **f** in a hom. medium with sound speed **c**.

$$u_{\text{total}} = u_{\text{inc}} + u_{\text{scatt}}.$$

1. PDE satisfied by u_{scatt} outside of the obstacles:

$$(-\Delta - \kappa^2)\, u_{\text{scatt}} = 0\,,\ \kappa = \frac{2\pi \mathbf{f}}{\mathbf{c}}.$$

2. Conditions on the boundary of the obstacles:

Dirichlet	$\gamma_0^+ u_{\text{total}} = 0$
Neumann	$\gamma_1^+ u_{\text{total}} = 0$
Impedance	$\gamma_1^+ u_{\text{total}} + \mathrm{i}\lambda\gamma_0^+ u_{\text{total}} = 0$

3. (Outgoing) Sommerfeld radiation condition at ∞:

$$\lim_{r\to\infty} \sqrt{r}\,(\partial_r\, u_{\text{scatt}} - \mathrm{i}\,\kappa u_{\text{scatt}}) = 0 \quad ; \quad r = |x|$$

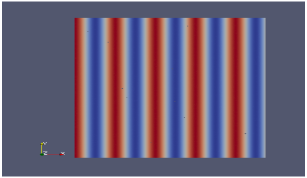
$\exists!$ solution for the exterior BVPs (all parameters > 0).

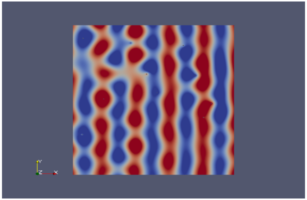
Time-harmonic Planewave :

$$u_{\text{pw}}(x)\ \exp(-i\,2\pi\,\mathbf{f}\,t)$$

$$u_{\text{pw}}(x) = \exp\left(\kappa\, x \cdot \binom{\cos\alpha_{\text{inc}}}{\sin\alpha_{\text{inc}}}\right)$$

$$\alpha_{\text{inc}} = 0^\circ\,,\ 2\pi\mathbf{f} = 1.0\,,\ \kappa = 1.0.$$





References: Hettlich, *Fréchet derivatives in inverse obstacle scattering*.
Colton, Kress, *Integral equation methods in scattering theory*.

Motivation

$$\mathbb{R}^2 = \underbrace{\Omega^-}_{\text{bounded}} \cup \Gamma \cup \Omega^+$$

v satisfies
$$\begin{aligned} &(-\Delta - \kappa^2)v = 0 \text{ in } \Omega^- \\ &(-\Delta - \kappa^2)v = 0 \text{ in } \Omega^+, \quad v \text{ outgoing} \end{aligned}$$

For $x \notin \Gamma$

$$v(x) = \underbrace{-\int_\Gamma G_\kappa(x,y)\,[\gamma_1 v](y)\,ds(y)}_{\mathcal{S}[\gamma_1 v]} + \underbrace{\int_\Gamma \frac{\partial}{\partial n(y)} G_\kappa(x,y)\,[\gamma_0 v](y)\,ds(y)}_{\mathcal{D}[\gamma_0 v]},$$

$$G_\kappa(x,y) = \frac{i}{4} H_0^{(1)}(\kappa|x-y|)$$

$$[\gamma_0 u] = \gamma_0^+ u - \gamma_0^- u \;;\; [\gamma_1 u] = \gamma_1^+ u - \gamma_1^- u$$

Choices of solution representation and trace operators

$$u_{\text{total}} = u + u_{\text{inc}}$$

Choice of ext. for u	Choice of trace op.	Dirichlet $\gamma_0^+ u_{\text{total}} = 0$	Neumann $\gamma_1^+ u_{\text{total}} = 0$
$u\vert_{\Omega^-} = 0$		$u\vert_{\Omega^-} = -\mathcal{S}\,\gamma_1^+ u + \mathcal{D}\,\gamma_0^+ u$	
$[\gamma_0 u] = 0$		$u = -\mathcal{S}\,[\gamma_1 u]$	
	Outer	Apply $\gamma_0^+ \to$ an equation $=$ EFIE	Apply γ_1^+
	Inner	**Null field method** : Extend $u_{\text{total}} = 0$ on Ω^- $\gamma_0^- \to$ Electric Field IE (EFIE) $\gamma_1^- \to$ Magnetic Field IE (MFIE) $\gamma_1^- + \eta\gamma_0^- \to$ Combined Field IE (CFIE)	
$[\gamma_1 u] = 0$		$u = \mathcal{D}\,[\gamma_0 u]$	
	Outer	Apply γ_0^+	Apply γ_0^+
	Inner		**Null field method** $u_{\text{total}} = 0$ on Ω^- $\gamma_1^- \to$ EFIE 2 $\gamma_0^- \to$ MFIE 2 $\eta\gamma_1^- + \gamma_0^- \to$ CFIE 2
Brackhage - Werner	Outer	$u\vert_{\Omega^+} = (\eta\mathcal{S} + \mathcal{D})\phi$ Apply γ_0^+	$u\vert_{\Omega^+} = (\mathcal{S} + \eta\mathcal{D})\phi$ Apply γ_1^+

Properties of the outgoing Green kernel

$$G_\kappa(x,y) = \tfrac{\mathrm{i}}{4}\,\mathrm{H}_0^{(1)}(\kappa|x-y|) \text{ is } \begin{cases} \text{smooth off the diagonal } \{x=y\} \\ \text{weakly singular around the diagonal} \end{cases} .$$

$$|G_\kappa(x,y)| \leq C|x-y|^{-1+\epsilon} \qquad 0<\epsilon<1\,.$$

$$\mathrm{H}_0^{(1)}(z) \;=\; \tfrac{2\mathrm{i}}{\pi}\left(\ln\tfrac{|z|}{2} + \substack{\text{Euler}\\ \text{constant}} - \tfrac{\pi \mathrm{i}}{2}\right) + \mathbf{O}\left(|z|^2 \ln\tfrac{1}{|z|}\right), \quad |z|\to 0\,.$$

$$\mathcal{S} : H^s(\Gamma) \to H^{s+\frac{3}{2}}_{\text{loc}}(\mathbb{R}^2) \quad \text{is bounded for } \begin{cases} -1<s<0 & ,\ \Gamma \text{ Lipschitz} \\ -1<s & ,\ \Gamma\ \mathcal{C}^\infty \end{cases}$$

The definition $\quad \mathcal{S}\phi := \int_\Gamma G_\kappa(x,y)\,\phi(y)\,,\,ds(y)\ ,\ x\notin\Gamma\ ,\ \phi\in L^1(\Gamma)$

extended to $\quad \mathcal{S} \;:=\; \mathcal{N}\,\gamma_0'$

$\mathcal{N}$ is the Newton potential $\mathcal{N}f := \int_{\mathbb{R}^2} G_\kappa(x,y)\,f(y)\,dy\ ,\ f\in L^2_{\text{comp}}(\mathbb{R}^2)\,.$

Jump of single-layer potential

Γ Lipschitz, $\phi \in H^{-1/2}(\Gamma)$

$$[\gamma_0 \mathcal{S}\phi] \;=\; 0 \quad , \quad \text{in} \;\; H^{1/2}(\Gamma) \quad ; \quad [\gamma_1 \mathcal{S}\phi] \;=\; -\phi \quad , \quad \text{in } H^{-1/2}(\Gamma)$$

Zero-th trace of single-layer potential

$$\mathrm{S} := \gamma_0 \mathcal{S} \;:\; H^s(\Gamma) \to H^{s+1}(\Gamma) \quad \text{is bounded for} \begin{cases} -1 < s < 0 & , \; \Gamma \text{ Lipschitz} \\ -1 < s < r + \frac{1}{2} & , \; \Gamma \in \mathcal{C}^{r+1,1} \end{cases}$$

Integral presentation (for $\Gamma \in \mathcal{C}^2$)
$$(\mathrm{S}\phi)(x) \;:=\; \int_\Gamma G_\kappa(x,y)\,\phi(y)\,ds(y)\;, x \in \Gamma \;,\; \phi \in L^\infty(\Gamma)\,.$$

Conormal derivative of single-layer potential

$$\gamma_1^{\pm} \mathcal{S} = \mp \tfrac{1}{2}\mathrm{Id} + \mathrm{D}' \;,\; \mathrm{D}' : H^s(\Gamma) \to H^s(\Gamma) \text{ bounded for } \begin{cases} -1 < s < 0 & , \; \Gamma \text{ Lipschitz} \\ -1 < s < r + \frac{1}{2} & , \; \Gamma \in \mathcal{C}^{r+1,1} \end{cases}$$

Integral presentation (for $\Gamma \in \mathcal{C}^2$)
$$(\mathrm{D}'\phi)(x) := \int_\Gamma \phi(y)\,\frac{\partial}{\partial n(x)} G_\kappa(x,y)\,ds(y)\;, x \in \Gamma \;,\; \phi \in L^\infty(\Gamma)\,.$$

Single layer potential formulation for multi-scattering .

Exterior Boundary Value Prob

$u_{\text{total}} = u_{\text{inc}} + u_{\text{scatt}};$

$$u_{\text{scatt}} = \sum_{J=1}^{N} u_{\text{scatt},J}\,;$$

$$u_{\text{scatt},J} := \mathcal{S}_J v_J = \int_{\Gamma_J} G_\kappa(x,y)\, v_J(y)\, ds(y).$$

Impose BC

Integral Eqn : Find densities v_J s.t. :

For $I = 1, \ldots, N$

$$\sum_{J=1}^{N} \gamma_{0,I}\, \mathcal{S}_J\, v_J = -\gamma_{0,I}\, u_{\text{inc}};\ \text{Dirichlet}$$

$$\sum_{J=1}^{N} \gamma_{1,I}^{+}\, \mathcal{S}_J\, v_J = -\gamma_{1,I}\, u_{\text{inc}};\ \text{Neumann}$$

BEM

Choose finite-dim subspaces approximating $H^{-1/2}(\Gamma_I)$ (Dir), $H^{1/2}(\Gamma_I)$ (Neu)

Fourier Series Single Layer

Other methods of discretization

2π-periodic parametrization $\phi_J : [0, 2\pi)_\theta \rightarrow$ Boundary of obstacle J.

Basis element $\mathbf{w}_{Jl}(\phi(\theta)) = e^{\mathrm{i}\, l\, \theta}$, $1 \leq J \leq N$, $l \in \mathbb{Z}$.

Nÿstrom, Collocation, Galerkin collocation, etc

Disc-shaped obstacles

0th order approx.

Multipole

0th order approx

Foldy isotropic point scattering

Fourier Series Single Layer method.

The scattered and approx. wave

$$u_{\textbf{scatt}} = \sum_{J=1}^{N} u_{\textbf{scatt};J} \,,$$

$$u_{\textbf{scatt},h} = \sum_{J=1}^{N} u_{h,\textbf{scatt};J} \,.$$

The exact and app. wave scattered by Obs J

$$u_{\textbf{scatt};J} = \mathcal{S}_J v_J \;;\; u_{h,\textbf{scatt};J} = \mathcal{S}_J v_{h,J} \,.$$

In basis elements

$$\mathbf{w}_{J,k}(x) = e^{i\,k\,\theta_J(x)},$$

$$u_{\text{scatt};J} = \sum_{k\in\mathbb{Z}} V_{J,k}\,\mathcal{S}_J\,\mathbf{w}_{J,k}$$

$$u_{h,\text{scatt};J} = \sum_{k=-\mathbf{m}}^{\mathbf{m}} V_{J,k}\,\mathcal{S}_J\,\mathbf{w}_{J,k} \,.$$

The unknowns are the Fourier coefficients of density v_J

$$V = (V_{J,k}) \,,\; k \in \mathbb{Z}, 1 \le J \le N,$$

and the truncated ones for the approx. $v_{h,J}$.

$$V_h = (V_{J,k}), -\mathbf{m} \le k \le \mathbf{m} \,,\; 1 \le J \le N \,.$$

For α = D, N, Im, they solve

$$\mathbf{A}_\alpha\, V = F_\alpha \quad , \quad \mathbf{A}_{h,\alpha}\, V_h = F_{\alpha,h} \,.$$

$$\mathbf{A}_\alpha = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1(\mathbf{N}-1)} & \mathbf{A}_{1\mathbf{N}} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2(\mathbf{N}-1)} & \mathbf{A}_{2\mathbf{N}} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \mathbf{A}_{(\mathbf{N}-1)1} & \mathbf{A}_{(\mathbf{N}-1)2} & \cdots & \mathbf{A}_{(\mathbf{N}-1)(\mathbf{N}-1)} & \mathbf{A}_{(\mathbf{N}-1)\mathbf{N}} \\ \mathbf{A}_{\mathbf{N}1} & \mathbf{A}_{\mathbf{N}2} & \cdots & \mathbf{A}_{\mathbf{N}(\mathbf{N}-1)} & \mathbf{A}_{\mathbf{NN}} \end{pmatrix}$$

$\mathbf{A}_{h,\alpha}$ square matrix of size $(2\mathbf{m}+1)\times N$.

$\mathbf{A}_{\alpha,I}$ self-interaction of obstacle I

$\mathbf{A}_{\alpha,IJ}$ diffraction by obs. I of wave emitted by obs. J

Multi-scattering with circular obstacles.

Single-layer potential with density $\mathbf{w}_{J,k}$ can be written in multipole expansions,

$$(\mathcal{S}_J\, \mathbf{w}_{J,k})\,(x) = \frac{\mathrm{i}\pi\, \mathrm{r}_J}{2}\, \mathrm{J}_k(\kappa\, \mathrm{r}_J)\ \underbrace{\mathrm{H}_k^{(1)}(\kappa\, r_J(x))\ e^{\mathrm{i}\, k\, \theta_J(x)}}_{\substack{\text{multiple pole of order k} \\ \text{placed at the center of } \mathcal{O}_J}}\,.$$

Same obstacle interaction

$$(\mathbf{A}_I)_{kl} \;=\; \mathrm{i}\pi\, \mathrm{r}_I\ \mathrm{J}_k(\kappa\, \mathrm{r}_I)\ \delta_{kl} \begin{cases} \mathrm{H}_k^{(1)}(\kappa\, \mathrm{r}_I) & \text{Dirichlet} \\ \kappa\ \mathrm{H}_k^{(1)\prime}(\kappa\, \mathrm{r}_I) & \text{Neumann} \end{cases}, \qquad k, l \in \mathbb{Z}\,.$$

Interaction between two different obstacles $I \neq J$

$$(\mathbf{A}_{IJ})_{kl} = \mathrm{i}\,\pi\, \mathrm{r}_J\, e^{\mathrm{i}(l-k)\theta_{\mathbf{x}_J}(\mathbf{x}_I)}\ \mathrm{H}_{l-k}^{(1)}\,(\kappa\, \mathrm{d}_{IJ})\ \mathrm{J}_k(\kappa\ \mathrm{r}_I) \begin{cases} \mathrm{J}_l(\kappa\, \mathrm{r}_J) & \text{Dirichlet} \\ \kappa\, \mathrm{J}_l'(\kappa \mathrm{r}_I) & \text{Neumann} \end{cases},$$

$$\mathrm{d}_{IJ} = |\mathbf{x}_I - \mathbf{x}_J| \quad ; \quad k, l \in \mathbb{Z}.$$

Obstacle I of radius r_I .
Relative polar coordinates $(r_J(\cdot), \theta_J(\cdot))$ with respect to obstacle $\mathbf{x}_J$

$$x = \mathbf{x}_J + r_J(x)(\cos\theta_J(x), \sin\theta_J(x))$$

Well-posedness

$$0 \leq \kappa < \infty \quad ; \quad \lambda \in \mathbb{R} \, .$$

If κ^2 is not a Dirichlet eigenvalues (EV) of $-\Delta$ for $\mathcal{O}_l$ for $1 \leq l \leq N$, then the following maps are injective

$$\mathbf{A}_\alpha : \mathbb{H}^{1/2}(\Gamma_{\text{Obs}}) \longrightarrow \mathbb{H}^{1/2}(\Gamma_{\text{Obs}}) \ , \ \text{Impedance, Neumann}$$

$$\mathbf{A}_\alpha : \mathbb{H}^{-1/2}(\Gamma_{\text{Obs}}) \longrightarrow \mathbb{H}^{1/2}(\Gamma_{\text{Obs}}) \ , \ \text{Dirichlet}$$

$$\mathbb{H}^s(\Gamma_{\text{Obs}}) = H^s(\Gamma_1) \times \ldots \times H^s(\Gamma_N)$$

Well-posedness and small obstacles

Circular obstacles

Dirichlet EV : $\lambda_{n,m} = \left(\frac{j_{n,m}}{\mathrm{r}}\right)^2$,

$j_{n,m}$ m-th positive root of $J_n(r) = 0$,

$\mathrm{r} =$ radius of obstacle.

Injectivity : $\kappa_e^2\, \mathrm{r}^2 \neq j_{n,m}$.

The first 4 roots :

$j_{0,1} \sim 2.40$, $j_{1,1} \sim 3.83$, $j_{2,1} \sim 5.13$, $j_{1,2} \sim 5.52$.

General shape obstacles

Isoperimetric inequality gives

$$\lambda_1(\mathcal{O}) \;\geq\; \frac{\pi}{\text{Area}\,(\mathcal{O})}\; j_{0,1}^2 \,.$$

Injectivity for small obstacles

$$\kappa_e \;\; \mathrm{r}_{\text{circumvent}}(\mathcal{O}) \;<\; 2$$

Multiple Scattering Literature

- **Single-layer method**

Thierry, Bertrand. *Analyse et simulations numériques du retournement temporel et de la diffraction multiple*. Diss. Université Henri Poincaré-Nancy I, 2011.

Thierry, Bertrand, et al. *μ-diff: an open-source Matlab toolbox for computing multiple scattering problems by disks*. (2015): 348-362.

- **Modified single-layer method** (single + double layer)

Ganesh, Mahadevan, and Stuart Collin Hawkins. *An efficient algorithm for simulating scattering by a large number of two dimensional particles*. (2011).

- **T-matrix method**

Amirkulova, Feruza A., and Norris, Andrew. *Acoustic multiple scattering using recursive algorithms*. (2015).

- **Approximation methods for small obstacles**

Challa, D.P., and Sini, Mourad. *On the justification of the Foldy–Lax approximation for the acoustic scattering by small rigid bodies of arbitrary shapes*. (2014).

Bendali, A., Cocquet, P-H and Tordeux, S. *Approximation by Multipoles of the Multiple Acoustic Scattering by Small Obstacles in Three Dimensions and Application to the Foldy Theory of Isotropic Scattering*. (2016).

Plan

2 Comparison with Montjoie

Feature of Direct Simulation Codes

- Written in `Fortran90`.
- Parallelized using MPI,
- Runs on the platform Plafrim of Inria.
- Multi-frequency option.
- Choices of both direct and iterative linear system solvers.

`Mumps` , `Lapack` , `Scalapack`

`GMRES with restart`[3] with various preconditioners

- Validated and compared with highly optimized Montjoie.

[3] GMRES with restart without preconditioner was developped by Luc Giraud's team (Cerfacs). L. Giraud, et al. , *A set of GMRES routines for real and complex arithmetics on high performace computers*, Technical report, CERFACS, tR/PA/03/3 (1997).

Calculation time costs (CPU time)

$$u_{h,\mathbf{scatt}}(x) = \frac{i\pi}{2} \sum_{J=1}^{N_{\mathrm{Obs}}} \mathbf{r}_J \sum_{l=-\mathbf{m}}^{\mathbf{m}} V_{J,l} \quad H_k^{(1)}(\kappa\, r_J(x)) \quad e^{i\, l\, \theta_J(x)} \qquad (\star)$$

Pre-processing time = Time to resolve the linear system for V_h.
Linear system is dense but small : $N_{\mathrm{Obs}} \times (2\mathbf{m}+1)$.

Post-processing time = $\begin{array}{c}\text{Eval. time of LHS of } (\star)\\ \text{at each point of visualization grid}\end{array}$.

Evaluation of Hankel is costly (∼ 540 times more expensive than '+' operation).

$$\text{Cost} \quad \sim \quad N_{\mathrm{Obs}} \;\times\; (\sharp \text{ points of visualization grid}).$$

⋆ Reduce the cost (associated with second factor) by

parallelization and $\begin{array}{l}\textbf{interpolation}\\ \text{e.g. Hermite interpolation} \subset \text{cubic spline}\end{array}$.

Experiment 1: Small obstacles on medium domain

Soft-scattering of PW with angle 90°

of wavelength $\kappa = 10$, $\lambda \sim 0.63$

by 200 obstacles

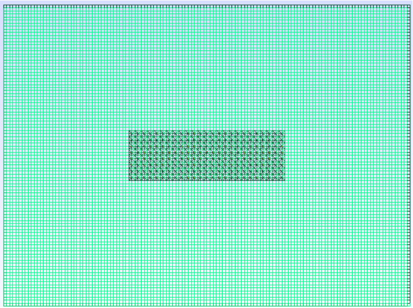
of radius $= 0.03$, with distanced by 0.3.

Domain size : $31\lambda \times 23\lambda$

$\kappa \times (\text{Obs Rad}) = 0.3$,

$\frac{\lambda}{\text{Obs Rad}} \sim 21$, $\frac{\lambda}{\text{Obs. Dist.}} \sim 2$,

$\frac{\text{Obs. Dist.}}{\text{Obs. Rad.}} \sim 10$.



Montjoie initial mesh has mesh size of 0.13.

Montjoie
(`montjoie.gforge.inria.fr`)

Bases: Curved finite element (FE) with Lagrange polynomials based on Gauss-Lobatto points.

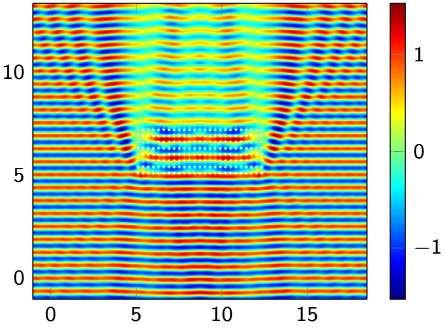
`Q-n` denotes the n^{th} order FE on quadrangular meshes.

Domain truncation: Perfectly Matched Layers.

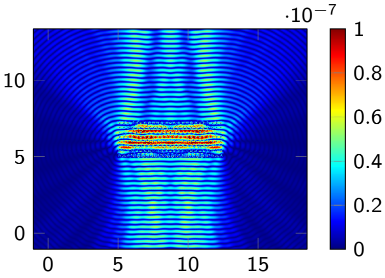
Experiment 1: Reference solutions

Soft-scattering of 200 obstacles on domain of size : $31\lambda \times 23\lambda$

$$\kappa \times (\text{Obs Rad}) = 0.3\ ,\ \frac{\lambda}{\text{Obs Rad}} \sim 21\ ,\ \frac{\lambda}{\text{Obs. Dist.}} \sim 2\ ,\ \frac{\text{Obs. Dist.}}{\text{Obs. Rad.}} \sim 10.$$

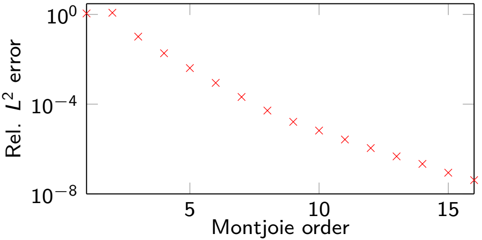


(a) Real part of FSSL 14 total wave

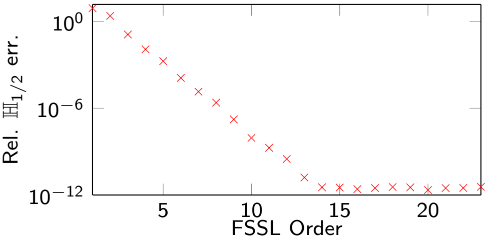


(b) Abs. difference compared with Montjoie Q17. Relative L^2 err. $= 3.38 \times 10^{-8}$.

Experiment 1: Convergence curve



(c) Rel. consecutive err. : Montjoie



(d) Rel. consecutive err : FSSL densities

Candidates for comparison at precision 10^{-3}

Compare between		Rel. L^2 error
FSSL 14	FSSL 2	4.65×10^{-5}
MJ Q17	MJ Q6	6.52×10^{-4}
MJ Q6	FSSL2	6.84×10^{-4}

Hermite interp. precision is 10^{-6}.

Compare between		Rel. L^2 error
FSSL 2 Inter	FSSL 2	1.76×10^{-5}
FSSL 2 Inter	MJ Q6	6.85×10^{-4}

Solvers for both Montjoie and FSSL are Mumps.

Experiment 1: Comparison at precision 10^{-3}

Pre-processing by Mumps	FSSL Order 2	MJ Q6
Size of lin. sys.	1000	842677
Task	**Time (s)**	
Construction	0.055	1.97
Factorization	0.44	29.8
Resolution	0.003	0.35
Total time	**0.498**	**32.12**

Evaluation on 400 $\times$ 400 grid

	Exact eval	Inter. eval	MJ Q6
Post-proc.	26.2	4.30	0.72
Pre-proc. + Post-proc.	**26.70**	**4.80**	**33.82**

At precision 10^{-3}, FSSL using Hermite interpolation takes 7 times less than MJ.

Experiment 2: sizable obstacles on a large domain

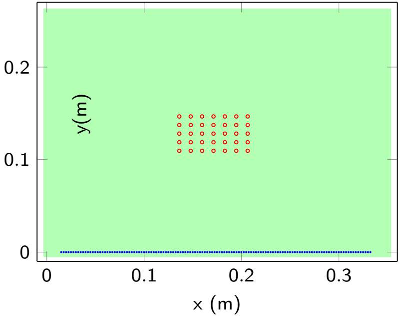
Acoustic vibration, produced by

a block transducer ·,

is diffracted by 35 thin aluminum

wires ◦ (of radius 0.5 mm)

immersed in water.



Horizontal-view cut.

The phenomenon is approximated by the hard scattering of acoustic sound in fluid.

- The incident wave (from the transducer) is simulated by a PW of angle 90°.
- Input pulse's central freq. = 500 kHz.
- The sound speed in water = $1478\ \mathrm{m\,s^{-1}}$.
- The wavenumber $\kappa = 2125.57\ \mathrm{m}^{-1}$.
- The spatial wavelength $\lambda = 2.96 \times 10^{-3}$ m.

$$\text{Domain size} = 117\lambda \times 87\lambda\ .$$

$$\kappa \times (\text{Obs Rad}) \sim 1.1\ , \quad \frac{\text{Obs Dist}}{\text{Obs Rad}} \sim (23, 19)\ ,$$

$$\frac{\lambda}{\text{Obs Rad}} \sim 5.91\ , \quad \frac{\lambda}{\text{Obs. Dist.}} \sim 0.3\ .$$

Exp 2: Computational time comparison at precision 10^{-4}

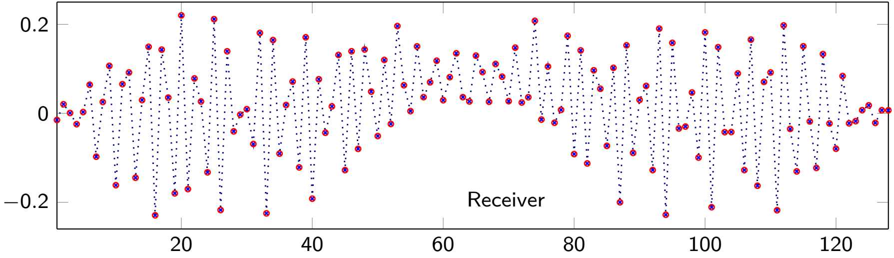
Regarding the value of the diffracted wave at 128 receptors,

Rel. L^2 error : FSSL 12 and FSSL 4 $= 2.82 \times 10^{-6}$,

Rel. L^2 error : MJ Q12 and MJ Q8 Ref 2 $= 1.42 \times 10^{-4}$.

Rel. L^2 error : FSSL 4 and MJ Q8 Ref 2 $= 1.48 \times 10^{-4}$.

Q8 Ref 2 = Q8 with one time mesh refinement.



Real of part of diffracted wave at 128 receptors : FSSL 4 and MJ Q8 Ref2.

Exp 2: Candidates for comparison at precision 10^{-4}

	Size of LS	Pre-proc. Time (s)	Post-proc. Time at 128 receivers (s)	Total time (s)
FSSL 4	315	0.024	6.58×10^{-3}	0.031
MJ Q8 Ref 2	993870	61.27	0.13	61.4

FSSL (with exact evaluation) is 2046 times faster than MJ.

Plan

3. Solver's robustness comparison
   - Closely spaced obstacles
   - Far away obstacles

Restart GMRES (generalized minimal residual method)

Restart with Krylov space size m.

Initial guess x_0.

Initial residue $r_0 = b - A_0$.

No preconditioning : $A\, p_\star \; = \; r_0$.

- Use Arnoldi process, to find, approximate sol. p_j in Krylov space $K_j(A, r_0)$, $j \leq m$, minimizes

$$p_j = \underset{p \in K_j(A\,,\, r_0)}{\operatorname{argmin}} \; \|A\, p - r_0\|_2 \quad (*).$$

- **Stop** if p_j satisfies the residue error criteria.

If not, and if $j = m$, *restart the process* with initial guess $r_0 = p_m$.

- **Final stop criteria** : NiterMax .

Right preconditioning : $(A\mathcal{P}^{-1})(\mathcal{P} p_\star) = r_0$.

Left preconditioning : $(\mathcal{P}^{-1}A)p_\star = \mathcal{P}^{-1} r_0$.

GMRES Preconditioners

L = strictly lower part of matrix A
D = diagonal of matrix A
U = strictly upper part of A

$M_u = U + D$, $N_u = -L$
$M_l = L + D$, $N_l = -U$,
$R = -L - U$.

Splitings of A :

$$A = L + D + U = M_u - N_u = M_l - N_l = D - R.$$

The **backward Gauss-Seidel (BGS)** preconditioner is $\mathcal{P} = M_u$.

The **Jacobi** preconditioner is $\mathcal{P} = D$.

The **2nd-order Jacobi (2Jacobi)** preconditioner is

$$\mathcal{P} = D(R + D)^{-1}D.$$

Formally, $\mathcal{P}^{-1}$ is *the 2nd approx. of the Neumann series of* $A^{-1} = (D - R)^{-1}$.

The **2nd-order Forward Gauss-Seidel (2FGS)** preconditioner is

$$\mathcal{P} = M_l\,(N_l + M_l)^{-1}\,M_l\,.$$

Formally, $\mathcal{P}^{-1}$ is *the 2nd approx. of the Neumann series of* $A^{-1} = (M_l - N_l)^{-1}$.

The **forward Gauss-Seidel (FGS)** preconditioner is $\mathcal{P} = M_l$.

The **Symmetric Gauss-Seidel (SGS)** preconditioner is

$$\mathcal{P} = M_u\,D^{-1}\,M_l\,.$$

Interpretation: $u = \mathcal{P}^{-1}f$ solves

$$M_u\,\tilde{u} = f\,,\; M_l\,u = N_l\tilde{u} + f\,.$$

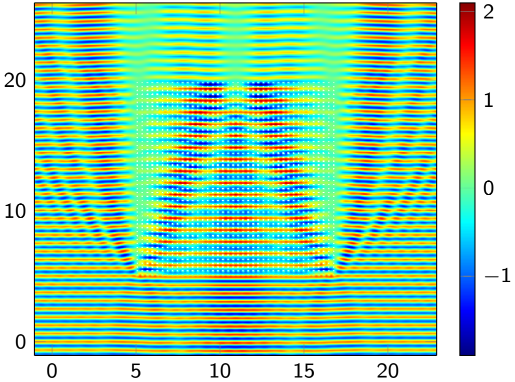
The **Lower-Upper Symmetric Gauss-Seidel (LUSGS)** preconditioner is

$$\mathcal{P} = M_l\,D^{-1}\,M_u\,.$$

Interpretation: $u = \mathcal{P}^{-1}f$ solves

$$M_l\,\tilde{u} = f\,,\; M_u\,u = N_u\,\tilde{u} + f\,.$$

Closely-spaced obstacles comparison



FSSL order 2 with Mumps for 2000 obstacles.

Planewave (PW) with 90°.

Wavenumber $\kappa = 10$.

Radius of obstacle 0.03.

Distance btwn obs 0.3.

$$\kappa \times (\text{Obs Rad}) = 0.3;$$

$$\frac{\lambda}{\text{Obs. Rad}} \sim 21 \;; \quad \frac{\lambda}{\text{Obs Dis}} \sim 2$$

$$\frac{\text{Obs Dist}}{\text{Obs Rad}} = 10.$$

GMRES stop criteria : Residue error tolerance, Niter Max, Size of Krylov.

Exp 3a: Closely-spaced obstacles comparison (Dirichlet)

	Case 200 obstacles				Case 1616 obstacles			
Name Method	**Cv**	δ_{err} in $\mathbb{H}_{1/2}$	$\sharp$ **Iter**	**Time** (s)	**Cv**	δ_{err} in $\mathbb{H}_{1/2}$	$\sharp$ **Iter**	**Time** (s)
Mumps	n/a	0	n/a	0.5	n/a	0	n/a	130
Lapack	n/a	10^{-12}	n/a	0.1	n/a	10^{-10}	n/a	42.7
	GMRES stop criteria (10^{-6}, 2000,100)				GMRES stop criteria (10^{-6}, 2000,150)			
NoPreCond	Y	5×10^{-3}	820	0.9	N	n/a	n/a	n/a
L_Jacobi	Y	5×10^{-3}	656	0.8	N	n/a	n/a	n/a
L_FGS	Y	2×10^{-3}	239	0.5	N	n/a	n/a	n/a
L_BGS	Y	4×10^{-3}	197	0.4	N	n/a	n/a	n/a
L_2Jacobi	Y	5×10^{-3}	594	2.2	N	n/a	n/a	n/a
L_2FGS	Y	1×10^{-3}	169	1.0	N	n/a	n/a	n/a
L_SGS	Y	2×10^{-3}	76	0.3	Y	4×10^{-1}	757	274
L_LUSGS	Y	1×10^{-3}	77	0.3	Y	1×10^{-1}	897	325
R_Jacobi	Y	4×10^{-3}	660	1.1	N	n/a	n/a	n/a
R_FGS	Y	3×10^{-3}	199	0.5	N	n/a	n/a	n/a
R_BGS	Y	3×10^{-3}	198	0.4	N	n/a	n/a	n/a
R_2Jacobi	Y	4×10^{-3}	600	1.7	N	n/a	n/a	n/a
R_2FGS	Y	3×10^{-3}	155	0.9	N	n/a	n/a	n/a
R_SGS	Y	3×10^{-3}	75	0.3	Y	2×10^{-1}	886	321
R_LUSGS	Y	3×10^{-3}	74	0.3	Y	2×10^{-1}	897	325

Exp 3b: Closely-spaced obstacles comparison (Dirichlet)

FSSL order $=2$; Size matrix $= 10^4 \times 10^4$; GMRES stop criteria $(10^{-6}, 5000, 400)$

Solver	Post-proc (n16)	Rel $\mathbb{H}_{1/2}$ diff	Rel L^2 diff	♯ iter	Preproc. time (s)	Postproc. time (s)	Total (s)
Mumps (n16)	Exact	3×10^{-10}	8×10^{-14}	n/a	242	96.0	338
Mumps (n16)	Inter	3×10^{-10}	9×10^{-6}	n/a	242	36.0	278
Lapack (n1)	Exact	0	0	n/a	80.4	96.0	176
Lapack (n1)	Inter	0	9×10^{-6}	n/a	80.4	37.5	118
R_LUSGS (n1)	Exact	1×10^{-1}	4×10^{-5}	1146	573	95.8	669
R_LUSGS (n1)	Inter	1×10^{-1}	4×10^{-5}	1146	573	36.2	609
R_SGS (n1)	Exact	1×10^{-1}	4×10^{-5}	1151	598	95.8	694
R_SGS (n1)	Inter	1×10^{-1}	4×10^{-5}	1151	598	36.2	635
Scala (n16)	Exact	3×10^{-10}	8×10^{-14}	n/a	34.6	95.6	130
Scala (n16)	Inter	3×10^{-10}	9×10^{-6}	n/a	34.6	36.1	70.9

PW of 90° ; $\kappa = 10.0$; $N_{\text{Obs}} = 2000$; Obs. Rad. $= 0.03$; Obs. Dist. $= 0.30$;

$$\kappa \times (\text{Obs Rad}) = 0.3 \ , \ \frac{\lambda}{\text{Obs. Rad}} \sim 21 \ , \ \frac{\lambda}{\text{Obs Dis}} \sim 2 \ , \ \frac{\text{Obs Dist}}{\text{Obs Rad}} = 10.$$

Exp 4: Far apart obstacles (Dirichlet)

FSSL order 2 ; Size matrix = 10000×10000;

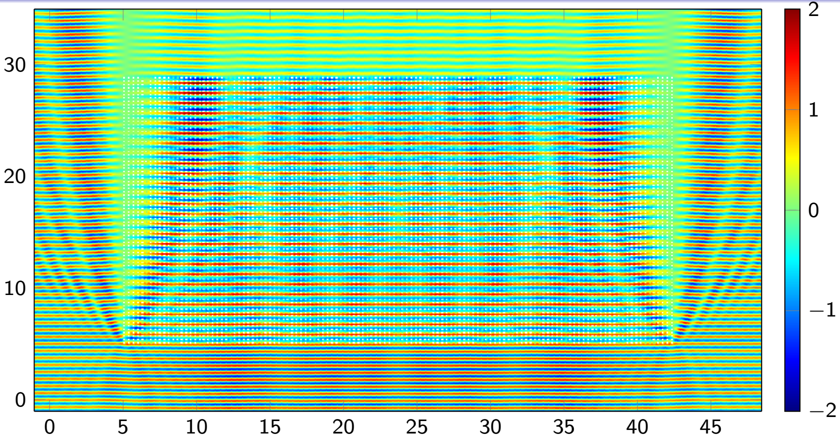
Post-processing on $800\times$ 800 regular grid; GMRES stop criteria (10^{-7}, 5000,500).

Solver	**Post-proc** (n16)	**Rel** $\mathbb{H}_{1/2}$ **diff**	**Rel** L^2 **diff**	**♯ iter**	**Pre-pro. time** (s)	**Post-pro. time** (s)	**Total** (s)
Mumps (n1)	Exact	0.0	0.0	n/a	251	96.0	347
Mumps (n1)	Inter	0.0	1×10^{-5}	n/a	251	37.5	289
Lapack (n1)	Exact	4×10^{-12}	2×10^{-15}	n/a	79.9	96.0	176
Lapack (n1)	Inter	4×10^{-12}	1×10^{-5}	n/a	79.9	37.5	118
R_LUSGS (n1)	Exact	3×10^{-4}	1×10^{-7}	57	37.5	96.0	134
R_LUSGS (n1)	Inter	3×10^{-4}	1×10^{-5}	57	37.5	37.5	75.3
R_SGS (n1)	Exact	4×10^{-4}	1×10^{-7}	56	37.0	96.0	133
R_SGS (n1)	Inter	4×10^{-4}	1×10^{-5}	56	37.0	37.5	74.6
Scala (n16)	Exact	1×10^{-11}	4×10^{-15}	n/a	34.9	96.0	131
Scala (n16)	Inter	1×10^{-11}	1×10^{-5}	n/a	34.9	37.5	72.5

PW of 90.0°; $\kappa = 10.0$; ♯ obs = 2000; Obs. Rad. = 0.01; Obs. Dist. = 2.00;

$\kappa \times$ (Obs Rad) = 0.1 , $\frac{\lambda}{\text{Obs. Rad.}} \sim 63$, $\frac{\lambda}{\text{Obs Dist.}} \sim 0.3$, $\frac{\text{Obs Dist}}{\text{Obs Rad.}} = 200$.

Exp 5



Soft scattering of a planewave coming from the south by 10^4 **obstacles**.

$\kappa \times \mathrm{r} = 0.03 \quad , \quad \frac{\lambda}{\mathrm{r}} \sim 21 \quad , \quad \frac{\mathrm{d}}{\mathrm{r}} = 10 \quad , \quad \frac{\lambda}{\mathrm{d}} \sim 2.$

Preprocessing uses FSSL order 2 + Scalapack. Dense matrix of size 50000×50000.

Post-processing on 800×800 grid of size $79\lambda \times 57\lambda$ uses Hermit inter

Total simulation time = 24 mins 40 secs on 48 processors (of Plafrim).

Plan

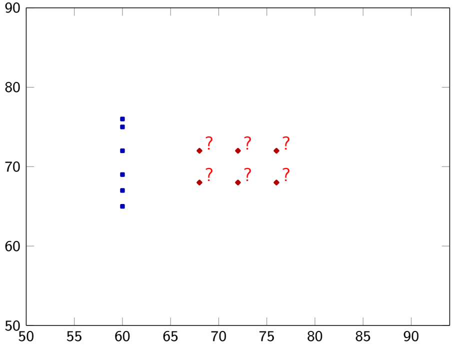
4. Discussion of the inverse problems
   - An example of an localization problem and data
   - Discussion of reconstruction method

An example of localization problem

On $[51, 93]_x \times [51, 89]_y$, locate 6 hard-scattering obstacles of radius 0.5 positioned at ◆

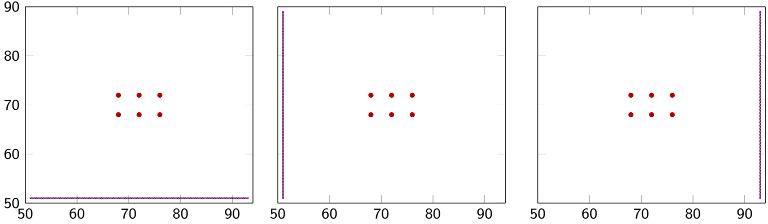
$$(68\,,\,68)\ ,\ (68\,,\,72)\ ,\ (72\,,\,68)$$
$$(72\,,\,72)\ ,\ (76\,,\,68)\ ,\ (76\,,\,72)\,.$$

Initial guesses are placed at ▪.



Synthetic data

The positions of 128 equally-spaced receivers vary with the angle of incidence.



(a) $\alpha_{\text{inc}} = 90^o$ (b) $\alpha_{\text{inc}} = 0^o$ (c) $\alpha_{\text{inc}} = 180^o$

Synthetic Data is produced by FSSL order 12 with solver Lapack.

Complex Gaussian white noise is added by Matlab routine

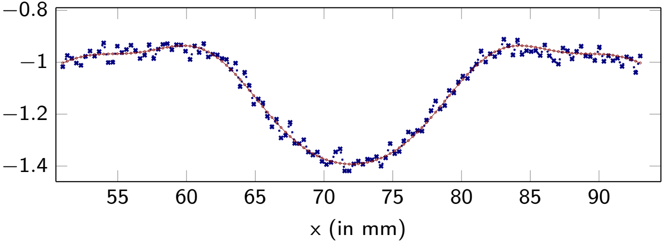
$$\texttt{awgn}(\texttt{data}\,,\ \texttt{SNR}_{\texttt{dB}}\,,\ \text{'}\texttt{measured}\text{'}).$$

$\texttt{SNR}_{\texttt{dB}}$ = signal-to-noise ratio per sample in decibel.

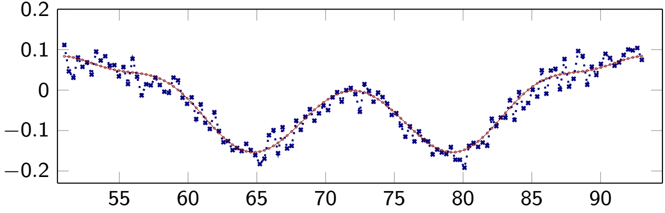
$$\texttt{SNR}_{\texttt{dB}} = 10\log_{10}\frac{\|\text{Data Vector}\|}{\|\text{Noise Vector}\|}.$$

Noise Vector is generated using Gaussian probability distribution.

Synthetic data (cont) : Noise - 30dB



(d) Real part of total wave at 128 receivers at $\kappa = 0.8$ with PW 90°



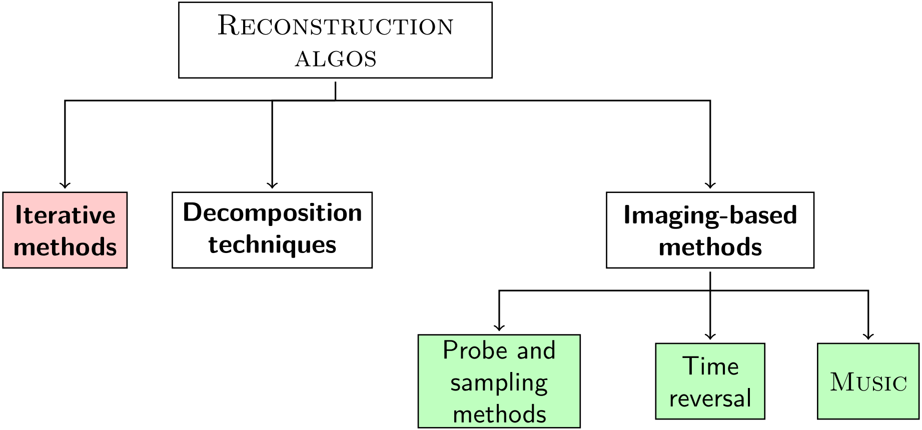
(e) Imaginary part .

Rel. error in norm

$l^2 = 5\%$,

$l^\infty = 8\ \%$

Inversion Literature



Quantitative gradient-based inversion

Find the minimizer of the (reduced) cost function $\widehat{\mathcal{J}}$,

$$\widehat{\mathcal{J}}(\mathbf{m}) \;=\; \frac{1}{2}\,\|\,\Phi(\mathbf{m}) - \mathbf{d_{obs}}\,\|^2 \,.$$

Trace operator at the receptors $\mathcal{R}_{\mathbf{rec}}$: $u|_{\text{receptor}}$, $\partial_n u|_{\text{receptor}}$, etc.

Observed data at receptors : $\mathbf{d_{obs}}$.

Forward map Φ : model $\mathbf{m} \mapsto$ simulated data at receptors.

Main features

Use line-search optimization strategy.

Calculate **gradient** $\nabla_p \widehat{\mathcal{J}}$ by adjoint method (with FSSL formulation).

Use frequency-hopping to escape from stagnation in local minima.

Motivation

- Approximate f by second-order Taylor poly $\mathfrak{M}$.

$$f(\mathbf{m}+\mathbf{s}) \;=\; \mathfrak{M}(\mathbf{s}) \;+\; o(\|\mathbf{s}\|^2).$$
$$\mathfrak{M}(\mathbf{s}) \;:=\; f(\mathbf{m}) \;+\; \mathbf{s}^t\,\nabla f(\mathbf{m}) \;+\; \tfrac{1}{2}\,\mathbf{s}^t\,\nabla^2 f(\mathbf{m})\,\mathbf{s}\;.$$

- Rate of change of f along direction $\mathbf{s}$ at $\mathbf{m}$ is : $\mathbf{s}^t\,\nabla f(\mathbf{m})$.
- Direction $\mathbf{s}$ is called a **descent direction** at $\mathbf{m}$ if $\mathbf{s}^t\;\nabla f(\mathbf{m}) < 0$.

Steepest descent

$\mathbf{s} = -\nabla f(\mathbf{m})$

⋆ **Pros** : does not require second derivatives ; ⋆ **Cons**: slow convergence.

Newton

Newton direction is defined by the minimum of $\mathfrak{M}$ (Assuming $\nabla^2 f_k$ pos def.)

$$\nabla\mathfrak{M} = 0 \;\Leftrightarrow\; \nabla f \;+\; \nabla^2 f\,\mathbf{s} = 0 \;\Leftrightarrow\; \mathbf{s}_{k\,,\,\mathbf{Newton}} := -(\nabla^2 f_k)^{-1}\,\nabla f_k.$$

Search Dir. $\mathbf{s}_{k\,,\,\mathbf{Newton}}$ is a descent direction, if $\nabla f_k \neq 0$ and $\nabla^2 f_k$ pos. def.

⋆ **Pros**: fast rate of local convergence ; ⋆ **Cons**: needs Hessian.

Search directions

- ⋆ Do not need the Hessian ;
- ⋆ faster than Steepest Descent.

Quasi-Newton

Use an approximation B_k (positive and definite) of the Hessian $\nabla^2 f_k$

$$\mathbf{s}_k = -B_k^{-1} \nabla f_k \, .$$

A popular formula is by BFGS (Broyden, Fletcher, Goldfarb and Shannon)

⋆ require storage of matrix.

Nonlinear Conjugate gradient

$$\mathbf{s}_{k+1} = -\nabla f_k + \beta_k \, \mathbf{s}_k$$

A popular formula for β_k is by Polak-Ribière

$$\beta_k = \frac{\nabla f_k^t \, (\nabla f_k - \nabla f_{k-1})}{\nabla f_{k-1}^t \, \nabla f_{k-1}} .$$

⋆ storage of matrix not required.

Adjoint method for calculating the gradient

$$u_{h,\mathbf{scatt}}(x) \;=\; \frac{\mathrm{i}\,\pi}{2} \sum_{J=1}^{N_{\mathrm{Obs}}} \mathrm{r}_J \sum_{l=-\mathrm{n}}^{\mathrm{n}} V_{J,l} \quad \mathrm{H}_k^{(1)}(\kappa\, r_J(x)) \quad e^{\mathrm{i}\, l\, \theta_J(x)} \qquad (\star)$$

$$= \; T(\mathbf{m})^t \; V(\mathbf{m}) \; .$$

Forward map Φ : Model space $\longrightarrow$ Simulated data space

$\mathbf{m} \mapsto u_{h,\text{scatt}}\big|_{\text{receivers}}$

$$\Phi(\mathbf{m}) \;=\; \mathcal{R}_{\mathbf{rec}}\; u_{h,\text{scatt}} \;=\; \mathcal{R}_{\mathbf{rec}}\; T(\mathbf{m})^t\; V(\mathbf{m}) \;=\; \mathfrak{R}(\mathbf{m})\; V(\mathbf{m}).$$

$$\nabla_{\mathbf{m}}\, \widehat{\mathcal{J}} \;=\; \mathrm{Re}\;\left[\; \partial_{\mathbf{m}}\Phi^{\star} \;\left(\,\Phi(\mathbf{m}) \;-\; \mathbf{d}_{\text{obs}}\;\right)\,\right]$$

Adjoint method for calculating the gradient (cnt)

- Avoid calculating the Jacobian $\partial_{\mathbf{m}}\Phi$
- Avoid calculating $\partial_{\mathbf{m}}\mathbf{A}^{-1}$.

Forward linear system	Adjoint linear system
$\mathbf{A}V = F$	$\mathbf{A}^{\star}\gamma_1 = -\mathfrak{R}^{\star}(\mathbf{m})\,(\Phi(\mathbf{m}) - \mathbf{d_{obs}})\,.$

$$\widehat{\mathcal{J}}'(\mathbf{m}) = \boxed{\text{expression in terms of } \gamma_1\ ,\ V\ ,\ \mathbf{d_{obs}}}$$

Inexact Line search algorithm

$$\mathbf{m}_{k+1} \;=\; \mathbf{m}_k \;+\; \overset{?}{\alpha_k}\; \mathbf{s}_k .$$

Exact minimization can be expensive

$\Rightarrow$ use a **line-search algorithm** to obtain an approximate minimum of

$$\min_{\alpha > 0} \quad \phi(\alpha) \;,$$

$$\phi(\alpha) \;=\; \widehat{\mathcal{J}} \left(\mathbf{m}_k \;+\; \alpha\, \mathbf{s}_k\right) .$$

Strategy : Adequate reduce in $\widehat{\mathcal{J}}$ with minimal cost.

Make a 'trade off'

$\star$ Choose α_k so that ϕ reduces substantially;

$\star$ Minimize the time making that choice.

Algo 1: Simple backtracking

$$\phi(\alpha) \;<\; \phi(0)$$

$$\alpha \mapsto 0.5\,\alpha$$

Algo 2: Sufficient decrease

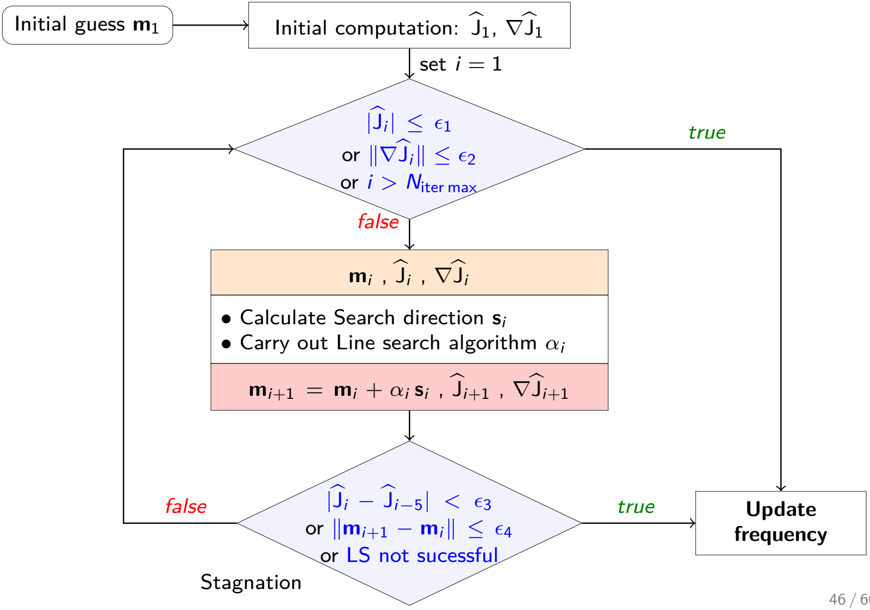
$$\phi(\alpha) \;<\; \phi(0) \;+\; c_1\, \alpha\, \phi'(0)$$

$\alpha \mapsto$ quadratic interpolation on $[0,\alpha]$

Algo 3 :

Strong Wolfe

Optimization algorithm at a frequency



Plan

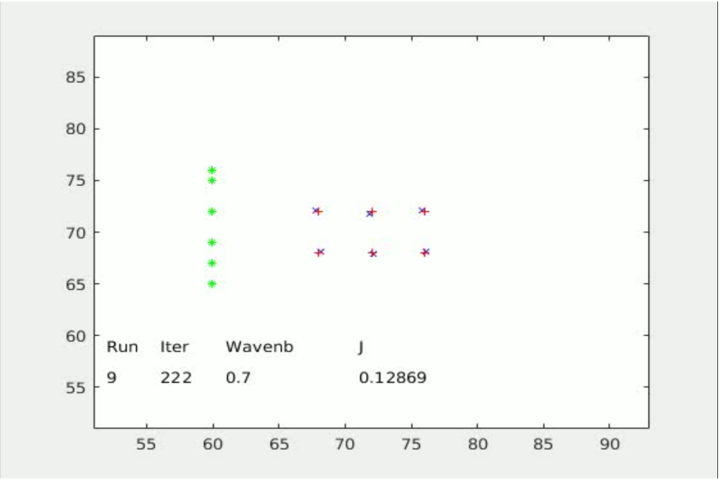
5. Numerical inversion experiments
   - Periodic configuration of 6 obstacles with 30dB
   - Periodic configuration of 12 obstacles with 25dB
   - Random configuration of 12 obstacles with 30 dB

Features of Inverse Problem codes

Latest version

- written in `Fortran90`
- offers choices of different optimization schemes.
- currently uses Mumps.
- integrates a copy of the principal part of the direct simulation codes.

Group 1 - Exp 1 - 6 obstacles with 30dB data



Run time = 2.27s .

Nb of iter = 224.

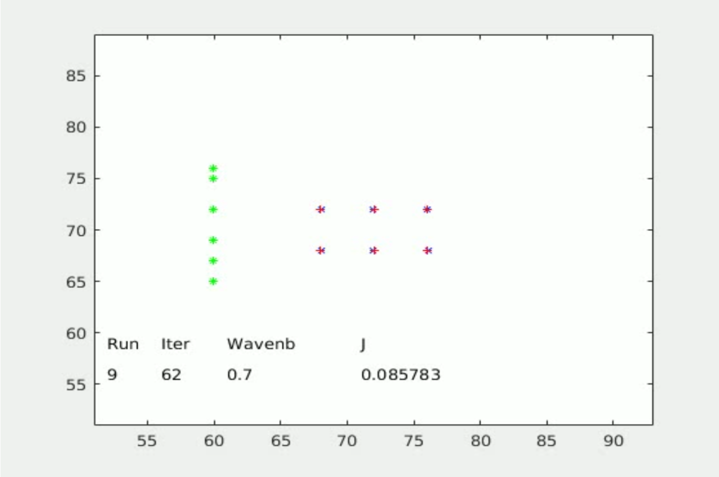
2 incidence angles: 90°, 0°

9 wavenumbers used: 0.08, 0.09 , 0.1- 0.7.

Quasi-Newton and Simple backtracking.

	Err Pos	scaled Err Pos
Initial guess	20.49	53.9%.
Final position	0.108	0.2%.

Group 1 - Exp 2 - 6 obstacles with 30dB data



Run time = 0.30s .

Nb of iter =57 .

2 incidence angle: 90°, 0°

NL Conjugate gradient and Sufficient decrease Backtracking.

	Err Pos	scaled Err Pos
Initial guess	20.49	53.9%
Final position	0.125	0.33%.

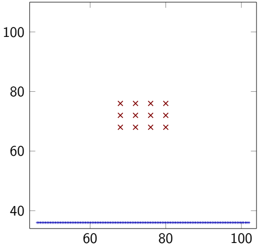
Comparison among the methods

Search Dir (SD)	Line Search (LS)	Linesearch parameters	♯ wn	♯ Iter	Final ErrPos	Final Scaled ErrPos	Run time (secs)
1	1	n/a	9	224	0.108	0.28 %	2.27
1	2	(0.0001, n/a)	9	147	0.130	0.34 %	0.58
1	3	(0.0001, 0.4)	9	46	0.090	0.24%	0.34
1	3	(0.0001, 0.9)	9	64	0.131	0.34 %	0.43
2	1	n/a	10	84	0.171	0.45 %	0.58
2	2	(0.0001, n/a)	9	57	0.125	0.33 %	0.30
2	3	(0.0001, 0.4)	9	61	0.104	0.27 %	0.42
2	3	(0.0001, 0.9)	9	73	0.142	0.37 %	0.41

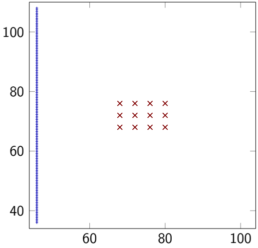
SD 1 : Quasi-Newton ;
SD 2 : NL Conjugate gradient.

LS 1: simple backtracking;
LS 2: backtracking with sufficient descent and quadratic interpolation.
LS 3 : Strong Wolfe.

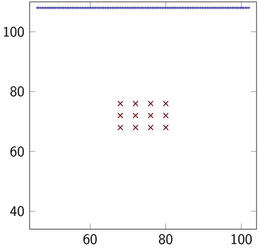
Inversion Exp 2 :



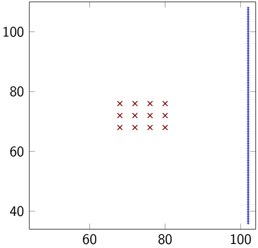
(f) 90° incidence



(g) 0° incidence



(h) 270° incidence



(i) 180° incidence

Locate **12 soft-scattering obs** of radius 0.5 placed at

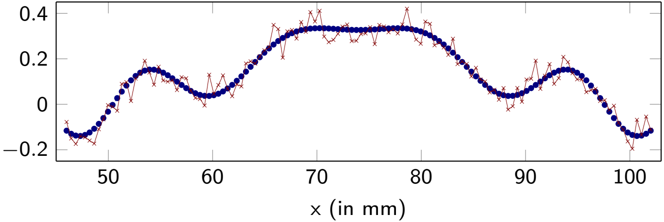
between 68 and 80 (in x);
between 68 and 76 (in y)

For each angle of incidence:

128 receivers on

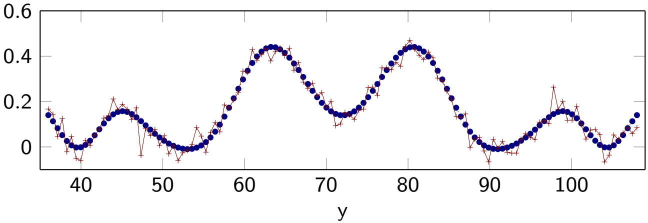
one corresponding side of $[46, 102]_x \times [36, 108]_y$.

Noisy Data at 25dB



(j) Real part of data for 270° incidence. Total relative error in $l^2 = 5.43$ % and $l^\infty = 9.92$ %.

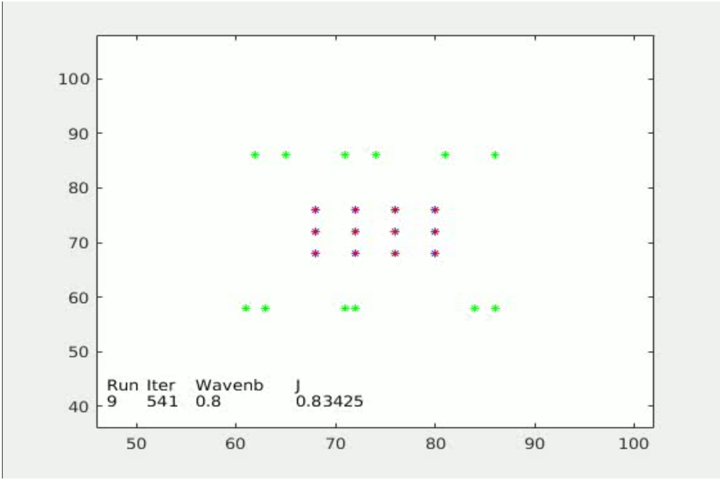
White Gaussian noise is added by using wgn in Matlab.



(k) Imag. part for 180°. Rel. error in l^2 5.24 % and in $l^\infty = 10.95\%$.

Rel. error in norm
l^2 : 4.9 - 6.2% ,
l^∞ : 7.3 - 15.4 %

Inversion result for data with 25dB noise



Run time = 17.2s .

Nb of iter = 552.

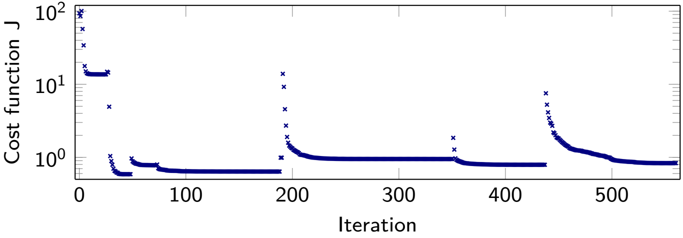
Four angles of acquisitions: 90°, 0°, 180°, 270°

9 wavenumbers used: 0.08, 0.09,0.1-0.6, 0.8.

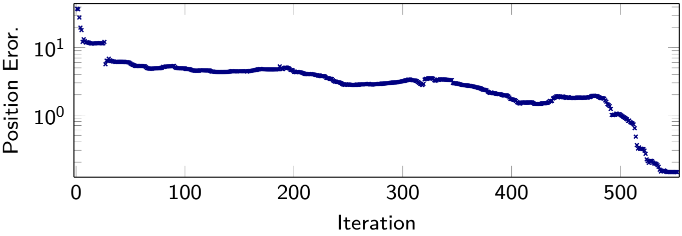
Quasi-Newton and strong Wolfe linesearch.

	Err Pos	scaled Err Pos
Initial Guess	37.6	67.15%
Final position	0.14	0.25%

Inversion result for data with 25dB noise (cnt)



Initial J $=$93.45 at $\kappa = 0.08$; Final J $=$ 0.83 at $\kappa = 0.8$



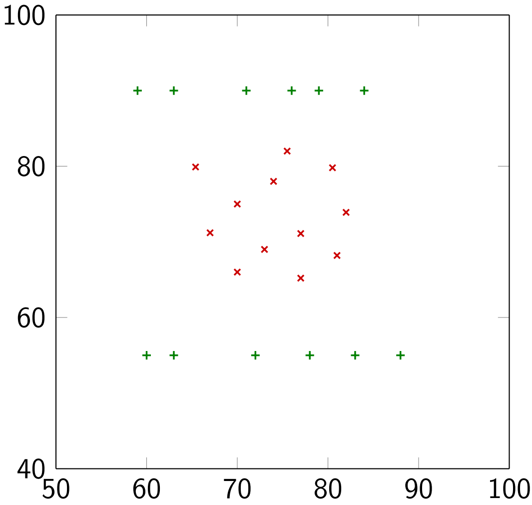
Initial guess: Err Pos $=$ 37.6; rel. err. $=$ 67.15%; Final construction: Err Pos $=$ 0.14; Rel. err $=$ 0.25%.

- $\star$ Four angles of acquisitions:

  $90^\circ, 0^\circ, 180^\circ, 270^\circ$.

- $\star$ Niter total $=$ 552;
- $\star$ Use 9 freqs : 0.08, 0.09, 0.1, . . . , 0.6, 0.8
- $\star$ Run time: 17.2 s

Numerical exp 3: 12 ran Obs and data w/ 30 dB noise



⋆ Locate 12 hard-scattering obstacles of radius 0.5 on domain $[50,100]_x \times [40,100]_y$.

⋆ Ratios

$$0.04 \leq \kappa \mathrm{r} \leq 0.25$$
$$0.34 \leq \kappa \mathrm{d}_{\mathrm{min}} \leq 2.12$$
$$1.56 \leq \kappa \mathrm{d}_{\mathrm{max}} \leq 9.75$$

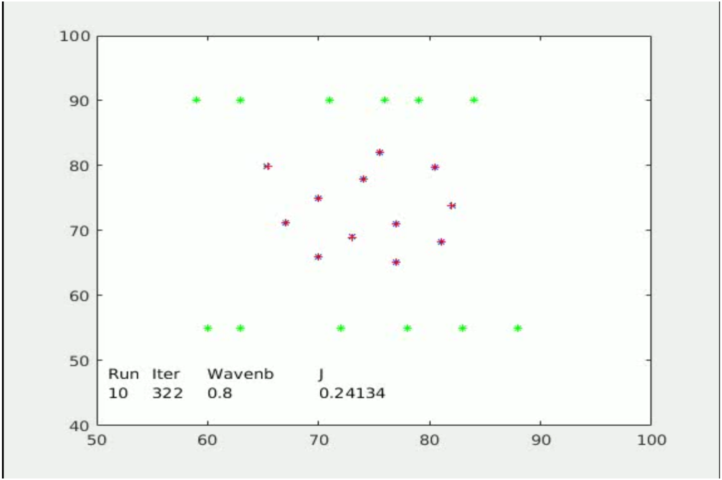
⋆ Four angles of acquisitions:

$$90^\circ, 0^\circ, 180^\circ, 270^\circ$$

⋆ 128 receivers for each angle of incidence, equally on a corresponding side of the domain.

⋆ **Noise** :
$2.7\% \leq l^2$ rel. err $\leq 3.6\%$,
$3.9\% \leq l^\infty \leq 8.8\%$.

Inversion exp 3 results.



Run time = 7.7 s .

Nb of iter = 317.

4 incidence angles:

10 wavenumbers used: 0.08-0.09, 0.1-0.8.

Quasi-Newton and strong Wolfe linesearch.

	Err Pos	scaled Err Pos
Initial guess	39.7	79.4%
Final position	0.17	0.3%

Conclusions

- FSSL is robust in simulating the multi-scattering by small circular obstacles in large homogeneous media.

- Direct Solvers (Lapack and Scalapack) are more efficient when the obstacles are close together.
- Iterative solvers are more preferable when the obstacles are far apart.

In particular, GMRES with LUSGS and SGS are faster than Lapack and as fast as Scalapack.

- LUSGS and SGS are the most robust among the preconditioners considered.

- Direct problem resolution using FSSL and direct solvers are robust in FWI.
- Successful reconstruction in presence of noise.
- Although NL conjugate gradient with cheaper linesearch can be faster in some cases, the more reliable method is **Quasi-Newton** with **strong Wolfe linesearch**.

Future directions

- Compare with other optimization, e.g. Newton-like methods, 2nd order.

- Compare with imaging-based methods, in particular MUSIC.

- Use in combination with such methods for a good initial guess (even without knowledge of the number of obstacles), and then use the current method for precise reconstruction.

- Other inverse problems: determining material parameters within the obstacles.

- Extension to elastic inclusions.

Thank you for your attention !

Questions?

Parameters for experiment : 6 Obs 30dB Exp 1

- Error tolerance and stagnation parameters:

$$\epsilon_{\text{Stag Pos}} = \begin{cases} 0.00005 \text{ for run 1-6} \\ 0.0005 \text{ for run 7-9} \end{cases} \qquad \begin{array}{c} \epsilon_J = \epsilon_{\nabla J} = \epsilon_{\text{Stag LS}} = 0.00001 \\ \epsilon_{\text{Stag J}} = 0.1 \end{array} .$$

- Order FSSL = 3.
- Niter max = 50 ; Niter LS max = 10.

Run	κ	**Order FSSL**	**Step size**
1	0.08	3	15
2	0.09	3	12
3	0.1	3	12
4	0.2	3	12
5	0.3	3	8

Run	κ	**Order FSSL**	**Step size**
6	0.4	3	8
7	0.5	3	5
8	0.6	3	5
9	0.7	3	3

Parameters for experiment : 6 Obs 30dB Exp 2

- Error tolerance and stagnation parameters:

$$\epsilon_{\text{Stag Pos}} = \begin{cases} 0.5 \text{ for run 1-6} \\ 0.001 \text{ for run 7-9} \end{cases}, \quad \begin{aligned} \epsilon_J = \epsilon_{\nabla J} = \epsilon_{\text{Stag LS}} &= 0.00001 \\ \epsilon_{\text{Stag J}} &= 0.1 \end{aligned}$$

- Wolfe Line search parameters : $c_1 = 0.0001$.
- $\sharp$ Iter Max $= 300$, $\sharp$ Iter Linesearch Max $= 30$.
- Order FSSL $= 3$.

Run	κ	$\epsilon_{\text{Stag Pos}}$	Init. Step size
1	0.08	0.5	10
2	0.09	0.5	10
3	0.1	0.5	10
4	0.2	0.5	8
5	0.3	0.5	8

Run	κ	$\epsilon_{\text{Stag Pos}}$	Init. Step size
6	0.4	0.5	5
7	0.5	0.01	5
8	0.6	0.01	3
9	0.7	0.01	3

Parameters of experiment : 12 structured Obs 25dB

- Error tolerance and stagnation parameters

$$\epsilon_J = 0.5 \qquad \epsilon_{\nabla J} = 5.0 \times 10^{-4}$$

$$\epsilon_{\text{Stag LS}} = 1.0 \times 10^{-3} \qquad \epsilon_{\text{Stag J}} = 0.1 \qquad \epsilon_{\text{Stag Pos}} = 1.0 \times 10^{-8}$$

- Wolfe Line search parameters : $c_1 = 0.0001$, $c_2 = 0.4$.
- $\sharp$ Iter Max $= 300$, $\sharp$ Iter Linesearch Max $= (30 , 30)$.

Run	κ	**Order FSSL**	**Step size**
1	0.08	3	1
2	0.09	3	24
3	0.1	3	24
4	0.2	3	19
5	0.3	3	18

Run	κ	**Order FSSL**	**Step size**
6	0.4	3	16
7	0.5	4	16
8	0.6	4	10
9	0.8	6	8

Parameters of experiment : 12 rand obs 30dB

- Error tolerance and stagnation parameters

$$\epsilon_J = 0.05 \qquad \epsilon_{\nabla J} = 5.0 \times 10^{-4}$$
$$\epsilon_{\text{Stag LS}} = 0.05 \qquad \epsilon_{\text{Stag J}} = 0.1 \qquad \epsilon_{\text{Stag Pos}} = 1.0 \times 10^{-8}$$

- Wolfe Line search parameters : $c_1 = 0.0001$, $c_2 = 0.4$.

- $\sharp$ Iter Max $= 300$, $\sharp$ Iter Linesearch Max $= (30\ ,\ 30)$.

Run	κ	**Ord FSSL**	**Init. Step size**
1	0.08	3	50
2	0.09	3	30
3	0.1	3	10

Run	κ	**Ord FSSL**	**Init. Step size**
$4-6$	$0.2-0.4$	3	10
$7-10$	$0.5-0.8$	4	10