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Efficient decoding of random errors for quantum expander codes

Antoine Grospellier & Anthony Leverrier & Omar Fawzi

10 Novembre 2017

Alice

Bob

m
k Q-bits

Alice

Bob

Content of the talk

The hypergraph product of an expander code :

- is an LDPC quantum code
- has a constant rate
- has a minimal distance : $d = \Theta(\sqrt{n})$

The decoding algorithm :

- has a capacity of correction : $\Theta(\sqrt{n})$
- **corrects the error with high probability for the depolarizing channel**

Content of the talk

- 1 Classical expander codes
- 2 Quantum expander codes
- 3 Our contribution

Plan

- 1 Classical expander codes
- 2 Quantum expander codes
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Tanner graph of a code

Classical expander codes [Sipser & Spielman, '96]

Theorem [Sipser & Spielman, '96]

We can construct a good family $(\mathcal{C}_n)_{n \in \mathbb{N}}$ of $[n, k, d]$ -error correcting codes. Here "Good" means :

- This family is LDPC
- k and d are linear in n
- There exists an efficient correcting algorithm

Remark

Good expander codes can be found efficiently by picking a random biregular graph

Decoding algorithm

- Error :
 $e_0 = \{0, 1, 2\}$
- Unsatisfied
check-nodes
(syndrome) :
 $\{10, 12, 14, 19\}$
- Satisfied
check-nodes :
 $\{11, 13, 15, 16, 17, 18\}$

Decoding algorithm

- INPUT : syndrome
- The error e_0 is unknown
- OUTPUT : e
a set of bits
- GOAL : $e = e_0$
- The algorithm flips a bit when it decreases the syndrome

Decoding algorithm : first example

Decoding algorithm : first example

Decoding algorithm : first example

Decoding algorithm : first example

Decoding algorithm : second example

Decoding algorithm : second example

Decoding algorithm : second example

Decoding algorithm : second example

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From classical to quantum error correcting codes

Hypergraph product [Tillich & Zémor, '09]

Using a classical code $\mathcal{C}$, we can construct a $[[n, k, d]]$ -CSS code with :

- $k = \Theta(n)$
- $d = \Theta(\sqrt{n})$: we can correct any error on d Q-bits

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Theorem [Leverrier & Tillich & Zémor, '15]

For the hypergraph product of an expander code ($\epsilon < 1/6$) :

There is an efficient decoding algorithm for this code.

This algorithm corrects any error of size $\leq \Theta(\sqrt{n})$

This algorithm is very close to the algorithm of Sipser and Spielman

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Our work

- **Question** : What happens for random errors of size $\Theta(n)$?
- **Depolarizing channel** : each Q-bit has an X-type error (resp. Y,Z-type error) with probability p independently

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Theorem : what we proved

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Idea : The algorithm is local :

- If two errors are far they don't interact
- The initial error can be decomposed in clusters of size $O(\ln(n))$
- If there is no cluster of size $\Theta(\sqrt{n})$ during the algorithm, the error will be corrected

Locality of the algorithm : the adjacency graph

In the following, we will say whp P (with high probability the property P holds) if : $\lim_{n \rightarrow +\infty} \mathbb{P}(P) = 1$

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Percolation Theorem

For a probability of error $p < \frac{1}{d-1}$, whp :

- The size of any connected components is $\leq \Theta(\ln(n))$

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Problem : Some clusters can merge during the decoding

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For a probability of error $p < \frac{1}{d-1}$, whp :

- if $|X \cap E(p)| \geq 1 \times |X|$ then $|X| < \Theta(\ln(n))$

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Percolation theorem, generalisation

$\forall \alpha > 0$, if $p < cst(\alpha, d)$, whp :

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Complexity of the decoding algorithm

The number of flips is linear in the size of the initial error

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Ideas to improve this bound :

- ① Improve the bound in the percolation theorem :
What is the critical probability ?
- ② Restrict the proof to interesting clusters :
 - * The diameter of an interesting cluster is $\leq \Theta(\ln(\ln(n)))$
 - * The number of edges inside an interesting cluster is large
(ideas from bootstrap percolation)

Conclusion

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Futur work :

- improve our bound
- run simulations
- apply this result to fault tolerant quantum computation (Gottesman)

Thank you for your attention

A motivation : fault-tolerant quantum computation

Threshold Theorem [Ben-Or & Aharonov, '97]

We can simulate a quantum circuit with perfect gates by a circuit with noisy gates of size **quasi-linear**

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$$\lim_{n \rightarrow +\infty} \mathbb{P}(\mathcal{A} \text{ corrects the error}) = 1$$

What we hope to prove using [Gottesman, '13]

We can simulate a quantum circuit with perfect gates by a circuit with noisy gates of size **linear**

$$n = 10, m = 5, d_1 = 2, d_2 = \frac{n \times d_1}{m} = 4$$

0

1

2

3

4

5

6

7

8

9

$$n = 10, m = 5, d_1 = 2, d_2 = \frac{n \times d_1}{m} = 4$$

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