Effect of time scales on stability of coupled systems involving the wave equation

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***Abstract*— This article considers systems coupling an ordinary differential equation (ODE) with a wave equation through its boundary data. The main focus is put on the role of different time scales for each equation on the stability of the coupled system. A fast wave equation coupled to an ODE is proven to be stable if each subsystem is stable. However, we show examples of stable subsystems generating an unstable full system when coupling a wave equation to a fast ODE.**

I. Introduction

For many physical considerations, it is needed to consider together systems with different time scales. As an example we can mention the Saint-Venant–Exner equations described in [9] and in [1, Section 1.5]. This hyperbolic system is used to study the dynamics of the flow in a reach, coupled with the sediment dynamics. The sediment dynamics has, by nature, a very slow dynamic with respect to the velocity flow in the fluid. Thus this model is a singularly perturbed hyperbolic system, as studied in [13] (see also [4] for control results on this system). Other examples of systems with different time scales appear when considering infinite-dimensional control systems with dynamics at the boundaries, as introduced in e.g., [1, Section 3.4]. One naturally obtain PDEs coupled to ODEs at different time scales. In [18, Chapter 2] a slow ODE coupled with a fast PDE appears, and in [15] a fast ODE coupled with a slow PDE is studied.

For these kinds of linear infinite-dimensional systems, we naturally think that the behavior of the full-system is defined by the one of each subsystem. However, this is not always the case. The goal of this paper is to report on some special features appearing in coupled systems involving different time scales even for very simple equations. We consider as a toy model the wave equation coupled to an ordinary differential equation through boundary data, as studied e.g. in [3]. More precisely, for all $t \geq 0$, and $0 < x < 1$,

$$\left\{\begin{array}{l} \delta_1 w_{tt}(t,x) - w_{xx}(t,x) = 0, \\ w(t,0) = cz(t), \\ w_x(t,1) = -d\delta_2 w_t(t,1), \\ \delta_3 \dot{z}(t) = az(t) + \delta_4 b w(t,1) + \delta_5 b w_t(t,1), \end{array}\right. \tag{1}$$

with a, b and c constant values and $d > 0$. Moreover, the coefficients δ_k will define different time scales and couplings.

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We consider usual initial condition for (1) given by w^0, w^1 and z^0, that is

$$\left\{\begin{array}{ll} w(0,x) = w^0(x), & 0 < x < 1, \\ w_t(0,x) = w^1(x), & 0 < x < 1, \\ z(0) = z^0. & \end{array}\right. \tag{2}$$

As usual, the literature on singularly perturbed systems has first grown up for finite-dimensional systems (see in particular the seminal works [7], [11]). For infinite-dimensional systems we find [5], [6] where delay systems are studied. Closer to the present contribution, let us cite [2] where a parabolic singularly perturbed PDE is considered. Regading coupled hyperbolic PDEs, we mention [16] and [17] dealing with conservation laws and balance laws, respectively. In both papers, Lyapunov function approaches are useful to analyze stability properties. The problem under consideration in this paper is related to [15], where coupled systems of conservation laws and ODE are considered, with some specific boundary conditions.

We first consider the fast PDE/slow ODE case.

Theorem 1: *In* (1) *take* $\delta_1 = \varepsilon^2 > 0$, $\delta_2 = \varepsilon > 0$, $\delta_3 = \delta_4 = 1$ *and* $\delta_5 = 0$. *Let* $d > 0$ *and* a, b, c *such that* $a + bc < 0$. *There exists* $\varepsilon^* > 0$ *such that for any* $\varepsilon \in (0, \varepsilon^*)$ *the full system* (1) *is exponentially stable, that is, there exists* $C > 0$ *and* $\mu > 0$ *such that, for all* (w^0, w^1, z^0) *in* $H^1(0,1) \times L^2(0,1) \times \mathbb{R}$, *there exists a unique solution with* $z \in C([0,+\infty);\mathbb{R})$ *and* $w \in C^0([0,+\infty); H^1(0,1)) \cap C^1([0,+\infty); L^2(0,1))$ *such that for all* $t \geq 0$,

$$\begin{aligned} &\|(w(t), w_t(t), z(t))\|_{H^1\times L^2(0,1)\times\mathbb{R}} \\ &\qquad \leq Ce^{-\mu t}\|(w^0, w^1, z^0)\|_{H^1\times L^2(0,1)\times\mathbb{R}}. \end{aligned}$$

Let us notice that the hypothesis in Theorem 1 are linked to the stability of subsystems obtained when applying the singular perturbation approach to study the stability of the full-system. In fact, as it will be explained in Section II below, the reduced system is

$$\tfrac{d}{dt}\bar{z} = (a + bc)\bar{z}, \quad t \geq 0, \tag{3}$$

while the boundary layer system for $\tau = t/\varepsilon$ is

$$\left\{\begin{array}{ll} \bar{w}_{\tau\tau}(\tau,x) - \bar{w}_{xx}(\tau,x) = 0, & \tau \geq 0,\, 0 < x < 1, \\ \bar{w}(\tau,0) = 0, & \tau \geq 0, \\ \bar{w}_x(\tau,1) = -d\bar{w}_\tau(\tau,1), & \tau \geq 0. \end{array}\right. \tag{4}$$

Under conditions in Theorem 1 ($d > 0$ and $a + bc < 0$), both subsystems are stable as well as the full system (1) in this regime when ε is small enough.

We turn now our attention to the case slow PDE/fast ODE, where something unexpected happens.

Theorem 2: *In* (1) *take* $\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = \varepsilon$, $\delta_4 = 0$ *and* $\delta_5 = 1$. *For any* $\varepsilon > 0$, $d > 0$, *and* $a < 0$, *there exist* $b, c \in \mathbb{R}$ *such that* (1) *is not asymptotically stable. Thus, there exists a solution to (1)-(2) with initial condition in* $H^2(0,1) \times H^1(0,1) \times \mathbb{R}$ *that does not converge to* 0.

In this case, as it will be explained in Section III below, the subsystems obtained by applying the singular perturbation approach are

$$\left\{ \begin{array}{ll} \bar{w}_{tt}(t,x) - \bar{w}_{xx}(t,x) = 0, & t \geq 0,\, 0 < x < 1, \\ \bar{w}(t,0) = 0, & t \geq 0, \\ \bar{w}_x(t,1) = -d\bar{w}_t(t,1), & t \geq 0, \end{array} \right. \tag{5}$$

for the reduced system, and

$$\tfrac{d}{d\tau}\bar{z}(\tau) = a\bar{z}(\tau), \quad \tau \geq 0, \tag{6}$$

for the boundary layer system. We see that both systems are exponentially stable under conditions in Theorem 2 ($d > 0$ and $a < 0$). However, the full system (1) is not always asymptotically stable.

The paper is organized as follows. In Section II we prove Theorem 1. Section III is devoted to the proof of Theorem 2. Section IV contains a numerical simulation illustrating Theorem 2, i.e., the lack of asymptotic stability of the full system even when both subsystems are asymptotically stable. Finally, we give in Section V some conclusions.

II. Fast PDE/slow ODE. Proof of Theorem 1

The goal of this section is to study the first case, namely a fast wave equation coupled with a slow ODE. We obtain subsystems (3) and (4), and prove Theorem 1 by developing a Lyapunov function approach.

To prove Theorem 1, let us particularize (1) with $\delta_1 = \varepsilon^2 > 0$, $\delta_2 = \varepsilon > 0$, $\delta_3 = \delta_4 = 1$ and $\delta_5 = 0$. This yields the following system of equations combining a wave equation with fast velocities and a slow ODE:

$$\left\{ \begin{array}{ll} \varepsilon^2 w_{tt} - w_{xx} = 0, & t \geq 0,\, 0 < x < 1, \\ w(t,0) = cz(t), & t \geq 0, \\ w_x(t,1) = -d\varepsilon w_t(t,1), & t \geq 0, \\ \dot{z}(t) = az(t) + bw(t,1), & t \geq 0, \end{array} \right. \tag{7}$$

with a, b and c constant values and $d > 0$ a positive value. We consider usual initial condition for (7) given by w^0, w^1 and z^0, namely (2).

Remark 1: By rewriting system (7) in Riemann coordinates, we get a system of conservation laws coupled with an ODE. The dynamical equations obtained from the first and the fourth lines in (7) are similar to the ones considered in [15] for fast PDE with slow ODE. However, in Riemann coordinates, the boundary conditions obtained from second and four lines in (7) are different to the ones in [15]. In this way, we can see that the results in [15] do not apply for (7). ∘

To formally compute the reduced order system, let $\varepsilon = 0$ in (7). We get from the boundary condition at $x = 1$ that $w_x(t,1) = 0$ which gives $w_x = 0$ when using $w_{xx} = 0$ (coming from the PDE). From the boundary condition at $x = 0$, it follows that $w(t,x) = cz(t)$ for all $t \geq 0$ and for all $x \in (0,1)$. Thus, the reduced order system is

$$\tfrac{d}{dt}\bar{z} = (a + bc)\bar{z}, \quad t \geq 0. \tag{8}$$

Let us compute the boundary layer system. We introduce $\tau = t/\varepsilon$ and the new variable $\bar{w}(\tau,x) = w(\tau,x) - cz(\tau)$. We compute $\frac{d}{d\tau}\bar{w} = \frac{d}{d\tau}w - c\varepsilon\frac{d}{dt}z = \frac{d}{d\tau}w$ by letting $\varepsilon = 0$ and by using the z dynamics. Moreover, $\frac{d^2}{d\tau^2}\bar{w} = \frac{d^2}{d\tau^2}w$, $\frac{d}{dx}\bar{w} = \frac{d}{dx}w$, and $\frac{d^2}{dx^2}\bar{w} = \frac{d^2}{dx^2}w$. Therefore $\bar{w}_{\tau\tau} - \bar{w}_{xx} = 0$. To compute the boundary conditions for the variable $\bar{w}$, let us note that $\bar{w}_x(\tau,1) = w_x(\tau,1) = -d\varepsilon w_t(\tau,1) = -dw_\tau(\tau,1) = -d\bar{w}_\tau(\tau,1)$ by approximating ε by 0 in the last equation. To sum up, the boundary layer system is written as follows:

$$\left\{ \begin{array}{ll} \bar{w}_{\tau\tau}(\tau,x) - \bar{w}_{xx}(\tau,x) = 0, & \tau \geq 0,\, 0 < x < 1, \\ \bar{w}(\tau,0) = 0, & \tau \geq 0, \\ \bar{w}_x(\tau,1) = -d\bar{w}_\tau(\tau,1), & \tau \geq 0. \end{array} \right. \tag{9}$$

The boundary layer system is exponentially stable. To check it, let us consider the following Lyapunov function candidate

$$\bar{V}_1(\bar{w}) = \int_0^1 e^{\mu x}(\bar{w}_x + \bar{w}_\tau)^2 dx + \int_0^1 e^{-\mu x}(\bar{w}_x - \bar{w}_\tau)^2 dx, \tag{10}$$

with μ to be fixed later. Along the solutions to (9), it holds

$$\begin{array}{rcl} \frac{d}{d\tau}\bar{V}_1 & = & 2\int_0^1 e^{\mu x}(\bar{w}_\tau + \bar{w}_x)(\bar{w}_{\tau\tau} + \bar{w}_{x\tau})dx \\ & & +2\int_0^1 e^{-\mu x}(\bar{w}_\tau - \bar{w}_x)(\bar{w}_{\tau\tau} - \bar{w}_{x\tau})dx \ , \\ & = & 2\int_0^1 e^{\mu x}(\bar{w}_\tau + \bar{w}_x)(\bar{w}_{xx} + \bar{w}_{x\tau})dx \\ & & -2\int_0^1 e^{-\mu x}(\bar{w}_\tau - \bar{w}_x)(\bar{w}_{x\tau} - \bar{w}_{xx})dx \ , \\ & = & -\mu\int_0^1 e^{\mu x}(\bar{w}_\tau + \bar{w}_x)^2 dx \\ & & +[e^{\mu x}(\bar{w}_\tau + \bar{w}_x)^2]_{x=0}^{x=1} \\ & & -\mu\int_0^1 e^{-\mu x}(\bar{w}_\tau - \bar{w}_x)^2 dx \\ & & -[e^{-\mu x}(\bar{w}_\tau - \bar{w}_x)^2]_{x=0}^{x=1} \ . \end{array}$$

Now, note that the boundary condition in the second line of (9) implies that $\bar{w}_\tau(\tau,0) = 0$ and thus, for all $\tau \geq 0$,

$$\begin{array}{r} [e^{\mu x}(\bar{w}_\tau + \bar{w}_x)^2](\tau,0) - [e^{-\mu x}(\bar{w}_\tau - \bar{w}_x)^2](\tau,0) \\ = \bar{w}_x^2(\tau,0) - \bar{w}_x^2(\tau,0) = 0. \end{array}$$

Therefore, we get

$$\begin{array}{rcl} \dfrac{d}{d\tau}\bar{V}_1 & = & -\mu\bar{V}_1 + e^{\mu}(\bar{w}_\tau(\tau,1) + \bar{w}_x(\tau,1))^2 \\ & & -e^{-\mu}(\bar{w}_\tau(\tau,1) - \bar{w}_x(\tau,1))^2 \ , \end{array} \tag{11}$$

and thus with the boundary condition in the last line of (9):

$$\begin{array}{rcl} \dfrac{d}{d\tau}\bar{V}_1 & = & -\mu\bar{V}_1 + e^{\mu}(\bar{w}_\tau(\tau,1) - d\bar{w}_\tau(\tau,1))^2 \\ & & -e^{-\mu}(\bar{w}_\tau(\tau,1) + d\bar{w}_\tau(\tau,1))^2 \ , \\ & = & -\mu\bar{V}_1 + \Big(e^{\mu}(1-d)^2 - e^{-\mu}(1+d)^2\Big)\bar{w}_\tau(\tau,1)^2 \ . \end{array}$$

We obtain the exponential stability by choosing μ such that $e^{\mu}(1-d)^2 < e^{-\mu}(1+d)^2$, which is possible due to $d > 0$.

Let us now define the following variable $\tilde{w} = w - cz$. We compute successively

$$\begin{array}{rcl} w_t &=& \tilde{w}_t + (a+bc)cz + bc\tilde{w}(t,1)\ , \\ w_{tt} &=& \tilde{w}_{tt} + (abc + b^2c^2)\tilde{w}(t,1) \\ && + bc\tilde{w}_t(t,1) + (a^2c + 2abc^2 + b^2c^3)z\ , \\ \tilde{w}_x &=& w_x\ , \\ \tilde{w}_{xx} &=& w_{xx}. \end{array}$$

Therefore we get the following dynamics, equivalent to (7):

$$\left\{ \begin{array}{l} \varepsilon^2\tilde{w}_{tt} - \tilde{w}_{xx} + \varepsilon^2(abc + b^2c^2)\tilde{w}(t,1) \\ \qquad + \varepsilon^2 bc\tilde{w}_t(t,1) + \varepsilon^2(a^2c + 2abc^2 + b^2c^3)z = 0, \\ \tilde{w}(t,0) = 0, \\ \tilde{w}_x(t,1) = \varepsilon bcd\tilde{w}(t,1) - d\varepsilon\tilde{w}_t(t,1) - \varepsilon d(a+bc)cz, \\ \dot{z}(t) = (a+bc)z(t) + b\tilde{w}(t,1). \end{array} \right. \tag{12}$$

Let us see if this last system is exponentially stable. We consider the following Lyapunov function candidate $V = V_1 + V_2$ where

$$V_1(\tilde{w}) = \int_0^1 e^{\mu x}(\tilde{w}_x + \varepsilon\tilde{w}_t)^2 dx + \int_0^1 e^{-\mu x}(\tilde{w}_x - \varepsilon\tilde{w}_t)^2 dx$$

and $V_2(z) = z^2$. Along the solutions to (12), we compute, using integrations by parts,

$$\begin{array}{rcl} \frac{d}{dt}V_1 &=& 2\int_0^1 e^{\mu x}(\tilde{w}_{xt} + \varepsilon\tilde{w}_{tt})(\tilde{w}_x + \varepsilon\tilde{w}_t)dx \\ && +2\int_0^1 e^{-\mu x}(\tilde{w}_{xt} - \varepsilon\tilde{w}_{tt})(\tilde{w}_x - \varepsilon\tilde{w}_t)dx \\ &=& \frac{2}{\varepsilon}\int_0^1 e^{\mu x}(\varepsilon\tilde{w}_{xt} + \tilde{w}_{xx})(\tilde{w}_x + \varepsilon\tilde{w}_t)dx \\ && -\frac{2}{\varepsilon}\int_0^1 e^{-\mu x}(-\varepsilon\tilde{w}_{xt} + \tilde{w}_{xx})(\tilde{w}_x - \varepsilon\tilde{w}_t)dx \\ && -2\varepsilon[(abc + b^2c^2)\tilde{w}(t,1) + bc\tilde{w}_t(t,1) \\ && \quad +(a^2c + 2abc^2 + b^2c^3)z]\int_0^1 e^{\mu x}(\tilde{w}_x + \varepsilon\tilde{w}_t)dx \\ && +2\varepsilon[(abc + b^2c^2)\tilde{w}(t,1) + bc\tilde{w}_t(t,1) \\ && +(a^2c + 2abc^2 + b^2c^3)z]\int_0^1 e^{-\mu x}(\tilde{w}_x - \varepsilon\tilde{w}_t)dx \end{array}$$

and thus

$$\begin{array}{rcl} \frac{d}{dt}V_1 &=& -\frac{\mu}{\varepsilon}\int_0^1 e^{\mu x}(\tilde{w}_x + \varepsilon\tilde{w}_t)^2 dx \\ && -\frac{\mu}{\varepsilon}\int_0^1 e^{-\mu x}(\tilde{w}_x - \varepsilon\tilde{w}_t)^2 dx \\ && + \left[\frac{e^{\mu x}}{\varepsilon}(\tilde{w}_x + \varepsilon\tilde{w}_t)^2 - \frac{e^{-\mu x}}{\varepsilon}(\tilde{w}_x - \varepsilon\tilde{w}_t)^2\right]_0^1 \\ && -2\varepsilon[(abc + b^2c^2)\tilde{w}(t,1) + bc\tilde{w}_t(t,1) \\ && +(a^2c + 2abc^2 + b^2c^3)z]\int_0^1 e^{\mu x}(\tilde{w}_x + \varepsilon\tilde{w}_t)dx \\ && +2\varepsilon[(abc + b^2c^2)\tilde{w}(t,1) + bc\tilde{w}_t(t,1) \\ && +(a^2c + 2abc^2 + b^2c^3)z]\int_0^1 e^{-\mu x}(\tilde{w}_x - \varepsilon\tilde{w}_t)dx\ . \end{array}$$

Now using the boundary conditions in lines two and three in (12) and the inequalities $2\alpha\beta \leq \frac{\alpha^2}{k} + k\beta^2$ and $(\alpha+\beta+\gamma)^2 \leq 3(\alpha^2+\beta^2+\gamma^2)$ (for any values α, β and γ and any positive value k), we get

$$\begin{array}{rcl} \frac{d}{dt}V_1 &\leq& -\frac{\mu}{\varepsilon}V_1 + \frac{e^{\mu}}{\varepsilon}\Big(\tilde{w}_x(t,1) + \varepsilon\tilde{w}_t(t,1)\Big)^2 \\ && -\frac{1}{\varepsilon}\Big(\tilde{w}_x(t,0) + \varepsilon\tilde{w}_t(t,0)\Big)^2 \\ && -\frac{e^{-\mu}}{\varepsilon}\Big(\tilde{w}_x(t,1) - \varepsilon\tilde{w}_t(t,1)\Big)^2 \\ && +\frac{1}{\varepsilon}\Big(\tilde{w}_x(t,0) - \varepsilon\tilde{w}_t(t,0)\Big)^2 \\ && +\frac{2\varepsilon}{\kappa_1}[(abc + b^2c^2)\tilde{w}(t,1) + bc\tilde{w}_t(t,1) \\ && +(a^2c + 2abc^2 + b^2c^3)z]^2 \\ && +\kappa_1\varepsilon\int_0^1 e^{2\mu x}(\tilde{w}_x + \varepsilon\tilde{w}_t)^2 dx \\ && +\kappa_1\varepsilon\int_0^1 e^{-2\mu x}(\tilde{w}_x - \varepsilon\tilde{w}_t)^2 dx \end{array}$$

and then

$$\begin{aligned} &\frac{d}{dt}V_1 \leq -\frac{\mu}{\varepsilon}V_1 \\ &+ \frac{e^{\mu}}{\varepsilon}\Big(\varepsilon bcd\tilde{w}(t,1) + \varepsilon(1-d)\tilde{w}_t(t,1) - \varepsilon d(a+bc)cz\Big)^2 \\ &- \frac{e^{-\mu}}{\varepsilon}\Big(\varepsilon bcd\tilde{w}(t,1) - \varepsilon(1+d)\tilde{w}_t(t,1) - \varepsilon d(a+bc)cz\Big)^2 \\ &\qquad + \frac{6\varepsilon}{\kappa_1}\Big[(abc + b^2c^2)^2\tilde{w}(t,1)^2 + b^2c^2\tilde{w}_t(t,1)^2 \\ &\qquad\qquad + (a^2c + 2abc^2 + b^2c^3)^2 z^2\Big] \\ &\quad + \kappa_1\varepsilon\int_0^1 (e^{2\mu x}(\tilde{w}_x + \varepsilon\tilde{w}_t)^2 + e^{-2\mu x}(\tilde{w}_x - \varepsilon\tilde{w}_t)^2)dx \end{aligned}$$

for any positive value κ_1. By developing the computations, using $2\alpha\beta \leq \frac{\alpha^2}{k} + k\beta^2$ (for any values α and β and any positive value k), and the existence (thanks to Poincaré inequality) of $C_1 > 0$ such that $|\tilde{w}(t,1)|^2 \leq \frac{C_1}{2}e^{\mu}V_1(\tilde{w})$, we can write

$$\begin{array}{rcl} \frac{d}{dt}V(t) &\leq& -\frac{\mu}{\varepsilon}V_1(t) + 2(a+bc)V_2(t) \\ && +\varepsilon\Big(e^{\mu}(1-d)^2 - e^{-\mu}(1+d)^2\Big)\tilde{w}_t(t,1)^2 \\ && +R, \end{array}$$

where the remainder term R is given by

$$\begin{aligned} &R = e^{\mu}\varepsilon\Big[b^2c^2d^2\tilde{w}(t,1)^2 + d^2(a+bc)^2c^2z^2 \\ &+ 2(1-d)bcd\tilde{w}(t,1)\tilde{w}_t(t,1) - 2d^2bc^2(a+bc)z\tilde{w}(t,1) \\ &\qquad - 2(1-d)d(a+bc)cz\tilde{w}_t(t,1)\Big] \\ &\qquad - e^{-\mu}\varepsilon\Big[b^2c^2d^2\tilde{w}(t,1)^2 + d^2(a+bc)^2c^2z^2 \\ &- 2(1+d)bcd\tilde{w}(t,1)\tilde{w}_t(t,1) - 2bc^2d^2(a+bc)z\tilde{w}(t,1) \\ &\qquad + 2d(1+d)(a+bc)cz\tilde{w}_t(t,1)\Big] \\ &\qquad + \frac{6\varepsilon}{\kappa_1}\Big[(abc + b^2c^2)^2\tilde{w}(t,1)^2 + b^2c^2\tilde{w}_t(t,1)^2 \\ &\qquad + (a^2c + 2abc^2 + b^2c^3)^2z^2\Big] + 2b\tilde{w}(t,1)z \\ &+ \kappa_1\varepsilon\int_0^1 (e^{2\mu x}(\tilde{w}_x + \varepsilon\tilde{w}_t)^2 + e^{-2\mu x}(\tilde{w}_x - \varepsilon\tilde{w}_t)^2)dx. \end{aligned}$$

We see that for any positive k_2, k_3, k_4, we have

$$\begin{aligned} &R \leq \varepsilon V_1(t)\Big[e^{2\mu}b^2c^2d^2C_1 + \frac{3C_1}{\kappa_1}e^{\mu}(abc + b^2c^2)^2C_1 \\ &\qquad + \frac{C_1}{2\kappa_2}|bc|dC_1\big[|1-d|e^{2\mu} + (1+d)\big] \\ &+ (e^{\mu} + e^{-\mu})|b|c^2d^2|a+bc|\frac{C_1}{2} + \kappa_4|b|\frac{C_1}{2\varepsilon}e^{\mu} + \kappa_1 e^{\mu}\Big] \\ &+\varepsilon V_2(t)\Big[\frac{6(a^2c + 2abc^2 + b^2c^3)^2}{\kappa_1} + (e^{\mu} + e^{-\mu})d^2c^2(a+bc)^2 \\ &\qquad + (e^{\mu} + e^{-\mu})|b|c^2d^2|a+bc| \\ &\qquad + [e^{\mu}|1-d| + e^{-\mu}(1+d)]d|a+bc||c| + \frac{|b|}{\varepsilon\kappa_4}\Big] \\ &\qquad + \varepsilon|\tilde{w}_t(t,1)|^2\Big[\frac{6}{\kappa_1}b^2c^2 \\ &+ |c|d(|1-d|e^{\mu} + (1+d)e^{-\mu})(\kappa_2|b| + |a+bc|\kappa_3)\Big]. \end{aligned}$$

Using these estimates we arrive to

$$
\begin{aligned}
&\frac{d}{dt}V(t) \le -\frac{\mu}{2\varepsilon}V_1(t) + (a+bc)V_2(t) \\
&+\varepsilon V_1(t)\Big[-\frac{\mu}{2\varepsilon^2} + e^{2\mu}b^2c^2d^2C_1 + \frac{3C_1}{\kappa_1}e^{\mu}(abc+b^2c^2)^2C_1 \\
&\quad + \frac{C_1}{2\kappa_2}|bc|dC_1\big[|1-d|e^{2\mu} + (1+d)\big] \\
&\quad + (e^{\mu}+e^{-\mu})|b|c^2d^2|a+bc|\frac{C_1}{2} + \kappa_4|b|\frac{C_1}{2\varepsilon}e^{\mu} + \kappa_1 e^{\mu}\Big] \\
&+\varepsilon V_2(t)\Big[\frac{(a+bc)}{\varepsilon} + \frac{6(a^2c+2abc^2+b^2c^3)^2}{\kappa_1} \\
&+ (e^{\mu}+e^{-\mu})d^2c^2(a+bc)^2 + (e^{\mu}+e^{-\mu})|b|c^2d^2|a+bc| \\
&\quad + [e^{\mu}|1-d| + e^{-\mu}(1+d)]d|a+bc||c| + \frac{|b|}{\varepsilon\kappa_4}\Big] \\
&+\varepsilon|\tilde{w}_t(t,1)|^2\Big[\Big(e^{\mu}(1-d)^2 - e^{-\mu}(1+d)^2\Big) + \frac{6}{\kappa_1}b^2c^2 \\
&+ \big(|1-d|e^{\mu} + (1+d)e^{-\mu}\big)\big(\kappa_2|bc|d + d|a+bc||c|\kappa_3\big)\Big]
\end{aligned} \tag{13}
$$

and then to

$$
\begin{aligned}
\frac{d}{dt}V(t) &\le -\frac{\mu}{2\varepsilon}V_1(t) + (a+bc)V_2(t)\ , \\
&\le -\min\left\{\frac{\mu}{2\varepsilon}, |a+bc|\right\}V(t)\ \le -|a+bc|V(t)\ ,
\end{aligned}
$$

by choosing κ_1, κ_2, κ_3 and κ_4 in an appropriate way for all $\varepsilon \in (0, \varepsilon^\star)$ with a sufficiently small positive value $\varepsilon^\star$. In fact, to prove (14), we first inspect the term $|\tilde{w}_t(t,1)|^2$ in (13) and then we consider the terms multiplying V_1 and V_2. To be more specific:

- We first choose κ_2 small enough so that

$$
\begin{aligned}
&\kappa_2|bc|d\big(|1-d|e^{\mu} + (1+d)e^{-\mu}\big) \\
&\qquad \le \frac{1}{4}\Big|e^{\mu}(1-d)^2 - e^{-\mu}(1+d)^2\Big|.
\end{aligned} \tag{14}
$$

  By doing so, the term in (13) multiplying $\varepsilon|\tilde{w}_t(t,1)|^2$ is smaller than

$$
\begin{aligned}
&\Big[\tfrac{3}{4}\Big(e^{\mu}(1-d)^2 - e^{-\mu}(1+d)^2\Big) \\
&+\big[|1-d|e^{\mu} + (1+d)e^{-\mu}\big]d|a+bc||c|\kappa_3 \\
&+\tfrac{6}{\kappa_1}b^2c^2\Big].
\end{aligned}
$$

- We then choose κ_3 small enough so that

$$
\begin{aligned}
&\kappa_3 d|c||a+bc|\big[|1-d|e^{\mu} + (1+d)e^{-\mu}\big] \\
&\quad \le \tfrac{1}{4}\Big|e^{\mu}(1-d)^2 - e^{-\mu}(1+d)^2\Big|.
\end{aligned}
$$

  By doing so, with the previous item, the term in (13) multiplying $\varepsilon|\tilde{w}_t(t,1)|^2$ is smaller than

$$
\Big[\frac{1}{2}\Big(e^{\mu}(1-d)^2 - e^{-\mu}(1+d)^2\Big) + \frac{6}{\kappa_1}b^2c^2.\Big]
$$

- We then choose κ_1 large enough so that

$$
\frac{6}{\kappa_1}b^2c^2 \le \frac{1}{2}\Big|\Big(e^{\mu}(1-d)^2 - e^{-\mu}(1+d)^2\Big)\Big|.
$$

  By doing so, with the previous item, the term in (13) multiplying $|\tilde{w}_t(t,1)|^2$ is smaller than 0.

- Now we select κ_4 large enough so that

$$
\frac{|b|}{\kappa_4} < \frac{|a+bc|}{2}.
$$

- With these positive values for κ_1, κ_2, κ_3 and κ_4, we finally pick $\varepsilon^\star$ small enough and positive to satisfy, for all $\varepsilon \in (0, \varepsilon^\star)$,

$$
\begin{aligned}
&\frac{6(a^2c+2abc^2+b^2c^3)^2}{\kappa_1} + (e^{\mu}+e^{-\mu})d^2c^2(a+bc)^2 \\
&\quad + (e^{\mu}+e^{-\mu})|b|c^2d^2|a+bc| \\
&\quad + [e^{\mu}|1-d| + e^{-\mu}(1+d)]d|a+bc||c| \\
&\qquad \le \frac{|a+bc|}{2\varepsilon}
\end{aligned} \tag{15}
$$

  and

$$
\begin{aligned}
&e^{2\mu}b^2c^2d^2C_1 + \frac{3C_1}{\kappa_1}e^{\mu}(abc+b^2c^2)^2C_1 \\
&\quad + \frac{C_1}{2\kappa_2}|bc|dC_1\big[|1-d|e^{2\mu} + (1+d)\big] \\
&\quad + (e^{\mu}+e^{-\mu})|b|c^2d^2|a+bc|\frac{C_1}{2} \\
&\quad + \kappa_4|b|\frac{C_1}{2\varepsilon}e^{\mu} + \kappa_1 e^{\mu} \le \frac{\mu}{2\varepsilon^2}
\end{aligned} \tag{16}
$$

All these conditions can be satisfied under the hypothesis in Theorem 1:

- $d > 0$ (thus, $\exists \mu > 0$ such that $e^{2\mu}(1-d)^2 < (1+d)^2$).
- $(a+bc) < 0$.

This ends the proof of Theorem 1. □

III. Slow PDE/fast ODE. Proof of Theorem 2

The goal of this section is to study the second case, namely a slow wave equation coupled with a fast ODE. We obtain subsystems (5) and (6), and prove Theorem 2. This is done by proving the existence of some full systems which are not asymptotical stable even if both subsystems are. This will be independent of the value of ε.

To prove Theorem 2, let us particularize (1) with $\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = \varepsilon$, $\delta_4 = 0$ and $\delta_5 = 1$. This yields the following system of equations combining a slow wave equation and a fast ODE:

$$
\left\{
\begin{array}{ll}
w_{tt}(t,x) - w_{xx}(t,x) = 0, & t \ge 0,\ 0 < x < 1, \\
w(t,0) = cz(t), & t \ge 0, \\
w_x(t,1) = -dw_t(t,1), & t \ge 0, \\
\varepsilon\dot{z}(t) = az(t) + \varepsilon bw_t(t,1), & t \ge 0,
\end{array}
\right. \tag{17}
$$

with a, b, c and d constant values and initial condition (2) given by some w^0, w^1 and z^0.

Letting $\varepsilon = 0$ in (17), we formally get the reduced order system

$$
\left\{
\begin{array}{ll}
\bar{w}_{tt}(t,x) - \bar{w}_{xx}(t,x) = 0, & t \ge 0,\ 0 < x < 1, \\
\bar{w}(t,0) = 0, & t \ge 0, \\
\bar{w}_x(t,1) = -d\bar{w}_t(t,1), & t \ge 0,
\end{array}
\right. \tag{18}
$$

that is asymptotically stable as soon as $d > 0$ (to prove that, consider the Lyapunov function candidate (10) as done in

Section II for system (9)). Moreover, letting $\tau = t/\varepsilon$ and $\bar{z}(\tau) = z(\tau)$, we can write

$$\tfrac{d}{d\tau}\bar{z}(\tau) = a\bar{z}(\tau) + b\varepsilon\bar{w}_t(t,1) \tag{19}$$

and therefore (after taking $\varepsilon = 0$) the boundary layer system is, for all $\tau \geq 0$,

$$\tfrac{d}{d\tau}\bar{z}(\tau) = a\bar{z}(\tau). \tag{20}$$

Let us rewrite system (17) in Riemann coordinates. To do that, denote $v_1 = w_t + w_x$ and $v_2 = w_x - w_t$. Thus, we may deduce from (17) the following dynamics and the following boundary conditions, for all $t \geq 0$, and for all $0 < x < 1$,

$$\begin{cases} v_{1t} - v_{1x} = 0\ ,\ v_{2t} + v_{2x} = 0, \\ 2c\dot{z}(t) = v_1(t,0) - v_2(t,0), \\ v_1(t,1) + v_2(t,1) = -dv_1(t,1) + dv_2(t,1), \\ \varepsilon\dot{z}(t) = az(t) + \varepsilon b/2v_1(t,1) - \varepsilon b/2v_2(t,1). \end{cases} \tag{21}$$

Denoting $\gamma(t) = (v_1(t,1), v_2(t,0), z(t))^\top$, we may rewrite the dynamics and the boundary conditions with γ. To do that, we first remark that the second line of (21) implies, for all $t \geq 1$,

$$\dot{z} = \frac{1}{2c}v_1(t-1,1) - \frac{1}{2c}v_2(t,0) \tag{22}$$

and that the third line of (21) implies

$$(1+d)v_1(t,1) + (1-d)v_2(t-1,0) = 0. \tag{23}$$

Now combining the fourth line of (21) with (22) yields

$$\begin{aligned} bv_1(t,1) + \tfrac{1}{c}v_2(t,0) - \tfrac{1}{c}v_1(t-1,1) \\ -bv_2(t-1,0) + \tfrac{2a}{\varepsilon}z(t) = 0. \end{aligned} \tag{24}$$

Therefore, with (22) and deriving (23) and (24), we get

$$\frac{d}{dt}\Big(M\gamma(t) + N\gamma(t-1)\Big) = P\gamma(t) + Q\gamma(t-1) \tag{25}$$

with the following matrices $M = \begin{pmatrix} 1+d & 0 & 0 \\ b & \frac{1}{c} & \frac{2a}{\varepsilon} \\ 0 & 0 & 1 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 1-d & 0 \\ -\frac{1}{c} & -b & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2c} & 0 \end{pmatrix}$, and $Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2c} & 0 & 0 \end{pmatrix}$.

Due to [8, Lemma 4.1] (see also [12, Section 1.4]) we know that a necessary condition for (25) to be asymptotically stable is that the discrete system

$$M\gamma(t) + N\gamma(t-1) = 0 \tag{26}$$

is also asymptotically stable. The last system is rewritten as a difference equation

$$\gamma(t) = \begin{pmatrix} 0 & \frac{d-1}{1+d} & 0 \\ 1 & \frac{2bc}{1+d} & 0 \\ 0 & 0 & 0 \end{pmatrix}\gamma(t-1). \tag{27}$$

We easily find that the eigenvalues of $(-M^{-1}N)$ are 0, and $\frac{bc}{1+d} \pm \sqrt{\frac{b^2c^2}{(1+d)^2} + \frac{d-1}{1+d}}$. We see that we have to choose b, c such that $|bc| \geq |1+d|$ in order to have at least one eigenvalue with modulus larger than 1. Therefore with this choice of parameters neither systems (25) nor (26) are exponentially stable, even if $a < 0$ and $d > 0$. (As an example, pick $a = -1$, $b = 1$, $c = 2$ and $d = 1$ in (17).)

To deduce the desired property for the primitive system (17), let us pick up a solution $t \mapsto \gamma(t)$ to (25) which does not converge to 0 (the origin). From the first and the second components of $\gamma(t)$, we may define $v_1(t,x)$ and $v_2(t,x)$ solution of the first line of (21), just by noting that the first line of (21) gives two transport equations that could be solved from $v_1(t,1)$ and $v_2(t,0)$ respectively. Denoting by $z(t)$ the third component of $\gamma(t)$, we get (21). We thus get a solution to (21) that does not converge to 0.

Define the function $(t,x) \mapsto w(t,x)$ such that $w_x(t,x) = \frac{1}{2}(v_1(t,x) + v_2(t,x))$, $w_t(t,x) = \frac{1}{2}(v_1(t,x) - v_2(t,x))$ and such that the second line of (17) holds. Since $v_{1t}(t,x) + v_{2t}(t,x) = v_{1x}(t,x) - v_{2x}(t,x)$, such a function exists by applying Poincaré theorem (see e.g. [10, Theorem 9.4.1]). We thus get a solution (w,z) to (17) that does not converge to the origin. This ends the proof of Theorem 2. ☐

To conclude this section, let us study another case, easier than the one considered in Theorem 2. To be more specific, we give another example of a not asymptotically stable system coupling a slow PDE and a fast ODE, for which both the reduced and the boundary layer subsystems are stable. Let us introduce the following system:

$$\begin{cases} w_{tt}(t,x) - w_{xx}(t,x) = 0, & t \geq 0,\ 0 < x < 1, \\ w(t,0) = cz(t), & t \geq 0, \\ w_x(t,1) = -dw_t(t,1), & t \geq 0, \\ \varepsilon\dot{z}(t) = az(t) + \varepsilon bw_t(x=0), & t \geq 0, \end{cases} \tag{28}$$

with a, b, c and d constant values. We consider (2) as given initial condition for (28) for some w^0, w^1 and z^0. By applying the same approach as before, we obtain the reduced order system

$$\begin{cases} \bar{w}_{tt}(t,x) - \bar{w}_{xx}(t,x) = 0, & t \geq 0,\ 0 < x < 1, \\ \bar{w}(t,0) = 0, & t \geq 0, \\ \bar{w}_x(t,1) = -d\bar{w}_t(t,1), & t \geq 0, \end{cases} \tag{29}$$

and the boundary layer system

$$\tfrac{d}{d\tau}\bar{z}(\tau) = a\bar{z}(\tau), \quad \tau \geq 0. \tag{30}$$

We see that both subsystems are asymptotically stable if $a < 0$ and $d > 0$.

On the other hand, to study directly (28) we can compute the z-dynamics. To be more specific, it follows from (28)

$$\varepsilon\dot{z}(t) = az(t) + \varepsilon bc\dot{z}(t)$$

which gives

$$z(t) = \exp\Big(\frac{a}{\varepsilon(1-bc)}t\Big)z^0.$$

Therefore $\frac{a}{\varepsilon(1-bc)} < 0$ is equivalent to the convergence of the z-variable of (28). We obtain thus that the full-system (28) is unstable if $bc > 1$, even if both the reduced and the boundary layer subsystems (29) and (30) are exponentially stable (pick for instance $a = -1$, $b = 1$, $c = 2$, and $d = 1$).

IV. Numerical simulations

In this section, we illustrate Theorem 2. To do that, we simulate system (17) with the numerical values considered in the proof of Theorem 2 for a, b, c, and d. Pick for the time scale $\varepsilon = 0.1$. As initial conditions consider, for all $x \in [0,1]$, $w(0,x) = \sin(2\pi x) + 2$, $w_t(0,x) = 0$ and $z(0) = 2$. By applying a Lax-Friedrichs (LxF) method [14], we get the numerical solutions of Figures 1 and 2 for w and z respectively, for $t \in [0, 0.5]$.It is observed that the solution to (17) starts growing even if the reduced and boundary layer subsystems (18) and (20) are asymptotically stable with these values for a and d.

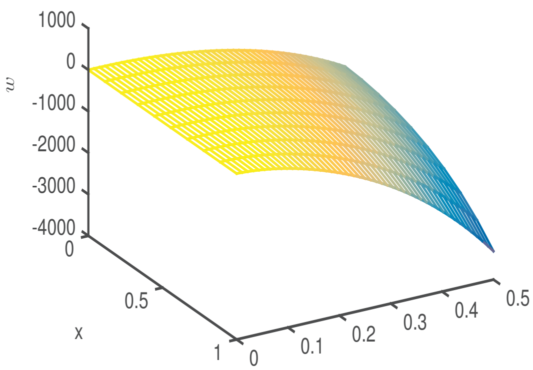

Fig. 1. Time evolution of the component w of the solution to (17)

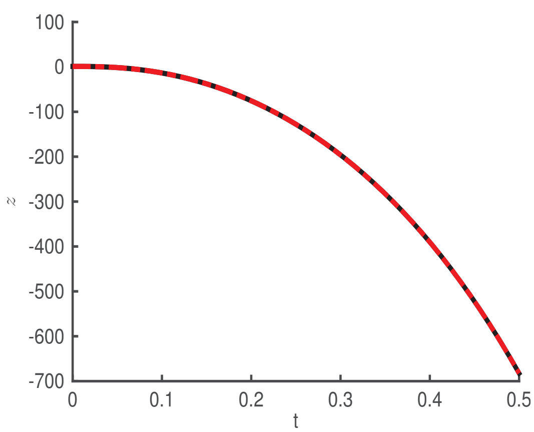

Fig. 2. Time evolution of the component z of the solution to (17)

V. Conclusions

In this paper, we consider an infinite-dimensional system obtained as the coupling of a wave equation with a linear ODE, with two different time scales. We have investigated the relation between the asymptotic stability of the subsystems with the asymptotic stability of the full dynamic. Two kinds of results have been obtained in this context. First, for a fast wave equation and a slow ODE, assuming that the reduced order system and the boundary layer system are exponentially stable, it has been proven that there exists a time scale ratio so that the full system is also exponentially stable. This first main result has been proven by employing a Lyapunov function approach. The second contribution of this paper is to show that the symmetric case is not so nice. To be more specific, when coupling a slow wave equation and a fast ODE, a system, which is not asymptotically stable, can be obtained, even if the reduced order system and the boundary layer system are both asymptotically stable. This latter result has been proving by using a necessary condition due to [8] together with an explicit example where such a phenomenon is proven to appear.

This paper arises many open questions. In particular, Tikhonov theorems for this systems and others with different couplings are under investigation.

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