NEUKIRCH-UCHIDA THEOREM FOR PURELY INSEPARABLE FIELD EXTENSIONS

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Abstract. In this paper we prove an analogue of the Neukirch-Uchida Theorem for purely inseparable field extensions. The main tool we are using is the Tannakian description of the local Nori fundamental gerbe developed in [TZ].

Introduction

Grothendieck's birational anabelian conjectures predict that a certain type of fields K should be "characterized" by their absolute Galois groups $\mathrm{Gal}(K)$. In a more categorical language the conjecture is about the functor $F : K \mapsto \mathrm{Gal}(K)$ from the category of fields of a certain type to the category of profinite groups. Is this functor F fully faithful? What is the essential image? To make the problem more precise, one first has to define the functor F. The topological group $\mathrm{Gal}(K)$, as an object in the category of profinite groups, depends not only on the field K but also on the choice of the algebraic closure $\bar{K}$ of K, as two different choices of $\bar{K}$ (say $\bar{K}_1$ and $\bar{K}_2$) would yield an isomorphism of $\mathrm{Gal}(K, \bar{K}_1) \cong \mathrm{Gal}(K, \bar{K}_2)$ which is only "canonical" up to an inner automorphism of $\mathrm{Gal}(K, \bar{K}_1)$ or $\mathrm{Gal}(K, \bar{K}_2)$. This suggests us to replace the target category of F by the category whose objects are profinite groups and whose morphisms between two objects G_1, G_2 are defined by the set of continuous group homomorphisms modulo the inner automorphisms of G_2:

$$\mathrm{Hom}_{\mathrm{cont}}(G_1, G_2)/\mathrm{Inn}(G_2)$$

Here is a more precise version of Grothendieck's birational anabelian conjectures.

Grothendieck's birational anabelian conjectures ([Pop, Part II, A)]):

(1) There is a group theoretic recipe by which one can recover the perfect closure K^{perf} of K from $\mathrm{Gal}(K)$ for every finitely generated infinite field K. In particular, if for such fields K and L one has $\mathrm{Gal}(K) \cong \mathrm{Gal}(L)$, then $K^{\mathrm{perf}} \cong L^{\mathrm{perf}}$.

(2) Moreover, given two such fields K and L, one has the following.
   - **Isom-form** Every isomorphism $\Phi\colon \mathrm{Gal}(K) \longrightarrow \mathrm{Gal}(L)$ is defined by a field isomorphism $\varphi\colon \bar{L} \longrightarrow \bar{K}$ via $\Phi(g) = \varphi^{-1} \circ g \circ \varphi$ for $g \in \mathrm{Gal}(K)$, and φ is unique up to Frobenius twists. In particular, one has $\varphi(L^{\mathrm{perf}}) = K^{\mathrm{perf}}$.

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- **Hom-form** Every open homomorphism $\Phi\colon \mathrm{Gal}(K) \longrightarrow \mathrm{Gal}(L)$ is defined by a field embedding $\varphi\colon \bar{L} \hookrightarrow \bar{K}$ which satisfies $\varphi(L^{\mathrm{perf}}) \subseteq K^{\mathrm{perf}}$, and φ is unique up to Frobenius twists. In particular, one has $\varphi \circ \Phi(g) = g \circ \varphi$ for $g \in \mathrm{Gal}(K)$.

Denote $\mathrm{Hom}(\bar{L}/L^{\mathrm{perf}}, \bar{K}/K^{\mathrm{perf}})$ the set of ring homomorphisms $\bar{L} \longrightarrow \bar{K}$ which send L^{perf} to K^{perf}, and let $\sim$ be the equivalence relation among morphisms which satisfies $f \sim g$ if and only if f and g differ by Frobenius twists of the target. The Hom-form (resp. Isom-form) conjecture actually claims that the canonical map of sets

$$\mathrm{Hom}(\bar{L}/L^{\mathrm{perf}}, \bar{K}/K^{\mathrm{perf}})/\sim \;\xrightarrow{\cong}\; \mathrm{Hom}^{\mathrm{open}}_{\mathrm{cont}}(\mathrm{Gal}(K), \mathrm{Gal}(L))$$

is an isomorphism. Note that there are natural $\mathrm{Gal}(L)$-actions on both sides of the above isomorphism and the isomorphism is $\mathrm{Gal}(L)$-equivariant. By taking the $\mathrm{Gal}(L)$-quotient on both sides one gets the following isomorphism:

$$\mathrm{Hom}_{\mathrm{Rings}}(L^{\mathrm{perf}}, K^{\mathrm{perf}})/\sim \;\longrightarrow\; \mathrm{Hom}^{\mathrm{open}}_{\mathrm{cont}}(\mathrm{Gal}(K), \mathrm{Gal}(L))/\mathrm{Inn}(\mathrm{Gal}(L))$$

The first results concerning the conjectures were proved by Neukirch and Uchida:

Theorem. (Neukirch-Uchida Theorem) *If K, L are number fields, then the canonical map of sets*

$$\mathrm{Isom}(L, K) \longrightarrow \mathrm{Isom}(\mathrm{Gal}(K), \mathrm{Gal}(L))/\mathrm{Inn}(\mathrm{Gal}(L))$$

is a bijection.

This theorem was later on generalized by Uchida ([Uch]) to all global fields, and then by Pop ([Pop94, Pop02]) to infinite fields that are finitely generated over their prime field. There is also a Hom-form result due to Mochizuki ([Mo]), but the result is for finitely generated regular extensions of a fixed base field.

The main purpose of this paper is to give an analogue of the Neukirch-Uchida theorem for purely inseparable field extensions. We first replace the Galois group by the *local Nori fundamental group scheme*. We work over a perfect field k of positive characteristic p (e.g. $k = \mathbb{F}_p$). Given a field extension K/k the local Nori fundamental group $\pi^{\mathrm{L}}(K/k)$ is defined as the Tannakian group scheme of a neutral k-Tannakian category attached to K(see 1.1).

We also denote by $\mathrm{PI}(K)$ the category of purely inseparable extensions of K. Since there exists at most one K-morphism between two purely inseparable extensions of K the category $\mathrm{PI}(K)$ is also equivalent to a partially ordered set. Finally we denote by $\mathrm{Gr.Sch}_k$ the category of affine group schemes over k. Here is our main result.

Theorem I. *Let k be a perfect field and K/k be a field extension. Then the contravariant functor*

$$\begin{array}{rcl} \mathrm{PI}(K) & \longrightarrow & \mathrm{Gr.Sch}_k/\pi^{\mathrm{L}}(K/k) \\ L/K & \longmapsto & (\pi^{\mathrm{L}}(L/k) \longrightarrow \pi^{\mathrm{L}}(K/k)) \end{array}$$

is fully faithful. More concretely, there is a well-defined map

$$\begin{array}{rcl} \mathrm{PI}(K) & \longrightarrow & \{\text{subgroups of } \pi^{\mathrm{L}}(K/k)\} \\ L/K & \longmapsto & \pi^{\mathrm{L}}(L/k) \end{array}$$

which is an order reverse embedding.

Part of Theorem I is to prove that, if L/K is purely inseparable, then $\pi^{\mathrm{L}}(L/k) \longrightarrow \pi^{\mathrm{L}}(K/k)$ is injective.

Notice that, by definition, the local Nori fundamental group π^{L} does not depend on the choice of an algebraic closure, so there is no need to mod out by the inner automorphisms.

In the case of Neukirch-Uchida Theorem there is implicitly a base $\mathbb{Q}$ everywhere. In particular for any number field K the Galois group $\mathrm{Gal}(K)$ should be considered as a profinite group equipped with a structure map $\mathrm{Gal}(K) \hookrightarrow \mathrm{Gal}(\mathbb{Q})$, but since we are in the category of profinite groups "up to inner automorphisms" and all automorphisms of $\mathrm{Gal}(\mathbb{Q})$ are inner [NSW, 12.2.3, pp.793], there is indeed no restriction.

The key property of $\pi^{\mathrm{L}}(-/k)$ which allows to prove Theorem I is the computation of its characters: if K/k is a field extension then there is a natural isomorphism

$$\mathrm{Hom}_k(\pi^{\mathrm{L}}(K/k), \mathbb{G}_m) \simeq (K^{\mathrm{perf}})^*/K^*$$

where K^{perf} is the perfect closure of K (see 1.3).

The second statement in Theorem I appears to be a reminiscent of the classical Galois correspondence between the separable extensions and the closed subgroups of the absolute Galois group. Unfortunately the analogy stops here: in our case the order reverse embedding in Theorem I is not surjective (see 1.6) unless K is perfect (which just means $\pi^{\mathrm{L}}(K/k) = 0$ by 1.4). Moreover, for a non trivial finite purely inseparable extension L/K the subgroup $\pi^{\mathrm{L}}(L/k)$ does not have finite index in $\pi^{\mathrm{L}}(K/k)$.

The proof of Theorem I follows directly from the Tannakian description of the local Nori fundamental group. On the other hand this Tannakian description is special case of a more general Tannakian construction, which also explains the use of the name *Nori* and of the adjective *local*. This connection is explained in the last chapter, whose purpose is to contextualize the notion of $\pi^{\mathrm{L}}(-/k)$.

The notion of the local Nori fundamental gerbe is developed in [TZ, §7]. Given a reduced algebraic stack $\mathcal{X}$ over k one can attach to it its local Nori fundamental gerbe $\Pi^{\mathrm{L}}_{\mathcal{X}/k}$ which essentially parametrizes maps from $\mathcal{X}$ to finite and local gerbes (see 2.1 and 2.2). What happens in this special case is that a gerbe which is finite and local is uniquely neutral. In other words, pro-local gerbes (that is projective limits of finite and local gerbes) correspond exactly to pro-local group schemes (projective limits of group schemes which are finite and connected over k). In particular, $\Pi^{\mathrm{L}}_{\mathcal{X}/k}$ corresponds to a pro-local group $\pi^{\mathrm{L}}(\mathcal{X}/k)$, which has the following universal property: morphisms from $\pi^{\mathrm{L}}(\mathcal{X}/k)$ to a finite and connected group G canonically correspond to G-torsors over $\mathcal{X}$ (see 2.8). We stress the fact that this does not require the choice of a rational or geometric point.

It is constructed in [TZ, §7] a k-Tannakian category (a posteriori uniquely neutralized) whose associated gerbe over k (a posteriori just a group scheme) is $\Pi^{\mathrm{L}}_{\mathcal{X}/k}$ (see 2.4). In the case when $\mathcal{X} = \operatorname{Spec} K$ this k-Tannakian category is nothing but a recast of the k-Tannakian category defining $\pi^{\mathrm{L}}(K/k)$. In other words, $\pi^{\mathrm{L}}(K/k) = \pi^{\mathrm{L}}((\operatorname{Spec} K)/k)$, which is therefore a projective limit of finite and connected group schemes and which has a universal property involving torsors under those groups (see 2.9).

The paper is divided as follows. In the first chapter we define the local Nori fundamental group and we prove Theorem I. In the second and last section we discuss the relationship with the local Nori fundamental group and the local Nori fundamental gerbe.

1. The Main Theorem

Let k be a perfect field of positive characteristic p. Given a field K we will denote by K^{perf} its perfect closure and by $\mathsf{Vect}(K)$ the category of finite dimensional K-vector spaces.

Definition 1.1. Let K be a field extension of k. We define $\mathcal{D}_\infty(K/k)$ as the category whose objects are (V, W, ψ), where $V \in \mathsf{Vect}(K)$, $W \in \mathsf{Vect}(k)$ and $\psi : K^{\mathrm{perf}} \otimes_K V \to K^{\mathrm{perf}} \otimes_k W$ is a K^{perf}-linear isomorphism. An arrow

$$(V, W, \psi) \to (V', W', \psi') \in \mathcal{D}_\infty(K/k)$$

is just a pair of maps (a, b), where a is K linear and b is k-linear, which is compatible with ψ and ψ'.

The category $\mathcal{D}_\infty(K/k)$ with its natural tensor product and k-linear structure is a neutral k-Tannakian category with the forgetful functor $\mathcal{D}_\infty(K/k) \longrightarrow \mathsf{Vect}(k)$ as the fiber functor. We define the *local Nori fundamental group* $\pi^{\mathrm{L}}(K/k)$ of K/k as the Tannakian group scheme associated with $\mathcal{D}_\infty(K/k)$.

Lemma 1.2. *If L/K is a purely inseparable extension of fields over k then the induced map of group schemes $\pi^{\mathrm{L}}(L/k) \to \pi^{\mathrm{L}}(K/k)$ is a closed embedding.*

Proof. By [DM82, Prop. 2.21 (b)] it is enough to show that the pullback functor

$$\mathcal{D}_\infty(K/k) \to \mathcal{D}_\infty(L/k)$$

is essentially surjective. Let $(V, W, \phi) \in \mathcal{D}_\infty(L/k)$. Clearly there is an isomorphism $(V, W, \phi) \cong (L^{\oplus m}, k^{\oplus m}, \varphi) \in \mathcal{D}_\infty(L/k)$. Now consider the isomorphism

$$\varphi \colon (L^{\mathrm{perf}})^{\oplus m} \longrightarrow (L^{\mathrm{perf}})^{\oplus m}$$

Since L/K is purely inseparable, we can identify K^{perf} with L^{perf}. In this case it is easy to see that $(K^{\oplus m}, k^{\oplus m}, \varphi) \in \mathcal{D}_\infty(K/k)$ is sent to $(L^{\oplus m}, k^{\oplus m}, \varphi)$. □

Proposition 1.3. *Let K/k be a field extension. Then there is a canonical isomorphism*

$$\mathrm{Pic}(\mathcal{D}_\infty(K/k)) \simeq \mathrm{Hom}_k(\pi^{\mathrm{L}}(K/k), \mathbb{G}_m) \simeq (K^{\mathrm{perf}})^*/K^*$$

Proof. The first isomorphism exists since both sides are the group of isomorphism classes of 1-dimensional representations of $\pi^{\mathrm{L}}(K/k)$. There is a group homomorphism

$$\begin{aligned} (K^{\mathrm{perf}})^* &\longrightarrow \mathrm{Pic}(\mathcal{D}_\infty(K/k)) \\ \phi &\longmapsto (K, k, \phi) \end{aligned}$$

It is easy to see that it is surjective and that its kernel is K^*. □

Proof of Theorem I. By 1.2 and the fact that $\mathrm{PI}(K)$ is an ordered set, it is enough to prove the second part of the statement. All elements of $\mathrm{PI}(K)$ has a unique embedding in K^{perf}. Thus we have to prove that, if $K \subseteq L_1, L_2 \subseteq K^{\mathrm{perf}}$ and $\pi^{\mathrm{L}}(L_1/k) \subseteq \pi^{\mathrm{L}}(L_2/k) \subseteq \pi^{\mathrm{L}}(K/k)$, then $L_2 \subseteq L_1$.

By 1.3 and the commutative diagram of group schemes

$$\begin{tikzcd} \pi^{\mathrm{L}}(L_1/k) \arrow[rr, "\phi"] \arrow[dr, "s_{L_1}"'] & & \pi^{\mathrm{L}}(L_2/k) \arrow[dl, "s_{L_2}"] \\ & \pi^{\mathrm{L}}(K/k) & \end{tikzcd}$$

we get a commutative diagram of abelian groups

$$\begin{tikzcd} (K^{\mathrm{perf}})^*/L_2^* \arrow[rr, "\varphi"] & & (K^{\mathrm{perf}})^*/L_1^* \\ & (K^{\mathrm{perf}})^*/K^* \arrow[ul, "a"] \arrow[ur, "b"'] & \end{tikzcd}$$

where s_{L_i} is the inclusion $\pi^{\mathrm{L}}(L_i/k) \subseteq \pi^{\mathrm{L}}(K/k)$. In particular we conclude that a and b are induced on the perfect closure by $K \subseteq L_2$ and $K \subseteq L_1$ respectively, that is a and b are induced by the identity map. Then it is clear that $b(L_2^*/K^*) = 1$, as $a(L_2^*/K^*) = 1$. Thus we have $L_2^*/K^* \subseteq L_1^*/K^*$. This shows the inclusion $L_2 \subseteq L_1$. □

Corollary 1.4. *Let K/k be a field extension. Then K is perfect if and only if $\pi^{\mathrm{L}}(K/k) = 0$.*

Proof. If $\pi^{\mathrm{L}}(K/k) = 0$, then from Theorem I we see that $\mathrm{PI}(K)$ has just one element, that is K is perfect. Now assume that K is perfect, that is $K^{\mathrm{perf}} = K$. If $(V, W, \phi) \in \mathcal{D}_\infty(K/k)$, then $\phi\colon V \longrightarrow W \otimes_k K$ is an isomorphism. It is easy to see that

$$(V, W, \phi) \xrightarrow{(\phi,\mathrm{id})} (W \otimes_k K, W, \mathrm{id})$$

is an isomorphism. This means that $\mathsf{Vect}(k) = \mathcal{D}_\infty(K/k)$, that is $\pi^{\mathrm{L}}(K/k) = 0$. □

Lemma 1.5. *If G is an affine group scheme then there exists a canonical surjective map*

$$G \longrightarrow \mathrm{D}(\mathrm{Hom}(G, \mathbb{G}_m))$$

where $\mathrm{D}(-)$ is the diagonalizable group associated with an abelian group, which is universal among all maps to diagonalizable group schemes.

Proof. Let $\mathcal{R}$ be the category of surjective maps $G \twoheadrightarrow H$ such that H is a diagonalizable group. It is easy to see that $\mathcal{R}$ is a cofiltered category, because products and subgroups of diagonalizable groups are diagonalizable. The group scheme

$$G' = \varprojlim_{(G \longrightarrow H) \in \mathcal{R}} H$$

is diagonalizable because it is a projective limit of diagonalizable groups. Moreover, there is a map $G \longrightarrow G'$, which is surjective because $k[G']$ is just a union of sub Hopf-algebras of $k[G]$ whose corresponding group schemes are diagonalizable. It is also easily seen to be universal among maps from G to diagonalizable groups, as a sub-group scheme of a diagonalizable group scheme is still diagonalizable. In particular $\mathrm{Hom}(G, \mathbb{G}_m) = \mathrm{Hom}(G', \mathbb{G}_m)$ and, since G' is diagonalizable, $G' = \mathrm{D}(\mathrm{Hom}(G', \mathbb{G}_m))$. □

Example 1.6. Let K be a field extension of k which is not perfect. We claim that:

(1) there are subgroups H of $\pi^{\mathrm{L}}(K/k)$ which are not of the form $\pi^{\mathrm{L}}(L/k)$ for some purely inseparable extension L/K;
(2) if L/K is a non trivial purely inseparable extension, then the quotient scheme $\pi^{\mathrm{L}}(K/k)/\pi^{\mathrm{L}}(L/k)$ is not finite over k.

From 1.3 and 1.5 we obtain a canonical surjective map $\pi^{\mathrm{L}}(K/k) \longrightarrow \mathrm{D}((K^{\mathrm{perf}})^*/K^*)$ which is universal among maps to a diagonalizable group scheme. Given a purely inseparable extension L/K we obtain a commutative diagram

$$\begin{tikzcd} \pi^{\mathrm{L}}(L/k) \arrow[r, hook] \arrow[d] & \pi^{\mathrm{L}}(K/k) \arrow[r, two heads] \arrow[d] & \pi^{\mathrm{L}}(K/k)/\pi^{\mathrm{L}}(L/k) \arrow[d] \\ \mathrm{D}((L^{\mathrm{perf}})^*/L^*) \arrow[r, hook] & \mathrm{D}((K^{\mathrm{perf}})^*/K^*) \arrow[r, two heads] & \mathrm{D}(L^*/K^*) \end{tikzcd}$$

In particular if $K^* \subseteq Q \subseteq (K^{\mathrm{perf}})^*$ is a subgroup not of the form L^* for some purely inseparable extension L of K then the inverse image of $\mathrm{D}((K^{\mathrm{perf}})^*/Q) \subseteq \mathrm{D}((K^{\mathrm{perf}})^*/K^*)$ along $\pi^{\mathrm{L}}(K/k) \longrightarrow \mathrm{D}((K^{\mathrm{perf}})^*/K^*)$ cannot be a local fundamental group. For (2) instead one just have to show that L^*/K^* is not finitely generated. Indeed one observes that the vertical map on the right is faithfully flat because all other surjective maps are affine and faithfully flat (see [SP, 036J]).

In order to have a concrete example and also show (2) it is enough to prove that, if $L = K[X]/(X^p - \lambda)$ with $\lambda \in K - K^p$, then L^*/K^* is not finitely generated. Since L^*/K^* is an $\mathbb{F}_p$-vector space it is enough to show that L^*/K^* is infinite. Set $v_n = 1 + \lambda^n X$ for $n \in \mathbb{N}$. We claim that they are all different in L^*/K^*. If they coincide for some $m \neq n \in \mathbb{N}$ then a direct computation shows that λ is a root of unity. In particular it is algebraic and thus separable over k. In conclusion X would be purely inseparable over the perfect field $k(\lambda)$, from which we find the contradiction $X \in k(\lambda) \subseteq K$.

2. The Arithmetic Local Nori Fundamental Group

In this section we fix a base field k of positive characteristic p.

Definition 2.1. A group scheme G over k is called *local* if it is finite and connected.

An affine gerbe Γ over k is called *finite* (resp. finite and *local*) if $\Gamma \times_k \overline{k} \simeq \mathrm{B}_{\overline{k}} G$, where G is a finite (resp. finite and local) group scheme over $\overline{k}$.

By a *pro-local gerbe* (resp. *pro-local group scheme*) over k we mean a small cofiltered limit of finite and local gerbes (resp. group schemes) over k.

Definition 2.2. Let $\mathcal{X}$ be a reduced algebraic stack over k such that $\mathrm{H}^0(\mathcal{O}_{\mathcal{X}})$ does not contain elements which are purely inseparable over k. The *local Nori fundamental gerbe* of $\mathcal{X}/k$ is a pro-local gerbe Π over k together with a morphism $\mathcal{X} \longrightarrow \Pi$ such that for all finite and local gerbes Γ over k the pullback functor

$$\mathrm{Hom}_k(\Pi, \Gamma) \longrightarrow \mathrm{Hom}_k(\mathcal{X}, \Gamma)$$

is an equivalence. If this gerbe exists it will be denoted by $\Pi^{\mathrm{L}}_{\mathcal{X}/k}$.

Remark 2.3. Notice that a gerbe Γ is finite (resp. finite and local) if and only it is finite (resp. finite and local) in the sense of [TZ, Definition 3.1, pp.10] (resp. [TZ, Definition 3.9, pp.12]). (See [TZ, Proposition B.6, pp.41])

By [TZ, Theorem 7.1, pp.36] a local Nori fundamental gerbe $\Pi^{\mathrm{L}}_{\mathcal{X}/k}$ exists and it coincides with the local Nori fundamental gerbe considered in [TZ, Definition 4.1, pp.12]. Moreover, the local Nori fundamental gerbe is unique up to a unique isomorphism because pro-local gerbes are projective limit of finite and local gerbes.

Theorem 2.4. [TZ, Theorem 7.1, pp.36] *Assume the same hypothesis of 2.2 and denote by $F\colon \mathcal{X} \longrightarrow \mathcal{X}$ the absolute Frobenius. For $i \in \mathbb{N}$ denote by $\mathcal{D}_i$ the category of triples $(\mathcal{F}, V, \lambda)$ where $\mathcal{F} \in \mathsf{Vect}(\mathcal{X})$, $V \in \mathsf{Vect}(k)$ and $\lambda\colon F^{i*}\mathcal{F} \longrightarrow V \otimes_k \mathcal{O}_{\mathcal{X}}$ is an isomorphism. Then the category $\mathcal{D}_i$ is k-Tannakian with k-structure $k \longrightarrow \mathrm{End}_{\mathcal{D}_i}(\mathcal{O}_{\mathcal{X}}, k, \mathrm{id})$, $x \longmapsto (x, x^{p^i})$. Moreover the association*

$$\mathcal{D}_i \longrightarrow \mathcal{D}_{i+1},\ (\mathcal{F}, V, \lambda) \longmapsto (\mathcal{F}, F_k^* V, F^*\lambda)$$

where F_k is the absolute Frobenius of k, is k-linear, monoidal and exact, and there is a natural equivalence of k-Tannakian categories:

$$\mathcal{D}_\infty := \varinjlim_{i \in \mathbb{N}} \mathcal{D}_i \xrightarrow{\simeq} \mathsf{Rep}(\Pi^{\mathrm{L}}_{\mathcal{X}/k})$$

Remark 2.5. Using the same notations from 2.4 and assuming that k is perfect, the functors $\mathcal{D}_n \longrightarrow \mathsf{Vect}(k)$, $(\mathcal{F}, V, \lambda) \longmapsto F_k^{-n*} V$, where F_k is the absolute Frobenius of k, are compatible when n varies, so they induce a functor $\mathcal{D}_\infty \longrightarrow \mathsf{Vect}(k)$. It is easy to check that this functor is k-linear, exact and tensorial. In particular $\mathcal{D}_\infty$ has a neutralization or, in other words, $\Pi^{\mathrm{L}}_{\mathcal{X}/k}(k) \neq \varnothing$.

In fact there is much more: over a perfect field a pro-local gerbe is neutral, and the neutralization is unique up to a unique isomorphism.

Lemma 2.6. *Let Γ be a pro-local gerbe over a perfect field k. Then $\Gamma(k)$ is equivalent to a set with one point, in other words, it is a non-empty groupoid in which between every two objects there exists exactly one isomorphism. Or equivalently, the Tannakian category $\mathsf{Vect}(\Gamma)$ has a neutral fiber functor which is unique up to a unique isomorphism.*

Proof. Since Γ is a profinite gerbe, we may write $\Gamma := \varprojlim_{i \in I} \Gamma_i$, where I is a cofiltered essentially small category and Γ_i are finite and local gerbes over k. It is easy to see that we can reduce the problem to the case where Γ is finite.

We first show that $\Gamma(k) \neq \varnothing$. The stack Γ is reduced because it has a faithfully flat map from a reduced scheme, namely the spectrum of some field. Moreover since Γ is local we clearly have $\Gamma = \Pi^{\mathrm{L}}_{\Gamma/k}$. Thus $\Gamma(k) = \Pi^{\mathrm{L}}_{\mathcal{X}/k}(k) \neq \varnothing$ by 2.5.

In particular $\Gamma = \mathrm{B}\, G$, where G is a finite and local group scheme over k. If P is a G-torsor over k then P is finite and geometrically connected. Thus $P = \operatorname{Spec} A$, where A is local, finite and with residue field purely inseparable over k. Thus $P(k)$ consists of one element. This is precisely what we wanted to show. $\square$

Since affine group schemes are the same as affine gerbes with a given rational section we obtain the following:

Corollary 2.7. *Let k be a perfect field. The functor*

$$\begin{array}{ccc} \{\text{pro-local group schemes over } k\} & \longrightarrow & \{\text{pro-local gerbes over } k\} \\ G & \longmapsto & \mathrm{B}\, G \end{array}$$

is an equivalence of categories (meaning that between two functors of pro-local gerbes there exists at most one natural isomorphism).

In particular if $\mathcal{X}$ is a reduced algebraic stack over a perfect field k, then $\Pi^{\mathrm{L}}_{\mathcal{X}/k} = \mathrm{B}\, \pi^{\mathrm{L}}(\mathcal{X}/k)$, where $\pi^{\mathrm{L}}(\mathcal{X}/k)$ is a pro-local group scheme over k with a $\pi^{\mathrm{L}}(\mathcal{X}/k)$-torsor over $\mathcal{X}$ such that the induced map

$$\mathrm{Hom}_k(\pi^{\mathrm{L}}(\mathcal{X}/k), G) \longrightarrow \{G\text{-torsors over } \mathcal{X}\}$$

is an equivalence for all finite and local group schemes G over k. This follows from the universal property of the local Nori gerbe and the equivalence in 2.7. In particular the category on the right is indeed a set, that is there exists at most one isomorphism between two G-torsors of $\mathcal{X}$.

Moreover, it also follows that surjective group homomorphisms $\pi^{\mathrm{L}}(\mathcal{X}/k) \longrightarrow G$ correspond to G-torsors which are minimal, that is the ones which are not induced by a torsor under a proper subgroup of G.

Definition 2.8. If $\mathcal{X}$ is a reduced algebraic stack over a perfect field k we call $\pi^{\mathrm{L}}(\mathcal{X}/k)$ the *local Nori fundamental group scheme* of $\mathcal{X}$ over k.

In this special situation of local group schemes there is no need for choosing a rational or geometric point. Similar phenomenon also appears in [Zh, Proposition 2.21 (ii) and Remark 2.22, pp.21]. The following proposition shows that this is indeed not a coincidence:

Proposition 2.9. *Let $\mathcal{X}$ be a reduced algebraic stack over a perfect field k and consider the category $\mathcal{N}(\mathcal{X}/k)$ of pairs $(G, \mathcal{P})$ where G is a finite and local group scheme over k and $f\colon \mathcal{P} \longrightarrow \mathcal{X}$ is a G-torsor. Then $\mathcal{N}(\mathcal{X}/k)$ is a small cofiltered category and there is a canonical isomorphism*

$$\pi^{\mathrm{L}}(\mathcal{X}/k) \simeq \varprojlim_{(G,\mathcal{P}) \in \mathcal{N}(\mathcal{X}/k)} G$$

Proof. By the above discussion we obtain that the category $\mathcal{N}(\mathcal{X}/k)$ is equivalent to the category $\mathrm{Hom}_k(\pi^{\mathrm{L}}(\mathcal{X}/k), -)$ of morphisms from $\pi^{\mathrm{L}}(\mathcal{X}/k)$ to finite and local group schemes. Notice that $\mathrm{Hom}_k(\pi^{\mathrm{L}}(\mathcal{X}/k), -)$ has fiber products and, in particular, it is cofiltered (see [Zh, Remark 1.3, (i), pp.8]). Moreover, if $\mathcal{N}'(\mathcal{X}/k)$ is the full subcategory of $\mathrm{Hom}_k(\pi^{\mathrm{L}}(\mathcal{X}/k), -)$

consisting of quotient maps, then $\varprojlim_{(G,\mathcal{P})\in\mathcal{N}(\mathcal{X}/k)} G$ equals to $\varprojlim_{(G,\mathcal{P})\in\mathcal{N}'(\mathcal{X}/k)} G$, which is the limit of all finite and local quotient of $\pi^{\mathrm{L}}(\mathcal{X}/k)$. Thus the Hopf-algebra of $\varprojlim_{(G,\mathcal{P})\in\mathcal{N}(\mathcal{X}/k)} G$ is contained in $k[\pi^{\mathrm{L}}(\mathcal{X}/k)]$. Since $\pi^{\mathrm{L}}(\mathcal{X}/k)$ is pro-local, it is a cofiltered limit of some of its finite and local quotients. So we obtained an inclusion of Hopf-algebras in the other direction, and this finishes the proof. □

Corollary 2.10. *If X is a reduced scheme over a perfect field k with a geomtric point $x\colon \operatorname{Spec}\Omega \longrightarrow X$, where Ω is an algebraically closed field, then $\pi^{\mathrm{L}}(X/k)$ coincides with the group scheme $\pi^{\mathrm{L}}(X/k,x)$ defined in* [Zh, Definition 5, (iv), pp.12].

Proposition 2.11. *If K/k is a field extension with k perfect, then we have an isomorphism $\pi^{\mathrm{L}}(K/k) \cong \pi^{\mathrm{L}}((\operatorname{Spec} K/k))$.*

Proof. Using notations from 2.4 for $\mathcal{X} = \operatorname{Spec} K$ we have to show that there is an equivalence of Tannakian categories $\mathcal{D}_\infty \simeq \mathcal{D}_\infty(K/k)$. If $(V,W,\phi) \in \mathcal{D}_n(K/k)$ with $V \in \mathsf{Vect}(K)$, $W \in \mathsf{Vect}(k)$ and $\phi\colon F^{n*}V \simeq W \otimes_k K$, then via the isomorphism of fields $K^{1/p^n} \to K$, $x \mapsto x^{p^n}$, we get $F^{n*}V \simeq V \otimes_K K^{1/p^n}$ and $W \otimes_k K \simeq (F_k^{-n*}W) \otimes_k K^{1/p^n}$. Then it is not difficult to show that, for $n \in \mathbb{N}$, the category $\mathcal{D}_n(K/k)$ is equivalent to the category of triples (M,N,φ) where $M \in \mathsf{Vect}(K)$, $N \in \mathsf{Vect}(k)$ and $\varphi\colon M \otimes_K K^{1/p^n} \simeq N \otimes_k K^{1/p^n}$ is an isomorphism. By passing to the limit we get $\mathcal{D}_\infty \simeq \mathcal{D}_\infty(K/k)$. □

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