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# Distance colouring without one cycle length

Ross J. Kang <sup>\*</sup>      François Pirot <sup>†</sup>

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## Abstract

We consider distance colourings in graphs of maximum degree at most  $d$  and how excluding one fixed cycle of length  $\ell$  affects the number of colours required as  $d \rightarrow \infty$ . For vertex-colouring and  $t \geq 1$ , if any two distinct vertices connected by a path of at most  $t$  edges are required to be coloured differently, then a reduction by a logarithmic (in  $d$ ) factor against the trivial bound  $O(d^t)$  can be obtained by excluding an odd cycle length  $\ell \geq 3t$  if  $t$  is odd or by excluding an even cycle length  $\ell \geq 2t + 2$ . For edge-colouring and  $t \geq 2$ , if any two distinct edges connected by a path of fewer than  $t$  edges are required to be coloured differently, then excluding an even cycle length  $\ell \geq 2t$  is sufficient for a logarithmic factor reduction. For  $t \geq 2$ , neither of the above statements are possible for other parity combinations of  $\ell$  and  $t$ . These results can be considered extensions of results due to Johansson (1996) and Mahdian (2000), and are related to open problems of Alon and Mohar (2002) and Kaiser and Kang (2014).

## 1 Introduction

For a positive integer  $t$ , the  $t$ -th power  $G^t$  of a (simple) graph  $G = (V, E)$  is the graph with vertex set  $V$  in which two distinct elements of  $V$  are adjacent in  $G^t$  if there is a path in  $G$  of length at most  $t$  between them. The *line graph*  $L(G)$  of a graph  $G = (V, E)$  is the graph with vertex set  $E$  in which two distinct elements are adjacent in  $L(G)$  if the corresponding edges of  $G$  have a common endpoint. The *distance- $t$  chromatic number*  $\chi_t(G)$ , respectively, *distance- $t$  chromatic index*  $\chi'_t(G)$ , of  $G$  is the chromatic number of  $G^t$ , respectively, of  $(L(G))^t$ . (So  $\chi_1(G)$  is the chromatic number  $\chi(G)$  of  $G$ ,  $\chi'_1(G)$  the chromatic index  $\chi'(G)$  of  $G$ , and  $\chi'_2(G)$  the strong chromatic index  $\chi'_s(G)$  of  $G$ .)

The goal of this work is to address the following basic question. What is the largest possible value of  $\chi_t(G)$  or of  $\chi'_t(G)$  among all graphs  $G$  with maximum degree at most  $d$  that do not contain the cycle  $C_\ell$  of length  $\ell$  as a subgraph? For both parameters, we are interested in finding those choices of  $\ell$  (depending on  $t$ ) for which there is an upper bound that is  $o(d^t)$  as  $d \rightarrow \infty$ . (Trivially  $\chi_t(G)$  and  $\chi'_t(G)$  are  $O(d^t)$  since the maximum degrees  $\Delta(G^t)$  and  $\Delta((L(G))^t)$  are  $O(d^t)$  as  $d \rightarrow \infty$ . Moreover, by probabilistic constructions [2, 8], these upper

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bounds must be  $\Omega(d^t/\log d)$  as  $d \rightarrow \infty$  regardless of the choice of  $\ell$ .) We first discuss some previous work.

For  $t = 1$  and  $\ell = 3$ , the question for  $\chi_t$  essentially was a long-standing problem of Vizing [13], one that provoked much work on the chromatic number of triangle-free graphs, and was eventually settled asymptotically by Johansson [7]. He used nibble methods to show that the largest chromatic number over all triangle-free graphs of maximum degree at most  $d$  is  $\Theta(d/\log d)$  as  $d \rightarrow \infty$ . It was observed in [9] that this last statement with  $C_\ell$ -free,  $\ell > 3$ , rather than triangle-free also holds, thus completely settling this question asymptotically for  $\chi_1 = \chi$ .

Regarding the question for  $\chi'_t$ , first notice that since the chromatic index of a graph of maximum degree  $d$  is either  $d$  or  $d + 1$ , there is little else to say asymptotically if  $t = 1$ .

For  $t = 2$  and  $\ell = 4$ , the question for  $\chi'_t$  was considered by Mahdian [10] who showed that the largest strong chromatic index over all  $C_4$ -free graphs of maximum degree at most  $d$  is  $\Theta(d^2/\log d)$  as  $d \rightarrow \infty$ . Vu [14] extended this to hold for any fixed bipartite graph instead of  $C_4$ , which in particular implies the statement for any  $C_\ell$ ,  $\ell$  even. Since the complete bipartite graph  $K_{d,d}$  satisfies  $\chi'_2(K_{d,d}) = d^2$ , the statement does not hold for  $C_\ell$ ,  $\ell$  odd. This completely settles the second question asymptotically for  $\chi'_2 = \chi'_s$ .

In this paper, we advance a systematic treatment of our basic question. Our main results are as follows, which may be considered as extensions of the results of Johansson [7] and Mahdian [10] to distance- $t$  vertex- and edge-colouring, respectively, for all  $t$ .

**Theorem 1.** *Let  $t$  be a positive integer and  $\ell$  an even positive integer.*

- (i) *For  $\ell \geq 2t + 2$ , the supremum of the distance- $t$  chromatic number over  $C_\ell$ -free graphs of maximum degree at most  $d$  is  $\Theta(d^t/\log d)$  as  $d \rightarrow \infty$ .*
- (ii) *For  $t \geq 2$  and  $\ell \geq 2t$ , the supremum of the distance- $t$  chromatic index over  $C_\ell$ -free graphs of maximum degree at most  $d$  is  $\Theta(d^t/\log d)$  as  $d \rightarrow \infty$ .*

**Theorem 2.** *Let  $t$  and  $\ell$  be odd positive integers such that  $\ell \geq 3t$ . The supremum of the distance- $t$  chromatic number over  $C_\ell$ -free graphs of maximum degree at most  $d$  is  $\Theta(d^t/\log d)$  as  $d \rightarrow \infty$ .*

This study was initiated by a conjecture of ours in [9], that the largest distance- $t$  chromatic number over all  $C_{2t+2}$ -free graphs of maximum degree at most  $d$  is  $\Theta(d^t/\log d)$  as  $d \rightarrow \infty$ . Theorem 1(i) confirms our conjecture.

In Section 2, we exhibit constructions to certify the following, so improved upper bounds are impossible for the parity combinations of  $t$  and  $\ell$  other than those in Theorems 1 and 2.

**Proposition 3.** *Let  $t$  and  $\ell$  be positive integers.*

- (i) *For  $t$  even and  $\ell$  odd, the supremum of the distance- $t$  chromatic number over  $C_\ell$ -free graphs of maximum degree at most  $d$  is  $\Theta(d^t)$  as  $d \rightarrow \infty$ .*
- (ii) *For  $t \geq 2$  and  $\ell$  odd, the supremum of the distance- $t$  chromatic index over  $C_\ell$ -free graphs of maximum degree at most  $d$  is  $\Theta(d^t)$  as  $d \rightarrow \infty$ .*

We have reason to suspect that the values  $2t + 2$  and  $2t$ , respectively, may not be improved to lower values in Theorem 1, but we do not go so far yet as to conjecture this. We also wonder whether the value  $3t$  in Theorem 2 is optimal — it might well only be a coincidence for  $t = 1$  — but we know that in general it may not be lower than  $t$ , as we show in Section 2.

Our basic question in fact constitutes refined versions of problems of Alon and Mohar [2] and of Kaiser and the first author [8], which instead asked about the asymptotically extremal distance- $t$  chromatic number and index, respectively, over graphs of maximum degree  $d$  and girth at least  $g$  as  $d \rightarrow \infty$ . Our upper bounds imply bounds given earlier in [2, 8, 9], and the lower bound constructions given there are naturally relevant here (as we shall see in Section 2).

It is worth pointing out that the basic question unrestricted, i.e. asking for the extremal value of the distance- $t$  chromatic number or index over graphs of maximum degree  $d$  as  $d \rightarrow \infty$ , is likely to be very difficult if we ask for the precise (asymptotic) multiplicative constant. This is because the question for  $\chi_t$  then amounts to a slightly weaker version of a well-known conjecture of Bollobás on the degree–diameter problem [3], while the question for  $\chi'_t$  then includes the notorious strong edge-colouring conjecture of Erdős and Nešetřil, cf. [5], as a special case.

Our proofs of Theorems 1 and 2 rely on direct applications of the following result of Alon, Krivelevich and Sudakov [1], which bounds the chromatic number of a graph with bounded neighbourhood density.

**Lemma 4** ([1]). *For all graphs  $G = (V, E)$  with maximum degree at most  $\Delta$  such that for each  $v \in V$  there are at most  $\Delta^2/f$  edges spanning  $N(v)$ , it holds that  $\chi(G) = O(\Delta/\log f)$  as  $\Delta \rightarrow \infty$ .*

The proof of this result in [1] invoked Johansson’s result for triangle-free graphs; using nibble methods directly instead, Vu [14] extended it to hold for list colouring. So Theorems 1 and 2 also hold with list versions of  $\chi_t$  and  $\chi'_t$ .

Section 3 is devoted to showing the requisite density properties for Lemma 4. In order to do so with respect to Theorem 1, we in part use some intermediary results that were employed in a recent improvement [12] upon the classic result of Bondy and Simonovits [4] that the Turán number  $\text{ex}(n, C_{2k})$  of the even cycle  $C_{2k}$ , that is, the maximum number of edges in a graph on  $n$  vertices not containing  $C_{2k}$  as a subgraph, satisfies  $\text{ex}(n, C_{2k}) = O(n^{1+1/k})$  as  $n \rightarrow \infty$ . It is natural that techniques used to show sparsity of  $C_{2k}$ -free graphs are helpful for Theorem 1, since the application of Lemma 4 demands the verification of a *local* sparsity condition.

We made little effort to optimise the multiplicative constants implicit in Theorems 1 and 2 and in Proposition 3, since we partly relied on a constant from Lemma 4 that as far as we know has yet to be optimised. More importantly, the constants we obtained depend on  $\ell$  or  $t$ , and it is left to future work to determine the correct dependencies. To be precise, in Theorems 1 and 2 the asymptotic (first letting  $d \rightarrow \infty$ ) multiplicative gaps between the best upper and lower bounds we know can be  $\Omega(t)$  as  $t \rightarrow \infty$ , while for Proposition 3 the gaps are often as large as  $2^{t+o(t)}$ .

## 2 Constructions

In this section, we describe some constructions that certify the conclusions of Theorems 1 and 2 are not possible with other parity combinations of  $t$  and  $\ell$ , in particular showing Proposition 3.

First we review constructions we used in previous work [9]. In combination with the trivial bound  $\chi_t(G) = O(d^t)$  if  $\Delta(G) \leq d$ , the following two propositions imply Proposition 3(i). The next result also shows that the value  $3t$  in Theorem 2 may not be reduced below  $t$ .

**Proposition 5.** *Fix  $t \geq 3$ . For every even  $d \geq 2$ , there exists a  $d$ -regular graph  $G$  with  $\chi_t(G) \geq d^t/2^t$  and  $\chi'_{t+1}(G) \geq d^{t+1}/2^t$ . Moreover,  $G$  is bipartite if  $t$  is even, and  $G$  does not contain any odd cycle of length less than  $t$  if  $t$  is odd.*

*Proof.* We define  $G = (V, E)$  as follows. The vertex set is  $V = \cup_{i=0}^{t-1} U^{(i)}$  where each  $U^{(i)}$  is a copy of  $[d/2]^t$ , the set of ordered  $t$ -tuples of symbols from  $[d/2] = \{1, \dots, d/2\}$ . For all  $i \in \{0, \dots, t-1\}$ , we join an element  $(x_0^{(i)}, \dots, x_{t-1}^{(i)})$  of  $U^{(i)}$  and an element  $(x_0^{(i+1 \bmod t)}, \dots, x_{t-1}^{(i+1 \bmod t)})$  of  $U^{(i+1 \bmod t)}$  by an edge if the  $t$ -tuples agree on all symbols except possibly at coordinate  $i$ , i.e. if  $x_j^{(i+1 \bmod t)} = x_j^{(i)}$  for all  $j \in \{0, \dots, t-1\} \setminus \{i\}$  (and  $x_i^{(i)}, x_i^{(i+1 \bmod t)}$  are arbitrary from  $[d/2]$ ).

It is easy to see that for any  $i \in [t]$ ,  $U^{(i)}$  is a clique in  $G^t$ , and the set of edges incident to  $U^{(i)}$  is a clique in  $(L(G))^{t+1}$ . This gives  $\chi_t(G) \geq |U^{(0)}| = (d/2)^t$  and  $\chi'_{t+1}(G) \geq d \cdot |U^{(0)}| = 2(d/2)^{t+1}$ . (In fact here it is easy to find a colouring achieving equality in both cases.)

Since  $G$  is composed only of bipartite graphs arranged in sequence around a cycle of length  $t$ , every odd cycle in  $G$  is of length at least  $t$ , and  $G$  is bipartite if  $t$  is even.  $\square$

As observed in [2] and [8], certain finite geometries yield bipartite graphs of prescribed girth giving better bounds than in Proposition 5 for a few cases.

**Proposition 6.** *Let  $d$  be one more than a prime power.*

- *There exists a bipartite, girth 6,  $d$ -regular graph  $\mathcal{P}_{d-1}$  with  $\chi_2(\mathcal{P}_{d-1}) = d^2 - d + 1$  and  $\chi'_3(\mathcal{P}_{d-1}) = d^3 - d^2 + d$ .*
- *There exists a bipartite, girth 8,  $d$ -regular graph  $\mathcal{Q}_{d-1}$  with  $\chi'_4(\mathcal{Q}_{d-1}) = d^4 - 2d^3 + 2d^2$ .*
- *There exists a bipartite, girth 12,  $d$ -regular graph  $\mathcal{H}_{d-1}$  with  $\chi'_6(\mathcal{H}_{d-1}) = d^6 - 4d^5 + 7d^4 - 6d^3 + 3d^2$ .*
- *If  $d$  is one more than a power of 2, then there exists a  $d$ -regular graph  $\tilde{\mathcal{Q}}_{d-1}$  with  $\chi_3(\tilde{\mathcal{Q}}_{d-1}) = d^3 - 2d^2 + 2d$ .*
- *If  $d$  is one more than a power of 3, then there exists a  $d$ -regular graph  $\tilde{\mathcal{H}}_{d-1}$  with  $\chi_5(\tilde{\mathcal{H}}_{d-1}) = d^5 - 4d^4 + 7d^3 - 6d^2 + 3d$ .*

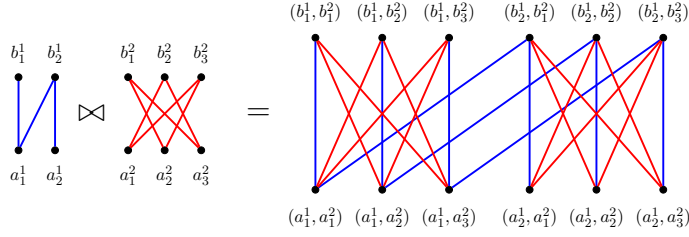


Figure 1: An illustration of the balanced bipartite product.

*Proof.* We let  $\mathcal{P}_{d-1}$  be the point-line incidence graph of the projective plane  $PG(2, d-1)$ ,  $\mathcal{Q}_{d-1}$  be that of a symplectic quadrangle with parameters  $(d-1, d-1)$ , and  $\mathcal{H}_{d-1}$  be that of a split Cayley hexagon with parameters  $(d-1, d-1)$ . Recall our definition of self-duality in [9] and let  $\tilde{\mathcal{Q}}_{d-1}$  (resp.  $\tilde{\mathcal{H}}_{d-1}$ ) be formed from a self-dual point-line incidence graph of a self-dual symplectic quadrangle (resp. split Cayley hexagon) with parameters  $(d-1, d-1)$ , the existence of which is guaranteed when  $d$  is one more than a power of 2 (resp. 3), by identifying those pairs of vertices which are in self-dual bijection. It is straightforward to check that these graphs satisfy the promised properties.  $\square$

In [9], we somehow combined Propositions 5 and 6 for other lower bound constructions having prescribed girth. This approach is built upon generalised  $n$ -gons, structures which are known not to exist for  $n > 8$  [6]. We refer the reader to [9] for further details.

Our second objective in this section is to introduce a different graph product applicable only to two regular balanced bipartite graphs. We use it to produce two bipartite constructions for  $\chi'_t$ , both of which settle the case of  $t$  even left open in Proposition 5, and the second of which also treats what could be interpreted as an edge version of the degree–diameter problem.

Let  $H_1 = (V_1 = A_1 \cup B_1, E_1)$  and  $H_2 = (V_2 = A_2 \cup B_2, E_2)$  be two balanced bipartite graphs with given vertex orderings, i.e.  $A_1 = (a_1^1, \dots, a_{n_1}^1)$ ,  $B_1 = (b_1^1, \dots, b_{n_1}^1)$ ,  $A_2 = (a_1^2, \dots, a_{n_2}^2)$ ,  $B_2 = (b_1^2, \dots, b_{n_2}^2)$  for some positive integers  $n_1, n_2$ . We define the *balanced bipartite product*  $H_1 \boxtimes H_2$  of  $H_1$  and  $H_2$  as the graph with vertex and edge sets defined as follows:

$$\begin{aligned}
 V_{H_1 \boxtimes H_2} &:= (A_1 \times A_2) \cup (B_1 \times B_2) \text{ and} \\
 E_{H_1 \boxtimes H_2} &:= \{(a_i^1, a^2)(b_i^1, b^2) \mid i \in \{1, \dots, n_1\}, a^2 b^2 \in E_2\} \cup \\
 &\quad \{(a^1, a_j^2)(b^1, b_j^2) \mid a^1 b^1 \in E_1, j \in \{1, \dots, n_2\}\}.
 \end{aligned}$$

See Figure 1 for an example of this product.

Usually the given vertex orderings will be of either of the following types. We say that a labelling  $A = (a_1, \dots, a_n)$ ,  $B = (b_1, \dots, b_n)$  of  $H = (V = A \cup B, E)$  is a *matching ordering* of  $H$  if  $a_i b_i \in E$  for all  $i \in \{1, \dots, n\}$ . We say it is a *comatching ordering* if  $a_i b_i \notin E$  for all  $i \in \{1, \dots, n\}$ . Note by Hall's theorem that every non-empty regular balanced bipartite graph admits a matching ordering, while every non-complete one admits a comatching ordering.

Let us now give some properties of this product relevant to our problem, especially concerning its degree and distance properties. The first of these propositions follow easily from the definition.

**Proposition 7.** *Let  $H_1$  and  $H_2$  be two balanced bipartite graphs that have part sizes  $n_1$  and  $n_2$ , respectively, and are regular of degrees  $d_1$  and  $d_2$ , respectively, for some positive integers  $n_1, n_2, d_1, d_2$ . Suppose  $H_1, H_2$  are given in either matching or comatching ordering. Then  $H_1 \bowtie H_2$  is a regular balanced bipartite graph with parts  $A_{H_1 \bowtie H_2} = A_1 \times A_2$  and  $B_{H_1 \bowtie H_2} = B_1 \times B_2$  each of size  $n_1 n_2$ . If both are in matching ordering, then  $H_1 \bowtie H_2$  has degree  $d_1 + d_2 - 1$ , otherwise it has degree  $d_1 + d_2$ .*

**Proposition 8.** *Let  $H_1 = (V_1 = A_1 \cup B_1, E_1)$  and  $H_2 = (V_2 = A_2 \cup B_2, E_2)$  be two regular balanced bipartite graphs.*

- (i) *Suppose that for every  $a^1, a'^1 \in X_1 \subseteq A_1$  there is a  $t_1$ -path between  $a^1$  and  $a'^1$  in  $H_1$  (for some  $t_1$  even). Suppose that for every  $a^2, a'^2 \in X_2 \subseteq A_2$  there is a  $t_2$ -path between  $a^2$  and  $a'^2$  in  $H_2$  (for some  $t_2$  even). Then for every  $(a^1, a^2), (a'^1, a'^2) \in X_1 \times X_2 \subseteq A_{H_1 \bowtie H_2}$ , there is a  $(t_1 + t_2)$ -path between  $(a^1, a^2)$  and  $(a'^1, a'^2)$  in  $H_1 \bowtie H_2$ .*
- (ii) *Suppose that for every  $a^1, a'^1 \in X_1 \subseteq A_1$  there is a  $t_1$ -path between  $a^1$  and  $a'^1$  in  $H_1$  (for some  $t_1$  even). Suppose that for every  $a^2 \in X_2 \subseteq A_2$  and  $b^2 \in Y_2 \subseteq B_2$  there is a  $t_2$ -path between  $a^2$  and  $b^2$  in  $H_2$  (for some  $t_2$  odd). Then for every  $(a^1, a^2) \in X_1 \times X_2 \subseteq A_{H_1 \bowtie H_2}$  and  $(b^1, b^2) \in Y_1 \times Y_2 \subseteq B_{H_1 \bowtie H_2}$  where  $Y_1 = \{b_i^1 \mid a_i^1 \in X_1\}$ , there is a  $(t_1 + t_2)$ -path between  $(a^1, a^2)$  and  $(b^1, b^2)$  in  $H_1 \bowtie H_2$ .*

*Proof.* We only show part (ii), as the other part is established in the same manner. Let  $(a^1, a^2) \in X_1 \times X_2$  and  $(b^1, b^2) \in Y_1 \times Y_2$ . Using the distance assumption on  $H_1$ , let  $a_{i_0}^1, b_{i_1}^1, a_{i_2}^1, \dots, b_{i_{t_1-1}}^1, a_{i_{t_1}}^1$  be a  $t_1$ -path in  $H_1$  between  $a^1 = a_{i_0}^1$  and  $a_{i_{t_1}}^1$ , where  $i_{t_1}$  is such that  $b^1 = b_{i_{t_1}}^1$ . Using the distance assumption on  $H_2$ , let  $a_{j_0}^2, b_{j_1}^2, a_{j_2}^2, \dots, a_{j_{t_2-1}}^2, b_{j_{t_2}}^2$  be a  $t_2$ -path in  $H_2$  between  $a^2 = a_{j_0}^2$  and  $b^2 = b_{j_{t_2}}^2$ . The following  $(t_1 + t_2)$ -path between  $(a^1, a^2)$  and  $(b^1, b^2)$  in  $H_1 \bowtie H_2$  traverses using one of the coordinates, then the other:

$$(a^1, a^2) = (a_{i_0}^1, a_{j_0}^2)(b_{i_1}^1, b_{j_0}^2)(a_{i_2}^1, a_{j_0}^2) \cdots (b_{i_{t_1-1}}^1, b_{j_0}^2)(a_{i_{t_1}}^1, a_{j_0}^2) \\ (b_{i_{t_1}}^1, b_{j_1}^2)(a_{i_{t_1}}^1, a_{j_2}^2) \cdots (a_{i_{t_1}}^1, b_{j_{t_2-1}}^2)(b_{i_{t_1}}^1, b_{j_{t_2}}^2) = (b^1, b^2). \quad \square$$

We use this product to show that no version of Theorem 2 may hold for  $\chi'_t$ . In combination with the trivial bound  $\chi'_t(G) = O(d^t)$  if  $\Delta(G) \leq d$ , we deduce Proposition 3(ii) from Proposition 5, the following result and the fact that  $\chi'_2(K_{d,d}) = d^2$ .

**Proposition 9.** *Fix  $t \geq 4$  even. For every  $d \geq 2$  with  $d \equiv 0 \pmod{2(t-2)}$ , there exists a  $d$ -regular bipartite graph  $G$  with  $\chi'_t(G) \geq d^t / (et2^{t-1})$ .*

*Proof.* Let  $t_1 = t - 2$  and  $d_1 = (t_1 - 1)d/t_1$ . Let  $G_1 = (V_1, E_1)$  be the construction promised by Proposition 5 for  $d_1$  and  $t_1$ . Since  $G_1$  is bipartite, we

can write  $V_1 = A_1 \cup B_1$  where  $A_1 = \cup\{U^{(i)} \mid i \in \{0, \dots, t_1 - 1\} \text{ even}\}$  and  $B_1 = \cup\{U^{(i)} \mid i \in \{0, \dots, t_1 - 1\} \text{ odd}\}$ . This is a  $d_1$ -regular balanced bipartite graph, and for every  $a_1, a'_1 \in U^{(0)} \subseteq A_1$  there exists a  $t_1$ -path between  $a_1$  and  $a'_1$ . Moreover, it is possible to label  $A_1$  and  $B_1$  so that the first  $|U^{(0)}|$  vertices of  $A_1$  are the ones of  $U^{(0)}$ , and the first  $|U^{(1)}|$  of  $B_1$  are those of  $U^{(1)}$ . We may also ensure that this labelling is in comatching ordering.

Let  $t_2 = 1$  and  $d_2 = d - d_1 = d/t_1$ . Let  $G_2 = (V_2 = A_2 \cup B_2, E_2) = K_{d_2, d_2}$ . This is a  $d_2$ -regular balanced bipartite graph, and for every  $a_2 \in A_2, b_2 \in B_2$ , there exists a  $t_2$ -path between  $a_2$  and  $b_2$ . Trivially any labelling of  $A_2$  and  $B_2$  gives rise to a matching ordering.

Let  $G = G_1 \bowtie G_2$ ,  $X = U^{(0)} \times A_2$  and  $Y = U^{(1)} \times B_2$ . Now  $G$  is a  $d$ -regular bipartite graph by Proposition 7, and by Proposition 8 for every  $(a_1, a_2) \in X$  and  $(b_1, b_2) \in Y$ , there exists a  $(t - 1)$ -path between  $(a_1, a_2)$  and  $(b_1, b_2)$ . Thus the edges of  $G$  that span  $X \times Y$  induce a clique in  $(L(G))^t$ . The number of such edges is (since  $t > 3$ ) at least

$$\left(\frac{d_1}{2}\right)^{t_1} d_2 \left(\frac{d_1}{2} + d_2\right) = \left(1 - \frac{1}{t-2}\right)^{t-2} \frac{(t-1)d^t}{(t-3)^2 2^{t-1}} \geq \frac{d^t}{et 2^{t-1}}. \quad \square$$

Alternatively, Proposition 3(ii) follows from the following result, albeit at the expense of a worse dependency on  $t$  in the multiplicative factor. For  $t \geq 2$ , we can take a  $(t-1)$ -th power of the product operation on the complete bipartite graph to produce a bipartite graph  $G$  of maximum degree  $d$  with  $\Omega(d^t)$  edges such that  $(L(G))^t$  is a clique.

**Proposition 10.** *Fix  $t \geq 2$ . For every  $d \geq 2$  with  $d \equiv 1 \pmod{t-1}$ , there exists a  $d$ -regular bipartite graph  $G = (V, E)$  with  $|E| = d \cdot \left(\frac{d-1}{t-1} + 1\right)^{t-1}$  and  $\chi'_t(G) = |E|$ .*

*Proof.* Let  $d' = (d-1)/(t-1) + 1$  and  $G = \bowtie^{t-1} K_{d', d'}$ , the  $(t-1)$ -th power of  $K_{d', d'}$  under the product  $\bowtie$ , where the factors are always taken in matching ordering. By Proposition 7,  $G$  is a  $d$ -regular bipartite graph and has  $d \cdot d'^{t-1}$  edges. By Proposition 8, there is a path of length at most  $t-1$  between every pair of vertices in the same part if  $t-1$  is even, or in different parts if  $t-1$  is odd. It follows that  $(L(G))^t$  is a clique.  $\square$

### 3 Proofs of Theorems 1 and 2

In this section we prove the main theorems. Before proceeding, let us set notation and make some preliminary remarks.

Let  $G = (V, E)$  be a graph. We will often need to specify the vertices at some fixed distance from a vertex or an edge of  $G$ . Let  $i$  be a non-negative integer. If  $x \in V$ , we write  $A_i = A_i(x)$  for the set of vertices at distance exactly  $i$  from  $x$ . If  $e \in E$ , we write  $A_i = A_i(e)$  for the set of vertices at distance exactly  $i$  from an endpoint of  $e$ . We shall often abuse this notation by writing  $A_{\leq j}$  for  $\cup_{i \leq j} A_i$  and so forth. We will write  $G_i = G[A_i, A_{i+1}]$  to be the bipartite subgraph induced by the sets  $A_i$  and  $A_{i+1}$



In proving the distance- $t$  chromatic number upper bounds in Theorems 1 and 2 using Lemma 4, given  $x \in V$ , we need to consider the number of pairs of distinct vertices in  $A_{<t}$  that are connected by a path of length at most  $t$ . It will suffice to prove that this number is  $O(d^{2t-\varepsilon})$  as  $d \rightarrow \infty$  for some fixed  $\varepsilon > 0$ . In fact, in our enumeration we may restrict our attention to paths of length *exactly*  $t$  whose endpoints are in  $A_t$  and whose vertices do not intersect  $A_{<t}$ . This is because  $|A_{\leq i}| \leq d^i$  for all  $i$  and the number of paths of length exactly  $j$  containing some fixed vertex is at most  $(j+1)d^j$  for all  $j$ .

Similarly, in proving the distance- $t$  chromatic index upper bound in Theorem 1 using Lemma 4, given  $e \in E$ , we need to consider the number of pairs of distinct edges that each have at least one endpoint in  $A_{<t}$  and that are connected by a path of length at most  $t-1$ . It will suffice to prove that this number is  $O(d^{2t-\varepsilon})$  as  $d \rightarrow \infty$  for some fixed  $\varepsilon > 0$ . Similarly as above, in our enumeration we may restrict our attention to paths of length exactly  $t-1$  whose endpoint edges both intersect  $A_{t-1}$  and whose vertices do not intersect  $A_{<t-1}$ .

As mentioned in the introduction, for Theorem 1 we are going to use two intermediate results of [12] concerning the presence of a  $\Theta$ -subgraph, defined to be any subgraph that is a cycle of length at least  $2k$  with a chord.

**Lemma 11** ([12]). *Let  $k \geq 3$ . Any bipartite graph of minimum degree at least  $k$  contains a  $\Theta$ -subgraph.*

**Lemma 12** ([12]). *If  $G = (V, E)$  is  $C_{2k}$ -free, then for  $i \in \{0, \dots, k-1\}$  and  $x \in V$ , neither  $G[A_i, A_{i+1}]$  nor  $G[A_i]$  contains a bipartite  $\Theta$ -subgraph, where  $A_i$  is defined based on  $G$  as above.*

*Proof of Theorem 1(i).* By the probabilistic construction described in [2], it suffices to prove only the upper bound in the statement. We may also assume that  $t \geq 2$ , since it was already observed in [9] that for any  $\ell \geq 3$  the chromatic number of any  $C_\ell$ -free graph of maximum degree  $d$  is  $O(d/\log d)$ .

Let  $\ell = 2k$  for some  $k \geq t+1$ , let  $G = (V, E)$  be a graph of maximum degree at most  $d$  such that  $G$  contains no  $C_\ell$  as a subgraph, and let  $x \in V$ . Let  $T$  denote the number of pairs of distinct vertices in  $A_t$  that are connected by a path of length exactly  $t$  that does not intersect  $A_{<t}$ . As discussed at the beginning of the section, it suffices for the proof to show that  $T \leq Cd^{2t-1}$  where  $C$  is a constant independent of  $d$ , by Lemma 4.

We define  $A'$  to be  $A_{t+1}$  if  $|A_{t+1}| \geq |A_t|$ , or  $A_t$  otherwise, and  $E_H$  to be the set of edges in  $A_t \times A_{t+1}$  whose endpoint in  $A'$  is of degree at least  $\ell$  in  $G_t = G[A_t, A_{t+1}]$ . If  $E_H$  is non-empty, then it induces some bipartite graph  $H = (X_H \cup Y_H, E_H)$  of average degree  $d(H)$ , such that  $X_H \subseteq A'$  and  $Y_H \subseteq (A_t \cup A_{t+1}) \setminus A'$ . It must hold that  $d(H) < \ell$ , or else from  $H$  it would be possible to extract a bipartite graph  $H'$  of minimum degree  $d(H)/2 \geq \ell/2 = k$ , which by Lemma 11 would contain a  $\Theta$ -subgraph. This contradicts Lemma 12 which says  $G_t$  contains no bipartite  $\Theta$ -subgraph. Therefore,

$$\ell > \frac{2|E_H|}{|X_H| + |Y_H|} \geq \frac{2|E_H|}{\frac{|E_H|}{\ell} + |Y_H|}$$

and so  $|E_H| < \ell|Y_H| \leq \ell d^t$ , where the last inequality follows from the definition of  $A'$ .

Moreover, the graph  $G[A_t]$  is of average degree  $d(G[A_t]) < 2\ell$ , for otherwise it would be possible to extract from  $G[A_t]$  a bipartite graph  $H'$  of average degree at least  $\ell$ . From  $H'$  it would then be possible to extract a bipartite graph of minimum degree at least  $\ell/2 = k$ , which contains a  $\Theta$ -subgraph by Lemma 11. This contradicts Lemma 12 which says  $G[A_t]$  contains no bipartite  $\Theta$ -subgraph. If we denote by  $E[A_t]$  the set of edges of  $G[A_t]$ , it means that  $|E[A_t]| < \frac{2\ell|A_t|}{2} \leq \ell d^t$ .

Let us count the possibilities for a path  $x_0 \dots x_t$  of length  $t$  between two distinct vertices  $x_0, x_t \in A_t$  that does not intersect  $A_{<t}$ . We discriminate based on the first edge  $e_0 = x_0x_1$  of this path, which can fall into three different cases.

- (i)  $e_0 \in E_H$ . We count the paths by first drawing  $e_0$  from the at most  $\ell d^t$  possible choices in  $E_H$ , then drawing the remaining  $t - 1$  vertices of the path one at a time, for which there are at most  $d$  choices each. So the number of paths in this case is at most  $\ell d^{2t-1}$ .
- (ii)  $e_0 \in (A_t \times A_{t+1}) \setminus E_H$ . It means that  $x_0$  (resp.  $x_1$ ) is of degree less than  $\ell$  in  $A_{t+1}$  (resp.  $A_t$ ) if  $|A_{t+1}| < |A_t|$  (resp. if  $|A_{t+1}| \geq |A_t|$ ). We count the paths by first drawing  $x_0$  (resp.  $x_t$ ) from the at most  $d^t$  possible choices in  $A_t$ , then drawing the other  $t$  vertices one at a time with  $d$  choices each, except for  $x_1$  (resp.  $x_0$ ) for which there are fewer than  $\ell$  possible choices. The number of paths in this case is therefore at most  $\ell d^{2t-1}$ .
- (iii)  $e_0 \in E[A_t]$ . We count the paths by first drawing  $e_0$  from the at most  $\ell d^t$  possible choices in  $E[A_t]$ , then drawing the remaining  $t - 1$  vertices of the path one at a time, for which there are at most  $d$  choices each. So the number of paths in this case is at most  $\ell d^{2t-1}$ .

Summing over the above cases, the overall number of choices for the path  $x_0 \dots x_t$  is at most  $3\ell d^{2t-1}$ , giving the required bound on  $T$ .  $\square$

*Proof of Theorem 1(ii).* By the probabilistic construction described in [8], it suffices to prove only the upper bound in the statement. To that end, let  $\ell \geq 2t$  be even, let  $G = (V, E)$  be a graph of maximum degree at most  $d$  such that  $G$  contains no  $C_\ell$  as a subgraph, and let  $e \in E$ . Let  $T$  denote the number of pairs of distinct edges in  $G[A_{t-1}]$  or  $G_{t-1} = G[A_{t-1}, A_t]$  that are connected by a path of length  $t - 1$  that does not intersect  $A_{<t-1}$ . As discussed at the beginning of the section, it suffices to show that  $T \leq C d^{2t-1}$  where  $C$  is a constant independent of  $d$ , by Lemma 4.

We define  $A'$  to be  $A_t$  if  $|A_t| \geq |A_{t-1}|$ , or  $A_{t-1}$  otherwise, and  $E_H$  to be the set of edges in  $A_{t-1} \times A_t$  whose endpoint in  $A'$  is of degree at least  $\ell$  in  $G_{t-1}$ . Exactly as in the proof of Theorem 1(i), it follows from Lemmas 11 and 12 that  $|E_H| < \ell d^{t-1}$  and  $|E[A_{t-1}]| < \ell d^{t-1}$ , where  $E[A_{t-1}]$  denotes the set of edges of  $G[A_{t-1}]$ .

Let us count the possibilities for a path  $x_0 \dots x_{t+1}$ , where  $x_1 \dots x_t$  is a path of length  $t - 1$  between two distinct edges  $x_0x_1$  and  $x_tx_{t+1}$  of  $G[A_{t-1}]$  or  $G_{t-1}$

that does not intersect  $A_{<t-1}$ . We discriminate based on the first edge  $e_0 = x_0x_1$  of this path, which can fall into three different cases.

- (i)  $e_0 \in E_H$ . We count the paths by first drawing  $e_0$  from the at most  $\ell d^{t-1}$  possible choices in  $E_H$ , then drawing the remaining  $t$  edges of the path one at a time, for which there are at most  $d$  choices each. So the number of paths in this case is at most  $\ell d^{2t-1}$ .
- (ii)  $e_0 = ab$  where  $a \in A_{t-1}, b \in A_t$ , and  $e_0 \notin E_H$ . It means that  $a$  (resp.  $b$ ) is of degree less than  $\ell$  in  $A_t$  (resp.  $A_{t-1}$ ) if  $|A_t| < |A_{t-1}|$  (resp. if  $|A_t| \geq |A_{t-1}|$ ). There are now three different possible subcases.
  - (a)  $b = x_1$ . We count the paths by first drawing  $x_0$  (resp.  $x_t$  if it is in  $A_{t-1}$  or  $x_{t-1} \in A_{t-1}$  otherwise) from the at most  $d^{t-1}$  possible choices in  $A_{t-1}$ , then drawing the other  $t+1$  vertices one at a time with  $d$  choices each, except for  $x_1$  (resp.  $x_0$ ) for which there are fewer than  $\ell$  possible choices. The number of paths in this subcase is therefore at most  $\ell d^{2t-1}$  (resp.  $2\ell d^{2t-1}$ ).
  - (b)  $a = x_1$  and  $x_2 \in A_{t-1}$ . We count the paths by first drawing  $e_1 = x_1x_2$  from the at most  $\ell d^{t-1}$  possible choices in  $E[A_{t-1}]$ , then drawing the other  $t$  edges one at a time with  $d$  choices each. The number of paths in this subcase is therefore at most  $\ell d^{2t-1}$ .
  - (c)  $a = x_1$  and  $x_2 \in A_t$ . We count the paths by first drawing  $x_t$  if it is in  $A_{t-1}$  or  $x_{t-1} \in A_{t-1}$  otherwise (resp.  $x_0$ ) from the at most  $d^{t-1}$  possible choices in  $A_{t-1}$ , then drawing the other  $t+1$  vertices one at a time with  $d$  choices each, except for  $x_0$  (resp.  $x_1$ ) for which there are fewer than  $2\ell$  possible choices. The number of paths in this subcase is therefore at most  $2\ell d^{2t-1}$  (resp.  $\ell d^{2t-1}$ ).
- (iii)  $e_0 \in E[A_{t-1}]$ . We count the paths by first drawing  $e_0$  from the at most  $\ell d^{t-1}$  possible choices in  $E[A_{t-1}]$ , then drawing the remaining  $t$  edges of the path one at a time, for which there are at most  $d$  choices each. So the number of paths in this case is at most  $\ell d^{2t-1}$ .

Summing over the above cases, the overall number of choices for the path  $x_0 \dots x_t$  is at most  $6\ell d^{2t-1}$ , giving the required bound on  $T$ .  $\square$

In the proof of Theorem 2, we use the following lemma, which bounds the number of vertices at distance at most  $t$  from some fixed vertex when we impose intersection conditions on certain paths. The proof of this lemma illustrates the two main methods we use to bound the local density as needed for Lemma 4.

**Lemma 13.** *Let  $G = (V, E)$  be a graph of maximum degree at most  $d$  and let  $x_0 \in V$ .*

- (i) *Let  $S$  be a set of vertices at distance exactly  $t$  from  $x_0$  such that any two paths of length  $t$  from  $x_0$  to distinct elements of  $S$  must intersect in at least one vertex other than  $x_0$ . Then  $|S| \leq d^{t-1}$ .*

- (ii) Let  $P$  be a path of length  $k > 0$  starting at  $x_0$ . Let  $S$  be a set of vertices at distance at most  $t$  from  $x_0$  such that for every  $s \in S$  there is a path of length at most  $t$  from  $x_0$  to  $s$  that intersects with  $P$  in at least one vertex other than  $x_0$ . Then  $|S| \leq kd^{t-1}$ .

*Proof of Lemma 13(i).* Suppose  $V$  is given with some ordering. As before, for each  $i > 0$  let  $A_i = A_i(x_0)$  denote the set of vertices at distance exactly  $i$  from  $x_0$  in  $G$ . We inductively construct a breadth-first search tree  $T = T_t$  as follows.

- $T_0$  consists only of the root  $x_0$ .
- If  $i > 0$ , then for every  $y \in A_i$  let  $a_y$  be the vertex in  $N(y) \cap A_{i-1}$  whose path from  $x_0$  in  $T_{i-1}$  is least in lexicographical order. Then  $T_i$  is obtained from  $T_{i-1}$  by adding each edge  $ya_y$ ,  $y \in A_i$ .

By assumption  $S \subseteq A_t$ . Let  $x_t$  be the vertex in  $S$  whose path in  $T$  from  $x_0$  is least in lexicographical order, and let  $P_x = x_0 \dots x_t$  be that path.

Let  $y_t \in S$  be distinct from  $x_t$  and moreover suppose for a contradiction that the lowest common ancestor of  $x_t$  and  $y_t$  in  $T$  is  $x_0$ . Then  $y_t$  is at distance at least  $t$  from  $x_1$ , or else it would have had  $x_1$  as an ancestor by the definition of  $T$  and the choice of  $P_x$ . Letting  $P_y = y_0 \dots y_t$  (where  $y_0 = x_0$ ) be the path from  $x_0$  to  $y_t$  in  $T$ , by assumption  $P_x$  and  $P_y$  must have a common vertex other than  $x_0$ . So there are  $i, j > 0$  such that  $x_i = y_j$ . It must be that  $j < i$ , for otherwise  $x_1 \dots x_i y_{j+1} \dots y_t$  would be a path of length  $i - 1 + t - j \leq t - 1$  between  $x_1$  and  $y_t$ , a contradiction. This means though that  $x_i \in A_i$  is at distance at most  $j < i$  from  $x_0$ , also a contradiction. We have shown that  $S$  is contained in the subtree of  $T$  rooted at  $x_1$ , which then implies that  $|S| \leq d^{t-1}$ .  $\square$

*Proof of Lemma 13(ii).* To each vertex in  $S$ , there is a path of length at most  $t - 1$  from some vertex of  $P$  other than  $x_0$ . There are at most  $d^{t-1}$  vertices within distance  $t - 1$  of a fixed vertex of  $P$ , so summing over all possible choices of such a vertex, this gives  $|S| \leq kd^{t-1}$ .  $\square$

*Proof of Theorem 2.* By the probabilistic construction described in [2], it suffices to prove only the upper bound in the statement. Moreover, we may assume  $t \geq 3$  due to Johansson's result [7] and our observation in [9].

Let  $\ell \geq 3t$  be odd, let  $G = (V, E)$  be a graph of maximum degree at most  $d$  such that  $G$  contains no  $C_\ell$  as a subgraph, and let  $x \in V$ . For convenience, let us call any path contained in  $A_{\geq t}$  *peripheral*. Let  $T$  denote the number of pairs of distinct vertices in  $A_t$  that are connected by a peripheral path of length  $t$  and are not connected by any path of length less than  $t$ . As discussed at the beginning of the section, it suffices for the proof to show that  $T \leq Cd^{2t-1}$  where  $C$  is a constant independent of  $d$ , by Lemma 4.

We specify a unique breadth-first search tree  $\text{BFS} = \text{BFS}(x)$  of  $G$ , rooted at  $x$ . Having fixed an ordering of  $V$ ,  $\text{BFS}$  is a graph on  $V$  whose edges are defined as follows. For every  $v \in A_i$ ,  $i > 0$ , we include the edge to the neighbour of  $v$  in  $A_{i-1}$  that is least in the vertex ordering.

Since  $\ell$  and  $t$  are odd, we know that  $\ell = 3t + 2k$  for some non-negative integer  $k$ . For  $j \in \{0, 1, \dots, 2k\}$ , let us call a vertex  $v \in A_t$  *j-implantable* if it is

the endpoint of some peripheral path of length  $j$ , the other endpoint of which is in  $A_t$ . In particular, any vertex of  $A_t$  is 0-implantable.

We first show that the number of pairs of vertices connected by a peripheral path of length  $t$  which has a  $2k$ -implantable endpoint is  $O(d^{2t-1})$ . Fix  $v$  to be a  $2k$ -implantable vertex and  $P = v_0v_1 \dots v_{2k}$  a path certifying its implantability, so that  $v_0 = v$  and (if  $k > 0$ )  $v_{2k} \in A_t \setminus \{v\}$ . By Lemma 13(ii) applied to  $G[A_{\geq t}]$  and  $P$ , the number of vertices connected by a peripheral path of length  $t$  starting at  $v$  which intersects  $P$  at another vertex is at most  $2kd^{t-1}$ . Now consider the set  $Y \subseteq A_t \setminus \{v\}$  such that there is a peripheral path of length  $t$  between  $v$  and  $y$  that does not intersect  $P$  except at  $v$  for all  $y \in Y$ . If  $a_Y$  is the ancestor of  $v_{2k}$  in BFS at layer  $A_1$ , then  $Y$  is contained in the subtree rooted at  $a_Y$ . Otherwise, there would be some  $y_1 \in Y$  such that its lowest common ancestor with  $v_{2k}$  in BFS is  $x$ , which gives rise to a cycle of length  $3t + 2k$  that contains  $x, v_{2k}, v, y_1$ , in that order, a contradiction. Thus  $|Y| \leq d^{t-1}$ , the number of pairs with  $v$  that are counted by  $T$  is at most  $(1 + 2k)d^{t-1}$ , and the number of pairs with a  $2k$ -implantable vertex that are counted by  $T$  is at most  $(1 + 2k)d^{2t-1}$ .

Observe that we are already done if  $k = 0$  since every vertex in  $A_t$  is 0-implantable by definition, so assume from here on that  $k > 0$ . It remains for us to (crudely) count the number of pairs  $(z_0, z_t) \in A_t^2$  of non- $2k$ -implantable vertices that are connected by a peripheral path  $z_0 \dots z_t$  of length  $t$  and are not connected by any shorter path.

First suppose  $k \leq t$ . Trivially the number of choices for  $z_0$  is at most  $d^t$  and the number of choices for the sub-path  $z_0 \dots z_{t-k}$  is  $d^{t-k}$ . Given  $z_{t-k}$ , the choice for the remainder sub-path  $z_{t-k} \dots z_t$  is restricted by the fact that  $z_t$  is not  $2k$ -implantable; in particular, all such sub-paths must intersect at a vertex other than  $z_{t-k}$ . By Lemma 13(i) applied to  $G[A_{\geq t}]$  and  $z_{t-k}$ , for a fixed choice of  $z_{t-k}$ , the number of possibilities for  $z_t$  is at most  $d^{k-1}$ , and so the number of pairs  $(z_0, z_t)$  in this case is at most  $d^t \cdot d^{t-k} \cdot d^{k-1} = d^{2t-1}$ .

Next suppose  $k > t$ . Then we discriminate based on the smallest possible value  $j \equiv 2k \pmod{t}$  such that  $z_0, z_t$  are both not  $j$ -implantable. Note that since we are in the case where  $z_0, z_t$  are not  $2k$ -implantable,  $j \leq 2k$ . More formally, we let  $\kappa_0 = t$  if  $k \bmod t = 0$ , or  $\kappa_0 = k \bmod t$  otherwise. Let  $j = \min\{2\kappa_0 + it \mid 0 \leq i \leq 2(k - \kappa_0)/t \text{ and } z_0, z_t \text{ are not } j\text{-implantable}\}$ . If  $j = 2\kappa_0 \leq 2t$ , then we can treat this just like the previous case, which means there are at most  $d^{2t-1}$  choices for the pair  $(z_0, z_t)$ .

So suppose that  $2\kappa_0 < j \leq 2k$ . By the definition of  $j$ , without loss of generality  $z_0$  is  $(j - t)$ -implantable, and  $z_0, z_t$  are not  $j$ -implantable. We fix  $z_0$  and let  $P$  be a path of length  $j - t$  certifying its  $(j - t)$ -implantability. First note that Lemma 13(ii) applied to  $G[A_{\geq t}]$  and  $P$  states that there are at most  $(j - t)d^{t-1}$  choices for those  $z_t$  such that there is a peripheral path of length  $t$  between  $z_0$  and  $z_t$  that intersects  $P$  in some vertex other than  $z_0$ . So consider the set  $Y \subseteq A_t \setminus \{z_0\}$  such that  $y$  is connected to  $z_0$  by a peripheral path  $P_y$  of length  $t$  that intersects  $P$  only in  $z_0$  for all  $y \in Y$ . Then every vertex  $y \in Y$  is  $j$ -implantable as certified by the path  $P$  concatenated with  $P_y$ . This means that no choice for  $z_t$  in  $Y$  is possible, and so the number of pairs  $(z_0, z_t)$  in this setting is at most  $(j - t)d^{2t-1}$ .

Summing over all possible  $j$ , the number of choices for  $(z_0, z_t)$  is at most  $\left(1 + \sum_{i=1}^{2(k-\kappa_0)/t} (2\kappa_0 + it - t)\right) d^{2t-1} = (2(k^2 - \kappa_0^2)/t)d^{2t-1}$  if  $k > t$ .

It therefore follows that  $T \leq (1 + 2k + 2(k^2 - \kappa_0^2)/t) d^{2t-1}$ , as required.  $\square$

Our impression is that it might be possible to improve upon the value  $3t$  in Theorem 2; however, in order to do so, it seems one would have to take a different approach. This is because of a simple construction of a  $d$ -regular graph  $G$  with no odd cycle of length less than  $3t$  such that  $G^t$  does not satisfy the density conditions demanded by Lemma 4. Roughly, we take the main example of Proposition 5 but around a cycle of length  $3t$  rather than of length  $t$ . More precisely, the vertex set is  $\cup_{i=0}^{3t-1} U^{(i)}$  where each  $U^{(i)}$  is a copy of  $[d/2]^t$ . For all  $i \in \{0, \dots, 3t-1\}$ , we join an element  $(x_0^{(i)}, \dots, x_{t-1}^{(i)})$  of  $U^{(i)}$  and an element  $(x_0^{(i+1 \bmod 3t)}, \dots, x_{t-1}^{(i+1 \bmod 3t)})$  of  $U^{(i+1 \bmod 3t)}$  by an edge if the  $t$ -tuples agree on all symbols except possibly at coordinate  $i \bmod t$ . It is straightforward to check that  $G^t$  is a graph in which all vertices have degree  $\Theta(d^t)$  and every neighbourhood is spanned by  $\Theta(d^{2t})$  edges, meaning that Lemma 4 is ineffective here. But neither is  $G$  an example to certify sharpness of the value  $3t$  in Theorem 2, since it is also straightforward to check that  $\chi_t(G) = o(d^t)$ .

## 4 Concluding remarks and open problems

Our goal was to address the question, what is the asymptotically largest value of  $\chi_t(G)$  or of  $\chi'_t(G)$  among graphs  $G$  with maximum degree at most  $d$  containing no cycle of length  $\ell$ , where  $d \rightarrow \infty$ ? The case  $t = 1$  for both parameters and the case  $t = 2$  for  $\chi'_t$  followed from earlier work, but we showed more generally that for each fixed  $t$  this question for both parameters can be settled apart from a finite number of cases of  $\ell$ . These exceptional cases are a source of mystery. We would be very interested to learn if the cycle length constraints  $2t$ ,  $2t + 2$  and  $3t$  in Theorems 1 and 2 could be weakened (or not).

More specifically, writing  $\chi_t(d, \ell) = \sup\{\chi_t(G) \mid \Delta(G) \leq d, G \not\supseteq C_\ell\}$  and  $\chi'_t(d, \ell) = \sup\{\chi'_t(G) \mid \Delta(G) \leq d, G \not\supseteq C_\ell\}$ , the following questions are natural, even if there is no manifest monotonicity in  $\ell$ .

- (i) For each  $t \geq 1$ , is there a critical even  $\ell_t^e$  such that for any even  $\ell$ , if  $\ell < \ell_t^e$  then  $\chi_t(d, \ell) = \Theta(d^t)$ , while if  $\ell \geq \ell_t^e$  then  $\chi_t(d, \ell) = \Theta(d^t / \log d)$ ?
- (ii) For each  $t \geq 2$ , is there a critical even  $\ell'_t$  such that for any even  $\ell$ , if  $\ell < \ell'_t$  then  $\chi'_t(d, \ell) = \Theta(d^t)$ , while if  $\ell \geq \ell'_t$  then  $\chi'_t(d, \ell) = \Theta(d^t / \log d)$ ?
- (iii) For each  $t \geq 1$  odd, is there a critical odd  $\ell_t^o$  such that for any odd  $\ell$ , if  $\ell < \ell_t^o$  then  $\chi_t(d, \ell) = \Theta(d^t)$ , while if  $\ell \geq \ell_t^o$  then  $\chi_t(d, \ell) = \Theta(d^t / \log d)$ ?

We knew from before that  $\ell_1^e = 4$ ,  $\ell_1^o = 3$ ,  $\ell_2^e = 6$ ,  $\ell_2^o = 4$ ,  $\ell_3^e = 6$ ,  $\ell_3^o = 8$ , and  $\ell_6^o = 12$ . In this paper, we showed that there are linear in  $t$  upper bounds on all these critical values, provided the values are well-defined.

The above three questions are natural analogues to open questions of Alon and Mohar [2] and of Kaiser and the first author [8] that ask for a critical

girth  $g_t$  (resp.  $g'_t$ ) for which there is an analogous decrease in the asymptotic extremal behaviour of the distance- $t$  chromatic number (resp. index). If these critical values all exist, it would be natural to think that  $g_t = \min\{\ell_t^e, \ell_t^o\}$  and  $g'_t = \ell'_t$ , and moreover, if  $t$  is odd, that  $|\ell_t^o - \ell_t^e| = 1$ . But there is limited evidence for the existence questions, let alone this stronger set of assertions. We have already established other lower bounds for these hypothetical critical values in [9], but for none of these critical values is there any general construction known to certify a lower bound that is unbounded as  $t \rightarrow \infty$ .

As mentioned in the introduction, Vu [14] proved that the exclusion of any fixed bipartite graph is sufficient for a  $O(d^2/\log d)$  upper bound on the strong chromatic index of graphs of maximum degree  $d$ . One might wonder, similarly, for each  $t \geq 2$  is there a natural wider class of graphs than sufficiently large cycles (of appropriate parity) whose exclusion leads to asymptotically non-trivial upper bounds on the distance- $t$  chromatic number or index?

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