M. Sghiar. International Journal of Engineering and Advanced Technology (IJEAT) ISNN:2249-8958, Volume- 7 Issue-2, December 2017

THE MERTENS FUNCTION AND THE PROOF OF THE RIEMANN'S HYPOTHESIS

Version of March 28, 2018

Deposited in the hal : <hal-01667383> version 4

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Abstract : I will prove that $M(n) = \mathcal{O}(n^{\frac{1}{2}+\epsilon})$ where M is the Mertens function, and I deduce a new proof of the Riemann's hypothesis.

Keywords : Prime Number, number theory , distribution of prime numbers, the law of prime numbers, the Riemann hypothesis, the Riemann zeta function, the Mertens function.

I- INTRODUCTION, RECALL, NOTATIONS AND DEFINITIONS

The Riemann's function ζ (see [2]) is a complex analytic function that has appeared essentially in the theory of prime numbers. The position of its complex zeros is related to the distribution of prime numbers and is at the crossroads of many other theories.

The Riemannn's hypothesis (see [5] and [6]) conjectured that all nontrivial zeros of ζ are in the line $x = \frac{1}{2}$.

The Mobius function generally designates a particular multiplicative function, defined on the strictly positive integers and with values in the set $\{-1, 0, 1\}$:

$$\mu(n) = \begin{cases} 0 \text{ if n has at least one repeated factor} \\ 1 \text{ if n=1} \\ (-1)^k \text{ if n is the product of k distinct prime factors} \end{cases}$$

In number theory, the function of Mertens is defined by $M(n) = \sum_{1 \le k \le n} \mu(k)$ and it has been falsely conjectured by Mertens that the absolute value of $M(n)$ is always less than $\sqrt{n}$, and that if we can prove that the absolute value of $M(n)$ is always less than $\sqrt{n}$, so Riemann's hypothesis is true.

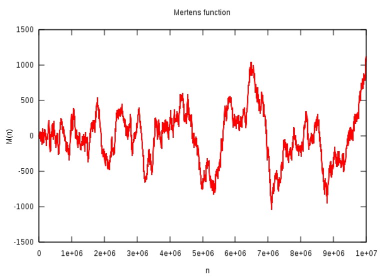


FIGURE 1 – Mertens function to n=10,000,000

This conjecture is passed for real for a long time. But in 1984, Andrew Odlyzko and Hermante Riele [1] show that there is a number greater than 1030 which invalidates it (See also [3] and [4]) However it has been shown that the Riemann's hypothesis [2] is equivalent to the following conjecture :
Conjecture : $M(n) = \mathcal{O}(n^{\frac{1}{2}+\epsilon})$.
The purpose of this paper is to prove this conjecture and to deduce a new proof of Riemann's hypothesis.
Recall also that in [7] it has been shown that prime numbers are defined by a function Φ , follow a law in their appearance and their distribution is not a coincidence.

II- THE PROOF OF THE CONJECTURE :

Proposition 1 : $M(n) = \mathcal{O}(n^{\frac{1}{2}+\epsilon})$
Lemma 1 : Let $\zeta_N(s) = \sum_{n=N}^{\infty} \frac{1}{n^s}$. If $0 < s < 1$ then : $\zeta_N(s) = o_N(1)$
Note : $\zeta_N(s)$ is defined by extension, and this means that the action of the force ζ_N on the particle s becomes weaker when N becomes large.

Proof of the lemma :

As a reminder, the ζ function of Riemann is a meromorphic and complex complex analytic function defined for s such that $Re(s) > 1$ by the Dirichlet series :$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

Since the Dirichlet eta function can be defined by $\eta(s) = (1 - 2^{1-s})\zeta(s)$ where : $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$ with $0 < Re(s)$ we have in particular :

$$\zeta(z) = \frac{1}{1 - 2^{1-z}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$$

for $z \neq 1$ and $0 < Re(z)$.

This extends the ζ function to the set $\{z \in \mathbb{C}\backslash\{1\} \text{tel que } 0 < Re(z)\}$.

Let : $\eta_N(s) = \sum_{n=N}^{\infty} \frac{(-1)^{n-1}}{n^s}$ with $0 < Re(s)$

And let : $\zeta_N(s) = \sum_{n=N}^{\infty} \frac{1}{n^s}$

First, by the uniqueness of the extensions of the analytic functions, $\zeta_N(s)$ is prolonged in $\{z \in \mathbb{C}\backslash\{1\} \text{such taht } 0 < Re(z)\}$ by the function $\zeta_N(s) = \zeta(s) - \sum_{n=1}^{N-1} \frac{1}{n^s}$

And as we have $\zeta_{2N}(s) = \eta_{2N}(s) + 2\sum_{n=N}^{\infty} \frac{1}{(2n)^s} = \eta_{2N}(s) + \frac{2}{2^s}\zeta_N(s)$

It follows that : $\zeta_{2N}(s) - \frac{1}{2^{s-1}}\zeta_N(s) = \eta_{2N}(s)$ is defined for any s such that $Re(s) > 0$.

So $\zeta_{2N}(s) - \frac{1}{2^{s-1}}\zeta_N(s) = o_N(1)$ because $\eta_{2N}(s) = o_N(1)$.

If s is such that $0 < s < 1$:

And if $\lim_{N \to +\infty} \zeta_{2N}(s) = -\infty$ then as $\zeta_N(s) = \sum_{i=N}^{\infty}(\zeta_i(s) - \zeta_{2i}(s))$, we can assume that $\zeta_N(s) < \zeta_{2N}(s)$ - because there is an infinity of such s -, so we will have $\zeta_{2N}(s) - \zeta_N(s) = o_N(1)$ because $|\zeta_{2N}(s) - \zeta_N(s)| < \left|\zeta_{2N}(s) - \frac{1}{2^{s-1}}\zeta_N(s)\right|$.

And consequently : $\zeta_N(s) - \frac{1}{2^{s-1}}\zeta_N(s) = \zeta_N(s) - \zeta_{2N}(s) + \zeta_{2N}(s) - \frac{1}{2^{s-1}}\zeta_N(s) = o_N(1)$. So $\zeta_N(s) = o_N(1)$. which is absurd.

It results that we have $\lim_{N \to +\infty} \zeta_{2N}(s) = \lim_{N \to +\infty} \zeta_N(s) = 0$

Hence the lemma is proved.

Proof of the proposition : Let ϵ such that $0 < \epsilon < \frac{1}{2}$. Note that we have one of the possibilities :

i- $M(n+1) = M(n)$

ii- $M(n+1) = M(n) + 1$

iii- $M(n+1) = M(n) - 1$

In any case we have :

$$\left|\frac{M(n+1)}{(n+1)^{\frac{1}{2}+\epsilon}}\right| \leq \left|\frac{M(n)}{n^{\frac{1}{2}+\epsilon}}\right| + \left|\frac{1}{(n+1)^{\frac{1}{2}+\epsilon}}\right|$$

It follows that :

$$\left|\frac{M(n+1)}{(n+1)^{\frac{1}{2}+\epsilon}}\right| \leq \left|\sum_{k=1}^{n+1}\frac{1}{k^{\frac{1}{2}+\epsilon}}\right| = \left|\sum_{k=1}^{+\infty}\frac{1}{k^{\frac{1}{2}+\epsilon}} - \sum_{k=n+2}^{+\infty}\frac{1}{k^{\frac{1}{2}+\epsilon}}\right|$$

And hence for n large enough and by using the **lemma** 1 we have :

$$\left|\frac{M(n+1)}{(n+1)^{\frac{1}{2}+\epsilon}}\right| \leq \left|\zeta(\frac{1}{2}+\epsilon)\right| + o(1)$$

And the result is deduced.

Lemma 2 :

$$\frac{1}{\zeta(z)} = z\int_{1}^{+\infty}\frac{M(x)}{x^{z+1}}dx$$

For z such taht $Re(z) > 1$

Proof : We have :

$$M(n) = \sum_{1\leq k\leq n}\mu(k)$$

where $\mu(k)$ is the Mobius function.

We have the link between the inverse of the Riemann zeta function, the

Möbius function and the Euler product :

$$\frac{1}{\zeta(z)} = \prod_{p_k}^{+\infty}(1 - \frac{1}{p_k^z}) = \sum_{n=1}^{+\infty}\frac{\mu(n)}{n^z}$$

where $Re(z) > 1$, p_k is the k-th prime number.

Mertens function, $M(x)$ is closely linked with the positions of zeros of the Riemann zeta-function $\zeta(z)$. If we define $M(0) = 0$, we get this relation :

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{+\infty}\frac{\mu(n)}{n^z} = \sum_{n=1}^{+\infty}\frac{M(n) - M(n-1)}{n^z}$$

$$= \sum_{n=1}^{+\infty}\frac{M(n)}{n^z} - \sum_{n=1}^{+\infty}\frac{M(n-1)}{n^z} = \sum_{n=1}^{+\infty}\frac{M(n)}{n^z} - \sum_{n=1}^{+\infty}\frac{M(n)}{(n+1)^z}$$

$$= \sum_{n=1}^{+\infty}M(n)(\frac{1}{n^z} - \frac{1}{(n+1)^z}) = \sum_{n=1}^{+\infty}M(n)\int_n^{n+1}\frac{z}{x^{z+1}}dx$$

$$= z\sum_{n=1}^{+\infty}\int_n^{n+1}\frac{M(x)}{x^{z+1}}dx = z\int_1^{+\infty}\frac{M(x)}{x^{z+1}}dx$$

Since $M(x)$ is constant on each interval [n, n+1[.

So :

$$\frac{1}{\zeta(z)} = z\int_1^{+\infty}\frac{M(x)}{x^{z+1}}dx$$

Corollary 1 (The Riemann hypothesis) : All non-trivial zeros of ζ are in the line $x = \frac{1}{2}$.

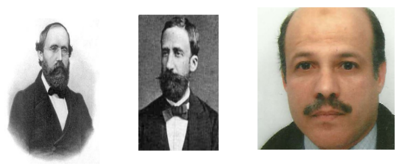
Proof : By posing $M(x) = \sum_{1\le k\le x}\mu(k)$, and by **lemma 2**, we can write :

$$\frac{1}{\zeta(z)} = z\int_1^{+\infty}\frac{M(x)}{x^{z+1}}dx$$

The Proposition 1 shows that this integral converges for $Re(z) > \frac{1}{2}$, implying that $\frac{1}{\zeta(z)}$ is defined for $Re(z) > \frac{1}{2}$. According to this result, it can define a function analytic in $Re(z) > 1/2$ and extend an analytic continuation of $\frac{1}{\zeta(z)}$ from $Re(z) > 1$ to $Re(z) > 1/2$ and by symmetry for $Re(z) < \frac{1}{2}$. Thus, the only non-trivial zeros of ζ satisfy $Re(z) = \frac{1}{2}$, which is the statement of the Riemann's hypothesis.

III- Conclusion :

To prove the Riemann's hypothesis we needed to prove and use a property of the Mertens function (Proposition 1). And curiously and inversely for the proof of this last proposition we have used properties of the Riemann function, which shows the close link between the Mertens function and the Riemann function, as shown by the equations above.



Riemann- Mertens- Sghiar

IV-Acknowledgement :

I thank Professor Fausto Galetto for his reading and his fruitful advice.

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