

Computing effectively stabilizing controllers for a class of nD systems

Yacine Bouzidi, Thomas Cluzeau, Guillaume Moroz, Alban Quadrat

► **To cite this version:**

Yacine Bouzidi, Thomas Cluzeau, Guillaume Moroz, Alban Quadrat. Computing effectively stabilizing controllers for a class of nD systems. The 20th World Congress of the International Federation of Automatic Control, Jul 2017, Toulouse, France. 50 (1), pp.1847 - 1852, 2017, <10.1016/j.ifacol.2017.08.200>. <hal-01667161>

HAL Id: hal-01667161

<https://hal.archives-ouvertes.fr/hal-01667161>

Submitted on 19 Dec 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Computing effectively stabilizing controllers for a class of n D systems

Yacine Bouzidi* Thomas Cluzeau** Guillaume Moroz***
Alban Quadrat*

* INRIA Lille-Nord Europe, Lille, France
(e-mail: {yacine.bouzidi,alban.quadrat}@inria.fr).

** University of Limoges ; CNRS ; XLIM UMR 7252, Limoges, France
(e-mail: thomas.cluzeau@unilim.fr).

*** INRIA Nancy-Grand Est, Nancy, France
(e-mail: guillaume.moroz@inria.fr)

Abstract:

In this paper, we study the internal stabilizability and internal stabilization problems for multidimensional (n D) systems. Within the fractional representation approach, a multidimensional system can be studied by means of matrices with entries in the integral domain of structurally stable rational fractions, namely the ring of rational functions which have no poles in the closed unit polydisc $\overline{\mathbb{U}}^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1| \leq 1, \dots, |z_n| \leq 1\}$. It is known that the internal stabilizability of a multidimensional system can be investigated by studying a certain polynomial ideal $I = \langle p_1, \dots, p_r \rangle$ that can be explicitly described in terms of the transfer matrix of the plant. More precisely the system is stabilizable if and only if $V(I) = \{z \in \mathbb{C}^n \mid p_1(z) = \dots = p_r(z) = 0\} \cap \overline{\mathbb{U}}^n = \emptyset$. In the present article, we consider the specific class of linear n D systems (which includes the class of 2D systems) for which the ideal I is zero-dimensional, i.e., the p_i 's have only a finite number of common complex zeros. We propose effective symbolic-numeric algorithms for testing if $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$, as well as for computing, if it exists, a stable polynomial $p \in I$ which allows the effective computation of a stabilizing controller. We illustrate our algorithms through an example and finally provide running times of prototype implementations for 2D and 3D systems.

Keywords: n D systems, stability, stabilization, polynomial ideals, symbolic-numeric methods.

1. INTRODUCTION

Multidimensional or n D systems (Bose (1984)) are systems of functional equations whose unknown functions depend on n independent variables. The stabilizability and stabilization problems are fundamental issues in the study of multidimensional systems in control theory. Nowadays, the problem is well-understood in the case of 1D systems whereas progress for n D systems with $n \geq 2$ are rather slow. One approach for handling stabilizability or stabilization issues in systems theory is the *fractional representation approach* (Vidyasagar (2011)) in which a plant is represented by its transfer matrix $P \in K^{q \times r}$ where $K = \mathbb{R}(z_1, \dots, z_n)$. This transfer matrix admits a left factorization $P = D^{-1}N$ (also called fractional representation of P), where the matrices $D \in A^{q \times q}$ satisfying $\det(D) \neq 0$ and $N \in A^{q \times r}$ have entries in the integral domain $A = \mathbb{R}(z_1, \dots, z_n)_S$ of structurally stable rational fractions, namely the ring of rational functions in z_1, \dots, z_n which have no poles in the closed unit polydisc of \mathbb{C}^n defined by:

$$\overline{\mathbb{U}}^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1| \leq 1, \dots, |z_n| \leq 1\}.$$

Introducing the matrix $R = \begin{pmatrix} D & -N \end{pmatrix} \in A^{q \times (q+r)}$, it is known (see Quadrat (2003b,a)) that the multidimensional system given by the transfer matrix $P = D^{-1}N$ is then internally stabilizable if and only if the A -module $A^{1 \times (q+r)} / A^{1 \times q} R$ is a projective A -module of rank r , where the closure $\overline{A^{1 \times q} R}$ of $A^{1 \times q} R$ in $A^{1 \times (q+r)}$ is defined by: $\overline{A^{1 \times q} R} = \{\lambda \in A^{1 \times (q+r)} \mid \exists a \in A \setminus \{0\} : a \lambda \in A^{1 \times q} R\}$. This projectivity condition is in turn equivalent to the fact that the reduced minors of the matrix R do not have common zeros in $\overline{\mathbb{U}}^n$ (see also Lin (1998)). In other terms, if we denote by p_1, \dots, p_r the *reduced minors* of R , i.e., the $q \times q$ minors of R divided by their gcd, by $I = \langle p_1, \dots, p_r \rangle$ the polynomial ideal generated by the p_i 's, and by $V(I) = \{z \in \mathbb{C}^n \mid p_1(z) = \dots = p_r(z) = 0\}$ the associated algebraic variety, then the system is internally stabilizable if and only if $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$.

The first contribution of the present paper is to provide an effective algorithm for testing the stabilizability condition $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$ for the class of n D systems for which the ideal I is zero-dimensional, i.e., the p_i 's have only a finite number of common complex zeros (i.e., $V(I)$ consists of a finite number of complex points). Note that this class includes the class of 2D systems. Our main idea is to take advantage of the univariate representation

* All but the third author were supported by the ANR-13-BS03-0005 (MS-DOS).

for zero-dimensional ideals (Canny (1988); Becker and Wörmann (1996); Alonso et al. (1996); Rouillier (1999)). This concept, which can be traced back to Kronecker (1882), yields a one-to-one correspondance between the elements of $V(I)$ and the zeros of a univariate polynomial f . Numerical techniques can thus be applied to compute certified numerical approximations of the roots of f and then of those of $V(I)$.

In the case of a stabilizable plant, the next step consists in computing a stabilizing controller which can be achieved by computing a stable (i.e., devoid from zeros in $\bar{\mathbb{U}}^n$) polynomial $s \in I$ (see Lin (1988)). The *polydisc Nullstellensatz*, proved by Bridges, Mines, Richman and Schuster (see Bridges et al. (2004)), shows that the existence of a stable polynomial $s \in I$ is equivalent to $V(I) \cap \bar{\mathbb{U}}^n = \emptyset$. Several proofs of this result have been investigated in the literature, mainly for the case where I is a zero-dimensional ideal (see Raman and Liu (1986); Lin (1988); Bisiacco et al. (1986); Guiver and Bose (1995) and Xu et al. (1994) for instance). Nevertheless none of them is effective in the sense that it provides an algorithm for computing s using calculations that can be performed in an exact way by a computer. Indeed, starting from a set of polynomials with rational coefficients ($I \subset \mathbb{Q}[z_1, \dots, z_n]$), these algorithms are built on *spectral factorization*, i.e., factorization of polynomials in $\mathbb{Q}[z]$ into stable and instable factors. For irreducible polynomials in $\mathbb{Q}[z]$, this factorization requires the explicit computation of the complex roots of the polynomials, which can be done only approximately. This leads to approximate (stable) polynomials that do not belong to the polynomial ideal. As a consequence, these algorithms are able to solve the aforementioned problem only for few simple systems (see Sections 4 and 5 for details).

Our second contribution is to provide an effective algorithm for computing a stable polynomial $s = \sum_{i=1}^r u_i p_i \in I$ for the class of systems for which $I \subset \mathbb{Q}[z_1, \dots, z_n]$ is a zero-dimensional ideal. Our symbolic-numeric method roughly follows the lines of that proposed in Xu et al. (1994) but once again we take advantage of the univariate representation of zero-dimensional ideals (Rouillier (1999)) to control the numeric precision required to achieve our goal.

The paper is organized as follows. In Section 2, we recall some classical computer algebra results on the complex zeros of polynomials and polynomial systems. We also introduce the univariate representation of zero-dimensional ideals which will be our main tool in what follows. In Section 3, we provide an effective stabilizability test, i.e., an algorithm for testing whether a zero-dimensional ideal intersects the closed unit polydisc. In Section 4, we provide an effective *polydisc Nullstellensatz* namely a symbolic-numeric method for computing, if it exists, a stable polynomial in a zero-dimensional polynomial ideal. Finally, in Section 5, we illustrate our methods on one example and show some running times of prototype implementations.

2. PRELIMINARIES ON ALGEBRAIC SYSTEMS

In this section, we introduce some notations and we recall some classical material about the computation of certified numerical approximations of the complex zeros of polynomials and polynomial systems.

The *bit-size* of an integer is the number of bits in its representation and for a rational number (resp., a polynomial with rational coefficients) the term *bit-size* refers to the maximum bit-size of its numerator and denominator (resp., of its coefficients). For a complex number $z \in \mathbb{C}$, we denote by $\Re(z) \in \mathbb{R}$ (resp., $\Im(z) \in \mathbb{R}$) its real (resp., imaginary) part. If $z_1, z_2 \in \mathbb{C}$, we write $z_1 < z_2$ if both $\Re(z_1) < \Re(z_2)$ and $\Im(z_1) < \Im(z_2)$. For $z_1, z_2 \in \mathbb{C}$ such that $z_1 < z_2$, we shall consider the *axes-parallel open box* or *box* for short $B = B(z_1, z_2) = \{z \in \mathbb{C} \mid z_1 < z < z_2\}$ and its *width* is defined by $w(B) = \max\{|\Re(z_2 - z_1)|, |\Im(z_2 - z_1)|\}$. We also introduce the non-negative real number $|B| = \max\{|\Re(z_1)|, |\Im(z_1)|, |\Re(z_2)|, |\Im(z_2)|\}$. The box B is said to be of *rational endpoints* if z_1, z_2 have rational real and imaginary parts, i.e. $\Re(z_1), \Im(z_1), \Re(z_2), \Im(z_2) \in \mathbb{Q}$. Finally, the box B is called *isolating* for a given polynomial $f \in \mathbb{Q}[z]$ if it contains exactly one complex zero of f .

The following result concerns the isolation of the complex zeros of a univariate polynomial. We refer, for instance, to Sagraloff and Yap (2009) for more details.

Lemma 1. Let $f \in \mathbb{Q}[z]$ be a squarefree polynomial of degree d . Then, for all $\epsilon > 0$, one can compute disjoint axes-parallel open boxes B_1, \dots, B_d , with rational endpoints such that each B_i contains exactly one complex root of f and satisfies $w(B_i) \leq \epsilon$.

In the algorithms given in Sections 3 and 4 below, we shall use a routine called `Isolate` which takes as input a univariate polynomial f , a box B , and a precision $\epsilon > 0$ and computes isolating boxes B_1, \dots, B_l with rational endpoints for the complex roots of f that lie inside the given box B and such that $\max_{i=1, \dots, l} w(B_i) \leq \epsilon$. If B (resp., ϵ) is not specified in the input, we consider all complex roots in \mathbb{C} (resp., the boxes are computed up to a sufficient precision for isolation).

Let us now recall a standard property about width expansion through interval arithmetic in polynomial evaluation. Here we consider *exact interval arithmetic*, that is, the arithmetic operations on the interval endpoints are considered exact (see Alefeld and Herzberger (2012)). If $f \in \mathbb{Q}[x_1, x_2]$ is a bivariate polynomial of two real variables x_1 and x_2 and B a box, we denote by $\square f(B)$ the interval that results from the evaluation of the polynomial f at the box B using interval arithmetic.

Lemma 2. (Cheng et al. (2010), Lemma 8). Let B be a box with rational endpoints satisfying $|B| \leq 2^\sigma$ and let $f \in \mathbb{Q}[x_1, x_2]$ be a bivariate polynomial of two real variables x_1 and x_2 of degree d with coefficients of bit-size τ . Then, f can be evaluated at the box B by interval arithmetic into an interval $\square f(B)$ of width at most $2^{\tau+d\sigma+1} d^3 w(B)$.

In particular, a direct consequence of Lemma 2 is that if $w(B) \leq \epsilon 2^{-\tau-d\sigma-1-3\log_2(d)}$, then we have $w(\square f(B)) \leq \epsilon$.

We now consider a set of polynomials p_1, \dots, p_r in $\mathbb{Q}[z_1, \dots, z_n]$. We denote by $I = \langle p_1, \dots, p_r \rangle$ the ideal generated by the p_i 's and by

$$V(I) = \{z \in \mathbb{C}^n \mid p_1(z) = \dots = p_r(z) = 0\} \subseteq \mathbb{C}^n,$$

the complex variety of their common zeros. In the sequel, we shall always assume that the ideal I under consideration is a *zero-dimensional* ideal, that is, that the p_i 's have only a finite number of common complex zeros, i.e., $V(I)$

consists of a finite number of complex points. Methods for computing certified numerical approximations for the elements of $V(I)$ usually proceed in two steps. First, a formal representation (as for instance, a *Gröbner basis*, a *triangular decomposition* or a *univariate representation*) of the set $V(I)$ is computed. Then, this formal representation is used to compute, more or less easily, numerical approximations of the elements of $V(I)$. The convenient representation of the variety of zero-dimensional ideals that we shall use in the sequel is the so-called *univariate representation* introduced by Rouillier in Rouillier (1999).

Definition 3. With the previous notation and assumptions, a *univariate representation* of $V(I)$ is the datum of a linear form $t = a_1 z_1 + \dots + a_n z_n$, with $a_1, \dots, a_n \in \mathbb{Q}$ as well as $n + 1$ univariate polynomials $f, g_{z_1}, \dots, g_{z_n} \in \mathbb{Q}[t]$ such that the following two applications

$$\begin{aligned} V(I) &\longrightarrow V(f) = \{z \in \mathbb{C} \mid f(z) = 0\} \\ z = (z_1, \dots, z_n) &\longmapsto a_1 z_1 + \dots + a_n z_n, \end{aligned}$$

and

$$\begin{aligned} V(f) &\longrightarrow V(I) \\ z &\longmapsto (g_{z_1}(z), \dots, g_{z_n}(z)), \end{aligned}$$

provide a one-to-one correspondence between the elements of $V(I)$ and the zeros of f .

The univariate representation then defines a bijection between the zeros of I and those of f that preserves the multiplicities and the real zeros. One of its main advantage is that it permits to study the variety $V(I)$, i.e., the set of common zeros of the polynomials p_i 's of n variables through the roots of the univariate polynomial f . In particular, if we are interested in computing certified numerical approximations of the complex elements of $V(I)$, a first step consists, in isolating the complex roots of the polynomial f (using for instance the algorithm in Sagraloff and Yap (2009)). Then, numerical approximations of the coordinates z_1, \dots, z_n of $z \in V(I)$ can be obtained by evaluating g_{z_1}, \dots, g_{z_n} at the obtained numerical boxes. Lemmas 1 and 2 above show that we can thus obtain a given precision for the elements of $V(I)$.

The computation of a univariate representation of $V(I)$ can be done by pre-computing a Gröbner basis G of the ideal I Cox et al. (1992), and then performing linear algebra calculations in the quotient vector space $\mathbb{Q}[z_1, \dots, z_n]/I$ which admits a basis composed of the monomials that are irreducible modulo the Gröbner basis G . For more details, see Rouillier (1999). For the specific case of bivariate algebraic systems, it should be stressed that very practically efficient algorithms exist for computing such a representation: see Bouzidi (2014). To achieve efficiency, these algorithms replace the “costly” computation of a Gröbner basis of I by *resultants* and *subresultants* computations, see Basu et al. (2006).

In the algorithms given in Sections 3 and 4 below, we shall use a routine called `UnivR` which takes as input the polynomials p_1, \dots, p_r and computes a univariate representation of the variety $V(I)$ of the zero-dimensional ideal $I = \langle p_1, \dots, p_r \rangle$.

3. AN EFFECTIVE STABILIZABILITY TEST

Using the fractional representation approach to multidimensional systems, in Section 1, we have seen that a plant

is stabilizable if and only if $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$ for a certain polynomial ideal $I = \langle p_1, \dots, p_r \rangle$ that can be explicitly described in terms of the transfer matrix of the plant.

Given polynomials $p_1, \dots, p_r \in \mathbb{Q}[z_1, \dots, z_n]$, the purpose of this section is to provide an effective algorithm to decide whether or not $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$ where $I = \langle p_1, \dots, p_r \rangle$ is a zero-dimensional ideal. To achieve this, we shall use a symbolic-numeric approach. We start by computing a univariate representation of $V(I)$. As explained in Section 2, such a representation allows to describe formally the elements $z = (z_1, \dots, z_n)$ of $V(I)$ as

$$\{f(t) = 0, \quad z_1 = g_{z_1}(t), \quad \dots, \quad z_n = g_{z_n}(t)\} \quad (1)$$

where $f, g_{z_1}, \dots, g_{z_n} \in \mathbb{Q}[t]$. In what follows, the degree of f is denoted by d , and those of g_{z_1}, \dots, g_{z_n} are then smaller than d (see Rouillier (1999)).

Using a univariate representation, one can compute a set of hypercubes in \mathbb{R}^{2n} isolating the elements of $V(I)$. Each coordinate is represented by a box in \mathbb{R}^2 obtained from the intervals containing its real and imaginary parts. Moreover, from Lemmas 1 and 2, these hypercubes can be refined up to an arbitrary precision. We shall now consider the intersection between those hypercubes and the closed unit polydisc of \mathbb{C}^n defined by:

$$\overline{\mathbb{U}}^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1| \leq 1, \dots, |z_n| \leq 1\}.$$

Below, for any $g_{z_i} \in \mathbb{Q}[t]$, we shall denote by $\mathcal{C}(g_{z_i})$ the bivariate polynomial $\Re(g_{z_i})^2 + \Im(g_{z_i})^2 - 1 \in \mathbb{Q}[x_1, x_2]$, where $\Re(g_{z_i})$ (resp., $\Im(g_{z_i})$) is the real (resp., complex) part of the polynomial resulting from $g_{z_i}(x_1 + i x_2)$.

From the definition of $\overline{\mathbb{U}}^n$, one can see that the situation is easier when $V(I)$ does not contain elements $z \in \mathbb{C}^n$ with $|z_i| = 1$ for some $i \in \{1, \dots, r\}$. Indeed, we have:

Theorem 4. With the previous notations, let us consider $z = (z_1, \dots, z_n) \in V(I)$ such that, for all $i \in \{1, \dots, n\}$, $|z_i| \neq 1$. Let B be an isolating box for the root of f corresponding to z in the univariate representation of $V(I)$. Then, there exists $\epsilon > 0$ such that if $w(B) \leq \epsilon$, then, for all $k \in \{1, \dots, n\}$, the interval $\square \mathcal{C}(g_{z_k})(B)$ does not contain zero.

Proof. Let d (resp., τ) denote an upper bound on the degree (resp., bit-size) of the polynomials g_{z_k} , $k = 1, \dots, n$. For $k \in \{1, \dots, n\}$, the real (resp., imaginary) part of $g_{z_k}(x_1 + i x_2)$ is a bivariate polynomial in x_1 and x_2 of degree (resp., bit-size) bounded by d (resp., $d + \tau$). Consequently, the bivariate polynomial $\mathcal{C}(g_{z_k}) \in \mathbb{Q}[x_1, x_2]$ has degree and bit-size respectively bounded by $2d$ and $(d + \tau)d$ respectively. Now, for $z = (z_1, \dots, z_n) \in V(I)$, let $m = \min_{k=1, \dots, n} ||z_k| - 1| > 0$ and let B be an isolating box for the root of f corresponding to z in the univariate representation of $V(I)$ (1), and such that $|B| \leq 2^\sigma$. From Lemma 2, if we refine B so that $w(B) \leq \epsilon$, where $\epsilon = m 2^{-(d+\tau)d-2d\sigma-1-3\log_2(2^d)}$, then, we have that for all $k \in \{1, \dots, n\}$, the interval $\square \mathcal{C}(g_{z_k})(B)$ satisfies $w(\square \mathcal{C}(g_{z_k})(B)) \leq m$ so that, by definition of m , it does not contain zero.

Therefore, if $V(I)$ does not contain elements $z \in \mathbb{C}^n$ with one coordinate z_i in the unit circle, one can easily test

the stabilizability condition $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$. Indeed, with the previous notations, we isolate the roots of f inside boxes B_1, \dots, B_d . Then, for $i \in \{1, \dots, d\}$, we refine B_i until, for all $k \in \{1, \dots, n\}$, the interval $\square\mathcal{C}(g_{z_k})(B_i)$ does not contain zero. If one of the intervals $\square\mathcal{C}(g_{z_k})(B_i)$ is included in \mathbb{R}_+ , we proceed to the next box B_i , otherwise we have found an element in $V(I) \cap \overline{\mathbb{U}}^n$ so that the system is certainly not stabilizable. After having investigated all the boxes B_i , we can then conclude about the stabilizability of the system.

We shall now consider the case where $V(I)$ contains (at least) one element having some coordinates on the unit circle. In this case, we cannot proceed numerically as before since if $|z_k| = 1$, then, using the above notations, we cannot fulfill the condition that the interval $\square\mathcal{C}(g_{z_k})(B)$ does not contain zero. To guarantee the termination of the algorithm, we shall then have to compute, for each variable z_k , the number of elements $z = (z_1, \dots, z_n) \in V(I)$ satisfying $|z_k| = 1$.

Lemma 5. Let $I \subset \mathbb{Q}[z_1, \dots, z_n]$ be a zero-dimensional ideal and $V(I)$ the associated algebraic variety. Then, for all $k \in \{1, \dots, n\}$, one can compute the non-negative integer $l_k = \#\{z = (z_1, \dots, z_n) \in V(I) \mid |z_k| = 1\}$.

Proof. Let $\{f(t), z_1 - g_{z_1}(t), \dots, z_n - g_{z_n}(t)\}$ be a univariate representation of $V(I)$ and $k \in \{1, \dots, n\}$. Computing the resultant of the polynomials f and $z_k - g_{z_k}$ with respect to the variable t we get a univariate polynomial that can be written $r_k = \prod_{\alpha \in V(I)} (z_k - \alpha_k)^{\mu_{\alpha_k}}$, where the multiplicity μ_{α_k} corresponds to $\#\{z \in V(I) \mid z_k = \alpha_k\}$. Then, using the classical *Bistritz* test (see *Bistritz* (2002)), one can compute the number of complex roots counted with multiplicity of r_k that lie on the unit circle and obtain the non-negative integer l_k .

Remark 1. An alternative to the *Bistritz* test, which we use in practice, consists in applying to the polynomial r_k , the Möbius transform $z_k \rightarrow \frac{z_k - i}{z_k + i}$, which maps the complex unit circle $\overline{\mathbb{U}}$ to the real line $\mathbb{R} \cup \{\infty\}$. The number of complex roots of r_k on the unit circle is then given as the number of real roots of the gcd of two polynomials \Re and \Im , where \Re (resp., \Im) is the real (resp., complex) part of the numerator of the rational fraction $r_k(\frac{z_k - i}{z_k + i})$.

Using Lemma 5, we can test the stabilizability condition $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$ as follows. We start with the variable z_1 . We refine the isolating boxes B_1, \dots, B_d for the roots of f until exactly l_1 intervals $\square\mathcal{C}(g_{z_1})(B_i)$ contain zero. We throw away the boxes B_i 's such that the interval $\square\mathcal{C}(g_{z_1})(B_i)$ is included in \mathbb{R}_+ and we proceed similarly with the next variable z_2 . If at some point we have thrown away all the boxes B_i 's, then the system is stabilizable. Otherwise the boxes which remain at the end of the process lead to elements of $V(I) \cap \overline{\mathbb{U}}^n$ so that the system is not stabilizable.

We summarize our symbolic-numeric method for testing stabilizability in the following `IsStabilizable` algorithm.

4. AN EFFECTIVE STABILIZATION ALGORITHM

When a multidimensional system is stabilizable, one is then interested in computing a stabilizing controller. Within the fractional representation approach, this prob-

Algorithm 1 IsStabilizable

Input: A set of r polynomials $p_1, \dots, p_r \subset \mathbb{Q}[z_1, \dots, z_n]$.

Output: True if $V(\langle p_1, \dots, p_r \rangle) \cap \overline{\mathbb{U}}^n = \emptyset$, else False.

Begin

◇ $\{f, g_{z_1}, \dots, g_{z_n}\} := \text{Univ_R}(\{p_1, \dots, p_r\});$

◇ $\{B_1, \dots, B_d\} := \text{Isolate}(f);$

◇ $L_B := \{B_1, \dots, B_d\}$ and $\epsilon := \min_{i=1, \dots, d} w(B_i);$

For k from 1 to n **do**

◇ $l_k := \#\{z \in V(I) \mid |z_k| = 1\}$ (see Lemma 5);

While $\#\{i \mid 0 \in \square\mathcal{C}(g_{z_k})(B_i)\} > l_k$ **do**

 ◇ $\epsilon := \epsilon/2;$

 ◇ For $i = 1, \dots, d$, set $B_i := \text{Isolate}(f, B_i, \epsilon);$

End While

◇ $L_B := L_B \setminus \{B_i \mid \square\mathcal{C}(g_{z_k})(B_i) \subset \mathbb{R}_+\};$

◇ If $L_B = \{\}$, then **Return True End If**;

End For

Return False.

End

lem reduces to the following task (see Section 1 or *Lin* (1988)): given a polynomial ideal $I = \langle p_1, \dots, p_r \rangle$ satisfying $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$, compute a polynomial $s \in \mathbb{Q}[z_1, \dots, z_n]$ such that s belongs to the ideal I and s is a *stable polynomial*, i.e., s is devoid from zeros in $\overline{\mathbb{U}}^n$.

For an ideal $I \subset \mathbb{Q}[z_1, \dots, z_n]$, the *polydisc Nullstellensatz* (see *Bridges et al.* (2004)) asserts that the existence of a stable polynomial $s \in I$ is equivalent to $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$.

Theorem 6. (Polydisc Nullstellensatz). Let us consider a polynomial ideal $I = \langle p_1, \dots, p_r \rangle \subset \mathbb{Q}[z_1, \dots, z_n]$ such that $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$. Then, there exist $r + 1$ polynomials s, u_1, \dots, u_r in $\mathbb{Q}[z_1, \dots, z_n]$ such that:

$$s = \sum_{i=1}^r u_i p_i \text{ and } V(s) \cap \overline{\mathbb{U}}^n = \emptyset.$$

Several proofs of Theorem 6 have been investigated in the literature. Nevertheless none of them is effective in the sense that it provides an algorithm for computing s and the *cofactors* u_i 's using calculations that can be performed in an exact way by a computer. In *Xu et al.* (1994), the authors study 2D systems (i.e., $n = 2$) for which the ideal I under consideration is zero-dimensional. The idea of their method for computing a stable polynomial $s \in I$ is to compute univariate elimination polynomials $r_{z_1} \in \mathbb{Q}[z_1]$ and $r_{z_2} \in \mathbb{Q}[z_2]$ with respect to each variable z_1 and z_2 and to factorize them into a stable and an unstable factor, i.e., for $i = 1, 2$, $r_{z_i} = r_{z_i}^{(s)} r_{z_i}^{(u)}$, where the roots of $r_{z_i}^{(s)}$ (resp., $r_{z_i}^{(u)}$) are outside (resp., inside) the closed unit disc $\overline{\mathbb{U}}$, and then, to compute the stable polynomial as $s = r_{z_1}^{(s)} r_{z_2}^{(s)}$. However, this approach presents a major drawback with respect to the effectiveness aspect. Indeed, when the elimination polynomial r_{z_1} (resp., r_{z_2}) is an irreducible polynomial in $\mathbb{Q}[z_1]$ (resp. $\mathbb{Q}[z_2]$), its stable factor $r_{z_1}^{(s)}$ (resp., $r_{z_2}^{(s)}$) could not be computed exactly since it will have coefficients in \mathbb{C} , and thus, only an approximation of this polynomial can be obtained. As a consequence, the polynomial $s = r_{z_1}^{(s)} r_{z_2}^{(s)}$ will not belong to the ideal I .

In the sequel, we present a symbolic-numeric algorithm for computing s and the u_i 's that follows roughly the approach

of Xu et al. (1994) while we provide a way for tackling the effectiveness issue. Our main ingredient is the univariate representation of zero-dimensional ideals which allows us to compute and refine approximate factorizations over \mathbb{Q} of the elimination polynomials r_{z_i} 's.

Let $I = \langle p_1, \dots, p_r \rangle \subset \mathbb{Q}[z_1, \dots, z_n]$ be a zero-dimensional ideal such that $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$. For simplicity reasons, in what follows, we further assume that the ideal I is *radical*, i.e., $\sqrt{I} = \{p \in \mathbb{Q}[z_1, \dots, z_n] \mid \exists m \in \mathbb{N}^*, p^m \in I\} = I$. The elements $z = (z_1, \dots, z_n)$ of $V(I)$ are given by a univariate representation

$$\{f(t) = 0, \quad z_1 = g_{z_1}(t), \quad \dots, \quad z_n = g_{z_n}(t)\},$$

where $t = a_1 z_1 + \dots + a_n z_n$, $a_k \in \mathbb{Q}$ for $k = 1, \dots, n$, $f, g_{z_1}, \dots, g_{z_n} \in \mathbb{Q}[t]$, and $\deg(f) = d$. Since I is a radical ideal, the polynomial f is a squarefree polynomial and $f(t) = \prod_{i=1}^d (t - \gamma_i)$ for distincts $\gamma_1, \dots, \gamma_d \in \mathbb{C}$. Moreover, from Definition 3, if we introduce the polynomial ideal $I_r = \langle f(t), z_1 - g_{z_1}(t), \dots, z_n - g_{z_n}(t) \rangle \subset \mathbb{Q}[t, z_1, \dots, z_n]$, then we have $I_r = I \cap \langle t - \sum_{k=1}^n a_k z_k \rangle$. In particular, if $p(t, z_1, \dots, z_n) \in I_r$, then $p(\sum_{k=1}^n a_k z_k, z_1, \dots, z_n) \in I$.

Let us first explain how we can compute approximations $\tilde{r}_{z_k}^{(s)}$ of the stable polynomials $r_{z_k}^{(s)}$ appearing in the method of Xu et al. (1994) sketched above. Using Lemma 1, we can compute a set of boxes B_1, \dots, B_d with rational endpoints, isolating the distinct complex roots $\gamma_1, \dots, \gamma_d$ of f . Then, according to the stabilizability condition $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$, for all $i \in \{1, \dots, d\}$, the box B_i can be refined so that there exists $k \in \{1, \dots, n\}$ satisfying $\square \mathcal{C}(g_{z_k})(B_i) \subset \mathbb{R}_+$. We then set $\tilde{\gamma}_i \in \mathbb{Q}$ to the midpoint of the refined box B_i and we add the factor $z_k - g_{z_k}(\tilde{\gamma}_i)$ to the polynomial $\tilde{r}_k^{(s)}$. We finally obtain a set of stable univariate polynomials $\tilde{r}_k^{(s)} \in \mathbb{Q}[z_k]$, $k = 1, \dots, n$ such that $\sum_{k=1}^n \deg(\tilde{r}_k^{(s)}) = d$.

Let us now introduce the polynomial $\tilde{s} = \prod_{k=1}^n \tilde{r}_k^{(s)}$. By construction $\tilde{s} \in \mathbb{Q}[z_1, \dots, z_n]$ has rational coefficients and it vanishes on $V(\tilde{I}_r)$, where the polynomial ideal \tilde{I}_r is defined by $\tilde{I}_r = \langle \tilde{f}(t), z_1 - g_{z_1}(t), \dots, z_n - g_{z_n}(t) \rangle$ with $\tilde{f}(t) = \prod_{i=1}^d (t - \tilde{\gamma}_i) \in \mathbb{Q}[t]$. Hence, according to the classical *Nullstellensatz theorem* (Cox et al. (1992)), \tilde{s} belongs to the ideal \tilde{I}_r so that there exist polynomials $\tilde{h}_0, \tilde{h}_1, \dots, \tilde{h}_n \in \mathbb{Q}[t, z_1, \dots, z_n]$ such that $\tilde{s} = \tilde{h}_0 \tilde{f} + \sum_{k=1}^n \tilde{h}_k (z_k - g_{z_k})$. Moreover \tilde{h}_0 can be explicitly computed as the quotient of the Euclidean division of $\tilde{s}(g_{z_1}(t), \dots, g_{z_n}(t))$ by $\tilde{f}(t)$ in $\mathbb{Q}[t]$.

We shall now show that if we refine enough the boxes B_i 's isolating the roots γ_i 's of f , then the stable polynomial $s \in I$ that we are seeking for can be obtained from the polynomials \tilde{s} and \tilde{h}_0 constructed as explained above. For $\epsilon > 0^1$, we denote $\tilde{\gamma}_{i,\epsilon}, \tilde{s}_\epsilon, \tilde{f}_\epsilon(t) = \prod_{i=1}^d (t - \tilde{\gamma}_{i,\epsilon})$, and $\tilde{h}_{i,\epsilon}$ the objects constructed by the previous process where the roots of f are isolated up to precision ϵ (i.e., $w(B_i) \leq \epsilon$, for all $i \in \{1, \dots, d\}$). Using the previous notations, the main result of this section can be stated as follows:

Theorem 7. The polynomial $s = \tilde{s}_\epsilon - \tilde{h}_{0,\epsilon}(\tilde{f}_\epsilon - f)$ belongs to the ideal I_r . Moreover, there exists $\epsilon > 0$ such that the polynomial $s(\sum_{i=1}^n a_i z_i, z_1, \dots, z_n)$ is a stable polynomial.

The proof of Theorem 7, given below, requires the following lemma.

Lemma 8. For $0 < \epsilon < 1$, the polynomial $\tilde{h}_{0,\epsilon}(\tilde{f}_\epsilon - f)$ has coefficients bounded by $\epsilon \rho$, where ρ is a positive real number that does not depend on ϵ .

Proof. Let $f = \sum_{i=0}^d a_i t^i$ and $\tilde{f}_\epsilon = \sum_{i=0}^d b_i t^i$, with $a_d = b_d = 1$, denote the expansion of the polynomials f and \tilde{f}_ϵ on the monomial basis. By the standard *Vieta's formulas*, for all $i \in \{1, \dots, d-1\}$, we have:

$$a_i - b_i = \sum_{1 \leq k_1 < \dots < k_i \leq d} (\gamma_{k_1} \cdots \gamma_{k_i}) - (\tilde{\gamma}_{k_1,\epsilon} \cdots \tilde{\gamma}_{k_i,\epsilon}).$$

By assumption, $|\gamma_i - \tilde{\gamma}_{i,\epsilon}| \leq \epsilon$ for all $i \in \{1, \dots, d\}$, so that

$$|a_i - b_i| \leq \left| \sum_{1 \leq k_1 < \dots < k_i \leq d} (\gamma_{k_1} \cdots \gamma_{k_i}) - ((\gamma_{k_1} - \epsilon) \cdots (\gamma_{k_i} - \epsilon)) \right|.$$

Now, we can write $(\gamma_{k_1} - \epsilon) \cdots (\gamma_{k_i} - \epsilon) = \sum_{l=0}^i \sigma_l \epsilon^l$, where the σ_l 's denote the *symmetric functions* associated to $\gamma_{k_1}, \dots, \gamma_{k_i}$ and, in particular, $\sigma_0 = \gamma_{k_1} \cdots \gamma_{k_i}$. Consequently, since $\epsilon < 1$, we get:

$$|a_i - b_i| \leq \epsilon \left| \sum_{1 \leq k_1 < \dots < k_i \leq d} \sum_{l=1}^i \sigma_l \epsilon^{l-1} \right| \leq \epsilon \underbrace{\left| \sum_{1 \leq k_1 < \dots < k_i \leq d} \sum_{l=1}^i \sigma_l \right|}_{\rho_1}.$$

On the other hand, the polynomial $\tilde{h}_{0,\epsilon}$ can be computed as the quotient of the Euclidean division of $\tilde{s}_\epsilon(g_{z_1}(t), \dots, g_{z_n}(t))$ by $\tilde{f}_\epsilon(t)$. Formally, these polynomials can be considered as polynomials in $\mathbb{Q}[\tilde{\gamma}_{1,\epsilon}, \dots, \tilde{\gamma}_{d,\epsilon}][t]$, where $\tilde{\gamma}_{1,\epsilon}, \dots, \tilde{\gamma}_{d,\epsilon}$ are considered as new indeterminates so that their quotient denoted by h_0 can be computed independently from ϵ . The coefficients of h_0 are thus bounded by a certain positive real number δ_1 that does not depend on ϵ (use, for instance, *Mignotte's bound* Mignotte (1989)). Now, for $i \in \{1, \dots, d\}$, since $\epsilon < 1$, we have $|\tilde{\gamma}_{i,\epsilon}| < |\gamma_i + \epsilon| < |\gamma_i + 1|$. Thus if we denote $\delta_2 = \max_{i=1, \dots, d} |\gamma_i + 1|$, then the coefficients of the evaluation $\tilde{h}_{0,\epsilon} \in \mathbb{Q}[t]$ of h_0 for particular values of $\tilde{\gamma}_{1,\epsilon}, \dots, \tilde{\gamma}_{d,\epsilon}$ in \mathbb{Q} are bounded by $\rho_2 = \delta_1 \delta_2^d$. Finally, we have proved that the coefficients of $h_{0,\epsilon}(\tilde{f}_\epsilon - f)$ are bounded by $\epsilon \rho_1 \rho_2$, which ends the proof.

We are now in position to give a proof of Theorem 7.

Proof. With the previous notations, for all $\epsilon > 0$, we have $s = \tilde{s}_\epsilon - \tilde{h}_{0,\epsilon}(\tilde{f}_\epsilon - f) = \sum_{k=1}^n \tilde{h}_{k,\epsilon}(z_k - g_{z_k}) + \tilde{h}_{0,\epsilon} f$ so that s vanishes on $V(I_r)$, which implies $s \in I_r$. Let us now prove that we can choose ϵ so that s , viewed as $s(\sum_{i=1}^n a_i z_i, z_1, \dots, z_n) \in \mathbb{Q}[z_1, \dots, z_n]$, is stable, i.e., $\forall \lambda \in \overline{\mathbb{U}}^n, |s(\lambda)| > 0$. According to Lemma 8, for $\epsilon < 1$, we have $|h_{0,\epsilon}(\lambda)(\tilde{f}_\epsilon(\lambda) - f(\lambda))| \leq \epsilon \rho \delta$, where ρ (resp., δ) denotes a positive real number (resp., the degree of the polynomial $h_{0,\epsilon}(\tilde{f}_\epsilon - f)$) that does not depend on ϵ . On the other hand, the polynomial $\tilde{s}_\epsilon \in \mathbb{Q}[z_1, \dots, z_n]$ can be written as $\tilde{s}_\epsilon = \prod_{k=1}^n \tilde{r}_k^{(s)} = \prod_{i=1}^d (z_{k_i} - g_{z_{k_i}}(\tilde{\gamma}_{i,\epsilon}))$ for some $k_i \in \{1, \dots, n\}$ not necessarily distinct. Thus, for all $\lambda \in \overline{\mathbb{U}}^n$, we have $|\tilde{s}_\epsilon(\lambda)| \geq \prod_{i=1}^d \left| |\lambda_{k_i}| - |g_{z_{k_i}}(\tilde{\gamma}_{i,\epsilon})| \right|$. Now, if we denote by $m = \min_{z \in V(I)} (|z|_\infty - 1)$, the minimum

distance between the elements of $V(I)$ and $\overline{\mathbb{U}}^n$, then we

¹ small enough so that the previous process can be applied.

have $|\lambda_{k_i}| - |g_{z_{k_i}}(\tilde{\gamma}_{i,\epsilon})| \geq m - \epsilon$, which yields:

$$\forall \lambda \in \overline{\mathbb{U}}^n, |\tilde{s}_\epsilon(\lambda)| \geq (m - \epsilon)^d.$$

Finally, for sufficiently small ϵ , we have $(m - \epsilon)^d > \epsilon \rho \delta$ so that:

$$\begin{aligned} \forall \lambda \in \overline{\mathbb{U}}^n, |s(\lambda)| &\geq |\tilde{s}_\epsilon(\lambda)| - |h_{0,\epsilon}(\lambda)(\tilde{f}_\epsilon(\lambda) - f(\lambda))| \\ &\geq (m - \epsilon)^d - \epsilon \rho \delta > 0, \end{aligned}$$

which ends the proof.

The following **StablePolynomial** algorithm summarizes our method for computing a stable polynomial in a zero-dimensional ideal I satisfying $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$. The routine **IsStable** is used to test if a polynomial $p \in \mathbb{Q}[z_1, \dots, z_n]$ is stable, i.e., if $V(p) \cap \overline{\mathbb{U}}^n = \emptyset$ (see Bouzidi et al. (2015); Bouzidi and Rouillier (2016)).

Algorithm 2 StablePolynomial

Input: $I := \langle p_1, \dots, p_r \rangle$ be such that $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$.

Output: $s \in I$ such that $V(s) \cap \overline{\mathbb{U}}^n = \emptyset$.

Begin

◊ $\{f, g_{z_1}, \dots, g_{z_n}\} := \text{Univ_R}(\{p_1, \dots, p_r\});$

◊ $\{B_1, \dots, B_d\} := \text{Isolate}(f);$

◊ $\epsilon := \min_{i=1, \dots, d} w(B_i);$

Do

◊ $[r_1, \dots, r_n] := [1, \dots, 1]$ and $\tilde{f} := 1;$

◊ **outside** := **False**;

For each B in $\{B_1, \dots, B_d\}$ **do**

While (**outside**=**False**) **do**

For i from 1 to n **do**

If $\square \mathcal{C}(g_{z_i})(B) \subset \mathbb{R}_+$ **then**

◊ $\gamma := \text{midpoint}(B);$

◊ $r_i := r_i(z_i - g_{z_i}(\gamma));$

◊ **outside** := **True** and **Break For**;

End If

End For

◊ $\epsilon := \epsilon/2;$

◊ $B := \text{Isolate}(f, B, \epsilon);$

End While

◊ $\tilde{f} := \tilde{f}(t - \gamma);$

◊ **outside** := **False**;

End ForEach

◊ $\tilde{s} := \prod_{i=1}^n r_i;$

◊ $\tilde{s}_t := \tilde{s}$ evaluated at $z_i = g_{z_i}(t);$

◊ $h_0 := \text{quotient}(\tilde{s}_t, \tilde{f})$ in $\mathbb{Q}[t];$

◊ $s := \tilde{s} - h_0(\tilde{f} - f)$ evaluated at $t = \sum_{k=1}^n a_k z_k;$

While (**IsStable**(s)=**False**)

◊ **Return** s .

End

5. EXAMPLES AND EXPERIMENTS

Let us illustrate the algorithm of Section 4 on the following simple example:

$$I = \langle p_1, p_2 \rangle, \quad p_1 = z_1^2 - 2z_1 - 2, \quad p_2 = z_1 + z_2 - 2.$$

The associated variety $V(I)$ contains two elements, namely, $(1 - \sqrt{3}, 1 + \sqrt{3})$ and $(1 + \sqrt{3}, 1 - \sqrt{3})$, so that the stabilizability condition $V(I) \cap \overline{\mathbb{U}}^2 = \emptyset$ is clearly fulfilled. Yet, p_1 and p_2 are both unstable polynomials. For $i \in \{1, 2\}$, the univariate elimination polynomials (i.e., the resultants of p_1 and p_2) $r_{z_i} \in \mathbb{Q}[z_i]$ is given by $r_{z_i} = z_i^2 - 2z_i - 2$.

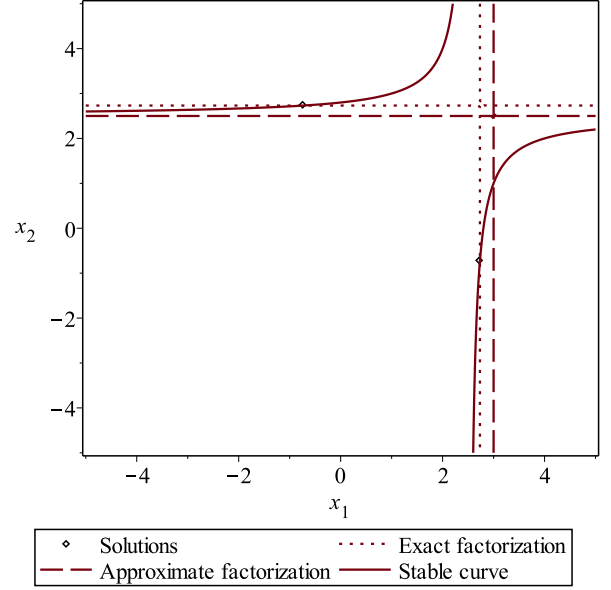


Fig. 1. Stabilizing polynomial for the variety $V(I)$ corresponding to $I = \langle z_1^2 - 2z_1 - 2, z_2^2 - 2z_2 - 2 \rangle$

The polynomial $z_i^2 - 2z_i - 2$ being irreducible in $\mathbb{Q}[z_i]$, this makes the approach of Xu et al. (1994) impracticable. Let us apply the algorithm of Section 4 for computing a stable polynomial $s \in I$. We start by computing a univariate representation of $V(I)$. We get:

$$f(t) := t^2 - 2t - 2 = 0, \quad z_1 = t, \quad z_2 = 2 - t.$$

The roots of $f(t)$ are given by $\gamma_1 \approx -0.73$ and $\gamma_2 \approx 2.73$ and choosing the precision $\epsilon = \frac{1}{2}$, we get the approximate roots (in \mathbb{Q}) $\tilde{\gamma}_1 = -\frac{1}{2}$ and $\tilde{\gamma}_2 = 3$. Consequently, the algorithm of Section 4 yields

$$\tilde{f}(t) = \left(t + \frac{1}{2}\right)(t - 3), \quad \tilde{s}(z_1, z_2) = (z_1 - 3) \left(z_2 - \frac{5}{2}\right),$$

which then leads to:

$$h_0(t) = -1, \quad (\tilde{f} - f)(t) = -\frac{1}{2}t + \frac{1}{2}.$$

Finally, after substituting $t = z_1$ in $\tilde{f} - f$, we get:

$$s(z_1, z_2) = z_1 z_2 - 3z_1 - 3z_2 + 8.$$

We can then check that this polynomial is stable so that we are done (see Bouzidi et al. (2015)). As a byproduct, we also obtain the corresponding cofactors, that is,

$$s = -p_1 + (z_1 - 3)p_2 \in I.$$

Figure 1 shows the stable polynomial obtained by an exact factorization of r_{z_1} and r_{z_2} in dots, the approximate factorization \tilde{s} used in our algorithm in dash, and finally the stable polynomial s obtained after adding the correcting term represented by the solid curve.

Data		Running time	
nbvar	$\#V(I)$	IsStabilizable	StablePolynomial
2	9	0.09	0.11
	25	1.23	0.50
	64	38.10	7.84
	100	244.91	49.49
3	8	0.13	0.11
	27	4.39	0.87
	36	11.83	1.98
	48	33.92	5.32
	64	118.28	24.09

Table 1. CPU times in seconds of `IsStabilizable` and `StablePolynomial` runned on sets of random polynomials in 2 or 3 variables with integer coefficients.

We have implemented two routines `IsStabilizable` and `StablePolynomial`, which correspond respectively to the algorithms given in Sections 3 and 4, in the computer algebra system MAPLE. These routines use the procedure `resultant` for computing the resultant of two polynomials (required in the computation of l_k in Algorithm 3), the procedure `RationalUnivariateRepresentation` of the MAPLE package `Groebner` for computing the *Rational Univariate Representation*² of a zero-dimensional ideals, and the procedure `fsolve` for computing the complex roots of a univariate polynomial³.

In Table 1 below, we report the running times (in seconds) of `IsStabilizable` and `StablePolynomial` applied to systems of randomly chosen polynomials in two or three variables with integer coefficients chosen uniformly at random between -100 and 100 . In order to get zero-dimensional systems, we choose as many polynomials as number of variables. Moreover, we use the change of variables $z_i = Z_i/10$, $i = 1, 2$ or $i = 1, 2, 3$ to increase the probability of the roots to be outside the unit polydisc. The experiments have been conducted on 2.10 GHz Core(TM) Intel i7-4600U with 4MB of L3 cache with MAPLE 2015 under windows platform.

Remark 2. From Table 1, one can notice that, in general, the running times of `IsStabilizable` are higher than those of `StablePolynomial`. This is, most likely, due to the additional cost induced by the computation of l_k in Algorithm 3, which requires the computation of elimination polynomials for each variable (resultants).

REFERENCES

- Alefeld, G. and Herzberger, J. (2012). *Introduction to interval computation*. Academic press.
- Alonso, M.E., Becker, E., Roy, M.F., and Wörmann, T. (1996). Zeros, multiplicities, and idempotents for zero-dimensional systems. In *Algorithms in algebraic geometry and applications*, 1–15. Springer.
- Basu, S., Pollack, R., and Roy, M.F. (2006). Algorithms in real algebraic geometry, volume 10 of algorithms and computation in mathematics.

- Becker, E. and Wörmann, T. (1996). Radical computations of zero-dimensional ideals and real root counting. *Mathematics and Computers in Simulation*, 42(4), 561–569.
- Bisiacco, M., Fornasini, E., and Marchesini, G. (1986). Controller design for 2D systems. *Frequency domain and state space methods for linear systems*, 99–113.
- Bistriz, Y. (2002). Zero location of polynomials with respect to the unit-circle unhampered by nonessential singularities. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 49(3), 305–314.
- Bose, N.K. (1984). *Multidimensional System Theory*. D. Reidel Publishing Comp.
- Bouzidi, Y., Quadrat, A., and Rouillier, F. (2015). Computer algebra methods for testing the structural stability of multidimensional systems. In *Proceedings of the 9th international Workshop on Multidimensional (nD) Systems (nDS’15), Vila Real (Portugal)*.
- Bouzidi, Y. and Rouillier, F. (2016). Certified algorithms for proving the structural stability of two-dimensional systems possibly with parameters. In *Proceedings of the 22nd international symposium on Mathematical Theory of Networks and Systems (MTNS’16)*.
- Bouzidi, Y. (2014). *Résolution de systèmes bivariés et topologie de courbes planes*. Ph.D. thesis, Université de Lorraine.
- Bridges, D., Mines, R., Richman, F., and Schuster, P. (2004). The polydisk nullstellensatz. *Proceedings of the American Mathematical Society*, 2133–2140.
- Canny, J. (1988). Some algebraic and geometric computations in pspace. In *Proceedings of the twentieth annual ACM symposium on Theory of computing*, 460–467. ACM.
- Cheng, J., Lazard, S., Peñaranda, L., Pouget, M., Rouillier, F., and Tsigaridas, E. (2010). On the topology of real algebraic plane curves. *Mathematics in Computer Science*, 4(1), 113–137.
- Cox, D., Little, J., and O’shea, D. (1992). *Ideals, Varieties, and Algorithms*, volume 3. Springer.
- Guiver, J. and Bose, N. (1995). Causal and weakly causal 2-D filters with applications in stabilization. In *Multidimensional systems theory and applications*, 35–78. Springer.
- Kronecker, L. (1882). *Grundzüge einer arithmetischen theorie der algebraischen grössen... von L. Kronecker*. G. Reimer.
- Lin, Z. (1988). Feedback stabilization of multivariable two-dimensional linear systems. *International Journal of Control*, 48(3), 1301–1317.
- Lin, Z. (1998). Feedback stabilizability of mimo nD linear systems. *Multidimensional Systems and Signal Processing*, 9(2), 149–172.
- Mignotte, M. (1989). Mathematiques pour le calcul formel.
- Quadrat, A. (2003a). The fractional representation approach to synthesis problems: An algebraic analysis viewpoint part ii: Internal stabilization. *SIAM Journal on Control and Optimization*, 42(1), 300–320.
- Quadrat, A. (2003b). The fractional representation approach to synthesis problems: An algebraic analysis viewpoint part i:(weakly) doubly coprime factorizations. *SIAM Journal on Control and Optimization*, 42(1), 266–299.

² This rational representation, which outputs expressions for the coordinates that are rational fractions, is post-processed in order to get polynomial expression for the coordinates as defined in Definition 3.

³ The detailed code along with the used testsuite can be found in ?

- Raman, V. and Liu, R.W. (1986). A constructive algorithm for the complete set of compensators for two-dimensional feedback system design. *IEEE transactions on automatic control*, 31(2), 166–170.
- Rouillier, F. (1999). Solving zero-dimensional systems through the rational univariate representation. *Applicable Algebra in Engineering, Communication and Computing*, 9(5), 433–461.
- Sagraloff, M. and Yap, C.K. (2009). An efficient and exact subdivision algorithm for isolating complex roots of a polynomial and its complexity analysis. *Draft, unpublished*.
- Vidyasagar, M. (2011). *Control system synthesis: a factorization approach*. Morgan & Claypool Publishers.
- Xu, L., Saito, O., and Abe, K. (1994). Output feedback stabilizability and stabilization algorithms for 2D systems. *Multidimensional Systems and Signal Processing*, 5(1), 41–60.