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Abstract

We consider stationary Navier-Stokes equations in $\mathbb{R}^3$ with a regular external force and we prove exponential frequency decay of the solutions. Moreover, if the external force is small enough, we give a pointwise exponential frequency decay for such solutions according to the K41 theory. If a damping term is added to the equation, a pointwise decay is obtained without the smallness condition over the force.

1 Introduction

Gevrey regularity for solutions of the Navier-Stokes equations has been studied in many different frameworks: for a periodic setting with external force see [1], [6]; for the stationary problem in $\mathbb{T}^3$ with frequency localized forces see [2]. For the evolution problem in $\mathbb{R}^3$ (with a null force) a pointwise analysis is obtained in [4].

In this article we generalize some of these previous results in the framework of stationary Navier-Stokes equations in $\mathbb{R}^3$

$$-\nu\Delta\overrightarrow{U} + \mathbb{P}(div(\overrightarrow{U}\otimes\overrightarrow{U})) = \overrightarrow{F}, \qquad div(\overrightarrow{U}) = 0, \quad div(\overrightarrow{F}), \tag{1}$$

where $\nu > 0$ is the fluid's viscosity parameter, $\overrightarrow{U} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the velocity, $\mathbb{P}$ is the Leray's projector and $\overrightarrow{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a time-independent external force.

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If the external force is regular enough we prove in Theorem 1.1 an exponential frequency decay. Moreover, if the external force is small enough, we give in Theorem 1.2 a pointwise exponential frequency decay for such solutions. Finally, if a damping term is added to the equation, a pointwise decay is obtained in Theorem 1.3 without the smallness condition over the force.

Theorem 1.1 *Let $\overrightarrow{F} \in \dot{H}^{-1}(\mathbb{R}^3)$ be such that for $\varepsilon_0 > 0$ we have*

$$\int_{\mathbb{R}^3} e^{2\varepsilon_0|\xi|}\left|\widehat{\overrightarrow{F}}(\xi)\right|^2|\xi|^{-2}d\xi < +\infty.$$

Then there exists $\overrightarrow{U} \in \dot{H}^1(\mathbb{R}^3)$ a solution to the stationary Navier-Stokes equations (1), such that $\overrightarrow{U}$ verifies the following exponential frequency decay:

$$\int_{\mathbb{R}^3} e^{2\varepsilon_1|\xi|}\left|\widehat{\overrightarrow{U}}(\xi)\right|^2|\xi|^{2}d\xi < +\infty, \quad \textit{where } \varepsilon_1 = \varepsilon_1(\varepsilon_0, \overrightarrow{F}, \nu) > 0. \tag{2}$$

In the laminar setting we obtain a sharper *pointwise* exponential frequency decay.
For $0 \leq a < 3$, we define the pseudo-measures space by

$$\mathcal{PM}^a = \left\{ \overrightarrow{g} \in \mathcal{S}'(\mathbb{R}^3) : \widehat{\overrightarrow{g}} \in L^1_{loc}(\mathbb{R}^3) \quad \text{and} \quad |\xi|^a\widehat{\overrightarrow{g}} \in L^\infty(\mathbb{R}^3) \right\},$$

which is a Banach space endowed with the norm $\|\overrightarrow{g}\|_{\mathcal{PM}^a} = \||\xi|^a\widehat{\overrightarrow{g}}\|_{L^\infty}$, for $a = 0$ we will simply denote the space $\mathcal{PM}^0$ by $\mathcal{PM}$.

Theorem 1.2 *Let $\overrightarrow{F} \in \mathcal{PM}$. There exists a (small) constant $\eta > 0$ such that if*

$$\sup_{\xi\in\mathbb{R}^3} e^{|\xi|}\left|\widehat{\overrightarrow{F}}(\xi)\right| < \eta,$$

then there exists $\overrightarrow{U} \in \mathcal{PM}^2$ a solution to the stationary Navier-Stokes equations (1) such that $\overrightarrow{U}$ verifies the following pointwise exponential frequency decay:

$$\left|\widehat{\overrightarrow{U}}(\xi)\right| \leq ce^{-|\xi|}|\xi|^{-2}, \quad \textit{for all} \quad \xi \neq 0. \tag{3}$$

If a damping term is added to the stationary Navier-Stokes system, we have the following result

Theorem 1.3 *Let $\overrightarrow{F} \in H^{-1}(\mathbb{R}^3)$ and for $\alpha > 0$ consider the damped stationary Navier-Stokes equations*

$$-\nu\Delta\overrightarrow{U} + \mathbb{P}(div(\overrightarrow{U} \otimes \overrightarrow{U})) = \overrightarrow{F} - \alpha\overrightarrow{U}, \qquad div(\overrightarrow{U}) = 0. \tag{4}$$

If the external force $\overrightarrow{F}$ is such that $\left|\widehat{\overrightarrow{F}}(\xi)\right| \leq e^{-\varepsilon_0|\xi|}$ for a fixed $\varepsilon_0 > 0$, then the stationary solution $\overrightarrow{U} \in H^1(\mathbb{R}^3)$ satisfies the following pointwise exponential frequency decay

$$\left|\widehat{\overrightarrow{U}}(\xi)\right| \leq ce^{-\varepsilon_1|\xi|}|\xi|^{-\frac{5}{2}}, \quad \textit{for all} \quad \xi \neq 0, \quad \textit{where } \varepsilon_1 = \varepsilon_1(\varepsilon_0, \overrightarrow{F}, \nu) > 0. \tag{5}$$

2 Proof of Theorem 1.1

Lemma 2.1 *If* $\overrightarrow{F} \in \dot{H}^{-1}(\mathbb{R}^3)$*, then there exists at least one solution* $\overrightarrow{U} \in \dot{H}^1(\mathbb{R}^3)$ *to the stationary Navier-Stokes equation (1).*

Lemma 2.2 *Let* $T_0 > 0$*. For* $\overrightarrow{u}_0 \in \dot{H}^1(\mathbb{R}^3)$ *a divergence-free initial data and a divergence-free external force* $\overrightarrow{f} \in \mathcal{C}([0, T_0[, \dot{H}^1(\mathbb{R}^3))$ *there exists a time* $0 < T_1 < T_0$ *and a function* $\overrightarrow{u} \in \mathcal{C}([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ *which is a unique solution to the Navier-Stokes equations*

$$\partial_t \overrightarrow{u} - \nu \Delta \overrightarrow{u} + \mathbb{P}(div(\overrightarrow{u} \otimes \overrightarrow{u})) = \overrightarrow{f}, \quad div(\overrightarrow{u}) = 0, \qquad \overrightarrow{u}(0, \cdot) = \overrightarrow{u}_0. \tag{6}$$

Existence and uniqueness issues are classical, see [5] for details.

In the following proposition we prove the frequency decay for the solution $\overrightarrow{u}$ obtained in Lemma 2.2.

Proposition 2.1 *Let* $\alpha > 0$ *and consider the Poisson kernel* $e^{\alpha\sqrt{t}\sqrt{-\Delta}}$*. Within the framework of Lemma 2.2, if the external force* $\overrightarrow{f}$ *is such that*

$$e^{\alpha\sqrt{t}\sqrt{-\Delta}} \overrightarrow{f} \in \mathcal{C}(]0, T_0[, \dot{H}^1(\mathbb{R}^3)),$$

then the unique solution of equations (6) satisfies $e^{\alpha\sqrt{t}\sqrt{-\Delta}} \overrightarrow{u} \in \mathcal{C}(]0, T_1[, \dot{H}^1(\mathbb{R}^3))$ *for all time* $t \in [0, T_1[$ *where* $0 < T_1 < T_0$ *is small enough.*

Proof. Consider the space

$$E = \left\{ \overrightarrow{u} \in \mathcal{C}(]0, T_1[, \dot{H}^1(\mathbb{R}^3)) : e^{\alpha\sqrt{t}\sqrt{-\Delta}} \overrightarrow{u} \in \mathcal{C}(]0, T_1[, \dot{H}^1(\mathbb{R}^3)) \right\},$$

endowed with the norm $\| \cdot \|_E = \| e^{\alpha\sqrt{t}\sqrt{-\Delta}} (\cdot) \|_{L^\infty_t \dot{H}^1_x}$. We study the quantity

$$\| \overrightarrow{u}_1 \|_E = \left\| h_{\nu t} * \overrightarrow{u}_0 + \int_0^t h_{\nu(t-s)} * \overrightarrow{f}(s, \cdot) ds - \int_0^t h_{\nu(t-s)} * \mathbb{P}(div(\overrightarrow{u}_1 \otimes \overrightarrow{u}_1))(s, \cdot) ds \right\|_E \tag{7}$$

where $h_{\nu t}$ is the heat kernel. The two first terms of this expression are easy to estimate and we have

$$\left\| h_{\nu t} * \overrightarrow{u}_0 + \int_0^t h_{\nu(t-s)} * \overrightarrow{f}(s, \cdot) ds \right\|_E \leq c(\nu, \alpha, T_0) \left(\| \overrightarrow{u}_0 \|_{\dot{H}^1_x} + \| e^{\alpha\sqrt{t}\sqrt{-\Delta}} \overrightarrow{f} \|_{L^\infty_t \dot{H}^1_x} \right). \tag{8}$$

For the last term of (7), by definition of the norm $\| \cdot \|_E$, by the Plancherel formula and by the boundedness of the Leray projector we have

$$\begin{aligned} (I) &= \left\| \int_0^t h_{\nu(t-s)} * \mathbb{P}(div(\overrightarrow{u}_1 \otimes \overrightarrow{u}_1)) ds \right\|_E \\ &= \sup_{0<t<T_1} \left\| e^{\alpha\sqrt{t}\sqrt{-\Delta}} \left(\int_0^t h_{\nu(t-s)} * \mathbb{P}(div(\overrightarrow{u}_1 \otimes \overrightarrow{u}_1)) ds \right) \right\|_{\dot{H}^1_x} \\ &\leq \sup_{0<t<T_1} c \left\| |\xi|^2 \int_0^t e^{-\nu(t-s)|\xi|^2} e^{\alpha\sqrt{t}|\xi|} \left| \left(\mathcal{F}[\overrightarrow{u}_1] * \mathcal{F}[\overrightarrow{u}_1] \right)(s, \cdot) \right| ds \right\|_{L^2_x}. \end{aligned}$$

Since we have the pointwise inequality

$$e^{\alpha\sqrt{t}|\xi|}\left|(\mathcal{F}[\overrightarrow{u}_1]*\mathcal{F}[\overrightarrow{u}_1])(s,\xi)\right| \leq \left[\left(e^{\alpha\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right)*\left(e^{\alpha\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right)\right](s,\xi), \tag{9}$$

due to the fact that $e^{\alpha\sqrt{t}|\xi|}\leq e^{\alpha\sqrt{t}|\xi-\eta|}e^{\alpha\sqrt{t}|\eta|}$ for all ξ, $\eta\in\mathbb{R}^3$, then we obtain

$$(I) \;\leq\; \sup_{0<t<T_1} c\int_0^t \left\| |\xi|^{\frac{3}{2}}e^{-\nu(t-s)|\xi|^2}|\xi|^{\frac{1}{2}}\left|\left[\left(e^{\alpha\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right)*\left(e^{\alpha\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right)\right]\right|\right\|_{L^2_x} ds.$$

Getting back to the spatial variable we can write

$$\begin{aligned}
(I) &\leq \sup_{0<t<T_1} c\int_0^t \Big\|(-\Delta)^{\frac{3}{4}}h_{\nu(t-s)}*(-\Delta)^{\frac{1}{4}}\Big\{\left(\mathcal{F}^{-1}\left[e^{\alpha\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right]\right)\otimes \\
&\qquad\qquad \left(\mathcal{F}^{-1}\left[e^{\alpha\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right]\right)\Big\}\Big\|_{L^2_x} ds \\
&\leq \left(c\int_0^{T_1}\left\|(-\Delta)^{\frac{3}{4}}h_{\nu(t-s)}\right\|_{L^1} ds\right)\Big\|\left(\mathcal{F}^{-1}\left[e^{\alpha\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right]\right)\otimes \\
&\qquad\qquad \left(\mathcal{F}^{-1}\left[e^{\alpha\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right]\right)\Big\|_{L^\infty_t\dot{H}^{\frac{1}{2}}_x} \\
&\leq c\frac{T^{\frac{1}{4}}}{\nu^{\frac{3}{4}}}\left\|\mathcal{F}^{-1}\left[e^{\alpha\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right]\right\|_{L^\infty_t\dot{H}^1_x}\left\|\mathcal{F}^{-1}\left[e^{\alpha\sqrt{t}|\xi|}|\mathcal{F}[\overrightarrow{u}_1]|\right]\right\|_{L^\infty_t\dot{H}^1_x} \\
&\leq c\frac{T_1^{\frac{1}{4}}}{\nu^{\frac{3}{4}}}\|\overrightarrow{u}_1\|_E\|\overrightarrow{u}_1\|_E.
\end{aligned} \tag{10}$$

With estimates (8) and (10) at hand, we fix T_1 small enough in order to apply Picard's contraction principle and we obtain a solution $\overrightarrow{u}_1\in E$ of (6). Since $E\subset\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$ we have $\overrightarrow{u}_1\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$ and by uniqueness of the solution $\overrightarrow{u}$ we have $\overrightarrow{u}_1=\overrightarrow{u}$, and thus $\overrightarrow{u}\in E$. ■

Now, we come back to the stationary Navier-Stokes equations (1) and we will prove that the solution $\overrightarrow{U}\in\dot{H}^1(\mathbb{R}^3)$ (given by Lemma 2.1) satisfies the exponential frequency decay given in (2). In the space $\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3))$ we consider the evolution problem (6) with the initial data $\overrightarrow{u}_0=\overrightarrow{U}$ where the external force $\overrightarrow{f}$ is now given by with the expression

$$\overrightarrow{f}=e^{-\alpha\sqrt{t}\sqrt{-\Delta}}(e^{\alpha\sqrt{t}\sqrt{-\Delta}}\overrightarrow{F}),$$

for the particular value $\alpha=\frac{2}{3}\varepsilon_0>0$ where $\varepsilon_0>0$ is given in the hypothesis of the force $\overrightarrow{F}$. To obtain a unique solution $\overrightarrow{u}\in\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3))$ to the equations (6) such that

$$e^{\alpha\sqrt{t}\sqrt{-\Delta}}\overrightarrow{u}\in\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3)),$$

we prove that the external force $\overrightarrow{f}$ verifies the hypotheses of Lemma 2.2 and Proposition 2.1 above:

$$\begin{aligned}\left\| e^{\alpha\sqrt{t}\sqrt{-\Delta}}\overrightarrow{F}\right\|^2_{L^\infty_t \dot{H}^1_x} &= \sup_{0<t<1}\int_{\mathbb{R}^3}|\xi|^2 e^{2\alpha\sqrt{t}|\xi|}\left|\widehat{\overrightarrow{F}}(\xi)\right|^2 d\xi\\ &\leq \frac{1}{\alpha^4}\int_{\mathbb{R}^3}(\alpha|\xi|)^4 e^{2\alpha|\xi|}\left|\widehat{\overrightarrow{F}}(\xi)\right|^2|\xi|^{-2}d\xi\\ &\leq \frac{1}{\alpha^4}\int_{\mathbb{R}^3} e^{3\alpha|\xi|}\left|\widehat{\overrightarrow{F}}(\xi)\right|^2|\xi|^{-2}d\xi\\ &\leq \frac{1}{\alpha^4}\int_{\mathbb{R}^3} e^{2\varepsilon_0|\xi|}\left|\widehat{\overrightarrow{F}}(\xi)\right|^2|\xi|^{-2}d\xi<+\infty.\end{aligned}$$

Thus, once we have $e^{\alpha\sqrt{t}\sqrt{-\Delta}}\overrightarrow{F}\in\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3))$, since the operator $e^{-\alpha\sqrt{t}\sqrt{-\Delta}}$ is bounded in the space $\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3))$ we have

$$\overrightarrow{f}=e^{-\alpha\sqrt{t}\sqrt{-\Delta}}(e^{\alpha\sqrt{t}\sqrt{-\Delta}}\overrightarrow{F})\in\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3)).$$

Moreover, we have

$$e^{\alpha\sqrt{t}\sqrt{-\Delta}}\overrightarrow{f}=e^{\alpha\sqrt{t}\sqrt{-\Delta}}\overrightarrow{F}\in\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3)).$$

By Lemma 2.2 there exists a time $0<T_1<1$ and a unique solution $\overrightarrow{u}\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$ to the equation (6). Moreover, since $e^{\alpha\sqrt{t}\sqrt{-\Delta}}\overrightarrow{f}\in\mathcal{C}(]0,1[,\dot{H}^1(\mathbb{R}^3))$ by Proposition 2.1 we have $e^{\alpha\sqrt{t}\sqrt{-\Delta}}\overrightarrow{u}\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$. Since the solution $\overrightarrow{U}\in\dot{H}^1(\mathbb{R}^3)$ of the stationary Navier-Stokes equations (1) is a constant in time, we have $\overrightarrow{U}\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$ and since $\partial_t\overrightarrow{U}\equiv 0$ and

$$\overrightarrow{f}=e^{-\alpha\sqrt{t}\sqrt{-\Delta}}(e^{\alpha\sqrt{t}\sqrt{-\Delta}}\overrightarrow{F})=\overrightarrow{F},$$

we find that $\overrightarrow{U}\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$ is also a solution to the equation (6) and thus, by uniqueness we get $\overrightarrow{U}=\overrightarrow{u}$. Then, since $e^{\alpha\sqrt{t}\sqrt{-\Delta}}\overrightarrow{u}\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3))$ we have

$$e^{\alpha\sqrt{t}\sqrt{-\Delta}}\overrightarrow{U}\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3)),$$

for all time $t\in[0,T_1[$. Thus, if $\varepsilon_1=\alpha\sqrt{\frac{T_1}{2}}>0$, we have

$$\int_{\mathbb{R}^3}e^{2\varepsilon_1|\xi|}|\widehat{\overrightarrow{U}}(\xi)|^2|\xi|^2d\xi=\left\|e^{\alpha\sqrt{\frac{T_1}{2}}\sqrt{-\Delta}}\overrightarrow{U}\right\|^2_{\dot{H}^1_x}\leq\sup_{0<t<T_1}\|e^{\alpha\sqrt{t}\sqrt{-\Delta}}\overrightarrow{U}\|^2_{\dot{H}^1_x}<+\infty,$$

and we obtain the frequency decay given in (2). ■

3 Proof of Theorem 1.2

We consider now the space $A = \left\{ \overrightarrow{U} \in \mathcal{PM}^2 : e^{\sqrt{-\Delta}} \overrightarrow{U} \in \mathcal{PM}^2 \right\}$, endowed with the norm

$$\|\cdot\|_A = \|e^{\sqrt{-\Delta}}(\cdot)\|_{\mathcal{PM}^2}, \tag{11}$$

and in this space we study the existence of a solution of equations (1) under the hypotheses of Theorem 1.2. For this we study the quantity

$$\|\overrightarrow{U}\|_A = \left\| \frac{1}{\nu}\mathbb{P}\left(\frac{1}{\Delta} div(\overrightarrow{U} \otimes \overrightarrow{U})\right) - \frac{1}{\nu}\frac{1}{\Delta}\overrightarrow{F} \right\|_A \leq \frac{1}{\nu}\left\| \mathbb{P}\left(\frac{1}{\Delta} div(\overrightarrow{U} \otimes \overrightarrow{U})\right)\right\|_A + \frac{1}{\nu}\left\|\frac{1}{\Delta}\overrightarrow{F}\right\|_A, \tag{12}$$

where, for the first term of the inequality above we have the following estimate:

$$\frac{1}{\nu}\left\| \mathbb{P}\left(\frac{1}{\Delta} div(\overrightarrow{U} \otimes \overrightarrow{U})\right)\right\|_A \leq \frac{c}{\nu}\|\overrightarrow{U}\|_A\|\overrightarrow{U}\|_A. \tag{13}$$

Indeed, by the expression (11) and by the continuity of the Leray projector we have

$$\begin{aligned}
\frac{1}{\nu}\left\| \mathbb{P}\left(\frac{1}{\Delta} div(\overrightarrow{U} \otimes \overrightarrow{U})\right)\right\|_A &= \frac{1}{\nu}\left\| |\xi|^2 e^{|\xi|}\mathcal{F}\left[\mathbb{P}\left(\frac{1}{\Delta} div(\overrightarrow{U} \otimes \overrightarrow{U})\right)\right]\right\|_{L^\infty} \\
&\leq \frac{c}{\nu}\left\| |\xi|^2 e^{|\xi|}\frac{1}{|\xi|}\left|\mathcal{F}\left[\overrightarrow{U}\right] * \mathcal{F}\left[\overrightarrow{U}\right]\right|\right\|_{L^\infty} \\
&\leq \frac{c}{\nu}\left\| |\xi| \left[\left(e^{|\xi|}\mathcal{F}\left[|\overrightarrow{U}|\right]\right) * \left(e^{|\xi|}\mathcal{F}\left[|\overrightarrow{U}|\right]\right)\right]\right\|_{L^\infty},
\end{aligned} \tag{14}$$

where the last inequality can be deduced from (9). Now we remark that

$$\begin{aligned}
\left[\left(e^{|\xi|}\mathcal{F}\left[|\overrightarrow{U}|\right]\right) * \left(e^{|\xi|}\mathcal{F}\left[|\overrightarrow{U}|\right]\right)\right](\xi) &= \int_{\mathbb{R}^3} e^{|\xi-\eta|}\mathcal{F}\left[|\overrightarrow{U}|\right](\xi-\eta)e^{|\eta|}\mathcal{F}\left[|\overrightarrow{U}|\right](\eta)d\eta \\
&\leq \|\overrightarrow{U}\|_A\,\|\overrightarrow{U}\|_A \int_{\mathbb{R}^3}\frac{d\eta}{|\xi-\eta|^2|\eta|^2} \leq \frac{c}{|\xi|}\|\overrightarrow{U}\|_A\,\|\overrightarrow{U}\|_A,
\end{aligned}$$

and thus, using this inequality in (14) we easily obtain the estimate (13). For the second term in the RHS of (12) we have

$$\frac{1}{\nu}\left\|\frac{1}{\Delta}\overrightarrow{F}\right\|_A = \frac{1}{\nu}\left\| e^{\sqrt{-\Delta}}\left(\frac{1}{\Delta}\overrightarrow{F}\right)\right\|_{\mathcal{PM}^2} = \frac{c_1}{\nu}\sup_{\xi\in\mathbb{R}^3}|\xi|^2 e^{|\xi|}\frac{1}{|\xi|^2}|\overrightarrow{F}(\xi)| = \frac{c_1}{\nu}\sup_{\xi\in\mathbb{R}^3} e^{|\xi|}|\overrightarrow{F}(\xi)|.$$

Thus, if the external force $\overrightarrow{F}$ satisfies $\sup_{\xi\in\mathbb{R}^3} e^{|\xi|}|\overrightarrow{F}(\xi)| < \eta$, for η small enough, we obtain $\overrightarrow{U} \in A$ a solution to the stationary Navier-Stokes equations (1) for which we have the pointwise estimate (3). ■

4 Proof of Theorem 1.3

For $\alpha > 0$ and under the hypotheses of Theorem 1.3, the existence of solutions of equation (4) is given by applying the Scheafer fixed point theorem. Now, for $\overrightarrow{u}_0 \in \mathcal{PM}^{\frac{5}{2}}$ we consider the non-stationary damped Navier-Stokes equations

$$\partial_t \overrightarrow{u} + \mathbb{P}(div(\overrightarrow{u} \otimes \overrightarrow{u})) - \nu\Delta\overrightarrow{u} = \overrightarrow{f} - \alpha\overrightarrow{u}, \quad div(\overrightarrow{u}) = 0, \quad \overrightarrow{u}(0,\cdot) = \overrightarrow{u}_0, \tag{15}$$

where the divergence-free external force $\overrightarrow{f}$ belongs to the space $\mathcal{C}([0,T_0[,\mathcal{PM}^{\frac{5}{2}})$. For this problem there exists a unique solution $\overrightarrow{u} \in \mathcal{C}([0,T_1[,\mathcal{PM}^{\frac{5}{2}})$ with $0 < T_1 < T_0$. For existence issues for equations (4) and (15) see the details in [5].

Following essentially the same lines of Proposition 2.1 above, we prove that if the external force is such that $e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{f} \in \mathcal{C}([0,T_0[,\mathcal{PM}^{\frac{5}{2}})$ then the unique solution of (15) is such that $e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{u} \in \mathcal{C}([0,T_1[,\mathcal{PM}^{\frac{5}{2}})$. As in the proof of Theorem 1.2, we consider

$$\overrightarrow{f} = e^{-\beta\sqrt{t}\sqrt{-\Delta}}\left(e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{F}\right) = \overrightarrow{F},$$

and for a suitable value of the parameter $\beta > 0$ we can prove that $\overrightarrow{f} \in \mathcal{C}([0,1[,\mathcal{PM}^{\frac{5}{2}})$ and $e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{f} \in \mathcal{C}([0,1[,\mathcal{PM}^{\frac{5}{2}})$.

In order to link the stationary solution to the non-stationary problem, we must prove that the solution $\overrightarrow{U} \in H^1(\mathbb{R}^3)$ of (4) is such that $\overrightarrow{U} \in \mathcal{PM}^{\frac{5}{2}}$, and in this step we use the extra damping term. Indeed, rewriting (4) we consider the equation

$$\overrightarrow{U} = \frac{-\nu\Delta}{\alpha Id - \nu\Delta}\left(\mathbb{P}\left(\frac{1}{\nu\Delta}div(\overrightarrow{U}\otimes\overrightarrow{U})\right)\right) + \frac{1}{\alpha Id - \nu\Delta}\left(\overrightarrow{F}\right), \tag{16}$$

and we obtain

$$\|\overrightarrow{U}\|_{\dot{H}^{\frac{3}{2}}} \leq \left\|\tfrac{-\nu\Delta}{\alpha I_d - \nu\Delta}\left(\mathbb{P}\left(\tfrac{1}{\nu\Delta}div(\overrightarrow{U}\otimes\overrightarrow{U})\right)\right)\right\|_{\dot{H}^{\frac{3}{2}}} + \left\|\tfrac{1}{\alpha I_d - \nu\Delta}\left(\overrightarrow{F}\right)\right\|_{\dot{H}^{\frac{3}{2}}}.$$

Since the operator $\frac{-\nu\Delta}{\alpha I_d - \nu\Delta}$ is bounded in $\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)$ and by the properties of $\overrightarrow{F}$ we can write

$$\begin{aligned}
\|\overrightarrow{U}\|_{\dot{H}^{\frac{3}{2}}} &\leq \left\|\frac{1}{\nu\Delta}div(\overrightarrow{U}\otimes\overrightarrow{U})\right\|_{\dot{H}^{\frac{3}{2}}} + \left\|\frac{1}{\alpha I_d - \nu\Delta}\left(\overrightarrow{F}\right)\right\|_{H^2} \\
&\leq c\|\overrightarrow{U}\otimes\overrightarrow{U}\|_{\dot{H}^{\frac{1}{2}}} + c(\alpha)\|\overrightarrow{F}\|_{L^2} \\
&\leq c\|\overrightarrow{U}\|_{H^1}\|\overrightarrow{U}\|_{H^1} + c(\alpha)\|\overrightarrow{F}\|_{L^2}.
\end{aligned}$$

We thus have $\overrightarrow{U} \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)$ and we prove now $\overrightarrow{U} \in \mathcal{PM}^{\frac{5}{2}}$: from equation (16) we obtain

$$\left|\mathcal{F}\left[\overrightarrow{U}\right](\xi)\right| \;\leq\; c\frac{1}{\nu|\xi|}|\left(\mathcal{F}\left[\overrightarrow{U}\right] * \mathcal{F}\left[\overrightarrow{U}\right]\right)(\xi)| + \frac{1}{\nu|\xi|^2}|\mathcal{F}\left[\overrightarrow{F}\right](\xi)|,$$

and then, multiplying by $|\xi|^{\frac{5}{2}}$ and by hypothesis on $\overrightarrow{F}$ we get the estimate

$$\begin{aligned} |\xi|^{\frac{5}{2}}\left|\mathcal{F}\left[\overrightarrow{U}\right](\xi)\right| &\leq \int_{\mathbb{R}^3} |\xi|^{\frac{3}{2}}\left|\mathcal{F}\left[\overrightarrow{U}\right](\xi-\eta)\right|\left|\mathcal{F}\left[\overrightarrow{U}\right](\eta)\right| d\eta + \frac{1}{\nu}|\xi|^{\frac{1}{2}}\left|\mathcal{F}\left[\overrightarrow{F}\right](\xi)\right| \\ &\leq 2\|\overrightarrow{U}\|_{\dot{H}^{\frac{3}{2}}}\|\overrightarrow{U}\|_{L^2} + \frac{1}{\nu}|\xi|^{\frac{1}{2}}e^{-\varepsilon_0|\xi|}, \end{aligned}$$

from which we deduce that $\overrightarrow{U} \in \mathcal{PM}^{\frac{5}{2}}$. Then, we study (15) with $\overrightarrow{u}_0 = \overrightarrow{U}$ and we have $\overrightarrow{U} \in \mathcal{C}([0,T_1[,\mathcal{PM}^{\frac{5}{2}})$, but since $\overrightarrow{U}$ verifies the equations (4), $\partial_t\overrightarrow{U} \equiv 0$ and $\overrightarrow{f} = \overrightarrow{F}$, we obtain that $\overrightarrow{U}$ is also a solution of (15) and by uniqueness we have $\overrightarrow{U} = \overrightarrow{u}$. Finally, we have $e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{U} \in \mathcal{C}([0,T_1[,\mathcal{PM}^{\frac{5}{2}})$ for $0 < t < T_1$ and if $\varepsilon_1 = \beta\sqrt{\frac{T_1}{2}}$ we can write

$$\|e^{\varepsilon_1\sqrt{-\Delta}}\overrightarrow{U}\|_{\mathcal{PM}^{\frac{5}{2}}} \leq \|e^{\beta\sqrt{t}\sqrt{-\Delta}}\overrightarrow{U}\|_{L^\infty([0,T_1[,\mathcal{PM}^{\frac{5}{2}})} < +\infty,$$

and we obtain the frequency decay stated in the formula (5). ■

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