On the stability conjecture for geodesic flows of manifold without conjugate points

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ON THE STABILITY CONJECTURE FOR GEODESIC FLOWS OF MANIFOLDS WITHOUT CONJUGATE POINTS

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Abstract. We study the C^2-structural stability conjecture from Mañé's viewpoint for geodesics flows of compact manifolds without conjugate points. The structural stability conjecture is an open problem in the category of geodesic flows because the C^1 closing lemma is not known in this context. Without the C^1 closing lemma, we combine the geometry of manifolds without conjugate points and a recent version of Franks' Lemma from Mañé's viewpoint to prove the conjecture for compact surfaces, for compact three dimensional manifolds with quasi-convex universal coverings where geodesic rays diverge, and for n-dimensional, generalized rank one manifolds.

1. Introduction

The motivations for the main results in this article come from two sources. First of all, the challenging problem of the C^1 closing lemma for geodesic flows that remains an open, very difficult problem. Recently, Rifford [27] proved a C^0 closing lemma for geodesic flows applying ideas of geometric control theory. The high technical difficulties involved in the proof of this fact give an idea of the considerable complexity of the problem in the C^1 level. However, C^0 perturbations are considered too rough by specialists in perturbative theory of dynamical systems. So the question of how far we can go in proving the C^1 stability conjecture for geodesic flows without a C^1 closing lemma is an interesting, appealing problem in Riemannian geometry and dynamical systems.

Secondly, some important results about the topological dynamics of the geodesic flow of compact manifolds without conjugate points and hyperbolic global geometry are known for nonpositive curvature manifolds. Eberlein [9] shows the topological transitivity of the geodesic flow of a visibility manifold, without restrictions on the sectional curvatures. The density of the set of periodic orbits, another important feature of topological dynamics, is known for visibility manifolds with nonpositive sectional curvatures (see for instance [3]). Notice that by one of the main results of [18], the C^2 structural stability of the geodesic flow of a compact manifold from Mañé's viewpoint implies the hyperbolicity of the closure of the set of periodic orbits. So it seems natural to ask whether the density of periodic orbits, a statement with a flavor of topological dynamics, really needs some extra assumptions on the geometry of the manifold (like nonpositive curvature) to hold. There are some known results of course, Anosov geodesic flows have this property, as well as

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expansive geodesic flows in compact manifolds without conjugate points [30]. But visibility manifolds of nonpositive curvature are examples of the so-called rank one manifolds, which may have non Anosov geodesic flows because of the presence of flat strips.

The goal of the paper is to deal with the stability conjecture of geodesic flows of compact manifolds without conjugate points, a geometric condition that is much weaker than nonpositive curvature but still ensures many important properties for the topological dynamics of the geodesic flow. Two kind of results are presented, the first one is for low dimensional manifolds.

Theorem 1.1. *Let (M, g) be a compact C^∞ manifold without conjugate points that is one of the following:*

(1) *A surface.*
(2) *A 3 dimensional manifold such that the universal covering is a quasi-convex space where geodesic rays diverge.*

Then, the geodesic flow is C^2 structurally stable from Mañé's viewpoint if and only if the geodesic flow is Anosov.

Item (1) is probably known but we did not find any records in the literature about the subject. The second result for higher dimensional manifolds introduces a generalized version of the rank one notion for manifolds without conjugate points and no restrictions on the sectional curvatures (Section 1).

Theorem 1.2. *Let (M, g) be a compact C^∞ manifold without conjugate points such that the universal covering is a quasi-convex space where geodesic rays diverge. If the set of generalized rank one points is dense in T_1M we have that the geodesic flow is C^2 structurally stable from Mañé's viewpoint if and only if the geodesic flow is Anosov.*

The paper is organized as follows: Section 2 is concerned with some preliminaries on manifolds without conjugate points. In Section 3, we investigate the strip issue for manifolds without conjugate points. In Section 4, we study the stability and hyperbolicity properties of the set of closed orbits. Section 5 is concerned with the density of closed orbits on manifolds with Gromov hyperbolic fundamental group. Section 6 is devoted to the proof of Theorem 1.1 and the proof of Theorem 1.2 is given in Section 8.

2. Preliminaries

Let us give some notations that will be used through the article. A pair (M, g) denotes a C^∞ complete Riemannan manifold, TM will denote its tangent space, T_1M denotes its unit tangent bundle. $\Pi : TM \longrightarrow M$ denotes the canonical projection $\Pi(p, v) = p$, the coordinates (p, v) for TM will be called canonical coordinates. The universal covering of M is $\tilde{M}$, the covering map is denoted by $\pi : \tilde{M} \longrightarrow M$, the pullback of the metric g by π is denoted by $\tilde{g}$. The geodesic $\gamma_{(p,v)}$ of (M, g) or $(\tilde{M}, \tilde{g})$ is the unique geodesic whose initial conditions are $\gamma_{(p,v)}(0) = p$, $\gamma'_{(p,v)}(0) = v$. All geodesics will be parametrized by arc length unless explicitly stated.

Definition 2.1. A C^∞ Riemannian manifold (M, g) has no conjugate points if the exponential map is nonsingular at every point.

Nonpositive curvature manifolds are well known examples of manifolds without conjugate points, but there are of course many examples of manifolds without conjugate points having sectional curvatures of variable sign.

To define a generalized rank one manifold without conjugate points we have to recall some basic notions of the theory of manifolds without conjugate points. Let (M, g) a C^∞ Riemannian manifold without conjugate points be fixed.

Definition 2.2. We say that geodesic rays diverge uniformly, or simply diverge in $(\tilde{M}, \tilde{g})$ if for given $\epsilon > 0$, $T > 0$, there exists $R > 0$ such that for any given $p \in \tilde{M}$ and two different geodesic rays $\gamma : \mathbb{R} \longrightarrow \tilde{M}$, $\beta : \mathbb{R} \longrightarrow \tilde{M}$ with $\gamma(0) = \beta(0) = p$, subtending an angle at p greater than ϵ, then $d(\gamma(t), \beta(t)) \geq T$ for every $t \geq R$.

This condition is quite common in all known categories of manifolds without conjugate points (nonpositive curvature, no focal points, bounded asymptote, compact surfaces without conjugate points), but it is a conjecture whether it is satisfied for every compact manifold without conjugate points.

Definition 2.3. Given $\theta = (p, v) \in T_1\tilde{M}$, the Busemann function $b^\theta : \tilde{M} \longrightarrow \mathbb{R}$ is given by

$$b^\theta(x) = \lim_{t\to+\infty}(d(x, \gamma_\theta(t)) - t).$$

Busemann functions of compact manifolds without conjugate points are C^{1+k}, namely, C^1 functions with k-Lipschitz first derivatives where the constant k depends on the minimum value of the sectional curvatures (see for instance [25], Section 6). The level sets of b^θ, $H_\theta(t)$, are called *horospheres*, the gradient ∇b^θ is a Lipschitz unit vector field and its flow preserves the foliation of the horospheres H_θ that is a continuous foliation by C^1, equidistant leaves. The integral orbits of the Busemann flow are geodesics of $(\tilde{M}, \tilde{g})$ and usually called *Busemann asymptotes* of γ_θ. When the curvature is nonpositive, Busemann functions and horospheres are C^2 smooth [10].

Definition 2.4. For $\theta = (p, v) \in T_1\tilde{M}$ let

$$\tilde{W}^s(\theta) = \{(x, -\nabla_x b^\theta),\ x \in H_\theta(0)\}$$

$$\tilde{W}^u(\theta) = \{(x, \nabla_x b^{(p,-v)}),\ x \in H_{(p,-v)}(0)\}.$$

If $\mathcal{P} : T_1\tilde{M} \longrightarrow T_1M$ is projection $\mathcal{P}(p, v) = (\pi(p), d_p\pi(v))$, let

$$W^s(\theta) = \mathcal{P}(\tilde{W}^s(\theta))$$

$$W^u(\theta) = \mathcal{P}(\tilde{W}^u(\theta)).$$

Let us denote by $\mathcal{F}^s$ the collection of the sets $W^s(\theta)$, $\theta \in T_1M$, and by $\mathcal{F}^u$ the collection of the sets $W^u(\theta)$, $\theta \in T_1M$.

The sets $W^s(\theta)$, $W^u(\theta)$, are continuous $n-1$ dimensional submanifolds of T_1M, and of course they coincide with the stable and the unstable sets of θ when the geodesic flow is Anosov. When (M, g) is a compact surface without conjugate points, the collections $\mathcal{F}^s$, $\mathcal{F}^u$, form continuous foliations by Lipschitz leaves. This is a well known consequence of the divergence of geodesic rays in the universal covering proved by Green [13] and the quasi-convexity of $(\tilde{M}, \tilde{g})$ shown by Morse [23].

Definition 2.5. The universal covering $(\tilde{M}, \tilde{g})$ of a complete Riemannian manifold (M, g) is called a (K, C)-quasi-convex space, or simply a quasi-convex space, if there exist constants $K > 0$, $C > 0$, such that given two pairs of points x_1, x_2, y_1, y_2 in $\tilde{M}$ and two minimizing geodesics $\gamma : [0,1] \longrightarrow \tilde{M}$, $\beta : [0,1] \longrightarrow \tilde{M}$ such that $\gamma(0) = x_1$, $\gamma(1) = y_1$, $\beta(0) = x_2$, $\beta(1) = y_2$, we have

$$d_H(\gamma, \beta) \leq K \max\{d(x_1, x_2), d(y_1, y_2)\} + C$$

where d_H is the Hausdorff distance.

The universal covering of manifold of nonpositive sectional curvature is $(1,0)$-quasi-convex. Most of the known categories of manifolds without conjugate points (no focal points, bounded asymptote) have quasi-convex universal coverings. Moreover, by the work of Gromov [12], the universal covering of every compact manifold whose fundamental group is hyperbolic is quasi-convex. Although geodesics in $(\tilde{M}\tilde{g})$ behave like hyperbolic geodesics when the dimension of M is 2, an example by Ballmann-Brin-Burns [2] shows that Jacobi fields may behave wildly compared with the quasi-convex behavior of geodesics.

Quasi-convexity is linked to the continuity of $\mathcal{F}^s$, $\mathcal{F}^u$, but it is not enough to grant this property. When the manifold has nonpositive curvature these sets form indeed continuous foliations by C^1 leaves (see [10] where it is shown the same statement for a larger class of manifolds satisfying the so-called bounded asymptote condition). Assuming no restrictions on either the curvatures or the asymptotic behavior of Jacobi fields it is true that $\mathcal{F}^s$, $\mathcal{F}^u$ form continuous foliations provided that geodesics in the universal covering satisfy the so-called Axiom of Asymptoticity (see [25]). The most general result known concerning the subject is proved in [31].

Theorem 2.6. *Let (M, g) be a compact manifold without conjugate points. Then geodesic rays diverge in $\tilde{M}$ if and only if $\mathcal{F}^s$, $\mathcal{F}^u$ are continuous foliations by Lipschitz leaves.*

Theorem 2.6 leads naturally to the following extension of the notion of rank one manifold.

Definition 2.7. A generalized rank one manifold is a compact manifold without conjugate points such that

(1) Geodesic rays diverge in $(\tilde{M}, \tilde{g})$.
(2) There exists $\theta \in T_1M$ and an open neighborhood $B(\theta)$ of θ in T_1M such that for each $\eta \in B(\theta)$, the connected component $W^s_{loc}(\eta)$ of $W^s(\eta) \cap B(\eta)$ containing η and the connected component $W^u_{loc}(\eta)$ of $W^u(\eta) \cap B(\eta)$ containing η satisfy

$$W^s_{loc}(\eta) \cap W^u_{loc}(\eta) = \eta.$$

The point θ will be called a generalized rank one point for the geodesic flow.

Notice that by definition, if the set of generalized rank one points is non empty then it is open. Recall that a manifold with nonpositive curvature is a rank one manifold if there exists a geodesic where the only parallel Jacobi field of the geodesic is the vector field tangent to the geodesic. Of course rank one manifolds of nonpositive curvature are generalized rank one manifolds since geodesic rays diverge in nonpositive curvature and the tangent space of $W^s_{loc}(\eta) \cap W^u_{loc}(\eta)$ is generated by parallel Jacobi fields. So along the orbit of a rank one point in a space of nonpositive curvature stable and unstable sets intersect transversally, and since these sets form continuous foliations by C^1 leaves the transversality between invariant submanifolds is an open property.

However, the set of generalized rank one points might include strictly the set of rank one points of manifolds with nonpositive curvature. This is the case of surfaces of nonpositive curvature where the curvature vanishes just at a finite set of closed geodesics. Every point of T_1M is a generalized rank one point while the set of rank one points is the complement of this finite set of flat geodesics. The expansivity of the geodesic flows of a compact manifold without conjugate points implies that every point in T_1M is a generalized rank one point (see [30]). We might expect that the presence of generalized rank one points would imply some kind of local expansivity, this will be the subject of the last section.

The study of the set of intersections $\tilde{W}^s(\eta) \cap \tilde{W}^u(\eta)$ is one of the most intriguing problems in the theory of manifolds without conjugate points. In the case of compact surfaces such a set is a connected compact curve with boundary (that might be a single point of course). The convexity properties of spaces of nonpositive curvature yield that such a set is a convex flat set (see for instance [4]) which generates a flat invariant set under the action of the geodesic flow. Such flat sets are called flat strips in the case of surfaces. Without restrictions on the sectional curvatures the intersections between invariant submanifolds might be non-flat strips as shown by Burns (see [7]), but still enjoy good topological properties. In higher dimensions this problem is much more difficult, we shall comme back to the subject in the next section.

2.1. **Busemann asymptotes versus asymptotes.** As we mentioned before, every orbit of the Busemann flow of b^θ is called a Busemann asymptote of γ_θ for $\theta \in T_1\tilde{M}$. However, the usual definition of asymptoticity is the following:

Definition 2.8. A geodesic $\beta \subset \tilde{M}$ is forward asymptotic to $\gamma \subset \tilde{M}$ if there exists $L > 0$ such that

$$d_H(\gamma[0,+\infty), \beta[0,+\infty)) \leq L$$

where $\gamma[0,+\infty) = \{\gamma(t), t \geq 0\}$. A geodesic $\sigma \subset \tilde{M}$ is backward asymptotic to γ if there exists $L > 0$ such that

$$d_H(\gamma(-\infty,0], \beta(-\infty,0]) \leq L.$$

Two geodesics γ, β in $\tilde{M}$ are bi-asymptotic if they are both forward and backward asymptotic.

While a Busemann asymptote of γ_θ might not be asymptotic to γ_θ what is true is

Lemma 2.9. *Let (M,g) be compact without conjugate points such that geodesic rays diverge in $\tilde{M}$. Then*

1. *Any geodesic γ_η forward asymptotic to γ_θ is a Busemann asymptote of γ_θ. Moreover, there exists $c > 0$ such that $b^\eta(x) = b^\theta(x) + c$ for every $x \in \tilde{M}$.*
2. *In particular, any geodesic that is bi-asymptotic to γ_θ is a Busemann asymptote of γ_θ and $\gamma_{-\theta}$ where $\theta = (p,v)$, $-\theta = (p,-v)$. In this case, if $\gamma_\eta(0) \in H_\theta(0)$ then $\gamma_\eta(0) \in H_\theta(0) \cap H_{-\theta}(0)$.*

Proof. We just sketch the proof since it is well known in the theory. Item (1) follows from the fact that the horosphere $H_\theta(0)$ is the limit of spheres $S_t(\gamma_\theta(t))$ of radius t centered at $\gamma_\theta(t)$. Indeed, let $x \in \tilde{M}$ and consider the geodesics $[x, \gamma_\theta(t)]$ joining x and $\gamma_\theta(t)$, and $[x, \gamma_\eta(t)]$ joining x and $\gamma_\eta(t)$. The angle subtended by these geodesics at x tends to zero as $t \to +\infty$ because of the divergence of geodesic rays. This yields that γ_η and γ_θ have the same Busemann asymptotes. Since the Busemann asymptotes of γ_θ define a flow by geodesics that is always orthogonal to the horospheres $H_\theta(s)$, we have that the horospheres $H_\theta(s)$ and $H_\eta(r)$ give rise to two foliations which are perpendicular to the same flow by geodesics. Hence the functions b^η and β^θ have the same gradients and since they are C^1 they differ by a constant. This proves item (1).

Item (2) follows from applying item (1) to $\gamma_{-\theta}$. □

A natural question arises from the previous Lemma: Is there any connected set of bi-asymptotic geodesics containing two given bi-asymptotic geodesics? This would be in many respects an analogous to the flat strip theorem. What we know is the following result proved by Croke-Schroeder [8].

Theorem 2.10. *Let (M, g) be a compact analytic manifold without conjugate points. Then the set of closed geodesics in a given nontrivial homotopy class is a connected, rectifiable set (each pair of points can be joined by a rectifiable curve in the set) of closed geodesics in the same homotopy class.*

3. The strip issue for manifolds without conjugate points

If we drop any assumption on the sectional curvatures or Jacobi fields, or even the analytic hypothesis considered by Croke and Schroeder, we can show the following result about the topology of the set of bi-asymptotic geodesics that is new in the theory and interesting in itself.

Lemma 3.1. *Let (M, g) be a compact C^∞ manifold without conjugate points such that $(\tilde{M}, \tilde{g})$ is (K, C)-quasi-convex where geodesic rays diverge. Then given $\theta = (p, v) \in T_1\tilde{M}$, and a geodesic $\beta = \gamma_\eta$ (with $\eta \in T_1\tilde{M}$) bi-asymptotic to $\gamma = \gamma_\theta$, there exists a connected set $\Sigma(\gamma, \beta) \subset H_\theta(0) \cap H_{-\theta}(0)$ containing p and $\beta \cap H_\theta(0)$, such that for every $x \in \Sigma(\gamma, \beta)$, the geodesic with initial conditions $(x, -\nabla_x b^\theta)$ is bi-asymptotic to both of them. In particular, the set*

$$S(\gamma, \beta) = \bigcup_{x \in \Sigma(\gamma,\beta),\, t \in \mathbb{R}} \gamma_{(x, -\nabla_x b^\theta)}(t)$$

is homeomorphic to $\Sigma(\gamma, \beta) \times \mathbb{R}$.

Before proving Lemma 3.1, let us demonstrate the following elementary lemma.

Lemma 3.2. *Let (M, g) be a compact C^∞ manifold without conjugate points and $\theta = (p, v) \in T_1\tilde{M}$ be fixed. Then for every $x \in \tilde{M}$, any $u \in \mathbb{R}$ such that*

$$d(x, \gamma) := \inf_{t \in \mathbb{R}} \{d(x, \gamma(t))\} = d(x, \gamma(u))$$

satisfies

$$\left| u + b^\theta(x) \right| \leq d(x, \gamma).$$

Proof. Let $x \in \tilde{M}$ and $u \in \mathbb{R}$ be such that $d(x,\gamma) = d(x,\gamma(u))$, for every $t > 0$ the triangle inequality yields

$$d\left(x,\gamma(t)\right) \le d\left(x,\gamma(u)\right) + d\left(\gamma(u),\gamma(t)\right) = d(x,\gamma) + |t-u|$$

and

$$|t-u| = d\left(\gamma(u),\gamma(t)\right) \le d\left(\gamma(u),x\right) + d\left(x,\gamma(t)\right) = d(x,\gamma) + d\left(x,\gamma(t)\right).$$

We conclude easily by letting t tend to $+\infty$ and using the definition of $b^\theta(x)$. $\square$

Proof of Lemma 3.1. We construct the set $\Sigma(\gamma,\beta)$ by hand. For every $t \in \mathbb{R}$, let $c_t : [0,1] \longrightarrow \tilde{M}$ be the geodesic with $c_t(0) = \gamma_\theta(t)$, $c_t(1) = \beta(t)$. For every positive integer n and every $s \in [0,1]$, we consider the geodesic segment $\alpha_n^s : [0,1] \to \tilde{M}$ joining $c_{-n}(s)$ to $c_n(s)$. Since geodesic rays diverge in $\tilde{M}$, by Lemma 2.9, there is $c \in \mathbb{R}$ such that $b^\eta = b^\theta + c$. Thus, for every integer n if $v, w \in \mathbb{R}$ satisfy

$$d\left(\beta(n),\gamma\right) = d\left(\beta(n),\gamma(v)\right) \quad \text{and} \quad d\left(\beta(-n),\gamma\right) = d\left(\beta(-n),\gamma(w)\right)$$

then by Lemma 3.2 there holds

$$|v-n-c| = |v + b^\eta(\beta(n)) - c| = \left|v + b^\theta(\beta(n))\right| \le d\left(\beta(n),\gamma\right) \le d_H\left(\gamma,\beta\right)$$

and

$$|v+n-c| = |v + b^\eta(\beta(-n)) - c| = \left|v + b^\theta(\beta(-n))\right| \le d\left(\beta(-n),\gamma\right) \le d_H\left(\gamma,\beta\right),$$

which shows that

$$\begin{aligned} d\left(\beta(n),\gamma(n)\right) &\le d\left(\beta(n),\gamma(v)\right) + d\left(\gamma(v),\gamma(n)\right) \\ &\le d\left(\beta(n),\gamma\right) + |v-n| \\ &\le 2d_H\left(\gamma,\beta\right) + |c|, \end{aligned}$$

and in the same way that $d\left(\beta(-n),\gamma(-n)\right) \le 2d_H\left(\gamma,\beta\right) + |c|$. Let $D := d_H(\gamma,\beta)$ and fix an integer $n > 2D + |c|$ and $s \in [0,1]$, we have

$$d\left(\alpha_n^s(1),\gamma(n)\right) \le d\left(\beta(n),\gamma(n)\right) \le 2d_H\left(\gamma,\beta\right) + |c|$$

and

$$d\left(\alpha_n^s(0),\gamma(-n)\right) \le d\left(\beta(-n),\gamma(-n)\right) \le 2d_H\left(\gamma,\beta\right) + |c|.$$

So that for every $t > 0$

$$\begin{aligned} d\left(\alpha_n^s(0),\gamma(t)\right) &\ge d\left(\gamma(t),\gamma(-n)\right) - d\left(\alpha_n^s(0),\gamma(-n)\right) \\ &\ge t + n - 2D - |c| > 0 \end{aligned}$$

and

$$\begin{aligned} d\left(\alpha_n^s(1),\gamma(t)\right) &\le d\left(\gamma(t),\gamma(n)\right) + d\left(\alpha_n^s(1),\gamma(n)\right) \\ &\le t - n + 2D + |c| < 0. \end{aligned}$$

Taking the limit as t tends to $+\infty$, we infer that $b^\theta(\alpha_n^s(0)) > 0$ and $b^\theta(\alpha_n^s(1)) < 0$. As a consequence, there is $r \in (0,1)$ such that $\alpha_n^s(r)$ belongs to $H_\theta(0)$. By Lemma 3.2, there is $u \in \mathbb{R}$ such that

$$d\left(\alpha_n^s(r),\gamma\right) = d\left(\alpha_n^s(r),\gamma(u)\right) \quad \text{and} \quad |u| \le d\left(\alpha_n^s(r),\gamma\right),$$

which by quasi-convexity together with the above inequalities gives

$$\begin{aligned} |u| \le d_H\left(\alpha_n^s,\gamma\right) &\le K \max\left\{d\left(c_{-n}(s),\gamma(-n)\right), d\left(c_n(s),\gamma(n)\right)\right\} + C \\ &\le K \max\left\{d\left(\beta(-n),\gamma(-n)\right), d\left(\beta(n),\gamma(n)\right)\right\} + C \\ &\le 2KD + K|c| + C, \end{aligned}$$

and in turn

$$\begin{aligned} d\left(\alpha_n^s(r),p\right) &\leq d\left(\alpha_n^s(r),\gamma(u)\right)+d\left(\gamma(u),\gamma(0)\right)\\ &= d\left(\alpha_n^s(r),\gamma\right)+|u|\\ &\leq 4KD+2K|c|+2C=:\tau. \end{aligned}$$

By the divergence of rays in $\tilde{M}$, the geodesics $[c_{-n}(s),c_n(s)]$ tend to be orthogonal to $H_\theta(0)$ at their points of intersection. So for n large, there is a unique $r_n^s\in(0,1)$ such that $\alpha_n^s(r_n^s)\in H_\theta(0)$ and the mapping

$$\Gamma_n\,:\,s\in[0,1]\longmapsto\alpha_n^s(r_n^s)\in H_\theta(0)\cap B_\tau(p)$$

is continuous (here $B_\tau(p)$ stands for the closed ball centered at p with radius τ). Let $\Sigma(\gamma,\beta)$ be the set of $q\in\tilde{M}$ for which there exists a sequence $\{n_k\}_k$ of positive integers tending to infinity such that

$$q=\lim_{k\to\infty}\Gamma_{n_k}\left(s_{n_k}\right).$$

By construction, $\Sigma(\gamma,\beta)$ is a closed subset of $H_\theta(0)$ contained in $B_\tau(p)$, which contains $p=\gamma(0)$ and $q:=\beta\cap H_\theta(0)$. We claim that $\Sigma(\gamma,\beta)$ is connected. As a matter of fact, if there are two disjoint open subsets A_1,A_2 of $H_\theta(0)$ such that $\Sigma(\gamma,\beta)\subset A_1\cup A_2$ with $p\in A_1$, then all points of $\Sigma(\gamma,\beta)$ must belong to A_1 because otherwise there is a sequence of continuous curves in $H_\theta(0)\cap B_\tau(p)$, given by the restrictions of some Γ_n, which connects p to A_2 that gives rise, by compactness, to an accumulation point in $\Sigma(\gamma,\beta)$ outside $A_1\cup A_2$, a contradiction.

To finish the proof of the lemma, we recall that by quasi-convexity, for every integer n and every $s\in[0,1]$,

$$d_H\left(\gamma,\alpha_n^s\right)\leq 2KD+K|c|+C,$$

so that any convergent subsequence of the points $\alpha_{n_k}(s_{n_k})\to q$ gives rise to a geodesic $\sigma^q(t)$ satisfying $\dot{\sigma}^q(0)=-\nabla_q b^\theta$ and

$$d_H(\gamma_\theta,\sigma^q)\leq K d_H(\gamma_\theta,\beta)+C,$$

meaning that σ^q is bi-asymptotic to γ_θ. □

Corollary 3.3. *Let (M,g) be a compact manifold without conjugate points such that $(\tilde{M},\tilde{g})$ is quasi-convex. Then given a geodesic $\gamma_\theta\subset\tilde{M}$ the set $S(\gamma_\theta)$ of geodesics which are bi-asymptotic to γ_θ is homeomorphic to a product $\Sigma(\gamma_\theta)\times\mathbb{R}$ where $\Sigma(\gamma_\theta)\subset H_\theta(0)\cap H_{-\theta}(0)$ is a connected set.*

Proof. This is straightforward from Lemma 3.1. □

Some remarks about Corollary 3.3. The set $S(\gamma)$ is a natural candidate to be "the strip" of γ. However, although it looks topologically as a usual strip (namely, a strip in nonpositive curvature), its topology and geometry might be very complicated.

4. Stability and hyperbolicity of the set of closed orbits

In this section we remind some of the main steps of the proof of the stability conjecture for diffeomorphisms that can be extended to geodesic flows, notably after a recent version of the Franks' Lemma for the so-called Mañé perturbations of a Riemannian metric.

We start by recalling some basic definitions concerning hyperbolic dynamics.

Definition 4.1. Let $\psi_t : N \longrightarrow N$ be a smooth flow without singularities acting on a complete C^∞ Riemannian manifold. An invariant set $Y \subset N$ is called hyperbolic if there exists $C > 0$, $r > 0$, and for every $p \in Y$ there exist subspaces $E^s(p)$, $E^u(p)$ such that

(1) $E^s(p) \oplus E^u(p) \oplus X(p) = T_pN$ where $X(p)$ is the subspace tangent to the flow.
(2) $\| d_p\psi_t(v) \| \leq Ce^{-rt} \| v \|$ for every $t \geq 0$ and $v \in E^s(p)$.
(3) $\| d_p\psi_t(v) \| \leq Ce^{rt} \| v \|$ for every $t \leq 0$ and $v \in E^u(p)$.

The subspace $E^s(p)$ is called stable susbpace, the subspace $E^u(p)$ is called the unstable subspace. When $Y = N$ the flow ψ_t is called Anosov. Replacing ψ_t by a diffeomorphism we get what is called an Anosov diffeomorphism.

The theory of hyperbolic sets of flows and diffeomorphisms is very rich, one of the main features of the dynamics is the existence of invariant submanifolds $\mathbf{W^s(p)}$, $\mathbf{W^u(p)}$ for every p in the hyperbolic set where asymptotic properties of orbits are counterparts of asymptotic properties of the differential of the system acting on stable and unstable subspaces (see for instance [16], [1]). The submanifold $\mathbf{W^s(p)}$ is always tangent to the bundle E^s, the submanifold $\mathbf{W^u(p)}$ is always tangent to the bundle E^u.

As we mentioned before, the invariant submanifols $\mathbf{W^s}(\theta)$, $\mathbf{W^u}(\theta)$ coincide with the sets $W^s(\theta)$, $W^u(\theta)$ locally when the geodesic flow of a manifold without conjugate points is Anosov. Actually, if the geodesic flow of a compact Riemannian manifold is Anosov then the manifold has no conjugate points by a celebrated theorem due to Klingenberg [17] (see also a nice generalization by R. Mañé [21]).

Systems with hyperbolic invariant sets are closely related to the theory of stable systems.

Definition 4.2. A smooth flow $\psi_t : N \longrightarrow N$ acting on a smooth manifold is C^k structurally stable if there exists $\epsilon > 0$ such that every flow ρ_t in the ϵ-neighborhood of ψ_t in the C^k topology is conjugate to ψ_t. Namely, there exists a homeomorphism $h_\rho : N \longrightarrow N$ such that

$$h(\psi_t(p)) = \rho_{s_p(t)}(h(p))$$

for every $t \in \mathbb{R}$, where $s_p(t)$ is a continuous injective function with $s_p(0) = 0$.

A series of results in the 60's, 70's and 80's characterize C^1 structurally stable systems (mainly [28], [19], [20]).

Theorem 4.3. *A diffeomorphism acting on a compact manifold is C^1 structurally stable if and only if it is Axiom A, namely, the closure of the set of periodic orbits is a hyperbolic set and the intersections of stable and unstable submanifolds are always transverse.*

This result characterizes as well C^1 structurally stable flows without singularities on compact manifolds. Newhouse [24] shows that a symplectic diffeomorphisms acting on a compact manifold is C^1 structurally stable if and only if it is Anosov. The extension of his proof to Hamiltonian flows acting on a level of energy is natural.

To give a context to our results we need to explain in some detail the main ideas of the proof of the so-called stability conjecture: C^1 structurally stable diffeomorphims are Axiom A and invariant submanifolds meet transversally, a result due to Mañé [19].

One of the main steps of the proof is that the C^1 structural stability implies that the closure of the set of periodic orbits is a hyperbolic invariant set. The key tool to prove this statement is the so-called Franks' lemma, we shall give an improved recent version of it for geodesic flows [18]. Then, it is natural to expect that under this condition, the set of nonwandering points, the set where the dynamics is nontrivial, is exactly the closure of the set of periodic orbits. To show this the essential tool is the C^1 closing lemma proved by Pugh [26], that is not available for geodesic flows up to date and is actually a very difficult problem in the theory of geodesic flows as we already mentioned in the Introduction (see [27]).

The step concerning the hyperbolicity of periodic orbits under stability assumptions has been extended and improved for geodesic flows in the context of the so-called *Mañé perturbations*. Recall that a C^∞ Hamiltonian $H : T^*M \longrightarrow \mathbb{R}$ defined in the cotangent bundle of M is called *Tonelli* if H is strictly convex and superlinear in each tangent space $T_\theta T^*M$, $\theta \in T^*M$.

Definition 4.4. A property P of the Hamiltonian flow of a Tonelli Hamiltonian $H : T^*M \longrightarrow \mathbb{R}$ is called C^k generic from Mañé's viewpoint if given $\epsilon > 0$ there exists a C^∞ function $f : M \longrightarrow \mathbb{R}$ whose C^k norm is less than ϵ such that the Hamiltonian flow of $H_f(q,p) = H(q,p)+f(q)$ has the property P. The Hamiltonian H_f is called a C^k Mañé perturbation of the Hamiltonian H.

By the Maupertuis principle of classical mechanics. given a Riemannian metric (M,g) every small C^k Mañé perturbation H_f of the Hamiltonian $H(q,p) = \frac{1}{2}g_q(p,p)$ defines the Riemannian Hamiltonian of a metric g_f that is conformal to g.

Definition 4.5. The geodesic flow ϕ_t of a compact Riemannian manifold (M,g) is C^k structural stable from Mañé's view point if there exists a C^k open neighborhood of g such that for each metric in the neighborhood the geodesic flow is conjugate to ϕ_t.

This notion of structural stability from Mañé's viewpoint is stronger than the usual one, since it requires persistence dynamics in a neighborhood of special type of perturbations of the metric, not all perturbations of a metric are Mañé type perturbations. As we commented above, perturbations of g which are not conformal to g do not belong to the family of Mañé perturbations.

Applying techniques of control theory we obtain an extension of the hyperbolicity of the set of periodic orbits just considering Mañé perturbations.

Theorem 4.6. *(see* [18]*) Let (M,g) be a compact manifold whose geodesic flow is C^2-structurally stable from Mañé's viewpoint. Then the closure of the set of periodic orbits is a hyperbolic set for the geodesic flow.*

As for the closing lemma, we have to rely on other kind of assumptions on the manifold to try to localize the set of periodic orbits. The natural domains to look for these assumptions are topology and global geometry.

5. On the density of periodic orbits for manifolds with Gromov hyperbolic fundamental groups

Let us recall that a metric space (X,d) is called **geodesic** if every pair of points xy can be joined by an isometric continuous embedding of an interval $c : [a,b] \longrightarrow X$

with $c(a) = x$, $c(b) = y$. The curve c will be called a geodesic of the metric space, it corresponds to minimizing geodesics if (X, d) is a Riemannian manifold. Let us denote by $[x, y]$ a minimizing geodesic joining x to y (there might be many).

Definition 5.1. Given $\delta > 0$ a complete geodesic metric space (X, d) is called δ-hyperbolic or Gromov hyperbolic if for every geodesic triangle $[x_0, x_1] \cup [x_1, x_2] \cup [x_2, x_0]$ we have that the distance from every point $p \in [x_i, x_{i+1}]$ to $[x_{i+1}, x_{i+2}] \cup [x_{i+2}, x_i]$ is bounded above by δ for every $i = 0, 1, 2$ (the indices are taken mod 3).

Definition 5.2. A complete Riemannian manifold (N, g) without conjugate points is called a visibility manifold if given $p \in N$ and $\epsilon > 0$ there exists $T > 0$ such that for every pair of points x, y in N, whenever the distance from p to the geodesic $[x, y]$ is larger than T, then the angle subtended by the geodesics $[p, x]$ and $[p, y]$ at p is less than ϵ. If T does not depend on p the manifold (N, g) is called a uniform visibility manifold.

There is a natural link between visibility manifolds and Gromov hyperbolic spaces [29], [32].

Theorem 5.3. *Let (M, g) be a compact manifold without conjugate points. The universal covering is a visibility manifold if and only if geodesic rays diverge and the fundamental group is Gromov hyperbolic.*

So the theory of Gromov hyperbolic spaces applies to visibility coverings of compact manifolds without conjugate points. Let us mention some of these results concerning the dynamics of the geodesic flow for the purposes of this article.

The following statement proved by Eberlein [9] for visibility manifolds and by Gromov [12] for Gromov hyperbolic spaces tells us that periodic orbits are "dense in the large".

Theorem 5.4. *Let (M, g) be a compact manifold without conjugate points whose universal covering is a visibility manifold. Then the action of the fundamental group at the ideal boundary of $\tilde{M}$ is dense in the cone topology.*

Since the notions of cone topology, ideal boundary and extended action of the fundamental group won't be needed in the article we refer the reader for details to the references above.

Theorem 5.5. *(see* [9]*)) Let (M, g) be a compact manifold without conjugate points such that $(\tilde{M}, \tilde{g})$ is a visibility manifold. Then the geodesic flow is topologically transitive, and given any geodesic $\gamma \in \tilde{M}$ there exists a geodesic $\beta \in \tilde{M}$ that is bi-asymptotic to γ and is in the closure of the set of axes.*

Theorem 5.6. *(see* [3]*) Let (M, g) be a compact manifold with nonpositive curvature and Gromov hyperbolic fundamental group, then the set of periodic orbits of the geodesic flow is dense in T_1M.*

An **axis** is any lift of a closed geodesic of (M, g) in the universal covering. Each axis is of course preserved by the action of an infinite cyclic subgroup generated by a covering isometry. Actually, Theorem 5.6 holds under a dynamical assumption introduced by Eberlein in [9] called duality, that is satisfied by Gromov hyperbolic fundamental groups. We refer the reader to the references above for details.

6. The stability problem in low dimensions: the proof of Theorem 1.1 for surfaces

The "easy" part of the proof of Theorem 1.1 is the converse, namely, if the geodesic flow is Anosov then the geodesic flow is C^1-structurally stable (which is precisely Anosov's work). In particular, the geodesic flow is C^2 structurally stable from Mañé's viewpoint since the geodesic flows of small C^2 neighborhoods of conformal metrics are contained in C^1 small neighborhoods the geodesic flow of (M,g).

Let us start the proof of the direct part of Theorem 1.1 with the case of surfaces. The main steps of the proof in the two dimensional case are a sort of paradigm of what we would like to do in general dimensions.

Let (M,g) be a compact surface without conjugate points whose geodesic flow is C^2 structurally stable from Mañé's viewpoint. The surface cannot be a sphere of course.

By Theorem 4.6, every closed orbit is a hyperbolic orbit of the geodesic flow. This implies that the surface has genus greater than one, since a torus without conjugate points is flat by the work of Hopf [15]. So the fundamental group of (M,g) is Gromov hyperbolic and by the work of Green [13] we know that geodesic rays diverge in $(\tilde{M},\tilde{g})$ according to Definition 2.2. Therefore, $(\tilde{M},\tilde{g})$ is a visibility manifold and by Theorem 5.5 every asymptotic class of geodesics in $(\tilde{M},\tilde{g})$ is in the closure of the set of axes, the lifts of closed geodesics in (M,g).

Let $\gamma_\theta \subset \tilde{M}$ be a geodesic having a bi-asymptotic geodesic γ_η in the closure of axes. By the work of Morse [23], the geodesics γ_θ and γ_η bound a strip in $\tilde{M}$ that is foliated by geodesics, all of them obviously bi-asymptotic to γ_θ. By Lemma 2.9, the intersection $H_\theta(0)\cap H_{-\theta}(0) = \hat{I}(\theta)$ consists of a connected compact curve with boundary, and we can suppose without loss of generality that this curve contains $\gamma_\eta(0)$ (which is true up to an affine reparametrization of this geodesic).

By definition, this implies that we have a curve $I(\theta)$ homeomorphic to an interval in the intersection of $\tilde{W}^s(\theta)\cap\tilde{W}^u(\theta)$, this curve contains the points θ and η in $T_1\tilde{M}$.

By Theorem 4.6, the closure of the set of orbits corresponding to axes is a hyperbolic set. So its dynamical local stable and unstable submanifolds coincide with, respectively, $\tilde{W}^s(\eta)$, $\tilde{W}^u(\eta)$. This means that these sets meet at η transversally. Since $\tilde{W}^s(\theta)=\tilde{W}^s(\eta)$, $\tilde{W}^u(\theta)=\tilde{W}^u(\eta)$, the curve $I(\theta)$ must be a single point.

Thus, the orbit of θ itself in the closure of the set of orbits corresponding to axes, and since this holds for every $\theta \in T_1\tilde{M}$ we conclude that the closure of the set of closed orbits is T_1M. Since this set is a hyperbolic set by Theorem 4.6, the geodesic flow of (M,g) is Anosov as claimed in Theorem 1.1.

7. The 3-dimensional case

In higher dimensions the divergence of geodesic rays in the universal covering of manifolds without conjugate points and no restrictions in the curvature or Jacobi fields is an open problem, this is the reason why we make this assumption (it is a reasonable hypothesis in the context of manifolds without conjugate points).

7.1. **Hyperbolicity and the fundamental group.** The roles of topology and hyperbolic global geometry of the manifold were crucial in the proof of Theorem 1.1 for surfaces. For higher dimensions we start with the following result linking hyperbolic closed geodesics and the fundamental group:

Lemma 7.1. *Let (M,g) be a compact C^∞ manifold without conjugate points. Assume that either*

(1) *(M,g) is analytic.*
(2) *Or $(\tilde{M},\tilde{g})$ is quasi-convex and geodesic rays diverge.*

Then, if every closed orbit is hyperbolic the fundamental group is a Preissmann group, namely, every nontrivial abelian subgroup is infinite cyclic.

Proof. If the manifold is analytic by the result due to Croke and Schroeder we conclude that the existence of a rank two abelian subgroup of the fundamental group implies that there are connected sets foliated by homotopic closed geodesics containing more than one geodesic, all of them with the same period. Since by assumption, every closed orbit is hyperbolic, such a set cannot exist.

If the manifold is not analytic, the assumption of item (2) implies the existence of a connected set of bi-asymptotic geodesics to a given axis γ_θ. This set contains at least two different axes (Corollary 3.3). By Lemma 2.9 this gives rise to a connected set containing θ in the intersection of local stable and unstable sets of θ, which is impossible by the hyperbolicity assumption of the orbit of θ. □

The Preissmann property is satisfied by Gromov hyperbolic groups, but it is not enough to characterize such groups. In the context of manifolds without conjugate points without restrictions on the dimension what it is known is the following.

Theorem 7.2. [5] *Let (M,g) be a compact analytic manifold with nonpositive curvature. Then $(\tilde{M},\tilde{g})$ is a visibility manifold if and only if there is no flat totally geodesic immersed torus in (M,g).*

In particular, the fundamental group of M is Gromov hyperbolic if and only if it satisfies the Preissmann property. Indeed, if the fundamental group does not satisfies the Preissmann property it exists a subgroup isomorphic to $\mathbb{Z}\times\mathbb{Z}$. Then, the global geometry of manifolds with nonpositive curvature yields the existence of a flat totally geodesic torus in (M,g) (see [4] for instance).

7.2. **Three dimensional manifolds without conjugate points and Thurston's geometrization.** The Preissmann property is actually a sufficient condition to characterize Gromov hyperbolic fundamental groups of compact 3-manifolds without conjugate points. The main result of the subsection is proved in [32], Chapter 8.

Theorem 7.3. *Let (M,g) be a compact C^∞ Riemannian manifold without conjugate points with dimension 3. Suppose that the fundamental group satisfies the Preissmann property. Then (M,g) admits a metric of constant negative curvature.*

The proof of Theorem 7.3 relies on the geometry of the fundamental group of a compact manifold without conjugate points and Thurston's geometrization work for 3-dimensional manifolds. We shall explain briefly the main steps of the proof for the sake of completeness. For the references and main fundamental results about 3-dimensional topology we refer to [14].

Let M be a C^∞ compact 3-manifold (closed without boundary). According to the work of J. Milnor [22] M is the connected sum of a finite collection of 3-manifolds. So M admits a *prime* decomposition as connected sum of 3-manifolds M_i, in the sense that if M_i is the connected sum of two 3-manifolds N^i_1, N^i_2 then either N^i_1 or N^i_2 is homeomorphic to a 3-sphere, in this case we say that the connected sum of

N_1^i and N_2^i is trivial. A prime decomposition of M is unique up to homeomorphism. We say that M is *prime* if every connected sum decomposition of M is trivial.

If every submanifold of M homeomorphic to a 2-sphere bounds a 3-open ball, then M is prime. Thus, a manifold whose universal covering is homeomorphic to $\mathbb{R}^3$ is prime, in particular Riemannian compact 3-manifolds without conjugate points (the exponential map is a covering map). Thurston's work about the classification of 3-manifolds asserts that every compact prime 3-manifold has to be modeled (namely, is homeomorphic to the quotient of a model space by a discrete subgroup of isometries) by one of the following 8 geometries:

1. The round sphere S^3 of curvature 1,
2. $\mathbb{R}^3$ endowed with the Euclidean metric,
3. The hyperbolic space $\mathbb{H}^3$,
4. $S^2 \times \mathbb{R}$ endowed with the product of the round metric and the Euclidean metric in the line,
5. $\mathbb{H}^2 \times \mathbb{R}$ endowed with the product of the 2-dimensional hyperbolic metric and the Euclidean metric in the line,
6. $SL(2,\mathbb{R})$ endowed with its natural left invariant metric,
7. The Heisenberg group Nil with its left invariant metric,
8. The group Sol endowed with its left invariant metric.

Thurston's geometrization result is equivalent to the Poincaré conjecture, solved by Perelman. So let (M,g) be a compact Riemannian manifold without conjugate points and Preissmann fundamental group. It has to fit in one of the above geometrical models, and therefore we can discard trivially some of them like S^3; $\mathbb{R}^3$ because a co-compact subgroup of isometries of Euclidean space has always a finite index abelian subgroup of rank 3; $S^2 \times \mathbb{R}$ because a manifold without conjugate points is covered by $\mathbb{R}^3$ (endowed with the pullback of g by the exponential map at some point of course); $\mathbb{H}^2 \times \mathbb{R}$ because co-compact subgroups of isometries have always abelian subgroups of rank 2.

The geometry of Nil can be discarded because of a well known result due to Croke and Schroeder [8]. The geometry of Sol can also be discarded after a famous result by P. Scott [33]: the fundamental group of a compact manifold covered by Sol has always a subgroup isomorphic to $\mathbb{Z}\times\mathbb{Z}$. Compact manifolds covered by the universal covering of $SL(2,\mathbb{R})$ are circle bundles or Seifert bundles, which have the property that the subgroup generated by the fiber of the bundle is normal in the fundamental group. In [32], Proposition 8.5 it is showed that if the fundamental group of a compact manifold without conjugate points has a normal infinite cyclic subgroup then the universal covering of the manifold is quasi-isometric to $S \times \mathbb{R}$ where S is a complete surface. This yields that the universal covering of $SL(2,\mathbb{R})$ cannot cover a compact manifold without conjugate points (otherwise it would be quasi-isometric to a product manifold). So the only possibility in the above list is hyperbolic 3-dimensional space, proving Theorem 7.3.

7.3. **Proof of Theorem 1.1 for 3-manifolds.** The previous subsection yields at once that

Corollary 7.4. *Let (M,g) be a compact Riemannian manifold without conjugate points whose fundamental group has the Preissmann property. Then the fundamental group is Gromov hyperbolic.*

Now we can prove Theorem 1.1:

Let (M, g) satisfy the assumptions of Theorem 1.1. $(\tilde{M}, \tilde{g})$ is quasi-convex and geodesic rays diverge. By the structural stability from Mañé's viewpoint, the closure of the set of periodic orbits is hyperbolic (Theorem 4.6). Then, every closed orbit is hyperbolic and by Lemma 7.1 and Corollary 7.4 the fundamental group is Gromov hyperbolic. By Theorem 5.3 the universal covering is a visibility manifold. Then combining the assumptions of Theorem 1.1 and Theorem 5.5 we conclude that in each bi-asymptotic class of geodesics in $\tilde{M}$ there exists one that is hyperbolic and hence, Corollary 3.3 allows us to conclude that each bi-asymptotic class of geodesics in $\tilde{M}$ must consist of just one geodesic. Therefore, every geodesic is in the closure of the set of hyperbolic geodesics, and the closure of the set of hyperbolic periodic orbits is T_1M. Since this set is hyperbolic, we conclude that the geodesic flow is Anosov.

8. The stability problem in higher dimensions

Now we proceed to prove Theorem 1.2. Let us resume what we already know about the statement of the theorem.

Let us start with the case of manifolds with nonpositive curvature. According to Theorem 4.6, C^2 structural stability from Mañé's viewpoint of the geodesic flow implies that periodic orbits are hyperbolic. The flat strip Theorem then yields that a compact manifold with nonpositive curvature whose geodesic flow is C^2 structurally stable from Mañé's viewpoint is a rank one manifold. By Theorem 5.6, periodic orbits of rank one manifolds are dense in the unit tangent bundle and therefore, Theorem 4.6 yields that the geodesic flow is Anosov.

The statement extends to compact manifolds without focal points with Gromov hyperbolic fundamental groups. Indeed, Theorem 5.5 yields that every geodesic γ in $\tilde{M}$ is bi-asymptotic to a geodesic β that is in the closure of the set of axes. By Theorem 4.6, this set is a hyperbolic set of geodesics. By the flat strip Theorem for manifolds without focal points, every pair of different bi-asymptotic geodesics is contained in a nontrivial flat strip. It is clear that a nontrivial flat strip cannot contain a hyperbolic geodesic. So γ and β coincide and hence, the closure of the set of axes is the whole collection of geodesics. This yields at once that the geodesic flow is Anosov.

The assumptions of Theorem 1.2 do not allow to apply directly the above results. We have no information about the fundamental group of the manifold. Moreover, there is no restriction a priori on the sectional curvatures of a generalized rank one manifold. The convexity of the geometry of nonpositive curvature manifolds, that plays an important role in the proof of Theorem 5.5, is our case is replaced by quasi-convexity. Nevertheless, as in all previous cases we shall show that

Proposition 8.1. *Under the assumptions of Theorem 1.2 the set of periodic orbits is dense in* T_1M.

To start the proof let us recall what we know about the geometry of asymptotic geodesics (see [32] for instance).

Lemma 8.2. *Let* (M, g) *be a compact manifold without conjugate points such that* $(\tilde{M}, \tilde{g})$ *is* K, C *quasi-convex. Then for every* $\theta \in T_1\tilde{M}$, $\theta = (p, v)$, *and every* $(q, w) \in H_\theta(0)$, *we have*

$$d(\gamma_\theta(t), \gamma_{(q,w)}(t)) \leq K d(p,q) + C$$

for every $t \geq 0$.

Next, we know that the set of recurrent orbits has total Liouville measure in T_1M. Since the set of points of T_1M where $W^s_{loc}(\theta) \cap W^u_{loc}(\theta) = \{\theta\}$ is open and dense by the assumption of Theorem 1.2, the set of recurrent points with this property is dense as well. So let η be such a point, let Σ_η be a local cross section of the geodesic flow containing η, and let

$$\mathfrak{W}^s(\sigma) = W^s_{loc}(\sigma) \cap \Sigma_\eta,$$
$$\mathfrak{W}^u(\sigma) = W^u_{loc}(\sigma) \cap \Sigma_\eta$$

for every $\sigma \in \Sigma_\eta$. By the definition of generalized rank one, we can shrink if necessary the section Σ_η in order to have that $\mathfrak{W}^s(\sigma)$, $\mathfrak{W}^u(\sigma)$ are both homeomorphic to $(n-1)$ dimensional open ball. Since η is recurrent (meaning forward and backward recurrent) there exists a sequence $t_n \to +\infty$ such that $\phi_{t_n}(\eta) \in \Sigma_\eta$ and converges to η. Let $P_n : \Sigma_\eta \longrightarrow \Sigma_\eta$ be the Poincaré return map of the secton Σ_η. By the above definitions we have that

$$P_k(\mathfrak{W}^s(\phi_{t_n}(\eta))) \subset \mathfrak{W}^s(\phi_{t_{n+k}}(\eta)))$$

for every $n, k > 0$.

Lemma 8.3. *There exists* $n_0 > 0$ *such that the closure of* $P_n(\mathfrak{W}^s(\eta))$ *is strictly contained in* $\mathfrak{W}^s(\phi_{t_n}(\eta)))$ *for every* $n > n_0$.

Proof. Indeed, otherwise for each $m > 0$ there exists $n_m \geq m$ and a point $\tau \in \mathfrak{W}^s(\eta)$ such that $P_{n_m}(\tau)$ is outside $\mathfrak{W}^s(\eta)$. Let $a > 0$ be the distance from η to the boundary of $\mathfrak{W}^s(\eta)$. Let us consider a lift $\gamma_{\bar{\eta}}$ of γ_η in $\tilde{M}$ and a lift $\gamma_{\bar{\tau}}$ of γ_τ in $\tilde{M}$. By Lemma 8.2 there exists constants $A, \bar{a}$ depending on K, C, a and η, τ such that

(1) $d(\gamma_{\bar{\eta}}(t), \gamma_{\bar{\tau}}(t)) \leq A$ for every $t \geq 0$,
(2) $d(\gamma_{\bar{\tau}}(t_{n_m}), \gamma_{\bar{\eta}}(t_{n_m})) \geq \bar{a}$ for every n_m where $\bar{\eta} = (q, w)$.

Since η is recurrent, there exists a sequence of covering isometries $T_m : \tilde{M} \longrightarrow \tilde{M}$ such that

(1) The pairs $(T_m(\gamma_{\bar{\eta}}(t_{n_m})), dT_m(\gamma'_{\bar{\eta}}(t_{n_m}))$ converge to $\bar{\eta}$.
(2) The sequence $T_m(\gamma_{\bar{\tau}}(t_m))$ is contained in a compact ball centered at p where $\bar{\eta} = (p, v)$,
(3) A convergent subsequence of the pairs $(T_m(\gamma_{\bar{\eta}}(t_{n_m})), dT_m(\gamma'_{\bar{\eta}}(t_{n_m}))$ gives rise to a geodesic β that is bi-asymptotic to $\gamma_{\bar{\eta}}$.
(4) $d(\beta(0), \gamma_{\bar{\eta}}) \geq \bar{a}$.

Thus, we get a geodesic β different from $\gamma_{\bar{\eta}}$ that is bi-asymptotic to $\gamma_{\bar{\eta}}$. But by Corollary 3.3 this would generate a connected subset in the intersection $W^s_{loc}(\eta) \cap W^u_{loc}(\eta)$ containing more than one point, which is impossible by the generalized rank one definition and the choice of η. This finishes the proof of the Claim. □

Lemma 8.4. *Let* (M, g) *be a generalized rank one manifold, let* $\eta \in T_1M$ *be a recurrent generalized rank one point for the geodesic flow, and let* Σ_η *be a local cross section containing* η. *Then there exists an open set* $U(\eta)$ *containing* η *such that for every* $\sigma, \tau \in U(\theta)$ *we have*

$$\mathfrak{W}^s(\sigma) \cap \mathfrak{W}^u(\tau)$$

is nonempty and consists of a single point.

Proof. By Theorem 2.6 the sets $W^s(\sigma)$, $W^u(\sigma)$ for $\sigma \in T_1M$ form continuous foliations $\mathcal{F}^s$ $\mathcal{F}^u$ by $n-1$-dimensional Lipschitz leaves. The central foliations of the geodesic flow $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$ are the saturates of $\mathcal{F}^s$, $\mathcal{F}^u$ respectively. Namely, let

$$W^{cs}(\sigma) = \cup_{t\in\mathbb{R}}\{\phi_t(W^s(\sigma))\}$$

$$W^{cu}(\sigma) = \cup_{t\in\mathbb{R}}\{\phi_t(W^u(\sigma))\}.$$

Both are continuous n-dimensional submanifolds and their collections form continuous foliations $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$ respectively of T_1M. The sets $W^{cs}(\sigma)$, $W^{cu}(\sigma)$ are locally graphs of the canonical projection: around every point θ of either $W^{cs}(\sigma)$ or $W^{cu}(\sigma)$ there exists an open set $\hat{U}(\theta)$ such that

$$\pi : \hat{U}(\theta) \longrightarrow \hat{W}^{cs}(\theta) \subset W^{cs}(\theta)$$
$$\pi : \hat{U}(\theta) \longrightarrow \hat{W}^{cu}(\theta) \subset W^{cu}(\theta)$$

are homeomorphisms.

Actually we can construct explicit parametrizations for these sets. For $\sigma = (p, v)$, $r > 0$ small, a local parametrization of $W^{cs}(\sigma)$ is given by

$$\psi_r^\sigma : B_r(p) \longrightarrow \hat{W}_r^{cs}(\sigma) \subset W^{cs}(\sigma)$$
$$\psi_r^\sigma(x) = (q, X^s(q))$$

where $X^s(q) = d\pi(-\nabla_{\bar{q}} b^{\bar{\sigma}})$, $\pi(\bar{q}) = q$, $\bar{\sigma}$ is a lift of σ in $T_1\tilde{M}$. An analogous parametrization can be given to local open subsets of $W^{cu}(\sigma)$, let us call $\hat{W}_r^{cu}(\sigma)$ the set arising form the above construction replacing the vector field X^s by the vector field $X^u = d\pi(\nabla b^{-\bar{\sigma}})$.

The sets $W^s(\sigma)$, $W^u(\sigma)$ can be parametrized similarly, by restricting the maps ψ_r^σ to sets of the form $B_r(p) \cap H_{(p,v)}(0)$, which are homeomorphic to open $n-1$ dimensional balls for r small. Let us denote by $\hat{W}_r^s(\sigma) = \psi_r^\sigma(B_r(p) \cap H_{(p,v)}(0))$, and $\hat{W}_r^u(\sigma)$ the set arising from the previous construction replacing X^s by X^u.

Let us consider the set

$$U_r^{su}(\eta) = \cup_{\sigma\in\hat{W}_r^{cs}(\eta)}\{\hat{W}_r^u(\sigma)\}.$$

By the definition of generalized rank one, each of the sets $\hat{W}_r^u(\sigma)$ meets $\hat{W}_r^{cs}(\eta)$ at just one point if σ is close enough to η.

Thus, for r small the set $U_r^{su}(\eta)$ is a continuous fibration over the set $\hat{W}_r^{cs}(\eta)$. The coordinates constructed above give us a homeomorphism from $U_r^{su}(\eta)$ and an open set in $\mathbb{R}^{2n-1}$, so $U_r^{su}(\eta)$ is an open subset of T_1M that is foliated by local unstable leaves.

Next, let us consider the set

$$U_r^{us}(\eta) = \cup_{\sigma\in\hat{W}_r^{u}(\eta)}\{\hat{W}_r^{cs}(\sigma)\}.$$

By an analogous reasoning, this set is an open neighborhood of η that is foliated by local center stable leaves. The set

$$U(\eta) = U_r^{su}(\eta) \cap U_r^{us}(\eta)$$

satisfies the statement of the lemma.

□

Proof of Theorem 1.2

Combining Lemmas 8.3 and 8.4 we get the following result:

Claim 1: Let $\eta \in T_1M$ be a recurrent generalized rank one point. There exist $n > n_0$ and $\sigma_0 \in \Sigma_\eta$ such that $P_n(W^u_{loc}(\sigma_0)) \subset W^u_{loc}(\sigma_0)$.

Indeed, to show the Claim let $\Pi_s : \Sigma_\eta \longrightarrow \mathfrak{W}^s(\eta)$ be the projection on $\mathfrak{W}^s(\eta)$ along the foliation defined by the sets $\mathfrak{W}^u(\sigma)$, namely, $\Pi_s(\sigma) = \mathfrak{W}^u(\sigma) \cap \mathfrak{W}^s(\eta)$.

By Lemma 8.4 the map Π_s is well defined in the neighborhood $U(\eta) \cap \Sigma_\eta$ of θ in Σ_η, and by Lemma 8.3 the map $\Pi_s \circ P_n$ is a contraction in the closure of $\mathfrak{W}^s(\eta)$. By the construction of the neighborhood $U^{su}_r(\eta)$ the set $W^{cs}_{loc}(\eta)$ is homeomorphic to an open n-dimensional ball, so we can assume without loss of generality that $\mathfrak{W}^s(\eta)$ is homeomorphic to an open $n-1$-dimensional ball. By Brower's fixed point theorem, there exists a fixed point of $\Pi_s \circ P_n$ which yields the Claim.

Claim 2: The set of periodic orbits accumulates η.

By inverting the orientation of the geodesic flow, we get by Lemma 8.3 and the same reasoning of the above Claim a fixed point σ_1 of $\Pi_u \circ P_{-m}$ for some $m > 0$ large enough, where Pi_u is the projection in $\mathfrak{W}^u(\eta)$ along the stable leaves. So

$$P_{-m}(W^s_{loc}(\sigma_1)) \subset W^s_{loc}(\sigma_1).$$

Since $\mathfrak{W}^u(\sigma_0) \cap \mathfrak{W}^s(\sigma_1)$ is nonempty in $U(\eta)$ and consists of a single point, we conclude that this intersection is a periodic point with period equal to some multiple of mn. Since we can choose $U(\eta)$ as small as we wish, this proves that η is accumulated by periodic orbits.

Finally, Theorem 1.2 follows from Claim 2 and the fact that the closure of the set of periodic orbits is hyperbolic.

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