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Abstract

We consider binary classification problems with positive definite kernels and square loss, and study the convergence rates of stochastic gradient methods. We show that while the excess testing *loss* (squared loss) converges slowly to zero as the number of observations (and thus iterations) goes to infinity, the testing *error* (classification error) converges exponentially fast if low-noise conditions are assumed.

1 Introduction

Stochastic gradient methods are now ubiquitous in machine learning, both from the practical side, as a simple algorithm that can learn from a single or a few passes over the data [1], and from the theoretical side, as it leads to optimal rates for estimation problems in a variety of situations [2, 3].

They follow a simple principle [4]: to find a minimizer of a function F defined on a vector space from noisy gradients, simply follow the negative stochastic gradient and the algorithm will converge to a stationary point, local minimum, global minimum of F (depending on the properties of the function F), with a rate of convergence that decays with the number of gradient steps n typically as $O(1/\sqrt{n})$, or $O(1/n)$ depending on the assumptions which are made on the problem (see, e.g., [3, 5, 6, 7, 8, 9, 10, 11]).

On the one hand, these rates are optimal for the estimation of the minimizer of a function given access to noisy gradients [2], which is essentially the usual machine learning set-up where the function F is the expected *loss*, e.g., logistic or hinge for classification, or least-squares for regression, and the noisy gradients are obtained from sampling a single pair of observations.

On the other hand, although these rates as $O(1/\sqrt{n})$ or $O(1/n)$ are optimal, there are a variety of extra assumptions that allow for faster rates, even exponential rates.

First, stochastic gradient from a finite pool, that is for $F = \frac{1}{k}\sum_{i=1}^{k} F_i$, a sequence of works starting from SAG [12], SVRG [13], SAGA [14], have shown explicit exponential convergence. However, these results, once applied to machine learning where the function F_i is the loss function associated with the i-th observation of a finite training data set of size k, say nothing about the loss on unseen data (test loss). The rates we present in this paper are on *unseen* data.

Second, assuming that at the optimum all stochastic gradients are equal to zero, then for strongly-convex problems (e.g., linear predictions with low-correlated features), linear convergence rates can be obtained for test losses [15, 16]. However, for supervised machine learning, this has limited relevance as having zero gradients for all stochastic gradients at the optimum essentially implies prediction problems with no uncertainty (that is, the output is a deterministic function of the input). Moreover, we can only get an exponential rate for strongly-convex problems, which imposes a parametric noiseless problem, which limits the applicability (even if the problem was noiseless, this can only reasonably be in a non-parametric way with neural networks or positive definite kernels). Our rates are on noisy problems and on infinite-dimensional problems where we can hope that we approach the optimal prediction function with large numbers of observations. For prediction functions described by a reproducing kernel Hilbert space, and for the square loss, the excess testing loss (equal to testing loss minus the minimal testing loss over all measurable prediction functions) is known to converge to zero at a subexponential rate typically above $O(1/n)$ [17, 11], these rates being optimal for the estimation of testing losses.

Going back to the origins of supervised machine learning with binary labels, we will not consider getting to the optimal testing *loss* (using a convex surrogate such as logistic, hinge or least-squares) but the testing *error* (number of mistakes in predictions), also referred to as the 0-1 loss.

It is known that the excess testing error (testing error minus the minimal testing error over all measurable prediction functions) is upper bounded by a function of the excess testing loss [18, 19], but always with a loss in the convergence rate (e.g., no difference or taking square roots). Thus a slow rate in $O(1/n)$ or $O(1/\sqrt{n})$ on the excess loss leads to a slow(er) rate on the excess testing error.

Such general relationships between excess loss and excess error have been refined with the use of *margin conditions*, which characterize how hard the prediction problems are [20]. Simplest input points are points where the label is deterministic (i.e., conditional probabilities of the label being zero or one), while hardest points are the ones where the conditional probabilities are equal to $1/2$. Margin conditions quantify the mass of input points which are hardest to predict, and leads to improved transfer functions from testing losses to testing errors, but still no exponential convergence rates [19].

In this paper, we consider the strongest margin condition, that is conditional probabilities are bounded away from $1/2$, but not necessarily equal to 0 or 1. This is an assumption on the learning problem which have been used in the past to show that regularized empirical (convex) risk minimization leads to exponential convergence rates [21, 22]. Our main contribution is to show that stochastic gradient descent also achieves similar rates. This requires several side contributions that are interesting on their own, that is, a new and simple formalization of the learning problem that allows exponential rates of estimation (regardless of the algorithms used to finding the estimator) and a new concentration result for averaged stochastic gradient descent (SGD) applied to least-squares, which is finer than existing work [10].

The paper is organized as follows: in Section 2, we present the main learning set-up, namely binary classification with positive definite kernels, with a particular focus on the relationship between errors and losses. Our main results rely on a generic condition for which we give concrete examples in Section 3. In Section 4, we present our version of stochastic gradient descent, with the use of tail averaging [23], and provide new deviation inequalities, which we apply in Section 5 to our learning problem, leading to exponential convergence rates for the testing errors. We present simulation results in Section 6, illustrating the different behaviors of excess testing errors (which converge exponentially fast) and excess training losses (which do not). We conclude in Section 7 by providing several avenues for future work.

2 Problem set-up

In this section, we present the general machine learning set-up, from generic assumptions to more specific assumptions.

2.1 Generic assumptions

We consider a measurable set $\mathcal{X}$ and a probability distribution ρ on data $(x, y) \in \mathcal{X} \times \{-1, 1\}$, we denote by $\rho_{\mathcal{X}}$ the marginal probability on x, and by $\rho(\pm 1|x)$ the conditional probability that $y = \pm 1$ given x. We have $\mathbb{E}(y|x) = \rho(1|x) - \rho(-1|x)$. Our main margin condition is the following (and independent of the learning framework):

(A 1) $|\mathbb{E}(y|x)| \geqslant \delta$ *almost surely for some* $\delta \in (0, 1]$.

This margin condition (often referred to as a low-noise condition) is commonly used in the study of binary classification [20, 21, 22], and usually takes the following form $\mathbb{P}(|\mathbb{E}(y|x)| < \varepsilon) = O(\varepsilon^{\alpha})$ for $\alpha > 0$. The smaller the α, the larger the mass of inputs with hard to predict labels. Our condition corresponds to $\alpha = +\infty$, and simply states that for all inputs, the problem is never totally ambiguous, and the degree of non-ambiguity is bounded from below by δ. When $\delta = 1$, then the label $y \in \{-1, 1\}$ is a deterministic function of x, but our results apply for all $\delta \in (0, 1]$ and thus to noisy problems (with low noise).

We will consider learning functions in a reproducing kernel Hilbert space (RKHS) $\mathcal{H}$ with kernel function $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and dot-product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. We make the following standard assumptions on $\mathcal{H}$:

(A 2) $\mathcal{H}$ *is a separable Hilbert space and there exists* $R > 0$*, such that for all* $x \in \mathcal{X}$*,* $K(x, x) \leqslant R^2$.

For $x \in \mathcal{X}$, we consider the function $K_x : \mathcal{X} \to \mathbb{R}$ defined as $K_x(x') = K(x, x')$. We have the classical reproducing property for $g \in \mathcal{H}$, $g(x) = \langle g, K_x \rangle_{\mathcal{H}}$ [24, 25].

We will consider other norms, beyond the RKHS norm $\|g\|_{\mathcal{H}}$, that is the L_2-norm (always with respect to $\rho_{\mathcal{X}}$), defined as $\|g\|_{L_2}^2 = \int_{\mathcal{X}} g(x)^2 d\rho_{\mathcal{X}}(x)$, as well as the L_∞-norm $\| \cdot \|_{L_\infty}$ on the support of $\rho_{\mathcal{X}}$. A key property is that **(A2)** implies $\|g\|_{L_\infty} \leqslant R\|g\|_{\mathcal{H}}$.

Since we want to perform non-parametric estimation, we assume:

(A 3) $\mathcal{H}$ *is dense in* L_2.

This density assumption is satisfied by a wide variety of kernels (see, e.g., [26]), and, as detailed in Section 3, by kernels leading to Sobolev spaces. Note that it could be relaxed by considering the closure $\overline{\mathcal{H}}$ of $\mathcal{H}$ in L^2 in subsequent developments.

Finally, we will consider observations with standard assumptions:

(A 4) *The observations* $(x_n, y_n) \in \mathcal{X} \times \{-1, 1\}, n \in \mathbb{N}$ *are independent and identically distributed with respect to the distribution* ρ.

2.2 Ridge regression

In this paper, we focus primarily on least-squares estimation to obtain estimators. We define g_* as the minimizer over L_2 of

$$\mathbb{E}(y - g(x))^2 = \int_{\mathcal{X} \times \{-1,1\}} (y - g(x))^2 d\rho(x, y).$$

We always have $g_*(x) = \mathbb{E}(y|x) = \rho(1|x) - \rho(-1|x)$, but we do not require $g_* \in \mathcal{H}$. We also consider the ridge regression problem and denote by g_λ the unique (when $\lambda > 0$) minimizer of $\mathcal{H}$ of

$$\mathbb{E}(y - g(x))^2 + \lambda \|g\|_{\mathcal{H}}^2.$$

The function g_λ always exists for $\lambda > 0$ and is always an element of $\mathcal{H}$. Our results will depend on the L_∞-error $\|g_\lambda - g_*\|_\infty$, which is weaker than $\|g_\lambda - g_\infty\|_{\mathcal{H}}$ which itself only exists when $g_* \in \mathcal{H}$ (which we do not assume).

Moreover our main technical assumption is:

(A 5) *There exists $\lambda > 0$ such that almost surely,* $\operatorname{sign}(\mathbb{E}(y|x)) g_\lambda(x) \geqslant \dfrac{\delta}{2}$.

In the assumption above, we could replace $\delta/2$ by any multiplicative constants in $(0, 1)$ times δ (instead of $1/2$). Moreover, for any estimator $\hat{g}$ such that $\|g_\lambda - \hat{g}\|_{L_\infty} < \delta/2$, the predictions from $\hat{g}$ (obtained by taking the sign of $\hat{g}(x)$ for any x), are the same as the sign of the optimal prediction $\operatorname{sign}(\mathbb{E}(y|x))$. Note that a sufficient condition is $\|g_\lambda - \hat{g}\|_{\mathcal{H}} < \delta/(2R)$ (which does not assume that $g_* \in \mathcal{H}$), see next subsection.

Note that more generally, for all problems for which ridge regression (in the population case) is so that $\|g_\lambda - g_*\|_{L_\infty}$ tends to zero as λ tends to zero then **(A5)** is satisfied, since $\|g_\lambda - g_*\|_{L_\infty} \leqslant \delta/2$ for λ small enough, together with **(A1)** then implies **(A5)**.

In Section 3, we provide concrete examples where **(A5)** is satisfied and we then present the SGD algorithm and our convergence results. Before we relate excess testing errors to excess testing losses.

2.3 From testing losses to testing error

Here we provide some results that will be useful to prove exponential rates for classification with squared loss and stochastic gradient descent. First we define the 0-1 loss defining the classification error:

$$\mathcal{R}(g) = \rho(\{(x, y) : \operatorname{sign}(g(x)) \neq y\}),$$

where $\operatorname{sign} u = +1$ for $u \geq 0$ and -1 for $u < 0$. In particular denote by $\mathcal{R}^*$ the so called *Bayes risk* $\mathcal{R}^* = \mathcal{R}(\mathbb{E}(y|x))$ which is the minimum classification error achievable [27].

A well known approach to bound the testing error by testing losses is via transfer functions. In particular we recall the following result [27, 19], let $g_*(x) := \mathbb{E}(y|x)$ a.e.,

$$\mathcal{R}(g) - \mathcal{R}^* \leq \phi(\|g - g_*\|_{L^2}), \qquad \forall g \in L^2(d\rho_X),$$

with $\phi(u) = \sqrt{u}$ (or $\phi(u) = u^\beta$, with $\beta \in [1/2, 1]$, depending on some properties of ρ [19]). While this result does not require **(A1)**, **(A5)**, it does not readily lead to exponential rates since the squared loss excess risk has minimax lower bounds that are polynomial in n (see [28]).

Here we follow a different approach, requiring via **(A5)** the existence of g_λ having the same sign of g_* and with absolute value uniformly bounded from below. Then we can bound the 0-1 error with respect to the distance in $\mathcal{H}$ of the estimator $\widehat{g}$ from g_λ as shown in the next lemma (proof in Appendix B). This leads to exponential rates when the distribution satisfies a margin condition **(A1)** as we prove in the next section and in Section 5.

Lemma 1 (From approximately correct sign to 0-1 error) *Let $q \in (0, 1)$. Under **(A1)**, **(A2)**, **(A5)**, let $\widehat{g} \in \mathcal{H}$ be a function such that*

$$\|\widehat{g} - g_\lambda\|_{\mathcal{H}} < \frac{\delta}{2R}, \text{ with probability at least } 1 - q.$$

Then

$$\mathcal{R}(\widehat{g}) = \mathcal{R}^*, \text{ with probability at least } 1-q,$$

and in particular

$$\mathbb{E}\left[\mathcal{R}(\widehat{g}) - \mathcal{R}^*\right] \leq q.$$

2.4 Exponential classification rates for kernel ridge regression

In this section, we first specialize some results already known in literature about the consistency of kernel ridge least-squares regression (KRLS) in $\mathcal{H}$-norm [28] and then we derive exponential classification learning rates. Let $(x_i, y_i)_{i=1}^n$ be n examples independently and identically distributed according to ρ, that is Assumption **(A4)**. Denote by $\Sigma, \widehat{\Sigma}$ the linear operators on $\mathcal{H}$ defined by

$$\widehat{\Sigma} = \frac{1}{n}\sum_{i=1}^n K_{x_i} \otimes K_{x_i}, \quad \Sigma = \int_{\mathcal{X}} K_x \otimes K_x d\rho_{\mathcal{X}}(x),$$

referred to as the covariance and empirical (non-centered) covariance operators (see [29] and references therein). We recall that the KRLS estimator $\widehat{g}_\lambda \in \mathcal{H}$ is defined as follows in terms of $\widehat{\Sigma}$,

$$\widehat{g}_\lambda = (\widehat{\Sigma} + \lambda I)^{-1}\left(\frac{1}{n}\sum_{i=1}^n y_i K_{x_i}\right).$$

Moreover we recall that the population regularized estimator g_λ is characterized by (see [28])

$$g_\lambda = (\Sigma + \lambda I)^{-1}\left(\mathbb{E}\left[yK_x\right]\right).$$

The following lemma bounds the empirical regularized estimator with respect to the population one in terms of λ, n and is essentially contained in [28], here we rederive it in a subcase (see Appendix C for the proof).

Lemma 2 *Under Assumptions **(A2)**, **(A4)** for any $\lambda > 0$ we have*

$$\|\widehat{g}_\lambda - g_\lambda\|_{\mathcal{H}} \leq \frac{u_n}{\lambda} + \frac{R v_n}{\lambda^2},$$

with $u_n = \|\frac{1}{n}\sum_{i=1}^n y_i K_{x_i} - \mathbb{E}\left[yK_x\right]\|_{\mathcal{H}}$ and $v_n = \|\Sigma - \widehat{\Sigma}\|_{\mathrm{op}}$.

By concentrating u_n, v_n in Lemma 2 and then applying Lemma 1, we obtain the following exponential bound for kernel ridge regression (see Appendix C for the complete proof):

Theorem 1 *Under **(A1)**,**(A2)**,**(A4)**,**(A5)** we have that for any $n \in \mathbb{N}$,*

$$\mathcal{R}(\widehat{g}_\lambda) - \mathcal{R}^* = 0 \text{ with probability at least } 1 - 4\exp\left(-\frac{C_0\lambda^4\delta^2}{R^8}n\right).$$

Moreover,

$$\mathbb{E}\left[\mathcal{R}(\widehat{g}_\lambda) - \mathcal{R}^*\right] \leq 4\exp\left(-\frac{C_0\lambda^4\delta^2}{R^8}n\right),$$

with $C_0 := \dfrac{1}{72(1+\lambda R^2)^2}$.

The result above is, to our knowledge, the first to prove exponential learning rates for classification via empirical risk minimization on the squared loss. Known results cover losses that are usually considered more suitable for classification, like the hinge or logistic loss and more generally losses that are non-decreasing (see [22]). With respect to this latter work, our analysis uses the explicit characterization of the kernel ridge regression estimator in terms of linear operators on $\mathcal{H}$ (see [28]). This, together with **(A5)**, allows us to use analytic tools specific to reproducing kernel Hilbert spaces, leading to proofs that are comparatively simpler, with explicit constants and a clearer problem setting (consisting essentially in **(A1)**, **(A5)** and no assumptions on $\mathbb{E}(y|x)$).

Finally note that the exponent of λ could be reduced by using a refined analysis under additional regularity assumption of $\rho_{\mathcal{X}}$ and $\mathbb{E}(y|x)$ (as *source condition* and *intrinsic dimension* from [28]), but it is beyond the scope of this paper.

In the next section we provide sufficient conditions and explicit settings naturally satisfying **(A5)**.

3 Concrete examples and related work

In this section we illustrate specific settings that naturally satisfy **(A5)**. We start by the following simple result showing that the existence of $g_* \in \mathcal{H}$ such that $g_*(x) = \mathbb{E}(y|x)$ a.e. on the support of $\rho_{\mathcal{X}}$, is sufficient to have **(A5)** (proof in Appendix D.1).

Proposition 1 *Under **(A1)**, assume that there exists $g_* \in \mathcal{H}$ such that $g_*(x) := \mathbb{E}(y|x)$ on the support of $\rho_{\mathcal{X}}$, then for any δ, there exists $\lambda > 0$ satisfying **(A5)**.*

We are going to use the proposition above to derive more specific settings. In particular we consider the case where the positive and negative classes are separated by a margin that is strictly positive. Let $\mathcal{X} \subseteq \mathbb{R}^d$ and denote by S the support of the probability $\rho_{\mathcal{X}}$ and by $S_+ = \{x \in \mathcal{X} : g_*(x) > 0\}$ the part associated to the positive class, and by S_- the one associated with the negative class.

(A6) *There exists $\mu > 0$ such that $\min_{x \in S_+, x' \in S_-} \|x - x'\| \geq \mu$.*

Denote by $W^{s,2}$ the Sobolev space of order s and L^2 norm, on $\mathbb{R}^d$ (see [30] and Appendix D.2). We introduce the following assumption:

(A7) *$\mathcal{X} \subseteq \mathbb{R}^d$ and the kernel is such that $W^{s,2} \subseteq \mathcal{H}$, with $s > d/2$.*

An example of kernel such that $\mathcal{H} = W^{s,2}$, with $s > d/2$ is the Abel kernel $K(x, x') = e^{-\frac{1}{\sigma}\|x - x'\|}$, for $\sigma > 0$. Now we are ready for the following corollary. In the following proposition we show that if there exist two functions in $\mathcal{H}$, one matching $\mathbb{E}(y|x)$ on S_+ and the second matching $\mathbb{E}(y|x)$ on S_- and the kernel satisfies **(A7)**, then **(A5)** is satisfied.

Proposition 2 *Under **(A1)**, **(A6)**, **(A7)**, if there exist two functions $g_+^*, g_-^* \in W^{s,2}$ such that $g_+^*(x) = \mathbb{E}(y|x)$ on S_+ and $g_-^*(x) = \mathbb{E}(y|x)$ on S_-, then **(A5)** is satisfied.*

Finally we are able to introduce another setting where **(A5)** is naturally satisfied (the proof of the proposition above and the example blow are given in Appendix D.2).

Example 1 (Independent noise on the labels) *Let $\rho_{\mathcal{X}}$ be a probability distribution on $\mathcal{X} \subseteq \mathbb{R}^d$ and let $S_+, S_- \subseteq \mathcal{X}$ be a partition of the support of $\rho_{\mathcal{X}}$ satisfying $\rho_{\mathcal{X}}(S_+), \rho_{\mathcal{X}}(S_-) > 0$ and **(A6)**. Let $n \in \mathbb{N}$. For $1 \leq i \leq n$, x_i independently sampled from $\rho_{\mathcal{X}}$ and the label y_i defined by the law*

$$y_i = \begin{cases} \zeta_i & \text{if } x_i \in S_+ \\ -\zeta_i & \text{if } x_i \in S_-, \end{cases}$$

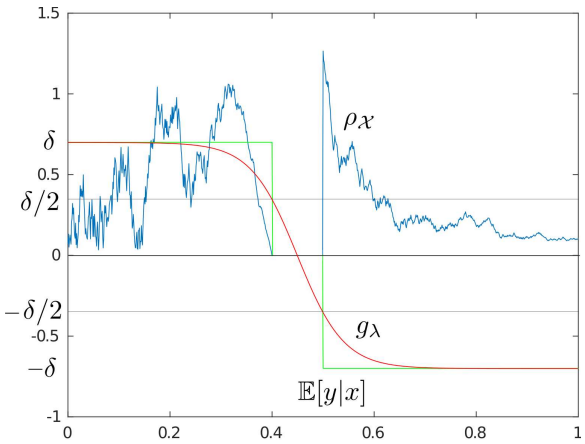


Figure 1: Pictorial representation of a model in 1D satisfying Example 1, ($p = 0.15$). Blue: $\rho_{\mathcal{X}}$, Green: $\mathbb{E}(y|x)$, Red: g_λ.

*with ζ_i independently distributed as $\zeta_i = -1$ with probability $p \in [0, 1/2)$ and $\zeta_i = 1$ with probability $1-p$. Then **(A1)** is satisfied with $\delta = 1-2p$ and **(A5)** is satisfied when the kernel is bounded **(A2)** and rich enough **(A7)**.*

4 Stochastic gradient descent

We now consider the stochastic gradient algorithm to solve the ridge regression problem with a fixed strictly positive regularization parameter λ. We consider solving the regularized problem with regularization $\|g - g_0\|^2_{\mathcal{H}}$ through stochastic approximation starting from a function $g_0 \in \mathcal{H}$ (typically 0).[1]

We can use the reproducing property for every function in $\mathcal{H}$ to write

$$F(g) = \mathbb{E}\big[(Y - g(X))^2\big] = \mathbb{E}\big[(Y - \langle K_X, g\rangle)^2\big].$$

As a function from $\mathcal{H}$ to $\mathbb{R}$, F has a the following gradient $\nabla F(g) = -2\mathbb{E}\left[(Y - \langle K_X, g\rangle)K_X\right]$. We consider $F_\lambda = F + \lambda\|\cdot - g_0\|^2_{\mathcal{H}}$, for which $\nabla F_\lambda(g) = \nabla F(g) + 2\lambda(g - g_0)$, and we have for each pair of observation **(A 4)** (x_n, y_n) that $F_\lambda(g) = \mathbb{E}\big[F_{n,\lambda}(g)\big] = \mathbb{E}\big[(\langle g, K_{x_n}\rangle - y_n)^2\big] + \lambda\|g - g_0\|^2_{\mathcal{H}}$, with $F_{n,\lambda}(g) = (\langle g, K_{x_n}\rangle - y_n)^2 + \lambda\|g - g_0\|^2_{\mathcal{H}}$.

Denoting $\Sigma = \mathbb{E}\big[K_{x_n} \otimes K_{x_n}\big]$ the covariance operator defined as a linear operator from $\mathcal{H}$ to $\mathcal{H}$ (see [29] and references therein), we have the optimality conditions for g_λ and g_*:

$$\Sigma g_\lambda - \mathbb{E}\left(y_n K_{x_n}\right) + \lambda(g_\lambda - g_0) = 0$$

and

$$\mathbb{E}\left[(y_n - g_*(x_n))\, K_{x_n}\right] = 0.$$

Let $(\gamma_n)_{n\geqslant 1}$ be a positive sequence; we consider the stochastic gradient recursion in $\mathcal{H}$ started at g_0:

$$g_n = g_{n-1} - \frac{\gamma_n}{2}\nabla F_{n,\lambda}(g_{n-1}) = g_{n-1} - \gamma_n\left[(\langle K_{x_n}, g_{n-1}\rangle - y_n)K_{x_n} + \lambda(g_{n-1} - g_0)\right]. \tag{1}$$

[1] Note that g_0 is the initialization of the recursion, and is not the limit of g_λ when λ tends to zero (this limit being g_*).

We are going to consider Polyak-Ruppert averaging, that is $\bar{g}_n = \dfrac{1}{n+1}\sum_{i=0}^{n} g_i$, as well as the tail-averaging estimate $\bar{g}_n^{\text{tail}} = \dfrac{1}{\lfloor n/2 \rfloor}\sum_{i=\lfloor n/2 \rfloor}^{n} g_i$.

As explained earlier (see Lemma 1), we need to show the convergence of g_n to g_λ in $\mathcal{H}$-norm. We are going to consider two cases :

- (γ_n) is a decreasing sequence, with the important particular case $\gamma_n = \gamma/n^\alpha$, for $\alpha \in [0,1]$, leading to results for the non-averaged recursion.
- The second one where (γ_n) is a constant sequence equal to γ, but for the averaged or tail-averaged functions.

For all the proofs of this section see Appendix E.

We first reformulate the recursion in Eq. (1) as least-squares recursion converging to g_λ.

4.1 Reformulation as noisy recursion

We can first derive the calculation of the SGD recursion equation in Eq. (1) as a regular least-squares SGD recursion with noise, with the notation $\xi_n = y_n - g_*(x_n)$, which satisfies $\mathbb{E}\big[\xi_n K_{x_n}\big] = 0$. This is the object of the following lemma:

Lemma 3 *The SGD recursion can be rewritten as follows:*

$$g_n - g_\lambda = \big[I - \gamma_n(K_{x_n} \otimes K_{x_n} + \lambda I)\big](g_{n-1} - g_\lambda) + \gamma_n \varepsilon_n, \tag{2}$$

with the noise term $\varepsilon_k = \xi_k K_{x_k} + (g_*(x_k) - g_\lambda(x_k))K_{x_k} - \mathbb{E}\left[(g_*(x_k) - g_\lambda(x_k))K_{x_k}\right] \in \mathcal{H}$.

Proof Let $n \geqslant 1$ and $g_0 \in \mathcal{H}$,

$$\begin{aligned} g_n &= g_{n-1} - \gamma_n\big[(\langle K_{x_n}, g_{n-1}\rangle - y_n)K_{x_n} + \lambda(g_{n-1} - g_0)\big] \\ &= g_{n-1} - \gamma_n\big[K_{x_n} \otimes K_{x_n} g_{n-1} - y_n K_{x_n} + \lambda(g_{n-1} - g_0)\big] \\ &= g_{n-1} - \gamma_n\big[K_{x_n} \otimes K_{x_n} g_{n-1} - g_*(x_n)K_{x_n} - \xi_n K_{x_n} + \lambda(g_{n-1} - g_0)\big], \end{aligned}$$

leading to (using the optimality conditions for g_λ and g_*):

$$\begin{aligned}
g_n - g_\lambda &= g_{n-1} - g_\lambda - \gamma_n \big[K_{x_n} \otimes K_{x_n}(g_{n-1} - g_\lambda) + \lambda(g_{n-1} - g_0) \\
&\qquad + (K_{x_n} \otimes K_{x_n}) g_\lambda - g_*(x_n) K_{x_n}\big] + \gamma_n \xi_n K_{x_n} \\
&= g_{n-1} - g_\lambda - \gamma_n \big[K_{x_n} \otimes K_{x_n}(g_{n-1} - g_\lambda) + \lambda(g_{n-1} - g_0) \\
&\qquad + (K_{x_n} \otimes K_{x_n} - \Sigma) g_\lambda + \Sigma g_\lambda - g_*(x_n) K_{x_n}\big] + \gamma_n \xi_n K_{x_n} \\
&= g_{n-1} - g_\lambda - \gamma_n \big[K_{x_n} \otimes K_{x_n}(g_{n-1} - g_\lambda) + \lambda g_{n-1} + (K_{x_n} \otimes K_{x_n} - \Sigma) g_\lambda \\
&\qquad - \lambda g_\lambda + \mathbb{E}\left[g_*(x_n) K_{x_n}\right] - g_*(x_n) K_{x_n}\big] + \gamma_n \xi_n K_{x_n} \\
&= g_{n-1} - g_\lambda - \gamma_n \big[(K_{x_n} \otimes K_{x_n} + \lambda I)(g_{n-1} - g_\lambda) + (K_{x_n} \otimes K_{x_n} - \Sigma) g_\lambda \\
&\qquad + \mathbb{E}\left[g_*(x_n) K_{x_n}\right] - g_*(x_n) K_{x_n}\big] + \gamma_n \xi_n K_{x_n} \\
&= \big[I - \gamma_n (K_{x_n} \otimes K_{x_n} + \lambda I)\big](g_{n-1} - g_\lambda) \\
&\qquad + \gamma_n \left[\xi_n K_{x_n} + (\Sigma - K_{x_n} \otimes K_{x_n}) g_\lambda + g_*(x_n) K_{x_n} - \mathbb{E}\left[g_*(x_n) K_{x_n}\right]\right] \\
&= \big[I - \gamma_n (K_{x_n} \otimes K_{x_n} + \lambda I)\big](g_{n-1} - g_\lambda) \\
&\qquad + \gamma_n \left[\xi_n K_{x_n} - (K_{x_n} \otimes K_{x_n}) g_\lambda + g_*(x_n) K_{x_n} + \Sigma g_\lambda - \mathbb{E}\left[g_*(x_n) K_{x_n}\right]\right] \\
&= \big[I - \gamma_n (K_{x_n} \otimes K_{x_n} + \lambda I)\big](g_{n-1} - g_\lambda) \\
&\qquad + \gamma_n \left[\xi_n K_{x_n} + (g_*(x_n) - g_\lambda(x_n)) K_{x_n} - \mathbb{E}\left[(g_*(x_n) - g_\lambda(x_n)) K_{x_n}\right]\right].
\end{aligned}$$

■

We are thus in presence of a least-squares problem in the Hilbert space $\mathcal{H}$, to estimate a function $g_\lambda \in \mathcal{H}$ with a specific noise ε_n in the gradient and feature vector K_x. In the next section, we will consider the generic recursion above, which will require some bounds on the noise.

We have the following almost sure bounds and the noise (see Lemma 9 of Appendix E):

$$\begin{aligned}
\|\varepsilon_n\|_{\mathcal{H}} &\leqslant R(1 + 2\|g_* - g_\lambda\|_{L_\infty}) \\
\mathbb{E}\big[\varepsilon_n \otimes \varepsilon_n\big] &\preccurlyeq 2\left(1 + \|g_* - g_\lambda\|_\infty^2\right)\Sigma,
\end{aligned}$$

where $\Sigma = \mathbb{E}\big[K_{x_n} \otimes K_{x_n}\big]$ is the covariance operator.

4.2 (Averaged) SGD for least-squares regression

We now consider results on (averaged) SGD for least-squares that are interesting on their own. As said before, we show results in two different settings depending on the step-size sequence. First, we consider (γ_n) as a decreasing sequence, second we take (γ_n) constant but prove the convergence of the (tail-)averaged iterates.

Since the results we need could be of interest (even for finite-dimensional models), in this section, we study the following general recursion:

$$\eta_n = (I - \gamma H_n)\eta_{n-1} + \gamma_n \varepsilon_n, \tag{3}$$

We make the following assumptions:

- (**H-a**) We start at some $\eta_0 \in \mathcal{H}$.
- (**H-b**) $(H_n, \varepsilon_n)_{n \geqslant 1}$ are i.i.d. and H_n is a positive self-adjoint operator so that almost surely $H_n \succcurlyeq \lambda I$, with $H = \mathbb{E}H_n$.

- (**H-c**) Noise: $\mathbb{E}\varepsilon_n = 0$, $\|\varepsilon_n\|_{\mathcal{H}} \leqslant c^{1/2}$ almost surely and $\mathbb{E}(\varepsilon_n \otimes \varepsilon_n) \preccurlyeq C$, with C commuting with H. Note that one consequence of this assumption is $\mathbb{E}\|\varepsilon_n\|_{\mathcal{H}}^2 \leqslant \operatorname{tr} C$.
- (**H-d**) For all $n \geqslant 1$, $\mathbb{E}\Big[H_n C H^{-1} H_n\Big] \preccurlyeq \gamma_0^{-1} C$ and $\gamma \leqslant \gamma_0$.
- (**H-e**) A is a positive self-adjoint operator which commutes with H.

We will later apply the results of this section to $H_n = K_{x_n} \otimes K_{x_n} + \lambda I$, $H = \Sigma + \lambda I$, $C = \Sigma$ and $A = I$. We first consider the non-averaged SGD recursion, then the (tail-)averaged recursion. The key difference with existing bounds is the need for precise probabilistic deviation results.

For least-squares, one can always separate the impact of the initial condition η_0 and of the noise terms ε_k, namely $\eta_n = \eta_n^{\text{bias}} + \eta_n^{\text{variance}}$, where η_n^{bias} is the recursion with no noise, and η_n^{variance} is the recursion started at $\eta_0 = 0$. The final performance will be bounded by the sum of the two separate performances (see, e.g., [31]). Hence all of our bounds will depend on these two. See more details in Appendix E.

4.2.1 Non-averaged SGD

In this section, we prove results for the recursion defined by Eq. (3) in the case where for $\alpha \in [0, 1]$, $\gamma_n = \gamma / n^\alpha$. These results extend the ones of [9] by providing deviation inequalities, but are limited to least-squares. For general loss functions and in the strongly-convex case, see [32].

Theorem 2 (SGD, decreasing step size: $\boldsymbol{\gamma_n = \gamma / n^\alpha}$) *Assume (**H-abc**), $\gamma_n = \gamma/n^\alpha$, $\gamma\lambda < 1$ and denote by $\eta_n \in \mathcal{H}$ the n-th iterate of the recursion in Eq.* (3)*. We have for $t > 0, n \geqslant 1$,*

- *for $\alpha = 1$ and $\gamma\lambda < 1/2$,*

$$\begin{aligned} \|g_n - g_\lambda\|_{\mathcal{H}} &\leqslant \frac{\|g_0 - g_\lambda\|_{\mathcal{H}}}{n^{\gamma\lambda}} + V_n, \quad \textit{almost surely, with} \\ \mathbb{P}\left(V_n \geqslant t\right) &\leqslant 2\exp\left(-\frac{t^2}{4^{3/2}(\operatorname{tr} C)\gamma^2/((1-2\gamma\lambda)n^{\gamma\lambda}) + 4tc^{1/2}\gamma/3} \cdot n^{\gamma\lambda}\right); \end{aligned}$$

- *for $\alpha = 0$,*

$$\begin{aligned} \|g_n - g_\lambda\|_{\mathcal{H}} &\leqslant (1-\gamma\lambda)^n \|g_0 - g_\lambda\|_{\mathcal{H}} + V_n, \quad \textit{almost surely, with} \\ \mathbb{P}\left(V_n \geqslant t\right) &\leqslant 2\exp\left(-\frac{t^2}{2\gamma(\operatorname{tr} C/\lambda + tc^{1/2}/3)}\right); \end{aligned}$$

- *for $\alpha \in (0, 1)$, for n large enough (see Appendix Section E Lemma 7),*

$$\begin{aligned} \|g_n - g_\lambda\|_{\mathcal{H}} &\leqslant \exp\left(-\frac{\gamma\lambda}{1-\alpha}\left((n+1)^{1-\alpha} - 1\right)\right)\|g_0 - g_\lambda\|_{\mathcal{H}} + V_n, \quad \textit{almost surely, with} \\ \mathbb{P}\left(V_n \geqslant t\right) &\leqslant 2\exp\left(-\frac{t^2}{\gamma(2^{\alpha+2}\operatorname{tr} C/\lambda + 2c^{1/2}t/3)} \cdot n^\alpha\right). \end{aligned}$$

We can make the following observations:

- The proof technique relies on the following scheme: first, we notice that η_n can be decomposed in two terms, (a) the bias: obtained from a product of n contractant operators, and (b) the variance: a sum of increments of martingale. We treat separately the two terms. For the second one, we prove almost sure bounds on the increments and on the variance that lead to a Bernstein-type concentration result on the tail $\mathbb{P}(V_n \geqslant t)$.

- Following the proof technique summed-up before, we see that coefficient in the exponential is composed of the variance bound plus the almost sure bound of the increments of martingale times t.
- There are three different regimes. For $\alpha = 0$ (constant step-size), the algorithm is not converging, as the tail probability bound on $\mathbb{P}(V_n \geqslant t)$ is not dependent on n. For $\alpha = 1$, confirming results from [9], there is no exponential forgetting of initial conditions. Finally, for $\alpha \in (0,1)$, the forgetting of initial conditions and the tail probability are converging to zero exponentially fast, respectively, as $\exp(-Cn^{1-\alpha})$ and $\exp(-Cn^{\alpha})$, for a constant C, hence the natural choice of $\alpha = 1/2$ in our experiments.

4.2.2 Averaged and Tail-averaged SGD with constant step-size

In the subsection, we take: $\forall n \geqslant 1,\ \gamma_n = \gamma$. We first start with a result on the variance term, whose proof extends the work of [11] to deviation inequalities.

Theorem 3 (Convergence of the variance term in averaged SGD) *Assume (**H-abcde**), and consider the average of the $n+1$ first iterates of the sequence defined in Eq. (3): $\bar{\eta}_n = \frac{1}{n+1}\sum_{i=0}^{n} \eta_i$. Assume $\eta_0 = 0$. We have for $t > 0, n \geqslant 1$:*

$$\mathbb{P}\left(\left\|A^{1/2}\bar{\eta}_n\right\|_{\mathcal{H}} \geqslant t\right) \leqslant 2\exp\left[-\frac{(n+1)t^2}{E_t}\right], \tag{4}$$

where E_t is defined with respect to the constants introduced in the assumptions:

$$E_t = 4\operatorname{tr}(AH^{-2}C) + \frac{2c^{1/2}\|A^{1/2}\|_{\mathrm{op}}}{3\lambda} \cdot t. \tag{5}$$

We could consider the regular averaged recursion, but in the strongly-convex case, following [23], we consider the tail-averaged recursion, $\bar{\eta}_n^{\mathrm{tail}} = \frac{1}{\lfloor n/2 \rfloor}\sum_{i=\lfloor n/2 \rfloor}^{n} \eta_i$.

For the bias term, we can simply use the fact that almost surely $\|\eta_i^{\mathrm{bias}}\|_{\mathcal{H}} \leqslant (1-\lambda\gamma)^i \|\eta_0\|_{\mathcal{H}}$, hence $\|\bar{\eta}_n^{\mathrm{tail,bias}}\|_{\mathcal{H}} \leqslant (1-\lambda\gamma)^{n/2}\|\eta_0\|_{\mathcal{H}}$. For the variance term, we can simply use the result above for n and $n/2$, as $\bar{\eta}_n^{\mathrm{tail}} = 2\bar{\eta}_n - \bar{\eta}_{n/2}$. This leads to:

Corollary 1 (Convergence of tailed averaged SGD) *Assume (**H-abcde**), and consider the tail-average of the sequence defined in Eq. (3): $\bar{\eta}_n^{\mathrm{tail}} = \frac{1}{\lfloor n/2 \rfloor}\sum_{i=\lfloor n/2 \rfloor}^{n} \eta_i$. We have for $t > 0, n \geqslant 1$:*

$$\left\|A^{1/2}\bar{\eta}_n^{\mathrm{tail}}\right\|_{\mathcal{H}} \leqslant (1-\gamma\lambda)^{n/2}\|A^{1/2}\|_{op}\|\eta_0\|_{\mathcal{H}} + L_n \quad , \textit{with} \tag{6}$$

$$\mathbb{P}(L_n \geqslant t) \leqslant 4\exp\left(-(n+1)t^2/(4E_t)\right). \tag{7}$$

We can make the following observations on the two previous results:

- The proof technique relies on concentration inequality of Bernstein type. Indeed, first, we notice that (in the setting of Theorem 3) $\bar{\eta}_n$ is a sum of increments of martingale. We prove almost sure bounds on the increments and on the variance (following the proof technique of [11]) that lead to a Bernstein type concentration result on the tail $\mathbb{P}(V_n \geqslant t)$.
- Following the proof technique summed-up before, we see that E_t is composed of the variance bound plus the almost sure bound times t.

- Classically, A and C are proportional to H for excess risk predictions. In the finite d-dimensional setting this leads us to the usual variance bound proportional to the dimension d: $\operatorname{tr}(AH^{-2}C) \cong \operatorname{tr} I = d$.
- Note that the result is general in the sense that we can apply it for all A matrices commuting with H (this can be used to prove results in L_2 or in $\mathcal{H}$).

5 Exponentially convergent SGD for classification error

In this section we want to show results on the error made (on unseen data) by the n-th iterate of the regularized SGD algorithm. Hence, we go back to the original SGD recursion defined in Eq. (2). Let us recall it:

$$g_n - g_\lambda = \left[I - \gamma_n(K_{x_n} \otimes K_{x_n} + \lambda I)\right](g_{n-1} - g_\lambda) + \gamma_n \varepsilon_n,$$

with the noise term $\varepsilon_k = \xi_k K_{x_k} + (g_*(x_k) - g_\lambda(x_k))K_{x_k} - \mathbb{E}\left[(g_*(x_k) - g_\lambda(x_k))K_{x_k}\right] \in \mathcal{H}$.

Like in the previous section we are going to state two results in two different settings, the first one for SGD with decreasing step-size ($\gamma_n = \gamma/n^\alpha$) and the second one for tail averaged SGD with constant step-size. For all the proofs of this section see the Appendix (section F).

5.1 SGD with decreasing step-size

In this section, we focus on decreasing step-sizes $\gamma_n = \gamma/n^\alpha$ for $\alpha \in (0,1)$, which leads to exponential convergence rates. Results for $\alpha = 1$ and $\alpha = 0$ can be derived in a similar way (but do not lead to exponential rates).

Theorem 4 *Assume* **(A-245)** *and $\gamma_n = \gamma/n^\alpha$, $\alpha \in (0,1)$ for any n and $\gamma\lambda < 1$. Let g_n be the n-th iterate of the recursion defined in Eq. (2), as soon as n satisfies $\exp\left(-\frac{\gamma\lambda}{1-\alpha}\left((n+1)^{1-\alpha} - 1\right)\right) \leqslant \dfrac{\delta}{5R\|g_0 - g_\lambda\|_{\mathcal{H}}}$, then*

$$\mathcal{R}(g_n) = \mathcal{R}^*, \text{ with probability at least } 1 - 2\exp\left(-\frac{\delta^2}{C_R} \cdot n^\alpha\right),$$

and in particular

$$\mathbb{E}\left[\mathcal{R}(g_n) - \mathcal{R}^*\right] \leqslant 2\exp\left(-\frac{\delta^2}{C_R} \cdot n^\alpha\right),$$

with $C_R = \gamma\left(\dfrac{2^{\alpha+7}R^2 \operatorname{tr}\Sigma\left(1 + \|g_* - g_\lambda\|_\infty^2\right)}{\lambda} + \dfrac{8R^2\delta(1 + 2\|g_* - g_\lambda\|_\infty)}{3}\right)$.

We can make the following observations:

- In other words, Theorem 4 shows that with probability at least $1 - 2\exp\left(-\frac{\delta^2}{C_R} \cdot n^\alpha\right)$, the previsions of g_n are perfect.
- The idea of the proof is the following: we know that as soon as $\|g_n - g_\lambda\|_{\mathcal{H}} \leqslant \delta/(2R)$, the predictions of g_n are perfect (Lemma 1). We just have to apply Theorem 2 for to the original SGD recursion and make sure to bound each term by $\delta/(4R)$.

- Similar results for non-averaged SGD could be derived beyond least-squares (e.g., hinge or logistic loss) using results from [32].
- The larger the α, the smaller the bound. However, it is only valid for n larger that a certain quantity depending of $\lambda\gamma$. A good trade-off is $\alpha = 1/2$, for which we get an excess error of $2\exp\left(-\frac{\delta^2}{C_R}n^{1/2}\right)$, which is valid as soon as $n \geqslant \log(10R\|g_0 - g_\lambda\|_{\mathcal{H}}/\delta)/(4\lambda^2\gamma^2)$. Notice that we should go for large $\gamma\lambda$ to increase the factor in the exponential and make the condition happen as soon as possible.
- Notice that the smaller δ, the faster the condition is satisfied but the slower the convergence will be because of the δ^2 in the exponential.
- Note that when the condition on n is not met, then we still have the usual bound obtained by taking directly the excess loss [19] (note the lack of square roots because of the improved margin condition but the extra factor δ^{-1}), but we lose exponential convergence:

$$\begin{aligned}
\mathbb{E}(R_\rho(g_n) - R_\rho^*) &= \mathbb{E}(R_\rho(g_n) - R_\rho(g_\lambda)) \leqslant \delta^{-1}\left[\mathbb{E}F_\lambda(g_n) - F_\lambda(g_\lambda)\right] = \delta^{-1}\mathbb{E}\|\Sigma^{1/2}(g_n - g_\lambda)\|_{\mathcal{H}}^2 \\
\mathbb{E}(R_\rho(g_n) - R_\rho^*) &\leqslant 2\delta^{-1}\exp\left(-\frac{2\gamma\lambda}{1-\alpha}\left((n+1)^{1-\alpha} - 1\right)\right)\|\Sigma\|_{\rm op}\|g_0 - g_\lambda\|_{\mathcal{H}}^2 + \delta^{-1}(\mathbb{E}V_n)^2, \\
&\leqslant 2\delta^{-1}\exp\left(-\frac{2\gamma\lambda}{1-\alpha}\left((n+1)^{1-\alpha} - 1\right)\right)\|\Sigma\|_{\rm op}\|g_0 - g_\lambda\|_{\mathcal{H}}^2 \\
&\qquad + \delta^{-1}\frac{2^{\alpha+2}(1 + \|g_* - g_\lambda\|_\infty^2)\gamma\,\mathrm{tr}\,\Sigma}{\lambda}\cdot\frac{1}{n^\alpha}.
\end{aligned}$$

5.2 Tail averaged SGD with constant step-size

We now consider the tail-averaged recursion, with the following result:

Theorem 5 *Assume* **(A-245)** *and $\gamma_n = \gamma$ for any n, $\gamma\lambda < 1$ and $\gamma \leqslant \gamma_0 = (R^2 + 2\lambda)^{-1}$. Let g_n be the n-th iterate of the recursion defined in Eq. (2), and $\bar{g}_n^{\rm tail} = \frac{1}{\lfloor n/2\rfloor}\sum_{i=\lfloor n/2\rfloor}^n g_i$, as soon as $n \geqslant \frac{2}{\gamma\lambda}\ln(\frac{5R\|g_0 - g_\lambda\|_{\mathcal{H}}}{\delta})$, then*

$$\mathcal{R}(\bar{g}_n^{\rm tail}) = \mathcal{R}^*, \text{ with probability at least } 1 - 4\exp\left(-\delta^2 K_R(n+1)\right),$$

and in particular

$$\mathbb{E}\left[\mathcal{R}(\bar{g}_n^{\rm tail}) - \mathcal{R}^*\right] \leqslant 4\exp\left(-\delta^2 K_R(n+1)\right),$$

with $K_R^{-1} = 2^9R^2\left(1 + \|g_* - g_\lambda\|_\infty^2\right)\mathrm{tr}(\Sigma(\Sigma+\lambda I)^{-2}) + \dfrac{32\delta R^2(1 + 2\|g_* - g_\lambda\|_\infty)}{3\lambda}$.

We can make the following observations:

- In other words, Theorem 5 shows that with probability at least $1 - 4\exp\left(-\delta^2 K_R(n+1)\right)$, the previsions of $\bar{g}_n^{\rm tail}$ are perfect.
- The idea of the proof is the following: we know that as soon as $\|\bar{g}_n^{\rm tail} - g_\lambda\|_{\mathcal{H}} \leqslant \delta/(2R)$, the predictions of $\bar{g}_n^{\rm tail}$ are perfect (Lemma 1). We just have to apply Corollary 1 to the original SGD recursion, and make sure to bound each term by $\delta/(4R)$.
- The condition on n is now logarithmic. Remark that we want to take $\gamma\lambda$ as big as possible to sastify quickly the condition and increase the coefficient in the exponential.

- For the dependence in λ, the first term in K_R^{-1} can be upperbounded by in $O(\lambda^{-2})$ but it could be made much smaller with assumptions on the decrease of eigenvalues of Σ (it has been shown [28] that if the decay happends at speed $1/n^\beta$: $\operatorname{tr}\Sigma(\Sigma+\lambda I)^{-2} \leqslant \lambda^{-1}\operatorname{tr}\Sigma(\Sigma+\lambda I)^{-1} \leqslant R^2/\lambda^{1+1/\beta}$).
- Notice that the smaller δ, the faster the condition is satisfied but the slower the convergence will be because of the δ^2 in the exponential.
- The dependence on n is also improved as the convergence is really an exponential of n (and not of some power of n as in the previous result).
- Results for the regular averaged recursion can be similarly derived, but the bias term will not converge exponentially fast almost surely as will appear the average of a geometric sum ($1/n$ convergence). To preserve exponential rates, we should certainly apply again a concentration inequality on this term.

6 Experiments

To illustrate our results, we consider one-dimensional synthetic examples ($\mathcal{X} = [0,1]$) for which our assumptions are easily satisfied. Indeed, we consider the following set-up that fulfils our assumptions:

- **(A1-A4)**. We consider here $X \sim U([0,(1-\varepsilon)/2] \cup [(1+\varepsilon)/2,1])$ and with the notations of Example 1, we take $S_+ = [0,(1-\varepsilon)/2]$ and $S_- = [(1+\varepsilon)/2,1]$. For $1 \leq i \leq n$, x_i independently sampled from $\rho_{\mathcal{X}}$ we define
$$y_i = \begin{cases} 1 & \text{if } x_i \in S_+ \\ -1 & \text{if } x_i \in S_-, \end{cases}$$

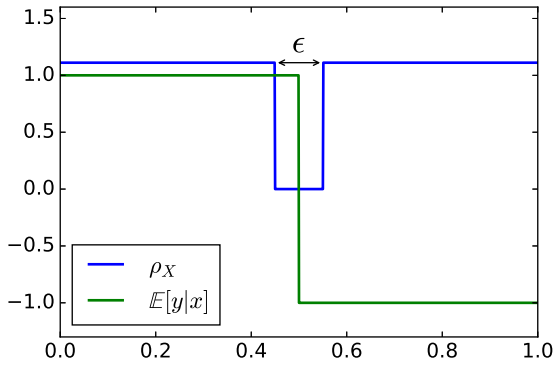


Figure 2: Representing the $\rho_{\mathcal{X}}$ density (uniform with ε-margin) and the best estimator, i.e., $\mathbb{E}(x|y)$ used for the simulations.

- **(A2-A3)**. We take the kernel to be the exponential kernel $K(x,x') = \exp(-|x-x'|)$ for which the RKHS is a Sobolev space $\mathcal{H} = W^{s,2}$, with $s > d/2$, which is indeed dense in L_2 [30].
- **(A-5)**. With this setting we could find a closed form for g_λ and checked that it verified. Indeed we could solve the optimality equation satisfied by g_λ :
$$\forall z \in [0,1],\ \int_0^1 K(x,z)g_\lambda(x)d\rho_X(x) + \lambda g_\lambda(z) = \int_0^1 K(x,z)g_\rho(x)d\rho_X(x),$$

the solution being a linear combination of exponentials in each set : $[0,(1-\varepsilon)/2]$, $[(1-\varepsilon)/2,(1+\varepsilon)/2]$ and $[(1+\varepsilon)/2,1]$.

In the case of SGD with decreasing step size, we computed only the test error. For tailed averaged SGD with constant step size, we computed the test error as well as the training error, the test loss (which corresponds to the L^2 loss : $\int_0^1 (g_n(x)-g_\lambda(x))^2 d\rho(x)$) and the training loss.

In all cases we computed the errors of the n-th iterate with respect to the calculated g_λ, taking $g_0=0$. For any $n \geqslant 1$,

$$g_n = g_{n-1} - \gamma_n \big[(g_{n-1}(x_n)-y_n)K_{x_n} + \lambda g_{n-1}\big].$$

We can use representants to find the recursion on the coefficients. Indeed, if

$$g_n = \sum_{i=1}^n a_i^n K_{x_i},$$

then the following recursion for the (a_i^n) reads :

$$\begin{aligned} \text{for } i \leqslant n-1,\ a_i^n &= (1-\gamma_n\lambda)a_i^{n-1} \\ a_n^n &= -\gamma_n(\sum_{i=1}^{n-1} a_i^{n-1}K(x_n,x_i) - y_n). \end{aligned}$$

From (a_i^n), we can also compute the coefficients of $\bar{g}_n$ and $\bar{g}_n^{\text{tail}}$ that we note $\bar{a}_i^n$ and $\bar{a}_i^{n,\text{tail}}$ respectively.

$$\begin{aligned} \bar{a}_i^n &= \sum_{k=i}^n \frac{a_i^k}{n+1} \\ \bar{a}_i^{n,\text{tail}} &= \frac{1}{\lfloor n/2 \rfloor} \sum_{k=\lfloor n/2 \rfloor}^n a_i^k. \end{aligned}$$

First let us recall the notation for the 0-1 loss defining the classification error:

$$\mathcal{R}_\rho(f) = \rho(\{(x,y) : \text{sign}(f(x)) \neq y\}).$$

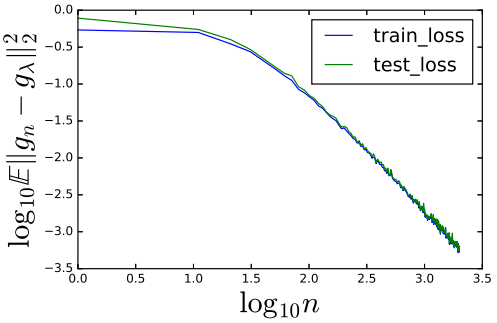
In particular denote with $\mathcal{R}_\rho^*$ the called *Bayes risk* $\mathcal{R}_\rho^* = \mathcal{R}_\rho(\mathbb{E}(y|x))$ which is the minimum classification error achievable. As there is no ambiguity here, we can note $\mathcal{R}(f) = \mathcal{R}_\rho(f)$.

For the plots where we plotted the expected excess errors, i.e., $\mathbb{E}(\mathcal{R}(g_n) - \mathcal{R}^*)$, we plotted the mean of the errors over 1000 replications for n from 1 to 200.

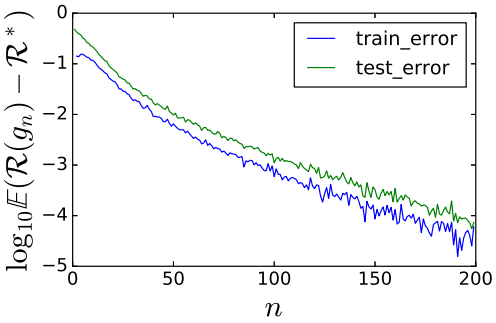
For the plots where we plotted the losses, i.e., a function of $\|g_n - g_*\|_2$, we plotted the mean of the errors over 100 replications for n from 1 to 2000.

We can make the following observations:

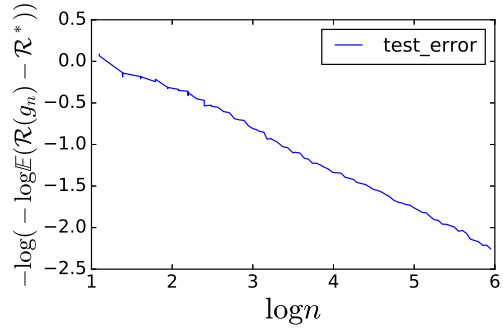
- First note that between plots of losses (Fig. 3a) and errors (Fig. 3b), there is a factor 10 between the numbers of samples (200 for errors and 2000 for losses) and another factor 10 between errors and losses (10^{-4} for errors and 10^{-3} for losses). That underlines well the difference between exponential rates of convergence of the excess error and $1/n$ rate of convergence of the loss.



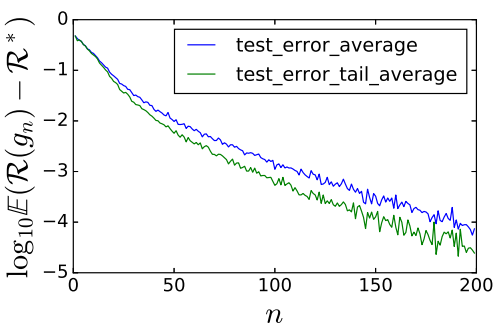
(a) Test and training losses in the averaged case, log scale versus n in log scale too.



(b) Test and training errors in the averaged case, log scale versus n.



(c) Error in the non-averaged case for $\gamma_n = \gamma/\sqrt{n}$ (i.e., $\alpha = 0.5$), $\log(\log \cdot)$ scale versus n in log scale.



(d) Comparison of the test error between the averaged and the tail averaged case, log scale versus n.

Figure 3: Showing linear convergence for the L^{01} errors in the case of margin of width ε. We took the following parameters : $\varepsilon = 0.05$, $\gamma = 0.25$, $\lambda = 0.01$.

- Second, for all plots, we adapted the scales to logarithmic ones to show lines (to illustrate our theoretical results):
  - To show exponential convergence for the averaged and tail averaged cases, we plotted $\log_{10}\mathbb{E}(\mathcal{R}(g_n) - \mathcal{R}^*))$ as a fonction of n (Figures 3b and 3d).
  - We recover the results of [11] that show convergence at speed $1/n$ for the loss (Figure 3a).
  - For Figure 3c, as the convergence of the excess error is of the form $\exp(-K\sqrt{n})$, we plotted $-\log(-\log(\mathbb{E}(\mathcal{R}(g_n) - \mathcal{R}^*)))$ of the excess error with respect to the log of n to show a line of slope $-1/2$.
- Moreover, we see that even if the excess error with tail averaging seems a bit faster, it seems that we have linear rates too for the convergence of the excess error in the averaged case.
- Finally, we remark that the error on the train set is always below the one for a unknown test set (of what seems to be close to a factor 2).

7 Conclusion

In this paper, we have shown that stochastic gradient could be exponentially convergent, once some margin conditions are assumed. This is obtained by running averaged stochastic gradient on a least-squares problem, and proving new deviation inequalities.

Our work could be extended in several natural ways: (a) our work relies on new concentration results for the least-mean-squaeres algorithm (i.e., SGD for square loss), it is natural to extend it to other losses, such as the logistic or hinge loss; (b) going beyond binary classification is also natural with the square loss [33, 34] or without [35]; (c) exploring the intermediate margin conditions also naturally lead to intermediate results wihout exponential convergence, but rates faster than $O(1/n)$ [21]; (d) in our experiments, we use regularization, but we have experimented with unregularized recursions, which do exhibit fast convergence, but for which proofs are usually harder [17]; finally, (e) in order to avoid the $O(n^2)$ complexity, extending the results of [36] would lead to a subquadratic complexity.

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A Probabilistic lemmas

In this section we recall two fundamental results for concentration inequalities in Hilbert spaces shown in [37].

Proposition 3 *Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of vectors of $\mathcal{H}$ adapted to a non decreasing sequence of σ-fields $(\mathcal{F}_k)$ such that $\mathbb{E}[X_k|\mathcal{F}_{k-1}] = 0$, $\sup_{k\leqslant n}\|X_k\| \leqslant a_n$ and $\sum_{k=1}^n \mathbb{E}\left[\|X_k\|^2|\mathcal{F}_{k-1}\right] \leqslant b_n^2$ for some sequences $(a_n), (b_n) \in \left(\mathbb{R}_+^*\right)^{\mathbb{N}}$. Then, for all $t \geqslant 0$, $n \geqslant 1$,*

$$\mathbb{P}\left(\left\|\sum_{k=1}^n X_k\right\| \geqslant t\right) \leqslant 2\exp\left(\frac{t}{a_n} - \left(\frac{t}{a_n} + \frac{b_n^2}{a_n^2}\right)\ln\left(1 + \frac{ta_n}{b_n}\right)\right). \tag{8}$$

Proof As $\mathbb{E}[X_k|\mathcal{F}_{k-1}] = 0$, the $\mathcal{F}_j$-adapted sequence (f_j) defined by $f_j = \sum_{k=1}^j X_k$ is a martingale and so is the stopped-martingale $(f_{j\wedge n})$. By applying Theorem 3.4 of [37] to the martingale $(f_{j\wedge n})$, we have the result. ∎

Corollary 2 *Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of vectors of $\mathcal{H}$ adapted to a non decreasing sequence of σ-fields $(\mathcal{F}_k)$ such that $\mathbb{E}[X_k|\mathcal{F}_{k-1}] = 0$, $\sup_{k\leqslant n}\|X_k\| \leqslant a_n$ and $\sum_{k=1}^n \mathbb{E}\left[\|X_k\|^2|\mathcal{F}_{k-1}\right] \leqslant b_n^2$ for some sequences $(a_n), (b_n) \in \left(\mathbb{R}_+^*\right)^{\mathbb{N}}$. Then, for all $t \geqslant 0$, $n \geqslant 1$,*

$$\mathbb{P}\left(\left\|\sum_{k=1}^n X_k\right\| \geqslant t\right) \leqslant 2\exp\left(-\frac{t^2}{2\left(b_n^2 + a_n t/3\right)}\right). \tag{9}$$

Proof We apply 3 and simply notice that

$$\begin{aligned}\frac{t}{a_n} - \left(\frac{t}{a_n} + \frac{b_n^2}{a_n^2}\right)\ln\left(1 + \frac{ta_n}{b_n}\right) &= -\frac{b_n^2}{a_n^2}\left(\left(1 + \frac{a_n t}{b_n^2}\right)\ln\left(1 + \frac{a_n t}{b_n^2}\right) - \frac{a_n t}{b_n^2}\right) \\ &= -\frac{b_n^2}{a_n^2}\phi\left(\frac{a_n t}{b_n^2}\right),\end{aligned}$$

where $\phi(u) = (1+u)\ln(1+u) - u$ for $u > 0$. Moreover $\phi(u) \geqslant \dfrac{u^2}{2\left(1 + u/3\right)}$, so that:

$$\frac{t}{a_n} - \left(\frac{t}{a_n} + \frac{b_n^2}{a_n^2}\right)\ln\left(1 + \frac{ta_n}{b_n}\right) \leqslant -\frac{b_n^2}{a_n^2}\frac{(a_n t/b_n^2)^2}{2\left(1 + a_n t/3b_n^2\right)} = -\frac{t^2}{2\left(b_n^2 + a_n t/3\right)}.$$

∎

B From $\mathcal{H}$ to 0-1 loss

In this section we prove Lemma 1. Note that **(A5)** requires the existence of g_λ having the same sign of g_* almost everywhere on the support of $\rho_{\mathcal{X}}$ and with absolute value uniformly bounded from below. In Lemma 1 we prove that we can bound the 0-1 error with respect to the distance in $\mathcal{H}$ of the estimator $\widehat{g}$ form g_λ.

Proof of Lemma 1 Denote by W the event such that $\left\|\widehat{g}-g_\lambda\right\|_{\mathcal{H}} < \delta/(2R)$. Note that for any $f \in \mathcal{H}$,

$$f(x) = \langle f, K_x\rangle_{\mathcal{H}} \leq \left\|K_x\right\|_{\mathcal{H}}\left\|f\right\|_{\mathcal{H}} \leq R\left\|f\right\|_{\mathcal{H}},$$

for any $x \in \mathcal{X}$. So for $\widehat{g} \in W$, we have

$$|\widehat{g}(x) - g_\lambda(x)| \leq R\left\|\widehat{g}-g_\lambda\right\|_{\mathcal{H}} < \delta/2 \quad \forall x \in \mathcal{X}.$$

Let x be in the support of $\rho_{\mathcal{X}}$. By **(A5)** $|g_\lambda(x)| \geq \delta/2$ a.e.. Let $\widehat{g} \in W$ and $x \in \mathcal{X}$ such that $g_\lambda(x) > 0$, we have

$$\widehat{g}(x) = g_\lambda(x) - (g_\lambda(x) - \widehat{g}(x)) \geq g_\lambda(x) - |g_\lambda(x) - \widehat{g}(x)| > 0,$$

so $\operatorname{sign}(\widehat{g}(x)) = \operatorname{sign}(g_\lambda(x)) = +1$. Similarly let $\widehat{g} \in W$ and $x \in \mathcal{X}$ such that $g_\lambda(x) < 0$, we have

$$\widehat{g}(x) = g_\lambda(x) + (\widehat{g}(x) - g_\lambda(x)) \leq g_\lambda(x) + |g_\lambda(x) - \widehat{g}(x)| < 0,$$

so $\operatorname{sign}(\widehat{g}(x)) = \operatorname{sign}(g_\lambda(x)) = -1$. Finally note that for any $\widehat{g} \in \mathcal{H}$, by **(A5)**, either $g_\lambda(x) > 0$ or $g_\lambda(x) < 0$ a.e., so $\operatorname{sign}(\widehat{g}(x)) = \operatorname{sign}(g_\lambda(x))$ a.e.

Now note that by **(A1)**, **(A5)** we have that $\operatorname{sign}(g_*(x)) = \operatorname{sign}(g_\lambda(x))$ a.e., where $g_*(x) := \mathbb{E}\,(y|x)$. So when $\widehat{g} \in W$, we have that $\operatorname{sign}(\widehat{g}(x)) = \operatorname{sign}(g_\lambda(x)) = \operatorname{sign}(g_*(x))$ a.e., so

$$\mathcal{R}(\widehat{g}) = \rho(\{(x,y) : \operatorname{sign}(\widehat{g}(x)) \neq y\}) = \rho(\{(x,y) : \operatorname{sign}(g_*(x)) \neq y\}) = \mathcal{R}^*.$$

Finally note that

$$\mathbb{E}\left[\mathcal{R}(\widehat{g})\right] = \mathbb{E}\left[\mathcal{R}(\widehat{g})\mathbf{1}_W\right] + \mathbb{E}\left[\mathcal{R}(\widehat{g})\mathbf{1}_{W^c}\right],$$

where $\mathbf{1}_W$ is 1 on the set W and 0 outside, W^c is the complement set of W. So, when $\widehat{g} \in W$, we have

$$\mathbb{E}\left[\mathcal{R}(\widehat{g})\mathbf{1}_W\right] = \mathcal{R}^*\mathbb{E}\left[\mathbf{1}_W\right] \leq \mathcal{R}^*,$$

while

$$\mathbb{E}\left[\mathcal{R}(\widehat{g})\mathbf{1}_{W^c}\right] \leq \mathbb{E}\left[\mathbf{1}_{W^c}\right] \leq q.$$

■

C Proofs of Exponential rates for Kernel Ridge Regression

Here we prove that Kernel Ridge Regression achieves exponential classification rates under assumptions **(A1)**, **(A5)**. In particular by Lemma 2 we bound $\left\|\widehat{g}_\lambda - g_\lambda\right\|_{\mathcal{H}}$ in high probability and then we use Lemma 1 that gives exponential classfication rates when $\left\|\widehat{g}_\lambda - g_\lambda\right\|_{\mathcal{H}}$ is small enough in high probability.

Proof of Lemma 2 Denote by $\widehat{\Sigma}_\lambda$ the operator $\widehat{\Sigma} + \lambda I$ and with Σ_λ the operator $\Sigma + \lambda I$. We have

$$\begin{aligned}\widehat{g}_\lambda - g_\lambda &= \widehat{\Sigma}_\lambda^{-1}\left(\frac{1}{n}\sum_{i=1}^n y_i K_{x_i}\right) - \Sigma_\lambda^{-1}(\mathbb{E}\left[yK_x\right]) \\ &= \widehat{\Sigma}_\lambda^{-1}\left(\frac{1}{n}\sum_{i=1}^n y_i K_{x_i} - \mathbb{E}\left[yK_x\right]\right) \;+\; (\widehat{\Sigma}_\lambda^{-1} - \Sigma_\lambda^{-1})\mathbb{E}\left[yK_x\right].\end{aligned}$$

For the first term, since $\left\|\widehat{\Sigma}_\lambda^{-1}\right\|_{\text{op}} \leq \lambda^{-1}$, we have

$$\begin{aligned}\left\|\widehat{\Sigma}_\lambda^{-1}\left(\frac{1}{n}\sum_{i=1}^n y_i K_{x_i} - \mathbb{E}\left[yK_x\right]\right)\right\|_{\mathcal{H}} &\leq \left\|\widehat{\Sigma}_\lambda^{-1}\right\|_{\text{op}}\left\|\frac{1}{n}\sum_{i=1}^n y_i K_{x_i} - \mathbb{E}\left[yK_x\right]\right\|_{\mathcal{H}} \\ &\leq \frac{1}{\lambda}\left\|\frac{1}{n}\sum_{i=1}^n y_i K_{x_i} - \mathbb{E}\left[yK_x\right]\right\|_{\mathcal{H}}.\end{aligned}$$

For the second term, since $\|\Sigma_\lambda^{-1}\|_{\text{op}} \leq \lambda^{-1}$ and $\|\mathbb{E}\left[yK_x\right]\| \leq \mathbb{E}\left[\|yK_x\|\right] \leq R$, we have

$$\begin{aligned}\left\|(\widehat{\Sigma}_\lambda^{-1} - \Sigma_\lambda^{-1})\mathbb{E}\left[yK_x\right]\right\|_{\mathcal{H}} &= \left\|\widehat{\Sigma}_\lambda^{-1}(\Sigma - \widehat{\Sigma})\Sigma_\lambda^{-1}\mathbb{E}\left[yK_x\right]\right\|_{\mathcal{H}} \\ &\leq \left\|\widehat{\Sigma}_\lambda^{-1}\right\|_{\text{op}}\left\|\Sigma - \widehat{\Sigma}\right\|_{\text{op}}\left\|\Sigma_\lambda^{-1}\right\|_{\text{op}}\left\|\mathbb{E}\left[yK_x\right]\right\|_{\mathcal{H}} \leq \frac{R}{\lambda^2}\left\|\Sigma - \widehat{\Sigma}\right\|_{\text{op}}.\end{aligned}$$

■

Proof of Theorem 1 Let $\tau > 0$. By Lemma 2 we know that

$$\|\widehat{g}_\lambda - g_\lambda\|_{\mathcal{H}} \leq \frac{u_n}{\lambda} + \frac{Rv_n}{\lambda^2},$$

with $u_n = \|\frac{1}{n}\sum_{i=1}^n (y_i K_{x_i} - \mathbb{E}\left[yK_x\right])\|_{\mathcal{H}}$ and $v_n = \|\Sigma - \widehat{\Sigma}\|_{\text{op}}$. For u_n we can apply Pinelis inequality [37, Thm. 3.5], since $(x_i, y_i)_{i=1}^n$ are sampled independently according to the probability ρ and that $y_i K_{x_i} - \mathbb{E}\left[yK_x\right]$ is zero mean. Since

$$\left\|\frac{1}{n}(y_i K_{x_i} - \mathbb{E}\left[yK_x\right])\right\|_{\mathcal{H}} \leq \frac{2R}{n}$$

a.e. and $\mathcal{H}$ is a Hilbert space, then we apply Pinelis inequality with $b_*^2 = \frac{4R^2}{n}$ and $D = 1$, obtaining

$$u_n \leq \sqrt{\frac{8R^2\tau}{n}},$$

with probability at least $1 - 2e^{-\tau}$. Now, denote by $\left\|\cdot\right\|_{HS}$ the Hilbert-Schmidt norm and recall that $\left\|\cdot\right\| \leq \left\|\cdot\right\|_{HS}$. To bound v_n we apply again the Pinelis inequality (see also [38]) considering that the space of Hilbert-Schmidt operators is again a Hilbert space and that $\widehat{\Sigma} = \frac{1}{n}\sum_{i=1}^n K_{x_i} \otimes K_{x_i}$, that $(x_i)_{i=1}^n$ are independently sampled from $\rho_{\mathcal{X}}$ and that $\mathbb{E}\left[K_{x_i} \otimes K_{x_i}\right] = \Sigma$. In particular we apply it with $D = 1$ and $b_*^2 = \frac{4R^4}{n}$, so

$$v_n = \left\|\Sigma - \widehat{\Sigma}\right\| \leq \left\|\Sigma - \widehat{\Sigma}\right\|_{HS} \leq \sqrt{\frac{8R^4\tau}{n}},$$

with probability $1 - 2e^{-\tau}$. Finally we take the intersection bound of the two events obtaining, with probability at least $1 - 4e^{-\tau}$,

$$\|\widehat{g}_\lambda - g_\lambda\|_{\mathcal{H}} \leq \sqrt{\frac{8R^2\tau}{\lambda^2 n}} + \sqrt{\frac{8R^6\tau}{\lambda^4 n}}.$$

By selecting $\tau = \frac{\delta^2}{9R^2(\sqrt{\frac{8R^2}{\lambda^2 n}} + \sqrt{\frac{8R^6}{\lambda^4 n}})^2}$, we obtain $\|\widehat{g}_\lambda - g_\lambda\|_{\mathcal{H}} \leq \frac{\delta}{3R}$, with probability $1 - 4e^{-\tau}$. Now we can apply Lemma 1 to have the exponential bound for the classification error. ■

D Proofs and additional results about concrete examples

In the next subsection we prove that $g_* \in \mathcal{H}$ is sufficient to satisfy **(A5)**, while in subsection D.2 we prove that specific settings naturally satisfy **(A5)**.

D.1 From $g_* \in \mathcal{H}$ to (A5)

Here we assume that there exists $g_* \in \mathcal{H}$ such that $g_*(x) = \mathbb{E}(y|x)$ a.e. on the support of $\rho_{\mathcal{X}}$. First we introduce $A(\lambda)$, that is a quantity related to the approximation error of g_λ with respect to g_* and we study its behavior when $\lambda \to 0$. Then we express $\|g_\lambda - g_*\|_{\mathcal{H}}$ in terms of $A(\lambda)$. Finally we prove that for any δ given by **(A1)**, there exists λ such that **(A5)** is satisfied.

Let $(\sigma_t, u_t)_{t\in\mathbb{N}}$ be an eigenbasis of Σ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$, and let $\alpha_j = \langle g_*, u_j\rangle$ we introduce the following quantity

$$A(\lambda) = \sum_{t:\sigma_t \leq \lambda} \alpha_t^2.$$

Lemma 4 *Under **(A2)**, $A(\lambda)$ is decreasing for any $\lambda > 0$ and*

$$\lim_{\lambda\to 0} A(\lambda) = 0.$$

Proof Under **(A2)** and the linearity of trace, we have that

$$\sum_{j\in N} \sigma_j = \mathrm{tr}(\Sigma) = \int \mathrm{tr}\left(K_x \otimes K_x\right) d\rho_{\mathcal{X}}(x) = \int \langle K_x, K_x\rangle_{\mathcal{H}}\, d\rho_{\mathcal{X}}(x) = \int K(x,x) d\rho_{\mathcal{X}}(x) \leq R^2.$$

Denote by $t_\lambda \in \mathbb{N}$, the number $\min\{t \in \mathbb{N} \mid \sigma_t \leq \lambda\}$. Since the $(\sigma_j)_{j\in\mathbb{N}}$ is a non-decreasing summable sequence, then it converges to 0, then

$$\lim_{\lambda\to 0}\ t_\lambda = \infty.$$

Finally, since $(\alpha_j^2)_{j\in\mathbb{N}}$ is a summable sequence we have that

$$\lim_{\lambda\to 0} A(\lambda) = \lim_{\lambda\to 0} \sum_{t:\sigma_t\leq\lambda} \alpha_t^2 = \lim_{\lambda\to 0} \sum_{j=t_\lambda} \alpha_j^2 = \lim_{t\to\infty} \sum_{j=t}^{\infty} \alpha_j^2 = 0.$$

■

Here we express $\|g_\lambda - g_*\|_{\mathcal{H}}$ in terms of $\|g_*\|_{\mathcal{H}}$ and of $A(\sqrt{\lambda})$.

Lemma 5 *Under **(A2)**, for any $\lambda > 0$ we have*

$$\|g_\lambda - g_*\|_{\mathcal{H}} \leq \sqrt{\sqrt{\lambda}\|g_*\|_{\mathcal{H}}^2 + A(\sqrt{\lambda})}.$$

Proof Denote by Σ_λ the operator $\Sigma + \lambda I$. Note that since $g_* \in \mathcal{H}$, then

$$\mathbb{E}\left[yK_x\right] = \mathbb{E}\left[g_*(x)K_x\right] = \mathbb{E}\left[(K_x \otimes K_x)g_*\right] = \mathbb{E}\left[K_x \otimes K_x\right] g_* = \Sigma g_*,$$

then $g_\lambda = \Sigma_\lambda^{-1}\mathbb{E}\left[yK_x\right] = \Sigma_\lambda^{-1}\Sigma g_*$. So we have

$$\|g_\lambda - g_*\|_{\mathcal{H}} = \|\Sigma_\lambda^{-1}\Sigma g_* - g_*\|_{\mathcal{H}} = \|(\Sigma_\lambda^{-1}\Sigma - I)g_*\|_{\mathcal{H}} = \lambda\|\Sigma_\lambda^{-1} g_*\|_{\mathcal{H}}.$$

Moreover

$$\lambda\big\|(\Sigma+\lambda I)^{-1}g_*\big\|_{\mathcal{H}} \leq \sqrt{\lambda}\big\|(\Sigma+\lambda I)^{-1/2}\big\|\ \sqrt{\lambda}\big\|(\Sigma+\lambda I)^{-1/2}g_*\big\|_{\mathcal{H}} \leq \sqrt{\lambda}\big\|(\Sigma+\lambda I)^{-1/2}g_*\big\|_{\mathcal{H}}.$$

Now we express $\sqrt{\lambda}\big\|(\Sigma+\lambda I)^{-1/2}g_*\big\|_{\mathcal{H}}$ in terms of $A(\lambda)$. We have that

$$\lambda\big\|(\Sigma+\lambda I)^{-1/2}g_*\big\|_{\mathcal{H}}^2 = \lambda\left\langle g_*, (\Sigma+\lambda I)^{-1}g_*\right\rangle = \lambda\left\langle g_*, \left(\sum_{j\in\mathbb{N}}(\sigma_j+\lambda)^{-1}u_j\otimes u_j\right)g_*\right\rangle = \sum_{j\in\mathbb{N}}\frac{\lambda\alpha_j^2}{\sigma_j+\lambda}.$$

Now divide the series in two parts

$$\sum_{j\in\mathbb{N}}\frac{\lambda\alpha_j^2}{\sigma_j+\lambda} = S_1(\lambda)+S_2(\lambda), \quad S_1(\lambda) = \sum_{j:\sigma_j\geq\sqrt{\lambda}}\frac{\lambda\alpha_j^2}{\sigma_j+\lambda}, \quad S_2(\lambda) = \sum_{j:\sigma_j<\sqrt{\lambda}}\frac{\lambda\alpha_j^2}{\sigma_j+\lambda}.$$

For each term in S_1, since j is selected such that $\sigma_j \geq \sqrt{\lambda}$ we have that $\lambda(\sigma_j+\lambda)^{-1} \leq \lambda(\sqrt{\lambda}+\lambda)^{-1} \leq \lambda/\sqrt{\lambda} \leq \sqrt{\lambda}$, so

$$S_1(\lambda) \leq \sqrt{\lambda}\sum_{j:\sigma_j\geq\sqrt{\lambda}}\alpha_j^2 \leq \sqrt{\lambda}\sum_{j\in\mathbb{N}}\alpha_j^2 = \sqrt{\lambda}\|g_*\|^2.$$

For S_2, we have that $\lambda(\sigma_j+\lambda)^{-1} \leq 1$, so

$$S_2(\lambda) \leq \sum_{j:\sigma_j<\sqrt{\lambda}}\alpha_j^2 = A(\sqrt{\lambda}).$$

■

Proof of Proposition 1 By Lemma 5 we have that

$$\|g_\lambda - g_*\|_{\mathcal{H}} \leq \sqrt{\sqrt{\lambda}\|g_*\|_{\mathcal{H}}^2 + A(\sqrt{\lambda})}.$$

Now note that the r.h.s. is non-decreasing in λ, and is 0 when $\lambda \to 0$, due to Lemma 4. Then there exists λ such that $\big\|g_\lambda - g_*\big\|_{\mathcal{H}} < \frac{\delta}{2R}$.

Since $|f(x)| \leq R\big\|f\big\|_{\mathcal{H}}$ for any $f \in \mathcal{H}$ when the kernel satisfies **(A2)** and moreover **(A1)** holds, we have that for any $x \in \mathcal{X}$ such that $g_*(x) > 0$ we have

$$g_\lambda(x) = g_*(x) - (g_*(x) - g_\lambda(x)) \geq g_*(x) - |g_*(x) - g_\lambda(x)| \geq \delta - R\big\|g_\lambda - g_*\big\| \geq \delta/2,$$

so $\operatorname{sign}(g_*(x)) = \operatorname{sign}(g_\lambda(x)) = +1$ and $\operatorname{sign}(g_*(x))g_\lambda(x) \geq \delta/2$. Analogously for any $x \in \mathcal{X}$ such that $g_*(x) < 0$ we have

$$g_\lambda(x) = g_*(x) + (g_\lambda(x) - g_*(x)) \leq g_*(x) + |g_*(x) - g_\lambda(x)| \leq -\delta + R\big\|g_\lambda - g_*\big\| \leq -\delta/2,$$

so $\operatorname{sign}(g_*(x)) = \operatorname{sign}(g_\lambda(x)) = -1$ and $\operatorname{sign}(g_*(x))g_\lambda(x) \geq \delta/2$. Note finally that $g_*(x) = 0$ on a zero measure set by **(A5)**. ■

D.2 Examples

In this subsection we first introduce some notation and basic results about Sobolev spaces, then we prove Prop. 2 and Example 1.

In what follows denote by A_t the t-fattening of a set $A \subseteq \mathbb{R}^d$, that is $A_t = \bigcup_{x\in P} B_t(x)$ where $B_t(x)$ is the open ball of ray t centered in x. We denote by $W^{s,2}(\mathbb{R}^d)$ the Sobolev space endowed with norm

$$\|f\|_{W^{s,2}} = \left\{ f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \;\middle|\; \int_{\mathbb{R}^d} \mathcal{F}(f)(\omega)^2 (1+\|\omega\|^2)^{s/2} d\omega < \infty \right\}.$$

Finally we define the function $\phi_{s,t} : \mathcal{X} \to \mathbb{R}$, that will be used in the proofs as follows

$$\phi_{s,t}(x) \;=\; q_{d,\delta}\; t^{-d}\; 1_{\{0\}_t}(x)\; (1-\|x/t\|^2)^{s-d/2},$$

with $q_{d,s} = \pi^{-d/2}\Gamma(1+s)/\Gamma(1+s-d/2)$ and $t > 0, s \geq d/2$. Note that $\phi_{s,t}(x)$ is supported on $\{0\}_{\epsilon/2}$, satisfies

$$\int_{\mathbb{R}^d} \phi_{s,t}(y)dy = 1$$

and it is continuous and belongs to $W^{s,2}(\mathbb{R}^d)$.

Proposition 4 *Let P, N two compact subsets of $\mathbb{R}^d$ with Hausdorff distance at least $\epsilon > 0$. There exists $g_{P,N} \in W^{s,2}$ such that*

$$g_{P,N}(x) = 1, \quad \forall\; x \in P, \qquad q_{P,N}(x) = 0, \quad \forall\; x \in N.$$

In particular $g_{P,N} = 1_{P_{\epsilon/2}} * \phi_{s,\epsilon/2}$.

Proof Denote by $v_{\epsilon,s}$ the function $(1-\|2x/\epsilon\|^2)^{s-d/2}$. We have

$$\begin{aligned} g_{P,N}(x) &= q_{d,s}(\epsilon/2)^{-d} \int_{\mathbb{R}^d} 1_{P_{\epsilon/2}}(x-y)\; 1_{\{0\}_{\epsilon/2}}(y)\; v_{\epsilon,s}(y)\; dy \\ &= q_{d,s}(\epsilon/2)^{-d} \int_{\{0\}_{\epsilon/2}} 1_{P_{\epsilon/2}}(x-y)\; v_{\epsilon,s}(y)\; dy \\ &= q_{d,s}(\epsilon/2)^{-d} \int_{\{x\}_{\epsilon/2}} 1_{P_{\epsilon/2}}(y)\; v_{\epsilon,s}(y-x)\; dy \end{aligned}$$

Now when $x \in P$, then $\{x\}_{\epsilon/2} \subseteq P_{\epsilon/2}$, so

$$\begin{aligned} g_{P,N}(x) &= q_{d,s}(\epsilon/2)^{-d} \int_{\{x\}_{\epsilon/2}} 1_{P_{\epsilon/2}}(y)\; v_{\epsilon,s}(y-x)\; dy \\ &= q_{d,s}(\epsilon/2)^{-d} \int_{\{x\}_{\epsilon/2}} v_{\epsilon,s}(y-x)dy \;=\; q_{d,s}\epsilon^{-d} \int_{\{0\}_{\epsilon/2}} v_{\epsilon,s}(y)dy \\ &= q_{d,s}(\epsilon/2)^{-d} \int_{\mathbb{R}^d} 1_{\{0\}_{\epsilon/2}}(y) v_{\epsilon,s}(y)dy \;=\; \int_{\mathbb{R}^d} \phi_{s,\epsilon/2}(y)dy \;=\; 1. \end{aligned}$$

Conversely, when $x \in N$, then $\{x\}_{\epsilon/2} \cap P_{\epsilon/2} = \emptyset$, so

$$g_{P,N}(x) \;=\; q_{d,s}(\epsilon/2)^{-d} \int_{\{x\}_{\epsilon/2}} 1_{P_{\epsilon/2}}(y)\; v_{\epsilon,s}(y-x)\; dy \;=\; 0.$$

Now we prove that $g_{P,N} \in W^{s,2}$. First note that $P_{\epsilon/2}$ is compact whenever P is compact. This implies that $1_{P_{\epsilon/2}}$ is in $L^2(\mathbb{R}^d)$. Since g_δ is the convolution of an $L^2(\mathbb{R}^d)$ function and a $W^{s,2}$, then

it belongs to $W^{s,2}$. ■

Proof of Proposition 2 Since we are under **(A6)**, we can apply Prop. 4 that prove the existence two functions $q_{S_+,S_-}, q_{S_-,S_+} \in W^{s,2}$ with the property to be respectively equal to 1 on S_+, 0 on S_-, and 1 on S_-, 0 on S_+. Since $W^{s,2}$ is a Banach algebra (see [30]), then $gh \in W^{s,2}$ for any $g, h \in W^{s,2}$. So in particular

$$g_* \;=\; g_+^* q_{S_+,S_-} \;-\; g_-^* q_{S_-,S_+},$$

belongs to $W^{s,2}$ (and so to $\mathcal{H}$) and is equal to $\mathbb{E}\,(y|x)$ a.e. on the support of $\rho_{\mathcal{X}}$ by definition. Finally, **(A5)** is satisfied, by Prop. 1. ■

Proof of Example 1 By definition of y, we have that

$$\mathbb{E}\,(y|x) = (1-2p)g(x), \quad g(x) = \mathbf{1}_{S_+} - \mathbf{1}_{S_-}.$$

In particular note that **(A1)** is satisfied with $\delta = 1 - 2p > 0$ since $p \in [0, 1/2)$. Moreover note that $\mathbb{E}\,(y|x)$ is constant δ on S_+ and $-\delta$ on S_-. Note now that there exists two functions in $W^{s,2} \subseteq \mathcal{H}$ (due to **(A7)**) that are, respectively δ on S_+ and $-\delta$ on S_-. They are exactly $g_+^* := \delta q_{S_+,S_-}$ and $g_-^* = -\delta q_{S_-,S_+}$, from Prop. 4. So we can apply Prop. 2, that given g_+^*, g_-^* guarantees that **(A5)** is satisfied. ■

E Proof of stochastic gradient descent results

Let us recall for the Appendix the SGD recursion defined in Eq. (3):

$$\eta_n = (I - \gamma H_n)\eta_{n-1} + \gamma_n \varepsilon_n,$$

for which we assume **H-abcde**.

Notations. We define the following notations, which will be useful during all the proofs of the section:

- the following contractant operators: for $i \geqslant k$,

$$M(i,k) = (I - \gamma H_i) \cdots (I - \gamma H_k), \text{ and } M(i, i+1) = I,$$

- the following sequences $Z_k = M(n, k+1)\varepsilon_k$ and $W_n = \sum_{k=1}^n \gamma_k Z_k$.

then,

$$\eta_n = M(n,n)\eta_{n-1} + \gamma_n \varepsilon_n \tag{10}$$

$$\eta_n = M(n,1)\eta_0 + \sum_{k=1}^{n} \gamma_k M(n, k+1)\varepsilon_k, \tag{11}$$

Note that in all this section, when there is no ambiguity, we will use $\|\cdot\|$ instead of $\|\cdot\|_{\mathcal{H}}$.

E.1 Non-averaged SGD - Proof of Theorem 2

In this section, we define the two following sequences:

- $\alpha_n = \prod_{i=1}^{n}(1-\gamma_i\lambda)$,
- $\beta_n = \sum_{k=1}^{n}\gamma_k^2 \prod_{i=k+1}^{n}(1-\gamma_i\lambda)^2$,
- $\zeta_n = \sup_{k\leqslant n}\left[\gamma_k \prod_{i=k+1}^{n}(1-\gamma_i\lambda)\right]$.

We can decompose η_n in two terms:

$$\eta_n = \underbrace{M(n,1)\eta_0}_{\textbf{Biais term}} + \underbrace{W_n}_{\textbf{Noise term}} \quad , \tag{12}$$

- The biais term represents the speed at which we forget initial conditions. It is the product of n contracting operators

$$\|M(n,1)\eta_0\| \leqslant \prod_{i=1}^{n}(1-\gamma_i\lambda)\|\eta_0\| = \alpha_n\|\eta_0\|.$$

- The noise term W_n which is a martingale. We are going to show by using a concentration inequality that the probability of the event $\{\|W_n\| \geq t\}$ goes to zero exponentially fast.

E.1.1 General result for all (γ_n)

As $W_n = \sum_{k=1}^{n}\gamma_k Z_k$, we want to apply Corollary 2 of section A to $(\gamma_k Z_k)_{k\in\mathbb{N}}$ that is why we need the following lemma:

Lemma 6 *We have the following bounds:*

$$\sup_{k\leqslant n}\|\gamma_k Z_k\| \leqslant c^{1/2}\zeta_n, \text{ and} \tag{13}$$

$$\sum_{k=1}^{n}\mathbb{E}\left[\|\gamma_k Z_k\|^2|\mathcal{F}_{k-1}\right] \leqslant \operatorname{tr} C\beta_n, \tag{14}$$

where c and C are defined in Lemma 9.

Proof First, $\|\gamma_k Z_k\| = \gamma_k\|M(n,k+1)\varepsilon_k\| \leq \gamma_k\|M(n,k+1)\|_{\text{op}}\|\varepsilon_k\| \leq \gamma_k\dfrac{\alpha_n}{\alpha_k}\|\varepsilon_k\| \leqslant \zeta_n c^{1/2}$.

Second,

$$\begin{aligned}\sum_{k=1}^{n}\mathbb{E}\left[\|\gamma_k Z_k\|^2|\mathcal{F}_{k-1}\right] &\leqslant \sum_{k=1}^{n}\frac{\alpha_n^2}{\alpha_k^2}\gamma_k^2\,\mathbb{E}\|\varepsilon_k\|^2 \\ &\leqslant \sum_{k=1}^{n}\frac{\alpha_n^2}{\alpha_k^2}\gamma_k^2\,\operatorname{tr} C.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{k=1}^{n}\mathbb{E}\left[\|\gamma_k Z_k\|^2|\mathcal{F}_{k-1}\right] &\leqslant \sum_{k=1}^{n}\gamma_k^2\prod_{i=k+1}^{n}(1-\gamma_i\lambda)^2\,\mathrm{tr}\,C\\ &= \mathrm{tr}\,C\beta_n.\end{aligned}$$

■

Proposition 5 *We have the following inequality: for $t>0, n\geqslant 1$,*

$$\|\eta_n\| \leqslant \alpha_n\|\eta_0\| + V_n, \quad \textit{with} \tag{15}$$

$$\mathbb{P}\left(V_n \geqslant t\right) \leqslant 2\exp\left(-\frac{t^2}{2(\mathrm{tr}\,C\beta_n + c^{1/2}\zeta_n t/3)}\right). \tag{16}$$

Proof We just need to apply Lemma 6 and Corollary 2 to the martingale W_n and $V_n = \|W_n\|$ for all n. ■

E.1.2 Result for all $\gamma_n = \gamma/n^\alpha$

We now derive estimates of α_n, β_n and ζ_n to have explicit bound for the previous result in the case where $\gamma_n = \dfrac{\gamma}{n^\alpha}$ for $\alpha\in[0,1]$.

Lemma 7 *In the interesting particular case where $\gamma_n = \dfrac{\gamma}{n^\alpha}$ for $\alpha\in[0,1]$:*

- *for $\alpha = 1$, i.e $\gamma_n = \dfrac{\gamma}{n}$, then $\zeta_n = \dfrac{\gamma}{1-\gamma\lambda}\alpha_n$, and we have the following estimations for $\gamma\lambda < 1/2$:*
  (i) $\alpha_n \leqslant \dfrac{1}{n^{\gamma\lambda}}$, *(ii)* $\beta_n \leqslant \dfrac{2(1-\gamma\lambda)}{1-2\gamma\lambda}\dfrac{4^{\gamma\lambda}\gamma^2}{n^{2\gamma\lambda}}$, *(iii)* $\zeta_n \leqslant \dfrac{\gamma}{(1-\lambda\gamma)n^{\gamma\lambda}}$.
- *for $\alpha = 0$, i.e $\gamma_n = \gamma$, then $\zeta_n = \gamma$, and we have the following:*
  (i) $\alpha_n = (1-\gamma\lambda)^n$, *(ii)* $\beta_n \leqslant \dfrac{\gamma}{\lambda}$, *(iii)* $\zeta_n = \gamma$.
- *for $\alpha\in\,]0,1[$, $\zeta_n = \max\left\{\gamma_n, \dfrac{\gamma}{1-\gamma\lambda}\alpha_n\right\}$, and we have the following estimations:*
  (i) $\alpha_n \leqslant \exp\left(-\dfrac{\gamma\lambda}{1-\alpha}\left((n+1)^{1-\alpha}-1\right)\right)$,
  (ii) Denoting $L_\alpha = \frac{2\lambda\gamma}{1-\alpha}2^{1-\alpha}\left(1-\left(\frac{3}{4}\right)^{1-\alpha}\right)$, *we distinguish three cases:*
  - $\alpha > 1/2, \quad \beta_n \leqslant \gamma^2\frac{2\alpha}{2\alpha-1}\exp\left(-L_\alpha n^{1-\alpha}\right) + \frac{2^\alpha\gamma}{\lambda n^\alpha}$,
  - $\alpha = 1/2, \quad \beta_n \leqslant \gamma^2\ln(3n)\exp\left(-L_\alpha n^{1-\alpha}\right) + \frac{2^\alpha\gamma}{\lambda n^\alpha}$,
  - $\alpha < 1/2, \quad \beta_n \leqslant \gamma^2\frac{n^{1-2\alpha}}{1-2\alpha}\exp\left(-L_\alpha n^{1-\alpha}\right) + \frac{2^\alpha\gamma}{\lambda n^\alpha}$.

  (iii) $\zeta_n \leqslant \max\left\{\frac{\gamma}{1-\gamma\lambda}\exp\left(-\frac{\gamma\lambda}{1-\alpha}\left((n+1)^{1-\alpha}-1\right)\right), \frac{\gamma}{n^\alpha}\right\}$.

Note that in this case for n large enough we have the following estimations:

(i) $\alpha_n \leqslant \exp\left(-\dfrac{\gamma\lambda}{2^{1-\alpha}(1-\alpha)}n^{1-\alpha}\right)$, *(ii)* $\beta_n \leqslant \dfrac{2^{\alpha+1}\gamma}{\lambda n^\alpha}$, *(iii)* $\zeta_n \leqslant \dfrac{\gamma}{n^\alpha}$.

Proof First we show for $\alpha \in [0,1]$ the equality for ζ_n. Denote $a_k = \gamma_k \prod_{i=k+1}^n (1-\gamma_i\lambda)$, we want to find $\zeta_n = \sup_{k\leqslant n} a_k$. We show for $\gamma_n = \dfrac{\gamma}{n^\alpha}$ that $(a_k)_{k\geqslant 1}$ decreases then increases so that $\zeta_n = \max\{a_1, a_n\}$. Let $k \leqslant n-1$,

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= \frac{\gamma_{k+1}}{\gamma_k}\frac{1}{(1-\gamma_{k+1}\lambda)} \\ &= \frac{1}{\frac{\gamma_k}{\gamma_{k+1}} - \gamma_k\lambda}\end{aligned}$$

Hence, $\dfrac{a_k}{a_{k+1}} - 1 = \dfrac{\gamma_k}{\gamma_{k+1}} - \gamma_k\lambda - 1$. Take $\alpha \in]0,1[$, in this case where $\gamma_n = \dfrac{\gamma}{n^\alpha}$,

$$\frac{a_k}{a_{k+1}} - 1 = \left(1+\frac{1}{k}\right)^\alpha - \frac{\gamma\lambda}{k^\alpha} - 1.$$

A rapid study of the function $f_\alpha(x) = \left(1+\dfrac{1}{x}\right)^\alpha - \dfrac{\gamma\lambda}{x^\alpha} - 1$ in $\mathbb{R}_+^\star$ shows that it decreases until $x_\star = (\gamma\lambda)^{\frac{1}{(\alpha-1)}} - 1$ then increases. This concludes the proof for $\alpha \in]0,1[$. By a direct calculation for $\alpha = 1$, $\dfrac{a_k}{a_{k+1}} - 1 = \dfrac{1-\gamma\lambda}{k} \geqslant 0$ thus a_k is non increasing and $\zeta_n = a_1 = \dfrac{\gamma}{1-\gamma\lambda}\alpha_n$. Similarly, for $\alpha = 0$, $\dfrac{a_k}{a_{k+1}} - 1 = \gamma\lambda < 0$ thus a_k is increasing and $\zeta_n = a_n = \gamma_n$.

We show now the different estimations we have for α_n, β_n and ζ_n for the three cases above.

- for $\alpha = 1$,

$$\begin{aligned}\ln\alpha_n &= \sum_{i=1}^n \ln\left(1-\frac{\gamma\lambda}{i}\right) \leqslant -\gamma\lambda\sum_{i=1}^n\frac{1}{i} \leqslant -\gamma\lambda\ln n \\ \alpha_n &\leqslant \frac{1}{n^{\gamma\lambda}}.\end{aligned}$$

Then,

$$\begin{aligned}
\beta_n &= \gamma^2 \sum_{k=1}^{n} \frac{1}{k^2} \prod_{i=k+1}^{n} \left(1 - \frac{\gamma\lambda}{i}\right)^2 \\
\beta_n &\leqslant \gamma^2 \sum_{k=1}^{n} \frac{1}{k^2} \exp\left(-2\gamma\lambda \sum_{i=k+1}^{n} \frac{1}{i}\right) \\
&\leqslant \gamma^2 \sum_{k=1}^{n} \frac{1}{k^2} \exp\left(-2\gamma\lambda \ln\left(\frac{n+1}{k+1}\right)\right) \\
&\leqslant \gamma^2 \sum_{k=1}^{n} \frac{1}{k^2} \left(\frac{k+1}{n+1}\right)^{2\gamma\lambda} \\
&\leqslant 4^{\gamma\lambda}\gamma^2 \sum_{k=1}^{n} \frac{1}{k^2} \left(\frac{k}{n}\right)^{2\gamma\lambda} \\
&\leqslant \frac{4^{\gamma\lambda}\gamma^2}{n^{2\gamma\lambda}} \sum_{k=1}^{n} k^{2\gamma\lambda-2},
\end{aligned}$$

Moreover for $\gamma\lambda < \frac{1}{2}$, $\sum_{k=1}^{n} k^{2\gamma\lambda-2} \leqslant 1 - \frac{1}{2\gamma\lambda - 1} = \frac{2(1-\gamma\lambda)}{1-2\gamma\lambda}$, hence,

$$\beta_n \leqslant \frac{2(1-\gamma\lambda)}{1-2\gamma\lambda} \frac{4^{\gamma\lambda}\gamma^2}{n^{2\gamma\lambda}}$$

Finally,

$$\zeta_n = \frac{\gamma}{1-\gamma\lambda}\alpha_n \leqslant \frac{\gamma}{1-\gamma\lambda}\frac{1}{n^{\gamma\lambda}}.$$

- for $\alpha = 0$,

$$\alpha_n = \prod_{i=1}^{n} (1-\gamma\lambda) = (1-\gamma\lambda)^n.$$

Then,

$$\beta_n = \gamma^2 \sum_{k=1}^{n} \prod_{i=k+1}^{n} (1-\gamma\lambda)^2 = \gamma^2 \sum_{k=1}^{n} (1-\gamma\lambda)^{2(n-k)} \leqslant \frac{1}{1-(1-\lambda\gamma)^2} \leqslant \frac{\gamma}{\lambda}.$$

Finally,

$$\zeta_n = \gamma_n = \gamma.$$

- for $\alpha \in]0,1[$,

$$\begin{aligned}
\ln \alpha_n &= \sum_{i=1}^{n} \ln\left(1 - \frac{\gamma\lambda}{i^\alpha}\right) \leqslant -\gamma\lambda \sum_{i=1}^{n} \frac{1}{i^\alpha} \leqslant -\gamma\lambda \frac{(n+1)^{1-\alpha} - 1}{1-\alpha} \\
\alpha_n &\leqslant \exp\left(-\frac{\gamma\lambda}{1-\alpha}\left((n+1)^{1-\alpha} - 1\right)\right).
\end{aligned}$$

To have an estimation on β_n, we are going to split it into two sums. Let $m \in [\![1, n]\!]$,

$$\begin{aligned}
\beta_n &= \sum_{k=1}^{n} \gamma_k^2 \prod_{i=k+1}^{n} (1-\gamma_i\lambda)^2 = \sum_{k=1}^{m} \gamma_k^2 \prod_{i=k+1}^{n} (1-\gamma_i\lambda)^2 + \sum_{k=m+1}^{n} \gamma_k^2 \prod_{i=k+1}^{n} (1-\gamma_i\lambda)^2 \\
\beta_n &\leqslant \sum_{k=1}^{m} \gamma_k^2 \exp\left(-2\lambda \sum_{i=m+1}^{n} \gamma_i\right) + \frac{\gamma_m}{\lambda} \sum_{k=m+1}^{n} \prod_{i=k+1}^{n} (1-\gamma_i\lambda)^2 \lambda\gamma_k \\
&\leqslant \sum_{k=1}^{n} \gamma_k^2 \exp\left(-2\lambda \sum_{i=m+1}^{n} \gamma_i\right) + \frac{\gamma_m}{\lambda} \sum_{k=m+1}^{n} \left[\prod_{i=k+1}^{n} (1-\gamma_i\lambda)^2 - \prod_{i=k+1}^{n} (1-\gamma_i\lambda)^2 (1-\gamma_k\lambda)\right] \\
&\leqslant \sum_{k=1}^{n} \gamma_k^2 \exp\left(-2\lambda \sum_{i=m+1}^{n} \gamma_i\right) + \frac{\gamma_m}{\lambda} \sum_{k=m+1}^{n} \left[\prod_{i=k+1}^{n} (1-\gamma_i\lambda)^2 - \prod_{i=k}^{n} (1-\gamma_i\lambda)^2\right] \\
&\leqslant \sum_{k=1}^{n} \gamma_k^2 \exp\left(-2\lambda \sum_{i=m+1}^{n} \gamma_i\right) + \frac{\gamma_m}{\lambda} \left(1 - \prod_{i=m+1}^{n} (1-\gamma_i\lambda)^2\right) \\
&\leqslant \sum_{k=1}^{n} \gamma_k^2 \exp\left(-2\lambda \sum_{i=m+1}^{n} \gamma_i\right) + \frac{\gamma_m}{\lambda}.
\end{aligned}$$

By taking $\gamma_n = \dfrac{\gamma}{n^\alpha}$ and $m = \lfloor \dfrac{n}{2} \rfloor$, we get:

$$\begin{aligned}
\beta_n &\leqslant \gamma^2 \sum_{k=1}^{n} \frac{1}{k^{2\alpha}} \exp\left(-2\lambda\gamma \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n} \frac{1}{i^\alpha}\right) + \frac{2^\alpha \gamma}{\lambda n^\alpha} \\
&\leqslant \gamma^2 \sum_{k=1}^{n} \frac{1}{k^{2\alpha}} \exp\left(-\frac{2\lambda\gamma}{1-\alpha}\left((n+1)^{1-\alpha} - \left(\frac{n}{2}+1\right)^{1-\alpha}\right)\right) + \frac{2^\alpha \gamma}{\lambda n^\alpha} \\
&\leqslant \gamma^2 \sum_{k=1}^{n} \frac{1}{k^{2\alpha}} \exp\left(-\frac{2\lambda\gamma}{1-\alpha} n^{1-\alpha}\left(\left(1+\frac{1}{n}\right)^{1-\alpha} - \left(\frac{1}{2}+\frac{1}{n}\right)^{1-\alpha}\right)\right) + \frac{2^\alpha \gamma}{\lambda n^\alpha} \\
&\leqslant \gamma^2 \sum_{k=1}^{n} \frac{1}{k^{2\alpha}} \exp\left(-\frac{2\lambda\gamma}{1-\alpha} n^{1-\alpha} 2^{1-\alpha}\left(1 - \left(\frac{3}{4}\right)^{1-\alpha}\right)\right) + \frac{2^\alpha \gamma}{\lambda n^\alpha}.
\end{aligned}$$

Calling $S_n^\alpha = \sum_{k=1}^{n} \frac{1}{k^{2\alpha}}$ and noting that: for $\alpha > 1/2,\ S_n^\alpha \leqslant \frac{2\alpha}{2\alpha-1}$, $\alpha = 1/2,\ S_n^\alpha \leqslant \ln(3n)$ and $\alpha < 1/2,\ S_n^\alpha \leqslant \frac{n^{1-2\alpha}}{1-2\alpha}$ we have the expected result.
Finally,

$$\zeta_n \leqslant \max\left\{\frac{\gamma}{1-\gamma\lambda} \exp\left(-\frac{\gamma\lambda}{1-\alpha}\left((n+1)^{1-\alpha} - 1\right)\right), \frac{\gamma}{n^\alpha}\right\}.$$

■

Proof [Proof of Theorem 2] We apply Proposition 5, and the bound found on α_n, β_n and ζ_n in Lemma 7 to get the results. ■

E.2 Averaged SGD - Proof of Theorem 3

We consider the same recursion but with $\gamma_n = \gamma$:

$$\eta_n = (I - \gamma H_n)\eta_{n-1} + \gamma\varepsilon_n,$$

started at $\eta_0 = 0$ and with assumptions **H-abcde**.

However, in this section, we consider the averaged:

$$\bar{\eta}_n = \frac{1}{n+1}\sum_{i=0}^{n}\eta_i.$$

Thus, we get

$$\bar{\eta}_n = \frac{1}{n+1}\sum_{i=0}^{n}\gamma\sum_{k=1}^{i}M(i,k+1)\varepsilon_k = \frac{\gamma}{n+1}\sum_{k=1}^{n}\Big(\sum_{i=k}^{n}M(i,k+1)\Big)\varepsilon_k = \frac{\gamma}{n+1}\sum_{k=1}^{n}\bar{Z}_k.$$

Our the goal is to bound $\mathbb{P}\left(\|\bar{\eta}_n\| \geqslant t\right)$ using Propostion 3 that is going to lead us to some Bernstein concentration inquality. Calling, as above, $\bar{Z}_k = \sum_{i=k}^{n}M(i,k+1)\varepsilon_k$, and as $\mathbb{E}\left[\bar{Z}_k|\mathcal{F}_{k-1}\right] = 0$ we just need to bound, $\sup_{k\leqslant n}\|\bar{Z}_k\|$ and $\sum_{k=1}^{n}\mathbb{E}\left[\|\bar{Z}_k\|^2|\mathcal{F}_{k-1}\right]$. For a more general result, we consider in the following lemma $(A^{1/2}\bar{Z}_k)_k$.

Lemma 8 *Assuming **H-abcde**, we have the following bounds for $\bar{Z}_k = \sum_{i=k}^{n}M(i,k+1)\varepsilon_k$:*

$$\sup_{k\leqslant n}\|A^{1/2}\bar{Z}_k\| \leqslant \frac{c^{1/2}\|A\|_{op}^{1/2}}{\gamma\lambda} \tag{17}$$

$$\sum_{k=1}^{n}\mathbb{E}\left[\|A^{1/2}\bar{Z}_k\|^2|\mathcal{F}_{k-1}\right] \leqslant n\frac{1}{\gamma^2}\frac{1}{1-\gamma/2\gamma_0}\operatorname{tr}\left(AH^{-2}\cdot C\right). \tag{18}$$

Proof First $\|A^{1/2}\bar{Z}_k\| \leqslant \|A\|_{op}^{1/2}\|\bar{Z}_k\|$ and we have, almost surely, $\|\varepsilon_k\| \leqslant c^{1/2}$ and $H_n \succcurlyeq \lambda I$, thus for all k, as $\gamma\lambda \leqslant 1$, $I - \gamma H_k \preccurlyeq (1-\gamma\lambda)I$. Hence, $\|M(i,k+1)\|_{\rm op} \leqslant (1-\gamma\lambda)^{i-k}$ and,

$$\|\bar{Z}_k\| \leqslant \|\varepsilon_k\|\sum_{i=k}^{n}\|M(i,k+1)\|_{\rm op} \leqslant c^{1/2}\sum_{i=k}^{n}(1-\gamma\lambda)^{i-k} \leqslant \frac{c^{1/2}}{\gamma\lambda}$$

Second, we need an upper bound on $\mathbb{E}\left[\|A^{1/2}\bar{Z}_k\|^2|\mathcal{F}_{k-1}\right]$, we are going to find it in two steps:

- **Step 1:** we first show that the upper bound depends of the trace of some operator involving H^{-1}.

$$\mathbb{E}\left[\|A^{1/2}\bar{Z}_k\|^2|\mathcal{F}_{k-1}\right] \leqslant 2\sum_{i=k}^{n}\operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M(i,k+1)CM(i,k+1)^*\right]\right),$$

- **Step 2:** we then upperbound this sum to a telescopic one involving H^{-2} to finally show:

$$\mathbb{E}\left[\|A^{1/2}\bar{Z}_k\|^2|\mathcal{F}_{k-1}\right] \leqslant \frac{1}{\gamma^2}\frac{1}{1-\gamma/2\gamma_0}\operatorname{tr}\left(AH^{-2}C\right).$$

Step 1: We write,

$$\begin{aligned}
\mathbb{E}\left[\|A^{1/2}\bar{Z}_k\|^2|\mathcal{F}_{k-1}\right] &= \mathbb{E}\left[\sum_{k\leqslant i,j\leqslant n}\left\langle A^{1/2}M(i,k+1)\varepsilon_k, A^{1/2}M(j,k+1)\varepsilon_k\right\rangle|\mathcal{F}_{k-1}\right] \\
&= \mathbb{E}\left[\sum_{k\leqslant i,j\leqslant n}\langle M(i,k+1)\varepsilon_k, AM(j,k+1)\varepsilon_k\rangle|\mathcal{F}_{k-1}\right] \\
&= \sum_{k\leqslant i,j\leqslant n}\mathbb{E}\left[\mathrm{tr}\left(M(i,k+1)^*AM(j,k+1)\cdot\varepsilon_k\otimes\varepsilon_k\right)\right] \\
&= \sum_{k\leqslant i,j\leqslant n}\mathrm{tr}\left(\mathbb{E}\left[M(i,k+1)^*AM(j,k+1)\right]\cdot\mathbb{E}\left[\varepsilon_k\otimes\varepsilon_k\right]\right).
\end{aligned}$$

We have $\mathbb{E}\left[\varepsilon_k\otimes\varepsilon_k\right]\preccurlyeq C$ so that as every operators are positive semi-definite,

$$\mathbb{E}\left[\|A^{1/2}\bar{Z}_k\|^2|\mathcal{F}_{k-1}\right] \leqslant \sum_{k\leqslant i,j\leqslant n}\mathrm{tr}\left(\mathbb{E}\left[M(i,k+1)^*AM(j,k+1)\right]\cdot C\right).$$

We now bound the last expression by dividing it into two terms, noting $M(i,k)=M_k^i$ for more compact notations (only until the end of the proof),

$$\sum_{k\leqslant i,j\leqslant n}\mathrm{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^*AM_{k+1}^j\right]\cdot C\right) = \sum_{i=k}^n\mathrm{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^*AM_{k+1}^i\right]\cdot C\right)+2\sum_{k\leqslant i<j\leqslant n}\mathrm{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^*AM_{k+1}^j\right]\cdot C\right).$$

Moreover,

$$\begin{aligned}
&\sum_{k\leqslant i<j\leqslant n}\mathrm{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^*AM_{k+1}^j\right]\cdot C\right) \\
=\ &\sum_{k\leqslant i<j\leqslant n}\mathrm{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^*A\left(I-\gamma H\right)^{j-i}M_{k+1}^i\right]\cdot C\right) \\
=\ &\sum_{i=k}^n\mathrm{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^*A\sum_{j=i+1}^n\left(I-\gamma H\right)^{j-i}M_{k+1}^i\right]\cdot C\right) \\
=\ &\sum_{i=k}^n\mathrm{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^*A\left[\left(I-\gamma H\right)\left(I-\left(I-\gamma H\right)^{n-i}\right)(\gamma H)^{-1}\right]M_{k+1}^i\right]\cdot C\right) \\
\leqslant\ &\sum_{i=k}^n\mathrm{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^*A\left[(\gamma H)^{-1}-I\right]M_{k+1}^i\right]\cdot C\right) \\
\leqslant\ &\sum_{i=k}^n\mathrm{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^*A\left(\gamma H\right)^{-1}M_{k+1}^i\right]\cdot C\right)-\sum_{i=k}^n\mathrm{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^*AM_{k+1}^i\right]\cdot C\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{k\leqslant i,j\leqslant n} \operatorname{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^* A M_{k+1}^j\right]\cdot C\right) &= \sum_{i=k}^{n} \operatorname{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^* A M_{k+1}^i\right]\cdot C\right) + 2\sum_{k\leqslant i<j\leqslant n} \operatorname{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^* A M_{k+1}^j\right]\cdot C\right)\\
&\leqslant 2\sum_{i=k}^{n} \operatorname{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^* A\left(\gamma H\right)^{-1} M_{k+1}^i\right]\cdot C\right) - \sum_{i=k}^{n} \operatorname{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^* A M_{k+1}^i\right]\cdot C\right)\\
&\leqslant 2\sum_{i=k}^{n} \operatorname{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^* A\left(\gamma H\right)^{-1} M_{k+1}^i\right]\cdot C\right)\\
&\leqslant 2\sum_{i=k}^{n} \operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^i C {M_{k+1}^i}^*\right]\right)
\end{aligned}$$

This concludes step 1.

Step 2: Let us now try to bound $\sum_{i=k}^{n} \operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^i C {M_{k+1}^i}^*\right]\right)$. We will do so by bounding it by a telescopic sum. Indeed,

$$\begin{aligned}
\mathbb{E}\left[M_{k+1}^{i+1} C\left(\gamma H\right)^{-1} {M_{k+1}^{i+1}}^*\right] &= \mathbb{E}\left[M_{k+1}^i\left(I-\gamma H_{i+1}\right) C\left(\gamma H\right)^{-1}\left(I-\gamma H_{i+1}\right) {M_{k+1}^i}^*\right]\\
&= \mathbb{E}\left[M_{k+1}^i \mathbb{E}\left[C\left(\gamma H\right)^{-1} - CH^{-1}H_{i+1} - H_{i+1}CH^{-1} + \gamma H_{i+1}CH^{-1}H_{i+1}\right] {M_{k+1}^i}^*\right]\\
&= \mathbb{E}\left[M_{k+1}^i C\left(\gamma H\right)^{-1} {M_{k+1}^i}^*\right] - 2\mathbb{E}\left[M_{k+1}^i C {M_{k+1}^i}^*\right]\\
&\qquad +\gamma\mathbb{E}\left[M_{k+1}^i \mathbb{E}\left[H_{i+1}CH^{-1}H_{i+1}\right] {M_{k+1}^i}^*\right],
\end{aligned}$$

such that, by multiplying the previous equality by $A\left(\gamma H\right)^{-1}$ and taking the trace we have,

$$\begin{aligned}
\operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^{i+1} C\left(\gamma H\right)^{-1} {M_{k+1}^{i+1}}^*\right]\right) &= \operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^i C\left(\gamma H\right)^{-1} {M_{k+1}^i}^*\right]\right)\\
&- 2\operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^i C {M_{k+1}^i}^*\right]\right)\\
&+ \gamma\operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^i \mathbb{E}\left[H_{i+1}CH^{-1}H_{i+1}\right] {M_{k+1}^i}^*\right]\right),
\end{aligned}$$

And as $\mathbb{E}\left[H_k C H^{-1} H_k\right] \preccurlyeq \gamma_0^{-1} C$ we have,

$$\gamma\operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^i \mathbb{E}\left[H_{i+1}CH^{-1}H_{i+1}\right] {M_{k+1}^i}^*\right]\right) \leqslant \gamma/\gamma_0 \operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^i C {M_{k+1}^i}^*\right]\right),$$

thus,

$$\begin{aligned}
\operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^{i+1} C\left(\gamma H\right)^{-1} {M_{k+1}^{i+1}}^*\right]\right) &\leqslant \operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^i C\left(\gamma H\right)^{-1} {M_{k+1}^i}^*\right]\right)\\
&- 2\operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^i C {M_{k+1}^i}^*\right]\right)\\
&+ \gamma/\gamma_0 \operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^i C {M_{k+1}^i}^*\right]\right)
\end{aligned}$$

$$\begin{aligned}
&\operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^i C {M_{k+1}^i}^*\right]\right)\\
\leqslant\ & \frac{1}{2-\gamma/\gamma_0}\left(\operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^i C\left(\gamma H\right)^{-1} {M_{k+1}^i}^*\right]\right) - \operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^{i+1} C\left(\gamma H\right)^{-1} {M_{k+1}^{i+1}}^*\right]\right)\right).
\end{aligned}$$

If we take all the calculations from the beginning,

$$\begin{aligned}
\mathbb{E}\left[\|A^{1/2}\bar{Z}_k\|^2|\mathcal{F}_{k-1}\right] &\leqslant \sum_{k\leqslant i,j\leqslant n} \operatorname{tr}\left(\mathbb{E}\left[{M_{k+1}^i}^* A M_{k+1}^j\right]\cdot C\right) \\
&\leqslant 2\sum_{i=k}^n \operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^i C {M_{k+1}^i}^*\right]\right) \\
&\leqslant \frac{2}{2-\gamma/\gamma_0}\sum_{i=k}^n \operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^i C\left(\gamma H\right)^{-1}{M_{k+1}^i}^*\right]\right) \\
&\qquad - \operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^{i+1} C\left(\gamma H\right)^{-1}{M_{k+1}^{i+1}}^*\right]\right) \\
&\leqslant \frac{2}{2-\gamma/\gamma_0}\operatorname{tr}\left(A\left(\gamma H\right)^{-1}\mathbb{E}\left[M_{k+1}^k C\left(\gamma H\right)^{-1}{M_{k+1}^k}^*\right]\right) \\
&\leqslant \frac{1}{\gamma^2}\frac{1}{1-\gamma/2\gamma_0}\operatorname{tr}\left(AH^{-2}\cdot C\right),
\end{aligned}$$

which concludes the proof if we sum this inequality from 1 to n.

■

Proof [Proof of Theorem 3] We apply Corollary 2 to the sequence $\left(\frac{\gamma}{n+1}A^{1/2}Z_k\right)_{k\leqslant n}$ thanks to Lemma 8. We have:

$$\begin{aligned}
\sup_{k\leqslant n}\|\frac{\gamma}{n+1}A^{1/2}Z_k\| &\leqslant \frac{c^{1/2}\|A^{1/2}\|}{(n+1)\lambda} \\
\sum_{k=1}^n \mathbb{E}\left[\|\frac{\gamma}{n+1}A^{1/2}Z_k\|^2|\mathcal{F}_{k-1}\right] &\leqslant \frac{1}{n+1}\frac{1}{1-\gamma/2\gamma_0}\operatorname{tr}\left(AH^{-2}\cdot C\right),
\end{aligned}$$

so that,

$$\begin{aligned}
\mathbb{P}\left(\left\|A^{1/2}\bar{\eta}_n\right\|\geqslant t\right) &= \mathbb{P}\left(\left\|\sum_{k=1}^n \frac{\gamma}{n+1}A^{1/2}Z_k\right\|\geqslant t\right) \leqslant 2\exp\left(-\frac{t^2}{2\left(\frac{\operatorname{tr}(AH^{-2}\cdot C)}{(n+1)(1-\gamma/2\gamma_0)}+\frac{c^{1/2}\|A^{1/2}\|t}{3\lambda(n+1)}\right)}\right) \\
\mathbb{P}\left(\left\|A^{1/2}\bar{\eta}_n\right\|\geqslant t\right) &\leqslant 2\exp\left(-\frac{(n+1)t^2}{\frac{2\operatorname{tr}(AH^{-2}\cdot C)}{(1-\gamma/2\gamma_0)}+\frac{2\|A^{1/2}\|c^{1/2}t}{3\lambda}}\right).
\end{aligned}$$

■

E.3 Tail-averaged SGD - Proof of Corollary 1

We now prove the result for tail-averaging that allow us to include relax the assumption that $\eta_0 = 0$.

Proof [Proof of Corollary 1]

Let $n \geqslant 1$ and n an even number for the sake of clarity (the case where n is an odd number can be solved similarly),

$$\begin{aligned} A^{1/2}\bar{\eta}_n^{\text{tail}} &= \frac{1}{n/2}\sum_{k=n/2}^{n} A^{1/2}\eta_k \\ &= \frac{1}{n/2}\sum_{k=n/2}^{n} A^{1/2}M(k,1)\eta_0 + \frac{1}{n/2}\sum_{k=n/2}^{n} A^{1/2}W_k \\ &= \frac{1}{n/2}\sum_{k=n/2}^{n} A^{1/2}M(k,1)\eta_0 + 2A^{1/2}\overline{W}_n - A^{1/2}\overline{W}_{n/2}. \end{aligned}$$

Hence,

$$\begin{aligned} \left\|A^{1/2}\bar{\eta}_n^{\text{tail}}\right\| &\leqslant \left\|\frac{1}{n/2}\sum_{k=n/2}^{n} A^{1/2}M(k,1)\eta_0\right\| + 2\left\|A^{1/2}\overline{W}_n\right\| + \left\|A^{1/2}\overline{W}_{n/2}\right\| \\ &\leqslant \frac{1}{n/2}\sum_{k=n/2}^{n}\left\|A^{1/2}M(k,1)\right\|_{op}\|\eta_0\| + 2\left\|A^{1/2}\overline{W}_n\right\| + \left\|A^{1/2}\overline{W}_{n/2}\right\|, \end{aligned}$$

Let $L_n = 2\left\|A^{1/2}\overline{W}_n\right\| + \left\|A^{1/2}\overline{W}_{n/2}\right\|$,

$$\begin{aligned} \left\|A^{1/2}\bar{\eta}_n^{\text{tail}}\right\| &\leqslant \frac{1}{n/2}\sum_{k=n/2}^{n}\|A^{1/2}\|_{op}(1-\gamma\lambda)^k\|\eta_0\| + L_n \\ \left\|A^{1/2}\bar{\eta}_n^{\text{tail}}\right\| &\leqslant (1-\gamma\lambda)^{n/2}\|A^{1/2}\|_{op}\|\eta_0\| + L_n, \end{aligned}$$

And finally for $t \geqslant 0$,

$$\begin{aligned} \mathbb{P}(L_n \geqslant t) &= \mathbb{P}(2\left\|A^{1/2}\overline{W}_n\right\| + \left\|A^{1/2}\overline{W}_{n/2}\right\| \geqslant t) \\ &\leqslant \mathbb{P}\left(2\left\|A^{1/2}\overline{W}_n\right\| \geqslant t\right) + \mathbb{P}\left(\left\|A^{1/2}\overline{W}_{n/2}\right\| \geqslant t\right) \\ &\leqslant 2\left[\exp\left(-\frac{(n+1)(t/2)^2}{E_{t/2}}\right) + \exp\left(-\frac{(n/2+1)t^2}{E_t}\right)\right]. \end{aligned}$$

Let us remark that $E_{t/2} \leqslant E_t$. Hence,

$$\begin{aligned} \mathbb{P}(L_n \geqslant t) &\leqslant 2\left[\exp\left(-\frac{(n+1)t^2}{4E_t}\right) + \exp\left(-\frac{(n+1)t^2}{2E_t}\right)\right] \\ &\leqslant 4\exp\left(-\frac{(n+1)t^2}{4E_t}\right). \end{aligned}$$

■

F Exponentially convergent SGD for classification error

In this section we prove the results for the error in the case of SGD. Let us recall the recursion:

$$g_n - g_\lambda = \left[I - \gamma_n(K_{x_n} \otimes K_{x_n} + \lambda I)\right](g_{n-1} - g_\lambda) + \gamma_n \varepsilon_n,$$

with the noise term $\varepsilon_k = \xi_k K_{x_k} + (g_*(x_k) - g_\lambda(x_k))K_{x_k} - \mathbb{E}\left[(g_*(x_k) - g_\lambda(x_k))K_{x_k}\right] \in \mathcal{H}$.
This is the same recursion as in Eq (3):

$$\eta_n = (I - \gamma H_n)\eta_{n-1} + \gamma_n \varepsilon_n,$$

with $H_n = K_{x_n} \otimes K_{x_n} + \lambda I$ and $\eta_n = g_n - g_\lambda$.
First we begin by showing that for this recursion and assuming **(A-24)**, we can show **(H-abcd)**.

Lemma 9 (Showing (H-abcd) for SGD recursion.) *Let us assume* **(A-24)**,

- ***(H-a)*** *We start at some $g_0 - g_\lambda \in \mathcal{H}$.*
- ***(H-b)*** *(H_n, ε_n) i.i.d. and H_n is a positive self-adjoint operator so that almost surely $H_n \succcurlyeq \lambda I$, with $H = \mathbb{E}H_n = \Sigma + \lambda I$.*
- ***(H-c)*** *We have the two following bounds on the noise:*

$$\begin{aligned} \|\varepsilon_n\| &\leqslant R(1 + 2\|g_* - g_\lambda\|_{L_\infty}) = c^{1/2} \\ \mathbb{E}\varepsilon_n \otimes \varepsilon_n &\preccurlyeq 2\left(1 + \|g_* - g_\lambda\|_\infty^2\right)\Sigma = C \\ \mathbb{E}\|\varepsilon_n\|^2 &\leqslant 2\left(1 + \|g_* - g_\lambda\|_\infty^2\right)\operatorname{tr}\Sigma = \operatorname{tr} C. \end{aligned}$$

- ***(H-d)*** *We have:*

$$\mathbb{E}\Big[H_k C H^{-1} H_k\Big] \preccurlyeq \left(R^2 + 2\lambda\right) C = \gamma_0^{-1} C \ .$$

Proof (**H-ab**) are obviously satisfied.

Let us show (**H-c**):

$$\begin{aligned} \|\varepsilon_n\| &= \|\xi_n K_{x_n} + (g_*(x_n) - g_\lambda(x_n))K_{x_n} - \mathbb{E}\left[(g_*(x_n) - g_\lambda(x_n))K_{x_n}\right]\| \\ &\leqslant (|\xi_n| + |g_*(x_n) - g_\lambda(x_n)|)\|K_{x_n}\| + \mathbb{E}\left[|g_*(x_n) - g_\lambda(x_n)|\|K_{x_n}\|\right] \\ &\leqslant (1 + \|g_* - g_\lambda\|_\infty)R + \|g_* - g_\lambda\|_\infty R \\ &= R(1 + 2\|g_* - g_\lambda\|_\infty) \end{aligned}$$

We have [2]:

$$\begin{aligned} \varepsilon_n \otimes \varepsilon_n &\preccurlyeq 2\xi_n K_{x_n} \otimes \xi_n K_{x_n} + 2\left((g_*(x_n) - g_\lambda(x_n))K_{x_n} - \mathbb{E}\left[(g_*(x_n) - g_\lambda(x_n))K_{x_n}\right]\right) \\ &\otimes \left((g_*(x_n) - g_\lambda(x_n))K_{x_n} - \mathbb{E}\left[(g_*(x_n) - g_\lambda(x_n))K_{x_n}\right]\right) \end{aligned}$$

Moreover, $\mathbb{E}[\xi_n K_{x_n} \otimes \xi_n K_{x_n}] = \mathbb{E}[\xi_n^2 K_{x_n} \otimes K_{x_n}] \preccurlyeq \Sigma$,

[2] We use the following inequality: for all a and $b \in \mathcal{H}$, $(a+b) \otimes (a+b) \preccurlyeq 2a \otimes a + 2b \otimes b$. Indeed, for all $x \in \mathcal{H}$, $\langle x, (a+b) \otimes (a+b)x\rangle = (\langle a+b, x\rangle)^2 = (\langle a, x\rangle + \langle b, x\rangle)^2 \leqslant 2\langle a, x\rangle^2 + 2\langle b, x\rangle^2 = 2\langle x, (a \otimes a)x\rangle + 2\langle x, (b \otimes b)x\rangle$.

And,

$$\begin{aligned}
&\mathbb{E}\left[((g_*(x_n)-g_\lambda(x_n))K_{x_n}-\mathbb{E}\left[(g_*(x_n)-g_\lambda(x_n)K_{x_n}\right])\otimes((g_*(x_n)-g_\lambda(x_n))K_{x_n}-\mathbb{E}\left[(g_*(x_n)-g_\lambda(x_n))K_{x_n}\right])\right]\\
&= \mathbb{E}\left[(g_*(x_n)-g_\lambda(x_n))^2(x_n)K_{x_n}\otimes K_{x_n}\right]-\mathbb{E}\left[(g_*(x_n)-g_\lambda(x_n))K_{x_n}\right]\otimes\mathbb{E}\left[(g_*(x_n)-g_\lambda(x_n))K_{x_n}\right]\\
&\preccurlyeq \mathbb{E}\left[(g_*(x_n)-g_\lambda(x_n))^2(x_n)K_{x_n}\otimes K_{x_n}\right]\\
&\preccurlyeq \|g_*-g_\lambda\|_\infty^2\Sigma.
\end{aligned}$$

So that,

$$\mathbb{E}\varepsilon_n\otimes\varepsilon_n \preccurlyeq 2\left(1+\|g_*-g_\lambda\|_\infty^2\right)\Sigma$$

Finally, as $\mathbb{E}\varepsilon_n\otimes\varepsilon_n \preccurlyeq 2\left(1+\|g_*-g_\lambda\|_\infty^2\right)\Sigma$, we have $\operatorname{tr}\mathbb{E}\varepsilon_n\otimes\varepsilon_n \leqslant 2\left(1+\|g_*-g_\lambda\|_\infty^2\right)\operatorname{tr}\Sigma$, such that

$$\operatorname{tr}\mathbb{E}\varepsilon_n\otimes\varepsilon_n = \mathbb{E}\operatorname{tr}\varepsilon_n\otimes\varepsilon_n = \mathbb{E}\|\varepsilon_n\|^2 \leqslant 2\left(1+\|g_*-g_\lambda\|_\infty^2\right)\operatorname{tr}\Sigma.$$

To conclude the proof of this lemma, let us show (**H-d**):

We have:

$$\begin{aligned}
\mathbb{E}\Big[(K_{x_k}\otimes K_{x_k}+\lambda I)\Sigma(\Sigma+\lambda I)^{-1}(K_{x_k}\otimes K_{x_k}+\lambda I)\Big] &= \mathbb{E}\Big[K_{x_k}\otimes K_{x_k}\Sigma(\Sigma+\lambda I)^{-1}K_{x_k}\otimes K_{x_k}\Big]\\
&+ \lambda\Sigma\Sigma(\Sigma+\lambda I)^{-1}+\lambda\Sigma
\end{aligned}$$

Moreover, $\lambda\Sigma\Sigma(\Sigma+\lambda I)^{-1} = \lambda\Sigma(\Sigma+\lambda I-\lambda I)(\Sigma+\lambda I)^{-1} = \lambda\Sigma-\lambda^2\Sigma(\Sigma+\lambda I)^{-1} \preccurlyeq \lambda\Sigma$, and similarly, $\mathbb{E}\Big[K_{x_k}\otimes K_{x_k}\Sigma(\Sigma+\lambda I)^{-1}K_{x_k}\otimes K_{x_k}\Big] = \mathbb{E}\Big[(K_{x_k}\otimes K_{x_k})^2\Big]-\lambda\mathbb{E}\Big[K_{x_k}\otimes K_{x_k}(\Sigma+\lambda I)^{-1}K_{x_k}\otimes K_{x_k}\Big] \preccurlyeq R^2\Sigma$.

Finally we obtain $\mathbb{E}\Big[(K_{x_k}\otimes K_{x_k}+\lambda I)\Sigma(\Sigma+\lambda I)^{-1}(K_{x_k}\otimes K_{x_k}+\lambda I)\Big] \preccurlyeq R^2\Sigma+\lambda\Sigma+\lambda\Sigma = (R^2+2\lambda)\Sigma$.

■

F.1 SGD with decreasing step-size

Proof [Proof of Theorem 4]

Let us apply Theorem 2 to g_n-g_λ. We assume **(A-24)** and $A=I$, such that **(H-abcde)** are verified (Lemma 9). Let δ correspond to the one of Assumption 5. We have for $t=\delta/(4R), n\geqslant 1$:

$$\begin{aligned}
\|g_n-g_\lambda\|_{\mathcal{H}} &\leqslant \exp\left(-\frac{\gamma\lambda}{1-\alpha}\left((n+1)^{1-\alpha}-1\right)\right)\|g_0-g_\lambda\|_{\mathcal{H}}+\|W_n\|_{\mathcal{H}}, \quad \text{almost surely, with}\\
\mathbb{P}\left(\|W_n\|_{\mathcal{H}}\geqslant\delta/(4R)\right) &\leqslant 2\exp\left(-\frac{\delta^2}{C_R}n^\alpha\right), \quad C_R=\gamma(2^{\alpha+6}R^2\operatorname{tr}C/\lambda+8Rc^{1/2}\delta/3).
\end{aligned}$$

Then if n is such that $\exp\left(-\frac{\gamma\lambda}{1-\alpha}\left((n+1)^{1-\alpha}-1\right)\right)\leqslant\dfrac{\delta}{5R\|g_0-g_\lambda\|_{\mathcal{H}}}$,

$$\begin{aligned}
\|g_n-g_\lambda\|_{\mathcal{H}} &\leqslant \frac{\delta}{5R}+\frac{\delta}{4R}, \quad \text{with probability } 1-2\exp\left(-\frac{\delta^2}{C_R}n^\alpha\right),\\
\|g_n-g_\lambda\|_{\mathcal{H}} &< \frac{\delta}{2R}, \quad \text{with probability } 1-2\exp\left(-\frac{\delta^2}{C_R}n^\alpha\right).
\end{aligned}$$

Now assume **(A-5)**, we simply apply Lemma 1 to g_n with $q = 2\exp\left(-\frac{\delta^2}{C_R}n^\alpha\right)$ And

$$C_R = \gamma(2^{\alpha+6}R^2 \operatorname{tr} C/\lambda + 8Rc^{1/2}\delta/3) = \gamma\left(\frac{2^{\alpha+7}R^2 \operatorname{tr}\Sigma\left(1+\|g_*-g_\lambda\|_\infty^2\right)}{\lambda} + \frac{8R^2\delta(1+2\|g_*-g_\lambda\|_\infty)}{3}\right).$$

■

F.2 Tail averaged SGD with constant step-size

Proof [Proof of Theorem 5]

Let us apply Corollary 1 to $g_n - g_\lambda$. We assume **(A-24)** and $A = I$, such that **(H-abcde)** are verified (Lemma 9). Let δ correspond to the one of Assumption5. We have for $t = \delta/(4R), n \geqslant 1$:

$$\begin{aligned} \left\|\bar{g}_n^{\text{tail}} - g_\lambda\right\|_{\mathcal{H}} &\leqslant (1-\gamma\lambda)^{n/2}\|g_0 - g_\lambda\|_{\mathcal{H}} + L_n \quad \text{, with} \\ \mathbb{P}(L_n \geqslant t) &\leqslant 4\exp\left(-(n+1)t^2/(4E_t)\right). \end{aligned}$$

Then as soon as $(1-\gamma\lambda)^{n/2} \leqslant \dfrac{\delta}{5R\|g_0 - g_\lambda\|_{\mathcal{H}}}$,

$$\begin{aligned} \left\|\bar{g}_n^{\text{tail}} - g_\lambda\right\|_{\mathcal{H}} &\leqslant \frac{\delta}{5R} + \frac{\delta}{4R}, \quad \text{with probability } 1 - 4\exp\left(-(n+1)\delta^2/(64R^2E_{\delta/(4R)})\right), \\ \left\|\bar{g}_n^{\text{tail}} - g_\lambda\right\|_{\mathcal{H}} &< \frac{\delta}{2R}, \quad \text{with probability } 1 - 4\exp\left(-(n+1)\delta^2/(64R^2E_{\delta/(4R)})\right). \end{aligned}$$

Now assume **(A-5)**, we simply apply Lemma 1 to $\bar{g}_n^{\text{tail}}$ with $q = 4\exp\left(-(n+1)\delta^2/K_R)\right)$. And

$$\begin{aligned} K_R = 64R^2E_{\delta/(4R)} &= 64R^2\left(4\operatorname{tr}(H^{-2}C) + \frac{2c^{1/2}}{3\lambda}\cdot\frac{\delta}{4R}\right) \\ &= 512R^2\left(1+\|g_*-g_\lambda\|_\infty^2\right)\operatorname{tr}((\Sigma+\lambda I)^{-2}\Sigma) + \frac{32R^2(1+2\|g_*-g_\lambda\|_\infty)}{3\lambda}. \end{aligned}$$

■

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