A cubical Squier's theorem

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Abstract

The homotopical Squier's theorem relates rewriting properties of a presentation of a monoid with homotopical invariants of this monoid. Lately, this theorem has been extended, enabling one to build a so-called polygraphic resolution of a monoid starting from a presentation with suitable rewriting properties.

It is currently a work in progress to get a better understanding of these results. We argue that cubical categories are a more natural setting in which to express and possibly extend those results. As a proof-of-concept, we give in this paper a new proof of Squier's homotopical theorem using cubical categories.

1 Introduction

Convergent rewriting systems are well-known tools in the study of the word-rewriting problem. In particular, a presentation of a monoid by a finite convergent rewriting system gives an algorithm to decide the word problem for this monoid. In (Squier, 1987) and (Squier, Otto and Kobayashi, 1994), the authors proved that there exists a finitely presented monoid whose word problem was decidable but which did not admit a finite convergent presentation. To do so, they constructed, for any convergent presentation (G, R) of a monoid M, a set of syzygies S corresponding to relations between the relations. Let us make this result a bit more precise.

We start from a presentation of a monoid. For example a presentation of the braid monoid B_3^+ is given by:

$$\langle a, s, t | ta = as, sa = a, sas = aa, saa = aat \rangle$$

Presenting B_3^+ by a monoidal polygraph (or computad) Σ consists in choosing a name and an orientation for the relations, giving for exaple:

$$\Sigma := \langle a, s, t | \alpha : ta \to as, \beta : sa \to a, \gamma : sas \to aa, \delta : saa \to aat \rangle$$

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Such a monoidal polygraph Σ generates a free (strict) monoidal groupoid, denoted $\Sigma^\top$, and a free (strict) monoidal catgeory, denoted Σ^*. The 0-cells are the free monoid generated by a, s and t, and the 1-cells are generated by α, β, γ and δ.

What we now want to do is to find a *coherent* presentation of B_3^+. This corresponds to extending Σ into a monoidal 2-polygraph such that the monoidal 2-groupoid it generates (denoted $\Sigma^\top$) satisfies the following property: for any couple of parallel 1-cells f and g in $\Sigma^\top$, there exists a 2-cell $A : f \Rightarrow g$ in $\Sigma^\top$. Squier's homotopical theorem consists in finding sufficient conditions for an extension of Σ into a 2-polygraph to be a coherent presentation of B_3^+:

Theorem 1.1 (Squier)**.** *Let Σ be a convergent monoidal 2-polygraph. Suppose that for every critical pair (f_1, f_2) of Σ, there exist two 1-cells g_1 and g_2 in Σ^* and a 2-cell A in $\Sigma^\top$ of the following shape:*

$$\begin{tikzcd}
\cdot \arrow[r, "f_1"] \arrow[d, "f_2"'] \arrow[dr, phantom, "\swArrow A"] & \cdot \arrow[d, dotted, "g_1"] \\
\cdot \arrow[r, dotted, "g_2"'] & \cdot
\end{tikzcd}$$

Then Σ is coherent.

Let us delay the definition of convergence for now (see Definition 3.10). A branching is a pair of 1-cells in Σ_1^* with the same source. Critical branchings are a special case of branching (see Definition 3.8). In our case of Σ is convergent and there are four critical branchings $(\beta a, s\alpha)$, $(\gamma t, sa\beta)$ $(\gamma as, sa\gamma)$ and $(\gamma aa, sa\gamma)$. So what Squier's theorem says is that the following extension of Σ into a 2-polygraph is coherent:

$$\begin{tikzcd}
sta \arrow[rr, bend left, "\beta a"] \arrow[dr, bend right, "s\alpha"'] & \Downarrow A & aa \\
& sas \arrow[ur, bend right, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
sast \arrow[rr, bend left, "\gamma t"] \arrow[dr, bend right, "sa\beta"'] & \Downarrow B & aat \\
& saa \arrow[ur, bend right, "\delta"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
sasas \arrow[rrr, bend left=10, "\gamma as"] \arrow[dr, bend right, "sa\gamma"'] & & \Downarrow C & aaas \\
& saaa \arrow[r, "\delta a"'] & aata \arrow[ur, bend right, "aa\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
sasaa \arrow[rrrr, bend left=10, "\gamma aa"] \arrow[dr, bend right, "sa\delta"'] & & \Downarrow D & & aaaa \\
& saaat \arrow[r, "\delta at"'] & aatat \arrow[r, "aa\alpha t"'] & aaast \arrow[ur, bend right, "aaa\beta"'] &
\end{tikzcd}$$

Squier's theorem has recently been expanded in higher dimensions (see Guiraud and Malbos, 2012b), where critical pairs are replaced by critical n-tuples. However, the natural shape of the confluence diagram of an n-branching is an n-cube, which is hard to express in a globular ω-category. This makes a lot of calculations from Guiraud and Malbos, 2012b very complicated.

This is a problem because, although Squier's theorem has been extended to various structures other than monoids (see Guiraud and Malbos, 2012a for example), an extension of the full resolution constructed in Guiraud and Malbos, 2012b seems more complicated. As a consequence, new tools are needed in order to get a better understanding of this construction. In this paper, we argue in favour of cubical categories as a good setting where this construction would be more natural.

Cubical categories were introduced in Al-Agl, Brown and Steiner, 2002a. Although they are equivalent to globular ω-categories, their combinatorics makes them a good candidate to improve the proof of Guiraud and Malbos, 2012b for two reasons. First as said earlier the confluence diagram of an n-branching is an n-cube. Secondly, in Guiraud and Malbos, 2012b the authors rely on the construction of an ω-natural transformation, an object that is once again easily described in cubical terms. Although we come short to proving the full result of Guiraud and Malbos, 2012b, we show how to prove the Squier's homotopical theorem in our new framework:

Theorem 4.2. *Let Σ be a convergent cubical 2-polygraph. Suppose that for every critical pair (f_1, f_2) of Σ, there exists (up to exchange of f_1 and f_2) a 2-cell in $\Sigma_2^\top$ whose shell is of the form:*

$$\begin{tikzcd} \cdot \arrow[r, "f_1"] \arrow[d, "f_2"'] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

Then Σ is coherent.

In Section 2 we introduce cubical categories in low dimensions. In Section 3 we recall some standard notions from word rewriting. Finally in Section 4 we prove our version of Squier's theorem.

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2 Cubical 2-categories

The equivalence between globular and cubical ω-groupoids was proven in Brown and Higgins, 1977 and Brown and Higgins, 1981. The case of ω-categories was covered in Al-Agl et al., 2002a. Finally the description of cubical (ω, p)-categories and their equivalence with their globular counterparts was done in Lucas, 2016. Here we focus on cubical 2-categories, $(2, 1)$-categories and 2-groupoids.

Definition 2.1. A cubical 2-set consists of:

- Sets $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_2$, whose objects are respectively called the 0, 1 and 2-cells.
- Applications $\partial^+, \partial^- : \mathcal{C}_1 \to \mathcal{C}_0$.

- Applications $\partial_1^+, \partial_1^-, \partial_2^+, \partial_2^- : \mathcal{C}_2 \to \mathcal{C}_1$.

Satisfying the following relations for any $\alpha, \beta \in \{+, -\}$:

$$\partial^\alpha \partial_2^\beta = \partial^\beta \partial_1^\alpha.$$

Notation 2.2. We represent a 1-cell f in the following way : $\partial^- f \xrightarrow{f} \partial^+ f$, and a 2-cell A as:

$$\begin{tikzcd} \cdot \arrow[r, "\partial_1^- A"] \arrow[d, "\partial_2^- A"'] \arrow[dr, phantom, "A"] & \cdot \arrow[d, "\partial_2^+ A"] \\ \cdot \arrow[r, "\partial_1^+ A"'] & \cdot \end{tikzcd}$$

Cubical 2-categories. A cubical 2-category is a cubical 2-set equipped with extra structure. See Al-Agl, Brown and Steiner, 2002b for a formal definition. We give here a run-down of the structure.

- An operation $\star$ sending any two 1-cells $x \xrightarrow{f} y \xrightarrow{g} z$ to a 1-cell $x \xrightarrow{f \star g} z$.
- An operation ϵ sending any 0-cell x to a 1-cell $x \xrightarrow{\epsilon x} x$, which we usually represent by $x = x$.
- An operation $\star_1$ (resp. $\star_2$) associating, to any 2-cells $\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ and $\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "B"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ satisfying $\partial_1^+ A = \partial_1^- B$ (resp. $\partial_2^+ A = \partial_2^- B$), 2-cells

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, phantom, "A \star_1 B"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[phantom, d, "A \star_2 B"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

- Operations $\epsilon_1, \epsilon_2 : \mathcal{C}_1 \to \mathcal{C}_2$ sending any 1-cell $\xrightarrow{f}$ to 2-cells $\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\epsilon_1 f"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$ and $\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\epsilon_2 f"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$.
- Operations $\Gamma^-, \Gamma^+ : \mathcal{C}_1 \to \mathcal{C}_2$ sending any 1-cell $\xrightarrow{f}$ to 2-cells $\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\Gamma^- f"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$ and $\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Gamma^+ f"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$.

Those operations have to satisfy a number of axioms. In particular, $(\mathcal{C}_0, \mathcal{C}_1, \partial^-, \partial^+, \star, \epsilon)$ and $(\mathcal{C}_1, \mathcal{C}_2, \partial_i^-, \partial_i^+, \star_i, \epsilon_i)$ (for $i = 1, 2$) are categories.

Remark 2.3. The cells Γ^α and ϵ_i are completely characterised by their faces. Hence we will omit them when the context is clear in the rest of this paper.

Cubical $(2,1)$-categories. A cubical $(2,1)$-category is given by a cubical 2-category $\mathcal{C}$ equipped with an operation $T : \mathcal{C}_2 \to \mathcal{C}_2$ sending any 2-cell $\begin{tikzcd} \cdot \arrow[r, "\partial_1^- A"] \arrow[d, "\partial_2^- A"'] \arrow[dr, phantom, "A"] & \cdot \arrow[d, "\partial_2^+ A"] \\ \cdot \arrow[r, "\partial_1^+ A"'] & \cdot \end{tikzcd}$ to a 2-cell of shape $\begin{tikzcd} \cdot \arrow[r, "\partial_2^- A"] \arrow[d, "\partial_1^- A"'] \arrow[dr, phantom, "TA"] & \cdot \arrow[d, "\partial_1^+ A"] \\ \cdot \arrow[r, "\partial_2^+ A"'] & \cdot \end{tikzcd}$ such that $T^2 = \mathrm{id}_{\mathcal{C}_2}$ and:

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, equal] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "TA"] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d, equal] \\ \cdot \arrow[r] & \cdot \arrow[r, equal] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[r, equal] \arrow[d, equal] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, equal] \arrow[d, equal] & \cdot \arrow[r, equal] \arrow[d, equal] & \cdot \arrow[d] \\ \cdot \arrow[r, equal] & \cdot \arrow[r] & \cdot \end{tikzcd} \tag{1}$$

Remark 2.4. The operation $A \mapsto TA$ corresponds to the operation $A \mapsto A^{-1}$ in a globular setting. The equation $T^2 = \mathrm{id}_{\mathcal{C}_2}$ corresponds to the equality $(A^{-1})^{-1}$ and the axiom (1) corresponds to the relation $A \star_1 A^{-1} = 1$. See Lucas, 2016 for more details.

Cubical 2-groupoid. A cubical 2-groupoid is a cubical 2-category such that $(\mathcal{C}_0, \mathcal{C}_1)$ is a groupoid (we note $\xleftarrow{f}$ the inverse of a cell $\xrightarrow{f}$) and equipped with operations $S_1, S_2 : \mathcal{C}_2 \to \mathcal{C}_2$, sending any 2-cell $\begin{tikzcd} \cdot \arrow[r, "\partial_1^- A"] \arrow[d, "\partial_2^- A"'] \arrow[dr, phantom, "A"] & \cdot \arrow[d, "\partial_2^+ A"] \\ \cdot \arrow[r, "\partial_1^+ A"'] & \cdot \end{tikzcd}$ to 2-cells of shape:

$$\begin{tikzcd} \cdot \arrow[r, "\partial_1^+ A"] \arrow[dr, phantom, "S_1 A"] & \cdot \\ \cdot \arrow[u, "\partial_2^- A"] \arrow[r, "\partial_1^- A"'] & \cdot \arrow[u, "\partial_2^+ A"'] \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[d, "\partial_2^+ A"'] \arrow[dr, phantom, "S_2 A"] & \cdot \arrow[l, "\partial_1^- A"'] \arrow[d, "\partial_2^- A"] \\ \cdot & \cdot \arrow[l, "\partial_1^+ A"] \end{tikzcd}$$

So that $(\mathcal{C}_1, \mathcal{C}_2, \partial_i^-, \partial_i^+, \star_i, \epsilon_i, S_i)$ is a groupoid for $i = 1, 2$.

Though the proof is not as straightforward as in the globular case, we still have the following expected result (see Lucas, 2016):

Proposition 2.5. *A cubical 2-groupoid is a cubical $(2,1)$-category.*

3 Word rewriting

In this section we redefine some of the standard concepts of higher-dimensional rewriting in our cubical setting (see Guiraud and Malbos, 2016 for a more detailed exposition). In what follows, by monoidal cubical (n,k)-categories, we mean monoid objects in the category of cubical (n,k)-categories.

Example 3.1. Let $\mathcal{C}$ be a monoidal cubical $(2,1)$-category (resp. 2-category). Let $f : u \to v$,and $g : u' \to v'$ be cells in $\mathcal{C}_1$. Then the monoid structure gives a 1-cell $fg : uu' \to vv'$ in $\mathcal{C}_1$. We write simply fv (resp. vf) for the cell $f(\epsilon v)$ (resp. $(\epsilon v)f$). There is also a product of 2-cells in a similar fashion.

Finally, these products are compatible with the identity maps which give for example the equation $\epsilon_i(fg) = (\epsilon_i f)(\epsilon_i g)$.

Polygraphs are presentations for higher-dimensional globular categories and were introduced by in Burroni, 1993 and by Street under the name of computads (see Street, 1976 and Street, 1987). We adapt them here to present monoidal cubical (n, k)-categories.

Definition 3.2. For any set E, we denote by E^* the free monoid on E. A monoidal 1-polygraph Σ is given by two sets Σ_0, Σ_1, together with maps $\partial^\alpha : \Sigma_1 \to \Sigma_0^*$ (for $\alpha = \pm$).

We denote by Σ^* (resp. $\Sigma^\top$) the free monoidal category (resp. groupoid) generated by Σ.

Definition 3.3. A monoidal cubical 2-polygraph (resp. $(2,0)$-polygraph) is given by three sets Σ_0, Σ_1 and Σ_2, together with maps $\partial^\alpha : \Sigma_1 \to \Sigma_0^*$ and $\partial_i^\alpha : \Sigma_2 \to \Sigma_1^*$ (resp. $\partial_i^\alpha : \Sigma_2 \to \Sigma_1^\top$).

We denote by Σ^* (resp. $\Sigma^\top$) the free monoidal cubical $(2,1)$-category (resp. 2-groupoid) generated by Σ.

Example 3.4. If Σ is a monoidal cubical 2-polygraph, the cells of Σ and Σ^* together with the faces operations can be visualized as follows (a similar diagram could be drawn for $\Sigma^\top$):

$$\begin{tikzcd}
\Sigma_0 \arrow[d, hook] & \Sigma_1 \arrow[dl, shift left] \arrow[dl, shift right] \arrow[d, hook] & \Sigma_2 \arrow[dl, shift left=3] \arrow[dl, shift left=1] \arrow[dl, shift right=1] \arrow[dl, shift right=3] \arrow[d, hook] \\
\Sigma_0^* & \Sigma_1^* \arrow[l, shift left] \arrow[l, shift right] & \Sigma_2^* \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3]
\end{tikzcd}$$

Example 3.5. In what follows we will use the following monoidal cubical 2-polygraph as our running example. Let Σ be the following monoidal cubical 2-polygraph:

$$\Sigma_0 = \{s, t, a\}$$

$$\Sigma_1 = \{\alpha : ta \to as, \beta : st \to a, \gamma : sas \to aa, \delta : saa \to aat\}$$

And finally Σ_2 consists of the following cells:

$$\begin{tikzcd}
sta \arrow[r, "\beta a"] \arrow[d, "s\alpha"'] \arrow[dr, phantom, "A"] & aa \arrow[d, equal] \\
sas \arrow[r, "\gamma"'] & aa
\end{tikzcd}
\qquad
\begin{tikzcd}
sast \arrow[r, "\gamma t"] \arrow[d, "sa\beta"'] \arrow[dr, phantom, "B"] & aat \arrow[d, equal] \\
saa \arrow[r, "\delta"'] & aat
\end{tikzcd}$$

$$\begin{tikzcd}
sasas \arrow[rr, "\gamma as"] \arrow[d, "sa\gamma"'] \arrow[drr, phantom, "C"] & & aaas \arrow[d, equal] \\
saaa \arrow[r, "\delta a"'] & aata \arrow[r, "aa\alpha"'] & aaas
\end{tikzcd}
\qquad
\begin{tikzcd}
sasaa \arrow[rrr, "\gamma aa"] \arrow[d, "sa\delta"'] \arrow[drrr, phantom, "D"] & & & aaaa \arrow[d, equal] \\
saaat \arrow[r, "\delta at"'] & aatat \arrow[r, "aa\alpha t"'] & aaast \arrow[r, "aaa\beta"'] & aaaa
\end{tikzcd}$$

Remark 3.6. The presentation in this article is slightly different from that of Guiraud and Malbos, 2016. Our monoidal (n, p)-categories are seen as one-object $(n+1, p+1)$-categories in Guiraud and Malbos, 2016. One advantage of this approach is that it makes the many-object generalisation very straightforward. Our approach on the other hand suggests a possible generalisation from monoidal objects in (n, p)-categories to $\mathcal{O}$-algebras in (n, p)-categories, where $\mathcal{O}$ would be some coloured set-theoretic non-symmetric operad.

Definition 3.7. Let Σ be a monoidal 1-polygraph. A *rewriting step* in Σ_1^* is a 1-cell of the form ufv, where f is in Σ_1, and u and v are elements of Σ_0^*.

Definition 3.8. Let Σ be a monoidal 1-polygraph. A *branching* is a pair of 1-cells $f, g \in \Sigma_1^*$ with the same source. It is said to be *local* if f and g are rewriting steps.

Up to permutation of f and g, there are three distinct types of local branchings:

- If $f = g$, (f, g) is said to be an *aspherical branching*.
- If there exists $f', g' \in \Sigma_1^*$ and $u, v \in \Sigma_0^*$ such that $f = f'v$ and $g = ug'$ with $\partial^- f' = u$ and $\partial^- g' = v$, (f, g) is said to be a *Peiffer branching*.
- Otherwise, (f, g) is said to be an *overlapping branching*.

Finally a *critical branching* is a minimal overlapping branching, where overlapping branchings are ordered by the (well-founded) relation: $(f, g) \leqslant (ufv, ugv)$ for $u, v \in \Sigma_0^*$

Example 3.9. Using our example, $sa\gamma$ and δa are rewriting steps, but not $sa\gamma \star \delta a$ or $\beta a\beta$. Finally, there are exactly four critical branchings: $(\beta a, s\alpha)$, $(\gamma t, sa\beta)$ $(\gamma as, sa\gamma)$ and $(\gamma aa, sa\gamma)$.

Definition 3.10. Let Σ be a monoidal 1-polygraph. A branching (f, g) is *confluent* if there exists 1-cells f' and g' in Σ_1^* with the same target and such that $\partial^+ f = \partial^- f'$ and $\partial^+ g = \partial^- g'$.

We say that Σ is *locally confluent* if any local branching is confluent, and Σ is *confluent* if any branching is confluent.

It is *terminating* if there is no infinite sequence of rewriting steps $f_1, \ldots, f_n, \ldots$ satisfying that $\partial^+ f_i = \partial^- f_{i+1}$ for all i.

It is *convergent* if it is both terminating and confluent.

Example 3.11. The shape of the cells of Σ_2 shows that all the critical branching of Σ are confluent. As a consequence Σ is locally confluent.

Moreover, Σ is terminating. To show this, we consider the order $t > a$ and $s > a$ on Σ_0, and extend it to Σ_0^* using the *deglex ordering scheme* (see Guiraud and Malbos, 2016). This is a well-founded ordering of Σ_0^* compatible with multiplication, and we can check that for any cell f of Σ_1, $\mathrm{s}(f) > t(f)$.

By Newman's Lemma, a terminating locally confluent rewriting system is confluent, and so Σ is actually convergent.

4 Squier's theorem

Before stating Squier's theorem, we need to define the cubical analogue to the notions of globe and of coherence.

Definition 4.1. Let $\mathcal{C}$ be a cubical 2-category. A *shell* over $\mathcal{C}_1$ is a family of cells f_i^α in $\mathcal{C}_1$, ($i = 1, 2$ and $\alpha = +, -$) satisfying $\partial^\alpha f_2^\beta = \partial^\beta f_1^\alpha$ for every α and β.

A *filler* in $\mathcal{C}_2$ of a shell $S = (f_i^\alpha)$ over $\mathcal{C}_1$ is a cell $A \in \mathcal{C}_2$ satisfying $\partial_i^\alpha A = f_i^\alpha$ for every i and α.

If Σ is a monoidal $(2,0)$-polygraph, we say that Σ is coherent if any shell over $\Sigma_1^\top$ admits a filler in $\Sigma_2^\top$.

The main result of this paper is the following:

Theorem 4.2 (Cubical Squier's theorem)**.** *Let Σ be a convergent cubical $(3,2)$-polygraph. Suppose that for every critical pair (f_1, f_2) of Σ, there exists a 2-cell in $\Sigma_2^\top$ whose shell is of the form:*

$$\begin{tikzcd} \cdot \arrow[r, "f_1"] \arrow[d, "f_2"'] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

Then Σ is coherent.

The proof of this result occupies the rest of this article and loosely follows the proof of the globular case from Guiraud and Malbos, 2016. Before that though, we show that this result applies to our example.

Example 4.3. We have already proven that Σ is convergent. We have also made the list of all possible critical branching, and we can check that each of them corresponds to a cell in Σ_2. Thus by Theorem 4.2, every shell over $\Sigma_1^\top$ admits a filler un $\Sigma_2^\top$.

Lemma 4.4. *For every local branching (f_1, f_2), there exists a cell A in $\Sigma_2^\top$ such that $\partial_1^- A = f_1$ and $\partial_2^- A = f_2$. So A is of the following shape:*

$$\begin{tikzcd} \cdot \arrow[r, "f_1"] \arrow[d, "f_2"'] \arrow[dr, phantom, "A"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

Proof. The proof is similar to the globular case, by distinguishing cases depending on the form of the branching (f_1, f_2). Note first that if A is a suitable cell for the branching (f_1, f_2), then TA satisfies the conditions for the branching (f_2, f_1), and uAv for the branching (uf_1v, uf_2v). So by hypothesis on Σ_2, it remains to show that the property holds for aspherical and Peiffer branchings.

If $(f_1, f_2) = (f, f)$ is an aspherical branching, then the 2-cell $\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\Gamma^- f"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$ satisfies the condition.

If $(f_1, f_2) = (fv, ug)$ is a Peiffer branching, then the 2-cell $(\epsilon_1 f)(\epsilon_2 g)$ satisfies the condition:

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, equal, "u"'] \arrow[dr, phantom, "\epsilon_1 f"] & \cdot \arrow[d, equal, "u'"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd} \cdot \begin{tikzcd} \cdot \arrow[r, equal, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\epsilon_2 g"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, equal, "v'"'] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "fv"] \arrow[d, "ug"'] \arrow[dr, phantom, "(\epsilon_1 f)(\epsilon_2 g)"] & \cdot \arrow[d, "u'g"] \\ \cdot \arrow[r, "fv'"'] & \cdot \end{tikzcd}$$

$\square$

Lemma 4.5. *For every $f, g \in \Sigma_1^*$ of same source and of target a normal form, the shell* $\begin{tikzcd} \cdot \arrow[r, "g"] \arrow[d, "f"'] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$ *admits a filler in Σ_1^*.*

Proof. Define the *origin* of a shell (f_i^α) as $\partial^- f_1^- \in \Sigma_0^*$. Let us prove that for any $u \in \Sigma_0^*$, any shell over Σ_1^* of origin u and of the form $\begin{tikzcd} \cdot \arrow[r, "g"] \arrow[d, "f"'] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$ admits a filler. We reason by induction on u. If u is a normal form, then $f = g = \epsilon u$ and $\epsilon_1 \epsilon u$ is a filler of the shell.

If u is not a normal form, then we can write $f = f_1 \star f_2$ and $g = g_1 \star g_2$ in Σ_1^*, where f_1 and g_1 are rewriting steps. Let A be a 2-cell in Σ_2^* such that $\partial_1^- A = f_1$ and $\partial_2^- A = g_1$ (which exists thanks to the previous Lemma). Denote $f' = \partial_1^+ A$ and $g' = \partial_2^+ A$. Then we can apply the induction hypothesis to both (f', g_2) and (f_2, g') defining 2-cells B_1 and B_2, and we conclude using the following composite:

$$\begin{tikzcd} \cdot \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "A"] & \cdot \arrow[r, "f_2"] \arrow[d, "g'"] \arrow[dr, phantom, "B_2"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, "f'"] \arrow[d, "g_2"'] \arrow[dr, phantom, "B_1"] & \cdot \arrow[r, equal] \arrow[d, equal] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, equal] & \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

$\square$

Lemma 4.6. *For every $f \in \Sigma_1^\top$, and every $g_1, g_2 \in \Sigma_1^*$ of target a normal form, the shell* $\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_1"'] & \cdot \arrow[d, "g_2"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$ *admits a filler in $\Sigma_2^\top$.*

Proof. To prove that the set of 1-cells f satisfying the Lemma is $\Sigma_1^\top$, we show that it contains Σ_1^*, and that it is closed under composition and inverses.

- It contains Σ_1^*. Indeed, let f, g_1 and g_2 be 1-cells in Σ_1^*. We can form the following composite, where the cell A is obtained by the previous Lemma:

$$\begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[d, equal] & \cdot \arrow[r, equal] \arrow[d, equal] & \cdot \arrow[d, "g_2"] \\
\cdot \arrow[r, "f"] \arrow[d, "g_1"'] & \cdot \arrow[r, "g_2"] \arrow[d, phantom, "A"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, equal] & \cdot \arrow[r, equal] & \cdot
\end{tikzcd}$$

- It is stable under composition. Indeed, let $f_1, f_2 \in E$ be two composable 1-cells, and $g_1, g_2 \in \Sigma_1^*$. Let $g_3 \in \Sigma_1^*$ be a 1-cell such that $\partial^- g_3 = \partial^+ f_1$, and whose target is a normal form. Then the following composite shows that $f_1 \star f_2$ is in E, where A_1 and A_2 exist since f_1 and f_2 are in E:

$$\begin{tikzcd}
\cdot \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "A_1"] & \cdot \arrow[r, "f_2"] \arrow[d, "g_3"] \arrow[dr, phantom, "A_2"] & \cdot \arrow[d, "g_2"] \\
\cdot \arrow[r, equal] & \cdot \arrow[r, equal] & \cdot
\end{tikzcd}$$

- It is stable under inverses. Indeed, let $f \in E$, and let $g_1, g_2 \in \Sigma_1^*$. We can construct the following cell, where A comes from the fact that f is in E, applied to the pair (g_2, g_1): $\begin{tikzcd} \cdot \arrow[d, "g_1"'] \arrow[dr, phantom, "S_2 B"] & \cdot \arrow[l, "f"'] \arrow[d, "g_2"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$

□

Proof of Theorem 4.2. Let us fix a shell (f_i^α) over $\Sigma_1^\top$. The following cell is a filler of f_i^α. The 1-cells g_1, g_2, g_3 and g_4 are arbitrary 1-cells in Σ_1^*, with the appropriate source, and a normal form as target. The cells B_1, B_2, B_3 and B_4 are obtained by the previous Lemma and rotated as needed using T, S_1 and S_2.

$$\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, equal] & \cdot \arrow[r, "f_1^-"] \arrow[d, "g_1"] \arrow[dr, phantom, "B_1"] & \cdot \arrow[r, equal] \arrow[d, "g_2"'] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "g_1"] \arrow[d, "f_2^-"'] \arrow[dr, phantom, "B_2"] & \cdot \arrow[r, equal] \arrow[d, equal] & \cdot \arrow[d, equal] \arrow[dr, phantom, "B_3"] & \cdot \arrow[l, "g_2"'] \arrow[d, "f_2^+"] \\
\cdot \arrow[r, "g_3"] \arrow[d, equal] & \cdot \arrow[r, equal] \arrow[dr, phantom, "B_4"] & \cdot & \cdot \arrow[l, "g_4"'] \arrow[d, equal] \\
\cdot \arrow[r, equal] & \cdot \arrow[r, "f_1^+"'] \arrow[u, "g_3"] & \cdot \arrow[r, equal] \arrow[u, "g_4"'] & \cdot
\end{tikzcd}$$

□

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