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Abstract: This article presents a complete second order theory for a large class of geometric functionals on homogeneous Poisson input. In particular, the results don't require the existence of a radius of stabilisation. Hence they can be applied to geometric functionals of spatial shot-noise fields excursions such as volume, perimeter, or Euler characteristic (the method still applies to stabilising functionals). More generally, it must be checked that a local contribution to the functional is not strongly affected under a perturbation of the input far away. In this case the exact asymptotic variance is given, as well as the likely optimal speed of convergence in the central limit theorem. This goes through a general mixing-type condition that adapts nicely to both proving asymptotic normality and that variance is of volume order.

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1. Introduction

Let $(\Omega, \mathscr{A}, \mathbf{P})$ be a probability space. Denote by ℓ^d the Lebesgue measure on $\mathbb{R}^d$. Let η be a homogeneous Poisson process on $\mathbb{R}^d$, and $\{F_W(\eta); W \subset \mathbb{Z}^d\}$ a family of geometric functionals. We give general conditions under which $F_W(\eta)$ has a variance asymptotically proportional to $\sigma_0^2|W|$ for some $\sigma_0 > 0$, and $\text{Var}(F_W(\eta))^{-1/2}(F_W - \mathbf{E}F_W(\eta))$ converges to a Gaussian variable, with a Kolmogorov distance decaying in $|W|^{-1/2}$, as $|W|$ goes to ∞.

Marked processes The model is even richer if one marks the input points by random independent variables, called *marks*, drawn from an external probability space $(\mathbf{M}, \mathscr{M}, \mu)$, the *marks space*. It can be used for instance to let the shape and size of grains be random in the boolean model, or to have a random impulse function for a shot noise process. For $A \subset \mathbb{R}^d$, denote by $\overline{A} = A \times \mathbf{M}$ the cylinder of marked points $\mathbf{x} = (x, m)$ with spatial coordinate $x \in A$. Endow $\overline{\mathbb{R}^d}$ with the product σ-algebra. The reader not familiar with such a setup can consider the case where $\mathbf{M}$ is a singleton, and all mark-related notation can be ignored (except in applications). By an abuse of notation, every spatial transformation applied to a couple $\mathbf{x} = (x, m) \in \overline{\mathbb{R}^d}$ is in fact applied to the spatial element, i.e. $\mathbf{x} - y = (x - y, m)$ for $y \in \mathbb{R}^d$, or for $A \subseteq \mathbb{R}^d \times \mathbf{M}, C \subset \mathbb{R}^d$, $A \cap C = \{(x, m) \in A : x \in C\}$. Denote for simplicity by $d\mathbf{x} = dx\mu(dm)$ the measure element on

$(\overline{\mathbb{R}^d}, \ell^d \times \mu)$. In all the paper, η denotes a Poisson measure on $\overline{\mathbb{R}^d}$ with intensity measure $\ell^d \times \mu$. We assume that η and all random variables introduced in the paper live on the probability space Ω, up to expanding it.

Functionals Let $\mathcal{A}$ be the class of locally finite sets of $\overline{\mathbb{R}^d}$ endowed with the topology induced by the mappings $\zeta \mapsto |\zeta \cap A|$ for compact sets $A \subset \overline{\mathbb{R}^d}$, where $|\cdot|$ denotes the cardinality of a set. Functionals of interest are not properly defined on every $\zeta \in \mathcal{A}$, so we restrict them to some $\mathscr{N}_0 \subset \mathcal{A}$ such that $\mathbf{P}(\eta \in \mathscr{N}_0) = 1$, and call $\mathscr{N}$ the class of configurations $\zeta \in \mathcal{A}$ such that $\zeta \subset \eta \cup \zeta'$ for some $\eta \in \mathscr{N}_0$ and finite set ζ'. Let $\mathcal{F}$ be the class of real measurable functionals on $\mathscr{N}$. Let $\tilde{Q}_a = [-a/2, a/2)^d, Q_a = \tilde{Q}_a \cap \mathbb{Z}^d, a > 0$. For $W \subset \mathbb{Z}^d$ finite, we consider a functional of the form

$$F_W(\zeta) = \sum_{k \in W} F_k^W(\zeta), \zeta \in \mathscr{N}, \text{ with } F_k^W(\zeta) = F_0(\zeta \cap \tilde{W} - k), k \in W, \quad (1.1)$$

where $F_0 \in \mathcal{F}$ and $\tilde{W} = \cup_{k \in W}(k + \tilde{Q}_1)$. It might also happen that all points of η have an influence but only contributions of the functional over $\tilde{W}$ are considered: introduce the infinite input version

$$F'_W(\zeta) = \sum_{k \in W} F_k(\zeta), \zeta \in \mathscr{N}, \text{ with } F_k(\zeta) = F_0(\zeta - k), k \in \mathbb{Z}^d. \quad (1.2)$$

A score function is a bi-measurable mapping $\xi : \mathbf{M} \times \mathscr{N} \to \mathbb{R}$ such that

$$F_0^\xi : \zeta \mapsto \sum_{\mathbf{x} = (x,m) \in \zeta \cap \tilde{Q}_1} \xi(m, \zeta - x), \quad (1.3)$$

is well defined on $\zeta \in \mathscr{N}$, which yields that $F_W(\zeta)$ is the sum of the scores of all points falling in $\tilde{W}$. Write $\xi(\zeta)$ instead of $\xi(m; \zeta)$ if no marking is involved (i.e. $\mathbf{M}$ is a singleton). It is explained later why some shot noise excursions functionals also obey representations (1.1)-(1.2). In this paper, we identify a functional $F : \mathscr{N} \to \mathbb{R}$ with the random variable that gives its value over $\eta : F = F(\eta)$, even if F will be applied to modified versions of η as well.

Non-degeneracy of the variance Define for $\zeta \subset \mathbb{R}^d, 0 \leqslant a < b$,

$$\zeta_a^b = \zeta \cap \tilde{Q}_b \cap \tilde{Q}_a^c, \ \zeta_a = \zeta \cap \tilde{Q}_a^c, \ \zeta^b = \zeta \cap \tilde{Q}_b.$$

A condition that seems necessary for the variance to be non-degenerate is that at least on a finite input and a bounded window, the functional is not trivial: for some $\delta > \rho > 0$, $\mathbf{P}(|F_{Q_\delta}(\eta^\rho) - F_{Q_\delta}(\emptyset)| > 0) > 0$. We actually need that this still holds if points are added far away from η^ρ:

Assumption 1.1. There is $\gamma > \rho > 0, c > 0, p > 0$ such that for $\delta > \gamma$ arbitrarily large

$$\mathbf{P}\left(|F_{Q_\delta}(\eta_\gamma) - F_{Q_\delta}(\eta^\rho \cup \eta_\gamma)| \geqslant c\right) \geqslant p.$$

Observation window In many works (e.g. [20],[14, Chapter 4]), the observation windows consist in a growing family of subsets $B_n, n \geqslant 1$ of $\mathbb{R}^d$, that satisfy the Van'Hoff condition: for all $r > 0$,

$$\ell^d(\partial B_n^{\oplus r})/\ell^d(B_n) \to 0, \tag{1.4}$$

as $n \to \infty$, where $B^{\oplus r} = \{x \in \mathbb{R}^d : d(x, B) \leqslant r\}$ for $B \subset \mathbb{R}^d$. We rather consider in this paper, like for instance in [24], a family $\mathscr{W}$ of bounded subsets of $\mathbb{Z}^d$ satisfying the regularity condition

$$\limsup_{W \in \mathscr{W}} \frac{|\partial_{\mathbb{Z}^d} W|}{|W|} = 0, \tag{1.5}$$

where $\partial_{\mathbb{Z}^d} W$ is the set of points of W at distance 1 from W^c, and consider a point process over $\tilde{W}$. In the large window asymptotics, condition (1.5) imposes the same type of restrictions as (1.4), and using subsets of the integer lattice eases certain estimates and is not fundamentally different. In the case where boundary effects occur (by observing $\eta \cap \tilde{W}$ instead of η), stronger geometric conditions will be required. To this end, let $B_r, r > 0$, be a family of measurable subsets of $\mathbb{R}^d$ such that for some $0 < a_- < a_+$, $B(0, a_- r) \subset B_r \subset B(0, a_+ r)$, where $B(x, r)$ is the Euclidean ball with center $x \in \mathbb{R}^d$ and radius $r > 0$. Let also $B_r(x) = x + B_r, x \in \mathbb{R}^d$. We set similarly as in [20, Section 2],

$$\begin{aligned}
\mathcal{B}_W^r =& \{\tilde{W} - k : \ k \in W, B_r^c \cap (\tilde{W} - k) \neq \emptyset\}, W \subset \mathbb{Z}^d, \\
\mathcal{B}_{\mathscr{W}}^r =& \bigcup_{W \in \mathscr{W}} \mathcal{B}_W^r \cup \{\mathbb{R}^d\}.
\end{aligned}$$

Background The family of functionals described above is quite general and covers large classes of statistics used in many application fields, from data analysis to ecology, see [14] for theory, models and applications. We study the variance, and Gaussian fluctuations, of such functionals, under the assumption that a modification of η far from 0 modifies slightly $F_0(\eta)$ (or $\xi(0, \eta)$). Most of the general results available require a *stabilization* or *localisation* radius : it consists in a random variable $R > 0$, with sufficiently fast decaying tail, such that any modification of η outside $B(0, R)$ does not affect $F_0(\eta)$ (or $\xi(0, \eta)$) at all. By stationarity this behaviour is transferred to any $F_k, k \in \mathbb{Z}^d$. This property is sometimes called quasi-locality in statistical physics [22]. In the Euclidean framework, the results of the present paper do not require stabilisation, but can still be applied to geometric functionals, see Section 1.1.

We give general conditions under which functionals of the form (1.1)-(1.2) have a volume order variance and undergo a central limit theorem, with a Kolmogorov distance to the normal given by the inverse square root of the variance. We recall that the Kolmogorov distance between two real variables U and V is defined as

$$d_{\mathscr{K}}(U, V) = \sup_{t \in \mathbb{R}} |\mathbf{P}(U \leqslant t) - \mathbf{P}(V \leqslant t)|. \tag{1.6}$$

Specified to the case where functionals are under the form (1.3) and the score function is stabilizing, our conditions demand that the tail of the stabilization radius R decays polynomially fast, with power strictly smaller than $-8d$, see Proposition 1.3.

Main result The main theoretical finding of this paper is condition (1.8), which is well suited for second order Poincaré inequalities in the Poisson space, i.e. bounds on the speed of convergence of a Poisson functional to the Gaussian law, and at the same time allows to prove non-degenerate asymptotic variance under Assumption 1.1. The application to shot-noise processes in the following section illustrates the versatility of the method. The results can be merged into the following synthetic result, whose proof is at Section 3.2. For two sequences $\{a_n; n \geqslant 1\}, \{b_n; n \geqslant 1\}$, write $a_n \sim b_n$ if $b_n \neq 0$ for n sufficiently large and $a_n b_n^{-1} \to 1$ as $n \to \infty$. Also, in all the paper, κ denotes a constant that depends on d, α, a_+, a_-, whose value may change from line to line, and which explicit optimal value in the main result could be traced through the different parts of the proof. If it is well defined, for $F_0 \in \mathcal{F}$, let

$$\sigma_0^2 := \sum_{k \in \mathbb{Z}^d} \operatorname{Cov}(F_0(\eta), F_k(\eta)). \tag{1.7}$$

Theorem 1.1. Let $F_0 \in \mathcal{F}$, F_W be defined as in (1.1), $\mathscr{W} = \{W_n; n \geqslant 1\}$ satisfying (1.5). Let M_1, M_2 be independent random elements of $\mathbf{M}$ with law μ. Assume that for some $C_0 > 0, \alpha > 2d$, for all $r \geqslant 0, B \in \mathcal{B}^r_{\mathscr{W}}, \ell^d - a.e.\ x_1, x_2 \in \mathbb{R}^d, \zeta \subset \{(x_1, M_1), (x_2, M_2)\}$

$$\left(\mathbf{E}\left|F_0((\eta \cup \zeta) \cap B_r \cap B) - F_0((\eta \cup \zeta) \cap B)\right|^4\right)^{1/4} \leqslant C_0(1+r)^{-\alpha}, \tag{1.8}$$

and Assumption 1.1 is satisfied. Then $0 < \sigma_0 < \infty$, and as $n \to \infty$,

$$\operatorname{Var}(F_{W_n}) \sim \sigma_0^2 |W_n|, \qquad (\sigma_0^2 |W_n|)^{-1/2}(F_{W_n} - \mathbf{E}F_{W_n}) \xrightarrow[n\to\infty]{\text{law}} N$$

where N is a standard Gaussian random variable. Furthermore, for n sufficiently large,

$$d_{\mathscr{K}}\left(\frac{F_{W_n} - \mathbf{E}F_{W_n}}{\operatorname{Var}(F_{W_n})^{1/2}}, N\right) \leqslant \kappa |W_n|^{-1/2}\left(\frac{C_0^2}{\sigma_0^2} + \frac{C_0^3}{\sigma_0^3} + \frac{C_0^4}{\sigma_0^4}\right). \tag{1.9}$$

Let us now give the version with infinite input, which is more simple to satisfy due to the absence of boundary effects, except for the power of the decay:

Theorem 1.2. Let $F_0 \in \mathcal{F}$, F'_W be defined as in (1.2), $\mathscr{W} = \{W_n; n \geqslant 1\}$ satisfying (1.5). Let M_1, M_2, be independent random elements of $\mathbf{M}$ with law μ. Assume that for some $C_0 > 0, \alpha > 5d/2$, for all $r \geqslant 0, \ell^d - a.e.\ x_1, x_2 \in \mathbb{R}^d, \zeta \subset \{(x_1, M_1), (x_2, M_2)\}$,

$$\left(\mathbf{E}\left|F_0((\eta \cup \zeta) \cap B_r) - F_0(\eta \cup \zeta)\right|^4\right)^{1/4} \leqslant C_0(1+r)^{-\alpha}, \tag{1.10}$$

and Assumption 1.1 is satisfied. Then $0 < \sigma_0 < \infty$ (defined in (1.7)), and

$$\mathrm{Var}(F'_{W_n}) \sim \sigma_0^2 |W_n|, \qquad (\sigma_0^2 |W_n|)^{-1/2}(F'_{W_n} - \mathbf{E}F'_{W_n}) \xrightarrow{\text{law}} N$$

as $n \to \infty$. Furthermore, for n sufficiently large

$$d_{\mathscr{K}}\left(\frac{F'_{W_n} - \mathbf{E}F'_{W_n}}{\mathrm{Var}(F'_{W_n})^{1/2}}, N\right) \leqslant \kappa |W_n|^{-1/2}\left(\frac{C_0^2}{\sigma_0^2} + \frac{C_0^3}{\sigma_0^3} + \frac{C_0^4}{\sigma_0^4}\right). \tag{1.11}$$

Remarks 1.1. 1. The application to score functionals (see (1.3)) goes as follows: let $M_i, 0 \leqslant i \leqslant 6$ be iid marks with law μ, and assume that $\xi : \mathbf{M} \times \mathscr{N} \to \mathbb{R}$ satisfies for all $r \geqslant 0, B \in \mathcal{B}^r_{\mathscr{W}}, x_0 \in \tilde{Q}_1, \zeta \subset \overline{\mathbb{R}^d}$ with at most 6 elements,

$$\left(\mathbf{E}\left|\xi(M_0, (\eta \cup \zeta) \cap B \cap B_r - x_0) - \xi(M_0, (\eta \cup \zeta) \cap B - x_0)\right|^4\right)^{1/4} \leqslant C_0(1+r)^{-\alpha}, \tag{1.12}$$

then the functional $F_0 = F_0^{\xi}$ defined in (1.3) satisfies (1.8). To see it, let $\mathbf{x}_i = (x_i, M_i)$ be the elements of ζ. Fix $\zeta_1 \subset \{(x_1, M_1), (x_2, M_2)\}$, apply Lemma 5.1 (with $r = 0$) to

$$\begin{aligned}\psi((x_0, M_0), \zeta') = \mathbf{1}_{\{x_0 \in \tilde{Q}_1\}} & |\xi(M_0, (\zeta' \cup \zeta_1) \cap B \cap B_r - x_0) \\ & - \xi(M_0, (\zeta' \cup \zeta_1) \cap B - x_0)|, \zeta' \in \mathscr{N}, x_0 \in \mathbb{R}^d.\end{aligned}$$

It yields

$$\begin{aligned}\left(\mathbf{E}\left|F_0((\eta \cup \zeta_1) \cap B \cap B_r) - F_0((\eta \cup \zeta_1) \cap B)\right|^4\right)^{1/4} &\leqslant \left(\mathbf{E}\left|\sum_{\mathbf{x} \in \eta \cap \tilde{Q}_1} \psi(\mathbf{x}, \eta)\right|^4\right)^{1/4} \\ &\leqslant \kappa C_0 (1+r)^{-\alpha}\end{aligned}$$

for some $C_0 \geqslant 0$, hence (1.8) is satisfied. In this framework the asymptotic variance can also be expressed as

$$\sigma_0^2 = \mathbf{E}\xi(M_0; \eta)^2 + \int_{\mathbb{R}^d} (\mathbf{E}[\xi(M_0, \eta \cup \{(x, M_1)\})\xi(M_1, \eta \cup \{(0, M_0)\} - x)] - [\mathbf{E}[\xi(M_0; \eta)]]^2) dx,$$

see for instance (4.10) in [14].

2. A variant of stabilisation, called *strong stabilisation*, occurs when the add-one cost version of the functional is stabilising instead of the functional itself. Penrose and Yukich derived variance asymptotics and asymptotic normality [20] in such a context. Let us indicate how the current approach could be adapted to strong stabilisation: let η' be an independent copy of η, and for $r > 0$, $\eta_r = (\eta \cap B_r) \cup (\eta' \cap B_r^c)$. Assume that a functional has a strong stabilisation radius with the tail decaying as a sufficiently low

power of r. In this case, (1.8) needs to hold with the left hand member replaced with $\mathbf{E}\left(|F_0((\eta\cup\zeta)\cap B) - F_0((\eta_r\cup\zeta)\cap B)|^4\right)$. Then it should be possible to adapt the proofs of Theorems 2.1 and 3.1 to be able to prove that the Berry-Esseen bounds and variance upper bounds still hold, under this new hypothesis.

3. Regarding variance asymptotics, recent results can be found in the literature, but the assumptions are of different nature, either dealing with different qualitative long range behaviour (i.e. strong stabilization in [20, 16]), or different non-degeneracy statements [18], whereas Assumption 1.1 is a mixture of non-triviality and continuity of the functional on large inputs. Penrose and Yukich [20] give a condition under which the asymptotic variance is strictly positive in Theorem 2.1. The condition is that the functional is *strongly stabilising*, and that the variable

$$\Delta(\infty) := \lim_{\delta\to\infty}[F_{Q_\delta}(\eta\cup\{0\}) - F_{Q_\delta}(\eta)]$$

is non-trivial. It roughly means that for δ sufficiently large, and ρ sufficiently small,

$$\mathrm{Var}(|F_{Q_\delta}(\eta_\rho\cup\eta^\rho) - F_{Q_\delta}(\eta_\rho) \mid |\eta^\rho| = 1\ ) > 0,$$

and this is very close to Assumption 1.1 in the particular case $\rho = \gamma$. This particular case seems more delicate to deal with that when γ is much larger than ρ, because in the latter case the interaction between η^ρ and η_γ hopefully becomes small.

4. Similar results where the input consists of m_n iid variables uniformly distributed in $\tilde{W}_n$, with $m_n = |W_n|$, should be within reach by applying the results of [15], following a route similar to [16].

Shot-noise excursions Let $\{g_m; m\in\mathbf{M}\}$ be a set of measurable functions $\mathbb{R}^d\to\mathbb{R}$ not containing the function $g\equiv 0$ indexed by some probability space $(\mathbf{M},\mathscr{M},\mu)$. Let η be a Poisson process with intensity measure $\ell^d\times\mu$ on $\overline{\mathbb{R}^d}$. Introduce the shot noise processes with *impulse distribution* μ by, for $\zeta\in\mathscr{N}$,

$$f_\zeta(y) = \sum_{\mathbf{x}=(x,m)\in\zeta} g_m(y-x), y\in\mathbb{R}^d. \tag{1.13}$$

Conditions under which f_ζ is well defined on Poisson input are discussed in Section 4, along with a proper choice for $\mathscr{N}_0$. Given some threshold $u\in\mathbb{R}$, we consider the excursion set $\{f_\zeta\geqslant u\} = \{x\in\mathbb{R}^d : f_\zeta(x)\geqslant u\}$ and the functionals $\zeta\mapsto\ell^d(\{f_\zeta\geqslant u\}\cap\tilde{W})$, $\zeta\mapsto\mathrm{Per}(\{f_\zeta\geqslant u\};\tilde{W})$, where for $A,B\subset\mathbb{R}^d; \mathrm{Per}(A;B)$ denotes the amount of perimeter of A contained in B in the variational sense, see Section 4.2. The total curvature, related to the Euler characteristic is also studied in Section 4.3 for a specific form of the kernels.

A shot noise field is the result of random functions translated at random locations in the space. It has been introduced by Campbell to model thermionic noise [10], and has been used since then under different names in many fields such as pharmacology, mathematical morphology [17, Section 14.1], image analysis [13], or telecommunication networks [2, 3]. Biermé and Desolneux [6, 7, 8] have computed the mean values for some geometric properties of excursions. More generally, the activity about asymptotic properties of random fields excursions has recently increased, with the notable recent contribution of Estrade and Léon [11], who derived a central limit theorem for the Euler characteristic of excursions of stationary Euclidean Gaussian fields. Bulinski, Spodarev and Timmerman [9] give general conditions for asymptotic normality of the excursion volume for quasi-associated random fields. Their results apply to shot-noise fields, under conditions of non-negativity and uniformly bounded marginal density, which can be verified in some specific examples. We give here the asymptotic variance and central limit theorems for volume and perimeter of excursions under weak assumptions on the density, as illustrated in Section 4. Still, a certain control of the distribution is necessary, and we provide in Lemma 4.2 a uniform bound on $\sup_{v\in\mathbb{R},\delta>0}(\delta\ln(\delta))^{-1}\mathbf{P}(f_\eta(0)\in[v-\delta,v+\delta])$ when f is of the form

$$f_\zeta(x)=\sum_{i\in I}g(\|x-x_i\|) \tag{1.14}$$

where $\zeta\in\mathscr{N}$, and $x_i,i\in I$, are the (random) spatial locations of its points, with g a smooth strictly non-increasing function $(0,\infty)\to(0,\infty)$ with a derivative not decaying too fast to 0. Our results allow to treat fields with singularities, such as those observed in astrophysics or telecommunications, see [2].

Let $\mathcal{M}_d$ be the space of measurable subsets of $\mathbb{R}^d$. The results of Section 4 also apply to processes that can be written under the form

$$f_\zeta(x)=\sum_{i\geqslant 1}L_i\mathbf{1}_{\{x-x_i\in A_i\}},x\in\mathbb{R}^d, \tag{1.15}$$

where the $(L_i,A_i),i\geqslant 1$ are iid couples of $\mathbb{R}\times\mathcal{M}_d$, endowed with a proper σ-algebra and probability measure, see Section 4.3. Such models are called *dilution functions* or *random token models* in mathematical morphology, see for instance [17, Section 14.1], where they are used to simulate random functions with a prescribed covariance.

To the best of our knowledge, the results about the perimeter or the Euler characteristic are the first of their kind for shot noise models, and the results about the volume improve existing results, see the beginning of Section 4.1 for more details.

1.1. Stabilization and nearest neighbour statistics

Let us transpose our results in the case where the functional stabilises.

Theorem 1.3. Let $\mathscr{W}=\{W_n;n\geqslant 1\}$ be a class of subsets of $\mathbb{Z}^d$. Let F_W be defined as in (1.1) (resp. as in (1.1)-(1.3) with $F_0=F_0^\xi$ for some score function ξ). Assume that for $x_i\in\mathbb{R}^d,M_i$ independent with law $\mu,i\geqslant 1,\zeta\subset\{(x_i,M_i);i=$

$1,\dots,6\}, \eta' = \eta \cup \zeta$, there is a random variable $R \geqslant 0$ such that almost surely, for $r \geqslant R, B \in \mathcal{B}^r_{\mathscr{W}}$,

$$F_0(\eta' \cap B_r \cap B) = F_0(\eta' \cap B). \tag{1.16}$$

$$(\text{resp. } \xi(m, \eta' \cap B_r \cap B - x) = \xi(m, \eta' \cap B - x), (x,m) \in \eta \cap \overline{\tilde{Q}_1}.) \tag{1.17}$$

Then (1.8) is satisfied if for some $p, q > 1$ with $1/p + 1/q = 1$, $\mathbb{P}(R > r) \leqslant Cr^{-8dp-\varepsilon}$ for some $C, \varepsilon > 0$, under the moment condition

$$\sup_{r \geqslant 0, B \in \mathcal{B}^r_{\mathscr{W}}} \mathbf{E} \left| F_0(\eta' \cap B \cap B_r) \right|^{4q} < \infty$$

$$(\text{resp. } \sup_{r \geqslant 0, B \in \mathcal{B}^r_{\mathscr{W}}, x_0 \in \tilde{Q}_1} \mathbf{E} \left| \xi(M_1, \eta' \cap B \cap B_r - x_0) \right|^{4q} < \infty). \tag{1.18}$$

For the infinite input version, "$\cap B$" should be removed from (1.16) (resp. (1.17)), the exponent $-8dp - \varepsilon$ should be replaced by $-10dp - \varepsilon$, and then (1.10) would hold.

Proof. For $r \geqslant 0, B \in \mathcal{B}^r_{\mathscr{W}}$, if (1.16) holds,

$$\begin{aligned}\mathbf{E} \left| F_0(\eta' \cap B) - F_0(\eta' \cap B \cap B_r) \right|^4 =& \mathbf{E} \mathbf{1}_{\{R > r\}} \left| F_0(\eta' \cap B) - F_0(\eta' \cap B \cap B_r)) \right|^4 \\ \leqslant& \mathbf{P}(R > r)^{1/p} \left(\mathbf{E} \left(|F_0(\eta' \cap B \cap B_r)| + |F_0(\eta' \cap B)| \right)^{4q} \right)^{1/q},\end{aligned}$$

hence (1.8) is satisfied. If $F_0 = F_0^\xi$, and (1.17) holds, for $r \geqslant R$

$$\begin{aligned} F_0^\xi(\eta' \cap B_r \cap B) =& \sum_{(x,m) \in \eta \cap \tilde{Q}_1} \xi(m, \eta' \cap B_r \cap B - x) \\ =& \sum_{(x,m) \in \eta \cap \tilde{Q}_1} \xi(m, \eta' \cap B - x) \\ =& F_0^\xi(\eta' \cap B), \end{aligned}$$

and (1.16) holds. □

Remarks 1.2.

1. The variance non-degeneracy is a disjoint issue, Assumption 1.1 has to be satisfied independently. Otherwise, if one is only interested in asymptotic normality, the above requirements can be weakened, see Theorem 3.1.
2. The definition of a stabilisation radius often involves stability under the addition of an external set, here denoted by ζ. A nice aspect of (1.16)-(1.17) with respect to classical results is that ζ does not depend on η, i.e. ζ does not in general achieve the worst case scenario given η. On the other hand, in the finite input version, one has to deal here with the intersection with $B \in \mathcal{B}^r_{\mathscr{W}}$. See Example 1.1 for an application to nearest neighbour statistics.

3. Asymptotic results for stabilizing functionals have been derived in numerous work, see the survey [14, Chapter 4] and references therein. In particular, Matthew Penrose first proved such results under polynomial decay for the stabilisation radius.

Example 1.1 (Nearest neighbours statistics)**.** Let us develop the example of nearest neighbour statistics for illustrative pruposes. Given $\zeta \in \mathscr{N}, x \in \mathbb{R}^d$, denote by $NN(x;\zeta)$ the *nearest neighbour* of x, i.e. the closest point of $\zeta \setminus \{x\}$ from x, with ties broken by the lexicographic order. Define recursively, for $k \geqslant 1$, $NN_k(x;\zeta) = NN(x;\zeta \setminus \cup_{i=0}^{k-1} NN_i(x;\zeta))$, with $x = NN_0(x;\zeta)$, and $NN_{\leqslant k}(x;\zeta) = \cup_{i=0}^{k} NN_i(x;\zeta)$. Fix $k \geqslant 1$ and call neighbours of x within ζ the set $N_k(x;\zeta)$ consisting of all points $y \in \zeta$ such that $x \in NN_{\leqslant k}(y, \zeta \cup \{x\})$ or $y \in NN_{\leqslant k}(x;\zeta)$.

Let then φ be a real functional defined on finite subsets of $\mathbb{R}^d$, and define the score function, for $\zeta \in \mathscr{N}$,

$$\xi(\zeta) = \begin{cases} \varphi(N_k(0;\zeta)) \\ 0 \text{ if } |\zeta| < k. \end{cases}$$

Assume that for each $j \geqslant k$, the induced mapping on $(\mathbb{R}^d)^j$, $\tilde{\varphi}_j : (x_1, \dots, x_j) \mapsto \varphi(\{x_1, \dots, x_j\})$, is measurable. The simplest example would be for $k = 1$ the functional $\varphi(A) = \frac{1}{2} \sum_{y \in A} \|y\|$, so that $F_W(\zeta) = \sum_{x \in \zeta} \xi(\zeta - x)$ gives the total length of the undirected nearest-neighbour graph for $\zeta \subset \tilde{W}$. Notice that no marking is involved in this setup. Such statistics are used in many applied fields, in nonparametric estimation procedures, or more recently in estimation of high-dimensional data sets [19]. Many asymptotic results have been established since the central limit theorem of Bickel and Breiman [4], see for instance [16, 18, 20].

Theorem 1.4. For $n \geqslant 1$, let

$$G_n = \sum_{x \in \eta \cap \tilde{Q}_{n^{1/d}}} \varphi(N_k(x; \eta \cap \tilde{Q}_{n^{1/d}})).$$

Assume that there is $C, c > 0, u < d/4$ such that for all $x_1, \dots, x_m \in \mathbb{R}^d$,

$$\varphi(\{x_1, \dots, x_m\}) \leqslant C \exp(c \max_i \|x_i\|^u) \tag{1.19}$$

and that φ is not degenerate: $\varphi(\{x_1, \dots, x_k\}) \neq 0$ for $(x_1, \dots, x_k)$ in a non-negligible subset of $(\mathbb{R}^d)^k$. Then $\mathrm{Var}(G_n) \sim n\sigma_0^2$, with $\sigma_0 > 0$ defined in Remark 1.1, and $n^{-1/2}(G_n - \mathbf{E}G_n)$ converges in law to a centred Gaussian variable with variance σ_0^2, with bounds on the Kolmogorov distance proportional to $n^{-1/2}$.

Proof. Call hypercube a set of the form $x + [-a,a]^d$ for some $x \in \mathbb{R}^d, a \geqslant 0$. For this proof we choose $B_r = [-r,r]^d, r \geqslant 0$ (hence $a_- = 1, a_+ = \sqrt{d}$). Let $a_0 \in (0, 1/4)$ and $Q_i = x_i + [-a_0, a_0]^d, i = 1, \dots, q$ be hypercubes contained in $B_1 \setminus B_{1/2\sqrt{d}}$ such that the following holds: for all hypercube B that touches $B_{1/2\sqrt{d}}$ and B_1^c and $y \in B \cap B(0,1)^c$, there is i such that $Q_i \subset (B \cap B(y, \|y\|))$. Let $Q_i' = x_i + [-a_0/2, a_0/2]^d$ and

$$R = \min\{r \geqslant 2\sqrt{d}(1 + 1/a_0) : |\eta \cap rQ_i'| \geqslant k \text{ for every } i = 1, \dots, q\}.$$

The fact that $R' := \sqrt{d}(R+1)$ is a stabilization radius in the sense of (1.17) is implied by the following claim:

Claim 1.1. Let $r \geqslant R'$, $B \in B^r_{\mathscr{W}}$, $x \in B_1$. All elements of $N_k(0, \eta' \cap B - x)$ are in $B(0, \sqrt{d}R)$.

Proof. Let $y \in \eta' \cap (B - x)$ be such that $0 \in NN_{\leqslant k}(y, (\eta' \cap B - x) \cup \{0\})$. Assume that $y \notin B(0, R)$, hence $y \in (B - x) \cap B(0, R)^c$. Since $B \cap B^c_r \neq \emptyset$, $(B - x) \cap B^c_{r-\sqrt{d}} \neq \emptyset$, and $(B - x) \cap B^c_R \neq \emptyset$. $0 \in B$ yields $(B - x) \cap B_t \neq \emptyset$ for $t \geqslant 1$, hence for $t = R/2\sqrt{d}$. It follows that there is i such that $B(y, \|y\|) \cap (B - x)$ contains RQ_i. Since η has (at least) k points in RQ'_i and $RQ'_i - x \subset RQ_i$ (using $Ra_0/2 \geqslant \sqrt{d}$), $\eta - x$ has k points in RQ_i, hence $(\eta' \cap B - x) \cap B(y, \|y\|)$ contains at least k points, and they are all closer from y than 0, which contradicts $0 \in NN_{\leqslant k}(y, (\eta' \cap B - x) \cup \{0\})$. This proves $y \in B(0, R)$.

For every i, RQ_i contains k points of η that are in B_R, hence in $B(0, R')$, hence $NN_{\leqslant k}(0, \eta' \cap B - x) \subset B_{R'}$. □

The claim implies that $N_k(0, \eta' \cap B - x) = N_k(0, \eta' \cap B \cap B_r - x)$ for $r \geqslant R'$. We have for $r \geqslant 0$,

$$\mathbf{P}(R \geqslant r) \leqslant \sum_{i=1}^{q} \mathbf{P}(|\eta \cap rQ'_i| \leqslant k - 1) \leqslant \lambda r^{(k-1)d} e^{-\lambda' r^d}$$

(for some $\lambda, \lambda' > 0$), and a similar bound holds for R'. For the moment condition, note that for $r > 0$, the neighbours of 0 in $\eta \cap B_r \cap B - x$ are at most at distance R', hence, in virtue of (1.19), uniformly in r, B, for $\varepsilon > 0$,

$$\mathbf{E}|\xi(\eta' \cap B_r \cap B - x)|^{4+\varepsilon} \leqslant C\mathbf{E}[\exp(cR')^{(4+\varepsilon)u}]$$

and this quantity is finite if ε is chosen such that $(4+\varepsilon)u < d$, and (1.17)-(1.18) hold, hence (1.8) holds.

Let us check Assumption 1.1. Note that every result giving variance lower bounds for such functionals requires some kind of non-triviality check, as in [20, Lemma 6.3], and the following result could likely be deduced from it. We prefer to present a self-contained proof since this example is supposed to illustrate the current method. Let $A \subset (\mathbb{R}^d)^k$ be such that $\tilde{\varphi}_k > 0$ on A and $A \subset \operatorname{int}(\tilde{Q}^k_\rho)$ for some $\rho > 1$. Hence there is $c > 0$ such that for $\delta > \rho$, $p := \mathbf{P}(|F_{Q_\delta}(\eta^\rho)| \geqslant c, |\eta^\rho| = k+1) > 0$ does not depend on δ. It is clear that for $\gamma > 3\rho, x \in \eta^\rho$, if $|\eta^\rho| = k+1$, $NN_{\leqslant k}(x; \eta^\rho \cup \eta_\gamma) \subset \eta^\rho$. Reciprocally, if for $x \in \eta_\gamma$, $|B(x, \|x\| - \rho) \cap \eta_\gamma| > k+1$, x has its k nearest neighbours in η_γ, and hence none in η^ρ. If the two latter conditions are satisfied, $F_{Q_\delta}(\eta^\rho \cup \eta_\gamma) = F_{Q_\delta}(\eta^\rho) + F_{Q_\delta}(\eta_\gamma)$, where $\delta > \gamma$. Hence

$$\begin{aligned}
\mathbf{P}(|F_{Q_\delta}(\eta_\gamma) - F_{Q_\delta}(\eta_\gamma \cup \eta^\rho)| \geqslant c) &\geqslant \mathbf{P}(|F_{Q_\delta}(\eta_\gamma) - F_{Q_\delta}(\eta_\gamma \cup \eta^\rho)| \geqslant c, |\eta^\rho| = k+1) \\
&\geqslant \mathbf{P}(|F_{Q_\delta}(\eta^\rho)| \geqslant c, |\eta^\rho| = k+1) - \sum_{j \in Q_\delta \setminus Q_\gamma} \mathbf{P}(|\eta \cap B(j, \|j\| - \rho + \sqrt{d})| \leqslant k) \\
&\geqslant p - \sum_{m=\gamma}^{\infty} \kappa m^{d-1} C_k (m - \rho + \sqrt{d})^k \exp(-\kappa(m - \rho + \sqrt{d})^d).
\end{aligned}$$

For $\gamma > 3\rho$ sufficiently large (and any $\delta > \gamma$), the last term is smaller than $p/2$, hence Assumption 1.1 is satisfied. $\square$

1.2. Further applications and perspectives

An important part of the paper is devoted to shot noise excursions, but the results should apply also to most stabilizing models studied in the literature (packing functionals, Voronoi tessellation, boolean models, proximity graphs), see the example of statistics on nearest neighbours graphs above.

In some models, the independent marking is replaced by *geostatistical marking*, also called dependent marking or external marking: let $m(x;\eta'), x \in \mathbb{R}^d$ be a random field measurable with respect to an independent homogeneous Poisson process η' on $\mathbb{R}^d$, and consider the marked process $\{(x, m(x,\eta')), x \in \eta\}$ instead of the independently marked process. Such a refinement is necessary to model a variety of random phenomena, such as gauge measurements for rainfalls or tree sizes in a sparse forest, see [23] and references therein. Labelling the points of η and η' with two different colors yields that $\eta \cup \eta'$ has the law of an independently marked Poisson process, hence our results could be applied to appropriate statistics.

In the non-marked setting ($\mathbf{M}$ is a singleton), let $a > 0$ be a scaling parameter, and consider the random field $X = (X_k)_{k\in\mathbb{Z}^d}$, where $X_k = \mathbf{1}_{\{a\eta\cap(k+[0,1)^d)=\emptyset\}}, k \in \mathbb{Z}^d$. X is an *independent spin-model* where the parameter $p = \mathbf{P}(X_0 = 1) = \exp(-a)$ can take any prescribed value. Then all the previous results can be applied to functionals of the form

$$F_W(X) = \sum_{k\in\mathbb{Z}^d} F_0(X \cap W - k) \text{ or } F'_W(X) = \sum_{k\in\mathbb{Z}^d} F_0(X - k),$$

where F_0 is some functional on the class of subsets of $\mathbb{Z}^d$, with finite second moment under iid Bernoulli input. Stabilising functionals and excursions functionals yield possible applications, our findings might apply for instance to the results of [22], where more general classes of discrete input than Bernoulli processes are also treated. Seeing F_W (or F'_W) as a functional of η, the variance and asymptotic normality results of Theorems 1.1-1.2 apply to F_W under conditions of the type

$$(\mathbf{E}\,|F_W(X' \cap B) - F_W(X' \cap B \cap B_r)|^4)^{1/4} \leqslant C_0(1 + r)^{-\alpha},$$

where B, B_r are like in (1.1), and X' is obtained from X by forcing up to 2 spins $X_k, X_{k'}$ to the value 1 (the bound has to be uniform over $k, k' \in \mathbb{Z}^d$).

2. Moment asymptotics

In this section, we give results for second and fourth moments of a geometric functional under general conditions of non-triviality and polynomial decay. The fourth order moment is useful for establishing Berry-Esseen bounds in the next section. The greek letter κ still denotes a constant depending on d, q, α, a_-, a_+ whose value may change from line to line.

Theorem 2.1. Let $\alpha > d, W \subset \mathbb{Z}^d$, $C_0 \geqslant 0$. Let $F_0 \in \mathcal{F}$.
Assume **(i)** that for $k \in W$, $G_k^W = F_k^W$, (resp. **(i')** for $k \in \mathbb{Z}^d, G_k^W = F_k$) and let $G_W = \sum_{k\in W} G_k^W = F_W$ (resp. $G_W = F'_W$) as defined in (1.1) (resp. (1.2)), and for all $r \geqslant 0, B \in \mathcal{B}_W^r \cup \{\mathbb{R}^d\}$,

$$\left(\mathbf{E}\left|F_0(\eta \cap B_r \cap B) - F_0(\eta \cap B)\right|^2\right)^{1/2} \leqslant C_0(1+r)^{-\alpha} \tag{2.1}$$

(resp. for all $r \geqslant 0$,

$$\left(\mathbf{E}\left|F_0(\eta \cap B_r) - F_0(\eta)\right|^2\right)^{1/2} \leqslant C_0(1+r)^{-\alpha}). \tag{2.2}$$

Then for $k, j \in W$ (resp. $k, j \in \mathbb{Z}^d$),

$$\mathrm{Cov}(G_j^W, G_k^W) \leqslant \kappa C_0^2 (1 + \|k-j\|)^{-\alpha}, \tag{2.3}$$

$$\sigma_0^2 := \sum_{k\in\mathbb{Z}^d} \mathrm{Cov}(F_0, F_k) < \infty,$$

and $\sigma_0 > 0$ if also Assumption 1.1 holds. If W is bounded and non-empty,

$$\left||W|^{-1}\mathrm{Var}(G_W) - \sigma_0^2\right| \leqslant \kappa C_0^2 (|\partial_{\mathbb{Z}^d} W|/|W|)^{1-d/\alpha}. \tag{2.4}$$

If furthermore $\alpha > 2d$

$$\mathbf{E}\left(G_W - \mathbf{E}G_W\right)^4 \leqslant \kappa C_0 (\mathbf{E}(F_0 - \mathbf{E}F_0)^4)^{3/4} |W|^2. \tag{2.5}$$

The proof is deferred to Section 5.1.

3. Asymptotic normality

We give bounds to the normal in terms of Kolmogorov distance, defined in (1.6), or Wasserstein distance, defined between two random variables U, V as

$$d_{\mathscr{W}}(U, V) = \sup_{h\in\mathrm{Lip}_1} |\mathbf{E}[h(U)] - \mathbf{E}[h(V)]|,$$

where Lip_1 is the set of 1-Lipschitz functions $h : \mathbb{R} \to \mathbb{R}$.

3.1. Malliavin derivatives

It has been shown in different frameworks [11, 15, 16, 18] that, through inequalities called *second-order Poincaré type* inequalities, Gaussian fluctuations of real functionals can be controlled by some second order difference operators defined on the random input. In the Poisson setting, this operator is incarnated by the Malliavin derivatives. We define it here as it is a central tool in the theory backing our results: for any functional $F \in \mathcal{F}, \zeta \in \mathscr{N}$, and $\mathbf{x} \in \overline{\mathbb{R}^d}$, define the first order Malliavin derivative $D_{\mathbf{x}} F \in \mathcal{F}$ by

$$D_{\mathbf{x}} F(\zeta) = F(\zeta \cup \{\mathbf{x}\}) - F(\zeta),$$

and for $\mathbf{x}, \mathbf{y} \in \overline{\mathbb{R}^d}, \zeta \in \mathscr{N}, F \in \mathcal{F}_0$, the second order Malliavin derivative is

$$D^2_{\mathbf{x},\mathbf{y}} F(\zeta) = D_{\mathbf{x}}(D_{\mathbf{y}} F(\zeta)) = F(\zeta \cup \{\mathbf{x},\mathbf{y}\}) - F(\zeta \cup \{\mathbf{x}\}) - F(\zeta \cup \{\mathbf{y}\}) + F(\zeta).$$

One can use this object to quantify the spatial dependency of the functional F: a point $\mathbf{y} \in \overline{\mathbb{R}^d}$ has a weak influence on a point $\mathbf{x} \in \overline{\mathbb{R}^d}$ for the functional F if its presence hardly affects the contribution of $\mathbf{x}$, i.e. $D_{\mathbf{x}} F(\eta) \approx D_{\mathbf{x}} F(\eta \cup \{y\})$, or in other words $D^2_{\mathbf{x},\mathbf{y}} F(\eta) = D_{\mathbf{y}}(D_{\mathbf{x}} F(\eta)) \approx 0$. The proof of the following theorem is based on the result of Last, Peccati and Schulte [18], that asserts that the functional F_W exhibits Gaussian behavior as $W \to \mathbb{R}^d$, as soon as $D_{\mathbf{x},\mathbf{y}} F_W$ is small when $\mathbf{x}, \mathbf{y}$ are far away, uniformly in W. The speed of decay actually yields a bound on the speed of convergence of F_W towards the normal.

Theorem 3.1. Let $W \subset \mathbb{Z}^d$ bounded. Let $G_W \in \{F_W, F'_W\}$ as defined in (1.1)-(1.2), with $F_0 \in \mathcal{F}$, and let $M, M' \sim \mu$ independent. Assume that for some $C_0 > 0$, either **(i)** $G_W = F_W$ and for some $\alpha > 2d$, for all $k \in W, a.a.\, x \in \tilde{W}, a.a.\, y \in \mathbb{R}^d$, $\eta' \in \{\eta, \eta \cup \{(y, M')\}\}$,

$$\left[\mathbf{E}|D_{(x,M)} F_0((\eta' \cap \tilde{W}) - k)|^4\right]^{1/4} \leqslant C_0(1 + \|x\|)^{-\alpha}, x \in \mathbb{R}^d, \tag{3.1}$$

or **(i')** $G_W = F'_W$ and for some $\alpha > 5d/2$, for a.a. $x, y \in \mathbb{R}^d$, $\eta' \in \{\eta, \eta \cup \{(y, M')\}\}$,

$$\left[\mathbf{E}|D_{(x,M)} F_0(\eta')|^4\right]^{1/4} \leqslant C_0(1 + \|x\|)^{-\alpha}, x \in \mathbb{R}^d. \tag{3.2}$$

Then, $\sigma^2 := \mathrm{Var}(G_W) < \infty$, and if $\sigma > 0$, with $\tilde{G}_W = \sigma^{-1}(G_W - \mathbf{E}G_W)$,

$$d_{\mathscr{W}}(\tilde{G}_W, N) \leqslant \kappa \left(C_0^2 \sigma^{-2} \sqrt{|W|} + C_0^3 \sigma^{-3} |W|\right) \left(1 + \left(\frac{|\partial_{\mathbb{Z}^d} W|}{|W|}\right)^a\right), \tag{3.3}$$

where $a = 0$ in case **(i)**, and $a = 2(\alpha/d - 2)$ in case **(i')**. Let $v := \sup_W (G_W - \mathbf{E}G_W)^4 |W|^{-2} \in \mathbb{R}_+ \cup \{\infty\}$, then

$$d_{\mathscr{K}}(\tilde{G}_W, N) \leqslant \kappa \left(C_0^2 \sigma^{-2} \sqrt{|W|} + C_0^3 \sigma^{-3} |W| + v^{1/4} C_0^3 \sigma^{-4} |W|^{3/2}\right) \left(1 + \left(\frac{|\partial_{\mathbb{Z}^d} W|}{|W|}\right)^a\right). \tag{3.4}$$

Recall that (2.1) (or (2.2) in case **(i')**) is a sufficient condition for $v < \infty$.

The proof is at Section 5.2

3.2. Proof of Theorems 1.1 and 1.2

We prove Theorem 1.1 (resp. Theorem 1.2) using Theorems 2.1 and 3.1.

Let $n \geqslant 1$ be such that $W = W_n$ is bounded and non-empty, $G_W = F_W$ (resp. $G_W = F'_W$), $\sigma^2 = \mathrm{Var}(G_W)$. Assumption (1.8) (resp. (1.10)) clearly implies (2.1) (resp. (2.2)), and therefore (2.4) holds:

$$\left||W|^{-1}\sigma^2 - \sigma_0^2\right| \leqslant \kappa C_0^2 (|\partial_{\mathbb{Z}^d} W|/|W|)^{1-d/\alpha}.$$

Let $y \in \mathbb{R}^d, k \in W, x \in \tilde{W} - k, \mathbf{x} = (x, M), \eta' \in \{\eta, \eta \cup \{(y, M')\}\}$ as in (3.1) (resp. (3.2)), $\eta'' = \eta' \cup \{\mathbf{x}\}, B = \tilde{W} - k$ (resp. $B = \mathbb{R}^d$), $r = \|x\|/a_+$. Note that $x \in B \setminus B_r$, hence

$$\begin{aligned} D_{\mathbf{x}} F_0(\eta' \cap B) =& F_0((\eta' \cap B) \cup \{\mathbf{x}\}) - F_0(\eta' \cap B) \\ =& F_0((\eta' \cup \{\mathbf{x}\}) \cap B) - F_0(\eta' \cap B) \\ =& F_0((\eta' \cup \{\mathbf{x}\}) \cap B) - F_0((\eta' \cup \{\mathbf{x}\}) \cap B \cap B_r) + F_0((\eta' \cup \{\mathbf{x}\}) \cap B \cap B_r) - F_0(\eta' \cap B) \\ =& F_0(\eta'' \cap B) - F_0(\eta'' \cap B \cap B_r) + F_0(\eta' \cap B \cap B_r) - F_0(\eta' \cap B). \end{aligned}$$

Applying (1.8) (resp. (1.10)) twice with $x_1 = x, x_2 = y$ yields

$$\left(\mathbf{E}|D_{\mathbf{x}} F_0(\eta' \cap B)|^4\right)^{1/4} \leqslant C_0(1 + r)^{-\alpha},$$

hence (3.1) (resp. (3.2)) holds, and (3.3) holds. Since furthermore Assumption 1.1 holds, Theorem 2.1 yields $\sigma_0 > 0$, and for n sufficiently large, $\sigma^{-2} \leqslant 2|W|^{-1}\sigma_0^{-2}$, hence, with $\tilde{G}_W := (G_W - \mathbf{E}G_W)(\mathrm{Var} G_W)^{-1/2}$, for n sufficiently large, using also (1.5),

$$d_{\mathscr{W}}(\tilde{G}_W, N) \leqslant \kappa |W|^{-1/2} \left(C_0^2 \sigma_0^{-2} + C_0^3 \sigma_0^{-3}\right).$$

Since (1.8) (resp. (1.10)) holds with $\alpha > 2d$, we have furthermore by (2.5):

$$v = \limsup_{n \geqslant 1} \mathbf{E}\left(G_{W_n} - \mathbf{E}G_{W_n}\right)^4 / |W_n|^2 \leqslant \kappa C_0 (\mathbf{E}(F_0 - \mathbf{E}F_0)^4)^{3/4}.$$

Applying (1.8) with $r = 0, B = \mathbb{R}^d$ gives $\mathbf{E}(F_0 - \mathbf{E}F_0)^4 \leqslant \kappa C_0^4$. The bound on Kolmogorov distance (1.9) (resp. (1.11)) follows easily by (3.4).

It remains to prove that $G'_W := (\sigma_0^2|W|)^{-1/2}(G_W - \mathbf{E}G_W)$ follows a central limit theorem. We achieve it by proving that its Wasserstein distance to the normal goes to 0. The triangular inequality yields

$$\begin{aligned} d_{\mathscr{W}}(G'_W, N) \leqslant& \mathbf{E}\left|G'_W - \tilde{G}_W\right| + d_{\mathscr{W}}(\tilde{G}_W, N) \\ \leqslant& \left|\frac{1}{\sigma_0\sqrt{|W|}} - \frac{1}{\sqrt{\mathrm{Var}(G_W)}}\right| \mathbf{E}\left|G_W - \mathbf{E}G_W\right| + d_{\mathscr{W}}(\tilde{G}_W, N) \end{aligned}$$

which indeed goes to 0 by (2.4).

4. Application to shot-noise processes

Let the notation of the introduction prevail. For the process f_η (see (1.13)) to be well defined, assume throughout the section that for some $\tau > 0$,

$$\int_{\mathbf{M}} \int_{B(0,\tau)^c} |g_m(x)| dx \mu(dm) < \infty, \tag{4.1}$$

and let $\mathscr{N}_0$ be the class of locally finite ζ such that $\sum_{(x,m)\in\zeta} |g_m(x)| < \infty, x \in \mathbb{R}^d$. The fact that $\eta \in \mathscr{N}_0$ a.s. follows from the Campbell-Mecke formula.

We study in this section the behaviour of functionals of the excursion set $\{f_\eta \geqslant u\}, u \geqslant 0$. We use the general framework of random measurable sets. A *random measurable set* is a random variable taking values in the space $\mathcal{M}_d$ of measurable subsets of $\mathbb{R}^d$, endowed with the Borel σ-algebra $\mathcal{B}(\mathcal{M}_d)$ induced by the local convergence in measure, see Section 2 in [12]. Regarding the more familiar setup of random closed sets, in virtue of Proposition 2 in [12], a random measurable set which realisations are a.s. closed can be assimilated to a random closed set.

4.1. Volume of excursions

For $u \in \mathbb{R}$ fixed, $W \subset \mathbb{Z}^d, \zeta \in \mathscr{N}$, define

$$F_W(\zeta) = \ell^d(\{f_{\zeta\cap\tilde{W}} \geqslant u\} \cap \tilde{W}), \;\; F'_W(\zeta) = \ell^d(\{f_\zeta \geqslant u\} \cap \tilde{W}).$$

A central limit theorem for the volume of a certain family of shot noise excursions has been derived in [9], under the assumption that $f_\eta(0)$ has a uniformly bounded density and $\int |g_m(x)|\mu(dm)$ decreases sufficiently fast as $\|x\| \to \infty$, using the associativity properties of non-negative shot-noise fields. In some specific cases, the bounded density can be checked manually with computations involving the Fourier transform. In this section, we refine this result in several ways:

- A general model of random function is treated, it can in particular take negative values, allowing for compensation mechanisms (see [17]). For $u > 0$, to avoid trivial cases we assume

$$\mu(\{m \in \mathbf{M} : g_m \geqslant 0\}) \neq 0. \tag{4.2}$$

- The precise variance asymptotics are derived.
- Weaker conditions are required for the results to hold, in particular bounded density is not needed.
- The likely optimal rate of convergence in Kolmogorov distance towards the normal is given.
- Boundary effects under finite input are considered, in the sense that only points falling in a bounded window (growing to infinity) contribute to the field. The case of infinite input is also treated.

The application to shot noise excursions is a nice illustration of the versatility of the general method derived in this article. We give examples of fields with no marginal density to which the results apply, such as sums of indicator functions, or of kernels with a singularity in 0. Controlling the density of shot-noise fields is in general crucial for deriving results on fixed-level excursions. The case of indicator kernels is treated in Section 4.3.

Assumption 4.1. Let f_η be of the form (1.14) with g such that $|g(x)| \leqslant c\|x\|^{-\lambda}, \|x\| \geqslant 1$ for some $\lambda > 11d, c > 0$. Assume that there is $\varepsilon > 0, c > 0$ such that

$$\int_0^r \frac{\rho^{-2} \wedge \rho^{2(d-1)}}{-g'(\rho)} d\rho \leqslant c \exp(cr^{d-\varepsilon}), r > 0. \tag{4.3}$$

Lemma 4.2 below yields that if f_η satisfies this assumption, we can somehow control its density: for $a \in (0,1)$ there is $c_a > 0$ such that

$$\sup_{v \in \mathbb{R}, \delta > 0} \mathbf{P}(f_\eta(0) \in (v-\delta, v+\delta)) \leqslant c_a \delta^a. \tag{4.4}$$

This result might be of independent interest, and is proved after Lemma 4.2. Here are examples of functions fulfilling Assumption 4.1 (and hence satisfying (4.4)), note that nothing prevents g from having a singularity in 0.

Example 4.1. Theorem 4.1 below applies in any dimension to $g(\rho) = C\rho^{-\nu}\mathbf{1}_{\{\rho \leqslant 1\}} + g_1(\rho)\mathbf{1}_{\{\rho > 1\}}, \rho > 0$ and $g_1(\rho)$ is for instance of the form $\exp(-a\rho^\gamma)$ or $\rho^{-\lambda}$, with $a, \nu > 0, \lambda > 11d, \gamma < d, C > 0$. Such fields don't necessarily have a finite first-order moment, and are used for instance in [2] to approximate stable fields, or for modeling telecommunication networks.

To give results in the case where boundary effects are considered, we need an additional hypothesis on the geometry of the underlying family of windows $\mathscr{W} = \{W_n; n \geqslant 1\}$. For $\theta > 0$, let $\mathcal{C}_\theta$ be the family of cones $C \subset \mathbb{R}^d$ with apex 0 and aperture θ, i.e. such that $\mathcal{H}^{d-1}(C \cap \mathcal{S}^{d-1}) \geqslant \theta$. Let $\mathcal{C}_{\theta,R} = \{C \cap B(0,R) : C \in \mathcal{C}_\theta\}$ for $R \geqslant 0$. Say that $\mathscr{W}$ has aperture $\theta > 0$ if for all $W \in \mathscr{W}$ with diameter $r > 0$, W has *aperture* θ : for $x \in \tilde{W}$, there is $C \in \mathcal{C}_{\theta, \ln(r)^{1/2d}}$ such that $(x + C) \subset \tilde{W}$.

Theorem 4.1. Let $u > 0$. Let $G_W = F'_W$, or $G_W = F_W$ if $\mathscr{W}$ is assumed to have aperture $\theta > 0$. Assume that Assumption 4.1 holds. Then as $|\partial_{\mathbb{Z}^d} W|/|W| \to 0$, $\operatorname{Var}(G_W) \sim \sigma_0^2|W|$, $(G_W - \mathbf{E}G_W)(\sigma_0\sqrt{|W|})^{-1}$ satisfies a central limit theorem, with

$$\sigma_0^2 = \int_{\mathbb{R}^d} \left[\mathbf{P}(f_\eta(0) \geqslant u, f_\eta(x) \geqslant u) - \mathbf{P}(f_\eta(0) \geqslant u)^2\right] dx > 0. \tag{4.5}$$

Also, the convergence rate (3.3) in Kolmogorov distance holds for $\tilde{G}_W$.

This result requires f to be under the form (1.14) mainly because of the density estimates provided by Lemma 4.2, but under general density assumptions, it could apply to more general models of the form (1.13). Let us state a lemma that will be required in the proof, and in other results concerning the non-triviality of shot-noise excursions.

Lemma 4.1. Let f_η be of the form (1.13). Assume that

$$\text{for some } M \subset \{m \in \mathbf{M} : \ell^d(g_m^{-1}((0,\infty))) > 0\}, \quad \mu(M) > 0. \tag{4.6}$$

Then there is $\rho > 1$ such that for $\beta \geqslant 1$, $\mathbf{E}(\ell^d(\{f_{\eta^\rho} > u\} \cap \tilde{Q}_\beta)) > 0$.

Proof. Basic measure theory yields $\varepsilon > 0, \rho > 1$ such that

$$\mu(\{m : \ell^d(g_m^{-1}((\varepsilon,\infty)) \cap \tilde{Q}_{\rho-1}) > 0\}) > 0.$$

Let $t \in \tilde{Q}_1, k > u/\varepsilon$, and $X_i = (Y_i, M_i), i \leqslant k$ iid couples of $\tilde{Q}_\rho \times \mathbf{M}$, and $U_i := g_{M_i}(t - Y_i)$. We have $\mathbf{P}(U_1 \geqslant \varepsilon) > 0$, hence

$$\mathbf{P}(f_{\eta^\rho}(t) \geqslant u) \geqslant \mathbf{P}(f_{\eta^\rho}(t) \geqslant k\varepsilon) \geqslant \mathbf{P}(|\eta^\rho \cap \tilde{Q}_\rho| = k)\mathbf{P}(U_1 \geqslant \varepsilon)^k > 0.$$

Then Fubini's theorem yields for $\beta \geqslant 1$

$$\begin{aligned}&\mathbf{E}\ell^d(\{t \in \tilde{Q}_\beta : f_{\eta^\rho}(t) \geqslant u\})\\ &\quad\geqslant \mathbf{E}\ell^d(\{t \in \tilde{Q}_1 : f_{\{X_1,\dots,X_k\}}(t) \geqslant u\})\mathbf{P}(|\eta^\rho| = k) > 0.\end{aligned}$$

□

Proof of Theorem 4.1. The decay assumption on g yields that (4.1) holds for $\tau = 1$, and the left hand member of (1.10) is uniformly bounded for $r \leqslant 2\sqrt{d}$. From now on we take $r > 2\sqrt{d}$. We wish to prove the the conditions of Theorems 1.1 and 1.2 are satisfied with the functional $F_0(\zeta) = \int_{\tilde{Q}_1} \mathbf{1}_{\{f_\zeta(t) \geqslant u\}} dt$. Let us start by proving Assumption 1.1. Let $M = \{m \in \mathbf{M} : g_m \geqslant 0\}$. For $\rho > 0$, let Γ_ρ be the event that $\eta^\rho \subset \tilde{Q}_\rho \times M$ (i.e. all functions of η^ρ are non-negative). Since $\mu(M) > 0$, $\mathbf{P}(\Gamma_\rho) > 0$. Lemma 4.1 yields $p > 0, \rho > 1$ such that for $t \in \tilde{Q}_1$, $\mathbf{P}(f_{\eta^\rho}(t) > 2u|\Gamma_\rho) \geqslant p$. Also $\mathbf{E}|f_{\eta_\gamma}(t)| \to 0$ as $\gamma \to \infty$ uniformly in $t \in \tilde{Q}_1$, hence for γ sufficiently large, $t \in \tilde{Q}_1, \mathbf{P}(|f_{\eta_\gamma}(t)| < u) > \frac{1}{2}$. Conditionaly on Γ_ρ, $f_{\eta_\gamma \cup \eta^\rho} = f_{\eta^\rho} + f_{\eta_\gamma} \geqslant f_{\eta_\gamma}$. Hence, for $\delta > \gamma > \rho$,

$$\mathbf{1}_{\{\Gamma_\rho\}} |F_{Q_\delta}(\eta_\gamma \cup \eta^\rho) - F_{Q_\delta}(\eta_\gamma)| = \mathbf{1}_{\{\Gamma_\rho\}} \int_{\tilde{Q}_\delta} \mathbf{1}_{\{f_{\eta^\rho \cup \eta_\gamma}(t) > u, f_{\eta_\gamma}(t) < u\}} dt \geqslant \mathbf{1}_{\{\Gamma_\rho\}} G$$

$$\text{where } G := \int_{\tilde{Q}_1} \mathbf{1}_{\{|f_{\eta_\gamma}(t)| < u, f_{\eta^\rho}(t) > 2u\}} dt.$$

$$\mathbf{E}[G|\Gamma_\rho] \geqslant \int_{\tilde{Q}_1} \mathbf{P}(f_{\eta^\rho}(t) > 2u|\Gamma_\rho)\mathbf{P}(|f_{\eta_\gamma}(t)| < u) dt \geqslant \frac{p}{2}.$$

Since $G \leqslant 1$, $\mathbf{P}(G \geqslant p/4|\Gamma_\rho) \geqslant p/4 > 0$. Hence $\mathbf{P}(|F_{Q_\delta}(\eta_\gamma \cup \eta^\rho) - F_{Q_\delta}(\eta^\rho)| > c) \geqslant \mathbf{P}(\Gamma_\rho)\mathbf{P}(G \geqslant p/4|\Gamma_\rho) =: p' > 0$, hence Assumption 1.1 is satisfied.

Let us now prove that (1.8) holds in the case $G_W = F_W$ (or (1.10) in the case $G_W = F'_W$). Let $x_1, x_2 \in \mathbb{R}^d$, M_1, M_2 independent marks with law μ, $r \geqslant 0$,

$\zeta \subset \{(x_1, M_1), (x_2, M_2)\}, \eta' = \eta \cup \zeta$. Let $B = \mathbb{R}^d$ in the case of infinite input ($G_W = F'_W$), and let $B \in \mathcal{B}^r_{\mathscr{W}}$ otherwise ($G_W = F_W$). Jensen's inequality yields

$$|F_0(\eta' \cap B) - F_0(\eta' \cap B_r \cap B)|^4 = \left[\int_{\tilde{Q}_1} \left(\mathbf{1}_{\{f_{\eta'\cap B}(t) \geqslant u\}} - \mathbf{1}_{\{f_{\eta'\cap B_r \cap B}(t) \geqslant u\}}\right) dt\right]^4$$
$$\leqslant \int_{\tilde{Q}_1} \left|\mathbf{1}_{\{f_{\eta'\cap B}(t) \geqslant u\}} - \mathbf{1}_{\{f_{\eta'\cap B_r \cap B}(t) \geqslant u\}}\right| dt,$$

and for $t \in \tilde{Q}_1, r > 2\sqrt{d}$,

$$|f_{\eta'\cap B}(t) - f_{\eta'\cap B_r\cap B}(t)| = \left|\sum_{\mathbf{x}=(x,m)\in(\eta'\cap B)\setminus B_r} g_m(t-x)\right|$$
$$\leqslant \delta_{r,t} := \sum_{\mathbf{x}=(x,m)\in\eta'\setminus B(t, a_-(r-\sqrt{d}))} |g_m(t-x)|.$$

Note that $\delta_{r,t}$ is independent from $\eta \cap B(t, a_- r/2)$ and its law does not depend on $t \in \tilde{Q}_1$. Since $B = \tilde{Z}$ for some $Z \subset \mathbb{Z}^d$ and $0 \in B, t \in B$. Let $R = 1 \wedge \ln(a_- r)^{\frac{1}{d-\varepsilon/2}}$, where ε is from Assumption 4.1. Since B intersects B_r^c, it has diameter at least $a_- r$ and since W has aperture θ, there is a solid cone $C_t \in \mathcal{C}_{\theta,R}$ such that, with $D_t = (C_t + t)$, $D_t \subset B$. In the infinite input case, the latter trivially holds with $B = \mathbb{R}^d, \theta = \sigma_{d-1} := \mathcal{H}^{d-1}(\mathcal{S}^{d-1}), D_t = B(t, R)$. We have

$$\mathbf{E}\left|F_0(\eta' \cap B) - F_0(\eta' \cap B_r \cap B)\right|^4 \leqslant \sup_{t \in \tilde{Q}_1} \mathbf{P}(f_{\eta'\cap B}(t) \in [u - \delta_{r,t}, u + \delta_{r,t}])$$
$$\leqslant \sup_{t\in\tilde{Q}_1} \mathbf{P}\left(f_{\eta\cap D_t}(t) + f_{\eta\cap(B\setminus D_t)}(t) + f_{\zeta\cap B}(t) \in [u - \delta_{r,t}, u + \delta_{r,t}]\right)$$
$$\leqslant \sup_{t\in\tilde{Q}_1} \mathbf{E}(\mathbf{P}(f_{\eta\cap D_t}(t) \in [u - f_{\eta\cap(B\setminus D_t)}(t) - f_{\zeta\cap B}(t) \pm \delta_{r,t}] \mid \sigma(\zeta, \eta \cap (B \setminus D_t))))$$
$$\leqslant \sup_{t\in\tilde{Q}_1} \mathbf{E}\left(\sup_{v\in\mathbb{R}} \mathbf{P}(f_{\eta\cap D_t}(t) \in [v - \delta_{r,t}, v + \delta_{r,t}] \mid \sigma(\delta_{r,t}))\right)$$
$$\leqslant \mathbf{E}\left(\sup_{C\in\mathcal{C}_{\theta,R}} \sup_{v\in\mathbb{R}} [\mathbf{P}(f_{\eta\cap C}(0) \in [v - \delta_{r,0}, v + \delta_{r,0}]) \mid \sigma(\delta_{r,0}))]\right). \quad (4.7)$$

To bound this quantity we need to study the density of the shot-noise field.

Lemma 4.2. Assume that f_η is of the form (1.14). Let $\delta > 0, R \geqslant 1$. Then for $v \in \mathbb{R}, C \in \mathcal{C}_{\theta,R}$,

$$\mathbf{P}(f_{\eta\cap C}(0) \in [v - \delta, v + \delta]\ ,\ |\eta \cap C| \geqslant 2) \leqslant \kappa\delta \int_0^R \frac{(\rho^{-2} \wedge \rho^{2(d-1)})d\rho}{-g'(\rho)}.$$

Before proving this result, let us conclude the proof of Theorem 4.1. Assume without loss of generality $r \geqslant 2r_0/a_-$. By Assumption 4.1, (4.7) is bounded by

$\sup_{C\in\mathcal{C}_{\theta,R}}\mathbf{P}(|\eta\cap C|<2)+c\kappa(\mathbf{E}[\delta_{r,0}\exp(cR^{d-\varepsilon})])$. The decay assumption on g yields that $\mathbf{E}(\delta_{r,0})\leqslant\kappa(1+r)^{-\lambda+d}$. Hence (4.7) is bounded by

$$(1+\kappa R^d)\exp(-\kappa\theta R^d)+c\kappa\exp(c\kappa\ln(r)^{\frac{d-\varepsilon}{d-\varepsilon/2}})(1+r)^{-\lambda+d}\leqslant\kappa(1+r)^{-(\lambda'-d)}$$

for any $\lambda'\in(11d,\lambda)$. Hence (1.8) and (1.10) hold with $\alpha=(\lambda'-d)/4>5d/2$. □

Proof of Lemma 4.2. Let $\lambda=\frac{\sigma_{d-1}}{\theta\kappa_d}$, $n_R=|\eta\cap C|$ be the number of germs (Poisson variable with parameter $\ell^d(C)=R^d/\lambda$), and let $g_R(x)=g(\|x\|)\mathbf{1}_{\{x\in C\}}$, so that $f_{\eta\cap C}(0)=\sum_{i=1}^{n_R}g_R(X_i)$ where the X_i are uniform iid in C. Call μ_R the distribution of the $g_R(X_i)$. We have for every $b>a\geqslant g(R)$, since g is one-to-one and continuous

$$\begin{aligned}\mu_R([a,b))=&\frac{\lambda}{R^d}\int_C\mathbf{1}_{\{a\leqslant g(\|x\|)<b\}}dx=\frac{\lambda}{R^d}\int_{g^{-1}(b)}^{g^{-1}(a)}\theta\rho^{d-1}d\rho\\=&\frac{\sigma_{d-1}}{d\kappa_dR^d}(g^{-1}(a)^d-g^{-1}(b)^d),\end{aligned}$$

whence μ_R has density $\varphi_R(a)=\mathbf{1}_{\{a\geqslant g(R)\}}\frac{\sigma_{d-1}}{\kappa_dR^d}\left(\frac{g^{-1}(a)^{d-1}}{-g'(g^{-1}(a))}\right)$. Then, denoting by $\varphi_R^{\otimes n}$ the density φ_R convoluted with itself n times on the real line,

$$\begin{aligned}\mathbf{P}(f_{\eta\cap C}(0)\in[v-\delta,v+\delta]\ ,\ |\eta\cap C|\geqslant2)\leqslant&\sum_{n=2}^{\infty}\mathbf{P}(n_R=n)\mathbf{P}\left(\sum_{i=1}^{n}g_R(X_i)\in[v-\delta,v+\delta]\right)\\\leqslant&\sum_{n\geqslant2}\mathbf{P}(n_R=n)\|\varphi_R^{\otimes n}\|_\infty2\delta\\\leqslant&2\sup_{n\geqslant2}\|\varphi_R^{\otimes n}\|_\infty\delta.\end{aligned}\tag{4.8}$$

Due to convolution properties, for $n\geqslant2$,

$$\begin{aligned}\|\varphi_R^{\otimes n}\|_\infty\leqslant&\|\varphi_R^{\otimes2}\|_\infty\leqslant\int_{\mathbb{R}}\varphi_R^2(a)da=\mathbf{E}\varphi_R(g_R(X_1))\\=&\frac{\lambda}{R^d}\int_C\varphi_R(g(\|x\|))dx\\\leqslant&\frac{\lambda}{R^d}\int_C\frac{g^{-1}(g(\|x\|))^{d-1}}{-g'(g^{-1}(g(\|x\|)))}\frac{\sigma_{d-1}dx}{\kappa_dR^d}\\=&\left(\frac{\sigma_{d-1}}{\kappa_dR^d}\right)^2\int_0^R\frac{1}{-g'(\rho)}\rho^{2(d-1)}d\rho\\\leqslant&\left(\frac{\sigma_{d-1}}{\kappa_d}\right)^2\left(\frac{1}{R^{2d}}\int_0^1\frac{\rho^{2(d-1)}}{-g'(\rho)}d\rho+\int_1^R\frac{\rho^{-2}d\rho}{-g'(\rho)}\right),\end{aligned}$$

which concludes the lemma after reporting in (4.8). □

Proof of claim (4.4). Let $v > 0, \delta > 0$. Let $R = R_\delta := 1 \wedge |\ln(\delta)|^{\frac{1}{d-\varepsilon/2}}$. Introduce the events $A_{\delta,v} = \{f_{\eta^R}(0) \in (v-\delta, v+\delta)\}$, $A'_{\delta,v} = \{f_\eta(0) \in (v-\delta, v+\delta)\}$, $B_\delta = \{|\eta^R| \geqslant 2\}$. Since Assumption 4.1 holds, Lemma 4.2 yields $\mathbf{P}(A_{\delta,v} \cap B_\delta) \leqslant c_a \delta^a$ for all $v > 0$. Let $U_\delta = f_\eta(0) - f_{\eta^{R_\delta}}(0)$. Note that U_δ is independent from $f_{\eta^{R_\delta}}(0)$. We have

$$\begin{aligned}
\mathbf{P}(A_{\delta,v}) \leqslant& \mathbf{P}(A_{\delta,v} \cap B_\delta) + \mathbf{P}(B_\delta^c) \leqslant c_a \delta^a + o(\delta^a) \leqslant c'_a \delta^a. \\
\mathbf{P}(A'_{\delta,v}) =& \mathbf{E}\left[\mathbf{P}(f_{\eta^R}(0) + U_\delta \in (v-\delta, v+\delta) | U_\delta)\right] \\
\leqslant& \mathbf{E}(\mathbf{P}(A_{\delta, v-U_\delta} | U_\delta)) \leqslant \mathbf{E}(c'_a \delta^a) = c'_a \delta^a,
\end{aligned}$$

hence the claim is proved. □

4.2. Perimeter

We use in this section the variational definition of perimeter, following Ambrosio, Fusco and Pallara [1]. Define the *perimeter* of a measurable set $A \subset \mathbb{R}^d$ within $U \subset \mathbb{R}^d$ as the total variation of its indicator function

$$\mathrm{Per}(A; U) := \sup_{\varphi \in \mathcal{C}_c^1(U, \mathbb{R}^d) : \|\varphi\| \leqslant 1} \int_{\mathbb{R}^d} \mathbf{1}_A(x) \mathrm{div}\varphi(x) dx,$$

where $\mathcal{C}_c^1(U, \mathbb{R}^d)$ is the set of continuously differentiable functions with compact support in U. Note that for regular sets, such as $\mathcal{C}^1$ manifolds, or convex sets with non-empty interior, this notion meets the classical notion of $(d-1)$-dimensional Hausdorff surface measure [1, Exercise 3.10], even though the term *perimeter* is traditionally used for 2-dimensional objects. It is a possibly infinite quantity, that might also have counterintuitive features for pathological sets ([1, Example 3.53]). The main difference with the traditional perimeter is that the variational one obviously cannot detect the points of the boundary whose neighborhoods don't charge the volume of the set, such as in line segments for instance.

For any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ and level $u \in \mathbb{R}$, the perimeter of the excursion $\mathrm{Per}(\{f \geqslant u\}; U)$ within U is a well-defined quantity. To be able to compute it efficiently, we must make additional assumptions on the regularity of f. Following [8], we assume that f belongs to the space $BV(U)$ of functions with bounded variations, i.e. $f \in L^1(U)$ and its variation above U is finite:

$$V(f, U) := \sup_{\varphi \in \mathcal{C}_c^1(U, \mathbb{R}^d) : \|\varphi\| \leqslant 1} \int_U f(x) \mathrm{div}\varphi(x) dx < \infty.$$

The original (equivalent) definition states that $f \in L^1(U)$ is in $BV(U)$ if and only if the following holds ([1, Proposition 3.6]): there exists signed Radon measures $D_i f$ on $U, 1 \leqslant i \leqslant d$, called *directional derivatives* of f, such that for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\int_U f(x) \mathrm{div}\varphi(x) dx = -\sum_{i=1}^d \int_U \varphi_i(x) D_i f(dx).$$

Then there is a finite Radon measure $\|Df\|$ on U, called *total variation measure*, and a $\mathcal{S}^{d-1}$-valued function $\nu_f(x), x \in U$, such that $Df = \sum_i D_i f = \|Df\|\nu_f$. According to the Radon-Nikodym theorem, the total variation can be decomposed as

$$\|Df\| = \nabla f \ell^d + D^j f + D^c f \tag{4.9}$$

where ∇f is defined as the density of the continuous part of $\|Df\|$ with respect to ℓ^d, $D^c f + D^j f$ is the singular part of $\|Df\|$ with respect to Lebesgue measure, decomposed in the *Cantor part* $D^c f$, and the jump part $D^j f$, that we specify below, following [1, Section 3.7].

For $x \in U$, denote by H_x the affine hyperplane containing x with outer normal vector $\nu_f(x)$. For $r > 0$, denote by $B^+(x, r)$ and $B^-(x, r)$ the two components of $B(x, r) \setminus H_x$, with $\nu_f(x)$ pointing towards $B^+(x, r)$. Say that x is a *regular point* if there are two values $f^+(x) \geqslant f^-(x)$ such that

$$\lim_{r\to 0} r^{-d} \int_{B^+(x,r)} |f^+(x) - f(y)|dy = \lim_{r\to 0} r^{-d} \int_{B^-(x,r)} |f(y) - f^-(x)|dy = 0. \tag{4.10}$$

It turns out that the set of non-regular points has $\mathcal{H}^{d-1}$-measure 0 ([1, Th. 3.77]), and the set J_f of points where $f^+(x) > f^-(x)$, called *jump points*, has Lebesgue measure 0 ([1, Th. 3.83]). Then, the jump measure of f is represented by

$$D^j f(dx) = \mathbf{1}_{\{x\in J_f\}}(f^+(x) - f^-(x))\mathcal{H}^{d-1}(dx),$$

where $\mathcal{H}^{d-1}$ stands for the $(d-1)$-dimensional Hausdorff measure.

In the classical case where f is continuously differentiable on U, $Df = \nabla f \ell^d$, $\nu_f(x) = \|\nabla f(x)\|^{-1}\nabla f(x)$ (and takes an irrelevant arbitrary value if $\nabla f(x) = 0$), and $V(f; U) = \int_U \|\nabla f(x)\|dx$. If $f = \mathbf{1}_{\{A\}}$ for some $\mathcal{C}^1$ compact manifold A, $\nu_f(x)$ is the outer normal to A for $x \in \partial A$, $\nabla f = 0, D^c f = 0$, and $D^j f = \mathbf{1}_{\{\partial A\}}\mathcal{H}^{d-1}$.

Denote by $SBV(U)$ the functions $f \in BV(U)$ such that $D^c f = 0$. Assume here that for $m \in \mathbf{M}, g_m \in SBV(\mathbb{R}^d)$, and that

$$\int_{\mathbf{M}} \left[\int_{\mathbb{R}^d} (|g_m(t)| + \|\nabla g_m(t)\|)dt + \int_{J_{g_m}} |g_m^+(t) - g_m^-(t)|\mathcal{H}^{d-1}(dt)\right] \mu(dm) < \infty.$$

Let $\mathscr{N}_0$ be the class of configurations ζ such that the corresponding shot noise field f_ζ is of class $SBV(U)$ on every bounded set U, finite a.e. on $\mathbb{R}^d$, its gradient density defined by (4.9) is a vector-valued shot-noise field, defined a.s. and ℓ^d-a.e. by

$$\nabla f_\zeta(t) = \sum_{(x,m)\in\zeta} \nabla g_m(t - x),$$

and its jump set J_f is the union of the translates of the impulse jump sets: $J_f = \cup_{(x,m)\in\zeta}(x + J_{g_m})$, and the jumps of f are

$$f_\zeta^+(y) - f_\zeta^-(y) = \sum_{(x,m)\in\zeta} \mathbf{1}_{\{y\in x+J_{g_m}\}}(g_m^+(y-x) - g_m^-(y-x)), y \in J_f.$$

[8, Theorem 2] and the previous assumption yield that $\eta \in \mathscr{N}_0$ a.s.. Let h be a *test function*, i.e. a function $h : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^1$ with compact support. Let H be a primitive function of h. Biermé and Desolneux [8, Theorem 1] give for $W \subset \mathbb{Z}^d, \zeta \in \mathscr{N}$,

$$F_W^{h,Per}(\zeta) := \int_{\mathbb{R}} h(u)\mathrm{Per}(\{f_\zeta \geqslant u\}; \tilde{W})du = F_W^{h,cont}(\zeta) + F_W^{h,jump}(\zeta),$$

where

$$F_W^{h,cont}(\zeta) = \int_{\tilde{W}} h(f_\zeta(x))\|\nabla f_\zeta(x)\|dx,$$
$$F_W^{h,jump}(\zeta) = \int_{J_f\cap\tilde{W}} (H(f_\zeta^+(x)) - H(f_\zeta^-(x)))\mathcal{H}^{d-1}(dx).$$

Their expectations under η are computed in [8, Section 3] :

$$\mathbf{E}[F_W^{h,cont}(\eta)] = |W|\mathbf{E}\left[h(f_\eta(0))\|\nabla f_\eta(0)\|\right]$$
$$\mathbf{E}[F_W^{h,jump}(\eta)] = |W|\int_{\mathbf{M}}\int_{J_{g_m}}\left(\int_{g_m^-(y)}^{g_m^+(y)}\mathbf{E}[h(s+f_\eta(0))]ds\right)\mathcal{H}^{d-1}(dy)\mu(dm).$$

Let us now give their second order behaviour. It is difficult to give sharp necessary conditions for non-degeneracy of the variance if the function h changes signs, so we treat the case $h \geqslant 0$, but it is can clearly be extended.

Theorem 4.2. Let $\mathscr{W} = \{W_n; n \geqslant 1\}$ satisfying (1.5). Assume that (4.6) holds and that $\mathbf{P}(F_W^{h,Per}(\eta) \neq F_W^{h,Per}(\emptyset)) > 0$ for some $W \subset \mathbb{Z}^d$. Assume that for some $\alpha > 5d/2, c > 0$,

$$(\mathbf{E}|g_M(x)|^4)^{1/4} \leqslant c(1+\|x\|)^{-d-\alpha}, \quad (4.11)$$
$$(\mathbf{E}\|\nabla g_M(x)\|^4)^{1/4} \leqslant c(1+\|x\|)^{-d-\alpha}, \quad (4.12)$$
$$\left(\mathbf{E}\left[\int_{J_{g_M}\cap(x+[0,1)^d)} (1 \vee |g_M^+(t) - g_M^-(t)|)\mathcal{H}^{d-1}(dt)\right]^4\right)^{1/4} \leqslant c(1+\|x\|)^{-d-\alpha}. \quad (4.13)$$

Then the conclusions of Theorems 1.1,1.2,2.1,3.1 hold for $F_0 := F_{\{0\}}^{h,\mathrm{Per}}$. In particular, $F_W^{h,\mathrm{Per}}$ has a variance proportional to $|W|$ and follows a CLT.

Example 4.2. Assume $\mathbf{M} = \mathbb{R}$ is endowed with a probability measure μ with finite 4-th moment. Let f be a function of the form

$$f_\zeta(x) = \sum_{(y,m)\in\zeta} m g(\|x-y\|)$$

with $g \in SBV(\mathbb{R})$. Conditions (4.11) and (4.12) hold if $|g(r)| \leqslant C(1+r)^{-d-\alpha}$ and $|g'(r)| \leqslant C(1+r)^{-d-\alpha}, r > 0$. Then (4.13) holds if J_g is countable and for some $C > 0$, for every $r > 0$

$$\sum_{t\in J_g\cap[r,r+1)} (1 \vee |g^+(t) - g^-(t)|) \leqslant C(1+r)^{-d-\alpha}.$$

Proof. First, (4.11)-(4.12) imply that the shot noise process and its gradient measure are a.s. well defined. The functionals $F_W^{h,cont}, F_W^{h,jump}$ are under the form (1.1)-(1.2), with F_0 defined respectively by, for $\zeta \in \mathscr{N}$,

$$\begin{aligned} F_0^{h,cont}(\zeta) &= \int_{\tilde{Q}_1} h(f_\zeta(t))\|\nabla f_\zeta(t)\| dt \\ F_0^{h,jump}(\zeta) &= \int_{J_{f_\zeta}\cap\tilde{Q}_1} (H(f_\zeta^+(t)) - H(f_\zeta^-(t)))\mathcal{H}^{d-1}(dt), \end{aligned}$$

where H is a primitive function of h.

Let $\mathbf{x}_i = (x_i, m_i) \in \mathbb{R}^d, i = 1, \ldots, 6$. Let $r > 0, \zeta \subset \{\mathbf{x}_1, \mathbf{x}_2\}$, and let $\eta_j = \eta' \cap A_j, j = 1, 2$, for some $A_1 \subset A_2 \subset \mathbb{R}^d$ that coincide on B_r. By the triangular inequality,

$$\begin{aligned} \left|F_0^{h,cont}(\eta_1) - F_0^{h,cont}(\eta_2)\right| &\leqslant \int_{\tilde{Q}_1} \|h'\|_\infty |f_{\eta_1}(t) - f_{\eta_2}(t)| \|\nabla f_{\eta_1}(t)\| dt \\ &\quad + \int_{\tilde{Q}_1} \|h\|_\infty \|\nabla f_{\eta_1}(t) - \nabla f_{\eta_2}(t)\| dt \\ \leqslant \sum_{(x,m)\in\eta'\setminus B_r} \int_{\tilde{Q}_1} &[\|h'\|_\infty \|\nabla f_{\eta_1}(t)\| |g_m(x-t)| + \|h\|_\infty \|\nabla g_m(x-t)\|]\, dt. \end{aligned}$$

Define for $\zeta_0 \in \mathscr{N}$, $\mathbf{x} = (x, m) \in \overline{\mathbb{R}^d}$,

$$\psi(\mathbf{x}, \zeta_0) = \int_{\tilde{Q}_1} \left[\|h'\|_\infty \|\nabla f_{(\zeta_0\cup\zeta)\cap A_1}(t)\| |g_m(x-t)| + \|h\|_\infty \|\nabla g_m(x-t)\|\right] dt.$$

For $\zeta' \subset \{\mathbf{x}_i, 3 \leqslant i \leqslant 6\}$, Jensen's inequality yields for $\mathbf{x} = (x, m) \in \overline{\mathbb{R}^d}$

$$\mathbf{E}\psi(\mathbf{x}, \eta \cup \zeta')^4 \leqslant C \int_{\tilde{Q}_1} \mathbf{E}\left[|g_m(x-t)|^4 \mathbf{E}\|\nabla f_{\eta_1\cup\zeta'}(t)\|^4 + \mathbf{E}\|\nabla g_m(x-t)\|^4\right] dt.$$

An easy application of Lemma 5.1 with $\psi'(x, m) = \|\nabla g_m(x-t)\|, r = 0$ yields that $\mathbf{E}\|\nabla f_{\eta_1\cup\zeta'}(t)\|^4 \leqslant c < \infty$ where c does not depend on $t \in \mathbb{R}^d, A_1$ or the $\mathbf{x}_i$. Therefore, Assumptions (4.11) and (4.12) yield for $\mathbf{x} = (x, m) \in \overline{\mathbb{R}^d}$

$$\mathbf{E}[\psi(\mathbf{x}, \eta \cup \zeta')^4] \leqslant C(1 + \|x\|)^{-4(\alpha+d)},$$

and Lemma 5.1 with (4.14) yields that

$$\left(\mathbf{E}\left[\left|F_0^{h,cont}(\eta\cap A_1)-F_0^{h,cont}(\eta\cap A_2)\right|^4\right]\right)^{1/4}\leqslant C(1+r)^{-\alpha},\qquad(4.14)$$

where C does not depend on the A_i. Hence, (1.8) is satisfied by $F_0^{h,cont}$ (hypothetical points of $\zeta\setminus B_r$ have to be treated separately).

Let us now prove that it is satisfied by the jump functional $F_0^{h,jump}$. Since it has to hold only for ℓ^d-a.e. x_1,x_2, and the J_{g_1},J_{g_2} have finite $\mathcal{H}^{d-1}$ measure, we assume that $J_{g_{m_1}}-x_1$ and $J_{g_{m_2}}-x_2$ have a $\mathcal{H}^{d-1}$−negligible intersection. They also a.s. have a $\mathcal{H}^{d-1}$-negligible intersection with each $J_{g_m}-x$, $(x,m)\in\eta$. Call $f_1=f_{\eta_1},f_2=f_{\eta_2}$,

$$\begin{aligned}
&\left|F_0^{h,jump}(\eta_1)-F_0^{h,jump}(\eta_2)\right|\\
=&\Bigg|\sum_{(x,m)\in\eta_1}\int_{J_{g_m}\cap\tilde{Q}_1}\left[(H(f_1^+(t))-H(f_1^-(t)))-(H(f_2^+(t))-H(f_2^-)(t))\right]\mathcal{H}^{d-1}(dt)\\
&-\sum_{(x,m)\in\eta_2\setminus\eta_1}\int_{J_{g_m}\cap\tilde{Q}_1}\left[H(f_2^+(t))-H(f_2^-(t))\right]\mathcal{H}^{d-1}(dt)\Bigg|\\
\leqslant&\int_{J_{f_1}\cap\tilde{Q}_1}\|h\|(\left|f_2^+(t)-f_1^+(t)\right|+\left|f_2^-(t)-f_1^-(t)\right|)\mathcal{H}^{d-1}(dt)\\
&+\sum_{(x,m)\in\eta'\setminus B_r}\int_{\tilde{Q}_1\cap J_{g_m}}\|h\|\left|g_m^+(x-t)-g_m^-(x-t)\right|\mathcal{H}^{d-1}(dt)\\
\leqslant&\sum_{(x,m)\in\eta'\setminus B_r}\|h\|\left(2\underbrace{\int_{J_{f_{\eta'}}\cap\tilde{Q}_1}|g_m(x-t)|\mathcal{H}^{d-1}(dt)}_{=:\psi_1((x,m),\eta)}+\underbrace{\int_{\tilde{Q}_1\cap J_{g_m}}|g_m^+(x-t)-g_m^-(x-t)|\mathcal{H}^{d-1}(dt)}_{=:\psi_2(x,m)}\right).
\end{aligned}\qquad(4.15)$$

We have $\mathbf{E}[\psi_2(x,M_0)^4]\leqslant C(1+\|x\|)^{-4(\alpha+d)}$ by (4.13), and Jensen's inequality yields for $\zeta'\subset\{\mathbf{x}_3,\dots,\mathbf{x}_6\}$, $f_3=f_{\eta'\cup\zeta'}$, after expanding the 4-th power of the integral as a quadruple integral,

$$\begin{aligned}
\mathbf{E}\psi_1((x,M_0),\eta\cup\zeta')^4=&\mathbf{E}\left(\mathbf{E}\left[\left(\int_{J_{f_3}\cap\tilde{Q}_1}|g_{M_0}(x-t)|\mathcal{H}^{d-1}(dt)\right)^4\middle|\sigma(\eta,\zeta,\zeta')\right]\right)\\
\leqslant&\mathbf{E}\left(\left(\int_{J_{f_3}\cap\tilde{Q}_1}(\mathbf{E}g_{M_0}(x-t)^4)^{1/4}\mathcal{H}^{d-1}(dt)\right)^4\middle|\sigma(\eta,\zeta,\zeta')\right)\\
\leqslant&C(1+\|x\|)^{-4(d+\alpha)}\mathbf{E}[\mathcal{H}^{d-1}(J_{f_3}\cap\tilde{Q}_1)^4]
\end{aligned}$$

by Assumption (4.11). Then (4.13) yields $\mathbf{E}[\mathcal{H}^{d-1}(J_{f_3} \cap \tilde{Q}_1)^4] < \infty$ with an application of Lemma 5.1, whence Lemma 5.1 again yields that $F_0^{h,jump}$ also satisfies (1.8) (here again the points of $\zeta \cup \zeta'$ have to be considered separately). Hence $F_0 = F_0^{h,cont} + F_0^{h,Per}$ satisfies (1.8).

It remains to prove Assumption 1.1. Assume wlog $F_0(\emptyset) = 0$. Since a set with positive volume has positive perimeter, Lemma 4.1 and Assumption (4.6) yield $\rho > 1, c > 0, p > 0$ such that for $\beta > \rho$, $\mathbf{P}(|F_{Q_\beta}(\eta^\rho)| \geqslant c) \geqslant p$. Then for $\delta > \gamma > \beta$,

$$\begin{aligned} U &:= \left|F_{Q_\delta}(\eta_\gamma \cup \eta^\rho) - F_{Q_\delta}(\eta_\gamma) - F_{Q_\beta}(\eta^\rho)\right| \\ &\leqslant \left|F_{Q_\beta}(\eta_\gamma \cup \eta^\rho) - F_{Q_\beta}(\eta^\rho)\right| + \left|F_{Q_\delta \setminus Q_\beta}(\eta^\rho \cup \eta_\gamma) - F_{Q_\delta \setminus Q_\beta}(\eta_\gamma)\right| + \left|F_{Q_\beta}(\eta_\gamma)\right| \\ \mathbf{E}U &\leqslant \kappa\beta^d(\gamma-\beta)^{-\alpha} + \sum_{m=\beta}^{\delta} \kappa m^{d-1}(m-\rho)^{-\alpha} + \kappa\beta^d(\gamma-\beta)^{-\alpha} \leqslant \kappa\beta^d(\gamma-\beta)^{d-\alpha} + C_\rho(\beta-\rho)^{d-\alpha}, \end{aligned}$$

the last estimates are obtained by choosing adequately A_1, A_2 in (4.14),(4.15). We can arbitrarily increase β such that $C_\rho(\beta-\rho)^{d-\alpha} < pc/8$, and then for γ sufficiently large $\kappa\beta^d(\gamma-\beta)^{d-\alpha} < pc/8$ as well, from where

$$\begin{aligned} \mathbf{P}(|F_{Q_\delta}(\eta_\gamma \cup \eta^\rho) - F_{Q_\delta}(\eta_\gamma)| > c/2) &\geqslant \mathbf{P}(|F_{Q_\beta}(\eta^\rho)| > c) - \mathbf{P}(|U| > c/2) \\ &\geqslant p - \mathbf{E}U/(c/2) \geqslant p - p/2 = p/2 > 0. \end{aligned}$$

That proves Assumption 1.1 and concludes the proof. □

4.3. Fixed level perimeter and Euler characteristic

Let $\mathcal{B}$ be a measurable subset of $\mathcal{M}_d$, and let the marks space be $\mathbf{M} = (\mathbb{R} \setminus \{0\}) \times \mathcal{B}$, endowed with the product σ-algebra and some probability measure μ. This section is restricted to shot-noise fields of the form

$$f_\zeta(x) = \sum_{(y,(L,A)) \in \zeta} L\mathbf{1}_{\{x-y \in A\}}, \zeta \subset \mathbb{R}^d \times \mathbf{M}, x \in \mathbb{R}^d. \tag{4.16}$$

Such fields are used in image analysis [7, 8], or in mathematical morphology [17], sometimes with $L = const.$, and their marginals might not have a density. The article [5] uses the asymptotic normality result below for the Euler characteristic when $\mathcal{B}$ is the class of closed discs in $\mathbb{R}^2$ (Example 4.5).

The current framework allows to give general results for a fixed level $u \in \mathbb{R}$, for a large class of additive functionals, including the perimeter or the total curvature, related to the Euler characteristic. For the latter, the main difficulty is to properly define it on a typical excursion of the shot noise field, as it is obtained by locally adding and removing sets from $\mathcal{B}$. The general result only involves the marginal distribution $\mu_{\mathcal{B}}(\cdot) := \mu(\mathbb{R} \times \cdot)$.

We call $\mathcal{B}'$ the class of excursion sets generated by shot noise fields of the form (4.16) where all but finitely many points of ζ in general position have been removed. Formally, given a measurable subclass $\mathcal{B}' \subset \mathcal{M}_d$, a function

$V : \mathcal{B}' \to \mathbb{R}$ such that $V(A)$ only depends on $A \cap \tilde{Q}_1$, and a function $|V| : \mathcal{B} \to (0, \infty)$, say that $(\mathcal{B}, \mathcal{B}', V, |V|)$ is *admissible* if for $A_1, \dots, A_q \in \mathcal{B}$, for a.a. $y_1, \dots, y_q \in \mathbb{R}^d$, any set A obtained by sequentially removing, adding or intersecting the $A_i + y_i, i = 1, \dots, q$, belongs to $\mathcal{B}'$, and $|V(A)| \leqslant \sum_{i=1}^q |V|(A_i)$. We consider below the functionals, for $W \subset \mathbb{Z}^d, \zeta \in \mathscr{N}$,

$$F_W(\zeta) = \sum_{k \in W} V(\{f_{\zeta \cap \tilde{W}} \geqslant u\} - k), \quad F'_W(\zeta) = \sum_{k \in W} V(\{f_\zeta \geqslant u\} - k).$$

Theorem 4.3. Let $u \in \mathbb{R}$, $(\mathcal{B}, \mathcal{B}', V, |V|)$ be an admissible quadruple, let f be of the form (4.16), and let $\mathscr{W} = \{W_n; n \geqslant 1\}$ be a sequence of subsets of $\mathbb{Z}^d$ satisfying (1.5). Assume that for some $\rho, p, c > 0$, $\mathbf{P}(|F_{Q_\beta}(\eta^\rho)| \geqslant c) \geqslant p$ for $\beta > \rho$, that $\int_{\mathcal{B}} |V|(A)^8 \mu_{\mathcal{B}}(dA) < \infty$, and that for some $\lambda > 28d, C > 0$,

$$\mu_{\mathcal{B}}(\{A \in \mathcal{B} : (x + A) \cap \tilde{Q}_1 \neq \emptyset\}) \leqslant C(1 + \|x\|)^{-\lambda}, x \in \mathbb{R}^d. \tag{4.17}$$

Then the conclusions of Theorems 1.1 and 1.2 hold: F_W and F'_W have variance of volume order and undergo a CLT.

Remark that nothing prevents the typical grain of $\mathcal{B}$ to be unbounded with positive $\mu_{\mathcal{B}}$-probability.

Proof. In this proof, $\mathscr{N}_0$ is chosen to be the class of ζ such that for any bounded set D, $\zeta[D] := \{(y, (L, A)) \in \zeta : (y + A) \cap D \neq \emptyset\}$ is finite. Assumption (4.17) implies that $\eta \in \mathscr{N}_0$ a.s. Let the notation of (1.8) prevail. Let $r \geqslant 0$. Introduce the independent variables

$$S_r^- = \sum_{(y,(L,A)) \in (\eta' \cap B_r)[\tilde{Q}_1]} |V|(A), \quad S_r^+ = \sum_{(y,(L,A)) \in (\eta' \setminus B_r)[\tilde{Q}_1]} |V|(A).$$

We have a.s.

$$\begin{aligned} \left|F_0(\eta' \cap B) - F_0(\eta' \cap B \cap B_r)\right| &= |V(\{f_{\eta' \cap B \cap B_r} \geqslant u\}) - V(\{f_{\eta' \cap B} \geqslant u\})| \\ &\leqslant \mathbf{1}_{\{S_r^+ \neq 0\}} 2(S_r^- + S_r^+) \leqslant 2\mathbf{1}_{\{S_r^+ \neq 0\}} S_r^- + 2S_r^+. \end{aligned} \tag{4.18}$$

Define $\psi(y, (L, A)) = \mathbf{1}_{\{(y+A) \cap \tilde{Q}_1 \neq \emptyset\}} |V|(A)$. Let (L_0, A_0) be a random variable with law μ. We have by Cauchy-Schwarz, for $y \in \mathbb{R}^d$,

$$\mathbf{E}(\psi(y, (L_0, A_0)))^4 \leqslant \sqrt{\mathbf{E}|V|(A_0)^8} \sqrt{\mathbf{P}((y + A_0) \cap \tilde{Q}_1 \neq \emptyset)} \leqslant C(1 + \|y\|)^{-\lambda/2}.$$

Hence Lemma 5.1 yields $\sup_{r>0} \mathbf{E}(S_r^-)^4 \leqslant \mathbf{E}(S_\infty^-)^4 < \infty$. The same method yields $(\mathbf{E}(S_r^+)^4)^{1/4} \leqslant C(1 + r)^{-\lambda/8+d}$. The same method again but this time with $\psi(y, (L, A)) = \mathbf{1}_{\{(y+A) \cap \tilde{Q}_1 = \emptyset\}}$ yields $\mathbf{P}(S_r^+ \neq 0) \leqslant C(1 + r)^{-\lambda+d}$. Taking the fourth moment and plugging these estimates back in (4.18) yields that (1.8) and (1.10) hold.

Let us show that Assumption 1.1 holds. For $\beta > \rho$, $\mathbf{P}(|F_{Q_\beta}(\eta^\rho)| \geqslant c) > p$. If $\eta_\gamma[Q_\beta] = \emptyset$, $F_{Q_\beta}(\eta^\rho \cup \eta_\gamma) = F_{Q_\beta}(\eta^\rho)$ and if $\eta^\rho[Q_\beta^c] = \emptyset$, $F_{Q_\delta \setminus Q_\beta}(\eta^\rho \cup \eta_\gamma) = F_{Q_\delta \setminus Q_\beta}(\eta_\gamma)$. Hence, with $U_{\delta,\gamma} := F_{Q_\delta}(\eta^\rho) - F_{Q_\delta}(\eta^\rho \cup \eta_\gamma) - F_{Q_\beta}(\eta^\rho)$,

$$\begin{aligned}\mathbf{P}(|F_{Q_\delta}(\eta^\rho) - F_{Q_\delta}(\eta^\rho \cup \eta_\gamma)| > c/2) \geqslant & \mathbf{P}(|F_{Q_\beta}(\eta^\rho)| > c) - \mathbf{P}(|U_{\delta,\gamma}| > c/2) \\ \geqslant & p - \mathbf{P}(\eta^\rho[Q_\beta^c] \neq \emptyset) - \mathbf{P}(\eta_\gamma[Q_\beta] \neq \emptyset).\end{aligned}$$

Since at fixed ρ, $\mathbf{1}_{\{\eta^\rho[Q_\beta^c]\neq\emptyset\}} \to 0$ a.s. as $\beta \to \infty$, fix β such that $\mathbf{P}(\eta^\rho[Q_\beta^c] \neq \emptyset) < p/4$. Then for γ sufficiently large and any $\delta > \gamma$, $\mathbf{P}(\eta_\gamma[Q_\beta] \neq \emptyset) < p/4$, hence Assumption 1.1 is satisfied.

□

Example 4.3 (Volume)**.** The simplest example is the class $\mathcal{B} = \mathcal{M}_d$ of measurable subsets of $\mathbb{R}^d$, endowed with Lebesgue measure $V(A) = \ell^d(A \cap \tilde{Q}_1)$. We have $F_W(\eta) := \ell^d(\{f_{\eta\cap\tilde{W}} \geqslant u\} \cap \tilde{W})$ a.s.. This example has been treated in a different framework at Section 4.1.

Example 4.4 (Perimeter)**.** Let $\mathcal{B}$ be the class of $A \in \mathcal{M}_d$ such that $\mathcal{H}^{d-1}(\partial A) < \infty$. Define $V(A) = \mathcal{H}^{d-1}(\partial A \cap \tilde{Q}_1)$, we prove below that $F_W(\eta) = \mathcal{H}^{d-1}(\{f_{\eta\cap\tilde{W}} \geqslant u\} \cap \tilde{W})$ a.s.. Assume for the moment condition that $\int_{\mathcal{B}} \mathcal{H}^{d-1}(\partial A)^8 \mu_{\mathcal{B}}(dA) < \infty$.

Example 4.5 (Total curvature)**.** Let $d = 2$, $\mathcal{B}$ be the class of non-trivial closed discs of $\mathbb{R}^2$. A set $A \subset \mathbb{R}^2$ is an *elementary set* in the terminology of Biermé & Desolneux [7] if ∂A can be decomposed as a finite union of $\mathcal{C}^2$ open curves $C_j, j = 1, \ldots, p$ with respective constant curvatures $\kappa_j > 0$, separated by corners $x_i \in \partial A, i = 1, \ldots, q$, (with $0 \leqslant q \leqslant p$) with angle $\alpha(x_i, A) \in (-\pi, \pi)$. The *total curvature* of A within some open set U is defined by

$$TC(A; U) := \sum_{j=1}^p \kappa_j \mathcal{H}^1(C_j \cap U) + \sum_{i=1}^q \mathbf{1}_{\{x_i \in U\}} \alpha(x_i, A).$$

Therefore we define $V(A) = TC(A; \tilde{Q}_1)$. Via the Gauss-Bonnet theorem, for $W \subset \mathbb{Z}^d$, $TC(A; \mathrm{int}(\tilde{W}))$ is strongly related to the Euler characteristic of $A \cap \tilde{W}$, in the sense that they coincide if $A \subset \mathrm{int}(\tilde{W})$, and otherwise they only differ by boundary terms, see [7]. We will see that $F_W(\eta) = TC(\{f_{\eta\cap\tilde{W}} \geqslant u\}; \mathrm{int}(\tilde{W}))$ a.s.. Assume also that the typical radius has a finite moment of order $8d$.

Proposition 4.1. In the three previous examples, assume that (4.17) holds, and that $\mathbf{P}(f_\eta(0) \geqslant cu) \notin \{0, 1\}$ for some $c > 0$. Then the functionals F_W, F'_W satisfy the conclusions of theorems 1.1,1.2, in particular, they have variance of volume order and undergo a central limit theorem as $|\partial_{\mathbb{Z}^d} W|/|W| \to 0$.

Proof. All proofs rely on defining an admissible quadruple that satisfies the assumptions of Theorem 4.3, and show that the variance assumption holds. We only treat the case $u \geqslant 0$, the case $u \leqslant 0$ can be treated similarly. Let $\Gamma_k = k + \tilde{Q}_1, k \in \mathbb{Z}^d$.

(Volume) Defining $\mathcal{B}' = \mathcal{M}_d, |V|(A) = \ell^d(A)$ yields an admissible quadruple $(\mathcal{B}, \mathcal{B}', V, |V|)$. In the case $u > 0$, the fact that $\mathbf{1}_{\{f_\eta(0)>0\}}$ is not trivial yields that (4.6) holds, and hence using Lemma 4.1, $\mathbf{P}(F_{Q_\beta}(\eta^\rho) \geqslant c) \geqslant \mathbf{P}(F_{Q_\rho}(\eta^\rho) \geqslant c) =: p > 0$ holds for some $\rho, c > 0$, and for $\beta > \rho$. The case $u = 0$ can be treated directly and is left to the reader.

(Perimeter) Let $\mathcal{B}'$ be the class of $A \in \mathcal{B}$ such that $\mathcal{H}^{d-1}(\partial A \cap \partial \Gamma_k) = 0$ for $k \in \mathbb{Z}^d$. For $A \in \mathcal{B}$, for a.a. $y \in \mathbb{R}^d$, $\mathcal{H}^{d-1}(\partial(A+y) \cap \partial\Gamma_k) = 0$. Hence for $A_1, \dots, A_q \in \mathcal{B}$, for a.a. $y_1, \dots, y_q \in \mathbb{R}^d$, any set A obtained by sequentially adding, intersecting or removing the $A_i + y_i$ is in $\mathcal{B}'$, using $\partial A \subset \cup_{i=1}^n (\partial A_i + y_i)$. Defining $|V|(A) := \mathcal{H}^{d-1}(\partial A)$ yields an admissible quadruple $(\mathcal{B}, \mathcal{B}', V, |V|)$. The justification that $\mathrm{Var}(F_{Q_\beta}(\eta^\rho)) > 0$ holds is the same as for the volume (above), because a set with positive volume has positive boundary measure.

(Total curvature) Let $\mathcal{B}'$ be the class of sets obtained from finite unions, intersections and removals of discs $A_1, \dots, A_q$ such that for $i \neq j$, A_i and A_j are not tangent and $\partial A_i \cap \partial A_j \cap \partial \Gamma_k = \emptyset$ for $k \in \mathbb{Z}^d$. Every $A \in \mathcal{B}'$ is elementary, and defining $|V| \equiv 1$ yields that $(\mathcal{B}, \mathcal{B}', V, |V|)$ is an admissible quadruple. Let $X_i = (Y_i, (L_i, D_i)), i \geqslant 1$, iid marked couples of discs with iid uniform centers Y_i in $B(0,1)$. Let $k \in \mathbb{N}$ be such that the event $\Gamma = (\sum_{i=1}^k L_i \geqslant u, \sum_{i=1}^{k-1} L_i < u)$ has positive probability. Conditionally on Γ, $\{f_{\{X_1,\dots,X_k\}} \geqslant u\} = \cap_{i=1}^k (Y_i + D_i)$. Since the D_i have positive radii, the probability that the $Y_i, i = 1, ..., k$ are sufficiently close to 0 such that this set is non-empty is also positive. In this case it is the intersection of discs, hence its total curvature is equal to 1, and $\mathbf{P}(F_{Q_\beta}(\eta^\rho) \geqslant 1) \geqslant p > 0$ is satisfied for some $\rho > 0$ and $\beta > \rho$.

□

With a similar route, the previous example can likely be generalised to more general classes of sets $\mathcal{B}$ in higher dimensions, such as the polyconvex ring, provided one can estimate properly the curvature or the Euler characteristic on sets from $\mathcal{B}'$.

5. Proofs

Recall that κ denotes a constant which depends on d, α, a_-, a_+ and whose value might change from line to line. The following lemma is useful several times in the paper.

Lemma 5.1. Let $\alpha > d, C_0 \geqslant 0$, $M_i, 0 \leqslant i \leqslant 4$ be independent marks with law μ. Let $r > 0, \psi : \overline{\mathbb{R}^d} \times \mathscr{N} \to \mathbb{R}_+$ be a measurable function such that for ℓ^d-a.e. $x_i \in \mathbb{R}^d, 0 \leqslant i \leqslant 4$, and $\zeta \subset \{(x_i, M_i), i = 1, \dots, 4\}$, $\left(\mathbf{E}\psi((x_0, M_0), \eta \cup \zeta)^4\right)^{1/4} \leqslant C_0(1 + \|x_0\|)^{-\alpha-d}$. Then

$$\left(\mathbf{E}\left|\sum_{\mathbf{x} \in \eta \setminus B_r} \psi(\mathbf{x}; \eta)\right|^4\right)^{1/4} \leqslant C_0 \kappa (1+r)^{-\alpha}.$$

Proof. Let $\eta_r = \eta \setminus B_r$. Let $\mathbf{x}_i = (x_i, M_i)$. Let $\mathcal{P}_4$ be the family of ordered tuples of natural integers which sum is 4. The multi-variate Mecke formula yields

$$
\begin{aligned}
&\mathbf{E}\left[\sum_{\mathbf{x}\in\eta_r}\psi(\mathbf{x};\eta)\right]^4 \leqslant \kappa \sum_{(m_1,\dots,m_q)\in\mathcal{P}_4}\mathbf{E}\left[\sum_{(\mathbf{x}_1,\dots,\mathbf{x}_q)\in\eta_r^q}\psi(\mathbf{x}_1;\eta)^{m_1}...\psi(\mathbf{x}_q;\eta)^{m_q}\right]\\
\leqslant&\kappa \sum_{(m_1,\dots,m_q)\in\mathcal{P}_4}\int_{(\overline{B_r^c})^q}\mathbf{E}\left[\prod_{l=1}^q\psi(\mathbf{x}_l,\eta\cup\{\mathbf{x}_1,\dots,\mathbf{x}_q\})^{m_l}\right]d\mathbf{x}_1\dots d\mathbf{x}_q\\
\leqslant&\kappa \sum_{(m_1,\dots,m_q)\in\mathcal{P}_4}\int_{(\overline{B_r^c})^q}\prod_{l=1}^q(\mathbf{E}\psi(\mathbf{x}_l,\eta\cup\{\mathbf{x}_1,\dots,\mathbf{x}_q\})^4)^{m_l/4}d\mathbf{x}_1\dots d\mathbf{x}_q\\
\leqslant&\kappa \sum_{(m_1,\dots,m_q)\in\mathcal{P}_4}\prod_{l=1}^q\kappa\int_{B_r^c}C_0^{m_l}(1+\|x_l\|)^{-m_l(\alpha+d)}dx_l\\
\leqslant&\kappa \sum_{(m_1,\dots,m_q)\in\mathcal{P}_4}C_0^4\prod_{l=1}^q\kappa\int_{a_-r}^\infty(1+t)^{-m_l(\alpha+d)}t^{d-1}dt\\
\leqslant&\kappa C_0^4 \sum_{(m_1,\dots,m_q)\in\mathcal{P}_4}(1+r)^{-4(\alpha+d)+qd}\leqslant\kappa C_0^4(1+r)^{-4\alpha}.
\end{aligned}
$$

□

5.1. Proof of Theorem 2.1

We prove (2.3) under Assumption (2.1) (i.e. in case **(i)**). Remark first that (2.1) trivially holds also for $B\in\mathcal{B}_W^s\setminus\mathcal{B}_W^r, s<r$. Also, if (2.1) is satisfied, with $B=\mathbb{R}^d$, (2.2) is also satisfied. Assume without loss of generality $F_0(\emptyset)=0$, then (2.1) with $r=0$ yields

$$m_2:=\sup_{k\in W}\mathbf{E}|F_k^W(\eta)|^2=\sup_{k\in W}\mathbf{E}|F_0((\eta\cap\tilde{W})-k)-F_0(((\eta\cap\tilde{W})-k)\cap B_0)|^2\leqslant\kappa C_0^2<\infty.$$

The following inequality is useful several times in the proof: given some square-integrable random variables $Y_i,Z_i,i=1,2$ on Ω, and a σ-algebra $\mathcal{Z}\subset\mathscr{A}$,

$$
\begin{aligned}
&\mathbf{E}\left|\mathrm{Cov}(Y_1,Y_2|\mathcal{Z})-\mathrm{Cov}(Z_1,Z_2|\mathcal{Z})\right|\\
\leqslant&\mathbf{E}\left(\sqrt{2\mathbf{E}(Y_1^2|\mathcal{Z})}\sqrt{2\mathbf{E}((Z_2-Y_2)^2|\mathcal{Z})}+\sqrt{2\mathbf{E}(Z_2^2|\mathcal{Z})}\sqrt{2\mathbf{E}((Z_1-Y_1)^2|\mathcal{Z})}\right)\\
\leqslant&2\left(\sqrt{\mathbf{E}Y_1^2}\sqrt{\mathbf{E}(Z_2-Y_2)^2}+\sqrt{\mathbf{E}Z_2^2}\sqrt{\mathbf{E}(Z_1-Y_1)^2}\right). \qquad (5.1)
\end{aligned}
$$

Let $B_r(k)=k+B_r$ for $k\in\mathbb{Z}^d, r\geqslant 0$. Let $k,j\in W, r=\|k-j\|/(3a_+)$, η',η'' independent copies of η, and

$$\eta_k=(\eta\cap B_r(k))\cup(\eta'\cap B_r(k)^c),\quad \eta_j=(\eta\cap B_r(j))\cup(\eta''\cap B_r(j)^c),$$

which are processes distributed as η, independent since $B_r(k)\cap B_r(j)=\emptyset$. Since $\eta\cap B_r(k)=\eta_k\cap B_r(k)$, (2.1) yields

$$\begin{aligned}F_k^W(\eta)-F_k^W(\eta_k)=&F_k^W(\eta)-F_k^W(\eta\cap B_r(k))+F_k^W(\eta_k\cap B_r(k))-F_k^W(\eta_k)\\ \mathbf{E}\left|F_k^W(\eta)-F_k^W(\eta_k)\right|^2\leqslant&2\Big(\mathbf{E}\left|F_0((\eta-k)\cap(\tilde W-k))-F_0((\eta-k)\cap(\tilde W-k)\cap B_r)\right|^2\\ &+\mathbf{E}\left|F_0((\eta_k-k)\cap(\tilde W-k)\cap B_r)-F_0((\eta_k-k)\cap(\tilde W-k))\right|^2\Big)\\ \leqslant&\kappa C_0^2(1+r)^{-2\alpha},\end{aligned}$$

because $\eta_k-k\overset{(d)}{=}\eta-k\overset{(d)}{=}\eta$. A similar bound holds for F_j^W. Then, (5.1) yields

$$\begin{aligned}&\left|\mathrm{Cov}(F_k^W(\eta),F_j^W(\eta))-\underbrace{\mathrm{Cov}(F_k^W(\eta_k),F_j^W(\eta_j))}_{=0}\right|\leqslant\kappa\sqrt{\mathbf{E}F_j^W(\eta)^2}\sqrt{\mathbf{E}\left|F_k^W(\eta)-F_k^W(\eta_k)\right|^2}\\ &\qquad\qquad+\kappa\sqrt{\mathbf{E}(F_k^W(\eta_k))^2}\sqrt{\mathbf{E}\left|F_j^W(\eta)-F_j^W(\eta_j)\right|^2}\\ &\qquad\leqslant\kappa\sqrt{m_2}\sqrt{C_0^2(1+r)^{-2\alpha}}\leqslant\kappa C_0^2(1+\|k-j\|)^{-\alpha}.\end{aligned}\tag{5.2}$$

Hence (2.3) is proved in case **(i)**. If $G_k^W=F_k$ and (2.2) is assumed instead of (2.1) (case **(i')**), replacing W by $\mathbb{Z}^d$ in the computation above yields the same bound for $\mathrm{Cov}(F_k,F_j)$. The finiteness of σ_0 follows from $\alpha>d$.

Let us now assume $|W|<\infty$ and show (2.4). Let $k\in W, r=d(k,\tilde W^c)/a_+$, so that $B_r\cap(\tilde W-k)=B_r$. We have if (2.1) holds

$$\begin{aligned}F_k^W-F_k&=F_0((\eta-k)\cap(\tilde W-k))-F_0((\eta-k)\cap(\tilde W-k)\cap B_r)+F_0((\eta-k)\cap B_r)-F_0(\eta-k)\\ \mathbf{E}|F_k^W-F_k|^2&\leqslant\kappa C_0^2(1+r)^{-2\alpha}\leqslant\kappa C_0^2(1+d(k,\tilde W^c))^{-2\alpha}.\end{aligned}$$

We hence have by (5.1), for $k,j\in W$, recalling also (5.2),

$$\begin{aligned}\left|\mathrm{Cov}(F_k^W,F_j^W)-\mathrm{Cov}(F_k,F_j)\right|\leqslant&\kappa C_0^2(1+\min(d(k,\tilde W^c),d(j,\tilde W^c)))^{-\alpha}\\ \leqslant&\kappa C_0^2(1+\max(\|k-j\|,\min(d(k,\tilde W^c),d(j,\tilde W^c))))^{-\alpha}.\end{aligned}\tag{5.3}$$

Denote by $[x]$ the integer part of $x\in\mathbb{R}$. Let $d_W\in\mathbb{N}\setminus\{0\}$, $W_m=\{k\in W:[d(k,\tilde W^c)]=m\}$ for $m\in\mathbb{N}, W_\partial=\{k\in W:[d(k,\tilde W^c)]\leqslant d_W\}=$

$\cup_{m=0}^{d_W} W_m, W_{int} = W \setminus W_\partial$. We have, using (2.3) and (5.3),

$$\begin{aligned}|\mathrm{Var}(F_W) - \sigma_0^2|W|| &= \left| \sum_{k\in W, j\in W} \mathrm{Cov}(F_k^W, F_j^W) - \sum_{k\in W, j\in \mathbb{Z}^d} \mathrm{Cov}(F_k, F_j) \right| \\ &\leqslant \sum_{k\in W, j\notin W} |\mathrm{Cov}(F_k, F_j)| + 2 \sum_{k,j\in W: d(k,\tilde{W}^c)\leqslant d(j,\tilde{W}^c)} \left|\mathrm{Cov}(F_k^W, F_j^W) - \mathrm{Cov}(F_k, F_j)\right| \\ &\leqslant \sum_{m=0}^{\infty} \sum_{k\in W_m} \left[\sum_{j\in W^c} \kappa C_0^2 (1 + \|k-j\|)^{-\alpha} + 2 \sum_{j\in W} \kappa C_0^2 (1 + \max(\|k-j\|, m))^{-\alpha} \right] \\ &\leqslant \kappa C_0^2 \sum_{m=0}^{\infty} \sum_{k\in W_m} \left(3 \sum_{j\in B(k,m)^c} (1 + \|k-j\|)^{-\alpha} + 2 \sum_{j\in B(k,m)} (1+m)^{-\alpha} \right) \\ &\leqslant \kappa C_0^2 \sum_{m=0}^{\infty} \sum_{k\in W_m} \left(3\kappa (1+m)^{-\alpha+d} + 2\kappa m^d (1+m)^{-\alpha} \right) \\ &\leqslant \kappa C_0^2 \left(|W_\partial| + d_W^{-\alpha+d} |W_{int}| \right)\end{aligned}$$

hence $\left|\frac{\mathrm{Var}(F_W)}{|W|} - \sigma_0^2\right| \leqslant \kappa C_0^2 \left(\frac{d_W^d |\partial_{\mathbb{Z}^d} W|}{|W|} + d_W^{-\alpha+d} \right)$. Equation (2.4) follows by taking $d_W = [(|W|/|\partial_{\mathbb{Z}^d} W|)^{\frac{1}{\alpha}}]$. The same computation where F_k^W is replaced by F_k (hence with no second term on the second line), treats the case **(i')**, without requiring (2.1).

Let us now prove that under the current assumptions, Assumption 1.1 implies $\sigma_0 > 0$. Recall the notation $\eta_a = \eta \cap \tilde{Q}_a^c, \eta_a^b = \eta_a \cap \tilde{Q}_b, a, b > 0$. Let $\delta > 0$, and decompose W in the finite disjoint union of subparts with sidelength δ: $W = \cup_{k\in\mathbb{Z}^d} W^{(k)}$ where $W^{(k)} = W \cap (\delta k + Q_\delta)$. Decompose accordingly $F_W = \sum_{k\in\mathbb{Z}^d} F^{(k)}$ where $F^{(k)} = \sum_{j\in W^{(k)}} F_j$. Let $\gamma < \delta$, and condition by the points of η γ-close to the boundary of a $W^{(k)}$: $\eta_\gamma^* = \eta \cap \tilde{Q}_\gamma^*$ where $\tilde{Q}_\gamma^* = \mathbb{R}^d \setminus (\cup_{k\in\mathbb{Z}^d} (\delta k + \tilde{Q}_\gamma))$. Denote by $\mathbf{E}_{\eta_\gamma^*}, \mathrm{Var}_{\eta_\gamma^*}$ and $\mathrm{Cov}_{\eta_\gamma^*}$ the conditional expectation, variance, and covariance with respect to η_γ^*. We have

$$\mathrm{Var}(F_W) \geqslant \mathbf{E}[\mathrm{Var}_{\eta_\gamma^*}(F_W)] \geqslant \sum_{k\in\mathbb{Z}^d} \mathbf{E}[\mathrm{Var}_{\eta_\gamma^*}(F^{(k)})] - \sum_{k\neq j} \mathbf{E}|\mathrm{Cov}_{\eta_\gamma^*}(F^{(k)}, F^{(j)})|. \tag{5.4}$$

We claim (and prove later) that for $k \in \mathbb{Z}^d$

$$\mathbf{E} \sum_{j\neq k} |\mathrm{Cov}_{\eta_\gamma^*}(F^{(k)}, F^{(j)})| \leqslant C' \delta^{2d} (\delta - \gamma)^{-\alpha}. \tag{5.5}$$

For the first term of (5.4), among the $k \in \mathbb{Z}^d$ such that $W^{(k)} \neq \emptyset$, call $W^{\delta,int}$ those such that $W^{(k)} - k\delta = Q_\delta$, and $W^{\delta,\partial}$ the others. We have, using also (5.1),

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} \mathbf{E}[\mathrm{Var}_{\eta_\gamma^*}(F^{(k)})] &\geqslant \sum_{k\in W^{\delta,int}} \mathbf{E}[\mathrm{Var}_{\eta_\gamma^*}(F^{(k)})] - 2 \sum_{k\in W^{\delta,\partial}} \mathbf{E}[(F^{(k)})^2] \\ &\geqslant |W^{\delta,int}| \mathbf{E}[\mathrm{Var}_{\eta_\gamma^*}(F_{Q_\delta})] - 2|W^{\delta,\partial}| \delta^d m_2 \end{aligned} \tag{5.6}$$

because by stationarity, for $k \in W^{\delta,int}$, $\mathbf{E}[\mathrm{Var}_{\eta_\gamma^*}(F^{(k)})] = \mathbf{E}[\mathrm{Var}_{\eta_\gamma^*}(F_{Q_\delta})]$. Recall that any real random variable U satisfies $\mathrm{Var}(U) = \inf_{z\in\mathbb{R}} \mathbf{E}(U-z)^2$. Since $\tilde{Q}_\gamma^* \subset Q_\gamma^c$, $\eta_\gamma^* \in \sigma(\eta_\gamma)$, hence for $\rho \in (0,\gamma)$,

$$\begin{aligned}\mathbf{E}[\mathrm{Var}_{\eta_\gamma^*}(F_{Q_\delta})] \geqslant &\mathbf{E}[\mathrm{Var}_{\eta_\gamma}(F_{Q_\delta})] = \mathbf{E}\left[\inf_{z\in\mathbb{R}} \mathbf{E}_{\eta_\gamma}(F_{Q_\delta}-z)^2\right]\\ \geqslant &\mathbf{E}\left[\inf_{z\in\mathbb{R}} \mathbf{E}_{\eta_\gamma}[1_{\eta_\rho^\gamma=\emptyset}(F_{Q_\delta}-z)^2]\right] = \mathbf{P}(\eta_\rho^\gamma=\emptyset)\mathbf{E}\left[\inf_{z\in\mathbb{R}} \mathbf{E}_{\eta_\gamma}[(F_{Q_\delta}(\eta^\rho\cup\eta_\gamma)-z)^2]\right]\\ = &\mathbf{P}(\eta_\rho^\gamma=\emptyset)\mathbf{E}\left[\mathrm{Var}_{\eta_\gamma}[F_{Q_\delta}(\eta^\rho\cup\eta_\gamma)]\right]\end{aligned}$$

where the second equality is true because $\eta_\rho^\gamma, \eta^\rho, \eta_\gamma$ are independent and $1_{\{\eta_\rho^\gamma=\emptyset\}}F_{Q_\delta} = \mathbf{1}_{\{\eta_\rho^\gamma=\emptyset\}}F_{Q_\delta}(\eta^\rho\cup\eta_\gamma)$. Up to increasing δ, let $0<\rho<\gamma$ be like in Assumption 1.1, which yields $v_\gamma>0$ such that for arbitrarily large $\delta>\gamma$, $\mathbf{E}[\mathrm{Var}_{\eta_\gamma}(F_{Q_\delta}(\eta^\rho\cup\eta_\gamma))] \geqslant v_\gamma$. By (5.5), (5.4) and (5.6) for $\delta>\gamma$ sufficiently large

$$\begin{aligned}\mathrm{Var}(F_W) \geqslant &|W^{\delta,int}|\mathbf{P}(\eta_\rho^\gamma=\emptyset)v_\gamma - 2|W^{\delta,\partial}|\delta^d m_2 - (|W^{\delta,int}|+|W^{\delta,\partial}|)C'\delta^{2d}(\delta-\gamma)^{-\alpha}\\ \geqslant &|W^{\delta,int}|(\mathbf{P}(\eta_\rho^\gamma=\emptyset)v_\gamma - C'\delta^{2d}(\delta-\gamma)^{-\alpha}) - |W^{\delta,\partial}|(2\delta^d m_2 + C'\delta^{2d}(\delta-\gamma)^{-\alpha}).\end{aligned}$$

Since $\alpha>2d$, given any γ, one can choose $\delta =: \delta_\gamma$ such that $C'\delta^{2d}(\delta-\gamma)^{-\alpha} < \varepsilon_\gamma := \mathbf{P}(\eta_\rho^\gamma=\emptyset)v_\gamma/2$. Hence $\mathrm{Var}(F_W) \geqslant |W^{\delta,int}|\varepsilon_\gamma - |W^{\delta,\partial}|(2\delta^d m_2+\varepsilon_\gamma)$. To conclude, let a sequence $\{W_n; n\geqslant 1\}$ be such that $\lim_n |\partial_{\mathbb{Z}^d}W_n|/|W_n| = 0$. Since $|\partial_{\mathbb{Z}^d}W_n|/|W_n| \geqslant |W_n^{\delta,\partial}|/(\delta^d(|W_n^{\delta,int}|+|W_n^{\delta,\partial}|))$, (2.4) yields

$$\sigma_0 = \liminf_n |W_n|^{-1}\mathrm{Var}(F_{W_n}) \geqslant \liminf_n (\delta^d|W_n^{\delta,int}|)^{-1}\mathrm{Var}(F_{W_n}) > 0.$$

Let us finally prove (5.5). Let $k\neq j\in\mathbb{Z}^d, l\in W^{(k)}, m\in W^{(j)}, r = \|j-k\|(\delta-\gamma)/(2a_+)$. Let η',η'' independent copies of η, and define

$$\eta_l = (\eta\cap B_r(l))\cup(\eta'\cap B_r(l)^c), \quad \eta_m = (\eta\cap B_r(m))\cup(\eta''\cap B_r(m)^c).$$

Since $B_r(l)\cap B_r(m) \subset \tilde{Q}_\gamma^*$, η_l and η_m are independent conditionally to η_γ^*, and we have by (5.1), with a computation similar to (5.2), $\mathbf{E}|\mathrm{Cov}_{\eta_\gamma^*}(F_l,F_m) - \mathrm{Cov}_{\eta_\gamma^*}(F_l(\eta_l),F_m(\eta_m))| \leqslant \kappa C_0^2(1+r)^{-\alpha}$. It follows that

$$\mathbf{E}\left|\mathrm{Cov}_{\eta_\gamma^*}(F^{(k)},F^{(l)})\right| \leqslant \mathbf{E}\sum_{l\in W^{(k)},m\in W^{(j)}} |\mathrm{Cov}_{\eta_\gamma^*}(F_l,F_m)| \leqslant \kappa C_0^2\delta^{2d}(1+r)^{-\alpha}$$

and, for some C' not depending on W, for $k\in W$,

$$\mathbf{E}\sum_{j\in\mathbb{Z}^d\setminus\{k\}} \left|\mathrm{Cov}_{\eta_\gamma^*}(F^{(k)},F^{(j)})\right| \leqslant \kappa C_0^2\delta^{2d}\sum_{p=1}^\infty p^{d-1}(\|p\|(\delta-\gamma))^{-\alpha} \leqslant C'\delta^{2d}(\delta-\gamma)^{-\alpha}.$$

This concludes the proof of (5.5) and hence of $\sigma_0>0$.

It remains to prove (2.5). Assume that (2.2) holds with $\alpha>2d$. The proof when instead (2.1) holds is exactly the same with F_k^W instead of F_k, and it is omitted. For $k \in \mathbb{Z}^d$, let $\bar{F}_k = F_k(\eta) - \mathbf{E}F_k(\eta)$. We have $\mathbf{E}(F_W - \mathbf{E}F_W)^4 = \sum_{i,j,k,l \in W} \mathbf{E}\bar{F}_i\bar{F}_j\bar{F}_k\bar{F}_l$. Let $I = \{i,j,k,l\} \subset W$. Assume that i is I*-isolated*, i.e. $\delta := [d(i, I \setminus \{i\})] = \max_{m \in I}[d(m, I \setminus \{m\})]$ (let this quantity be 0 if $i = j = k = l$). Let $\eta_m, m \in I$, be independent copies of η, and $H_m = B_{\delta/2a_+}(m)$,$\eta'_m = (\eta \cap H_m) \cup (\eta_m \cap H_m^c)$. Note that η'_m is distributed as η, and that for $m \in I \setminus \{i\}$, $H_i \cap H_m = \emptyset$, hence η'_i is independant from $\{\eta'_j, \eta'_k, \eta'_l\}$. Introduce $\bar{F}'_m = F_m(\eta'_m) - \mathbf{E}F_m$, $\bar{F} = \bar{F}_j\bar{F}_k\bar{F}_l$, $\bar{F}' = \bar{F}'_j\bar{F}'_k\bar{F}'_l$, independent of $\bar{F}'_i$. We have, using Holder's inequality,

$$
\begin{aligned}
\Big|\mathbf{E}\bar{F}_i\bar{F} - \underbrace{\mathbf{E}\bar{F}'_i\bar{F}'}_{=0}\Big| \leqslant & \mathbf{E}\Big[|(\bar{F}_i - \bar{F}'_i)\bar{F}_j\bar{F}_k\bar{F}_l| + |\bar{F}'_i(\bar{F}_j - \bar{F}'_j)\bar{F}_k\bar{F}_l| \\
& + |\bar{F}'_i\bar{F}'_j(\bar{F}_k - \bar{F}'_k)\bar{F}_l| + |\bar{F}'_i\bar{F}'_j\bar{F}'_k(\bar{F}_l - \bar{F}'_l)|\Big] \\
\leqslant & 4\sum_{m \in I}(\mathbf{E}\bar{F}_0^4)^{3/4}(\mathbf{E}|\bar{F}_m - \bar{F}'_m|^4)^{1/4} \\
\leqslant & \kappa C_0(\mathbf{E}\bar{F}_0^4)^{3/4}(1+\delta)^{-\alpha}
\end{aligned}
$$

by (2.2) (or (2.1) for the proof with the F_k^W). Notice that one point among $\{j,k,l\}$ is between distance δ and $\delta + 1$ from i, call it a, and there are at most $\kappa\delta^{d-1}$ possible values for a, given i. If there are two points remaining in $\{j,k,l\} \setminus a$, they are at mutual distance at most 3δ. We have

$$
\begin{aligned}
\mathbf{E}(F_W - \mathbf{E}\bar{F}_W)^4 \leqslant & 4\sum_{i,j,k,l \in W} \mathbf{1}_{\{i \text{ isolated}\}}\kappa C_0(\mathbf{E}\bar{F}_0^4)^{3/4}(1 + [d(i, \{j,k,l\})])^{-\alpha} \\
\leqslant & \kappa C_0(\mathbf{E}\bar{F}_0^4)^{3/4}\sum_{\delta=0}^{\infty}(1+\delta)^{-\alpha}\sum_{i,j,k,l \in W}\mathbf{1}_{\{i \text{ isolated and } [d(i,\{j,k,l\})] = \delta\}} \\
\leqslant & \kappa C_0(\mathbf{E}\bar{F}_0^4)^{3/4}\sum_{\delta=0}^{\infty}|W|^2(1+\delta)^{-\alpha}\kappa\delta^{d-1}(3\delta)^d \leqslant \kappa C_0(\mathbf{E}\bar{F}_0^4)^{3/4}|W|^2
\end{aligned}
$$

where $\kappa < \infty$ because $\alpha > 2d$.

5.2. Proof of Theorem 3.1

W is fixed. For simplicity, in all the proof we use the notation $G = G_W, \tilde{G} = \tilde{G}_W$. If (3.2) is satisfied, put $G_k = F_k$ and $A = \mathbb{R}^d$. If instead (3.1) is satisfied, put $G_k = F_k^W$ and $A = \tilde{W}$. Assume without loss of generality that F_0 is centered. Theorem 1.2 from [18] gives general Berry-Esseen bounds on the Poisson functional $\tilde{G}$: provided $\int_A \mathbf{E}(D_{\mathbf{x}}G)^2 d\mathbf{x} < \infty$ (implied here by Assumption (3.2) or (3.1) and $\alpha > d$), $d_{\mathscr{W}}(\tilde{G}, N) \leqslant \sum_{i=1}^3 \gamma_i$, $d_{\mathscr{K}}(\tilde{G}, N) \leqslant \sum_{i=1}^6 \gamma_i$, where the γ_i are quantities depending on the first and second-order Malliavin derivatives of $\tilde{G}$, whose values are recalled later. Let $x, y \in A, \mathbf{x} = (x, M), \mathbf{y} = (y, M')$. Call

$\eta^{\mathbf{x}} = \eta \cup \{\mathbf{x}\}, \eta^{\mathbf{y}} = \eta \cup \{\mathbf{y}\}$. We have, using Hölder's inequality at the last line,

$$|D_{\mathbf{x},\mathbf{y}}G(\eta)| \leqslant \sum_{k\in W} \min\left(|D_{\mathbf{x}}G_k(\eta)| + |D_{\mathbf{x}}G_k(\eta^{\mathbf{y}})|, |D_{\mathbf{y}}G_k(\eta)| + |D_{\mathbf{y}}G_k(\eta^{\mathbf{x}})|\right).$$

$$\begin{aligned}\mathbf{E}|D^2_{\mathbf{x},\mathbf{y}}G(\eta)|^4 \leqslant& \mathbf{E}\left|2\sum_{k\in W}\min\left(\sup_{\eta'\in\{\eta,\eta^{\mathbf{y}}\}}|D_{\mathbf{x}}G_k(\eta')|, \sup_{\eta'\in\{\eta,\eta^{\mathbf{x}}\}}|D_{\mathbf{y}}G_k(\eta')|\right)\right|^4\\ \leqslant& 2^4\sum_{k_1,\dots,k_4\in W}\mathbf{E}\prod_{i=1}^4\min\left(\sup_{\eta'\in\{\eta,\eta^{\mathbf{y}}\}}|D_{\mathbf{x}}G_{k_i}(\eta')|, \sup_{\eta'\in\{\eta,\eta^{\mathbf{x}}\}}|D_{\mathbf{y}}G_{k_i}(\eta')|\right)\\ \leqslant& 2^4\left(\sum_{k\in W}\left(\mathbf{E}\min\left(\sup_{\eta'\in\{\eta,\eta^{\mathbf{y}}\}}|D_{\mathbf{x}}G_k(\eta')|^4, \sup_{\eta'\in\{\eta,\eta^{\mathbf{x}}\}}|D_{\mathbf{y}}G_k(\eta')|^4\right)\right)^{1/4}\right)^4.\end{aligned}$$

Let $k \in W$. Note that, with $B = \tilde{W} - k$, for $x \in \tilde{W}, y \in \mathbb{R}^d$,

$$\begin{aligned}D_{\mathbf{x}}F_k^W(\eta') =& F_k^W(\eta' \cup \{\mathbf{x}\}) - F_k^W(\eta') = F_0((\eta' \cup \{\mathbf{x}\}) \cap \tilde{W} - k) - F_0(\eta' \cap \tilde{W} - k)\\ =& F_0(((\eta' - k) \cap B) \cup \{\mathbf{x} - k\}) - F_0((\eta' - k) \cap B) = D_{\mathbf{x}-k}F_0((\eta' - k) \cap B).\end{aligned}$$

Since $\eta - k \stackrel{(d)}{=} \eta$, applying either (3.2) or (3.1) with $y - k$ instead of y yields

$$\mathbf{E}|D^2_{\mathbf{x},\mathbf{y}}G(\eta)|^4 \leqslant \kappa C_0^4\left(\sum_{k\in W}\min((1 + \|x - k\|)^{-\alpha}, (1 + \|y - k\|)^{-\alpha})\right)^4.$$

Consider case **(i')** (the following is valid but irrelevant in case **(i)**). Summing in a radial manner around x yields that the previous sum is bounded by $\kappa C_0^4(\sum_{m=[d(x,W)]}^{\infty} m^{d-1}(1+m)^{-\alpha})^4 \leqslant \kappa C_0^4(1 + d(x, W))^{4(d-\alpha)}$, and the same holds for y. We can also work on the first order derivative with a similar technique:

$$\mathbf{E}|D_{\mathbf{x}}G|^4 \leqslant \kappa C_0^4\left(\sum_{k\in W}(1 + \|x - k\|)^{-\alpha}\right)^4 \leqslant \kappa C_0^4(1 + d(x, W))^{4(d-\alpha)}.$$

Noting $I_{x,y} = \{k \in W : \|k - x\| \geqslant \|k - y\|\}$,

$$\begin{aligned}\mathbf{E}|D_{\mathbf{x},\mathbf{y}}G(\eta)|^4 \leqslant& \kappa C_0^4\left[\left(\sum_{k\in I_{x,y}}(1 + \|x - k\|)^{-\alpha}\right)^4 + \left(\sum_{k\in I_{y,x}}(1 + \|y - k\|)^{-\alpha}\right)^4\right]\\ \leqslant& \kappa C_0^4\left[\sum_{k\in\mathbb{Z}^d\setminus B(x,\|y-x\|/2)}(1 + \|x - k\|)^{-\alpha} + \sum_{k\in\mathbb{Z}^d\setminus B(y,\|x-y\|/2)}(1 + \|y - k\|)^{-\alpha}\right]^4\\ \leqslant& C_0^4\kappa(1 + \|x - y\|/2)^{4(d-\alpha)},\end{aligned}$$

whence finally

$$\mathbf{E}|D_{\mathbf{x},\mathbf{y}}G(\eta)|^4 \leqslant \kappa C_0^4(1+\max(\|x-y\|, d(x,W), d(y,W)))^{4(d-\alpha)}. \tag{5.7}$$

Let us start with a few geometric estimates, useful in the case **(i')**.

Lemma 5.2. Let $W \subset \mathbb{Z}^d$, bounded and non-empty, $d_W = (|W|/|\partial_{\mathbb{Z}^d} W|)^{1/d}$, $W' = \{k \in \mathbb{Z}^d : d(k,W) \leqslant d_W\}$. We have

$$|W'| \leqslant \kappa |W| \tag{5.8}$$

$$\int_{(\tilde{W}')^c} (1+d(x,\tilde{W}))^a dx \leqslant \kappa_a |W| d_W^a,\ a < -d \tag{5.9}$$

$$I(x) := \int_{\mathbb{R}^d} (1+\max(d(x,W), \|x-y\|))^{d-\alpha} dy \leqslant \kappa (1+d(x,W))^{2d-\alpha}, x \in \mathbb{R}^d. \tag{5.10}$$

Proof. Since each point of $W' \setminus W$ is in a ball with radius d_W centered in $\partial_{\mathbb{Z}^d} W$, (5.8) is proved via

$$|W'| \leqslant |W| + |\partial_{\mathbb{Z}^d} W| \kappa d_W^d \leqslant \kappa |W|.$$

Let $\psi(x) = d(x,\tilde{W}), x \in \mathbb{R}^d, h(t) = \mathbf{1}_{\{t \geqslant d_W\}}(1+t)^a, t \geqslant 0$. The Federer co-area formula yields

$$\int_{\mathbb{R}^d} h(\psi(x)) \|\nabla \psi(x)\| dx = \int_{\mathbb{R}_+} h(t) \mathcal{H}^{d-1}(\psi^{-1}(\{t\})) dt.$$

We have $\|\nabla \psi(x)\| = 1$ for a.a. $x \in \tilde{W}^c$. According to [21, Lemma 4.1], for almost all $t > 0$,

$$\mathcal{H}^{d-1}(\psi^{-1}(\{t\})) \leqslant t^{d-1} \mathcal{H}^{d-1}(\psi^{-1}(\{1\})),$$

and the latter is bounded by $\kappa t^{d-1} |\partial_{\mathbb{Z}^d} W|$. Since $a + d < 0$

$$\int_{(\tilde{W}')^c} h(\psi(x)) dx \leqslant \kappa \int_{d_W}^{\infty} (1+t)^a t^{d-1} |\partial_{\mathbb{Z}^d} W| dt \leqslant \kappa_a |\partial_{\mathbb{Z}^d} W| d_W^a d_W^d = \kappa_a |W| d_W^a,$$

which yields (5.9). The left hand member of (5.10) is equal to

$$\begin{aligned} I(x) =& \ell^d(B(x, d(x,W)))(1+\max(d(x,W)))^{d-\alpha} + \int_{B(x,d(x,W))^c} (1+\|x-y\|)^{d-\alpha} dy \\ \leqslant& \kappa(1+d(x,W))^{2d-\alpha} + \int_{d(x,W)}^{\infty} (1+r)^{d-\alpha} \kappa r^{d-1} dr, \end{aligned}$$

from which the result follows. $\square$

Writing $\mathbf{x}_1 = (x_1, M_1), \mathbf{x}_2 = (x_2, M_2), \mathbf{x}_3 = (x_3, M_3)$, with M_1, M_2, M_3 iid distributed as μ, denote by $\tilde{\mathbf{E}}$ the expectation with respect to (M_1, M_2, M_3), and $\mathbf{E}_\eta$ the expectation with respect to η, such that $\mathbf{E} = \tilde{\mathbf{E}}\mathbf{E}_\eta$. We have, bounding $\mathbf{E}D^4_{\mathbf{x}_1}G$ by κC_0^4 and using Cauchy-Scwharz inequality several times,

$$\begin{aligned}
\gamma_1 =& 4\sigma^{-2}\left[\int_{A^3}\tilde{\mathbf{E}}\left[\sqrt{\mathbf{E}_\eta\left[(D_{\mathbf{x}_1}G)^2(D_{\mathbf{x}_2}G)^2\right]}\sqrt{\mathbf{E}_\eta\left[(D^2_{\mathbf{x}_1,\mathbf{x}_3}G)^2(D^2_{\mathbf{x}_2,\mathbf{x}_3}G)^2\right]}\right]dx_1dx_2dx_3\right]^{1/2}\\
\leqslant& 4\sigma^{-2}\left[\int_{A^3}\sqrt{\tilde{\mathbf{E}}\left[\mathbf{E}_\eta\left[(D_{\mathbf{x}_1}G)^2(D_{\mathbf{x}_2}G)^2\right]\right]}\sqrt{\tilde{\mathbf{E}}\left[\mathbf{E}_\eta\left[(D^2_{\mathbf{x}_1,\mathbf{x}_3}G)^2(D^2_{\mathbf{x}_2,\mathbf{x}_3}G)^2\right]\right]}dx_1dx_2dx_3\right]^{1/2}\\
\leqslant& \kappa C_0\sigma^{-2}\left[\int_{A^3}\left(\mathbf{E}(D^2_{\mathbf{x}_1,\mathbf{x}_3}G)^4\right)^{1/4}\left(\mathbf{E}(D^2_{\mathbf{x}_2,\mathbf{x}_3}G)^4\right)^{1/4}dx_1dx_2dx_3\right]^{1/2}\\
\leqslant& \kappa C_0^2\sigma^{-2}\sqrt{\int_A\left(\int_A(1+\max(d(x,\tilde{W}),\|x-x_3\|))^{d-\alpha}dx\right)^2dx_3}
\end{aligned}$$

using (5.7). Similar techniques to integrate out the marks yield the same bound

$$\begin{aligned}
\gamma_2 \leqslant& \kappa C_0\sigma^{-2}\left[\int_{A^3}\left(\mathbf{E}(D^2_{\mathbf{x}_1,\mathbf{x}_3}G)^4\right)^{1/4}\left(\mathbf{E}(D^2_{\mathbf{x}_2,\mathbf{x}_3}G)^4\right)^{1/4}dx_1dx_2dx_3\right]^{1/2}\\
\leqslant& \kappa C_0^2\sigma^{-2}\sqrt{\int_A\left(\int_A(1+\max(d(x,\tilde{W}),\|x-x_3\|))^{d-\alpha}dx\right)^2dx_3}
\end{aligned}$$

In the case **(i)**, $A = \tilde{W}$ and $\alpha > 2d$. We have

$$\max(\gamma_1,\gamma_2) \leqslant \kappa C_0^2\sigma^{-2}\sqrt{\ell^d(\tilde{W})\left(\int_{\mathbb{R}^d}(1+\|x\|)^{d-\alpha}dx\right)^2} \leqslant \kappa C_0^2\sigma^{-2}\sqrt{|W|}.$$

In the case $(\mathbf{i'}), A = \mathbb{R}^d, \alpha > 5d/2$. Using successively (5.10),(5.8) and (5.9) yield, with $2(2d-\alpha) < 2(-d/2) = -d$,

$$\begin{aligned}
\max(\gamma_1,\gamma_2) \leqslant& \kappa C_0^2\sigma^{-2}\sqrt{\int_A I(x_3)^2dx} \leqslant \kappa C_0^2\sigma^{-2}\sqrt{\kappa\ell^d(\tilde{W}') + \int_{(\tilde{W}')^c}(1+d(x,W))^{2(2d-\alpha)}dx}\\
&\leqslant \kappa C_0^2\sigma^{-2}\sqrt{\kappa|W| + \kappa_{2(2d-\alpha)}|W|d_W^{2(2d-\alpha)}} \leqslant \kappa C_0^2\sigma^{-2}\sqrt{|W|}(1+d_W^{2(2d-\alpha)}),
\end{aligned}$$

which gives the power a in (3.3)-(3.4). Let us keep assuming we are in case **(i')**. Since $A = \mathbb{R}^d$ and $\alpha > 2d$, (5.9) yields

$$\begin{aligned}
\gamma_3 \leqslant& \sigma^{-3}\int_{\mathbb{R}^d}\left(C_0^4\kappa(1+d(x,W))^{4(d-\alpha)}\right)^{3/4}dx \leqslant \quad C_0^3\kappa\sigma^{-3}\left(\ell^d(\tilde{W}') + \int_{(\tilde{W}')^c}(1+d(x,W))^{3(d-\alpha)}dx\right)\\
\leqslant& \kappa C_0^3\sigma^{-3}|W|(1+d_W^{3(d-\alpha)}).
\end{aligned}$$

In case **(i)**, the same bound holds after removing $d_W^{3(d-\alpha)}$. Reporting back gives (3.3).

Introduce $\overline{G} = G - \mathbf{E}G$. Using (5.8) and (5.9),

$$\begin{aligned}\gamma_4 \leqslant& \frac{1}{2}\sigma^{-1}(\mathbf{E}\overline{G}^4)^{1/4}\int_{\mathbb{R}^d}\sigma^{-3}\left(C_0^4\kappa(1+d(x,W))^{4(d-\alpha)}\right)^{3/4}dx\\ \leqslant&\kappa\sigma^{-4}v^{1/4}\sqrt{|W|}C_0^3\left(\ell^d(\tilde{W}')+\int_{(\tilde{W}')^c}(1+d(x,W))^{3(d-\alpha)}dx\right)\\ \leqslant&\sigma^{-4}C_0^3\kappa|W|^{3/2}v^{1/4}(1+d_W^{3(d-\alpha)})\end{aligned}$$

where $v := \sup_{|W|\to\infty}\frac{\mathbf{E}((G-\mathbf{E}G)^4)}{|W|^2}$. Let us conclude the proof: (5.9) yields

$$\gamma_5 \leqslant \left[\int_{\mathbb{R}^d}\sigma^{-4}C_0^4\kappa(1+d(x,W))^{4(d-\alpha)}dx\right]^{1/2} \leqslant \sigma^{-2}C_0^2\kappa\sqrt{|W|}\left(1+d_W^{4(d-\alpha)}\right)^{1/2}$$

$$\begin{aligned}\gamma_6 \leqslant& \Big[\int_{(\mathbb{R}^d)^2}\sigma^{-4}\Big(6C_0^4\kappa(1+d(x_1,W))^{2(d-\alpha)}(1+\|x_1-x_2\|)^{2(d-\alpha)}\\ &\qquad + 3C_0^4\kappa(1+d(x_1,W))^{2(d-\alpha)}(1+\|x_1-x_2\|)^{2(d-\alpha)}\Big)dx_1dx_2\Big]^{1/2}\\ \leqslant&\sigma^{-2}C_0^2\kappa\left[\int_{\mathbb{R}^d}(1+d(x_1,W))^{2(d-\alpha)}\left(\int_{\mathbb{R}^d}(1+\|x_1-x_2\|)^{2(d-\alpha)}dx_2\right)dx_1\right]^{1/2}\\ \leqslant&\sigma^{-2}C_0^2\kappa\sqrt{|W|}\left(1+d_W^{2(d-\alpha)}\right)^{1/2}.\end{aligned}$$

In case **(i)**, $A = \tilde{W}$, all the same inequalities still hold after removing terms of the form d_W^a. Reporting back gives (3.4).

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