Shot-noise excursions and non-stabilizing Poisson functionals

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Abstract

This article presents an asymptotic study of shot noise processes excursions, and of a more general class of statistics on marked spatial Poisson processes. A particularity of shot noise excursions, with respect to many popular objects of stochastic geometry, is that they are not stabilizing in general, but still a modification of the point process far from 0 will not modify the excursion set close to the origin by much. We shall present a complete second order theory that is applicable to stabilizing functionals, as well as non-stabilizing ones that satisfy this principle. This goes through a general mixing-type condition that adapts nicely to both proving asymptotic normality and volumic variance.

Keywords Poisson functionals, Shot noise fields, random excursions, central limit theorem, stabilisation, Berry-Esseen bounds

1 Introduction

Denote by ℓ the Lebesgue measure in $\mathbb{R}^d$. Let η be a homogeneous Poisson process on $\mathbb{R}^d$, and $\{F_B(\eta); B \subset \mathbb{R}^d\}$ a family of geometric functionals. We give a general conditions under which $\{F_B(\eta)\}$ has a variance proportional to $\ell(B)$, and $\mathrm{Var}(F_B(\eta))^{-1/2}(F_B - \mathbf{E}F_B(\eta))$ converges to a Gaussian variable, with a Kolmogorov distance decaying in $\ell(B)^{-1/2}$, as $\ell(B)$ goes to ∞.

Marked processes The model is even richer if one marks the input points by random independent variables, called *marks*, drawn from an external probability space $(\mathbf{M}, \mu)$, the *marks space*. It can be used for instance to let the shape and size of grains be random in the boolean model, or to have a random impulse function for a shot noise process. For $A \subset \mathbb{R}^d$, denote by $\overline{A} = A \times \mathbf{M}$ the cylinder of marked points $\mathbf{x} = (x, m)$ with spatial coordinate $x \in A$. Endow $\overline{\mathbb{R}^d}$ with the product σ-algebra. The reader not familiar with such a setup can consider the case where $\mathbf{M}$ is a singleton, and all mark-related notation can be ignored (except in applications). By an abuse of notation, every spatial transformation applied to a couple $\mathbf{x} = (x, m) \in \overline{\mathbb{R}^d}$ is in fact applied to the spatial element, i.e. $\mathbf{x} - y = (x - y, m)$ for $y \in \mathbb{R}^d$, or for $A \subset \mathbb{R}^d \times \mathbf{M}, C \subset \mathbb{R}^d$, $A \cap C = \{(x, m) \in A : x \in C\}$. Denote for simplicity by $d\mathbf{x} = dx\mu(dm)$ the measure element on $(\overline{\mathbb{R}^d}, \ell \times \mu)$. In all the paper, η denotes a Poisson measure on $\overline{\mathbb{R}^d}$ with intensity measure $\ell \times \mu$.

Functionals Let $\mathcal{A}$ be the class of locally finite sets of $\overline{\mathbb{R}^d}$ endowed with the topology induced by the mappings $\zeta \mapsto |\zeta \cap A|$ for compact sets $A \subset \overline{\mathbb{R}^d}$, where $|\cdot|$ denotes the cardinality of a set. Let $\mathscr{N}_0 \subset \mathcal{A}$ be such that $\mathbf{P}(\eta \in \mathscr{N}_0) = 1$, and $\mathscr{N}$ the class of configurations $\zeta \in \mathcal{A}$ such that $\zeta \subset \eta \cup \zeta'$ for some $\eta \in \mathscr{N}_0$ and finite set ζ'. Let $\mathcal{F}$ be the class of real measurable functionals on $\mathscr{N}$. For $W \subset \mathbb{Z}^d$, we consider a functional of the form

$$F_W(\zeta) = \sum_{k \in W} F_k^W(\zeta), \zeta \in \mathscr{N}, \text{ with } F_k^W(\zeta) = F_0(\zeta \cap \tilde{W} - k), k \in W, \tag{1.1}$$

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where $F_0 \in \mathcal{F}$ and $\tilde{W} = \cup_{k\in W}(k+[0,1)^d)$. It might also happen that η occurs on all the space but only contributions over W are considered: introduce the infinite input version

$$F'_W(\zeta) = \sum_{k\in W} F_k(\zeta), \zeta \in \mathscr{N}, \text{ with } F_k(\zeta) = F_0(\zeta - k), k \in \mathbb{Z}^d. \tag{1.2}$$

A score function is a mapping $\xi : \mathbf{M} \times \mathscr{N} \to \mathbb{R}$ such that

$$F_0^\xi : \zeta \mapsto \sum_{\mathbf{x}=(x,m)\in\zeta\cap[0,1)^d} \xi(m, \zeta - x), \tag{1.3}$$

is well defined on $\zeta \in \mathscr{N}$, which yields that $F_W(\zeta)$ is the sum of the scores of all points falling in $\tilde{W}$. Use $\xi(\zeta)$ instead of $\xi(m;\zeta)$ if no marking is involved (i.e. $\mathbf{M}$ is a singleton). It is explained later why some shot noise excursions functionals also obey representations (1.1)-(1.2). In this paper, we identify a functional $F : \mathscr{N} \to \mathbb{R}$ with the random variable that gives its value over $\eta : F = F(\eta)$, even if F will be applied to modified versions of η as well.

In many works (e.g. [19],[13, Chapter 4]), the observation windows consist in a growing family of subsets $B_n, n \geqslant 1$ of $\mathbb{R}^d$, that satisfy the Van'Hoff condition: for all $r > 0$,

$$\ell(\partial B_n^{\oplus r})/\ell(B_n) \to 0, \tag{1.4}$$

as $n \to \infty$, where $B^{\oplus r} = \{x \in \mathbb{R}^d : d(x, B) \leqslant r\}$ for $B \subset \mathbb{R}^d$. We rather consider in this paper, like for instance in [24], a family $\mathscr{W}$ of bounded subsets of $\mathbb{Z}^d$ satisfying the regularity condition

$$\limsup_{W\in\mathscr{W}} \frac{|\partial_{\mathbb{Z}^d} W|}{|W|} = 0, \tag{1.5}$$

where $\partial_{\mathbb{Z}^d} W$ is the set of points of W at distance 1 from W^c, and consider a point process over $\tilde{W}$. In the large window asymptotics, condition (1.5) imposes the same type of restrictions as (1.4), and using subsets of the integer lattice eases certain estimates and is not fundamentally different. In the case where boundary effects occur (by observing $\eta \cap \tilde{W}$ instead of η), stronger geometric conditions will be required. To this end, let $B_r, r > 0$, be a family of subsets of $\mathbb{R}^d$ such that for some $0 < a_- < a_+$, $B(0, a_- r) \subset B_r \subset B(0, a_+ r)$, where $B(x, r)$ is the Euclidean ball with center $x \in \mathbb{R}^d$ and radius $r > 0$. Let also $B_r(x) = x + B_r, x \in \mathbb{R}^d$. We set similarly as in [19, Section 2],

$$\begin{aligned}
\mathcal{B}_W^r =& \{\tilde{W} - k : \ k \in W, B_r^c \cap (\tilde{W} - k) \neq \emptyset\}, W \subset \mathbb{Z}^d, \\
\mathcal{B}_{\mathscr{W}}^r =& \bigcup_{W\in\mathscr{W}} \mathcal{B}_W^r \cup \{\mathbb{R}^d\}.
\end{aligned}$$

Background The family of functionals described above is quite general and covers large classes of statistics used in many application fields, from data analysis to ecology, see [13] for theory, models and applications. We study the variance, and Gaussian fluctuations, of such functionals, under the assumption that a modification of η far from 0 modifies slightly $F_0(\eta)$ (or $\xi(0, \eta)$). Most of the general results available require a *stabilization* property : roughly speaking, it means that there exists a random radius $R > 0$, with sufficiently fast decaying tail, such that any modification of η outside $B(0, R)$ does not affect $F_0(\eta)$ (or $\xi(0, \eta)$) at all. By stationarity this behaviour is transferred to any $F_k, k \in \mathbb{Z}^d$. The recent paper [15] establishes presumably optimal rates of convergence in a very broad setting, under exponential decay of the stabilization radius, both on Euclidean and non-Euclidean metric spaces, with the extension to functionals of binomial processes.

We give in this article general conditions under which functionals of the form (1.1)-(1.2) have a volume order variance and undergo a central limit theorem, with a Kolmogorov distance to the normal in the inverse square root of the variance. We recall that the Kolmogorov distance between two real variables U and V is defined as

$$d_{\mathscr{K}}(U, N) = \sup_{t\in\mathbb{R}} |\mathbf{P}(U \leqslant t) - \mathbf{P}(V \leqslant t)|. \tag{1.6}$$

Specified to the case where functionals are under the form (1.3) and the score function is stabilizing, our conditions demand that the tail of the stabilization radius R decays polynomially fast, with power strictly smaller than $-2d$, see Proposition 1.1.

Main result The main theoretical finding of this paper is condition (1.8), which is well suited for second order Poincaré inequalities in the Poisson space, i.e. bounds on the speed of convergence of a Poisson functional to the Gaussian law, and at the same time allows to prove non-degenerate asymptotic variance under the elementary non-triviality assumption (1.9). The application to shot-noise processes in the following section illustrates the versatility of the method. The results can be merged into the following synthetic result, which proof is at Section 3.2. For two sequences $\{a_n; n \geqslant 1\}, \{b_n; n \geqslant 1\}$, write $a_n \sim b_n$ if $b_n \neq 0$ for n sufficiently large and $a_n b_n^{-1} \to 1$ as $n \to \infty$. Also, in all the paper, κ denotes a constant that depends on d, α, a_+, a_-, which value may change from line to line, and which explicit optimal value in the main result could be traced through the different parts of the proof. If it is well defined, for $F_0 \in \mathcal{F}$, let

$$\sigma_0^2 := \sum_{k \in \mathbb{Z}^d} \operatorname{Cov}(F_0(\eta), F_k(\eta)). \tag{1.7}$$

Define, for $a > 0$, $W_a = [-a/2, a/2]^d \cap \mathbb{Z}^d$.

Theorem 1.1. Let $F_0 \in \mathcal{F}$, F_W be defined as in (1.1), $\mathscr{W} = \{W_n; n \geqslant 1\}$ satisfying (1.5). Let M_1, M_2 be independent random elements of $\mathbf{M}$ with law μ. Assume that for some $C_0 > 0, \alpha > 2d$, for all $r \geqslant 0, B \in \mathcal{B}_{\mathscr{W}}^r, \ell - a.e.\ x_1, x_2 \in \mathbb{R}^d, \zeta \subset \{(x_1, M_1), (x_2, M_2)\}$

$$\left(\mathbf{E}\left|F_0((\eta \cup \zeta) \cap B_r \cap B) - F_0((\eta \cup \zeta) \cap B)\right|^4\right)^{1/4} \leqslant C_0(1+r)^{-\alpha}, \tag{1.8}$$

and for some bounded set $A \subset \mathbb{R}^d, \sup_{k \in \mathbb{Z}^d} \mathbf{E} F_k(\eta \cap A)^2 < \infty$ and

$$\limsup_{a \to \infty} \operatorname{Var}(F_{W_a}(\eta \cap A)) > 0. \tag{1.9}$$

Then $0 < \sigma_0 < \infty$, and as $n \to \infty$, $\operatorname{Var}(F_{W_n}) \sim \sigma_0^2 |W_n|$, $(\sigma_0^2 |W_n|)^{-1/2}(F_{W_n} - \mathbf{E} F_{W_n})$ converges in law towards a standard Gaussian random variable N. Furthermore, for n sufficiently large,

$$d_{\mathscr{K}}\left(\frac{F_{W_n} - \mathbf{E} F_{W_n}}{\operatorname{Var}(F_{W_n})^{1/2}}, N\right) \leqslant \kappa |W_n|^{-1/2}\left(\frac{C_0^2}{\sigma_0^2} + \frac{C_0^3}{\sigma_0^3} + \frac{C_0^4}{\sigma_0^4}\right). \tag{1.10}$$

Let us now give the version with infinite input, which is more simple to satisfy due to the absence of boundary effects, except for the power of the decay:

Theorem 1.2. Let $F_0 \in \mathcal{F}$, F'_W be defined as in (1.2), $\mathscr{W} = \{W_n; n \geqslant 1\}$ satisfying (1.5). Let M_1, M_2, be independent random elements of $\mathbf{M}$ with law μ. Assume that for some $C_0 > 0, \alpha > 5d/2$, for all $r \geqslant 0, \ell - a.e.\ x_1, x_2 \in \mathbb{R}^d, \zeta \subset \{(x_1, M_1), (x_2, M_2)\}$,

$$\left(\mathbf{E}\left|F_0((\eta \cup \zeta) \cap B_r) - F_0(\eta \cup \zeta)\right|^4\right)^{1/4} \leqslant C_0(1+r)^{-\alpha}, \tag{1.11}$$

and (1.9) holds. Then $0 < \sigma_0 < \infty$ (defined in (1.7)), and as $n \to \infty$, $\operatorname{Var}(F'_{W_n}) \sim \sigma_0^2 |W_n|$, $(\sigma_0^2 |W_n|)^{-1/2}(F'_{W_n} - \mathbf{E} F'_{W_n})$ converges in law towards a standard Gaussian random variable N. Furthermore, for n sufficiently large

$$d_{\mathscr{K}}\left(\frac{F'_{W_n} - \mathbf{E} F'_{W_n}}{\operatorname{Var}(F'_{W_n})^{1/2}}, N\right) \leqslant \kappa |W_n|^{-1/2}\left(\frac{C_0^2}{\sigma_0^2} + \frac{C_0^3}{\sigma_0^3} + \frac{C_0^4}{\sigma_0^4}\right). \tag{1.12}$$

Remarks 1.1. 1. The application to score functionals (see (1.3)) goes as follows: let $M_i, 0 \leqslant i \leqslant 6$ be iid marks with law μ, and assume that $\xi : \mathbf{M} \times \mathscr{N} \to \mathbb{R}$ satisfies for all $r \geqslant 0, B \in \mathcal{B}^r_{\mathscr{W}}, x_0 \in [0,1]^d, \zeta \subset \mathbb{R}^d$ with at most 6 elements,

$$\left(\mathbf{E}\left|\xi(M_0, (\eta \cup \zeta) \cap B \cap B_r - x_0) - \xi(M_0, (\eta \cup \zeta) \cap B - x_0)\right|^4\right)^{1/4} \leqslant C_0(1+r)^{-\alpha}, \tag{1.13}$$

then the functional $F_0 = F_0^\xi$ defined in (1.3) satisfies (1.8). To see it, let $\mathbf{x}_i = (x_i, M_i)$ be the elements of ζ. Fix $\zeta_1 \subset \{(x_1, M_1), (x_2, M_2)\}$, apply Lemma 5.1 (with $r = 0$) to

$$\begin{aligned}\psi((x_0, M_0), \zeta') = \mathbf{1}_{\{x_0 \in [0,1]^d\}} &\left|\xi(M_0, (\zeta' \cup \zeta_1) \cap B \cap B_r - x_0)\right. \\ &\left. - \xi(M_0, (\zeta' \cup \zeta_1) \cap B - x_0)\right|, \zeta' \in \mathscr{N}, x_0 \in \mathbb{R}^d.\end{aligned}$$

It yields

$$\begin{aligned}\left(\mathbf{E}\left|F_0((\eta \cup \zeta_1) \cap B \cap B_r) - F_0((\eta \cup \zeta_1) \cap B)\right|^4\right)^{1/4} \leqslant &\left(\mathbf{E}\left|\sum_{\mathbf{x} \in \eta \cap [0,1]^d} \psi(\mathbf{x}, \eta)\right|^4\right)^{1/4} \\ \leqslant &\kappa C_0(1+r)^{-\alpha}\end{aligned}$$

for some $C_0 \geqslant 0$, hence (1.8) is satisfied. In this framework the asymptotic volumic variance is the finite quantity

$$\sigma_0^2 = \mathbf{E}\xi(M_0; \eta)^2 + \int_{\mathbb{R}^d} (\mathbf{E}\xi(M_0, \eta \cup \{x\})\xi(M_1, \eta \cup \{-x\}) - \mathbf{E}\xi(M_0; \eta)^2) dx,$$

see for instance (4.10) in [13].

2. A variant of stabilisation, called *strong stabilisation*, occurs when the add-one cost version of the functional is stabilising instead of the functional itself. Penrose and Yukich derived variance asymptotics and asymptotic normality [19] in such a context. In our context, let η' be an independent copy of η, and for $r > 0$, $\eta_r = (\eta \cap B_r) \cup (\eta' \cap B_r^c)$. Assume that a functional has a strong stabilisation radius with the tail decaying as a sufficiently low power of r. In this case, (1.8) should hold with the left hand member replaced with $\mathbb{E}\left(|F_0((\eta \cup \zeta) \cap B) - F_0((\eta_r \cup \zeta) \cap B)|^4\right)$. Then, going through the proofs of Theorems 2.1 and 3.1, one can see that the Berry-Esseen bounds and variance upper bounds should still hold. On the other hand, for the variance lower bound to hold, condition (1.9) has to be replaced by a condition adapted to strong stabilisation.

3. For non-degenerate asymptotic volumic variance to hold ($\sigma_0 > 0$), one essentially needs to find a bounded set $A \subset \mathbb{R}^d$ such that

$$\limsup_{a > 0} \mathrm{Var}(F_{W_a}(\eta \cap A)) > 0.$$

In many cases there is $k \in \mathbb{N}$ such that, with $U_1, \ldots, U_k$ uniform independent variables in A, the lower bound $\mathrm{Var}(F_{W_a}(\{U_1, \ldots, U_k\}))\mathbf{P}(|\eta \cap A| = k)$ suffices, see Example 1.1 for an illustration. Recent similar result regarding the variance can be found in the literature, but the assumptions are of different nature, either dealing with different qualitative long range behaviour (i.e. strong stabilization in [19, 15]), or different non-degeneracy statements [17], whereas condition (1.8) is more dependent on the quantitative asymptotic independence.

4. Similar results where the input consists of m_n iid variables uniformly distributed in $\tilde{W}_n$, with $m_n = |W_n|$, should be within reach by applying the results of [14], following a route similar to [15].

Shot-noise excursions We develop here the application of the results to geometric functionals of Poisson shot noise processes excursions. Let $\{g_m; m \in \mathbf{M}\}$ be measurable functions indexed by some measurable space $\mathbf{M}$. Let μ be a probability measure on $\mathbf{M}$ and η a Poisson process with intensity measure $\ell \times \mu$ on $\mathbb{R}^d$. Introduce the shot noise processes with kernel distribution μ by, for $\zeta \in \mathscr{N}$,

$$f_\zeta(y) = \sum_{\mathbf{x}=(x,m)\in\zeta} g_m(y-x), y \in \mathbb{R}^d. \tag{1.14}$$

Conditions under which f_ζ is well defined on Poisson input are discussed in Section 4, along with a proper choice for $\mathscr{N}_0$. Given some threshold $u \in \mathbb{R}$, we consider the excursion set $\{f_\zeta \geqslant u\} = \{x \in \mathbb{R}^d : f_\zeta(x) \geqslant u\}$ and the functionals

$$\begin{aligned}\zeta \mapsto& \ell(\{f_\zeta \geqslant u\} \cap \tilde{W})\\ \zeta \mapsto& \mathrm{Per}(\{f_\zeta \geqslant u\}; \tilde{W}),\end{aligned}$$

where for $A, B \subset \mathbb{R}^d; \mathrm{Per}(A;B)$ denotes the amount of perimeter of A contained in B in the variational sense, see Section 4.2. The Euler characteristic is also studied in Section 4.3 for a specific form of the kernels.

A shot noise field is the result of random impulse kernels translated at random locations in the space. It has been introduced by Schottky in the context of electronics in 1918 [23], and has been used since then under different names in many fields such as pharmacology, mathematical morphology [16, Section 14.1], image analysis [12], or telecommunication networks [3, 2]. Biermé and Desolneux [6, 8, 7] have computed the mean values for some geometric properties of excursions. More generally, the activity about asymptotic properties of random fields excursions has recently increased, with the notable recent contribution of Estrade and Léon [10], who derived a central limit theorem for the Euler characteristic of excursions of stationary Euclidean Gaussian fields. Bulinski, Spodarev and Timmerman [9] give general conditions for asymptotic normality of the excursion volume for quasi-associated random fields. Their results apply to shot-noise fields, under conditions of non-negativity and uniformly bounded marginal density, which can be verified in some specific examples. We give here the asymptotic variance and central limit theorems for volume and perimeter of excursions under weak assumptions on the density, as illustrated in Section 4. Still, a certain control of the distribution is necessary, and we provide in Lemma 4.2 a uniform bound on $\sup_{v\in\mathbb{R},\delta>0}(\delta\ln(\delta))^{-1}\mathbf{P}(f_\eta(0) \in [v-\delta, v+\delta])$ when f is of the form

$$f_\zeta(x) = \sum_{i\in I} g(\|x - x_i\|) \tag{1.15}$$

where $\zeta \in \mathscr{N}$, and $x_i, i \in I$, are the spatial locations of its points, with g a strictly non-increasing function $(0,\infty) \to (0,\infty)$ with a derivative not decaying too fast to 0. Our results allow to treat fields with singularities, such as those observed in astrophysics or telecommunications, see [2].

Let $\mathcal{M}$ be the space of measurable subsets of $\mathbb{R}^d$. The results of Section 4 also apply to processes that can be written under the form

$$f_\zeta(x) = \sum_{i\geqslant 1} L_i \mathbf{1}_{\{x-x_i\in A_i\}}, x \in \mathbb{R}^d, \tag{1.16}$$

where the $(L_i, A_i), i \geqslant 1$ are iid couples of $\mathbb{R}_+ \times \mathcal{M}$, endowed with a proper σ-algebra and probability measure, see Section 4.3. Such models are called *dilution functions* or *random token models* in mathematical morphology, see for instance [16, Section 14.1], where they are used to simulate random functions with a prescribed covariance kernel. The results potentially also apply to the Euler characteristic, provided the latter is properly defined on the class of excursions arising from such shot-noise processes (Theorem 4.4).

1.1 Stabilization and nearest neighbour statistics

Let us transpose our results in the case where the functional stabilises.

Proposition 1.1. Let $\mathscr{W} = \{W_n; n \geqslant 1\}$ be a class of subsets of $\mathbb{Z}^d$. Let F_W be defined as in (1.1) (resp. as in (1.1)-(1.3) with $F_0 = F_0^\xi$ for some score function ξ). Assume that for $x_i \in \mathbb{R}^d, M_i$ independent with law $\mu, i \geqslant 1, \zeta \subset \{(x_1, M_1), (x_2, M_2)\}$ (resp. $\zeta \subset \{(x_i, M_i); i = 1, \ldots, 6\}$), $\eta' = \eta \cup \zeta$, there is a random variable $R \geqslant 0$ such that almost surely, for $r \geqslant R, B \in \mathcal{B}^r_{\mathscr{W}}$,

$$F_0(\eta' \cap B_r \cap B) = F_0(\eta' \cap B). \tag{1.17}$$

$$\text{(resp. } \xi(m, \eta' \cap B_r \cap B - x) = \xi(m, \eta' \cap B - x), (x, m) \in \eta \cap \overline{[0,1]^d}.\text{)} \tag{1.18}$$

Then (1.8) is satisfied if for some $p, q > 1$ with $1/p + 1/q = 1$, R's tail has a polynomial decay in $r^{-8dp-\varepsilon}$ for some $\varepsilon > 0$, under the moment condition

$$\sup_{r \geqslant 0, B \in \mathcal{B}^r_{\mathscr{W}}} \mathbf{E} \left| F_0(\eta' \cap B \cap B_r) \right|^{4q} < \infty$$

$$\text{(resp. } \sup_{r \geqslant 0, B \in \mathcal{B}^r_{\mathscr{W}}, x_0 \in [0,1]^d} \mathbf{E} \left| \xi(\eta' \cap B \cap B_r - x_0) \right|^{4q} < \infty). \tag{1.19}$$

For the infinite input version, "$\cap B$" should be removed from (1.17) (resp. (1.18)), and the exponent $-8dp - \varepsilon$ should be replaced by $-10dp - \varepsilon$, and then (1.11) would hold.

Remarks 1.2.

- The variance non-degeneracy is a disjoint issue, (1.9) has to be satisfied independently. If one is only interested in asymptotic normality, the above requirements can be weakened, see Theorem 3.1.

- The definition of a stabilisation radius often involves stability under the addition of an external set, here denoted by ζ. A nice aspect of (1.17)-(1.18) with respect to classical results is that ζ does not depend on η, i.e. ζ does not in general achieve the worst case scenario given η. On the other hand, in the finite input version, one has to deal here with the intersection with $B \in \mathcal{B}^r_{\mathscr{W}}$. See Example 1.1 for an application to nearest neighbour statistics.

- Asymptotic results for stabilizing functionals have been derived in numerous work, see the survey [13, Chapter 4] and references therein. In particular, Matthew Penrose first proved such results under polynomial decay for the stabilisation radius.

Proof. For $r \geqslant 0, B \in \mathcal{B}^r_{\mathscr{W}}$, if (1.17) holds,

$$\begin{aligned}\mathbf{E} \left| F_0(\eta' \cap B) - F_0(\eta' \cap B \cap B_r) \right|^4 =& \mathbf{E} \mathbf{1}_{\{R > r\}} \left| F_0(\eta' \cap B) - F_0(\eta' \cap B \cap B_r)) \right|^4 \\ \leqslant& \mathbf{P}(R > r)^{1/p} \left(\mathbf{E} \left(|F_0(\eta' \cap B \cap B_r)| + |F_0(\eta' \cap B)| \right)^{4q} \right)^{1/q}.\end{aligned}$$

If $F_0 = F_0^\xi$, and (1.18) holds, for $r \geqslant R$

$$\begin{aligned}F_0^\xi(\eta' \cap B_r \cap B) =& \sum_{(x,m) \in \eta \cap [0,1]^d} \xi(m, \eta' \cap B_r \cap B - x) \\ =& \sum_{(x,m) \in \eta \cap [0,1]^d} \xi(m, \eta' \cap B - x) \\ =& F_0^\xi(\eta' \cap B),\end{aligned}$$

and (1.17) holds. □

Example 1.1 (Nearest neighbours statistics)**.** Given $\zeta \in \mathscr{N}, x \in \mathbb{R}^d$,denote by $NN(x;\zeta)$ the *nearest neighbour* of x, i.e. the closest point of $\zeta \setminus \{x\}$ from x, with ties broken by the lexicographic order. Define recursively for $k \geqslant 1$ $NN_k(x;\zeta) = NN(x;\zeta \setminus \cup_{i=0}^{k-1} NN_i(x;\zeta))$, with $x = NN_0(x;\zeta)$, and $NN_{\leqslant k}(x;\zeta) = \cup_{i=0}^{k} NN_i(x;\zeta)$. Fix $k \geqslant 1$ and call neighbours of x within ζ the set $N_k(x;\zeta)$ consisting of all points $y \in \zeta$ such that $x \in NN_{\leqslant k}(y, \zeta \cup \{x\})$ or $y \in NN_{\leqslant k}(x;\zeta)$. Straightforward geometric considerations yield that the cardinality of $N_k(x;\zeta)$ for some $x \in \zeta$ is bounded by a deterministic number $\kappa_{k,d}$ not depending on ζ.

Let then φ be a real functional taking in argument finite subsets of $\mathbb{R}^d$, and define the score function, for $\zeta \in \mathscr{N}$,

$$\xi(\zeta) = \varphi(N_k(0;\zeta)).$$

The simplest example would be for $k = 1$ the functional $\varphi(\zeta) = \frac{1}{2}\sum_{y \in \zeta} \|y\|$, so that

$$F_W(\zeta) = \sum_{x \in \zeta} \xi(\zeta - x)$$

gives the total length of the nearest-neighbour graph of ζ. Notice that no marking is involved in this setup. Such statistics are used in many applied fields, in nonparametric estimation procedures [4], or more recently in estimation of high-dimensional data sets. Many asymptotic results have been established since the central limit theorem of Bickel and Breiman [4], see for instance [19, 17, **?**].

Theorem 1.3. For $n \geqslant 1$, let

$$G_n = \sum_{x \in \eta \cap \tilde{W}_{n^{1/d}}} \varphi(N_k(x;\eta \cap \tilde{W}_{n^{1/d}})).$$

Assume that there is $C, c > 0, u < d/4$ such that for all $x_1, \dots, x_{\kappa_{k,d}} \in \mathbb{R}^d$,

$$\varphi(\{x_1, \dots, x_{\kappa_{k,d}}\}) \leqslant C \exp(c \max_i \|x_i\|^u) \tag{1.20}$$

and for some bounded $A \subset \mathbb{R}^d$

$$\int_{A^{k+1}} \varphi(\{x_1, \dots, x_k\} - x_{k+1}) dx_1 \dots dx_{k+1} \neq 0. \tag{1.21}$$

Then $\mathrm{Var}(G_n) \sim n\sigma_0^2$, with $\sigma_0 > 0$ explicited in Remark 1.1, and $n^{-1/2}(G_n - \mathbf{E}G_n)$ converges in law to $\mathcal{N}(0, \sigma_0)$, with bounds on the Kolmogorov distance proportional to $n^{-1/2}$.

For power length functionals, up to the exact shape of the observation window, this theorem generalizes some results appearing in [19, 17, 15]. For a well chosen φ, our results also apply to the Levina-Bickel statistic [18], used to estimate intrinsic dimension of high dimensional data sets (see [14] and references therein for details in the binomial setting).

Proof. Call hypercube a set of the form $x + [-a, a]^d$ for some $x \in \mathbb{R}^d, a \geqslant 0$. For this proof we choose $B_r = [-r, r]^d, r \geqslant 0$ (hence $a_- = 1, a_+ = \sqrt{d}$). Let $a_0 \in (0, 1/4)$ and $Q_i = x_i + [-a_0, a_0]^d, i = 1, \dots, q$ be hypercubes contained in $B_1 \setminus B_{1/2\sqrt{d}}$ such that the following holds: for all hypercube B that touches $B_{1/2\sqrt{d}}$ and B_1^c and $y \in B \cap B(0,1)^c$, there is i such that $Q_i \subset (B \cap B(y, \|y\|))$. Let $Q_i' = x_i + [-a_0/2, a_0/2]^d$ and

$$R = \min\{r \geqslant 2\sqrt{d}(1 + 1/a_0) : |\eta \cap rQ_i'| \geqslant k \text{ for every } i = 1, \dots, q\}.$$

The fact that $R' := \sqrt{d}(R + 1)$ is a stabilization radius in the sense of (1.18) is implied by the following claim:

Claim 1.1. Let $r \geqslant R'$,$B \in B^r_{\mathscr{W}}, x \in B_1$. All elements of $N_k(0, \eta' \cap B - x)$ are in $B(0, \sqrt{d}R)$.

Proof. Let $y \in \eta' \cap (B - x)$ be such that $0 \in NN_{\leqslant k}(y, (\eta' \cap B - x) \cup \{0\})$. Assume that $y \notin B(0, R)$, hence $y \in (B - x) \cap B(0, R)^c$. Since $B \cap B_r^c \neq \emptyset$, $(B - x) \cap B^c_{r-\sqrt{d}} \neq \emptyset$, and $(B - x) \cap B_R^c \neq \emptyset$. $0 \in B$ yields $(B - x) \cap B_t \neq \emptyset$ for $t \geqslant 1$, hence for $t = R/2\sqrt{d}$. It follows that there is i such that $B(y, \|y\|) \cap (B - x)$ contains RQ_i. Since η has (at least) k points in RQ'_i and $RQ'_i - x \subset RQ_i$ (using $Ra_0/2 \geqslant \sqrt{d}$), $\eta - x$ has k points in RQ_i, hence $(\eta' \cap B - x) \cap B(y, \|y\|)$ contains at least k points, and they are all closer from y than 0, which contradicts $0 \in NN_{\leqslant k}(y, (\eta' \cap B - x) \cup \{0\})$. This proves $y \in B(0, R)$.

For every i, RQ_i contains k points of η that are in B_R, hence in $B(0, R')$, hence $NN_{\leqslant k}(0, \eta' \cap B - x) \subset B_{R'}$. □

In particular $N_k(0, \eta' \cap B - x) = N_k(0, \eta' \cap B \cap B_r - x)$ for $r \geqslant R'$. We have for $r \geqslant 0$,

$$\mathbf{P}(R \geqslant r) \leqslant \sum_{i=1}^{q} \mathbf{P}(|\eta \cap rQ'_i| \leqslant k - 1) \leqslant \lambda r^{(k-1)d} e^{-\lambda' r^d}$$

(for some $\lambda, \lambda' > 0$), which has subpolynomial decay, and a similar bound holds for R'. For the moment condition, note that for $r > 0$, the neighbours of 0 in $\eta \cap B_r \cap B - x$ are at most at distance R', hence, in virtue of (1.20), uniformly in r, B, for $\varepsilon > 0$,

$$\mathbf{E}|\xi(\eta' \cap B_r \cap B - x)|^{4+\varepsilon} \leqslant C\mathbf{E}\exp(cR')^{(4+\varepsilon)u}$$

and this quantity is finite if ε is chosen such that $(4 + \varepsilon)u < d$, and (1.18)-(1.19) hold, hence (1.8) holds.

For the non-triviality of the variance, let $a > 0$ be such that $A \subset [-a/2, a/2]^d$. Let $X_i, i \geqslant 1$, be iid variables uniformly distributed in A. Without loss of generality, assume that $\eta \cap A = \{X_1, \ldots, X_{N_A}\}$ where N_A is Poisson distributed with parameter $\ell(A)$. If $|\eta \cap A| = k + 1$, we have $N_k(X_i, \eta \cap A) = \eta \cap A \setminus \{X_i\} = \{X_j; 1 \leqslant j \leqslant k + 1, j \neq i\}$. Then

$$\begin{aligned} F_{W_a}(\eta \cap A) =& \sum_{i=1}^{k+1} \varphi(\{X_j; j \neq i\} - X_i) \\ \mathbf{E}[F_{W_a} || \eta \cap A| = k + 1] =& (k + 1)\mathbf{E}\varphi(\{X_1, \ldots, X_k\} - X_{k+1}) \neq 0 \end{aligned}$$

using Assumption (1.21). Hence $\mathbf{P}(F_{W_a}(\eta \cap A) \neq 0) \geqslant \mathbf{P}(F_{W_a}(\eta \cap A) \neq 0 \big| |\eta \cap A| = k + 1)\mathbf{P}(|\eta \cap A| = k + 1) > 0$. Since we also have $\mathbf{P}(F_{W_a}(\eta \cap A) = 0) \geqslant \mathbf{P}(\eta \cap A = \emptyset) > 0$, $F_{W_a}(A)$ is not trivial, and (1.9) is satisfied (noticing that the latter estimates do not depend on a).

□

1.2 Other applications

An important part of the paper is devoted to shot noise excursions, but the results should apply also to most stabilizing models studied in the literature (packing functionals, Voronoi tessellation, boolean models, proximity graphs), see the example of statistics on nearest neighbours graphs above. Checking asymptotic non-degeneracy of the variance is reduced to checking its non-triviality on finite input.

In some models, the independent marking is replaced by *geostatistical marking* , also called dependent marking or external marking: let $m(x; \eta'), x \in \mathbb{R}^d$ be a random field measurable with respect to an independent homogeneous Poisson process η' on $\mathbb{R}^d$. Such a refinement is necessary to model a variety of random phenomena, such as gauge measurements for rainfalls or tree sizes in a sparse forest, see [22] and references therein. Labelling the points of η and η' with two different colors yields that $\eta \cup \eta'$ has the law of an independently marked Poisson process, hence our results should apply to appropriate statistics.

In the non-marked setting ($\mathbf{M}$ is a singleton), let $a > 0$ be a scaling perimeter, and consider the random field $X = (X_k)_{k\in\mathbb{Z}^d}$, where $X_k = \mathbf{1}_{\{a\eta\cap(k+[0,1)^d)\}=0}, k \in \mathbb{Z}^d$. X is an *independent spin-model* where the parameter $p = \mathbf{P}(X_0 = 1) = \exp(-a)$ can take any prescribed value. Then all the previous results can be applied to functionals of the form

$$F_W(X) = \sum_{k\in\mathbb{Z}^d} F_0(X_W - k) \text{ or } F'_W(X) = \sum_{k\in\mathbb{Z}^d} F_0(X - k),$$

where F_0 is some functional on the class of subsets of $\mathbb{Z}^d$, with finite second moment under iid Bernoulli input. Examples include also stabilising functionals and excursions functionals, our findings might apply for instance to the results of [21], where more general classes of discrete input than Bernoulli processes are also treated. Seeing F_W (or F'_W) as a functional of η, the variance and asymptotic normality results of Theorems 1.1-1.2 apply to F_W under conditions of the type

$$(\mathbf{E}\,|F_W(X'\cap B) - F_W(X'\cap B\cap B_r)|^4)^{1/4} \leqslant C_0(1+r)^{-\alpha},$$

where B, B_r are like in (1.1), and X' is obtained from X by forcing up to 2 spins $X_k, X_{k'}$ to the value 1 (the bound has to be uniform over $k, k' \in \mathbb{Z}^d$).

2 Moment asymptotics

In this section, we give asymptotic results for second and fourth moments of a geometric functional under general conditions of non-triviality and polynomial decay. The fourth order moment is useful for establishing Berry-Esseen bounds in the next section. The greek letter κ still denotes a constant depending on d, q, α, a_-, a_+ which value may change from line to line.

Theorem 2.1. Let $\alpha > d, W \subset \mathbb{Z}^d$, $C_0 \geqslant 0$. Let $F_0 \in \mathcal{F}$.
Assume **(i)** that for $k \in W$, $G_k^W = F_k^W$, (resp. **(i')** for $k \in \mathbb{Z}^d, G_k^W = F_k$) and let $G_W = \sum_{k\in W} G_k^W = F_W$ (resp. $G_W = F'_W$) as defined in (1.1) (resp. (1.2)), and for all $r \geqslant 0, B \in \mathcal{B}_W^r \cup \{\mathbb{R}^d\}$,

$$\left(\mathbf{E}\,|F_0(\eta\cap B_r\cap B) - F_0(\eta\cap B)|^2\right)^{1/2} \leqslant C_0(1+r)^{-\alpha}, \tag{2.1}$$

(resp. for all $r \geqslant 0$,

$$\left(\mathbf{E}\,|F_0(\eta\cap B_r) - F_0(\eta)|^2\right)^{1/2} \leqslant C_0(1+r)^{-\alpha},) \tag{2.2}$$

then for $k, j \in W$ (resp. $\mathbb{Z}^d$),

$$\mathrm{Cov}(G_j^W, G_k^W) \leqslant \kappa C_0^2(1 + \|k-j\|)^{-\alpha}, \tag{2.3}$$

$$\sigma_0^2 := \sum_{k\in\mathbb{Z}^d} \mathrm{Cov}(F_0, F_k) < \infty,$$

and $\sigma_0 > 0$ if (1.9) holds. If W is bounded and non-empty,

$$\left||W|^{-1}\mathrm{Var}(G_W) - \sigma_0^2\right| \leqslant \kappa C_0^2(|\partial_{\mathbb{Z}^d} W|/|W|)^{1-d/\alpha}. \tag{2.4}$$

If furthermore $\alpha > 2d$

$$\mathbf{E}\left(G_W - \mathbf{E}G_W\right)^4 \leqslant \kappa C_0(\mathbf{E}(F_0 - \mathbf{E}F_0)^4)^{3/4}|W|^2. \tag{2.5}$$

The proof is deferred to Section 5.1.

3 Asymptotic normality

We give bounds to the normal in terms of Kolmogorov distance, defined in (1.6), or Wasserstein distance, defined between two random variables U, V as

$$d_{\mathscr{W}}(U,V) = \sup_{h \in \mathrm{Lip}_1} |\mathbf{E}[h(U) - h(V)]|,$$

where Lip_1 is the set of 1-Lipschitz functions $h : \mathbb{R} \to \mathbb{R}$.

3.1 Malliavin derivatives

It has been shown in different frameworks [17, 10, 15, 14] that Gaussian fluctuations of real functionals can be controlled by some second order difference operators defined on the random input. In the Poisson setting, this operator is incarnated by the Malliavin derivatives. We define it here as it is a central tool in the theory backing our results: for any functional $F \in \mathcal{F}, \zeta \in \mathscr{N}$, and $\mathbf{x} \in \overline{\mathbb{R}^d}$, define the first order Malliavin derivative $D_{\mathbf{x}} F \in \mathcal{F}$ by

$$D_{\mathbf{x}} F(\zeta) = F(\zeta \cup \{\mathbf{x}\}) - F(\zeta),$$

and for $\mathbf{x}, \mathbf{y} \in \overline{\mathbb{R}^d}, \zeta \in \mathscr{N}, F \in \mathcal{F}_0$, the second order Malliavin derivative (only required in the proof) is

$$D^2_{\mathbf{x},\mathbf{y}} F(\zeta) = D^2_{\mathbf{y},\mathbf{x}} F(\zeta) = D_{\mathbf{x}}(D_{\mathbf{y}} F(\zeta)) = F(\zeta \cup \{\mathbf{x},\mathbf{y}\}) - F(\zeta \cup \{\mathbf{x}\}) - F(\zeta \cup \{\mathbf{y}\}) + F(\zeta).$$

One can use this object to quantify the spatial dependency of the functional F: a point $\mathbf{y} \in \overline{\mathbb{R}^d}$ has a weak influence on a point $\mathbf{x} \in \overline{\mathbb{R}^d}$ for the functional F if its presence hardly affects the contribution of $\mathbf{x}$, i.e. $D_{\mathbf{x}} F(\eta) \approx D_{\mathbf{x}} F(\eta \cup \{y\})$, or in other words $D^2_{\mathbf{x},\mathbf{y}} F(\eta) = D_{\mathbf{y}}(D_{\mathbf{x}} F(\eta)) \approx 0$. The proof of the following theorem is based on the result of Last, Peccati and Schulte [17], that somehow asserts that the functional F_W exhibits Gaussian behavior as $W \to \mathbb{R}^d$, as soon as $D_{\mathbf{x},\mathbf{y}} F_W$ is small when $\mathbf{x}, \mathbf{y}$ are far away, uniformly in W. The speed of decay actually yields a bound on the speed of convergence of F_W towards the normal.

Theorem 3.1. Let $W \subset \mathbb{Z}^d$ bounded. Let $G_W \in \{F_W, F'_W\}$ as defined in (1.1)-(1.2), with $F_0 \in \mathcal{F}$, and let $M, M' \sim \mu$ independent. Assume that for some $C_0 > 0$, either **(i)** $G_W = F_W$ and for some $\alpha > 2d$, for $k \in W, a.a.\, x \in \tilde{W} - k, a.a.\, y \in \mathbb{R}^d$, $\eta' \in \{\eta, \eta \cup \{(y, M')\}\}$,

$$\left[\mathbf{E}|D_{(x,M)} F_0(\eta' \cap (\tilde{W} - k))|^4\right]^{1/4} \leqslant C_0 (1 + \|x\|)^{-\alpha}, x \in \mathbb{R}^d, \tag{3.1}$$

or **(i')** $G_W = F'_W$ and for some $\alpha > 5d/2$, for a.a. $x, y \in \mathbb{R}^d$, $\eta' \in \{\eta, \eta \cup \{(y, M')\}\}$,

$$\left[\mathbf{E}|D_{(x,M)} F_0(\eta')|^4\right]^{1/4} \leqslant C_0 (1 + \|x\|)^{-\alpha}, x \in \mathbb{R}^d. \tag{3.2}$$

Then, $\sigma^2 := \mathrm{Var}(G_W) < \infty$, and if $\sigma \neq 0$, with $\tilde{G}_W = \sigma^{-1}(G_W - \mathbf{E} G_W)$,

$$d_{\mathscr{W}}(\tilde{G}_W, N) \leqslant \kappa \left(C_0^2 \sigma^{-2} \sqrt{|W|} + C_0^3 \sigma^{-3} |W|\right) \left(1 + \left(\frac{|\partial_{\mathbb{Z}^d} W|}{|W|}\right)^a\right), \tag{3.3}$$

where $a = 0$ in case **(i)**, and $a = 2(\alpha/d - 2)$ in case **(i')**. Let $v := \sup_W (G_W - \mathbf{E} G_W)^4 |W|^{-2}$, then

$$d_{\mathscr{K}}(\tilde{G}_W, N) \leqslant \kappa \left(C_0^2 \sigma^{-2} \sqrt{|W|} + C_0^3 \sigma^{-3} |W| + v^{1/4} C_0^3 \sigma^{-4} |W|^{3/2}\right) \left(1 + \left(\frac{|\partial_{\mathbb{Z}^d} W|}{|W|}\right)^a\right). \tag{3.4}$$

Recall that (2.1) (or (2.2) in case **(i')**) is a sufficient condition for $v < \infty$.

The proof is at Section 5.2

3.2 Proof of Theorems 1.1 and 1.2

We prove Theorem 1.1 (resp. Theorem 1.2) using the previous results.

Let $n \geqslant 1$ be such that $W = W_n$ is bounded and non-empty, $G_W = F_W$ (resp. $G_W = F'_W$), $\sigma^2 = \mathrm{Var}(G_W)$. Assumption (1.8) (resp. (1.11)) clearly implies (2.1) (resp. (2.2)), and therefore (2.4) holds:

$$\left||W|^{-1}\sigma^2 - \sigma_0^2\right| \leqslant \kappa C_0^2(|\partial_{\mathbb{Z}^d}W|/|W|)^{1-d/\alpha}.$$

Let $y \in \mathbb{R}^d, k \in W, x \in \tilde{W} - k, \mathbf{x} = (x, M), \eta' \in \{\eta, \eta \cup \{(y, M')\}\}$ as in (3.1) (resp. (3.2)), $\eta'' = \eta' \cup \{\mathbf{x}\}, B = \tilde{W} - k$ (resp. $B = \mathbb{R}^d$), $r = \|x\|/a_+$. Note that $x \in B \setminus B_r$, hence

$$\begin{aligned}
D_{\mathbf{x}}F_0(\eta' \cap B) =& F_0((\eta' \cap B) \cup \{\mathbf{x}\}) - F_0(\eta' \cap B)\\
=& F_0((\eta' \cup \{\mathbf{x}\}) \cap B) - F_0(\eta' \cap B)\\
=& F_0((\eta' \cup \{\mathbf{x}\}) \cap B) - F_0((\eta' \cup \{\mathbf{x}\}) \cap B \cap B_r) + F_0((\eta' \cup \{\mathbf{x}\}) \cap B \cap B_r) - F_0(\eta' \cap B)\\
=& F_0(\eta'' \cap B) - F_0(\eta'' \cap B \cap B_r) + F_0(\eta' \cap B \cap B_r) - F_0(\eta' \cap B).
\end{aligned}$$

Applying (1.8) (resp. (1.11)) twice with $x_1 = x, x_2 = y$ yields

$$\left(\mathbf{E}|D_{\mathbf{x}}F_0(\eta' \cap B)|^4\right)^{1/4} \leqslant C_0(1 + r)^{-\alpha},$$

hence (3.1) (resp. (3.2)) holds, and (3.3) holds. Since furthermore (1.9) holds, Theorem 2.1 yields $\sigma_0 > 0$, and for n sufficiently large, $\sigma^{-2} \leqslant 2|W|^{-1}\sigma_0^{-2}$, hence, with $\tilde{G}_W := (G_W - \mathbf{E}G_W)(\mathrm{Var}G_W)^{-1/2}$, for n sufficiently large, using also (1.5),

$$d_{\mathscr{W}}(\tilde{G}_W, N) \leqslant \kappa|W|^{-1/2}\left(C_0^2\sigma_0^{-2} + C_0^3\sigma_0^{-3}\right).$$

Finally, since (1.8) (resp. (1.11)) holds with $\alpha > 2d$, we have furthermore (2.5):

$$v = \limsup_{n\geqslant 1} \mathbf{E}\left(G_{W_n} - \mathbf{E}G_{W_n}\right)^4/|W_n|^2 \leqslant \kappa C_0(\mathbf{E}(F_0 - \mathbf{E}F_0)^4)^{3/4}.$$

Applying (1.8) with $r = 0, B = \mathbb{R}^d$ gives $\mathbf{E}(F_0 - \mathbf{E}F_0)^4 \leqslant \kappa C_0^4$. The bound on Kolmogorov distance (1.10) (resp. (1.12)) follows easily.

It remains to prove that $G'_W := (\sigma_0^2|W|)^{-1/2}(G_W - \mathbf{E}G_W)$ follows a central limit theorem. We achieve it by proving that its Wasserstein distance to the normal goes to 0. The triangular inequality yields

$$\begin{aligned}
d_{\mathscr{W}}(G'_W, N) \leqslant& \mathbf{E}\left|G'_W - \tilde{G}_W\right| + d_{\mathscr{W}}(\tilde{G}_W, N)\\
\leqslant& \left|\frac{1}{\sigma_0\sqrt{|W|}} - \frac{1}{\sqrt{\mathrm{Var}(G_W)}}\right|\mathbf{E}\left|G_W - \mathbf{E}G_W\right| + d_{\mathscr{W}}(\tilde{G}_W, N)
\end{aligned}$$

which indeed goes to 0 by (2.4).

4 Application to shot-noise processes

Let the notation of the introduction prevail. For the process f_η (see (1.14)) to be well defined, assume that for some $\tau > 0$,

$$\int_{\mathbf{M}}\int_{B(0,\tau)^c}|g_m(x)|dx\mu(dm) < \infty, \tag{4.1}$$

and let $\mathscr{N}_0$ be the class of locally finite ζ such that $\sum_{(x,m)\in\zeta\cap B(0,\tau)^c}|g_m(x)| < \infty$. The fact that $\eta \in \mathscr{N}_0$ a.s. follows from the Mecke formula.

We use the general framework of random measurable sets. A *random measurable set* is a random variable taking values in the space $\mathcal{M}$ of measurable subsets of $\mathbb{R}^d$, endowed with the Borel σ-algebra $\mathcal{B}(\mathcal{M})$ induced by the local convergence in measure, see Section 2 in [11]. Regarding the more familiar setup of random closed sets, in virtue of Proposition 2 in [11], a random measurable set which realisations are a.s. closed can be assimilated to a random closed set.

4.1 Volume of excursions

For $u \in \mathbb{R}$ fixed, $W \subset \mathbb{Z}^d, \zeta \in \mathscr{N}$, define

$$F_W(\zeta) = \ell(\{f_{\zeta \cap \tilde{W}} \geqslant u\} \cap \tilde{W}), \;\; F'_W(\zeta) = \ell(\{f_\zeta \geqslant u\} \cap \tilde{W}).$$

A central limit theorem for the volume of a certain family of shot noise excursions has been derived in [9], under the assumption that $f_\eta(0)$ has a uniformly bounded density and $\mathbf{E}|g_M(x)|$ decreases sufficiently fast as $\|x\| \to \infty$, using the associativity properties of non-negative shot-noise fields. In some specific cases, the bounded density can be checked manually with computations involving the Fourier transform. In this section, we refine this result in several ways:

- A general model of random kernel is treated, it can in particular take negative values, allowing for compensation mechanisms (see [16]).
- The precise variance asymptotics are derived.
- Weaker conditions are required for the results to hold, in particular no bounded density is assumed.
- The likely optimal rate of convergence in Kolmogorov distance towards the normal is given.
- Boundary effects under finite input are considered, in the sense that only points falling in a bounded window (growing to infinity) contribute to the field. The case of infinite input is also treated.

The application to shot noise excursions is a nice illustration of the versatility of the general method derived in this article. We give examples of fields with no marginal density to which the results apply, such as sums of indicator functions, or of kernels with a singularity in 0. Controlling the density of shot-noise fields is in general crucial for deriving results on fixed-level excursions.

The decay assumption in 0 is of the following form: for some $\beta, c > 0$, for$\|x\| \geqslant 1$,

$$\mathbf{E}|g_M|(x) \leqslant c\|x\|^{-\beta}, \tag{4.2}$$

further assumptions are made on β later. We also assume that one of the two following assumptions is satisfied. The case of indicator kernels is treated in Section 4.3.

Assumption 4.1. There is $a \in (0,1]$ and $c > 0, r_0 > 0$, such that for $\delta > 0$,

$$\sup_{v \in \mathbb{R}\delta, r \geqslant r_0} \mathbf{P}(f_{\eta^r}(0) \in (v - \delta, v + \delta) \mid \eta^r \neq \emptyset) \leqslant c\delta^a, x \in B,$$

where $\eta^r = \eta \cap B(0, r)$, and $\mathbf{P}(f_\eta(0) > 0) > 0$.

(the condition $\eta^r \neq \emptyset$ is necessary as $\mathbf{P}(f_{\eta^r}(0) = 0) \geqslant \mathbf{P}(\eta^r = \emptyset) > 0$.) Lemma 4.2 below gives a bound in $\delta \ln(\delta)$ on the density of some shot-noise fields. The generality of the model can be used for instance to have the shape or the orientation of the random kernel to be random, see for instance the anisotropic models of *procedural noise* used in [12] for texture synthesis.

Assumption 4.2. Let f be of the form (1.15) with g satisfying the following condition: there is $\varepsilon > 0$ and $c > 0$ such that for $r > 0$,

$$\int_0^r \frac{\rho^{-2} \wedge \rho^{2(d-1)}}{-g'(\rho)} d\rho \leqslant c \exp(cr^{d-\varepsilon}). \tag{4.3}$$

Nothing prevents g from having a singularity in 0: Theorem 4.1 below applies in any dimension to $g(\rho) = C\rho^{-\lambda}\mathbf{1}_{\{\rho \leqslant 1\}} + g_1(\rho)\mathbf{1}_{\{\rho > 1\}}, \rho > 0$, where $\lambda > 0$ and $g_1(\rho)$ is for instance of the form $\exp(-a\rho^\gamma)$ or $\rho^{-\beta}$, with $a, \lambda > 0, \beta > 11d, \gamma < d$. Such fields don't necessarily have a finite first-order moment, and are used for instance in [2] to approximate stable fields, or for modeling telecommunication networks. Here is the result for infinite input shot-noise fields.

Theorem 4.1. Let $u > 0, G_W^u = \ell(\{x \in \tilde{W} : f(x, \eta) \geqslant u\}), W \subset \mathbb{Z}^d$. Assume that (4.2) is satisfied and either Assumption 4.1 and $\beta > d(10/a + 1)$ hold, or Assumption 4.2 with $\beta > 11d$ holds. Then $F_{\{0\}}^u$ satisfies (1.11),(1.9),(2.2),(3.2), and hence as $|\partial_{\mathbb{Z}^d} W|/|W| \to 0$, $\operatorname{Var}(G_W^u) \sim \sigma_0^2 |W|$, $(G_W^u - \mathbf{E} G_W^u)(\sigma_0\sqrt{|W|})^{-1}$ satisfies a central limit theorem, with

$$\sigma_0^2 = \int_{\mathbb{R}^d} \left[\mathbf{P}(f(0,\eta) \geqslant u, f(x) \geqslant u) - \mathbf{P}(f(0,\eta) \geqslant u)^2 \right] dx > 0. \tag{4.4}$$

Also, the convergence rate (3.4) in Kolmogorov distance holds for $\tilde{G}_W^u$.

To give results in the case where boundary effects are considered, we need an additional hypothesis on the geometry of the underlying family of windows $\mathscr{W} = \{W_n; n \geqslant 1\}$. For $\theta > 0$, let $\mathcal{C}_\theta$ be the family of cones $C \subset \mathbb{R}^d$ with apex 0 and aperture θ, i.e. such that $\mathcal{H}^{d-1}(C \cap \mathcal{S}^{d-1}) = \theta$. Let $\mathcal{C}_{\theta,r} = \{C \cap B(0,r) : C \in \mathcal{C}_\theta\}$ for $r \geqslant 0$. Say that $\mathscr{W}$ has aperture $\theta > 0$ if for all $W \in \mathscr{W}$ with diameter $r \geqslant 0$, W has *aperture* θ : for $x \in \tilde{W}$, there is a subset $C \in \mathcal{C}_{\theta, r/2}$ such that $(x + C) \subset \tilde{W}$. The factor 1/2 is arbitrary here and could be replaced by any strictly positive number (proofs would have to be adapted). Let us give an adapted version of Assumption 4.1.

Assumption 4.3. Let M be a random variable with law μ. There is $a \in (0,1], c > 0, r_0 > 0$, such that for $\delta > 0, r \geqslant r_0$,

$$\sup_{v \in \mathbb{R}, C \in \mathcal{C}_{\theta,r}} \mathbf{P}(f_{\eta \cap C}(0) \in [v - \delta, v + \delta] \mid \eta \cap C \neq \emptyset) \leqslant c\delta^a, x \in B,$$

and $\mathbf{P}(f_\eta(0) > 0) > 0$.

Theorem 4.2. Let $u > 0, G_W^u = \ell(\{x \in \tilde{W} : f_{\eta \cap \tilde{W}}(x) \geqslant u\}), W \subset \mathbb{Z}^d$. Assume that $\mathscr{W}$ has aperture θ and satisfies (1.5), that (4.2) holds and that either Assumption 4.3 and $\beta > d(8/a+1)$ hold, or Assumption 4.2 with $\beta > 9d$ holds. Then $F_{\{0\}}^u$ satisfies (1.8),(1.9),(2.1),(3.1), and hence as $|\partial_{\mathbb{Z}^d} W|/|W| \to 0$, $\operatorname{Var}(G_W^u) \sim \sigma_0^2 |W|$, $(G_W^u - \mathbf{E} G_W^u)(\sigma_0\sqrt{|W|})^{-1}$ satisfies a central limit theorem, with $\sigma_0 \in (0, \infty)$ defined in (4.4). Also, the convergence rate (3.4) in Kolmogorov distance holds for $\tilde{G}_W^u$.

Proof of Theorems 4.1 and 4.2. We put $F_0(\zeta) = F_W^u(\zeta \cap [0,1)^d), \zeta \in \mathscr{N}$. In all the examples proposed, we have $\mathbf{E}\|g_M(x)\| \leqslant C\|x\|^{-\beta}$ for $\|x\| \geqslant 1$, for some $\beta > d$, hence (4.1) holds for $\tau = 1$, and the left hand member of (1.11) is uniformly bounded for $r \leqslant 2\sqrt{d}$. From now on we take $r > 2\sqrt{d}$. Let us start by proving (1.9). We need the following lemma. Recall that $\eta^\rho = \eta \cap B(0,\rho), \rho \geqslant 0$.

Lemma 4.1. Assume that $u \geqslant 0$ and

$$\mu(\{m \in \mathbf{M} : g_m^{-1}((0,\infty)) > 0\}) > 0. \tag{4.5}$$

Then for some $\rho > 0$, for $a > 2$,

$$\mathbf{E}(\ell(\{f_{\eta^\rho} > u\} \cap \tilde{W}_a)) > 0.$$

Proof. Let $\varepsilon > 0, \rho > 0$ sufficiently large so that

$$\mu(\{m : \ell(g_m^{-1}(\varepsilon,\infty) \cap B(0,\rho-1)) > 0\}) > 0,$$

using the fact that $(0,\infty)$ and $\mathbb{R}^d$ are open. Let $t \in B(0,1)$, $M_i, i \geqslant 1$, iid random variables with law μ, $X_i, i \geqslant 1$, independant uniformly distributed on $B(0,\rho)$, and $U_i = g_{M_i}(t - X_i)$. Let $n > u/\varepsilon$. We have $\mathbf{P}(U_1 \geqslant \varepsilon) > 0$, hence

$$\begin{aligned}
\mathbf{P}(f_{\eta^\rho}(t) \geqslant u) \geqslant & \mathbf{P}(f_{\eta^\rho}(t) \geqslant n\varepsilon) \\
= & \sum_{k=0}^{\infty} \mathbf{P}(U_1 + \cdots + U_k \geqslant n\varepsilon)\mathbf{P}(|\eta^\rho| = k) \\
\geqslant & \sum_{k=n}^{\infty} \mathbf{P}(U_1 \geqslant \varepsilon)^n \mathbf{P}(|\eta^\rho| = k) > 0.
\end{aligned}$$

Fubini's theorem yields

$$\mathbf{E}\ell(\{t \in \tilde{W}_a : f_{\eta^\rho}(t) \geqslant u\}) \geqslant \mathbf{E}\ell(\{t \in B(0,1) : f_{\eta^\rho}(t) \geqslant u\}) > 0.$$

□

It is easy to prove in all cases that (4.5) holds. Since $\mathbf{P}(F_{W_a}(\eta^\rho) = 0) = \mathbf{P}(\ell(\{t \in \tilde{W}_a : f_{\eta^\rho}(t)\}) = 0) \geqslant \mathbf{P}(\eta^\rho = \emptyset) > 0$, $F_{W_a}(\eta^\rho)$ is not trivial for $\beta > 2$, and F_0 satisfies (1.9) with $A = B(0,\rho)$.

Let us now prove that (1.8) holds, or (1.11) for Theorem 4.2. Let $x_1, x_2 \in \mathbb{R}^d$, M_1, M_2 independent marks with law μ, $r \geqslant 0$, $\zeta \subset \{(x_1, M_1), (x_2, M_2)\}, \eta' = \eta \cup \zeta$. Let $B = \mathbb{R}^d$ in the cas of infinite input (Theorem 4.1), and let $B \in \mathcal{B}^r_{\mathscr{W}}$ otherwise (Theorem 4.2). Jensen's inequality yields

$$\begin{aligned}
|F_0(\eta' \cap B) - F_0(\eta' \cap B_r \cap B)|^4 = & \left[\int_{[0,1]^d} \left(\mathbf{1}_{\{f_{\eta' \cap B}(t) \geqslant u\}} - \mathbf{1}_{\{f_{\eta' \cap B_r \cap B}(t)\}}\right) dt\right]^4 \\
\leqslant & \int_{[0,1]^d} \left|\mathbf{1}_{\{f_{\eta' \cap B}(t) \geqslant u\}} - \mathbf{1}_{\{f_{\eta' \cap B_r \cap B}(t) \geqslant u\}}\right| dt,
\end{aligned}$$

and for $t \in [0,1]^d, r > 2\sqrt{d}$,

$$\begin{aligned}
|f_{\eta' \cap B}(t) - f_{\eta' \cap B_r \cap B}(t)| = & \left|\sum_{\mathbf{x}=(x,m) \in (\eta' \cap B) \setminus B_r} g_m(t-x)\right| \\
\leqslant & \delta_{r,t} := \sum_{\mathbf{x}=(x,m) \in \eta' \setminus B(t, a_-(r-\sqrt{d}))} |g_m(t-x)|,
\end{aligned}$$

and $\delta_{r,t}$ is independent from $\eta \cap B(t, a_- r/2)$ and its law does not depend on $t \in [0,1]^d$. Since $B = \tilde{Z}$ for some $Z \subset \mathbb{Z}^d$ and $0 \in B, t \in B$. Also, B contains a point of B_r^c, hence it has diameter at least $a_- r$. Since W has aperture θ, there is a solid cone $C_t \in \mathcal{C}_{\theta, a_- r/2}$ such that, with $D_t = (C_t + t)$, $D_t \subset B$. In the infinite input case, the latter trivially holds with

$B=\mathbb{R}^d, \theta=\sigma_{d-1}:=\mathcal{H}^{d-1}(\mathcal{S}^{d-1}), D_t=B(t,a_-r/2)$. We have

$$\begin{aligned}
\mathbf{E}\Big|F_0(\eta'\cap B)-F_0(\eta'\cap B_r\cap B)\Big|^4 &\leqslant \sup_{t\in[0,1]^d}\mathbf{P}(f_{\eta'\cap B}(t)\in[u-\delta_{r,t},u+\delta_{r,t}])\\
&\leqslant \sup_{t\in[0,1]^d}\mathbf{P}\left(f_{\eta\cap D_t)}(t)+f_{\eta\cap(B\setminus D_t}(t)\cup\zeta)\in[u-\delta_{r,t},u+\delta_{r,t}]\right)\\
&\leqslant \sup_{v\in\mathbb{R},t\in[0,1]^d}\mathbf{P}\left(f_{\eta\cap D_t}(t)\in[v-\delta_{r,t},v+\delta_{r,t}]\right))\\
&\leqslant \sup_{v\in\mathbb{R},C\in\mathcal{C}_{\theta,a_-r/2}}\mathbf{P}\left(f_{\eta\cap C}(0)\in[v-\delta_{r,0},v+\delta_{r,0}]\right))\\
&= \sup_{v\in\mathbb{R},C\in\mathcal{C}_{\theta,a_-r/2}}\mathbf{E}\left[\mathbf{P}(f_{\eta\cap C}(0)\in[v-\delta_{r,0},v+\delta_{r,0}])\,|\delta_{r,0})\right]. \qquad (4.6)
\end{aligned}$$

If Assumption 4.3 holds (or Assumption 4.1 in the infinite input case), assume without loss of generality $r\geqslant 2r_0/a_-$. The previous expression is bounded by

$$\begin{aligned}
\sup_{C\in\mathcal{C}_{\theta,a_-r/2}}\mathbf{P}(\eta\cap C=\emptyset)+c\mathbf{E}(\delta_{r,0}^a)\leqslant&\sup_{C\in\mathcal{C}_{\theta,a_-r/2}}\exp(-\ell(C))+c(\mathbf{E}[\delta_{r,0}])^a\\
&\leqslant\exp(-\kappa\theta r^d)+\kappa c\left(\int_{a_-(r-\sqrt{d})}^{\infty}\rho^{-\beta}\rho^{d-1}d\rho\right)^a\leqslant Cr^{-a(\beta-d)},
\end{aligned}$$

hence in the finite input case (1.8) holds with $\alpha=a(\beta-d)/4>2d$, and in the infinite input case (1.11) holds with $\alpha=a(\beta-d)/4>5d/2$.

Let us now assume that instead Assumption 4.2 holds. At this point we need to study the density of the shot-noise field.

Lemma 4.2. Assume that f is of the form (1.15). Let $\delta>0, R\geqslant 1$. Then for $v\in\mathbb{R}, C\in\mathcal{C}_{\theta,R}$,

$$\mathbf{P}(f_{\eta\cap C}(0)\in[v-\delta,v+\delta])\leqslant\kappa\delta\int_0^R\frac{(\rho^{-2}\wedge\rho^{2(d-1)})d\rho}{-g'(\rho)}+(1+\kappa\theta R^d)\exp(-\kappa\theta R^d).$$

Before proving this result, let us conclude the proof. Let $R\leqslant a_-r/2$. For $C\in\mathcal{C}_{\theta,r}$, with $C'=\frac{r}{R}C\in\mathcal{C}_{\theta,R}$, $f_{\eta\cap C}(0)$ is the independent sum of $f_{\eta\cap C'}(0)+f_{\eta\cap(C'\setminus C)}(0)$, hence the right hand member of (4.6) is bounded by, using Lemma 4.2 and hypothesis (4.3),

$$\sup_{v\in\mathbb{R},C\in\mathcal{C}_{\theta,R}}\mathbf{E}[\mathbf{P}(f_{\eta\cap C}(0)\in[v-\delta_{r,0},v+\delta_{r,0}])\;\delta_{r,0}]\leqslant\kappa\mathbf{E}\delta_{r,0}\exp(\kappa\theta R^{d-\varepsilon})+(1+\kappa\theta R^d)\exp(-\kappa\theta R^d).$$

With $R=[|\ln(r)|^{\frac{1}{d-\varepsilon/2}}\wedge(a_-r/2)]\vee 1$, $\exp(\kappa\theta R^{d-\varepsilon})<<(1+r)^\gamma<<\exp(\kappa\theta R^d)$ for $\gamma>0$ as $r\to\infty$, and we have for any $\gamma,\gamma'>0$

$$\begin{aligned}
\mathbf{E}\Big|F_0(\eta'\cap B)-F_0(\eta'\cap B_r\cap B)\Big|^4\leqslant&C\left[(1+r)^\gamma\int_{a_-(r-\sqrt{d})}^{\infty}g(\rho)\rho^{d-1}d\rho+(1+R^d)(1+r)^{-\gamma'}\right]\\
\leqslant&C'(1+r)^{-(\beta'-d)}.
\end{aligned}$$

for some $9d<\beta'<\beta$ (resp. $11d<\beta'<\beta$), hence (1.8) (resp. (1.11)) holds with $\alpha=(\beta'-d)/4$.

Proof of Lemma 4.2. Let $\lambda=\frac{\sigma_{d-1}}{\theta\kappa_d}, n_R=|\eta\cap C|$ be the number of germs (Poisson variable with parameter $\ell(C)=R^d/\lambda$), and let $g_R(x)=g(\|x\|)\mathbf{1}_{\{x\in C\}}$, so that $f_{\eta\cap C}(0)=\sum_{i=1}^{n_R}g_R(X_i)$ where the X_i are uniform iid in C. Call μ_R the distribution of the $g_R(X_i)$. We have for every $g(R)\leqslant a<b$, since g is one-to-one,

$$\begin{aligned}
\mu_R([a,b))=&\frac{\lambda}{R^d}\int_C\mathbf{1}_{\{a\leqslant g(\|x\|)<b\}}dx=\frac{\lambda}{R^d}\int_{g^{-1}(b)}^{g^{-1}(a)}\theta\rho^{d-1}d\rho\\
=&\frac{\sigma_{d-1}}{d\kappa_d R^d}(g^{-1}(a)^d-g^{-1}(b)^d),
\end{aligned}$$

whence μ_R has density $\varphi_R(a) = \mathbf{1}_{\{a\geqslant g(R)\}}\frac{\sigma_{d-1}}{\kappa_d R^d}\left(\frac{g^{-1}(a)^{d-1}}{g'(g^{-1}(a))}\right)$. Then, noting $\varphi_R^{\otimes n}$ the density φ_R convoluted with itself n times on the real line,

$$\begin{aligned}\mathbf{P}(f_{\eta\cap C}(0)\in[v-\delta,v+\delta]) \leqslant &\sum_{n=0}^{\infty}\mathbf{P}(n_R=n)\mathbf{P}\left(\sum_{i=1}^{n}g_R(X_i)\in[v-\delta,v+\delta]\right)\\ \leqslant&(1+R^d/\lambda)\exp(-R^d/\lambda)+\sum_{n\geqslant 2}\mathbf{P}(n_R=n)\|\varphi_R^{\otimes n}\|_\infty 2\delta\\ \leqslant&\kappa(1+R^d/\lambda)\exp(-R^d/\lambda)+2\sup_{n\geqslant 2}\|\varphi_R^{\otimes n}\|_\infty\delta. \end{aligned}\tag{4.7}$$

Due to convolution properties, for $n\geqslant 2$,

$$\begin{aligned}\|\varphi_R^{\otimes n}\|_\infty \leqslant&\|\varphi_R^{\otimes 2}\|_\infty\leqslant\int_{\mathbb{R}}\varphi_R^2(a)da=\mathbf{E}\varphi_R(g_R(X_1))\\ =&\frac{\lambda}{R^d}\int_C\varphi_R(g(\|x\|))dx\\ \leqslant&\frac{\lambda}{R^d}\int_C\frac{g^{-1}(g(\|x\|))^{d-1}}{g'(g^{-1}(g(\|x\|)))}\frac{\sigma_{d-1}dx}{\kappa_d R^d}\\ =&\left(\frac{\sigma_{d-1}}{\kappa_d R^d}\right)^2\int_0^R\frac{1}{g'(\rho)}\rho^{2(d-1)}d\rho\\ \leqslant&\left(\frac{\sigma_{d-1}}{\kappa_d}\right)^2\left(\frac{1}{R^{2d}}\int_0^1\frac{\rho^{2(d-1)}}{g'(\rho)}d\rho+\int_1^R\frac{\rho^{-2}d\rho}{g'(\rho)}\right),\end{aligned}$$

which concludes the claim after reporting in (4.7). □

□

Remarks 4.1. In the case $u<0$, the results above still holds except for Assumption 1.9. One needs to prove that $\{f(\cdot,\eta^\rho)\geqslant u\}\neq\mathbb{R}^d$ with positive probability. Assume that the $-g_m$ satisfy (4.5) (instead of the g_m satisfying it). Then by Lemma 4.1

$$\mathbf{P}(\ell(\{-f\geqslant -u\})>0)>0$$

which yields the same result as for $u\geqslant 0$.

4.2 Perimeter

We use in this section the variational definition of perimeter, following Ambrosio, Fusco and Pallara [1]. Define the *perimeter* of a measurable set $A\subset\mathbb{R}^d$ within $U\subset\mathbb{R}^d$ as the total variation of its indicator function

$$\mathrm{Per}(A;U):=\sup_{\varphi\in\mathcal{C}_c^1(U,\mathbb{R}^d):\|\varphi\|\leqslant 1}\int_{\mathbb{R}^d}\mathbf{1}_A(x)\mathrm{div}\varphi(x)dx,$$

where $\mathcal{C}_c^1(U,\mathbb{R}^d)$ is the set of continuously differentiable functions with compact support in U. Note that for regular sets, such as $\mathcal{C}^1$ manifolds, or convex sets with non-empty interior, this notion meets the classical notion of $(d-1)$-dimensional Hausdorff surface measure [1, Exercise 3.10], even though the term *perimeter* is traditionally used for 2-dimensional objects. It is a possibly infinite quantity, that might also have counterintuitive features for pathological sets ([1, Example 3.53]). The main difference with the traditional perimeter is that the variational one obviously cannot detect the points of the boundary whose neighborhoods don't charge the volume of the set, such as in line segments for instance.

For any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ and level $u \in \mathbb{R}$, the perimeter of the excursion $\mathrm{Per}(\{f \geqslant u\}; U)$ within U is a well-defined quantity. To be able to compute it efficiently, we must make additional assumptions on f's regularity. Following [8], we assume that f belongs to the space $BV(U)$ of functions with bounded variations, i.e. $f \in L^1(U)$ and its variation above U is finite:

$$V(f, U) := \sup_{\varphi \in \mathcal{C}_c^1(U, \mathbb{R}^d) : \|\varphi\| \leqslant 1} \int_U f(x) \mathrm{div}\varphi(x) dx < \infty.$$

The original (equivalent) definition states that $f \in L^1(U)$ is in $BV(U)$ if and only if the following holds ([1, Proposition 3.6]): there exists signed Radon measures $D_i f$ on $U, 1 \leqslant i \leqslant d$, called *directional derivatives* of f, such that for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\int_U f(x) \mathrm{div}\varphi(x) dx = -\sum_{i=1}^d \varphi_i(x) D_i f(dx).$$

Then there is a finite Radon measure $\|Df\|$ on U, called *total variation measure*, and a $\mathcal{S}^{d-1}$-valued function $\nu_f(x), x \in U$, such that $Df = \sum_i D_i f = \|Df\| \nu_f$. According to the Radon-Nikodym theorem, the total variation can be decomposed as

$$\|Df\| = \nabla f \ell + D^j f + D^c f \tag{4.8}$$

where ∇f is defined as the density of the continuous part of $\|Df\|$ with respect to ℓ, $D^c f + D^j f$ is the singular part of $\|Df\|$ with respect to Lebesgue measure, decomposed in the *Cantor part* $D^c f$, and the jump part $D^j f$, that we precise below, following [1, Section 3.7].

For $x \in U$,denote by H_x the affine hyperplane containing x with outer normal vector $\nu_f(x)$. For $r > 0$, denote by $B^+(x, r)$ and $B^-(x, r)$ the two components of $B(x, r) \setminus H_x$, with $\nu_f(x)$ pointing towards $B^+(x, r)$. Say that x is a *regular point* if there are two values $f^+(x) \geqslant f^-(x)$ such that

$$\lim_{r \to 0} r^{-d} \int_{B^+(x,r)} |f^+(x) - f(y)| dy = \lim_{r \to 0} r^{-d} \int_{B^-(x,r)} |f(y) - f^-(x)| dy = 0. \tag{4.9}$$

It turns out that the set of non-regular points has $\mathcal{H}^{d-1}$-measure 0, and the set J_f of points where $f^+(x) > f^-(x)$, called *jump points*, has Lebesgue measure 0. Then, the jump measure of f is represented by

$$D^j f(dx) = \mathbf{1}_{\{x \in J_f\}} (f^+(x) - f^-(x)) \mathcal{H}^{d-1}(dx),$$

where $\mathcal{H}^{d-1}$ stands for the $(d-1)$-dimensional Hausdorff measure.

In the classical case where f is continuously differentiable on U, $Df = \nabla f \ell$, $\nu_f(x) = \|\nabla f(x)\|^{-1} \nabla f(x)$ (and takes an irrelevant arbitrary value if $\nabla f(x) = 0$), and $V(f; U) = \int_U \|\nabla f(x)\| dx$. If $f = \mathbf{1}_{\{A\}}$ for some $\mathcal{C}^1$ compact manifold A, $\nu_f(x)$ is the outer normal to A for $x \in \partial A$, $\nabla f = 0, D^c f = 0$, and $D^j f = \mathbf{1}_{\{\partial A\}} \mathcal{H}^{d-1}$.

Denote by $SBV(U)$ the functions $f \in BV(U)$ such that $D^c f = 0$. Assume here that for $m \in \mathbf{M}, g_m \in SBV(\mathbb{R}^d)$, and that

$$\int_{\mathbf{M}} \left[\int_{\mathbb{R}^d} (|g_m(t)| + \|\nabla g_m(t)\|) d\mu(m) dt + \int_{J_{g_m}} |g_m^+(t) - g_m^-(t)| \mathcal{H}^{d-1}(dt) \right] < \infty.$$

It follows by [8, Theorem2] that for $\zeta \in \mathscr{N}$, $f_\zeta \in SBV(U)$, and for every bounded U, its gradient density defined by (4.8) is a vector-valued shot-noise field, defined a.s. and ℓ-a.e. by

$$\nabla f_\zeta(t) = \sum_{(x,m) \in \zeta} \nabla g_m(t - x),$$

its jump set J_f is the union of the translates of the kernel jump sets: $J_f = \cup_{(x,m)\in\zeta}(x + J_{g_m})$, and the jumps of f are

$$f_\zeta^+(y) - f_\zeta^-(y) = \sum_{(x,m)\in\zeta} \mathbf{1}_{\{y\in x+J_{g_m}\}}(g_m^+(y-x) - g_m^-(y-x)), y \in J_f.$$

Let h be a *test function*, i.e. a function $h : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^1$ with compact support. Let H be a primitive function of h. Biermé and Desolneux [8, Theorem 1] give for $W \subset \mathbb{Z}^d, \zeta \in \mathscr{N}$,

$$F_W^{h,Per}(\zeta) := \int_{\mathbb{R}} h(u)\mathrm{Per}(\{f_\zeta \geqslant u\}; \tilde{W})du = F_{\tilde{W}}^{h,cont}(\zeta) + F_{\tilde{W}}^{h,jump}(\zeta),$$

where

$$\begin{aligned} F_W^{h,cont}(\zeta) &= \int_{\tilde{W}} h(f_\zeta(x))\|\nabla f_\zeta(x)\|dx, \\ F_W^{h,jump}(\zeta) &= \int_{J_f\cap\tilde{W}} (H(f_\zeta^+(x)) - H(f_\zeta^-(x)))\mathcal{H}^{d-1}(dx). \end{aligned}$$

Their expectations under η are computed in [8, Section 3] :

$$\begin{aligned} \mathbf{E}F_W^{h,cont}(\eta) &= \ell(\tilde{W})\mathbf{E}\left[h(f_\eta(0))\|\nabla f_\eta(0)\|\right] \\ \mathbf{E}F_W^{h,jump}(\eta) &= \ell(\tilde{W})\int_{\mathbf{M}}\int_{J_{g_m}}\int_{g_m^-(y)}^{g_m^+(y)} \mathbf{E}h(s + f_\eta(0))ds\mathcal{H}^{d-1}(dy)\mu(dm). \end{aligned}$$

Let us now give their second order behaviour. It is difficult to give sharp necessary conditions with a general function h for non-degeneracy of the variance, but (1.9) is satisfied if for instance the mean $\mathbf{E}F_W^{h,Per}$ is not zero for some $W \subset \mathbb{Z}^d$, or in virtue of Lemma 4.1, if h has support in $\mathbb{R}_+$ and (4.5) holds, because a set with positive volume has positive perimeter.

Theorem 4.3. Let $\mathscr{W} = \{W_n; n \geqslant 1\}$ satisfying (1.5). Assume that for some $\alpha > 5d/2, c > 0$,

$$(\mathbf{E}|g_M(x)|^4)^{1/4} \leqslant c(1 + \|x\|)^{-d-\alpha}, \tag{4.10}$$

$$(\mathbf{E}\|\nabla g_M(x)\|^4)^{1/4} \leqslant c(1 + \|x\|)^{-d-\alpha}, \tag{4.11}$$

$$\left(\mathbf{E}\left[\int_{J_{g_M}\cap(x+[0,1]^d)} (1 \vee |g_M^+(t) - g_M^-(t)|)\mathcal{H}^{d-1}(dt)\right]^4\right)^{1/4} \leqslant c(1 + \|x\|)^{-d-\alpha} \tag{4.12}$$

and (1.9) holds (see above). Then the conclusions of Theorems 1.1,1.2,2.1,3.1 hold for $F_W^{h,\mathrm{Per}}$.

Example 4.1. Assume $\mathbf{M} = \mathbb{R}$ is endowed with a probability measure μ with finite 4-th moment. Let f be a function of the form

$$f_\zeta(x) = \sum_{(y,m)\in\zeta} mg(x - y)$$

with $g \in SBV(\mathbb{R})$. Conditions (4.10) (resp. (4.11)) holds if $g(r) \leqslant C(1 + r)^{-d-\alpha}, r \geqslant 0$ (resp. $|g'(r)| \leqslant C(1+r)^{-d-\alpha}, r > 0$). Then (4.12) holds if J_g is finitely countable and for some $C > 0$, for every $r > 0$

$$\sum_{x\in J_g\cap[r,r+1]} (1 \vee |g^+(t) - g^-(t)|)dt \leqslant C(1 + r)^{-d-\alpha}.$$

Proof. First, (4.10)-(4.11) imply that the shot noise process and its gradient measure are a.s. well defined. The functionals $F_W^{h,cont}, F_W^{h,jump}$ are under the form (1.1)-(1.2), with F_0 defined respectively by, for $\zeta \in \mathscr{N}$,

$$F_0^{h,cont}(\zeta) = \int_{[0,1)^d} h(f_\zeta(t))\|\nabla f_\zeta(t)\| dt$$
$$F_0^{h,jump}(\zeta) = \int_{J_{f_\zeta}\cap[0,1)^d} (H(f_\zeta^+(t)) - H(f_\zeta^-(t)))\mathcal{H}^{d-1}(t),$$

where H is a primitive function of h.

Let $\mathbf{x}_i = (x_i, m_i) \in \mathbb{R}^d, i = 1, \dots, 6$. Let $r > 0, B \in \mathcal{B}_{\mathscr{W}}^r, \zeta \subset \{\mathbf{x}_1, \mathbf{x}_2\}$, $\eta' = \eta \cup \zeta$. Then

$$\begin{aligned}\left|F_0^{h,cont}(\eta'\cap B_r\cap B) - F_0^{h,cont}(\eta'\cap B)\right| &\leqslant \int_{[0,1]^d} \|h'\|_\infty |f_{\eta'\cap B}(t) - f_{\eta'\cap B\cap B_r}(t)|\|\nabla f_{\eta'\cap B}(t)\| dt \\ &+ \int_{[0,1]^d} \|h\|_\infty \|\nabla f_{\eta'\cap B_r\cap B}(t) - \nabla f_{\eta'\cap B}(t)\| dt \\ &\leqslant \sum_{(x,m)\in\eta\setminus B_r} \int_{[0,1]^d} [\|h'\|_\infty \|\nabla f_{\eta'\cap B}(t)\||g_m(x-t)| + \|h\|_\infty \|\nabla g_m(x-t)\|]\, dt. \end{aligned} \tag{4.13}$$

Define for $\zeta_0 \in \mathscr{N}$, $\mathbf{x} = (x, m) \in \overline{\mathbb{R}^d}$,

$$\psi(\mathbf{x}, \zeta_0) = \int_{[0,1]^d} \left[\|h'\|_\infty \|\nabla f_{(\zeta_0\cup\zeta)\cap B}(t)\||g_m(x-t)| + \|h\|_\infty \|\nabla g_m(x-t)\|\right] dt.$$

For $\zeta' \subset \{\mathbf{x}_i, 3 \leqslant i \leqslant 6\}$, Jensen's inequality yields for $\mathbf{x} = (x, m) \in \overline{\mathbb{R}^d}$

$$\mathbf{E}\psi(\mathbf{x}, \eta\cup\zeta')^4 \leqslant C \int_{[0,1]^d} \mathbf{E}\left[|g_m(x-t)|^4 \mathbf{E}\|\nabla f(t, (\eta'\cup\zeta')\cap B)\|^4 + \mathbf{E}\|\nabla g_m(x-t)\|^4\right] dt.$$

An easy application of Lemma 5.1 with $\psi'(x, m) = \|\nabla g_m(x-t)\|, r = 0$ yields that $\mathbf{E}\nabla f_{(\eta'\cup\zeta')\cap B}(t)^4 \leqslant c < \infty$ where c does not depend on $t \in \mathbb{R}^d, B \in \cup_{r\geqslant 0}\mathcal{B}_{\mathscr{W}}^r$, or the $\mathbf{x}_i$. Therefore, Assumptions (4.10) and (4.11) yield for $\mathbf{x} = (x, m) \in \overline{\mathbb{R}^d}$

$$\mathbf{E}\psi(\mathbf{x}, \eta\cup\zeta')^4 \leqslant C(1 + \|x\|)^{-4(\alpha+d)},$$

and Lemma 5.1 with (4.13) yields that (1.8) is satisfied by $F_0^{h,cont}$.

Let us now prove that it is satisfied by the jump functional $F_0^{h,jump}$. Since it has to hold only for ℓ-a.e. x_1, x_2, and the J_{g_1}, J_{g_2} have finite $\mathcal{H}^{d-1}$ measure, we assume that $J_{g_{m_1}}(\cdot - x_1)$ and $J_{g_{m_2}}(\cdot - x_2)$ have a $\mathcal{H}^{d-1}-$negligible intersection. They also a.s. have a $\mathcal{H}^{d-1}$-negligible intersection with each $J_{g_m}(\cdot - x)$, $(x,m) \in \eta$. Call $f_1 = f_{\eta' \cap B}, f_2 = f_{\eta' \cap B_r \cap B}$,

$$\begin{aligned}
&\left| F_0^{jump}(\eta' \cap B) - F_0^{jump}(\eta' \cap B \cap B_r) \right| \\
=&\left| \sum_{(x,m)\in\eta'\cap B_r\cap B} \int_{J_{g_m}\cap[0,1]^d} \left[(H(f_1^+(t)) - H(f_1^-(t))) - (H(f_2^+(t)) - H(f_2^-)(t)) \right] \mathcal{H}^{d-1}(dt) \right. \\
&\left. + \sum_{(x,m)\in\eta'\cap B\setminus B_r} \int_{J_{g_m}\cap[0,1]^d} \left[H(f_1^+(t)) - H(f_1^-(t)) \right] \mathcal{H}^{d-1}(dt) \right| \\
\leqslant& \int_{J_{f_2}\cap[0,1]^d} \|h\| (\left| f_2^+(t) - f_1^+(t) \right| + \left| f_2^-(t) - f_1^-(t) \right|) \mathcal{H}^{d-1}(dt) \\
&+ \sum_{(x,m)\in\eta'\setminus B_r} \int_{[0,1]^d\cap J_{g_m}} \|h\| \left| g_m^+(x-t) - g_m^-(x-t) \right| \mathcal{H}^{d-1}(dt) \\
\leqslant& \sum_{(x,m)\in\eta\setminus B_r} \|h\| \left(\underbrace{2\int_{J_{f_2}\cap[0,1]^d} |g_m(x-t)| \mathcal{H}^{d-1}(dt)}_{=:\psi_1((x,m),\eta)} + \underbrace{\int_{[0,1]^d\cap J_{g_m}} |g_m^+(x-t) - g_m^-(x-t)| \mathcal{H}^{d-1}(dt)}_{=:\psi_2(x,m)} \right).
\end{aligned}$$

We have $\mathbf{E}\psi_2(x,M_0)^4 \leqslant C(1+\|x\|)^{-4(\alpha+d)}$ by (4.12), and Jensen's inequality yields for $\zeta' \subset \{\mathbf{x}_3,\dots,\mathbf{x}_6\}$, $f_3 = f_{(\eta\cup\zeta'\cup\zeta)\cap B_r\cap B}$, after expanding the 4-th power of the integral as a quadruple integral,

$$\begin{aligned}
\mathbf{E}\psi_1((x,M_0),\eta\cup\zeta')^4 =& \mathbf{E}\mathbf{E}\left[\left(\int_{J_{f_3}\cap[0,1]^d} |g_{M_0}(x-t)| \mathcal{H}^{d-1}(dt) \right)^4 \middle| \eta,\zeta,\zeta' \right] \\
\leqslant& \mathbf{E}\left(\left(\int_{J_{f_3}\cap[0,1]^d} (\mathbf{E} g_{M_0}(x-t)^4)^{1/4} \mathcal{H}^{d-1}(dt) \right)^4 \middle| \eta,\zeta,\zeta' \right) \\
\leqslant& C(1+\|x\|)^{-4(d+\alpha)} \mathbf{E}\mathcal{H}^{d-1}(J_{f_3}\cap[0,1]^d)^4
\end{aligned}$$

by Assumption (4.10). Then (4.12) yields $\mathbf{E}\mathcal{H}(J_{f_1}\cap[0,1]^d)^4 < \infty$ with an application of Lemma 5.1, whence Lemma 5.1 again yields that $F_0^{h,jump}$ also satisfies (1.8). Hence $F_0 := F_0^{h,cont} + F_0^{h,Per}$ satisfies (1.8), which yields the conclusion. □

4.3 Fixed level perimeter and Euler characteristic

Let $\mathcal{B}$ be a measurable subset of $\mathcal{M}$, and let the marks space be $\mathbf{M} = \mathbb{R}^* \times \mathcal{B}$, endowed with the product σ-algebra. This section is restricted to shot-noise fields of the form

$$f_\xi(x) = \sum_{(y,(L,A))\in\xi} L\mathbf{1}_{\{x-y\in A\}}, \xi \subset \mathbb{R}^d \times \mathbf{M}, x \in \mathbb{R}^d. \tag{4.14}$$

Such fields are used in image analysis [8, 7], or in mathematical morphology [16], sometimes with $L_1 = const.$ a.s, and their marginals might not have a density. The article [5] uses the asymptotic normality result below for the Euler characteristic when the A_i are random discs (Example 4.4).

The current framework allows to give general results for a fixed level $u \in \mathbb{R}$, for a large class of additive functionals, including the perimeter or the Euler characteristic. For the latter, the main difficulty is to properly define it on a typical excursion of the shot noise field, as it is obtained by locally adding and removing sets from $\mathcal{B}$. The general result only involves the marginal distribution $\mu_{\mathcal{B}}(\cdot) = \mu(\mathbb{R}^* \times \cdot)$.

We call $\mathcal{B}'$ the class of excursion sets generated by shot noise fields of the form (4.14) where all but finitely many points of ξ in general position have been removed. Formally, given a measurable subclass $\mathcal{B}' \subset \mathcal{M}$, a function $V : \mathcal{B}' \to \mathbb{R}$ such that $V(A)$ only depends on $A \cap [0,1)^d$, and a function $\tilde{V} : \mathcal{B} \to \mathbb{R}_+$, say that $(\mathcal{B}, \mathcal{B}', V, \tilde{V})$ is *admissible* if for $A_1, \dots, A_q \in \mathcal{B}$, for a.a. $y_1, \dots, y_q \in \mathbb{R}^d$, any set A obtained by sequentially removing, adding or intersecting the $A_i + y_i, i = 1, \dots, q$, belongs to $\mathcal{B}'$, and $|V(A)| \leqslant \sum_{i=1}^q \tilde{V}(A_i)$.

Theorem 4.4. Let $u \in \mathbb{R}$, $(\mathcal{B}, \mathcal{B}', V, \tilde{V})$ be an admissible quadruple, let f be of the form (4.14), and let $\mathscr{W} = \{W_n; n \geqslant 1\}$ be a sequence of subsets of $\mathbb{Z}^d$ satisfying (1.5). Let μ be a probability measure on $\mathbb{R}^* \times \mathcal{B}$. Assume that $\int_{\mathcal{B}} \tilde{V}(A)^4 \mu_{\mathcal{B}}(dA) < \infty$ and that for some $\beta > 14d, C > 0$,

$$\int_{\mathcal{B}} \mathbf{1}_{\{(x+A)\cap[0,1)^d \neq \emptyset\}} \mu_{\mathcal{B}}(dA) \leqslant C(1 + \|x\|)^{-\beta}, x \in \mathbb{R}^d. \tag{4.15}$$

Let $F_0(\xi) = V(\{f_\xi \leqslant u\})$. Then if the non-triviality assumption (1.9) holds, all conclusions of Theorems 1.1 and 1.2 hold.

Remark that this result cannot be formally deduced from stabilisation results, as condition (4.15) does not imply that the elements of $\mathcal{B}$ are bounded.

Proof. In this proof, $\mathscr{N}_0$ is chosen to be the class of ξ such that for all bounded set D, $\xi_D := \{(y,(L,A)) \in \xi : (y + A) \cap D \neq \emptyset\}$ is finite. Assumption (4.15) implies that $\eta \in \mathscr{N}_0$ a.s. Let $\xi = (\eta \cup \zeta) \cap B$ with the notation of (1.8). Let $r \geqslant 0$. We have a.s.

$$\begin{aligned} |F_0(\xi) - F_0(\xi \setminus B_r)| &= \left|V(\{f_\xi \leqslant u\}) - V(\{f_{\xi\setminus B_r} \leqslant u\})\right| \\ &\leqslant \mathbf{1}_{\{\xi_{[0,1)^d} \cap (\xi \setminus B_r) \neq \emptyset\}} 2 \underbrace{\sum_{(y,(L,A)) \in \xi_{[0,1)^d}} \tilde{V}(A)}_{U_\xi} \leqslant \sum_{\mathbf{x} \in \xi \setminus B_r} \psi(\mathbf{x}, \xi) \end{aligned}$$

where $\psi((y,(L,A)),\xi) = \mathbf{1}_{\{(y+A)\cap[0,1)^d \neq \emptyset\}} U_\xi$. $\mathbf{E} U_\xi^4 < \infty$ easily follows from Lemma 5.1. Using Assumption (4.15) and Lemma 5.1 again yields that (1.8) and (1.11) hold. $\square$

Example 4.2. The simplest example is the class of measurable subsets of $\mathbb{R}^d$ $\mathcal{B} = \mathcal{B}' = \mathcal{M}$, endowed with Lebesgue measure $V(A) = \ell(A \cap \tilde{[}0,1)^d), \tilde{V}(A) = \ell(A)$. We indeed have $F_W(\xi) := \ell(\{f_\xi \leqslant u\} \cap \tilde{W}) = \sum_{k \in W} V(\{f_\xi \leqslant u\} - k)$. It can be derived from Lemma 4.1 that the non-triviality condition (1.9) is satisfied if for instance

$$\mu(\{(L,A) \in \mathbf{M} : (\mathbf{1}_{\{u=0\}} + u)L\ell(A) > 0\}) \neq 0. \tag{4.16}$$

In the following, let $\Gamma_k = k + [0,1)^d, k \in \mathbb{Z}^d$.

Example 4.3 (Perimeter)**.** Let $\mathcal{B}$ be the class of $A \in \mathcal{M}$ such that $\mathcal{H}^{d-1}(\partial A) < \infty$. Let $\mathcal{B}'$ be the class of $A \in \mathcal{B}$ such that $\mathcal{H}^{d-1}(\partial A \cap \partial \Gamma_k) = 0$ for $k \in \mathbb{Z}^d$. For $A \in \mathcal{B}$, for a.a. $y \in \mathbb{R}^d$, $\mathcal{H}^{d-1}(\partial(A + y) \cap \partial \Gamma_k) = 0$. Hence for $A_1, \dots, A_q \in \mathcal{B}$, for a.a. $y_1, \dots, y_q \in \mathbb{R}^d$, any set A obtained by sequentially adding, intersecting or removing the $A_i + y_i$ is in $\mathcal{B}'$, using $\partial A \subset \cup_{i=1}^n (\partial A_i + y_i)$. Defining $V(A) := \mathcal{H}^{d-1}(\partial A \cap [0,1)^d), \tilde{V}(A) := \mathcal{H}^{d-1}(\partial A)$ yields an admissible quadruple $(\mathcal{B}, \mathcal{B}', V, \tilde{V})$. Non-triviality is for instance implied by (4.16). Hence the previous result describes the asymptotic behaviour of the functionals $F'_W(\eta) = \mathrm{Per}(\{f_\eta \leqslant u\} \cap \tilde{W}) = \sum_{k \in W} V(\{f_{\eta - k} \leqslant u\})$ and $F_W(\eta) = \mathrm{Per}(\{f_{\eta \cap \tilde{W}} \leqslant u\} \cap \tilde{W})$.

Example 4.4 (Euler characteristic for indicators of discs)**.** Let $d = 2$, $\mathcal{B}$ be the class of discs of $\mathbb{R}^2$, and $\mathcal{B}'$ be the class of sets obtained by finite unions, intersections and removals of discs $A_1, \dots, A_q$ such that for $i \neq j$, A_i and A_j are not tangent and $\partial A_i \cap \partial A_j \cap \partial \Gamma_k = \emptyset$ for

$k \in \mathbb{Z}^d$. The set A is an *elementary set* in the terminology of Biermé & Desolneux [7]: ∂A can be decomposed as a finite union of $\mathcal{C}^2$ open curves $C_j, j = 1, \ldots, p$ with respective constant curvatures $\kappa_j > 0$, separated by corners $x_i \in \partial A, i = 1, \ldots, q$, (with $0 \leqslant q \leqslant p$) with angle $\alpha(x_i, A) \in (-\pi, \pi)$. The total curvature of A within some open set U is defined by

$$TC(A; U) := \sum_{j=1}^{p} \kappa_j \mathcal{H}^1(C_j \cap U) + \sum_{i=1}^{q} \mathbf{1}_{\{x_i \in U\}} \alpha(x_i, A).$$

Therefore we define $V(A) = TC(A; (0,1)^d)$. For $A \in \mathcal{B}', TC(A; \text{int}(\tilde{W})) = \sum_{k \in W} V(A - k)$. Via the Gauss-Bonnet theorem, for $A \in \mathcal{B}'$, $TC(A; \text{int}(\tilde{W}))$ is strongly related to the Euler characteristic of $A \cap \tilde{W}$, in the sense that they coincide if $A \subset \text{int}(\tilde{W})$, and otherwise they only differ by boundary terms, see [7]. To obtain the second order behaviour of $F_W(\eta) = TC(\{f_{\eta \cap \tilde{W}} \leqslant u\}; \tilde{W})$ and $F'_W(\eta) = TC(\{f_\eta \leqslant u\}; \tilde{W})$, define $\tilde{V} \equiv 1$, and $(\mathcal{B}, \mathcal{B}', V, \tilde{V})$ is an admissible quadruple. Let $(L, D) \in \mathbb{R}^* \times \mathcal{B}$ be a random couple with the law μ of the typical mark, and let R be the radius of D. Then the volumic asymptotic variance and asymptotic normality hold (Theorems 1.1-1.2) if for some $\beta > 14d$, $\mathbb{P}(R \geqslant r) \leqslant C(1+r)^{-\beta}$, and $\mathbb{P}((\mathbf{1}_{\{u=0\}} + u)LR > 0) \neq 0$ (non-triviality assumption).

With a similar route, the previous example can likely be generalised to more general classes of sets $\mathcal{B}$ in higher dimensions, such as the polyconvex ring, provided one can estimate properly the curvature on sets from $\mathcal{B}'$.

5 Proofs

Recall that κ denotes a constant which depends on d, α, a_-, a_+ and which value might change from line to line. The following lemma is useful several times in the paper.

Lemma 5.1. Let $\alpha > d$, $M_i, 0 \leqslant i \leqslant 4$ independent marks with law μ. Let $r > 0, \psi : \overline{\mathbb{R}^d} \times \mathscr{N} \to \mathbb{R}_+$ a kernel such that for ℓ-a.e. $x_i \in \mathbb{R}^d, 0 \leqslant i \leqslant 4$, and $\zeta \subset \{(x_i, M_i), i = 1, \ldots, 4\}$

$$\left(\mathbf{E}\psi((x_0, M_0), \eta \cup \zeta)^4\right)^{1/4} \leqslant C_0(1 + \|x_0\|)^{-\alpha - d}.$$

Then

$$\left(\mathbf{E}\left|\sum_{\mathbf{x} \in \eta \setminus B_r} \psi(\mathbf{x}; \eta)\right|^4\right)^{1/4} \leqslant C_0 \kappa (1 + r)^{-\alpha}.$$

Proof. Let $\eta_r = \eta \setminus B_r$. Let $\mathbf{x}_i = (x_i, M_i)$. Let $\mathcal{P}_4$ be the family of ordered tuples of natural integers which sum is 4. The multi-variate Mecke formula yields

$$\begin{aligned}
&\mathbf{E}\left|\sum_{\mathbf{x}\in\eta_r}\psi(\mathbf{x};\eta)\right|^4 \leqslant 4!\mathbf{E}\sum_{\{\mathbf{x}_1,\dots,\mathbf{x}_4\}\subset\eta_r}\psi(\mathbf{x}_1;\eta)...\psi(\mathbf{x}_4;\eta)\\
\leqslant&4!\sum_{P=(m_1,\dots,m_q)\in\mathcal{P}_4}q!\mathbf{E}\sum_{(\mathbf{x}_1,\dots,\mathbf{x}_q)\in(\eta_r)^q_{\neq}}\psi(\mathbf{x}_1;\eta)^{m_1}...\psi(\mathbf{x}_q;\eta)^{m_q}\\
\leqslant&\kappa\sum_{P=(m_1,\dots,m_q)\in\mathcal{P}_4}\int_{(\overline{B_r^c})^q}\mathbf{E}\prod_{l=1}^q\psi(\mathbf{x}_l,\eta\cup\{\mathbf{x}_1,\dots,\mathbf{x}_q\})^{m_l}d\mathbf{x}_1\dots d\mathbf{x}_q\\
\leqslant&\kappa\sum_{P=(m_1,\dots,m_q)\in\mathcal{P}_4}\int_{(\overline{B_r^c})^q}\prod_{l=1}^q(\mathbf{E}\psi(\mathbf{x}_l,\eta\cup\{\mathbf{x}_1,\dots,\mathbf{x}_q\})^4)^{m_l/4}d\mathbf{x}_1\dots d\mathbf{x}_q\\
\leqslant&\kappa\sum_{P=(m_1,\dots,m_q)\in\mathcal{P}_4}\prod_{l=1}^q\kappa\int_{B_r^c}C_0^{m_l}(1+\|x_l\|)^{-m_l(\alpha+d)}dx_l\\
\leqslant&\kappa\sum_{P=(m_1,\dots,m_q)\in\mathcal{P}_4}C_0^4\prod_{l=1}^q\kappa\int_{a_-r}^\infty(1+t)^{-m_l(\alpha+d)}t^{d-1}dt\\
\leqslant&\kappa C_0^4\sum_{P=(m_1,\dots,m_q)\in\mathcal{P}_4}(1+r)^{-4(\alpha+d)+qd}\leqslant\kappa C_0^4(1+r)^{-4\alpha}.
\end{aligned}$$

□

5.1 proof of Theorem 2.1

We prove (2.3) under Assumption (2.1) (i.e. in case **(i)**). Remark first that (2.1) trivially holds also for $B\in\mathcal{B}_W^s\setminus\mathcal{B}_W^r, s<r$. Also, if (2.1) is satisfied, with $B=\mathbb{R}^d$ (2.2) is also satisfied. Assume without loss of generality $F_0(\emptyset)=0$, then (2.1) with $r=0$ yields

$$\begin{aligned}
m_2 :=& \sup_{k\in W}\mathbf{E}|F_k^W(\eta)|^2\\
=& \sup_{k\in W}\mathbf{E}|F_0((\eta\cap\tilde{W})-k)-F_0(((\eta\cap\tilde{W})-k)\cap B_0)|^2\leqslant\kappa C_0^2<\infty.
\end{aligned}$$

The following inequality is useful several times in the proof: given some square-integrable random variables $Y_i, Z_i, i=1,2$, and a σ-algebra $\mathcal{Z}$,

$$\begin{aligned}
&\mathbf{E}\left|\mathrm{Cov}(Y_1,Y_2|\mathcal{Z})-\mathrm{Cov}(Z_1,Z_2|\mathcal{Z})\right|\\
\leqslant&\mathbf{E}\left(\sqrt{2\mathbf{E}(Y_1^2|\mathcal{Z})}\sqrt{2\mathbf{E}((Z_2-Y_2)^2|\mathcal{Z})}+\sqrt{2\mathbf{E}(Z_2^2|\mathcal{Z})}\sqrt{2\mathbf{E}((Z_1-Y_1)^2|\mathcal{Z})}\right)\\
\leqslant&2\left(\sqrt{\mathbf{E}Y_1^2}\sqrt{\mathbf{E}(Z_2-Y_2)^2}+\sqrt{\mathbf{E}Z_2^2}\sqrt{\mathbf{E}(Z_1-Y_1)^2}\right). \qquad (5.1)
\end{aligned}$$

Recall that $B_r(k)=k+B_r$ for $k\in W, r\geqslant 0$. Let $k,j\in W, r=\|k-j\|/(3a_+)$, η',η'' independent copies of η, and

$$\begin{aligned}
\eta_k &= (\eta\cap B_r(k))\cup(\eta'\cap B_r(k)^c),\\
\eta_j &= (\eta\cap B_r(j))\cup(\eta''\cap B_r(j)^c),
\end{aligned}$$

which are processes distributed as η, independent since $B_r(k)\cap B_r(j)=\emptyset$. Since $\eta\cap B_r(k)=$

$\eta_k \cap B_r(k)$, (2.1) yields

$$
\begin{aligned}
F_k^W(\eta) - F_k^W(\eta_k) =& F_k^W(\eta) - F_k^W(\eta \cap B_r(k)) + F_k^W(\eta_k \cap B_r(k)) - F_k^W(\eta_k)\\
\mathbf{E}\left|F_k^W(\eta) - F_k^W(\eta_k)\right|^2 \leqslant& 2\big(\mathbf{E}\left|F_0((\eta-k)\cap(\tilde{W}-k)) - F_0((\eta-k)\cap(\tilde{W}-k)\cap B_r)\right|^2\\
&+\mathbf{E}\left|F_0((\eta_k-k)\cap(\tilde{W}-k)\cap B_r) - F_0((\eta_k-k)\cap(\tilde{W}-k))\right|^2\big)\\
\leqslant& \kappa C_0^2(1+r)^{-2\alpha},
\end{aligned}
$$

a similar bounds holds for F_j^W. Then, (5.1) yields

$$
\begin{aligned}
\left|\mathrm{Cov}(F_k^W(\eta), F_j^W(\eta)) - \underbrace{\mathrm{Cov}(F_k^W(\eta_k), F_j^W(\eta_j))}_{=0}\right| \leqslant& \kappa\sqrt{\mathbf{E}F_j^W(\eta)^2}\sqrt{\mathbf{E}\left|F_k^W(\eta) - F_k^W(\eta_k)\right|^2}\\
&+\kappa\sqrt{\mathbf{E}(F_k^W(\eta_k))^2}\sqrt{\mathbf{E}\left|F_j^W(\eta) - F_j^W(\eta_j)\right|^2}\\
\leqslant& \kappa\sqrt{m_2}\sqrt{C_0^2(1+r)^{-2\alpha}} \leqslant \kappa C_0^2(1+\|k-j\|)^{-\alpha}. \qquad (5.2)
\end{aligned}
$$

Hence (2.3) is proved in case **(i)**. If $G_k^W = F_k$ and (2.2) is assumed instead of (2.1) (case **(i')**), replacing W by $\mathbb{Z}^d$ in the computation yields the same bound for $\mathrm{Cov}(F_k, F_j)$. The finiteness of σ_0 follows from $\alpha > d$.

Let us now assume $|W| < \infty$ and show (2.4). Let $k \in W, r = d(k, \tilde{W}^c)/a_+$, so that $B_r \cap (\tilde{W} - k) = B_r$. We have if (2.1) holds

$$
\begin{aligned}
F_k^W - F_k =& F_0((\eta-k)\cap(\tilde{W}-k)) - F_0((\eta-k)\cap(\tilde{W}-k)\cap B_r) + F_0((\eta-k)\cap B_r) - F_0(\eta-k)\\
\mathbf{E}|F_k^W - F_k|^2 \leqslant& \kappa C_0^2(1+r)^{-2\alpha} \leqslant \kappa C_0^2(1+d(k,\tilde{W}^c))^{-2\alpha}.
\end{aligned}
$$

We hence have by (5.1), for $k, j \in W$, recalling also (5.2),

$$
\begin{aligned}
\left|\mathrm{Cov}(F_k^W, F_j^W) - \mathrm{Cov}(F_k, F_j)\right| \leqslant& \kappa C_0^2(1+\min(d(k,\tilde{W}^c), d(j,\tilde{W}^c)))^{-\alpha}\\
\leqslant& \kappa C_0^2(1+\max(\|k-j\|, \min(d(k,\tilde{W}^c), d(j,\tilde{W}^c))))^{-\alpha}. \qquad (5.3)
\end{aligned}
$$

Denote by $[x]$ the integer part of $x \in \mathbb{R}$. Let $d_W \in \mathbb{N}^*$, $W_m = \{k \in W : [d(k,\tilde{W}^c)] = m\}$ for $m \in \mathbb{N}, W_{bd} = \{k \in W : [d(k,\tilde{W}^c)] \leqslant d_W\} = \cup_{m=0}^{d_W} W_m, W_{int} = W \setminus W_{bd}$. We have, using (2.3) and (5.3),

$$
\begin{aligned}
|\mathrm{Var}(F_W) - \sigma_0^2|W|| =& \left|\sum_{k\in W, j\in W}\mathrm{Cov}(F_k^W, F_j^W) - \sum_{k\in W, j\in\mathbb{Z}^d}\mathrm{Cov}(F_k, F_j)\right|\\
\leqslant& \sum_{k\in W, j\notin W}\mathrm{Cov}(F_k, F_j) + 2\sum_{k,j\in W: d(k,\tilde{W}^c)\leqslant d(j,\tilde{W}^c)}\left|\mathrm{Cov}(F_k^W, F_j^W) - \mathrm{Cov}(F_k, F_j)\right|\\
\leqslant& \sum_{m=0}^{\infty}\sum_{k\in W_m}\left[\sum_{j\in W^c}\kappa C_0^2(1+\|k-j\|)^{-\alpha} + 2\sum_{j\in W}\kappa C_0^2(1+\max(\|k-j\|, m))^{-\alpha}\right]\\
\leqslant& \kappa C_0^2\sum_{m=0}^{\infty}\sum_{k\in W_m}\left(3\sum_{j\in B(k,m)^c}(1+\|k-j\|)^{-\alpha} + 2\sum_{j\in B(k,m)}(1+m)^{-\alpha}\right)\\
\leqslant& \kappa C_0^2\sum_{m=0}^{\infty}\sum_{k\in W_m}\left(3\kappa(1+m)^{-\alpha+d} + 2\kappa m^d(1+m)^{-\alpha}\right)\\
\leqslant& \kappa C_0^2\left(|W_{bd}| + d_W^{-\alpha+d}|W_{int}|\right)
\end{aligned}
$$

hence

$$\left|\frac{\mathrm{Var}(F_W)}{|W|} - \sigma_0^2\right| \leqslant \kappa C_0^2 \left(d_W^d \frac{|\partial_{\mathbb{Z}^d} W|}{|W|} + d_W^{-\alpha+d}\right).$$

Equation (2.4) follows by taking $d_W = [(|W|/|\partial_{\mathbb{Z}^d} W|)^{\frac{1}{\alpha}}]$. The same computation where F_k^W is replaced by F_k (hence with no second term on the second line), treats the case **(i')**, without requiring (2.1).

Let us now prove that (1.9) implies $\sigma_0 > 0$. Since (2.1) implies (2.2), assume without loss of generality that we are in case **(i')** and (2.2) holds, hence (2.4) holds. We recall and set here some notation and results required for the proofs. Recall that for $a > 0$, $W_a = [-a/2, a/2]^d \cap \mathbb{Z}^d$, and let $W_a(k) = k + W_a, k \in \mathbb{Z}^d$. For $b \geqslant a, \zeta \in \mathscr{N}$, let furthermore $\zeta_a^b = \zeta \cap (W_b \setminus W_a)$, $\zeta^b = \zeta_0^b$ and $\zeta_a = \zeta \cap W_a$. Let $\rho > 1$ be such that $A \subset \tilde{W}_\rho$. Introduce $\beta = 2\rho, \gamma = \beta + (\beta+\rho)\sqrt{d}a_+/2a_-, \delta = \tau\gamma$, where $\tau > 1$ will be set later. Let $v = \frac{1}{2}\limsup_{a>0} \mathrm{Var}(F_{W_a}(\eta \cap A)) > 0$, using (1.9). Increase ρ until $\mathrm{Var}(F_{W_\beta}(\eta \cap A)) \geqslant v$.

Put $A_{ext} = \mathbb{R}^d \setminus (\cup_{l \in \mathbb{Z}^d} \tilde{W}_\gamma(l\delta))$, and define $\eta_{ext} = \eta \cap A_{ext}$. Remark that $\eta_{ext} \subset \eta^\gamma$. Let $W \subset \mathbb{Z}^d$. Call $W_{int}^\delta = \{l \in \mathbb{Z}^d : W_\delta(l\delta) \subset W\}$, $W_{bd}^\delta = \{l \in \mathbb{Z}^d \setminus W_{int}^\delta : W_\delta(l\delta) \cap W \neq \emptyset\}$, $W^\delta = W_{int}^\delta \cup W_{bd}^\delta, W_l^\delta = W \cap W_\delta(l\delta)$. We decompose, for $\zeta \in \mathscr{N}$,

$$F_W(\zeta) = \sum_{l \in \mathbb{Z}^d} F_W^{(l)}(\zeta)$$

where $F_W^{(l)} = \sum_{k \in W_l^\delta} F_k$. Let $F_{\mathbb{Z}^d}^{(l)} = F^{(l)}$, and remark that if $l \in W_{int}^\delta, F_W^{(l)} = F^{(l)} \overset{(d)}{=} F^{(0)}$, whereas $l \notin W^\delta$ implies $F^{(l)} = 0$. Denote by $\mathbf{P}_{ext}, \mathrm{Var}_{ext}, \mathrm{Cov}_{ext}$ the conditional probability, variance and covariance with respect to η_{ext}. We have

$$\begin{aligned}
\mathrm{Var}(F_W(\eta)) \geqslant & \mathbf{E}\mathrm{Var}_{ext}(F_W(\eta)) \\
= & \sum_{l \in W^\delta} \mathbf{E}\mathrm{Var}_{ext}(F_W^{(l)}) + \sum_{l,m \in W^\delta}^{\neq} \mathbf{E}\mathrm{Cov}_{ext}(F_W^{(m)}, F_W^{(l)}) \\
\geqslant & \sum_{l \in W_{int}^\delta} \mathbf{E}\mathrm{Var}_{ext}(F^{(0)}) - \sum_{l \in W_{bd}^\delta} \mathbf{E}\mathrm{Var}_{ext}(F_W^{(l)}) - \sum_{m,l \in W^\delta}^{\neq} \mathbf{E}\left|\mathrm{Cov}_{ext}(F_W^{(m)}, F_W^{(l)})\right|.
\end{aligned} \tag{5.4}$$

First, $\mathbf{E}(F_W^{(l)})^2 \leqslant \kappa m_2 \delta^{2d}$ for $l \in \mathbb{Z}^d$. Let us prove that up to increasing ρ, $\mathbf{E}\mathrm{Var}_{ext}(F^{(0)}) > 0$. Let $\Omega = \mathbf{1}_{\{\eta^\gamma \setminus A = \emptyset\}}$. Since $\eta_{ext} \in \sigma(\eta_\gamma)$, we have $\mathbf{E}\mathrm{Var}_{ext}(F^{(0)}) \geqslant \mathbf{E}\mathrm{Var}(F^{(0)}|\Omega, \eta_\gamma) \geqslant \mathbf{P}(\Omega = 1)\mathbf{E}\mathrm{Var}(F^{(0)}|\Omega = 1, \eta_\gamma)$ because Ω is independent of η_γ. Recall that $B_r(x) = x + B_r$ for $x \in \mathbb{R}^d, r > 0$. Define

$$\begin{aligned}
\delta_1 &= \sum_{k \in W_\beta} (F_k(\eta) - F_k(\eta \cap B_{(\gamma-\beta)/a_+}(k))) \\
\delta_2 &= \sum_{k \in W_\delta \setminus W_\beta} (F_k(\eta) - F_k(\eta \cap B_{d(k,A)/a_+}(k))).
\end{aligned}$$

We have

$$\begin{aligned}
F^{(0)}(\eta) &= \sum_{k \in W_\beta} F_k(\eta) + \sum_{k \in W_\delta \setminus W_\beta} F_k(\eta) \\
&= \sum_{k \in W_\beta} F_k(\eta \cap B_{(\gamma-\beta)/a_+}(k)) + \sum_{k \in W_\delta \setminus W_\beta} F_k(\eta \cap B_{d(k,A)/a_+}(k)) + \delta_1 + \delta_2.
\end{aligned}$$

For $k \in \mathbb{Z}^d$, $B_{d(k,A)/a_+}(k) \subset B(k, d(k,A)) \subset A^c$, whence $\mathrm{Var}(F_k(\eta \cap B_{d(k,A)/a_+}(k))|\Omega = 1, \eta_\gamma) = 0$. Also, conditionally to $\Omega = 1$, for $k \in W_\beta$, since $(\gamma - \beta)a_-/a_+ \geqslant (\beta + \rho)\sqrt{d}/2$, $A \subset$

$B(k,(\rho+\beta)\sqrt{d}/2) \subset B_{(\gamma-\beta)/a_+}(k) \subset \tilde{W}_\gamma$. Hence $\eta \cap B_{(\gamma-\beta)/a_+}(k) = \eta \cap A$, and (5.1) yields, with $V := \sup_{k\in\mathbb{Z}^d} \mathbf{E}F_k(\eta\cap A)^2$,

$$\begin{aligned}\mathbf{E}|\mathrm{Var}(F^{(0)}(\eta)|\Omega=1,\eta_\gamma)-&\mathrm{Var}(F_{W_\beta}(\eta\cap A))| \\ &= \mathbf{E}\left|\mathrm{Var}(F_{W_\beta}(\eta\cap A)+\delta_1+\delta_2|\Omega=1,\eta_\gamma) - \mathrm{Var}(F_{W_\beta}(\eta\cap A)|\Omega=1,\eta_\gamma)\right| \\ &\leqslant \kappa\sqrt{\mathbf{E}(F_{W_\beta}(\eta\cap A)+\delta_1+\delta_2)^2+\mathbf{E}F_{W_\beta}(\eta\cap A)^2}\sqrt{\mathbf{E}(\delta_1+\delta_2)^2} \\ &\leqslant \kappa\sqrt{\mathbf{E}(2\beta^{2d}V+\delta_1^2+\delta_2^2}\sqrt{\mathbf{E}(\delta_1+\delta_2)^2}.\end{aligned}$$

Assumption (2.2) yields $\mathbf{E}\delta_1^2 \leqslant \kappa\beta^{2d}(1+\gamma-\beta)^{-2\alpha} \leqslant \kappa(1+\rho)^{2(d-\alpha)}$, and

$$\begin{aligned}\mathbf{E}\delta_2^2 &\leqslant \sum_{k,j\in W_\delta\setminus W_\beta}\sqrt{\mathbf{E}(F_k-F_k(\eta\cap B_{d(k,A)/a_+}(k)))^2}\sqrt{\mathbf{E}(F_j-F_j(\eta\cap B_{d(j,A)/a_+}(j)))^2} \\ &\leqslant \left(\sum_{m=\beta}^{\delta}\kappa m^{d-1}\sqrt{\kappa C_0^2(1+m-\rho)^{-2\alpha}}\right)^2 \\ &\leqslant \kappa C_0^2\left(\sum_{m=\beta}^{\infty}(m-\rho)^{d-1-\alpha}\right)^2 \leqslant \kappa C_0^2(\beta-\rho)^{2(d-\alpha)} \leqslant \kappa C_0^2(1+\rho)^{2(d-\alpha)}.\end{aligned}$$

Finally, for some $C>0$ not depending on ρ or W,

$$\mathbf{E}\mathrm{Var}_{ext}(F^{(0)}) \geqslant \mathbf{P}(\Omega=1)(v-C(1+\rho)^{d-\alpha}).$$

Now let $l\neq m\in\mathbb{Z}^d, k\in W_l^\delta, j\in W_m^\delta, r=\|l-m\|(\delta-\gamma)/a_+$. Let η',η'' independent copies of η, and define

$$\begin{aligned}\eta_k &= (\eta\setminus B_r(k))\cup(\eta'\cap B_r(k)^c) \\ \eta_j &= (\eta\setminus B_r(j))\cup(\eta''\cap B_j(r)^c).\end{aligned}$$

Since $B_r(k)\cap B_r(j)\subset A_{ext}$, η_k and η_j are independent conditionally to η_{ext}, and we have by (5.1), with a computation similar to (5.2),

$$|\mathrm{Cov}_{ext}(F_k,F_j) - \underbrace{\mathrm{Cov}_{ext}(F_k(\eta_k),F_j(\eta_j))}_{=0}| \leqslant \kappa C_0^2(1+r)^{-\alpha}.$$

It follows that

$$\left|\mathrm{Cov}_{ext}(F_W^{(l)},F_W^{(m)})\right| \leqslant \sum_{k\in W_l^\delta, j\in W_m^\delta}|\mathrm{Cov}_{ext}(F_k,F_j)| \leqslant \kappa C_0^2\delta^{2d}(1+r)^{-\alpha}$$

and, for some C' not depending on W,

$$\sum_{m\in\mathbb{Z}^d\setminus\{l\}}\left|\mathrm{Cov}_{ext}(F_W^{(m)},F_W^{(l)})\right| \leqslant \kappa C_0^2\delta^{2d}\sum_{p=1}^{\infty}p^{d-1}(\|p\|(\delta-\gamma))^{-\alpha} \leqslant C'\delta^{2d}(\delta-\gamma)^{-\alpha} \leqslant C'(\tau\rho)^{2d-\alpha}.$$

Reporting back in (5.4) yields

$$\mathrm{Var}(F_W) \geqslant |W_{int}^\delta|((v-C(1+\rho)^{(d-\alpha)})\mathbf{P}(\Omega=1)-C'(\tau\rho)^{2d-\alpha}) - |W_{bd}^\delta|C'' \tag{5.5}$$

where C,C',C'' don't depend on ρ,τ,W, and $\mathbf{P}(\Omega=1)$ depends on ρ but not τ. Let therefore ρ be such that $\mathrm{Var}(F_{W_\beta}(\eta\cap A))\geqslant v$ and $v-C(1+\rho)^{d-\alpha}>0$, and then τ such that $(v-C(1+\rho)^{2(d-\alpha)})\mathbf{P}(\Omega=1)-C'(\tau\rho)^{2d-\alpha}>0$. To conclude, let a sequence $\{W_n;n\geqslant 1\}$ be such that $\lim_n |\partial_{\mathbb{Z}^d} W_n|/|W_n| = 0$. Since $|\partial_{\mathbb{Z}^d} W|/|W| \geqslant |W^\delta_{bd}|/(\delta^d |W^\delta|)$, then we also have $|(W_n)^\delta_{bd}|/|(W_n)^\delta_{int}| \to 0$. Applying to (5.5) yields $\liminf_n |W_n|^{-1}\mathrm{Var}(F_{W_n}) > 0$, whence (2.4) implies $\sigma_0 > 0$.

It remains to prove (2.5). Assume that (2.2) holds with $\alpha > 2d$. The proof when instead (2.1) holds is exactly the same with F_k^W instead of F_k, and it is omitted. For $k \in \mathbb{Z}^d$, let $\bar{F}_k = F_k(\eta) - \mathbf{E}F_k(\eta)$. We have

$$\mathbf{E}(F_W - \mathbf{E}F_W)^4 = \sum_{i,j,k,l \in W} \mathbf{E}\bar{F}_i\bar{F}_j\bar{F}_k\bar{F}_l.$$

Let $I = \{i,j,k,l\} \subset W$. Assume that i is *I-isolated*, i.e. $\delta := [d(i, I \setminus \{i\})] = \max_{m \in I}[d(m, I \setminus \{m\})]$. Let $\eta_m, m \in I$, be independent copies of η, and

$$\begin{aligned} H_m =& B_{\delta/2a_+}(m) \\ \eta'_m =& (\eta \cap H_m) \cup (\eta_m \cap H_m^c), \end{aligned}$$

note that η'_m is distributed as η, and that for $m \in I \setminus \{i\}$, $H_i \cap H_m = \emptyset$, hence η'_i is independant from $\{\eta'_j, \eta'_k, \eta'_l\}$. Introduce $\bar{F}'_m = F_m(\eta'_m) - \mathbf{E}F_m$, $\bar{F} = \bar{F}_j\bar{F}_k\bar{F}_l$, $\bar{F}' = \bar{F}'_j\bar{F}'_k\bar{F}'_l$, independent of $\bar{F}'_i$. We have

$$\begin{aligned} \Big|\mathbf{E}\bar{F}_i\bar{F} - \underbrace{\mathbf{E}\bar{F}'_i\bar{F}'}_{=0}\Big| \leqslant& \mathbf{E}\Big[|(\bar{F}_i - \bar{F}'_i)\bar{F}_j\bar{F}_k\bar{F}_l| + |\bar{F}'_i(\bar{F}_j - \bar{F}'_j)\bar{F}_k\bar{F}_l| \\ & + |\bar{F}'_i\bar{F}'_j(\bar{F}_k - \bar{F}'_k)\bar{F}_l| + |\bar{F}'_i\bar{F}'_j\bar{F}'_k(\bar{F}_l - \bar{F}'_l)|\Big] & (5.6) \\ \leqslant& 4\sum_{m \in I} (\mathbf{E}\bar{F}_0^4)^{3/4}(\mathbf{E}|\bar{F}_m - \bar{F}'_m|^4)^{1/4} & (5.7) \\ \leqslant& \kappa C_0 (\mathbf{E}\bar{F}_0^4)^{3/4}(1+\delta)^{-\alpha} & (5.8) \\ & & (5.9) \end{aligned}$$

by (2.2) (or (2.1) for the proof with the F_k^W). Notice that one point among $\{j,k,l\}$ is between distance δ and $\delta + 1$ from i, call it a, and there are at most $\kappa\delta^{d-1}$ possible values for a, given i. If there are two points remaining in $\{j,k,l\} \setminus a$, they are at mutual distance at most 3δ. We have

$$\begin{aligned} \mathbf{E}(F_W - \mathbf{E}\bar{F}_W)^4 \leqslant& 4\sum_{i,j,k,l \in W} \mathbf{1}_{\{i \text{ isolated}\}}\kappa C_0(\mathbf{E}\bar{F}_0^4)^{3/4}(1 + [d(i,\{j,k,l\})])^{-\alpha} \\ \leqslant& \kappa C_0(\mathbf{E}\bar{F}_0^4)^{3/4}\sum_{\delta=1}^{\infty}(1+\delta)^{-\alpha}\sum_{i,j,k,l \in W}\mathbf{1}_{\{i \text{ isolated and } [d(i,\{j,k,l\})]=\delta\}} \\ \leqslant& \kappa C_0(\mathbf{E}\bar{F}_0^4)^{3/4}\sum_{\delta=1}^{\infty}|W|^2(1+\delta)^{-\alpha}\kappa\delta^{d-1}(3\delta)^d \\ \leqslant& \kappa C_0(\mathbf{E}\bar{F}_0^4)^{3/4}|W|^2 \end{aligned}$$

where $\kappa < \infty$ because $\alpha > 2d$.

5.2 Proof of Theorem 3.1

W is fixed. For simplicity, in all the proof we use the notation $G = G_W, \tilde{G} = \tilde{G}_W$. If (3.2) is satisfied, put $G_k = F_k$ and $A = \mathbb{R}^d$. If instead (3.1) is satisfied, put $G_k = F_k^W$ and $A = \tilde{W}$. Assume without loss of generality that F_0 is centered. Theorem 1.2 from [17] gives general

Berry-Esseen bounds on the Poisson functional $\tilde{G}$ in terms of integrals involving first and second-order Malliavin derivatives: provided $\int_A \mathbf{E}(D_{\mathbf{x}}G)^2 d\mathbf{x} < \infty$ (proved below), $d_{\mathscr{W}}(\tilde{G}, N) \leqslant \sum_{i=1}^3 \gamma_i$, $d_{\mathscr{K}}(\tilde{G}, N) \leqslant \sum_{i=1}^6 \gamma_i$, where

$$\begin{aligned}
\gamma_1 =& 4\left[\int_{\overline{A}^3}\left[\mathbf{E}(D_{\mathbf{x}_1}\tilde{G})^2\left(D_{\mathbf{x}_2}\tilde{G}\right)^2\right]^{1/2}\left[\mathbf{E}\left(D^2_{\mathbf{x}_1,\mathbf{x}_3}\tilde{G}\right)^2\left(D^2_{\mathbf{x}_2;\mathbf{x}_3}\tilde{G}\right)^2\right]^{1/2} d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3\right]^{1/2}\\
\gamma_2 =& \left[\int_{\overline{A}^3}\mathbf{E}\left(D^2_{\mathbf{x}_1,\mathbf{x}_3}\tilde{G}\right)^2\left(D^2_{\mathbf{x}_2,\mathbf{x}_3}\tilde{G}\right)^2 d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3\right]^{1/2}\\
\gamma_3 =& \int_{\overline{A}}\mathbf{E}|D_{\mathbf{x}}\tilde{G}|^3 d\mathbf{x}\\
\gamma_4 =& \frac{1}{2}\left[\mathbf{E}(\tilde{G}-\mathbf{E}\tilde{G})^4\right]^{1/4}\int_{\overline{A}}\left[\mathbf{E}(D_{\mathbf{x}}\tilde{G})^4\right]^{3/4} d\mathbf{x}\\
\gamma_5 =& \left[\int_{\overline{A}}\mathbf{E}(D_{\mathbf{x}}\tilde{G})^4 d\mathbf{x}\right]^{1/2}\\
\gamma_6 =& \left[\int_{(\overline{A})^2}\left(6\left[\mathbf{E}(D_{\mathbf{x}_1}(\tilde{G}))^4\right]^{1/2}\left[\mathbf{E}\left(D^2_{\mathbf{x}_1,\mathbf{x}_2}\tilde{G}\right)^4\right]^{1/2} + 3\mathbf{E}(D^2_{\mathbf{x}_1,\mathbf{x}_2}\tilde{G})^4\right) d\mathbf{x}_1 d\mathbf{x}_2\right]^{1/2}.
\end{aligned}$$

Using either (3.2) or (3.1),

$$\begin{aligned}
\int_{\overline{A}}\mathbf{E}(D_{\mathbf{x}}G)^2 d\mathbf{x} =& \int_A \mathbf{E}(D_{(x,M)}G)^2 dx\\
\leqslant& \int_A \sum_{k,k'\in W}\sqrt{\mathbf{E}D_{(x,M)}G_k^2}\sqrt{\mathbf{E}D_{(x,M)}G_{k'}^2}dx\\
\leqslant& \int_A \sum_{k,k'\in W}(1+\|x-k\|)^{-\alpha}(1+\|x-k'\|)^{-\alpha}dx\\
\leqslant& \left(\int_{B(0,2\mathrm{diam}(W))}C_0^2|W|^2 dx + \int_{B(0,2\mathrm{diam}(W))^c}C_0^2|W|^2(1+\|x\|-\mathrm{diam}(W))^{-2\alpha}dx\right)\\
\leqslant& C_0^2\left(\kappa|W|^2\mathrm{diam}(W)^d + |W|^2\int_{2\mathrm{diam}(W)}^\infty \kappa(1+r-\mathrm{diam}(W))^{-2\alpha}r^{d-1}dr\right) < \infty
\end{aligned}$$

using $\alpha > d \geqslant 1$. Let $x, y \in A, \mathbf{x} = (x, M), \mathbf{y} = (y, M')$. Call $\eta^{\mathbf{x}} = \eta\cup\{\mathbf{x}\}, \eta^{\mathbf{y}} = \eta\cup\{\mathbf{y}\}$. We have

$$|D_{\mathbf{x},\mathbf{y}}G(\eta)| \leqslant \sum_{k\in W}\min\left(|D_{\mathbf{x}}G_k(\eta)| + |D_{\mathbf{x}}G_k(\eta^{\mathbf{y}})|, |D_{\mathbf{y}}G_k(\eta)| + |D_{\mathbf{y}}G_k(\eta^{\mathbf{x}}|\right).$$

$$\begin{aligned}
\mathbf{E}|D^2_{\mathbf{x},\mathbf{y}}G(\eta)|^4 \leqslant& \mathbf{E}\left|2\sum_{k\in W}\min\left(\sup_{\eta'\in\{\eta,\eta^{\mathbf{y}}\}}|D_{\mathbf{x}}G_k(\eta')|, \sup_{\eta'\in\{\eta,\eta^{\mathbf{x}}\}}|D_{\mathbf{y}}G_k(\eta')|\right)\right|^4\\
\leqslant& 2^4\sum_{k_1,\dots,k_4\in W}\mathbf{E}\prod_{i=1}^4\min\left(\sup_{\eta'\in\{\eta,\eta^{\mathbf{y}}\}}|D_{\mathbf{x}}G_{k_i}(\eta')|, \sup_{\eta'\in\{\eta,\eta^{\mathbf{x}}\}}|D_{\mathbf{y}}G_{k_i}(\eta')|\right)\\
\leqslant& 2^4\left(\sum_{k\in W}\left(\mathbf{E}\min\left(\sup_{\eta'\in\{\eta,\eta^{\mathbf{y}}\}}|D_{\mathbf{x}}G_k(\eta')|^4, \sup_{\eta'\in\{\eta,\eta^{\mathbf{x}}\}}|D_{\mathbf{y}}G_k(\eta')|^4\right)\right)^{1/4}\right)^4.
\end{aligned}$$

Let $k \in W$. Note that, with $B = \tilde{W} - k$, for $x, y \in \tilde{W}, \eta' \in \mathscr{N}$,

$$\begin{aligned}
D_{\mathbf{x}}F_k^W(\eta') =& F_k^W(\eta'\cup\{\mathbf{x}\}) - F_k^W(\eta') = F_0((\eta'\cup\{\mathbf{x}\})\cap\tilde{W} - k) - F_0(\eta'\cap\tilde{W} - k)\\
=& F_0(((\eta'-k)\cap B)\cup\{\mathbf{x}-k\}) - F_0((\eta'-k)\cap B) = D_{\mathbf{x}-k}F_0((\eta'-k)\cap B).
\end{aligned}$$

Since $\eta - k \overset{(d)}{=} \eta$, applying either (3.2) or (3.1) with $y-k$ instead of y yields

$$\mathbf{E}|D^2_{\mathbf{x},\mathbf{y}}G(\eta)|^4 \leqslant \kappa C_0^4 \left(\sum_{k\in W} \min((1+\|x-k\|)^{-\alpha}, (1+\|y-k\|)^{-\alpha})\right)^4 .$$

Consider case **(i')** (the following is valid but irrelevant in case **(i)**). Summing in a radial manner around x yields that the previous sum is bounded by $\kappa C_0^4(\sum_{m=[d(x,W)]}^{\infty} m^{d-1}(1+m)^{-\alpha})^4 \leqslant \kappa C_0^4(1+d(x,W))^{4(d-\alpha)}$, and the same holds for y. We can also work on the first order derivative with a similar technique:

$$\mathbf{E}|D_{\mathbf{x}}G|^4 \leqslant \kappa C_0^4 \left(\sum_{k\in W}(1+\|x-k\|)^{-\alpha}\right)^4 \leqslant \kappa C_0^4(1+d(x,W))^{4(d-\alpha)}.$$

Noting $I_{x,y} = \{k \in W : \|k-x\| \geqslant \|k-y\|\}$,

$$\begin{aligned}
\mathbf{E}|D_{\mathbf{x},\mathbf{y}}G(\eta)|^4 \leqslant& \kappa C_0^4 \left[\left(\sum_{k\in I_{x,y}}(1+\|x-k\|)^{-\alpha}\right)^4 + \left(\sum_{k\in I_{y,x}}(1+\|y-k\|)^{-\alpha}\right)^4\right]\\
\leqslant& \kappa C_0^4 \left[\sum_{k\in\mathbb{Z}^d\setminus B(x,\|y-x\|/2)}(1+\|x-k\|)^{-\alpha} + \sum_{k\in\mathbb{Z}^d\setminus B(y,\|x-y\|/2)}(1+\|y-k\|)^{-\alpha}\right]^4\\
\leqslant& C_0^4\kappa(1+\|x-y\|/2)^{4(d-\alpha)},
\end{aligned}$$

whence finally

$$\mathbf{E}|D_{\mathbf{x},\mathbf{y}}G(\eta)|^4 \leqslant \kappa C_0^4(1+\max(\|x-y\|, d(x,W), d(y,W)))^{4(d-\alpha)}. \tag{5.10}$$

Let us start with a few geometric estimates, useful in the case **(i')**.

Lemma 5.2. Let $W \subset \mathbb{Z}^d$, bounded and non-empty, $d_W = (|W|/|\partial_{\mathbb{Z}^d}W|)^{1/d}, W' = \{k \in \mathbb{Z}^d : d(k,W) \leqslant d_W\}$.

$$|W'| \leqslant \kappa|W| \tag{5.11}$$

$$\int_{(\tilde{W}')^c}(1+d(x,\tilde{W}))^a dx \leqslant \kappa_a|W|d_W^a,\ a < -d \tag{5.12}$$

$$I(x) := \int_{\mathbb{R}^d}(1+\max(d(x,W), \|x-y\|))^{d-\alpha}dy \leqslant \kappa(1+d(x,W))^{2d-\alpha}, x\in\mathbb{R}^d. \tag{5.13}$$

Proof. Since each point of $W'\setminus W$ is in a ball with radius d_W centered in $\partial_{\mathbb{Z}^d}W$, (5.11) is proved via

$$|W'| \leqslant |W| + |\partial_{\mathbb{Z}^d}W|\kappa d_W^d \leqslant \kappa|W|.$$

Let $\psi(x) = d(x,\tilde{W}), x\in\mathbb{R}^d, h(t) = \mathbf{1}_{\{t\geqslant d_W\}}(1+t)^a, t \geqslant 0$. The Federer co-area formula yields

$$\int_{\mathbb{R}^d} h(\psi(x))\|\nabla\psi(x)\|dx = \int_{\mathbb{R}_+} h(t)\mathcal{H}^{d-1}(\psi^{-1}(\{t\}))dt.$$

We have $\|\nabla\psi(x)\| = 1$ for a.a. $x\in\tilde{W}^c$. According to [20, Lemma 4.1], for almost all $t > 0$,

$$\mathcal{H}^{d-1}(\psi^{-1}(\{t\})) \leqslant t^{d-1}\mathcal{H}^{d-1}(\psi^{-1}(\{1\})),$$

and the latter is bounded by $\kappa t^{d-1}|\partial_{\mathbb{Z}^d}W|$. Since $a+d<0$

$$\int_{(\tilde W')^c} h(\psi(x))dx \leqslant \kappa \int_{d_W}^{\infty}(1+t)^a t^{d-1}|\partial_{\mathbb{Z}^d}W|dt \leqslant \kappa_a|\partial_{\mathbb{Z}^d}W|d_W^a d_W^d = \kappa_a |W|d_W^a,$$

which yields (5.12). The left hand member of (5.13) is equal to

$$\begin{aligned} I(x) =&\ell(B(x,d(x,W)))(1+\max(d(x,W)))^{d-\alpha} + \int_{B(x,d(x,W))^c}(1+\|x-y\|)^{d-\alpha}dy \\ \leqslant&\kappa(1+d(x,W))^{2d-\alpha} + \int_{d(x,W)}^{\infty}(1+r)^{d-\alpha}\kappa r^{d-1}dr, \end{aligned}$$

from which the result follows. □

Writing $\mathbf{x}_1=(x_1,M_1),\mathbf{x}_2=(x_2,M_2),\mathbf{x}_3=(x_3,M_3)$, with M_1,M_2,M_3 iid distributed as μ,denote by $\tilde E$ the expectation with respect to (M_1,M_2,M_3), and $\mathbf{E}_\eta$ the expectation with respect to η, such that $\mathbf{E}=\tilde{\mathbf{E}}\mathbf{E}_\eta$. We have, bounding $\mathbf{E}D_{\mathbf{x}_1}^4$ by κC_0^4 and using Cauchy-Scwharz inequality several times,

$$\begin{aligned} \gamma_1 =&4\sigma^{-2}\left[\int_{A^3}\tilde{\mathbf{E}}\left[\sqrt{\mathbf{E}_\eta\left[(D_{\mathbf{x}_1}G)^2(D_{\mathbf{x}_2}G)^2\right]}\sqrt{\mathbf{E}_\eta\left[(D^2_{\mathbf{x}_1,\mathbf{x}_3}G)^2(D^2_{\mathbf{x}_2,\mathbf{x}_3}G)^2\right]}\right]dx_1dx_2dx_3\right]^{1/2} \\ \leqslant&4\sigma^{-2}\left[\int_{A^3}\sqrt{\tilde{\mathbf{E}}\left[\mathbf{E}_\eta\left[(D_{\mathbf{x}_1}G)^2(D_{\mathbf{x}_2}G)^2\right]\right]}\sqrt{\tilde{\mathbf{E}}\left[\mathbf{E}_\eta\left[(D^2_{\mathbf{x}_1,\mathbf{x}_3}G)^2(D^2_{\mathbf{x}_2,\mathbf{x}_3}G)^2\right]\right]}dx_1dx_2dx_3\right]^{1/2} \\ \leqslant&\kappa C_0\sigma^{-2}\left[\int_{A^3}\left(\mathbf{E}(D^2_{\mathbf{x}_1,\mathbf{x}_3}G)^4\right)^{1/4}\left(\mathbf{E}(D^2_{\mathbf{x}_2,\mathbf{x}_3}G)^4\right)^{1/4}dx_1dx_2dx_3\right]^{1/2}. \end{aligned}$$

Similar techniques to integrate out the marks yield the same bound

$$\begin{aligned} \gamma_2 \leqslant&\kappa C_0\sigma^{-2}\left[\int_{A^3}\left(\mathbf{E}(D^2_{\mathbf{x}_1,\mathbf{x}_3}G)^4\right)^{1/4}\left(\mathbf{E}(D^2_{\mathbf{x}_2,\mathbf{x}_3}G)^4\right)^{1/4}dx_1dx_2dx_3\right]^{1/2} \\ \leqslant&\kappa C_0^2\sigma^{-2}\sqrt{\int_A I(x_3)^2dx_3} \end{aligned}$$

using (5.10). In the case **(i)**, $A=\tilde W$ and $\alpha>2d$. We have

$$\max(\gamma_1,\gamma_2)\leqslant \kappa C_0^2\sigma^{-2}\sqrt{\ell(\tilde W)\left(\int_{\mathbb{R}^d}(1+\|x_3-x\|)^{d-\alpha}dx\right)^2}\leqslant \kappa C_0^2\sigma^{-2}\sqrt{|W|}.$$

In the case $(\mathbf{i}'), A=\mathbb{R}^d, \alpha>5d/2$. Using successively (5.13),(5.11) and (5.12) yield, with $2(2d-\alpha)<2(-d/2)<-d$,

$$\begin{aligned} \max(\gamma_1,\gamma_2) \leqslant&\kappa C_0^2\sigma^{-2}\sqrt{\ell(\tilde W')+\int_{(\tilde W')^c}(1+d(x,W))^{2(2d-\alpha)}dx} \\ \leqslant&\kappa C_0^2\sigma^{-2}\sqrt{\kappa|W|+\kappa_{2(2d-\alpha)}|W|d_W^{2(2d-\alpha)}}\leqslant \kappa C_0^2\sigma^{-2}\sqrt{|W|}(1+d_W^{2(2d-\alpha)}), \end{aligned}$$

which gives the power a in (3.3)-(3.4). Let us keep assuming we are in case **(i')**. Since $A=\mathbb{R}^d$, $\alpha>2d$ and (5.12) yield

$$\begin{aligned} \gamma_3 \leqslant&\sigma^{-3}\int_{\mathbb{R}^d}\left(C_0^4\kappa(1+d(x,W))^{4(d-\alpha)}\right)^{3/4}dx \leqslant \quad C_0^3\kappa\sigma^{-3}(\ell(\tilde W')+\int_{(\tilde W')^c}(1+d(x,W))^{3(d-\alpha)})dx \\ \leqslant&\kappa C_0^3\sigma^{-3}|W|(1+d_W^{3(d-\alpha)}). \end{aligned}$$

In case **(i)**, the same bound holds after removing $d_W^{3(d-\alpha)}$. Reporting back gives (3.3).

Introduce $\overline{G} = G - \mathbf{E}G$. Using (5.11) and (5.12),

$$\begin{aligned}\gamma_4 \leqslant& \frac{1}{2}\sigma^{-1}(\mathbf{E}\overline{G}^4)^{1/4}\int_{\mathbb{R}^d}\sigma^{-3}\left(C_0^4\kappa(1+d(x,W))^{4(d-\alpha)}\right)^{3/4}dx\\ \leqslant& \kappa\sigma^{-4}v^{1/4}\sqrt{|W|}C_0^3(\ell(\tilde{W}')+\int_{(\tilde{W}')^c}(1+d(x,W))^{3(d-\alpha)}dx)\\ \leqslant& \sigma^{-4}C_0^3\kappa|W|^{3/2}v^{1/4}(1+d_W^{3(d-\alpha)})\end{aligned}$$

where $v := \sup_{|W|\to\infty}\frac{\mathbf{E}((G-\mathbf{E}G)^4)}{|W|^2}$. Let us conclude the proof: (5.12) yields

$$\gamma_5 \leqslant \left[\int_{\mathbb{R}^d}\sigma^{-4}C_0^4\kappa(1+d(x,W))^{4(d-\alpha)}dx\right]^{1/2} \leqslant \sigma^{-2}C_0^2\kappa\sqrt{|W|}\left(1+d_W^{4(d-\alpha)}\right)^{1/2}$$

$$\begin{aligned}\gamma_6 \leqslant& \Big[\int_{(\mathbb{R}^d)^2}\sigma^{-4}\Big(6C_0^4\kappa(1+d(x_1,W))^{2(d-\alpha)}(1+\|x_1-x_2\|)^{2(d-\alpha)}\\ &\qquad + 3C_0^4\kappa(1+d(x_1,W))^{2(d-\alpha)}(1+\|x_1-x_2\|)^{2(d-\alpha)}\Big)dx_1dx_2\Big]^{1/2}\\ \leqslant& \sigma^{-2}C_0^2\kappa\left[\int_{\mathbb{R}^d}(1+d(x_1,W))^{2(d-\alpha)}\left(\int_{\mathbb{R}^d}(1+\|x-y\|)^{2(d-\alpha)}dy\right)dx\right]^{1/2}\\ \leqslant& \sigma^{-2}C_0^2\kappa\sqrt{|W|}\left(1+d_W^{2(d-\alpha)}\right)^{1/2}.\end{aligned}$$

In case **(i)**, $A = \tilde{W}$, all the same inequalities still hold after removing terms of the form d_W^a. Reporting back gives (3.4).

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