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A STUDY OF STUDENTS' PROVING PROCESSES AT THE JUNIOR HIGH SCHOOL LEVEL

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2017 caveat: this version differs from the 1988 one only by the correction of some typo (the English is as it was...)

I. INTRODUCTION

The teaching of mathematical proof appears to be a failure in almost all countries, no matter how this teaching is organized. One consequence has been to suppress proof as a content to be taught in most curricula; in France the word "*démonstration*" has disappeared from the statement of the official programs, in USA mathematical proofs are taught only to students who take the geometry course, etc.. In my opinion, such a consequence is drastic insofar as it first of all empties mathematics of its essence, and second because the validity of the mathematical knowledge is fundamentally based on the proof which establishes it.

An analysis of teaching materials indicates that there is too strong an emphasis on teaching the logical side of proof, while its social and practical importance in the mathematical activity remains hidden. It is too often forgotten that mathematical proofs are a means for communication among mathematicians; they play an essential role in establishing the validity of a statement and also in enlightening its meaning. And, as Manin[1] recalls: "a proof becomes a proof after the social act of 'accepting it as a proof'." This social nature of mathematical proofs is part of their practical value; it leads to acknowledge them as efficient and reliable tools for the mathematicians.

But, mathematical proof is taught without taking into account that students have had criteria for judging the validity and the relevance of their mathematical statements before being introduced to this new piece of knowledge. For Most students mathematical proofs appear ultimately to be a kind of rhetoric specific to the mathematical classroom (Balacheff 1982). They produce it because the teacher demands it, not because they recognize it as necessary in their practice; as a British student told to an interviewer: "to prove something in maths means that you have worked it out and it proves how good you are at working questions out and understanding them" (Galbraith 1979).

Then, the aim of our research is to identify the foundations of students' belief in the validity of a statement in their mathematical activity: what they recognize in practice as a proof, and how they treat a refutation. We have focused this study on the relationships among the students' proving process, the knowledge they have at their disposal, the language they can use and the role of the situational context.

[1] Manin quoted by Hanna 1983, p.71

II. COGNITIVE AND SOCIAL ASPECTS OF PROOF

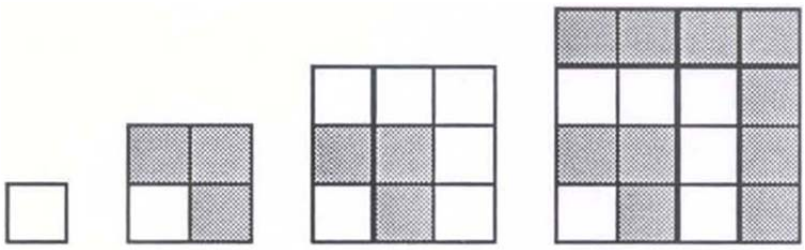
II.1 PRECISION IN TERMINOLOGY

In English, as well as in French, two terms are used in mathematics in a synonymous way: "proof" (French *preuve)* and "mathematical proof' (French *démonstration)*. This custom presents an obstacle to our study, for it hides different levels that should be differentiated. Thus, we propose starting with the following distinctions:

- We will use the term *explanation* to describe the discourse of an individual intending to establish for somebody else the validity of a statement. The validity of an explanation is initially related to the speaker who articulates it.
- We will use the term *proof* to refer to an explanation which is accepted by a community at a given time;
- We will designate as a *mathematical proof* a proof accepted by mathematicians. As a type of discourse, mathematical proofs nowadays have a specific structure and follow well-defined rules that have been formalized by logicians.

II.2. COGNITIVE ASPECTS OF PROOF

In accordance with what cognitive psychology tells us, the most elementary type of proof consists in merely showing, what Semadeni (1984) calls *proof in action.* Here is a classical example of such a proof of the property: "the sum of the n first odd numbers is n^2":



Such proofs rely upon the capacity of the person observing the schema to reconstruct the reasons that are embedded in it but not formulated.

At a higher level, reasons could be expressed but still strongly related to actions performed on some example. Insofar as the example is viewed not as a particular case but as a representative of a class of objects, we will recognize this as a proof by generic example. Such a proof requires the generality to be viewed beyond the particular. Let us give an example taken from Bezout (*Notes sur l'arithmétique*, 1832 p.23, our free translation):

> The remainder of the division of a number by 2x2 or by 5x5 is the same as the remainder of the division by 2x2 or by 5x5 of the number written with the two last digits on the right of this same number [...] Let us consider the number 43728 and the divisor 5x5. The number 43728 equals 47700+28. But 43700 is divisible by 5x5, because 47700 is the product of 437 by 100, and 100 equaling 10x10, or 5x2x5x2, or 5x5x2x2, the factor 100 is divisible by 5x5. The remainder of the division of 43728 by 5x5 or 25, is the same as the one of 28 by 25.

We call *pragmatic proofs* those proofs which rely upon action, and we call *intellectual proofs* those which use verbalizations of the properties of objects and of their relationships. This step towards intellectual proof does not consist in a mere translation of action into words; it requires a genuine construction of language means as an operative tool. The problem-solver must be able to use language and symbols as means to compute on statements and relations.

At the highest level, in mathematics, mathematical proofs require a specific status of knowledge which must be organized in a theory and recognized as such by a community: the validity of definitions, theorems, and deductive rules is socially shared.

II.3. Social Aspects of Proof

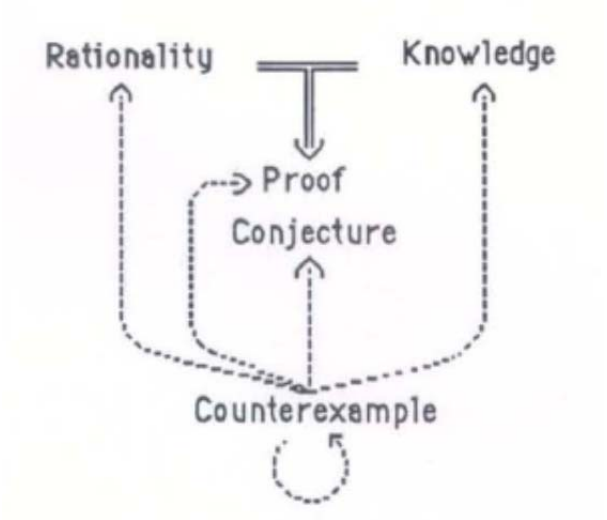
As Popper (1979, p.78) emphasizes it: “the 'certainty' of a belief is not so much a matter of its intensity, but of the *situation*: of our expectation of its possible consequences. Everything depends on the importance attached to the truth or falsity of the belief.” In other words, we will say that involvement at a given level of validation is a question of *economy of logic* "which wants that one does not bring into play more logic than what is necessary for practical needs" (Bourdieu 1980 p.145, our free translation).

Thus, the fact of presenting a problem to students does not guarantee that they will be committed to producing a proof; this is due not to a fundamental lack of awareness but to the fact that their reading of the situation does not call for producing a proof. In our research, therefore, we have paid much attention to providing students with a context promoting awareness of the need for proof – i.e. a context which holds some risk linked to uncertainty, and therefore something to gain by entering a proving process.

II.4. The Dialectic of Proofs and Refutations

A well-known interpretation of a counterexample in the mathematics classroom is that of a sort of catastrophe which implies the definitive rejection of what has been refuted. From this point of view the mathematics classroom ideology is more Manichaean than dialectic. Analysis of the activity of the mathematician suggests a quite different and, at least, less radical functioning. The decision on the validity of a proof depends on the quality of its critical analysis, which finally guarantees the absence of logical mistakes and of counterexamples. In this regard the proving process is based on the commitment of the problem-solver to take into account the possible existence of contradictions: the proving process is fundamentally *dialectic*. That is even more obvious in the context of social interactions, where tentative explanations, or refutations, of a given statement are elicited.

To take into account this dimension of the proving process, we have adopted the model of the dialectic of proofs and refutations proposed by Lakatos (1976). We should mention here that this model is consonant with the theories developed by the Piagetian school which has shown the central role played by contradiction in the genesis of cognitive structures. Using this model we can differentiate the implications of a counterexample, depending on whether we are considering the conjecture, its proof, or the related knowledge and rationality of the problem-solver himself.



This scheme showing the conjecture and its proof as the product of both the knowledge and rationality of a subject summarizes the main possible consequences of a counterexample, Actually, it merely evokes the range of these possible consequences, but it is sufficient to give an idea of what we call the openness of the treatment of a refutation.

The nature of the development of the mathematical knowledge as described by Lakatos suggests a question which Lakatos did not pose[2] but which is nonetheless essential for the teacher or researcher in mathematics education: *what determines the appropriateness of a choice for overcoming the contradiction brought by a counter-example*? We have investigated this question together with the question of the nature of students' proving processes.

III. TYPES AND HIERARCHY OF STUDENTS' PROOFS

III.1. AN EXPERIMENTAL APPROACH

In order to explore students' proving processes, we have used a situation of social interaction which encourages confrontation of differing views on the solution of a problem and hence a verbal exchange which make those views explicit.

Pairs of 13-14 year-old students were required to solve the following problem:

> *Give a way of calculating the number of diagonals of a polygon once the number of vertices is known.*

The answer to this question was to be expressed in a message addressed to, and to be used by, other 13-14 year-old students. The two students have access to as much paper as they want but to only one pencil. This constraint reinforces the co-operative nature of the situation; at the same time it gives us more direct access to the dynamic of the two confronted knowledge systems, especially in cases of decision making. The observer intervenes only after students claim that they have produced a final solution; at this stage he abandons his stance of neutrality and asks the students to deal with counterexamples that he proposes.

The chosen problem refers to student knowledge that is more cultural than scholastic, for while polygons may have been studied in the primary school (in activities related to geometric classification), they are no longer a part of the curriculum at the level with which we are concerned. As in the case of solids in the eighteenth century, students' conceptions about this mathematical content are not theorized. The context, therefore, is conducive to the emergence of processes resembling those described by Lakatos, so it permits us to analyze the dynamic of proving, the related treatment of refutations, and their relation to concept building.

In this experimental situation:

- The kind of communication encouraged structures the students' activity, and, more particularly, promotes a verbal formulation of the counting procedure. This is something that students do not normally do straight away, even if they are technically capable of it. At the same time, the desire to supply a reliable tool to the other group is likely to lead the pair of students to pay more attention to the formulation and the validity of its solution.
- The social interaction, through the exchanges it requires from the pair of students, helps elicit the students' conceptions and plans, and the basis for their decisions independently of any observer's intervention. The

[2] Actually, Lakatos gives examples which could lead to such a question. One of them is the case of Poinsot who, in 1809, considered some star-polyhedral as counterexamples to the Euler theorem. But about 50 years later, Poinsot changed his mind considering these polyhedral as having only triangular faces; "now he sees only examples where he previously saw counter-examples" (ibid p.31).

constraints of using only one pencil oblige a permanent confrontation and the elicitation of the choice of common criteria to accept or refuse any statement or strategy.

We have observed fourteen student pairs, each session lasting about 90 minutes. All sessions have been recorded. From these data, we made the analysis underlying the results we present here (a complete report on this research is presented in Balacheff 1988).

III.2. HIERARCHY AND INTERRELATION OF THE TYPES OF PROOFS

III.2.1. NAIVE EMPIRICISM

The origin of *naive empiricism* can be traced in two quite different phenomena: factual evidence and cognitive belief[3]. "The term 'belief' expresses the direct, the sympathetic, form of knowledge, the feeling of the implicit validity and reliability of the respective representations or interpretations and their extrapolative capacity." (Fishbein 1982, p.ll).

In the first case, the level of proof evidences a pragmatic empiricism which allows students to consider mere observation as sufficient. Such behavior could come from conflict between students or from a reading of the situation that leads them to prefer quickly submitting their solution to the observer instead of trying to enter a proving process on their own. It is for example the case of two students we observed, Pierre and Mathieu[4], who proposed a solution to the observer but telling him that "*it's a game... it's better to try something*" (even if they also claimed that it is of no use "*to try anything*").

The second case is quite different, as it relies upon a real belief in the validity of the proposed solution. This belief is strongly related to the students' conceptions which they are unable to express or analyze: "*an intrinsic type of conviction, directly imposed by the structure of the situation itself*" (Fishbein, 1982, p.ll, *situation* means here *mathematical situation*). By way of illustration we offer the following excerpt of one of our case studies:

> Pierre and Philippe[5] induce from the observation of polygons with 6 and 8 vertices the following numbers of diagonals: 6 x3 and 8x5 (despite the fact that this is contradicted by their drawings). Their conjecture, then, is that the number of diagonals is equal to the number of diagonals from one vertex multiplied by the number of vertices, but the students say, "*we don't know how to explain it*". When the observer proposes the case of a polygon with nine vertices, they compute 9x6 and say: "*They are 54, it is the same procedure*". Their belief in the validity of this solution is based on the evidence that the number of diagonals from each vertex of a polygon is the same. But they are unable to justify their solution, or the statement that it would be sufficient to multiply this number by the number of vertices.

We should mention here that we have done the same experiment, but provided students with a document presenting the definitions of *polygon* and *diagonal* together with some figures. This document was quite similar to what they could encounter in a text book. In this situation students offered more correct solutions to the problem, but in most cases, with a naive empiricist foundation. In the new context the examples used by students to check their solutions were examples taken from the given document. The students understood them as *prototypes*, so that no further testing appeared to be necessary, which is clearly stated by some of the students[6]: *"We will look whether it always work, on all the polygons* ". This kind of naive empiricism, which we would like to call a *prototypical effect,* is essential from the didactical point of view, as it calls under question the use of examples for teaching purposes.

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[3] This expression has been coined by Fishbein (1982); we render it in French by *évidence de raison*.

[4] Balacheff 1988 p.I08

[5] Balacheff 1988 p.109

[6] Ba1acheff 1988 p.250. What is meant here is: all the polygons proposed by the document.

Examples are indeed helpful for teaching purposes, but it appears that from the student point of view they can have a status which is likely to tum into an obstacle to proving processes.

III.2.2. THE CRUCIAL EXPERIMENT

The crucial experiment is an important step, as it identifies awareness of the problem of the validity of a mathematical statement, taking into account the problem of generalization. Its origin could be in the awareness of the insufficiency of a mere verification on few examples, but within cognitive and language limits which do not allow the student to go beyond.

A good example is the case of the solutions of the type : f(n)=(n-3)+(n-3)+(n-4)+ ... +2+1 . This requires expressing an iteration, which the students we observed were unable to do. Here is the solution formulated by Martine and Laura[7]:

> First vertex: number of diagonals = number of vertices -3
> Second vertex: number of diagonals = the same
> From the third vertex: number of diagonals just obtained – 1 diagonal
> Forth vertex: number of diagonals just obtained – 1 diagonal
>
> and so on
>
> - at the end we add all the numbers of diagonal at each vertex to find the number of diagonals of the polygon
>
> The two students decide "to draw a very large figure to verify", which they do with a polygon with ten vertices.

Let us take another example which gives clear evidence of this behavior, Nadine and Elisabeth[8] produced a solution which consists in a recurrent formula:

> They have found nine diagonals for a polygon with six vertices, then announced for a polygon with seven vertices: "*we should add five, normally ... if it works, then normally with seven we will find fourteen diagonals*"; which is confirmed by their experiment. But to accept the solution definitively they relied on a crucial experiment. As they stated it: "*try once with fifteen vertices, then if it works that means that it works with any other number*". Actually, the experiment was done with a polygon with ten vertices: *"so normally... with ten sides we should find thirty-five diagonals*". The conjecture is then accepted as true.

In contrast to the naive empiricism which disappears when students reach the level of intellectual proofs, the crucial experiment remains as an ultimate means to ground the students' conviction, especially in the case of a proof relying on a generic example. This is an example of the operative coexistence of empirical pragmatism and logical rationalism. It confirms the thesis of Fishbein (1982), who claims that these are two types of rationality with different practical value: «the two basic ways of proving – the empirical and the logical – are not symmetrical; they do not have the same weight in our practical activity» (ibid. p.17).

The crucial experiment takes quite a different significance in social interactions when it becomes a means of defusing a conflict about a statement between the two students. It is then no longer a genuine tool for proving. It supports one position against another without effectively asserting its validity. We note that in such circumstances, there were other explanations of the statement in the student pair. It is the case for Christophe and Bertrand[9]:

[7] Balacheff 1988 pp.98. 121

[8] Balacheff 1988 pp.122-123

[9] Balacheff 1988 p .118

Christophe does not accept the solution proposed by Bertrand. He claims that the number of diagonals is twice the number of vertices, like for P_7 the example he considered. Instead of explaining his solution (what he could have done, see this solution below) Bertrand proposed to Christophe to try with P_8 and see... after the observation of this case Christophe surrenders and then accepts the solution of Bertrand. The example P_8 played exactly the role of the Bacon "crucial experiment" which result allow to make a choice between two possibilities.

III.2.3. The Generic Example and the Thought Experiment

The thought experiment can be seen in the linkage of proofs relying on a generic example, throughout a process of de-contextualization which requires eliminating the particular. This process does not occur only at the level of language; it requires cognitive constructions of great complexity to elicit the objects involved in the proof and their relationships. This complexity is due to the fact that very often the thought experiment relies on intermediate steps at a lower level of proving.

From the thought experiment the process of de-contextualization, together with de-temporalization (obliteration of time, in a movement from dynamic to static) and that of de-personalization (obliteration of the actor), can develop towards proofs which consist in a real computation on relations. This has been observed very seldom, since it requires powerful language means and demands the knowledge to be theory-like.

The following example shows the transition from a proof by generic example to a thought experiment:

Christophe and Bertrand[10] deduced from a close analysis of a polygon with seven vertices that the number of diagonals to each vertex follows the pattern n-3. The following account they produced shows they arrive at this:

If we have a polygon with 7 vertices, each point has two neighbors, and from this point starts:

7 – (2 + 1) We do that for all points.

Number of vertices — number of neighbors — the point

Their formulation reflects the reasoning behind their solutions, but this occurs at the level of action and not really at that of computation involving relations:

Knowing the number of vertices of a polygon, it starts from each point the number of vertices – (its two neighbors + itself). What has been obtained then should be multiplied by the number of vertices (from each vertex starts the same number of diagonals).
But we count twice each diagonal: the obtained number of diagonal is thus to be divided by two to find the number of diagonals.

[10] Balacheff 1988 p.138-141

III.3. THE TREATMENT OF REFUTATIONS[11]

The analysis of students' treatment of refutations reflects most of the possibilities described by the model of Lakatos. The question on which we have focused is that of criteria of students' choice among all these possibilities.

Three types of analysis appear to determine the choice of treatment of a refutation:

- *Analysis with reference to the problem itself*. This type of analysis gives a central place to discussion of the nature, and thus the definition, of the objects involved in the problem. Potentially, it may lead to any of the possible types of treatment, none of which is privileged a priori. The choice students make can be understood only through local analysis of the problem-solving process or of the specific characteristic of each individual. The type of treatment can change in the course of the problem-solving process, for example, the students decision to modify the definition can be followed by that of introducing a condition or of modifying the initial conjecture when their conceptions have been stabilized. But the origins of the choice to treat the counterexample by introducing a condition or searching for a specific solution, or modifying the conjecture, cannot be traced with the data we have gathered. All we can suppose is that when a refutation might eliminate a wide range of polygons (with respect to students' conceptions), students prefer to modify the conjecture by adding a specific solution for the objects referred to in the counterexample (extension to odd polygons in case of f(n)=n/2, search for a solution for non-convex polygons, etc.).

- *Analysis with reference to a global conception of the nature of mathematics*. This could be a serious obstacle to some responses to a refutation and could lead to refusal to treat the counterexample as an exception, rejection of a solution which cannot be expressed by a unique formula, etc.

- *Analysis with reference to the situation*. This is essentially a question of the didactical contract[12] which leads students to favor certain treatments of the counterexample (for ex ample the students' reading of the situation as a «definition game» in which they shift from a definition to another to escape the counterexample) while it raises obstacle to others (refusal to introduce a condition because it has not been stated in the problem statement, cf. the example given below).

The following brief account of one of the case studies shows how these different analyses play a role in students' decision:

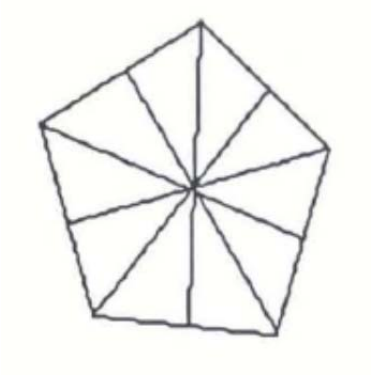
> Evelyne and Christine[13] consider the solution “the number of diagonals of a polygon is half the number of its vertices”, first on the basis of a naïve empiricism, and then on that of a thought experiment. The thought experiment is related to the conception of a diagonal they make explicit after a refutation by a polygon with seven vertices: "a diagonal is a straight line which belongs to a vertex of a polygon and which cuts its surface in two pieces ". But they hold different points of view: one of them would like to introduce a condition in the statement of the conjecture; the other would prefer to reconsider the definition. Finally, it is the latter that is chosen after Evelyne argues that "it could not be that because there [in the statement of the task] they don't tell that a polygon is... the number of vertices is even". In other words, students have not the right to restrict the set of polygons which fit the conjecture; they have to give a solution for all of them. Actually, the definition they choose restricts the set of the polygons to those for which the conjecture is valid: "if we say that a polygon is... a thing whose

[11] For a complete report on this aspect of the research see : N. Balacheff, "Treatment of refutations : aspects of the complexity of a constructivist approach of mathematics learning". *10 appear in* E. Von Glasersfeld (Eds.) : "Constructivism in mathematics education", D. Reidel Publishing Company

[12] In our study, the experimental contract

[13] Balacheff 1988 p.182, pp.192·194, pp.221·223

edges are always parallel two by two... then it is not necessary to be more precise. We need only to divide by two ". But the case of a triangle is not treatable under this definition. The triangle is rejected as a polygon because it has no diagonal. In the eyes of the students, it is a kind of monster. Later on, a counter-example (a polygon with five vertices), produced by the observer imposes itself as a polygon. Then the initial definition is rejected and a new one is considered: "for sure a polygon can have any number of diagonals... but it should be regular". So the only way to save the conjecture is to introduce a condition and to search for a solution specific to the odd polygons.



This last solution starts from a student's conception of a diagonal as an “axis of symmetry”: a diagonal is a line which passes through a vertex of a polygon and cuts it into two equal pieces. In case of a polygon with five vertices it corresponds to the students' drawing given above. The solution they conjecture is f(n)=n for odd polygons. They first envision proving it by means of a crucial experiment with a polygon of eleven vertices, but as it appears to be too complex, they prove it “by reasons”: *"it is obvious that it is* 11, as *there is* 1 *in each vertex* ". The uncertainty which remains is an uncertainty in the students' premises: it involves the definition of a polygon and of a diagonal. It is on the basis of these definitions that they then treat the counterexamples produced by the observer.
These definitions are even formulated in their message:

If the number of vertices of the polygon is even: you divide the number by two and you will obtain the number of its diagonals
If the number of vertices of the polygon is odd: the number of diagonals is equal to the number of vertices
A polygon is a geometrical figure which can have any number of vertices but whose edges must be equal

Finally, we have examined the question of a possible influence of the type of conjecture on the choice of treatment of a refutation. An initial hypothesis is that if the conjecture is false, then its rejection or modification, or the revision of the definition, should be dominant; on the other hand, if the conjecture is correct, then the rejection of the counterexample should be dominant. Actually, we have observed that conjectures like f(n)=n or f(n)=2n are abandoned after their refutation. But such conjectures are very fragile, insofar as they are verified only by one polygon. It is quite different when the conjecture is verified by a large set of polygons, like the false conjecture f(n)=n/2 whose strength comes from the fact that it is related (explicitly or not) to a conception of a polygon as a regular polygon and a diagonal as a diameter. For these conjectures no type of treatment appears to be favored. Even the type of foundation of the conjecture has no effective influence on the students' choice of treatment of the counterexample. For example, an ad hoc modification of the conjecture (what is indeed at the same level as the naive empiricism) can follow a counterexample whereas the foundation of this solution was at a higher level like that

of the thought-experiment. It is the case of Lionel and Laurent[14] who first established the solution f(n)=n(n-3), and then modified it to f(n)=n(n-3)/2 only because they noticed dividing by two as a relationships between the number of diagonals (5) of the counter-example they examined and the number they expected (10).

In the case of correct conjectures, whose construction refers to a "correct" conception of a polygon and a diagonal, whether they have been constructed deductively or as the result of a dialectic between successive attempts and their refutations, the treatment appears to be far less varied than in case of a false conjecture. One prevailing type of response is rejecting the counterexample after its analysis with reference to students' conceptions; a second dominant type of response entails considering the counterexample as an exception or introducing a condition (actually, the latter appears to be a way to avoid recognizing exception).

IV. CONCLUSIONS AND QUESTIONS ABOUT TEACHING

The type of proving processes students evidence do not intrinsically characterize what could be called their "rationality", in that different levels of proof could be observed in their problem-solving activity. The meaning of proving processes cannot be understood without a careful analysis of students' conceptions of the mathematical concepts involved and their reading of the situation within which they act. The characteristics of the situation appear to determine the level of proving, while the image students have of mathematics also plays an important role, particularly in their treatment of refutations.

The passage from pragmatic proofs to intellectual proofs requires a cognitive and linguistic base. Our disregard of the complexity of this passage could be one of the main reasons for the failure of the teaching of mathematical proof, since this passage is very often considered only at the logical level. In geometry in particular, this teaching takes place in a conceptual field which for students is not yet constituted as a theory; we should not forget that geometry was for them essentially restricted to observation and construction of geometrical objects with no requirement for proof. So the teaching of proof is associated to what could be described as a *cognitive break* in the student activity, related to the didactical break represented by new requirement for mathematical proofs.

Thus, we would like to say that the construction of the students' mathematical rationality should he considered at the same time and with the same priority as the construction of their mathematical knowledge. It might be possible to propose proving problems from the beginning of mathematics learning, provided that something other than strict mathematical proofs were acceptable. For that, we must take into account students' reasoning skills and consider the didactical conditions for their evolution. But when a proof has been accepted, the validity of the statement concerned is difficult to question later ; therefore, the teacher acceptance of a proof which is not a mathematical proof raises the problem of its eventual recurrence in a form which must appear reasonable to students.

The case of refutations raises specific didactical problems. We have shown the breadth of students' treatment of a refutation and the variety of rational for this treatment. So how can we deal with the fact that when the teacher produces a counterexample, students believe that it is a particular case, whereas what should actually be questioned is the naïve empiricism on which their conjecture is based or their understanding of the related mathematical knowledge?

If there is no strict cognitive determinism involved in the way a contradiction shown by a counterexample could be overcome, then what role should he played by the situation ? The interventions of the teacher – the way the interactions with students are manage – will be essential to helping students realize that a major goal is examining their knowledge, or the reasoning behind their conjecture, and not a mere ad hoc adaptation of their solution or even its total rejection.

[14] Balacheff 1988 p.170

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