TOPOLOGICAL PHASE TRANSITIONS I: QUANTUM PHASE TRANSITIONS

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To cite this version:

Tian Ma, Shouhong Wang. TOPOLOGICAL PHASE TRANSITIONS I: QUANTUM PHASE TRANSITIONS. 2017. hal-01651908

HAL Id: hal-01651908
https://hal.archives-ouvertes.fr/hal-01651908

Submitted on 29 Nov 2017

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The main objective of this paper is to provide a systematic theoretical study on quantum phase transitions associated with the Bose-Einstein condensates, the superfluidity and the superconductivity. First, we indicate that there are only two types of phase transitions: dynamical phase transitions and topological phase transitions (TPTs). A TPT refers to the transition in its topological structure in the physical space of the system, and quantum phase transitions (QPTs) belong to the category of TPTs. Second, the basic physical characteristics of a QPT are precisely formulated. In particular, a QPT is a transition between quantum states, which are described by the wave functions $\psi$, and the control parameters are non-thermal. Third, we derive the topological structure equations for QPTs, so that the basic geometric theory of incompressible flows can be readily applied. In particular, a topological index for the supercurrent can be defined, leading to precise information on the topological structure and its transitions in the physical space for the underlying system. Fourth, we derive then the basic theory for different TPTs, leading to transparent physical pictures of various condensates. Finally, we derive the microscopic mechanism for the Meissner effect of superconductivity: below the critical temperature, under the applied magnetic field, spin $J = 1$ Cooper pairs reversely parallel to the applied field can form and are stable, and the total magnetic moment of all Cooper pairs, together with the surface supercurrent, can cancel out the magnetism induced by the applied field and resists the applied field from entering its body.
1. Introduction

Phase transition is a universal phenomena of Nature. The central problem in statistical physics and in nonlinear sciences is on phase transitions. A phase transition refers to transitions of the system from one state to another, as the control parameter crosses certain critical threshold. Typical phase transitions include

1) equilibrium phase transitions of thermodynamical systems,
2) quantum phase transitions,
3) Kosterlitz–Thouless transitions, also called the topological order transitions,
4) metal–insulator transitions, and
5) superfluid–insulator transitions.

Recently the authors have developed the dynamic transition theory for dissipative systems; see [7] and the references therein. This is a new
notion of phase transitions, applicable to all dissipative systems, including nonlinear dissipative systems in statistical physics, fluid dynamics, atmospheric and oceanic sciences, biological and chemical systems etc. However, not all the transitions listed in the five categories above can be characterized as dynamic transitions of dissipative systems. In fact, only Category (1) belongs to dynamic phase transitions, and all other categories are not dynamic phase transitions. Clear definitions and systematic theories for those non-dynamic phase transitions are inevitably needed.

Based on an extensive investigation, all types of phase transitions in Nature that we have encountered, including those mentioned above, can be classified into the following two types:

(1) dynamical phase transitions, and
(2) topological phase transitions (TPTs), also called the pattern formation transitions.

The notion of TPTs is originated from the pioneering work by J. Michael Kosterlitz and David J. Thouless [1], where they identified a completely new type of phase transitions in two-dimensional systems where topological defects play a crucial role. With this work, they received 2016 Nobel prize in physics.

TPTs are entirely different from dynamic phase transitions. In fact, the five categories of typical phase transitions listed above are all TPTs, except the equilibrium phase transitions, which are dynamic phase transitions. Intuitively speaking, a TPT refers to the change of the topological structure in the physical space as certain system control parameter crosses a critical threshold.

There have been many attempts, but the basic theoretical understanding of TPTs is still largely open. This paper is on quantum phase transitions (QPTs), a special type of TPTs. Also, this paper is the first one in the series of papers on TPTs, including the forthcoming papers for the following science problems:

(1) galactic spiral structures,
(2) electromagnetic eruptions on solar surface,
(3) boundary-layer separation of fluid flows, and
(4) interior separation of fluid flows.

The main ingredients of the paper are as follows.

First, QPTs are TPTs for condensate systems, including the gaseous Bose-Einstein condensates, superconductivity, and superfluidity. The basic physical characteristics of a QPT are as follows:
(1) A QPT is a transition between quantum states, and the state quantities or the order parameters of the quantum system describing the transition should be the wave functions $\psi$ of the quantum states;
(2) the control parameters are non-thermal; and
(3) a QPT is a TPT of a condensate system, rather than a dynamical phase transition. Consequently, the state functions describing a QPT are $(\zeta, \varphi)$ in the wave function $\psi = \zeta e^{i\varphi}$.

Second, the field equations governing the condensation are determined by the principle of Hamiltonian dynamics (PHD) or equivalently by the principle of Lagrangian dynamics (PLD):

\begin{equation}
    i\hbar \frac{\partial \Psi}{\partial t} = \frac{\delta}{\delta \psi^*} \mathcal{H}(\psi, \lambda),
\end{equation}

where $\lambda$ is the control parameter. The associated phase transition equations are then the following topological structure equations:

\begin{equation}
    \frac{\delta}{\delta \psi^*} \mathcal{H}(\psi, \lambda) = \begin{cases}
        \mu \psi & \text{for particle number conserved systems}, \\
        0 & \text{for non-conserved systems}.
    \end{cases}
\end{equation}

where $\Psi = e^{-i\mu t/\hbar} \psi(x)$, $\psi = \zeta e^{i\varphi}$, and $\mu \in \mathbb{R}$ represents the chemical potential.

Third, based on the above three characteristics and the topological structure equations (1.2), the wave function of the condensate is a function of the control parameter $\lambda$, i.e. $\psi(\lambda) = \zeta e^{i\varphi}$. Then mathematically we say that the system undergoes a QPT at a critical $\lambda_c$ if the topological structure of $(\zeta(\lambda_c - \varepsilon), \varphi(\lambda_c - \varepsilon))$ is different from that of $(\zeta(\lambda_c + \varepsilon), \varphi(\lambda_c + \varepsilon))$.

Fourth, we deduce from the topological structure equations (1.2) that the supercurrents of superconductivity and superfluidity are given by

\begin{equation}
    J = \begin{cases}
        \frac{\hbar}{m} \zeta^2 \nabla \varphi & \text{for BEC and liquid He}, \\
        \frac{e_s m_s c}{e c} \zeta^2 \left( \hbar \nabla \varphi - \frac{e_s}{c} A \right) & \text{for superconductivity},
    \end{cases}
\end{equation}

which enjoy the divergence-free condition (incompressibility):

\begin{equation}
    \text{div} J = 0.
\end{equation}

Fifth, the divergence-free condition of $J$ in (1.4) shows the incompressibility of the supercurrents. The authors have developed a geometric theory for incompressible flows to study the structure and its stability and transitions of incompressible fluid flows in the physical
spaces. The complete account of this geometric theory is given in the authors’ research monograph [5]. This geometric theory can then be directly applied to study the transitions of topological structure associated with the quantum phase transitions of BEC, superfluidity and superconductivity. We derive in particular the basic theory for different TPTs, leading to transparent physical pictures of various condensates.

Sixth, after careful examination of the formation of Cooper pairs in superconductivity, we derive the following microscopic mechanism for the Meissner effect:

1. Below the critical temperature $T_c$, an applied magnetic field $H_a$ induces spin $J = 1$ Cooper pairs;
2. only the Cooper pairs reversely parallel to the applied $H_a$ are stable, leading to their physical formation; and
3. the total magnetic moment of all Cooper pairs with $s = -1$, together with the surface supercurrents, can cancel out the magnetism induced by the applied field $H_a$ in the superconductor, and resists the applied field $H_a$ from entering its body.

The paper is organized as follows. Section 2 provides precise definitions and the differences for dynamic phase transitions and topological phase transitions. The topological structure equations for quantum condensates are derived in Section 3, and the field equations for condensates are given in Section 5. Precise definition of quantum phase transitions is given in Section 4. The basic theory of quantum phase transitions is addressed in Section 6.

2. Dynamical Phase Transitions and Topological Phase Transitions

2.1. Dynamical phase transitions. Most dissipative systems in non-linear sciences are governed by differential equations, which can written as in the abstract form as follows

$$(2.1) \quad \frac{du}{dt} = K(u, \lambda),$$

where $u$ is the state function, $\lambda$ is the control parameter, and $K$ represents the differential operator.

The equation $(2.1)$ represents the natural law dictates the associated physical systems. The dynamical phase transition for this system is defined by the natural law $(2.1)$.

In science, a basic state often refers to a stable steady state solution $\bar{u}$ of $(2.1)$, namely $\bar{u}$ satisfies

$$K(\bar{u}, \lambda) = 0.$$
If we take the translation 
\[ u = \bar{u} + u' , \]
then the equation (2.1) can be rewritten as (dropping the primes):

\[ \frac{du}{dt} = L_\lambda u + G(u, \lambda) , \]

where \( L_\lambda \) is a linear operator, and \( G \) is a nonlinear operator with higher order terms, i.e.
\[ G(u, \lambda) = o(||u||) . \]

For all dynamical phase transitions of nonlinear dissipative systems in science, \( u = 0 \) represents the basic steady state \( \bar{u} \), and the phase transitions are transitions from the basic state \( u = 0 \) of (2.2) to another stable state. More precisely, the definition of the dynamical phase transitions is based on the following theorem; see [7] and [8].

**Theorem 2.1.** Consider a nonlinear dissipative system (2.2). Let \( \{ \beta_k(\lambda) \in \mathbb{C} \mid k = 1, 2, \cdots \} \) be all eigenvalues of \( L_\lambda \). If

\[ \begin{cases} 
Re\beta_j(\lambda) < 0 & \text{if } \lambda < \lambda_0 , \\
Re\beta_j(\lambda) = 0 & \text{if } \lambda = \lambda_0 , \quad 1 \leq i \leq m , \\
Re\beta_j(\lambda) > 0 & \text{if } \lambda > \lambda_0 , \\
Re\beta_j(\lambda_0) < 0 & \forall \ j \geq m + 1 , 
\end{cases} \]

then the system (2.2) always undergoes a transition from the basic steady state \( u = 0 \) to another stable state (not necessarily a steady state).

Then we have the following definition.

**Definition 2.2.** For the dissipative system (2.2), the transition described by Theorem 2.1 is defined as the dynamical phase transition.

### 2.2. Topological phase transitions

The dynamical phase transitions are based on the natural laws (2.2). However, there exist non-dissipative systems and systems whose topological structure undergoes transitions, rather than from the critical eigenvalues \( \beta_i(\lambda_0)(1 \leq i \leq m) \) as in (2.3). Therefore, it is necessary to introduce the notion of topological phase transitions, a new form of transition, different from the dynamic phase transitions for dissipative systems.

Let \( u \) be a state function of a physical system, and \( u \) depend on some parameter \( \lambda \):

\[ u = u(\lambda) , \quad \lambda \text{ is the time or other control parameters} . \]
Definition 2.3. For the state function $u(\lambda)$ of (2.4) possessing a topological structure (i.e. pattern), if there is a critical parameter $\lambda_0$, such that for any $\delta \lambda > 0$ sufficiently small, the topological structure of $u(\lambda_0 + \delta \lambda)$ is different from that of $u(\lambda_0 - \delta \lambda)$, then we say that the system undergoes a topological phase transition (or pattern formation transition) at $\lambda_0$.

From the above two definitions, Definitions 2.2 and 2.3, we see that a topological phase transition and a dynamical phase transition may occur at different critical points. For the example, the formation of the Taylor vortices for the Taylor-Couette-Poiseuille flow studied in [6] is a topological phase transition, which occurs at a different Taylor number than that for the dynamical transition of the problem.

However, it is quite often that both transitions may occur at the same critical point $\lambda_0$. In fact, assume that the system (2.2) has a phase transition from the basic state $u_0(u = 0)$ to the transition state $u_0$ at $\lambda_0$, if $u_0$ and $u_\lambda$ have different topological structures, then the system (2.2) undergoes both types of phase transitions at $\lambda_0$.

In this series of papers on topological phase transitions, we shall demonstrate the following:

- QPTs are TPTs;
- the transition leading to the galactic spiral structure is both topological and dynamical;
- the transition associated with the solar eruption is a TPT;
- the boundary layer separation of fluid flows is a TPT; and
- the interior separation of fluid flows is a TPT.

In addition, as mentioned in the Introduction, the five categories of typical phase transitions listed in the beginning of the Introduction are all TPTs, except the equilibrium phase transitions, which are dynamic phase transitions.

3. Topological Structure Equation of Quantum Condensates

3.1. New interpretation of wave functions in quantum mechanics. First we recall in this section the interpretation of quantum wave functions derived recently by the authors [9]. In classical quantum mechanics, a micro-particle is described by a complex-valued wave function $\psi : \Omega \to \mathbb{C}$, satisfying such a wave equation as the Schrödinger equation with external interaction potential $V(x)$:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V(x)\psi, \quad x \in \Omega,$$
where \( \Omega \subset \mathbb{R}^3 \) is the region that the particle occupies, and \( m \) is the mass of the particle. The Schrödinger equation conserves the energy, and the wave function \( \Psi \) can be expressed as

\[
\Psi = e^{-iEt/\hbar}\psi(x), \quad \psi = |\psi|e^{i\varphi},
\]

where \( E \) is the energy, and \( \psi \) is the time-independent wave function, satisfying

\[
-\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi = E\psi.
\]

The classical Born statistical interpretation of quantum mechanics amounts to saying that without constraints, the motion of a micro-particle is random and there is no definite trajectory of the motion. Also \( |\psi(x)|^2 \) stands for the probability density of the particle appearing at the particular point \( x \). The Born interpretation of the wave function is treated as a fundamental postulate of quantum mechanics. This leads to the classical Einstein-Bohr debates, and is the origin of absurdities associated with the interpretation of quantum mechanics.

The key observation for the new interpretation is that \( \hbar m \nabla \varphi(x) \), \( \varphi \) is the phase as in (3.2), can be regarded as the velocity field of the particles, and the wave function \( \psi \) is the field function for the motion of all particles with the same mass in the same class determined by the external potential \( V(x) \). More precisely, we have the following new interpretation of quantum mechanical wave functions:

1. Under the external potential field \( V(x) \), the wave function \( \psi \) is the field function for the motion of all particles with the same mass in the same class determined by the external potential \( V(x) \). In other words, it is not the wave function of a particular particle in the classical sense;
2. When a particle is observed at a particular point \( x_0 \in \Omega \), the motion of the particle is fully determined by the solution of the following motion equation with initial position at \( x_0 \):

\[
\frac{dx}{dt} = \frac{\hbar}{m}\nabla \varphi(x), \quad x(0) = x_0,
\]

where \( \varphi \) is the phase of the wave function \( \psi \) in (3.2);
3. With \( \psi \) being the field function,

\[
|\psi(x)|^2 = \text{distribution density of particles at } x;
\]
The energy $E$ in (3.2) represents the average energy level of the particles and can be written as

$$E = \int_{\Omega} \left[ \frac{\hbar^2}{2m} |\nabla|\psi||^2 + \frac{\hbar^2}{2m} |\nabla \varphi(x)||^2 + V(x)|\psi|^2 \right] dx,$$

where in the integrand on the right-hand side, the first term represents the non-uniform distribution potential of particles, the second term is the average kinetic energy, and the third term is the potential energy of the external field. Here $\nabla|\psi|$ is characteristic of quantum mechanics and there is no such a term in classical mechanics.

In summary, our new interpretation says that $\psi = |\psi|e^{i\varphi}$ is the common wave function for all particles in the same class determined by the external potential $V(x)$, $|\psi(x)||^2$ represents the distribution density of the particles, and $\frac{\hbar}{m} \nabla \varphi$ is the velocity field of the particles. The trajectories of the motion of the particles are then dictated by this velocity field. The observed particles are the particles in the same class described by the same wave function, rather than a specific particle in the sense of classical quantum mechanics.

This is an entirely different interpretation from the classical Bohr interpretation. Also this new interpretation of wave functions does not alter the basic theories of quantum mechanics, and instead offers new understanding of quantum mechanics, and plays a fundamental role for the quantum theory of condensed matter physics and quantum physics.

It is worth mentioning that the Landau school of physics was the first who noticed that the relation between the superfluid velocity $v_s$ and the wave function $\psi = |\psi|e^{i\varphi}$ of the condensate is given by $\frac{\hbar}{m} \nabla \varphi$; see [2] (26.12 on page 106). However they fail to make an important connection between $\frac{\hbar}{m} \nabla \varphi$ and the basic interpretation of quantum mechanics.

3.2. **Quantum condensation mechanism.** Also we need to recapitulate the basic quantum mechanism of condensates introduced in [9], based on the new interpretation of wave functions in quantum mechanics in the previous section. This mechanism serves as a unified physical picture and formation for superconductivity, superfluidity and the Bose-Einstein condensates (BEC).

1) **Physical picture of condensates.** The new interpretation of quantum mechanics shows that the wave function $\psi$ is a field governing the motion of particles. The motion of particles in the field of $\psi$ is similar to the motion of objects in a gravitational field. Hence, the physical
picture of a condensate is:

\begin{equation}
\textit{a large collection of Bosons is governed by a wave function field } \psi \textit{ determined by a common potential field, similar to the galactic motion that a large number of stars move permanently around a central gravitational field determined by the galactic nucleus.}
\end{equation}

2). \textit{Microscopic mechanism of condensation formation.} Condensation occurs at low temperatures. The main reason is that at higher temperature, particles possess relatively high kinetic energy, causing them to collide. Collisions are interactions, which make each particle to be situated at an independent interaction field with its own wave function, and consequently the picture (3.3) does not happen.

However, at a relative low temperature, the kinetic energy of particles is small. Hence the possibility of collision decreases, leading to the formation of a common external potential field. In addition, since particles with the same wave function field do not collide, the system energy decreases. By the minimum potential principle, the system has the tendency toward condensation of particles. This is the mechanism of the physical condensation depicted in (3.3).

3). \textit{Mechanism of superfluidity.} As a common potential field $V$ forms, the Bosons are governed by the wave function $\psi = |\psi|e^{i\varphi}$ determined by $V$, where the phase $\varphi$ provides the velocity field of the Bosons:

\begin{equation}
v = \frac{\hbar}{m} \nabla \varphi.
\end{equation}

This velocity field gives rise to a flow field:

\begin{equation}
\text{particle flow } = \rho v,
\end{equation}

where $\rho = |\psi|^2$ is the distribution density of particles and .

The equalities (3.4) and (3.5) give the mechanism of superconductive currents and superfluidity flows.

In summary, we have derived the following condensation formation mechanism for the superconductivity and superfluidity.

\textbf{Condensation Mechanism 3.1.} \hfill (1) \textit{The condition for condensation of bosonic particles is that their mutual interaction is sufficiently weak to ensure that they are in a state governed by the same wave function field under the common bounding potential $V$;}

\hfill (2) \textit{a large collection of bosonic particles occupy the same wave function } \psi \textit{, called condensation wave function, that is determined by the wave equation under the same potential $V$;}
(3) the condensation wave function $\psi$ is the smallest allowed energy state of the wave equation; and

(4) the motion of particles in a condensate system is determined by (3.4)–(3.5), which is energy-conserved, resembles the motion of stars in a galaxy, and can last permanently.

3.3. Topological structure equations. Let the wave function $\psi$ of a condensate system be expressed as

$$\psi = \zeta e^{i\varphi}, \quad \zeta = |\psi|.$$  

In Section 3.1 we have given the physical meaning of $\zeta$ and $\varphi$:

$\rho = \zeta^2$ is distribution density of particles in the condensate; and

$$\frac{\hbar}{m} \nabla \varphi$$ is the velocity field of the flow of particles.

Hence $\zeta$ and $\varphi$ describe the topological structure of condensates in the physical space, and the equations of $\zeta$ and $\varphi$ are called topological structure equations or pattern formation equations.

The topological structure equations for $\zeta$ and $\varphi$ can be derived from the field equations

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{\delta}{\delta \Psi^*} H(\Psi),$$

where $H(\Psi)$ is the Hamiltonian energy of the system. The energy of the Hamiltonian system (3.7) is conserved, i.e. both the energy and the number of particles are conserved. Hence the solution of (3.7) can be written as:

$$\Psi = e^{-i\mu t/\hbar} \psi(x), \quad \psi \text{ is as (3.6)},$$

where $\mu \in \mathbb{R}$ represents the chemical potential. We infer from (3.8) that

$$\mu \psi = \frac{\delta}{\delta \psi^*} H(\psi).$$

Hence (3.7) is reduced to the steady state form given by (3.9).

1). Superfluid systems. For gaseous and liquid systems, (3.9) can be written in a unified form as follows:

$$-\frac{\hbar^2}{2m} \Delta \psi + f(|\psi|) \psi = 0,$$

where $f(x)$ is a function of $x \in \mathbb{R}$. If $\psi$ is a spinor, then $f(x)$ is a matrix-valued function of $x \in \mathbb{R}^3$, and $\psi$ can be written as

$$\psi = \zeta e^{i\varphi}, \quad \zeta = (\zeta_+, \zeta_0, \zeta_-),$$
as the three types of particles in the system possess the same mass, and consequently share the same velocity field $-\frac{\hbar}{m} \nabla \varphi$. 

In view of (3.6), we infer from (3.10) the following topological structure equations for gaseous and liquid condensate systems:

\begin{equation}
\text{div}(\zeta^2 \nabla \varphi) = 0,
\end{equation}

\begin{equation}
-\frac{\hbar^2}{2m} [\Delta \zeta - \zeta |\nabla \varphi|^2] + f(\zeta) \zeta = 0.
\end{equation}

2). \textit{Superconducting systems.} For a superconducting system, (3.9) can be written in the following unified form:

\begin{equation}
\frac{1}{2m_s} \left( i \hbar \nabla + \frac{e_s}{c} A \right) \psi + f(|\psi|) \psi = 0,
\end{equation}

with superconducting currents:

\[ J_s = -\frac{e_s^2}{m_s c^2} |\psi|^2 A - \frac{e_s \hbar}{2m_s c} i (\psi^* \nabla \psi - \psi \nabla \psi^*) . \]

In view of (3.6), $J_s$ becomes

\begin{equation}
J_s = -\frac{e_s^2}{m_s c^2} \zeta^2 A + \frac{e_s \hbar}{m_s c} \zeta^2 \nabla \varphi .
\end{equation}

Also by (3.6), (3.13) is rewritten as

\begin{equation}
\text{div} \left( \hbar \zeta^2 \nabla \varphi - \frac{e_s}{c} \zeta^2 A \right) = 0,
\end{equation}

\begin{equation}
-\frac{\hbar^2}{2m_s} [\Delta \zeta - \zeta |\nabla \varphi|^2] + \frac{e_s^2}{2m_s c^2} A^2 \zeta + f(\zeta) \zeta = 0.
\end{equation}

In (3.15), we have used the Coulomb gauge:

\[ \text{div} A = 0 . \]

By (3.14) and (3.15), we obtain that $\text{div} J_s = 0$.

The above two systems of equations: (3.11)–(3.12), and (3.15)–(3.16) are the topological structure equations in the physical space for superfluidity, superconductivity and BEC. These are important equations. First, they dictate two important physical structures of condensate particles:

- physical picture of the distribution function $\rho = \zeta^2$, and
- topological structure of the flow determined by the velocity field $\frac{\hbar}{m} \nabla \varphi$.

Second, the topological structure equations are fundamental equations for studying quantum phase transitions. In fact, quantum phase
transition deals with the phenomena of structure changes in the physical space of the topological structure equations (3.11)–(3.12), and (3.15)–(3.16) of condensate systems at the critical control parameter.

4. **Quantum Phase Transitions (QPTs)**

4.1. **Classical QPTs.** To the best knowledge of the authors, there is no clear definition for QPTs in classical statistical physics. Usually, a QPT is regarded as a transition with the following three basic characteristics:

1. the transition occurs with temperature at or near absolute zero;
2. the control parameters are non-thermal; and
3. the change of the physical states at a threshold is induced by quantum fluctuations, rather than thermal fluctuations.

Here the quantum fluctuations mean that they are caused by the Heisenberg uncertainty relation.

A popular description for a QPT is given by Figure 4.1; see among others [10]. In this diagram, the horizontal axis represents the control parameter $\lambda$ used to tune the system through the QPT, and the vertical axis is the temperature. In Figure 4.1(a), the solid line separates the region into two parts D and E, where D is the region of ordered phase, E is the region of disordered phase, and the QPT occurs between the phases. In Figure 4.1(b), the region E is separated into three parts A, B, C by the dashed lines, where A represents the domain where the states are controlled by quantum fluctuations, C is the domain by thermal fluctuations, and B is the quantum critical region where both types of fluctuations are important.

![Figure 4.1](image)

**Figure 4.1**

Figure 4.1 offers the physical definition of the classical QPT: There are two ways that result in the transition, the first one is from region C to D, which is the thermodynamical phase transition, and the second one is from A through B to D, which is called the QPT.
Hence, from Figure 4.1 and the three basic characteristics of QPTs listed in the beginning of this section, we arrive at the following classical criterion for QPT:

(4.1) \[ \text{the QPT is caused by quantum fluctuations.} \]

4.2. Definition of QPTs. We note that the classical criterion of QPT is given by (4.1). However, it is hard to distinguish both thermal fluctuations and quantum fluctuations. In particular, we know that the reason causing the instability of a system is the state functions lose their stability, rather than the fluctuations, which only make the system deviate from the unstable states. Hence, the fluctuations are not the main characters of phase transitions, and can not be used to define QPTs.

A precise definition is crucial to establish a QPT theory. Based on the condensation theory developed in Section 3, a better description for QPTs can be derived. First, a QPT should possess the following three physical characteristics:

Characteristics of QPT 4.1. (1) A QPT is a transition between quantum states, and the state quantities of the quantum system describing the transition should be the wave functions \( \psi \) of the quantum states;

(2) the control parameters are non-thermal; and

(3) since the dynamical equations (3.7) of a QPT system are energy conserved, a QPT is a topological phase transition of a condensate system, rather than a dynamical phase transition. Consequently, the state functions describing a QPT are \( (\zeta, \varphi) \) in the wave function \( \psi = \zeta e^{i\varphi} \); see (3.6).

Thanks to the above three characteristics, we introduce the following definition of QPTs.

Definition 4.2. Let \( \psi(\lambda) = \zeta e^{i\varphi} \) be a wave function of a QPT system, where \( \lambda \) be the control parameter. If there is a critical value \( \lambda_c \), for any \( \Delta \lambda > 0 \) sufficiently small such that the topological structure of \( (\zeta(\lambda_c - \Delta \lambda), \varphi(\lambda_c - \Delta \lambda)) \) is different from that of \( (\zeta(\lambda_c + \Delta \lambda), \varphi(\lambda_c + \Delta \lambda)) \).

Remark 4.3. In the above definition, the topological structure of \( (\zeta, \varphi) \) is described as follows; see the description at the end of Section 3:

(1) the flow structure of velocity field \( \zeta^2 \nabla \varphi \), and

(2) the function structure of distribution density \( \rho = \zeta^2 \).

In quantum condensation systems, \( \zeta^2 \nabla \varphi \) represents the superfluid flows or superconducting currents.
4.3. Examples. Based on the definition of QPTs, Definition 4.2, we present in this section a few examples of QPTs.

1). Josephson superconductor-insulator transition. In superconductivity, the Josephson tunnel effect is that a superconducting current remains even if a thin insulating layer is sandwiched in a circular superconductor. The quantum effect shows a QPT behavior.

Consider a circular superconductor with an insulating section in the conductor, as shown in Figure 4.2. Let \( \lambda > 0 \) be the thickness of the insulating layer. It is easy to know that there exists critical thickness \( \lambda_c > 0 \), such that the superconducting current \( J_s \) has a transition at \( \lambda_c \), i.e.

\[
J_s \begin{cases} 
\neq 0 & \text{if } 0 < \lambda < \lambda_c, \\
= 0 & \text{if } \lambda_c < \lambda. 
\end{cases}
\]

The phenomena is the QPT, called the Josephson superconductor-insulator transition. If the wave function of the superconductivity is as \( \psi = \zeta e^{i\varphi} \), then \( J_s \) is as in (3.14):

\[
J_s = \frac{e\gamma_h}{m_s c^2} \zeta^2 \nabla \varphi(\lambda), \quad \nabla \varphi = a(\lambda) \sin(\varphi_2 - \varphi_1),
\]

where \( \varphi_2 \) and \( \varphi_1 \) are the phases of \( \psi \) on the two sides of the insulating section, \( a(\lambda) \) is phase factor with \( a(\lambda) = 0 \) for \( \lambda > \lambda_c \). Hence, (4.2) can be rewritten as

\[
\nabla \varphi(\lambda) \begin{cases} 
\neq 0 & \text{for } 0 < \lambda < \lambda_c, \\
= 0 & \text{for } \lambda_c < \lambda. 
\end{cases}
\]

It is clear that this is a typical QPT as defined by Definition 4.2.

\[\text{Figure 4.2}\]

2). Mott superfluid-insulator transition. The Mott superfluid-insulator transition is the QPT of a gaseous BEC system, which undergoes a condensation transition as the temperature \( T < T_c \). Similar to the liquid
He, under an external potential field, the gaseous BEC system also exhibits the superfluidity behavior. In this case, the wave function is

\[ \psi = \zeta e^{i\phi}, \]  
(4.3)

superfluidity: \( \frac{\hbar}{m} \zeta^2 \nabla \varphi \neq 0. \)

The superfluidity in (4.3) occurs in the system is exerted some potential barrier, then it will yield a damping for the superfluidity. The damping can be describe by a control parameter:

\[ \lambda = \frac{V}{J}, \]

where \( V \) represents the barrier intensity, and \( J \) is the coefficient of tunnel breakdown. Then, the experiments show that there exists a critical value \( \lambda_c \), such that

\[ \nabla \varphi \begin{cases} \neq 0 & \text{if } 0 \leq \lambda < \lambda_c, \\ = 0 & \text{if } \lambda_c < \lambda. \end{cases} \]  
(4.4)

The phenomena (4.4) is the Mott superfluid-insulator transition.

3). **Abrikosov vortex-Meissner phase transition.** In superconductivity, the Abrikosov vortex-Meissner phase transition is a typical QPT phenomena. Based on the Abrikosov theory, for a Type-II superconductor, there are two types of superconducting states:

the Meissner phase denoted by \( \psi_M \),

the vortex phase denoted by \( \psi_V \).

The control parameter \( \lambda \) is the external magnetic field:

\[ \lambda = \text{the applied magnetism } H_a. \]

Then there is a critical magnetic field \( \lambda_c \), such that

\[ \text{the superconducting wave function } = \begin{cases} \psi_M & \text{as } \lambda_c < \lambda < \lambda_N, \\ \psi_V & \text{as } 0 < \lambda < \lambda_c, \end{cases} \]

where \( \psi_M, \psi_V \) are as in (4.5), \( \lambda_N \) is the critical magnetic field for the normal state. The property (4.6) is called the Abrikosov vortex–Meissner phase transition.

4). **Liquid \(^3\text{He} \) A–B phase transitions.** In the absence of applied magnetic field, the superfluidity of liquid \(^3\text{He} \) has two phases:

A phase: \( \psi_A = \psi_+ + \psi_- \),

B phase: \( \psi_B = \psi_+ + \psi_- + \psi_0 \),

where \( \psi_+, \psi_- \), \( \psi_0 \) represent the wave functions for the particles with spin \( s = 1, -1, 0 \) respectively.
When a magnetic field $H$ is applied, the system has three phases:

$$
\text{(4.8) A, B phases and } A_1 \text{ phase: } \psi_{A_1} = \psi_+.
$$

For (4.7) and (4.8), the control parameters are

$$
\lambda = \begin{cases} 
\text{pressure } p & \text{for } H = 0, \\
\text{applied magnetic field } H & \text{for } H \neq 0.
\end{cases}
$$

Then there exist three types of QPTs for the $^3\text{He}$ superfluid as follows:

1. For $H = 0$, there are two critical temperatures $T_0$ and $T_1$. For each fixed $T$ with $T_0 < T < T_1$, there exists a critical pressure $p_c$, such that the following QPT occurs as $p$ crosses $p_c$ and is near $p_c$:

$$
\text{(4.9) the superfluid wave function } \psi = \begin{cases} 
\psi_A & \text{for } p < p_c, \\
\psi_B & \text{for } p < p_c,
\end{cases}
$$

where $\psi_A$, $\psi_B$ are as in (4.7).

2. For $H \neq 0$ with $H = |H|$, there are three critical temperatures $T_0 < T_1 < T_2$.

If $T_1 < T < T_2$, there is a critical magnetic field $H_{c_1}$, such that for $H$ near $H_{c_1}$,

$$
\text{(4.10) the superfluid wave function } \psi = \begin{cases} 
\psi_A & \text{if } H_{c_1} < H, \\
\psi_{A_1} & \text{if } H < H_{c_1}.
\end{cases}
$$

3. For $H \neq 0$ and for $T_0 < T < T_1$, there is an $H_{c_2}$ such that for $H$ near $H_{c_2}$,

$$
\text{(4.11) the superfluid wave function } \psi = \begin{cases} 
\psi_B & \text{if } H_{c_2} < H, \\
\psi_A & \text{if } H < H_{c_2},
\end{cases}
$$

where $\psi_{A_1}$ is as in (4.8).

The three transitions (4.9), (4.10) and (4.11) characterize the QPTs for the liquid $^3\text{He}$ superfluid.

5. Field Equations for Condensates

5.1. Field equations of superconductivity. The field equations governing the condensation are determined by the principle of Hamiltonian dynamics (PHD) and the principle of Lagrangian dynamics (PLD).

Let $\mathcal{H}(\psi, E, H, \lambda)$ be the Hamiltonian energy of a condensation system, where $\psi$ is the wave function, $(E, H)$ is the electromagnetic fields,
and $\lambda$ is the control parameter. Based on PHD, the Hamiltonian system is

$$
2\hbar \frac{\partial \psi_1}{\partial t} = \frac{\delta}{\delta \psi_2} \mathcal{H},
$$

$$
2\hbar \frac{\partial \psi_2}{\partial t} = -\frac{\delta}{\delta \psi_1} \mathcal{H},
$$

and by PLD, the Lagrangian dynamic equations read as

$$
i\hbar \frac{\partial \psi}{\partial t} = \frac{\delta}{\delta \psi^*} \mathcal{H}
$$

$$
\frac{\delta}{\delta \mathbf{E}} \mathcal{H} = 0, \quad \frac{\delta}{\delta \mathbf{H}} \mathcal{H} = 0,
$$

where $\psi = \psi_1 + i\psi_2$.

The equations (5.1) and (5.2) are equivalent, and both will be used to describe different properties of condensation systems. Because (5.1) and (5.2) are energy conserved systems, the quantum phase transitions described by them must be topological phase transitions.

We now present the explicit expression of the field equations of superconductivity. Based on the thermodynamic potential theory established in [3], the Hamilton energy of a superconductor is given by

$$
\mathcal{H} = \int_{\Omega} \left[ \frac{1}{2m_s} \left( -i\hbar \nabla - \frac{e_s}{c} \mathbf{A} \right) \psi \right]^2 - V |\psi|^2 + \frac{g}{2} |\psi|^4
$$

$$
+ \frac{1}{8\pi} \mathbf{E}^2 - \frac{1}{4\pi} \mathbf{E} \cdot \mathbf{E}_a - \varepsilon \cdot \mathbf{E} |\psi|^2
$$

$$
+ \frac{1}{8\pi} \mathbf{H}^2 - \frac{1}{4\pi} \mathbf{H} \cdot \mathbf{H}_a - \mu \cdot \mathbf{H} |\psi|^2 \right] \, dx.
$$

where ($\mathbf{E}, \mathbf{H}$) are the electromagnetism, ($\mathbf{E}_a, \mathbf{H}_a$) are the applied fields, $e_s$ is the charge of Cooper pair, $V$ is the bounding potential, and $\varepsilon$ is the polarization, $\mu$ is the induction magneton

$$
\mu = \frac{e_s \hbar}{m_s c} \mu_0 \hat{\mathbf{H}}_a,
$$

and $\hat{\mathbf{H}}_a$ is the unite vector in the direction of $\mathbf{H}_a$, $\mu_0$ is

$$
\mu_0 = \frac{N^+ - N^-}{N},
$$

where $N^+, N^-$ represent the numbers of the Cooper pairs with spin $s = 1$ and $s = -1$, $N$ is the total number of superconducting electrons.
The control parameter $\lambda$ is

$$\lambda = (E_a, H_a).$$

The Lagrangian dynamic equations (5.2) for superconducting systems (5.3) are written in the form

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m_s} \left( i\hbar \nabla + \frac{e_s}{c} A \right)^2 \psi - V\psi + g|\psi|^2\psi - (\mu \cdot H + \epsilon \cdot E)\psi,$$

together with the electromagnetic equilibrium equations

$$\frac{\delta}{\delta E} \mathcal{H} = 0, \quad \text{and} \quad \frac{\delta}{\delta H} \mathcal{H} = 0,$$

which are expressed as

$$E = E_a + 4\pi|\psi|^2\epsilon, \quad H = H_a + 4\pi|\psi|^2\mu,$$

**Remark 5.1.** For the magnetic field $H$ in (5.7) for a superconductor, we see that

$$H = 0 \iff H_a = -4\pi|\psi|^2\mu.$$ This is the Meissner effect. It shows that the demagnetic effect is caused by the Cooper pairs with spin $s = -1$, i.e. their orientation is reverse to $H_a$. The spins of Cooper pairs as shown in Figure 5.1.

![Figure 5.1](image-url)

**Figure 5.1.** Arrangements of Cooper pairs: (a) collinear orientation with spin $J = 1$; (b) anti-parallel orientation with spin $J = 0$; they lead to attractive magnetic moment force.
5.2. Field equations of superfluidity and BEC. In this section, we consider the Lagrangian dynamical equations for superfluid and BEC systems.

1). $^4$He superfluid and scalar BEC systems. The Hamiltonian energy for $^4$He superfluid and scalar BEC systems is the Gross-Pitaevskii functional, given by

$$\mathcal{H} = \int_{\Omega} \left[ \frac{\hbar^2}{2m} |\nabla \psi|^2 - V|\psi|^2 + \frac{g}{2} |\psi|^4 \right] dx,$$

where the control parameter is the bounding potential $\lambda = V$. The Lagrangian dynamic equation for (5.8) is given by

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi - V \psi + g|\psi|^2 \psi.$$

2). $^3$He systems without no applied magnetic fields and spinor BEC systems. These are spinor systems with 3-component wave function

$$\Psi = (\psi_+, \psi_0, \psi_-),$$

where $\psi_+, \psi_0, \psi_-\,$ represent the particles with spin $s = 1, 0, -1$ respectively. The energy functional is given by

$$\mathcal{H} = \int_{\Omega} \left[ \frac{\hbar^2}{2m} |\nabla \Psi|^2 - \Psi^\dagger V \Psi + \frac{g}{2} |\Psi|^4 + \frac{g_s}{2} |\Psi^\dagger \hat{F} \Psi|^2 \right] dx,$$

where $V$ is the matrix representing bounding potential, $g, g_s > 0$ are coupling parameters, $\hat{F}$ is the spin operator

$$\hat{F} = (F_1, F_2, F_3),$$

with $F_i \,(1 \leq i \leq 3)$ being Hermitian matrices:

$$F_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

and $\Psi^\dagger \hat{F} \Psi$ is written as

$$\Psi^\dagger \hat{F} \Psi = (\Psi^\dagger F_1 \Psi, \Psi^\dagger F_2 \Psi, \Psi^\dagger F_3 \Psi)^T,$$

$$\Psi^\dagger F_i \Psi = (\psi_+^*, \psi_0^*, \psi_-^*) F_i \begin{pmatrix} \psi_+ \\ \psi_0 \\ \psi_- \end{pmatrix}.$$

The Lagrangian field equation for (5.11) is

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi - V \Psi + g|\Psi|^2 \Psi + g_s (\Psi^\dagger \hat{F} \Psi) \cdot \hat{F} \Psi,$$
where \( V \) is the matrix as
\[
\begin{pmatrix}
V_+ & 0 \\
V_0 & V_-
\end{pmatrix}
\]
and \( V_+, V_0, V_- \) are the bounding potential acting on \( \psi_+, \psi_0, \psi_- \) respectively. The control parameter is \( \lambda = (V_+, V_0, V_-) \).

3. \(^3\)He superfluid systems with applied magnetic fields. When an external magnetic field \( H_a \) is exerted, the energy functional of a \(^3\)He superfluid system includes two parts
\[
\mathcal{H} = \mathcal{H}_\Psi + \mathcal{H}_M,
\]
where \( \mathcal{H}_\Psi \) is as in (5.11), and \( \mathcal{H}_M \) is the magnetic field energy given by
\[
\mathcal{H}_M = \int_\Omega \left[ \frac{1}{8\pi} H^2 - \frac{1}{4\pi} \mathbf{H} \cdot \mathbf{H}_a - \mu_0 \Psi^\dagger (\mathbf{H} \cdot \hat{\mathbf{F}}) \Psi \right] \, dx,
\]
where \( \mu_0 \) is the magnetization coefficient. Then the Hamiltonian energy of the \(^3\)He system is written as (see [3])
\[
\mathcal{H} = \int_\Omega \left[ \frac{\hbar^2}{2m} |\nabla \Psi|^2 - \Psi^+ V \Psi + \frac{g}{2} |\Psi|^4 + \frac{g_s}{2} |\Psi^\dagger \hat{\mathbf{F}} \Psi|^2 \\
+ \frac{1}{8\pi} H^2 - \frac{1}{4\pi} \mathbf{H} \cdot \mathbf{H}_a - \mu_0 \Psi^\dagger (\mathbf{H} \cdot \hat{\mathbf{F}}) \Psi \right] \, dx,
\]
where \( \Psi \) is as in (5.10), \( V \) is as in (5.15), \( \hat{\mathbf{F}} = (F_1, F_2, F_3) \) is as in (5.12). Then the Lagrangian system (5.2) for the Hamiltonian (5.16) is in the form
\[
i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \delta \Psi - V \Psi + g |\Psi|^2 \Psi + g_s (\Psi^\dagger \hat{\mathbf{F}} \Psi) \cdot \hat{\mathbf{F}} \Psi - \mu_0 (\mathbf{H} \cdot \hat{\mathbf{F}}) \Psi,
\]
with the magnetic field equation \( \frac{\delta}{\delta \mathbf{H}} \mathcal{H} = 0 \) as
\[
\mathbf{H} = \mathbf{H}_a + 4\pi \mu_0 (\mathbf{H} \cdot \hat{\mathbf{F}}) \Psi.
\]
The control parameter of the system (5.17)–(5.18) is
\[
\lambda = (V_0, V_1, V_2, H_a).
\]

6. Theory of Quantum Phase Transitions

6.1. Topological structure of quantum condensates. The four groups of equations (5.6)–(5.7), (5.9), (5.14), and (5.17)–(5.18) constitute the basic models to study topological phase transitions for quantum condensation systems. They can be in the unified abstract form
as

\[ i\hbar \frac{\partial \psi}{\partial t} = \frac{\delta}{\delta \psi^*} \mathcal{H}(\psi, \lambda). \]

There are two types of transitions for (6.1) given as follows:

1) **Condensate particle number conserved systems.** In this case, the equation (6.1) is equivalent to (3.9), i.e.

\[ \frac{\delta}{\delta \psi^*} \mathcal{H}(\psi, \lambda) = \mu \psi, \]

where \( \mu \) is the chemical potential.

2) **Non-conserved condensate particle number systems.** For this transition, the number of condensate particles is not conserved, and the model describing the transition is the steady state equation of (6.1) given by

\[ \frac{\delta}{\delta \psi^*} \mathcal{H}(\psi, \lambda) = 0. \]

As mentioned in Section 4.2, the QPT for systems (6.2) and (6.3) is the transition of topological phase transition (pattern formation) as defined by Definition 4.2. To establish the QPT theory for (6.2) and (6.3), we need to introduce the mathematical theory of the topological structure of \((\zeta, \varphi)\) for the solutions \(\psi\) of (6.2) and (6.3) written as

\[ \psi(\lambda) = \zeta(\lambda)e^{i\varphi(\lambda)}. \]

1). To justify Definition 4.2, it is necessary to ensure the topological structure given by \((\zeta(\lambda), \varphi(\lambda))\) in (6.4) is physically stable:

\[ \text{for a non-critical value } \lambda, \text{ the topological structure defined by } (\zeta(\lambda), \varphi(\lambda)) \text{ is the same (in equivalent sense) as that of } (\zeta(\lambda \pm \varepsilon), \varphi(\lambda \pm \varepsilon)) \text{ for any } \varepsilon > 0 \text{ sufficiently small.} \]

2). Both equations (6.2) and (6.3) can be equivalently expressed in the form of (3.11)–(3.12) for gaseous and liquid condensation systems, and in the form of (3.15)–(3.16) for superconducting systems. Hence we deduce from these equations that the supercurrents of the condensate particles given by

\[ J = \begin{cases} \frac{\hbar}{m} \zeta^2 \nabla \varphi & \text{for BEC and liquid He,} \\ \frac{e_s}{m_s c} \zeta^2 \left( \hbar \nabla \varphi - \frac{e_s}{c} A \right) & \text{for superconductivity} \end{cases} \]

satisfy the divergence-free condition

\[ \text{div } J = 0. \]
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3). The authors have developed a geometric theory for incompressible flows to study the structure and its stability and transitions of 2-D incompressible fluid flows in the physical spaces. The complete account of this theory is given in the authors’ research monograph [5].

The divergence-free condition of $J$ in (6.7) shows the incompressibility of the supercurrents. For the 2D divergence–free vector fields, we proved [5] the structural classification and structural stability theorems, which amount to saying that a 2D divergence–free regular vector field consists of finite number of vortices, and they are structurally stable as defined in (6.5).

Based on the authors’ geometric theory for 2D incompressible flows, we can define a topological index for 2D systems. For $(\zeta, \varphi)$ in (6.4) as follows

\begin{equation}
\text{Ind}(\psi) = \text{Ind}(\zeta, \varphi) = N,
\end{equation}

where $J$ is as in (6.6)–(6.7), and $N$ is the vortex number of $J$. The index (6.8) represents the topological structure of $(\zeta, \varphi)$ satisfying the stability defined by (6.5).

4). For 3D systems, the vector fields $J$ given by (6.6)–(6.7) are harmonic fields in essence. Hence, if $J \neq 0$ then in mathematics the harmonic field $J$ consists of finite numbers of columnar vortex flows and torus flows as shown in Figure 6.1(a) and (b), which are stable in the sense that their columnar and torus structures remain invariant.

Hence, we can define the topological index for the 3D system by

\begin{equation}
\text{Ind}(\psi) = \text{Ind}(\zeta, \varphi) = (N_1, N_2),
\end{equation}

where $N_1, N_2$ represent the numbers of columnar flows and torus flows of $J$. Such index possesses the stability stability as required by (6.5).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figures/6.1.png}
\caption{(a) Columnar vortex flows; (b) torus flows}
\end{figure}
5). For the case $\nabla \varphi = 0$, the wave function
\[
(6.10) \quad \psi = \text{real function, i.e. } \text{Im} \psi = 0.
\]
For the real function $\psi$, the domain $\Omega \subset \mathbb{R}^n$ ($1 \leq n \leq 3$) can be divided into three kinds of connected regions as
\[
\begin{align*}
\Omega^+ &= \{ x \in \Omega \mid \psi(x) > 0 \}, \\
\Omega^- &= \{ x \in \Omega \mid \psi(x) < 0 \}, \\
\Omega^0 &= \{ x \in \Omega \mid \psi(x) = 0 \}.
\end{align*}
\]
(6.11)

It is clear that the topological structure defined by the numbers of connected components of $\Omega^+$ and $\Omega^-$ is mathematically stable if and only if the following two conditions hold true:

(i) the numbers $N^+$ and $N^-$ of connected components of $\Omega^+$ and $\Omega^-$ are finite; and

(ii) the dimension of each $\Omega^0$ is less than the dimension $n$ of the domain $\Omega$:
\[
\dim \Omega^0 < n.
\]

For physical stability, it suffices that only (i) holds true, because in condensation systems, the normal state regions $\Omega^0$ with $\dim \Omega^0 = n$ are usually caused by the material properties, which possess stability under small perturbation of $\lambda$. Hence, $\psi$ is called the physically stable, if only condition (i) above holds true.

For physically stable field $\psi$, we define the index as
\[
(6.12) \quad \text{Ind}(\psi) = (N^+, N^-),
\]
where $N^+$ and $N^-$ are the numbers of connected regions $\Omega^+$ and $\Omega^-$ defined by (6.11).

**Remark 6.1.** In the case that $\nabla \varphi = 0$ and $\zeta \neq 0$, the system is in condensation state, but no particle superflows exist. The states of $\psi$ in $\Omega^+$ and $\Omega^-$ are different from phases each other, i.e.
\[
(6.13) \quad \begin{align*}
\varphi^+ - \varphi^- &= (2m + 1)\pi, \\
\varphi^+_i - \varphi^+_j &= 2m\pi, \quad i \neq j, \quad (m = 0, \pm 1, \pm 2, \cdots), \\
\varphi^-_k - \varphi^-_l &= 2m\pi, \quad k \neq l,
\end{align*}
\]
where $\varphi^+_i, \varphi^-_k$ represent the phases of $\varphi$ in $\Omega^+_i$ and $\Omega^+_k$ and $\Omega^+_i$ and $\Omega^+_j$ ($i \neq j$) are different connected domains.

Although the phase differences in (6.13) can not be detected by experiments, the different indices of (6.12) represent different states of $\psi$, and under some external driving forces, the states will illustrate their own superflow structure.
6.2. Microscopic mechanism of the Meissner effect. The Meissner effect is one of two main properties: superconductivity and diamagnetism. When the temperature is below \( T_c \), a superconductor not only excludes an external magnetic field to enter its body, but also removes the magnetism, which was originally present in its interior in the normal state. A superconducting state is called the Meissner phase if the superconductor lies in a completely diamagnetic state.

In Section 4.3, we considered the Abrikosov vortex–Meissner phase transition, which is a QPT from the Meissner phase to vortex state in a Type-II superconductor when an applied magnetic field \( H_a \) increases to exceed a critical field \( H_c \). To establish the QPT theory for the Abrikosov transition, we have to consider the microscopic mechanism for the Meissner effect.

In Figure 5.1 (a) and (b), we note that there are two arrangements for Cooper pairs, leading to different spins:

\[
\begin{align*}
\text{spin } J = 0 & \quad \text{in Figure 5.1(a),} \\
\text{spin } J = 1 & \quad \text{in Figure 5.1(b).}
\end{align*}
\]

For the case without an applied magnetic field, or for a moving Cooper pair, the spin \( J = 0 \) state is stable and the \( J = 1 \) state is unstable; see [9].

When an external magnetic field \( H_a \) is applied, the arrangement of the \( J = 1 \) Cooper pairs become stable since the electrons are held in their position by the applied magnetic field. There are two orientations for the \( J = 1 \) Cooper pairs (i.e, \( s = \pm 1 \)): parallel to the field \( H_a \) (\( s = 1 \)) and reversely parallel to \( H_a \) (\( s = -1 \)); see Figure 6.2.

**Figure 6.2.** (a) Cooper pairs parallel to \( H_a \) (\( s = 1 \)), and (b) Cooper pairs reversely parallel to \( H_a \) (\( s = -1 \)).
We know that for a magnetic body under a magnetic field \( H \), if the north-pole \( N \) points to the same direction as \( H \), then the force acting on the north-pole of this body points to the same direction as \( N \), and on the force acting on the south pole \( S \) points to the direction of \(-H\); see Figure 6.3(a). Conversely, if the north pole is in the direction of \(-H\), then the forces acting on both the N and S poles are reversed as shown in Figure 6.3(b).

![Figure 6.3](image)

**Figure 6.3.** Directions of the forces \( F \) acting on the two ends of the magnetic body under an applied magnetic field \( H \).

Based on this principle of magnetic interaction as described by Figure 6.3, it is easy to see that the Cooper pairs as shown in Figure 6.2(a) are exerted a stretching force by \( H_a \), and these as shown in Figure 6.2(b) are exerted a squeezing force. Consequently, the Cooper pairs parallel to \( H_a \) are unstable as the two electrons can break easily due to the stretching force. However, the Cooper pairs reversely parallel to \( H_a \) are stable.

In addition, each Cooper pair with spin \( J = 1 \) \((s = \pm 1)\) induces a magnetic moment \( \vec{\mu}_0 \) given by

\[
\vec{\mu}_0 = \frac{e_s \hbar}{m_s c} \vec{S},
\]

where \( \vec{S} \) is the unit vector at the spin direction. Because the orientation of \( \vec{\mu} \) is reversely parallel to that of \( H_a \), the total magnetic moment of all Cooper pairs with spin \( s = -1 \) can counteract the applied magnetic field \( H_a \). Hence, we derive the following microscopic mechanism for the Meissner effect.

**Mechanism of Meissner Effect 6.2.**

1. **Below the critical temperature** \( T_c \), **an applied magnetic field** \( H_a \) **induces spin** \( J = 1 \) **Cooper pairs**;
(2) Only the Cooper pairs with spin $s = -1$ (reversely parallel to $H_a$) are stable, leading to their physical formation;

(3) The total magnetic moment of all Cooper pairs with $s = -1$, together with the surface supercurrents, can cancel out the magnetism induced by the applied field $H_a$ in the superconductor, and resists the applied field $H_a$ from entering its body.

**Remark 6.3.** The current explanation of the Meissner effect is that the magnetic field $M$ induced by the surface supercurrents cancels out the applied field $H_a$. However, if the applied field $H_a$ is homogeneous, it is impossible to be completely eliminated by the nonhomogeneous magnetic field induced by the surface supercurrents. This fact shows that the magnetic field

$$M = -\frac{e_s h}{m_s c} N^\ast |\psi|^2 \hat{H}_a$$

plays an important role to counteract the applied field $H_a$ in the Meissner effect, where $N^\ast$, $N$ and $\hat{H}_a$ are as in (5.4).

6.3. **Abrikosov vortex–Meissner phase transition.** In Section 4.3, we briefly introduced the Abrikosov vortex–Meissner phase transition (A–M transition). Based on the mechanism of the Meissner effect and the superconducting field equations (5.6) and (5.7), we can derive the theory for this topological phase transition.

1). **Meissner state and vortex state.** In a Type-II superconductor, in general there are two superconducting states (or three states in some materials): the Meissner state and the vortex state (also called the mixed state). The Meissner state is that in the surface there is a penetrating magnetism, and in the interior the magnetism is zero:

the Meissner state = the interior magnetism $H = 0$.

The vortex state is a mixed state, i.e. the normal and the superconducting states coexist. In this phase, the magnetism penetrates the sample to form some columnar vortex supercurrents with central column being in the normal state, as shown in Figure 6.4 called the Abrikosov vortices.

2). **Phase diagram for critical magnetic fields.** For general Type-II superconductors, there are two (or three for certain materials) critical magnetic fields, denoted by $H_{c1}$, $H_{c2}$ and $H_{c3}$. The phase diagram for Type-II superconductors are given by Figure 6.5.

For the A–M transition, a Type-II superconductor enters the vortex phase from the Meissner phase as the magnetic field $H_a$ increase crossing the first critical magnetic field $H_{c1}$.
Figure 6.4. Schematic diagram for the Abrikosov vortex: the central column region marked by N is in the normal state, the magnetism penetrates the region, and $J_s$ is the supercurrents, reversely parallel to $H_a$.

Figure 6.5

3). Theory for A–M transition. The A–M transition is an equilibrium state transition process. The model describing this system is the steady state equations of (5.6) and (5.7) with $E = 0$:

$$\frac{1}{2m_s} \left( i\hbar \nabla + \frac{e_s}{c} \mathbf{A} \right)^2 \psi - V\psi + g|\psi|^2\psi - \mathbf{\mu} \cdot \mathbf{H}\psi = 0,$$

(6.14)

$$\mathbf{H} = \mathbf{H}_a + 4\pi|\psi|^2\mathbf{\mu}.$$

(6.15)

By (5.4) and the mechanism of the Meissner effect, the induction magneton $\mathbf{\mu}$ is as

$$\mathbf{\mu} = -\frac{e_s\hbar}{m_sc} N^\prime \mathbf{H}_a,$$

(6.16)

where $N^\prime$ is the number of the Cooper pairs with spin $s = -1$, $N$ is the number of Cooper pairs, and $\mathbf{H}_a$ is the unit vector in the direction
of $H_a$, the bounding potential $V$ is

$$V = g_0 - \alpha T \quad (V > 0).$$

Here $g_0$ is the particle interacting potential, $\alpha$ is the entropy coupling parameter, and

$$T_c = g_0/\alpha, \quad \text{is the critical temperature.}$$

For (6.17) and (6.18), see [9].

For a Type-II superconductor and for a fixed $T < T_c$, by (6.14) and (6.15), we have the following conclusions for the A–M transition:

$$\text{if } 0 \leq H_a < H_{c_1}, \text{ then the system is in the Meissner phase with the interior magnetic field } H = 0 \text{ or } H_a = -4\pi|\psi|^2\mu, \text{ and with } \psi = \sqrt{V/g};$$

and

$$\text{if } H_{c_1} < H_a < H_{c_2}, \text{ then the system is in the vortex phase with the interior magnetic field } H \neq 0 \text{ or } H_a > \frac{4\pi e^2}{m_s c} N|\psi|^2, \text{ and } \psi \text{ satisfies } (6.14) \text{ with } A \neq 0.$$

By using the topological index (6.8), the transition from (6.19) to (6.20) can be simply stated as

$$\text{Ind}(\psi) = \begin{cases} 0 & \text{for } 0 \leq H_a < H_{c_1}, \\ (N, 0) \text{ with } N \neq 0 & \text{for } H_{c_1} < H_a < H_{c_2}, \end{cases}$$

where $N$ is the number of Abrikosov vortices, $H_{c_1}$ and $H_{c_2}$ are as in Figure 6.5. In particular, the number $N$ can be computed from the equation (6.14)–(6.15), which are equivalently written as

$$\frac{1}{2m_s}
\left(ih\nabla + \frac{e_s}{c}A\right)^2\psi - \left(V - \frac{e_s}{m_s c} N H_a\right)\psi
\left(g - \frac{4\pi e_s^2}{m_s c^2} \left(\frac{N}{N}\right)^2\right)|\psi|^2\psi = 0,$$

supplemented with a physical boundary condition.

4). Critical induction magneton $\mu_c$. By the Abrikosov theory, $H_{c_1}$ can be expressed as

$$H_{c_1} = \frac{H_c}{\sqrt{2\kappa}} \ln \kappa,$$
where $\kappa > \sqrt{2}$ is the Ginzburg–Landau parameter, and $H_c$ is the critical $H_a$, which makes the coefficient of $\psi$ in (6.22) to be zero:

$$\mu_c H_c = V, \quad V \text{ as in (6.17).}$$

(6.25)

The $\mu_c$ in (6.25) is also the critical magneton which satisfies $H = 0$ of (6.15) at $H_a = H_{c_1}$, i.e.

$$H_{c_1} = 4\pi |\psi|^2 \mu_c.$$

By (6.19), $|\psi|^2 = V/g$. Then $\mu_c$ satisfies

$$H_{c_1} = \frac{4\pi V}{g} \mu_c.$$

(6.26)

From (6.24)–(6.26), we derive the critical induction magneton as

$$\mu_c^2 = \frac{g}{4\sqrt{2}\pi \kappa} \ln \kappa.$$

(6.27)

Equivalently,

$$\mu_c = -\mu_c \hat{H}_a.$$

In view of (6.16), we have

$$N_c^- = N \frac{m_a c}{e s \hbar} \mu_c, \quad N = |\Omega||\psi|^2,$$

(6.28)

where $|\Omega|$ is the volume of the superconductor, and $N_c^-$ represents the maximal value of $N^-$ in this system.

### 6.4. QPT theorem for scalar BEC systems

In this section, we introduce a QPT theorem for the scalar BEC system established in [4].

The dynamic equation governing the systems is given by (5.9), which is recalled here for convenience:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi - V \psi + g|\psi|^2 \psi,$$

(6.29)

supplemented with either the Neumann boundary condition:

$$\frac{\partial \psi}{\partial n} \bigg|_{\partial \Omega} = 0,$$

(6.30)

or the Dirichlet boundary condition:

$$\psi|_{\partial \Omega} = 0.$$

(6.31)

The physical significance of (6.30) is that $T < T_c$ on $\Omega$, and (6.31) is that $T < T_c$ in the interior of $\Omega$ and $T \geq T_c$ on $\partial \Omega$.

In the BEC state, the particle number is conserved:

$$\int_{\Omega} |\psi|^2 = \text{constant.}$$
For this case, the solutions of (6.29) can be written as

\[(6.32) \quad \psi = \varphi(x) e^{-i\mu t/\hbar},\]

where \(\mu\) is the chemical potential. Inserting (6.32) into (6.29), we deduce that the equations

\[(6.33) \quad -\frac{\hbar^2}{2m} \Delta \varphi + g|\varphi|^2 \varphi = (V + \mu) \varphi\]

with \(\varphi\) satisfying either the boundary conditions (6.30) or (6.31).

1). **Main mathematical theorem.** Consider the eigenvalue problem for the Laplace operator

\[(6.34) \quad -\Delta e_k = \lambda_k e_k, \quad e_k|_{\partial\Omega} = 0 \quad (\text{or } \frac{\partial e_k}{\partial n}|_{\partial\Omega} = 0).\]

Mathematically, (6.34) possess infinite number of eigenvalues (counting multiplicities):

\[(6.35) \quad 0 < \lambda_1 < \cdots \leq \lambda_k \leq \cdots \quad (\lambda_1 = 0 \text{ for (6.30)}),\]

where \(\lambda_1\) is simple. Let \(e_k\) be the eigenfunction corresponding to \(\lambda_k\). For \(n = 1\) case, the topological index of \(e_k\) is defined by (6.12) is given by

\[\text{Ind}(e_k) = \begin{cases} (N, N) & \text{if } k = 2N, \\ (N + 1, N) & \text{if } k = 2N + 1. \end{cases}\]

For \(n \geq 2\), the expression of \(\text{Ind}(e_k)\) is complex. Then we have the following theorem, which is an equivalent form as the theorem in [4].

**Theorem 6.4.** Let \(\lambda = (V + \mu)\) be the control parameter. For the system (6.33), we have the following assertions:

(1) For each eigenvalue \(\lambda_k\) of (6.34), the equation (6.33) bifurcates from \(\lambda_k \frac{\hbar^2}{2m}\) two nonzero solutions \(\pm \varphi_k(\lambda)\) on the side of \(\lambda > \lambda_k \frac{\hbar^2}{2m}\), and

\[\text{Ind}(\varphi_k(\lambda)) = \text{Ind}(e_k), \quad \forall \lambda > \lambda_k.\]

(2) If \(\lambda_k \frac{\hbar^2}{2m} < \lambda \leq \lambda_{k+1} \frac{\hbar^2}{2m}\), then (6.33) possesses at least \(2k\) solutions \(\pm \varphi_j(\lambda)\) (\(1 \leq j \leq k\)), where \(\pm \varphi_j(\lambda)\) are the solutions bifurcated from \(\lambda_j\).

2). **Physical conclusions.** Theorem 6.4 leads to the following physical conclusions:
(i) Let \( \varphi \) be a solution of (6.33). Then the Hamiltonian energy \( E \) and the chemical potential \( \mu \) for \( \varphi \) are

\[
E = \int_{\Omega} \left[ \frac{\hbar^2}{2m_0} |\nabla \varphi|^2 - V|\varphi|^2 + \frac{g}{2} |\varphi|^4 \right] dx,
\]

\[
\mu = \int_{\Omega} \left[ \frac{\hbar^2}{2m_0} |\nabla \varphi|^2 - V|\varphi|^2 + g|\varphi|^4 \right] dx,
\]

\[
\mu - E = \frac{g}{2} \int_{\Omega} |\varphi|^4 dx;
\]

(ii) The control parameters for this system are \((V, N, \Omega)\), where \( V \) is the bounding potential, \( N \) is the particle number, and \( \Omega \) is the container. The parameter \( \lambda = V + \mu \) can be viewed as a combined control parameter, which is determined by the parameters \( V, N \) and \( \Omega \);

(iii) Assertion (1) of Theorem 6.4 amounts to saying that if \( \lambda = V + \mu \) passes through \( \hbar^2 \lambda_j / 2m \) from the left side to the right side, then the system (6.33) will add 2\( k \) quantum states, where \( k \) is the multiplicity of \( \lambda_j \):

\[
(6.36) \quad 2k \text{ new quantum states for (6.33) emerge as } \lambda \text{ crosses } \frac{h^2 \lambda_k}{2m};
\]

(iv) If \( \lambda = V + \mu \) satisfies

\[
(6.37) \quad \frac{h^2 \lambda_k}{2m} < \lambda < \frac{h^2 \lambda_{k+1}}{2m},
\]

then the system (6.33) has at least 2\( k \) quantum states \( \pm \varphi_j \) (1 \( \leq \) \( j \) \( \leq \) \( k \)), such that \( \varphi_j \) has the same index as the \( j \)-th eigenfunction \( e_j \) of (6.34);

(v) If \( \lambda = V + \mu \) is fixed as in (6.37), then we can tune the physical parameters: \( N, \Omega, \) the electromagnetic fields \( \mathbf{E}, \mathbf{H} \) and the pressure \( p \) so that the system undergoes QPT between the 2\( k \) quantum states \( \pm \varphi_j \) (1 \( \leq \) \( j \) \( \leq \) \( k \)).

**References**


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