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Abstract. Boolean Region Connection Calculus is a formalism for reasoning about the topological relations between regions. In this paper, we provide computability results about unifiability in Boolean Region Connection Calculus and prove that elementary unification is finitary.

Keywords: Region connection calculus · Boolean terms · Unifiability problem · Computability · Unification type

1 Introduction

The Region Connection Calculus (RCC) is a formalism for reasoning about the topological relations between regions [19]. With $RCC8$, a variant of RCC based on 8 atomic relations [6,17], knowledge is represented by means of a conjunction of disjunctions of atomic relations between variables representing regions. Given such a formula, the main task is to know whether it is consistent—an NP-complete problem [20,21]. Consisting of a combination of $RCC8$ with Boolean reasoning, $BRCC8$ is a variant of $RCC8$ in which regions are represented by Boolean terms [24]. With $BRCC8$, showing the consistency of formulas is NP-complete in arbitrary topological spaces and $PSPACE$-complete in Euclidean spaces [14–16,24]. $BRCC8$ and its multifarious variants have attracted considerable interest both for their practical applications in spatial reasoning [6,20] and for the mathematical problems they give rise to [3–5,7–9,14–16,23,24].

We are interested in supporting a new inference capability: unifiability of formulas. The unifiability problem consists, given a finite set $\{(\varphi_1(x_1,\ldots,x_n),\psi_1(x_1,\ldots,x_n)),\ldots,(\varphi_m(x_1,\ldots,x_n),\psi_m(x_1,\ldots,x_n))\}$ of pairs of formulas, in determining whether there exists Boolean terms $a_1,\ldots,a_n$ such that $\varphi_1(a_1,\ldots,a_n)\leftrightarrow\psi_1(a_1,\ldots,a_n)$, ..., $\varphi_m(a_1,\ldots,a_n)\leftrightarrow\psi_m(a_1,\ldots,a_n)$ are valid. To explain our motivation for considering unifiability, consider a finite set of pairs of $BRCC8$-formulas representing desired properties about some regions. This set may contain non-equivalent formulas that can be made equivalent by applying to them appropriate substitutions. And if one is able to find such appropriate substitutions, then one is interested to find the maximal ones. An important question is then the following: when a set of $BRCC8$-formulas is unifiable, has it a minimal complete set of unifiers? When the answer is "yes", how large is this set? See [1,2] where such question is addressed for description logics.

The section-by-section breakdown of the paper is as follows. In Sect. 2, we define the syntax of $BRCC8$. Section 3 explains our motivation for considering unification in $BRCC8$. In Sect. 4, we present the semantics of $BRCC8$. Section 5 introduces the basic ideas involved in unification. In Sect. 6, we embark on the study of specific Boolean terms: monoms and polynoms. The main result we prove there is Proposition 4. Section 7 defines equivalence relations between tuples of terms. The main results we prove there are Propositions 5 and 7. In Sect. 8, we provide computability results about unifiability in $BRCC8$. Section 9 shows that unification in $BRCC8$ is finitary. Due to lack of space, we only consider the elementary case where the considered terms do not contain free constant symbols.

2 Syntax

Now, it is time to meet the language we are working with. We adopt the standard rules for omission of the parentheses.

Let VAR be a countable set of *propositional variables* (with typical members denoted x, y, etc.). Let $(x_1, x_2, \ldots)$ be an enumeration of VAR without repetitions. The *terms* (denoted a, b, etc.) are defined as follows:

- $a ::= x \mid 0 \mid a^\star \mid (a \cup b)$.

The other constructs for terms (for instance, 1 and $\cap$) are defined as usual. We use the following notations for terms:

- a^0 for $a^\star$,
- a^1 for a.

Reading terms as regions, the constructs 0, * and $\cup$ should be regarded as the empty region, the complement operation and the union operation. As a result, the constructs 1 and $\cap$ should be regarded as the full region and the intersection operation. In the sequel, we use $a(x_1, \ldots, x_n)$ to denote a term a whose variables form a subset of $\{x_1, \ldots, x_n\}$. For all nonnegative integers n, let $TER(x_1, \ldots, x_n)$ be the set of all terms whose variables form a subset of $\{x_1, \ldots, x_n\}$. Let TER be the set of all terms and $TER(\emptyset)$ be the set of all variable-free terms.

The *formulas* (denoted φ, ψ, etc.) are defined as follows:

- $\varphi ::= P(a, b) \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi)$.

Here, a and b are terms and P is one of the following 8 binary predicates corresponding to the 8 binary relations of $RCC8$:

- DC ("disconnected"),

- EC ("external contact"),
- PO ("partial overlap"),
- TPP ("tangential proper part"),
- $TPPI$ ("inverse of TPP"),
- $NTPP$ ("nontangential proper part"),
- $NTPPI$ ("inverse of $NTPP$"),
- EQ ("equal").

The other constructs for formulas (for instance, $\top$ and $\wedge$) are defined as usual. We say that a formula φ is *equational* iff EQ is the only binary predicate possibly occurring in φ. In the sequel, we use $\varphi(x_1, \ldots, x_n)$ to denote a formula φ whose variables form a subset of $\{x_1, \ldots, x_n\}$. For all nonnegative integers n, let $FOR(x_1, \ldots, x_n)$ be the set of all formulas whose variables form a subset of $\{x_1, \ldots, x_n\}$. Let FOR be the set of all formulas and $FOR(\emptyset)$ be the set of all variable-free formulas. An *inference rule* is a pair of the form $\frac{\varphi}{\psi}$ where φ and ψ are formulas.

A *substitution* is a function $\sigma : VAR \longrightarrow TER$ which moves at most finitely many variables, i.e. there exists at most finitely many variables x such that $\sigma(x) \neq x$. Given a substitution σ, let $\bar{\sigma} : TER \cup FOR \longrightarrow TER \cup FOR$ be the endomorphism such that for all variables x, $\bar{\sigma}(x) = \sigma(x)$. Obviously, for all substitutions σ, τ, the function $\sigma \circ \tau$ such that for all $x \in VAR$, $(\sigma \circ \tau)(x) = \bar{\tau}(\sigma(x))$ is a substitution called the *composition* of the substitutions σ and τ.

3 Motivation for Considering Unifiability in $BRCC8$

Our motivation for considering unifiability in $BRCC8$ comes from the following three facts: $BRCC8$ is a formalism both with theoretical merits and with practical relevance; unification in Boolean algebras has attracted considerable interest; there is a wide variety of situations where unifiability problems in formalisms like $BRCC8$ arise.

$BRCC8$ is the result of the combination of $RCC8$ with Boolean reasoning. Within the context of $RCC8$, formulas would just be quantifier-free first-order formulas in a constant-free function-free language based on the 8 binary predicates of $RCC8$. For instance, $TPP(x,y) \wedge TPP(x,z) \rightarrow TPP(y,z) \vee TPP(z,y)$. By allowing to apply the 8 binary predicates of $RCC8$ not only to propositional variables but also to Boolean terms, Wolter and Zakharyaschev [24] have strictly extended their expressive capacity. For instance, in the class of all topological spaces, the $BRCC8$ formula $EQ(x \cup y, z)$ has no equivalent formula in a pure $RCC8$-based language. As well, with this enriched language, one becomes able by using the $BRCC8$ formula $DC(x, x^\star) \rightarrow EQ(x, 0) \vee EQ(x^\star, 0)$ to distinguish between connected and non-connected topological spaces.

Unification in Boolean algebras has attracted considerable interest and several algorithms for computing solutions to Boolean equations are known, some of them going back to Boole and Löwenheim. But the most important result is that unification is unitary: given an equation $a(x_1, \ldots, x_n) = b(x_1, \ldots, x_n)$, either it possesses no solution, or it possesses a single most general unifier. See [2,18] for an introduction to the unifiability problem in Boolean algebras. So, it is natural to ask whether unification in $BRCC8$ inherits the unitariness character of Boolean unification. In this paper, we refute this idea by proving that unification in $BRCC8$ is finitary.

There is a wide variety of situations where unifiability problems arise. We will explain our motivation for considering them within the context of geographical information systems. Suppose $\varphi(p_1, \ldots, p_m)$ is a formula representing our knowledge about regions denoted $p_1, \ldots, p_m$ in some geographical universe and $\psi(x_1, \ldots, x_n)$ is a formula representing a desirable property about regions denoted $x_1, \ldots, x_n$. It may happen that $\psi(x_1, \ldots, x_n)$ is not a logical consequence of $\varphi(p_1, \ldots, p_m)$ in the considered geographical universe whereas some of its instances are. Hence, one may wonder whether there are n-tuples $(a_1, \ldots, a_n)$ of terms for which the property represented by $\psi(x_1, \ldots, x_n)$ becomes a logical consequence of $\varphi(p_1, \ldots, p_m)$ in the considered geographical universe. And if one is able to decide such question, then one may be interested to obtain n-tuples $(b_1, \ldots, b_n)$ as general as possible. Central to unification theory are the questions of the computability of unifiability and the unification type. Within the context of $BRCC8$, these questions will be addressed in Sects. 8 and 9.

4 Semantics

The best way to understand the meaning of the binary predicates is by interpreting terms and formulas in topological spaces [14–16, 24]. More precisely, in a topological space (X, τ), if $Int_\tau(\cdot)$ denotes its interior operation then to each binary predicate P, one usually associates a binary relation $P^{(X,\tau)}$ on the set of all regular closed subsets of X:

- $DC^{(X,\tau)}(A, B)$ iff $A \cap B = \emptyset$,
- $EC^{(X,\tau)}(A, B)$ iff $A \cap B \neq \emptyset$ and $Int_\tau(A) \cap Int_\tau(B) = \emptyset$,
- $PO^{(X,\tau)}(A, B)$ iff $Int_\tau(A) \cap Int_\tau(B) \neq \emptyset$, $Int_\tau(A) \not\subseteq B$ and $Int_\tau(B) \not\subseteq A$,
- $TPP^{(X,\tau)}(A, B)$ iff $A \subseteq B$, $A \not\subseteq Int_\tau(B)$ and $B \not\subseteq A$,
- $TPPI^{(X,\tau)}(A, B)$ iff $B \subseteq A$, $B \not\subseteq Int_\tau(A)$ and $A \not\subseteq B$,
- $NTPP^{(X,\tau)}(A, B)$ iff $A \subseteq Int_\tau(B)$ and $B \not\subseteq A$,
- $NTPPI^{(X,\tau)}(A, B)$ iff $B \subseteq Int_\tau(A)$ and $A \not\subseteq B$,
- $EQ^{(X,\tau)}(A, B)$ iff $A = B$.

This topological semantics is considered in [14–16, 24]. Obviously, these relations are jointly exhaustive and pairwise disjoint on the set of all nonempty regular closed subsets of X. We say that a topological space (X, τ) is *indiscrete* iff $\tau = \{\emptyset, X\}$. We say that a topological space (X, τ) is *connected* iff for all $A, B \in \tau$, either $A \cap B \neq \emptyset$, or $A \cup B \neq X$.

A relational perspective is suggested by Galton [11] who introduces the notion of adjacency space. Galton's spaces are frames (W, R) where W is a nonempty set of cells and R is an adjacency relation between cells. Galton defines regions to be sets of cells. He also defines two regions A and B to be connected iff some cell in A is adjacent to some cell in B. This definition relates Galton's adjacency spaces to the relational semantics of modal logic which makes it possible to use methods from modal logic for studying region-based theories of space. The truth is that the above-mentioned topological semantics and the relational perspective suggested by Galton are equivalent [23].

In this paper, we adopt a relational perspective by interpreting terms and formulas in frames. A *frame* is a structure of the form (W, R) where W is a nonempty set (with typical members denoted s, t, etc.) and R is a reflexive and symmetric relation on W. A frame (W, R) is *indiscrete* iff $R = W \times W$. A frame (W, R) is connected iff $R^+ = W \times W$ where R^+ denotes the transitive closure of R. Let (W, R) be a frame. We associate to each binary predicate P a binary relation $P^{(W,R)}$ on 2^W as follows:

- $DC^{(W,R)}(A, B)$ iff $R \cap (A \times B) = \emptyset$,
- $EC^{(W,R)}(A, B)$ iff $R \cap (A \times B) \neq \emptyset$ and $A \cap B = \emptyset$,
- $PO^{(W,R)}(A, B)$ iff $A \cap B \neq \emptyset$, $A \not\subseteq B$ and $B \not\subseteq A$,
- $TPP^{(W,R)}(A, B)$ iff $A \subseteq B$, $R \cap (A \times (W \setminus B)) \neq \emptyset$ and $B \not\subseteq A$,
- $TPPI^{(W,R)}(A, B)$ iff $B \subseteq A$, $R \cap (B \times (W \setminus A)) \neq \emptyset$ and $A \not\subseteq B$,
- $NTPP^{(W,R)}(A, B)$ iff $R \cap (A \times (W \setminus B)) = \emptyset$ and $B \not\subseteq A$,
- $NTPPI^{(W,R)}(A, B)$ iff $R \cap (B \times (W \setminus A)) = \emptyset$ and $A \not\subseteq B$,
- $EQ^{(W,R)}(A, B)$ iff $A = B$.

This relational semantics is considered in [3–5, 23]. Obviously, these binary relations are jointly exhaustive and pairwise disjoint on $2^W \setminus \{\emptyset\}$, i.e. for all nonempty subsets A, B of W, there exists exactly one binary predicate P such that $P^{(W,R)}(A, B)$. The truth is that for all binary predicates P and for all subsets A, B of W, if either $A = \emptyset$, or $B = \emptyset$ then $P^{(W,R)}(A, B)$ iff either $P = DC$, or $P = NTPP$ and $B \neq \emptyset$, or $P = NTPPI$ and $A \neq \emptyset$, or $P = EQ$ and $A = B$.

A *valuation* on W is a map $\mathcal{V}$ associating with every variable x a subset $\mathcal{V}(x)$ of W. Given a valuation $\mathcal{V}$ on W, we define

- $\bar{\mathcal{V}}(x) = \mathcal{V}(x)$,
- $\bar{\mathcal{V}}(0) = \emptyset$,
- $\bar{\mathcal{V}}(a^\star) = W \setminus \bar{\mathcal{V}}(a)$,
- $\bar{\mathcal{V}}(a \cup b) = \bar{\mathcal{V}}(a) \cup \bar{\mathcal{V}}(b)$.

Thus, every term is interpreted as a subset of W. A valuation $\mathcal{V}$ on W is *balanced* iff for all terms a, either $\bar{\mathcal{V}}(a) = \emptyset$, or $\bar{\mathcal{V}}(a) = W$, or $\bar{\mathcal{V}}(a)$ is infinite and coinfinite.

A *model* on (W, R) is a structure $\mathcal{M} = (W, R, \mathcal{V})$ where $\mathcal{V}$ is a valuation on W. The *satisfiability* of a formula φ in $\mathcal{M}$ (in symbols $\mathcal{M} \models \varphi$) is defined as follows:

- $\mathcal{M} \models P(a, b)$ iff $P^{(W,R)}(\bar{\mathcal{V}}(a), \bar{\mathcal{V}}(b))$,
- $\mathcal{M} \not\models \bot$,
- $\mathcal{M} \models \neg\varphi$ iff $\mathcal{M} \not\models \varphi$,
- $\mathcal{M} \models \varphi \vee \psi$ iff either $\mathcal{M} \models \varphi$, or $\mathcal{M} \models \psi$.

A formula φ is *valid* in (W, R) iff for all valuations $\mathcal{V}$ on W, $(W, R, \mathcal{V}) \models \varphi$. A formula φ is *satisfiable* in (W, R) iff there exists a valuation $\mathcal{V}$ on W such that $(W, R, \mathcal{V}) \models \varphi$. Let $\mathcal{C}$ be a class of frames. We say that a formula φ is $\mathcal{C}$*-valid* iff for all frames (W, R) in $\mathcal{C}$, φ is valid in (W, R). We say that a formula φ is $\mathcal{C}$*-satisfiable* iff there exists a frame (W, R) in $\mathcal{C}$ such that φ is satisfiable in (W, R). The $\mathcal{C}$ *-satisfiability problem* consists in determining whether a given formula is $\mathcal{C}$-satisfiable. We say that $\mathcal{C}$ *agrees with unions* iff for all disjoint frames $(W, R), (W', R')$ in $\mathcal{C}$, there exists a frame (W'', R'') in $\mathcal{C}$ such that $W \cup W' = W''$. Note that if $\mathcal{C}$ contains frames of arbitrary cardinality then $\mathcal{C}$ agrees with unions. We say that $\mathcal{C}$ is *determined* iff there exists a set of formulas such that $\mathcal{C}$ is the class of all frames validating each formula in that set. We say that $\mathcal{C}$ is *balanced* iff for all formulas φ, if φ is $\mathcal{C}$-satisfiable then there exists a countable frame (W, R) in $\mathcal{C}$ and there exists a balanced valuation $\mathcal{V}$ on W such that $(W, R, V) \models \varphi$.

As illustrative examples of classes of frames, let $\mathcal{C}_{all}$ denote the class of all frames, $\mathcal{C}_{ind}$ denote the class of all indiscrete frames and $\mathcal{C}_{con}$ denote the class of all connected frames. The topological counterparts of these classes are the class of all topological spaces, the class of all indiscrete spaces and the class of all connected spaces. The following formulas are $\mathcal{C}_{all}$-valid:

– $DC(x, x) \rightarrow EQ(x, 0)$,
– $DC(x, y) \rightarrow DC(y, x)$.

In an indiscrete frame (W, R), any two points are R-related. Hence, for all subsets A, B of W, if $DC^{(W,R)}(A, B)$ then either $EQ^{(W,R)}(A, \emptyset)$, or $EQ^{(W,R)}(B, \emptyset)$. Thus, the following formula is $\mathcal{C}_{ind}$-valid:

– $DC(x, y) \rightarrow EQ(x, 0) \vee EQ(y, 0)$.

In a connected frame (W, R), any two points are R^+-related. Hence, for all subsets A of W, if $DC^{(W,R)}(A, W \setminus A)$ then either $EQ^{(W,R)}(A, \emptyset)$, or $EQ^{(W,R)}(W \setminus A, \emptyset)$. Thus, the following formula is $\mathcal{C}_{con}$-valid:

– $DC(x, x^\star) \rightarrow EQ(x, 0) \vee EQ(x^\star, 0)$.

Proposition 1. *$\mathcal{C}_{all}$, $\mathcal{C}_{ind}$ and $\mathcal{C}_{con}$ agree with unions.*

Proof. By the definition of what it means for classes of frames to agree with unions.

Proposition 2. *$\mathcal{C}_{all}$, $\mathcal{C}_{ind}$ and $\mathcal{C}_{con}$ are determined.*

Proof. It suffices to note that $\mathcal{C}_{all}$ is determined by $\emptyset$, $\mathcal{C}_{ind}$ is determined by $\{DC(x, y) \rightarrow EQ(x, 0) \vee EQ(y, 0)\}$ and $\mathcal{C}_{con}$ is determined by $\{DC(x, x^\star) \rightarrow EQ(x, 0) \vee EQ(x^\star, 0)\}$.

Proposition 3. *$\mathcal{C}_{all}$, $\mathcal{C}_{ind}$ and $\mathcal{C}_{con}$ are balanced.*

Proof. By Proposition 2 and [5, Theorem 4.1], $\mathcal{C}_{all}, \mathcal{C}_{ind}$ and $\mathcal{C}_{con}$ admit filtration. Now, consider an arbitrary finite frame (W, R). We define the countable frame (W', R') as follows:

– $W' = W \times \mathbb{Z}$,

– for all $(s,i),(t,j) \in W'$, $(s,i)R'(t,j)$ iff sRt.

Obviously, if (W,R) is indiscrete (respectively, connected) then (W',R') is indiscrete (respectively, connected) too. Moreover, according to [4, Definition 3.1], (W,R) is a bounded morphic image of (W',R'). Thus, by [4, Proposition 3.1], for all formulas φ, if φ is satisfiable in (W,R) then there exists a balanced valuation $\mathcal{V}'$ on (W',R') such that $(W',R',\mathcal{V}') \models \varphi$. Since (W,R) was arbitrary and $\mathcal{C}_{all}$, $\mathcal{C}_{ind}$ and $\mathcal{C}_{con}$ admit filtration, therefore $\mathcal{C}_{all}$, $\mathcal{C}_{ind}$ and $\mathcal{C}_{con}$ are balanced.

As for the satisfiability problem, it is known to be NP-complete in $\mathcal{C}_{all}$ and $\mathcal{C}_{ind}$ and $PSPACE$-complete in $\mathcal{C}_{con}$ [5,14–16,24].

5 Unifiability

Let $\mathcal{C}$ be a class of frames.

We say that a substitution σ is *$\mathcal{C}$-equivalent* to a substitution τ (in symbols $\sigma \simeq_{\mathcal{C}} \tau$) iff for all variables x, $EQ(\sigma(x),\tau(x))$ is $\mathcal{C}$-valid. We say that a substitution σ is *more $\mathcal{C}$-general than* a substitution τ (in symbols $\sigma \preceq_{\mathcal{C}} \tau$) iff there exists a substitution υ such that $\sigma \circ \upsilon \simeq_{\mathcal{C}} \tau$.

We say that a finite set $\{(\varphi_1,\psi_1),\ldots,(\varphi_n,\psi_n)\}$ of pairs of formulas is *$\mathcal{C}$ -unifiable* iff there exists a substitution σ such that $\bar{\sigma}(\varphi_1) \leftrightarrow \bar{\sigma}(\psi_1)$, ..., $\bar{\sigma}(\varphi_n) \leftrightarrow \bar{\sigma}(\psi_n)$ are $\mathcal{C}$-valid. As a consequence of the classical interpretation of the constructs for formulas, this is equivalent to $\bar{\sigma}((\varphi_1 \leftrightarrow \psi_1) \wedge \ldots \wedge (\varphi_n \leftrightarrow \psi_n))$ is $\mathcal{C}$-valid. This means that we can restrict our attention to a simpler kind of unifiability problems consisting of exactly one formula. We say that a formula φ is *$\mathcal{C}$ -unifiable* iff there exists a substitution σ such that $\bar{\sigma}(\varphi)$ is $\mathcal{C}$-valid. In that case, we say that σ is a *$\mathcal{C}$-unifier* of φ. For instance, $EQ(0,x) \vee EQ(1,x)$ is unifiable in $\mathcal{C}_{all}$, $\mathcal{C}_{ind}$ and $\mathcal{C}_{con}$. As we will prove it with Proposition 15, its unifiers are the substitutions σ such that considered as a formula in Classical Propositional Logic (CPL), $\sigma(x)$ is either equivalent to 0, or equivalent to 1. The *elementary $\mathcal{C}$ -unifiability problem* consists in determining whether a given formula is $\mathcal{C}$-unifiable. See [1,2,12,13] for an introduction to the unifiability problem in modal and description logics.

We say that a set of $\mathcal{C}$-unifiers of a formula φ is *complete* iff for all $\mathcal{C}$-unifiers σ of φ, there exists a $\mathcal{C}$-unifier τ of φ in that set such that $\tau \preceq_{\mathcal{C}} \sigma$. As we will prove it with Proposition 15, the substitutions σ_0 and σ_1 such that $\sigma_0(x) = 0$, $\sigma_1(x) = 1$ and for all variables y, if $x \neq y$ then $\sigma_0(y) = y$ and $\sigma_1(y) = y$ constitute a complete set of $\mathcal{C}$-unifiers of $EQ(0,x) \vee EQ(1,x)$. An important question is: when a formula is $\mathcal{C}$-unifiable, has it a minimal complete set of $\mathcal{C}$-unifiers? When the answer is "yes", how large is this set?

We say that a $\mathcal{C}$-unifiable formula φ is *$\mathcal{C}$-nullary* iff there exists no minimal complete set of $\mathcal{C}$-unifiers of φ. We say that a $\mathcal{C}$-unifiable formula φ is *$\mathcal{C}$-infinitary* iff there exists a minimal complete set of $\mathcal{C}$-unifiers of φ but there exists no finite one. We say that a $\mathcal{C}$-unifiable formula φ is *$\mathcal{C}$-finitary* iff there exists a finite minimal complete set of $\mathcal{C}$-unifiers of φ but there exists no with cardinality 1. We say that a $\mathcal{C}$-unifiable formula φ is *$\mathcal{C}$ -unitary* iff there exists a minimal complete set of $\mathcal{C}$-unifiers of φ with cardinality 1. We say that elementary unification in $\mathcal{C}$ is *nullary* iff there exists a $\mathcal{C}$-nullary formula. We say that elementary unification in $\mathcal{C}$ is *infinitary* iff every $\mathcal{C}$-unifiable formula is either $\mathcal{C}$-infinitary, or $\mathcal{C}$-finitary, or $\mathcal{C}$-unitary and there exists a $\mathcal{C}$-infinitary formula. We say that elementary unification in $\mathcal{C}$ is *finitary* iff every $\mathcal{C}$-unifiable formula is either $\mathcal{C}$-finitary, or $\mathcal{C}$-unitary and there exists a $\mathcal{C}$-finitary formula. We say that elementary unification in $\mathcal{C}$ is *unitary* iff every $\mathcal{C}$-unifiable formula is $\mathcal{C}$-unitary. See [10] for an introduction to the unification types in logics.

An axiomatic system for $\mathcal{C}$ consists of axioms and rules. Its theorems are all formulas which can be derived from the axioms by means of the rules. See [5,23] for systems of axioms and rules characterizing validity with respect to different classes of frames. In order to make stronger an axiomatic system for $\mathcal{C}$, we can add new axioms and new rules to it. Concerning new axioms, they should always consist of $\mathcal{C}$-valid formulas. About new rules, they should always consist of rules that preserve $\mathcal{C}$-validity. We say that an inference rule $\frac{\varphi}{\psi}$ is *$\mathcal{C}$-admissible* iff for all substitutions σ, if $\sigma(\varphi)$ is $\mathcal{C}$-valid then $\sigma(\psi)$ is $\mathcal{C}$-valid. The *elementary $\mathcal{C}$-admissibility problem* consists in determining whether a given inference rule is $\mathcal{C}$-admissible. See [22] for an introduction to the admissibility problem in logics.

6 Monoms and Polynoms

Before we provide, in Sect. 8, computability results about unifiability and admissibility in $BRCC8$ and prove, in Sect. 9, that elementary unification is finitary, we introduce the notions of monom and polynom (this section) and define some equivalence relations (next section).

Let k, n be nonnegative integer and $f : \{0,1\}^k \longrightarrow \{0,1\}^n$ be a function. An *n-monom* is a term of the form

– $x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n}$

where $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$. Considering the terms $x_1^{\beta_1}$, ..., $x_n^{\beta_n}$ as literals in CPL, n-monoms are just conjunctions of literals. Considering a term a in $TER(x_1, \ldots, x_n)$ as a formula in CPL, let $mon(n, a)$ be the set of all n-monoms $x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n}$ such that a is a tautological consequence of $x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n}$. An *$n$ -polynom* is a term of the form

– $(x_1^{\beta_{11}} \cap \ldots \cap x_n^{\beta_{1n}}) \cup \ldots \cup (x_1^{\beta_{m1}} \cap \ldots \cap x_n^{\beta_{mn}})$

where m is a nonnegative integer and $(\beta_{11}, \ldots, \beta_{1n}), \ldots, (\beta_{m1}, \ldots, \beta_{mn}) \in \{0,1\}^n$. Considering the terms $x_1^{\beta_{11}} \cap \ldots \cap x_n^{\beta_{1n}}$, ..., $x_1^{\beta_{m1}} \cap \ldots \cap x_n^{\beta_{mn}}$ as conjunctions of literals in CPL, n-polynoms are just disjunctive normal forms. Note that for all terms a in $TER(x_1, \ldots, x_n)$, $\bigcup mon(n, a)$ is an n-polynom. For all positive integers i, if $i \leq n$ then let $\pi_i : \{0,1\}^n \longrightarrow \{0,1\}$ be the function such that for all $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$, $\pi_i(\beta_1, \ldots, \beta_n) = \beta_i$. For all $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$, we define

– $f^{-1}(\beta_1, \ldots, \beta_n) = \{(\alpha_1, \ldots, \alpha_k) \in \{0,1\}^k : f(\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_n)\}$.

Obviously, for all $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$, $f^{-1}(\beta_1, \ldots, \beta_n) \subseteq \{0,1\}^k$. For all positive integers i, if $i \leq n$ then we define:

- $\Delta_i = \{(\alpha_1, \ldots, \alpha_k) \in \{0,1\}^k : \ \pi_i(f(\alpha_1, \ldots, \alpha_k)) = 1\}$,
- $c_i = \bigcup\{x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} : \ (\alpha_1, \ldots, \alpha_k) \in \Delta_i\}$.

Obviously, for all positive integers i, if $i \leq n$ then $\Delta_i \subseteq \{0,1\}^k$ and c_i is a k-polynom. Note that Δ_i and c_i depend on f too. Lemma 1 is a consequence of the definition of $mon(n, a)$.

Lemma 1. *Let $a(x_1, \ldots, x_n) \in TER(x_1, \ldots, x_n)$. Considered as formulas in CPL, the terms a and $\bigcup mon(n, a)$ are equivalent.*

Proposition 4. *For all $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$, considered as formulas in CPL, the terms $\bigcup\{x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} : \ (\alpha_1, \ldots, \alpha_k) \in f^{-1}(\beta_1, \ldots, \beta_n)\}$ and $c_1^{\beta_1} \cap \ldots \cap c_n^{\beta_n}$ are equivalent.*

Proof. Let $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$. It suffices to show that considered as formulas in CPL, for all $\theta_1, \ldots, \theta_k \in \{0,1\}$, if x_1 is interpreted by θ_1, ..., x_k is interpreted by θ_k then $\bigcup\{x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} : \ (\alpha_1, \ldots, \alpha_k) \in f^{-1}(\beta_1, \ldots, \beta_n)\}$ is equivalent to 1 iff $c_1^{\beta_1} \cap \ldots \cap c_n^{\beta_n}$ is equivalent to 1. Let $\theta_1, \ldots, \theta_k \in \{0,1\}$. Let x_1 be interpreted by θ_1, ..., x_k be interpreted by θ_k.

Suppose $\bigcup\{x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} : \ (\alpha_1, \ldots, \alpha_k) \in f^{-1}(\beta_1, \ldots, \beta_n)\}$ is equivalent to 1. Hence, $(\theta_1, \ldots, \theta_k) \in f^{-1}(\beta_1, \ldots, \beta_n)$. Thus, $f(\theta_1, \ldots, \theta_k) = (\beta_1, \ldots, \beta_n)$. For the sake of the contradiction, suppose $c_1^{\beta_1} \cap \ldots \cap c_n^{\beta_n}$ is equivalent to 0. Let i be a positive integer such that $i \leq n$ and $c_i^{\beta_i}$ is equivalent to 0. Since either $\beta_i = 0$, or $\beta_i = 1$, therefore we have to consider two cases. In the former case, $\beta_i = 0$ and therefore $\bigcup\{x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} : \ (\alpha_1, \ldots, \alpha_k) \in \Delta_i\}$ is equivalent to 1. Consequently, $(\theta_1, \ldots, \theta_k) \in \Delta_i$. Hence, $\pi_i(f(\theta_1, \ldots, \theta_k)) = 1$. Since $f(\theta_1, \ldots, \theta_k) = (\beta_1, \ldots, \beta_n)$, therefore $\beta_i = 1$: a contradiction. In the latter case, $\beta_i = 1$ and therefore $\bigcup\{x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} : \ (\alpha_1, \ldots, \alpha_k) \in \Delta_i\}$ is equivalent to 0. Thus, $(\theta_1, \ldots, \theta_k) \not\in \Delta_i$. Hence, $\pi_i(f(\theta_1, \ldots, \theta_k)) = 0$. Since $f(\theta_1, \ldots, \theta_k) = (\beta_1, \ldots, \beta_n)$, therefore $\beta_i = 0$: a contradiction.

Suppose $c_1^{\beta_1} \cap \ldots \cap c_n^{\beta_n}$ is equivalent to 1. Let i be an arbitrary positive integer such that $i \leq n$. Since $c_1^{\beta_1} \cap \ldots \cap c_n^{\beta_n}$ is equivalent to 1, therefore $c_i^{\beta_i}$ is equivalent to 1. Since either $\beta_i = 0$, or $\beta_i = 1$, therefore we have to consider two cases. In the former case, $\beta_i = 0$ and therefore c_i is equivalent to 0. Hence, $(\theta_1, \ldots, \theta_k) \not\in \Delta_i$. Thus, $\pi_i(f(\theta_1, \ldots, \theta_k)) = 0$. Since $\beta_i = 0$, therefore $\pi_i(f(\theta_1, \ldots, \theta_k)) = \beta_i$. In the latter case, $\beta_i = 1$ and therefore c_i is equivalent to 1. Consequently, $(\theta_1, \ldots, \theta_k) \in \Delta_i$. Hence, $\pi_i(f(\theta_1, \ldots, \theta_k)) = 1$. Since $\beta_i = 1$, therefore $\pi_i(f(\theta_1, \ldots, \theta_k)) = \beta_i$. In both cases, $\pi_i(f(\theta_1, \ldots, \theta_k)) = \beta_i$. Since i was arbitrary, therefore $f(\theta_1, \ldots, \theta_k) = (\beta_1, \ldots, \beta_n)$. Thus, $(\theta_1, \ldots, \theta_k) \in f^{-1}(\beta_1, \ldots, \beta_n)$. Consequently, $\bigcup\{x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} : \ (\alpha_1, \ldots, \alpha_k) \in f^{-1}(\beta_1, \ldots, \beta_n)\}$ is equivalent to 1.

7 Some Equivalence Relations

Let k, n be nonnegative integers and $\mathcal{C}$ be a class of frames.

Given $(a_1, \ldots, a_n) \in TER(x_1, \ldots, x_k)^n$, we define on $\{0,1\}^k$ the equivalence relation $\sim^k_{(a_1,\ldots,a_n)}$ as follows:

– $(\alpha_1, \ldots, \alpha_k) \sim^k_{(a_1,\ldots,a_n)} (\alpha'_1, \ldots, \alpha'_k)$ iff for all positive integers i, if $i \leq n$, then $x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon(k, a_i)$ iff $x_1^{\alpha'_1} \cap \ldots \cap x_k^{\alpha'_k} \in mon(k, a_i)$.

Lemma 2 is a consequence of its definition.

Lemma 2. *For all $(a_1, \ldots, a_n) \in TER(x_1, \ldots, x_k)^n$, $\sim^k_{(a_1,\ldots,a_n)}$ has at most 2^n equivalence classes on $\{0,1\}^k$.*

Let $f : \{0,1\}^k \longrightarrow \{0,1\}^n$ be a function such that for all $(\alpha_1, \ldots, \alpha_k), (\alpha'_1, \ldots, \alpha'_k) \in \{0,1\}^k$, if $f(\alpha_1, \ldots, \alpha_k) = f(\alpha'_1, \ldots, \alpha'_k)$ then $(\alpha_1, \ldots, \alpha_k) \sim^k_{(a_1,\ldots,a_n)} (\alpha'_1, \ldots, \alpha'_k)$. By means of the function f, we define the n-tuple $(b_1, \ldots, b_n)$ of n-polynoms as follows:

– $b_i = \bigcup\{x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n} : x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon(k, a_i)$ and $f(\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_n)\}$.

We say that $(b_1, \ldots, b_n)$ is the n-tuple of n-polynoms *properly obtained* from the given n-tuple $(a_1, \ldots, a_n)$ in $TER(x_1, \ldots, x_k)^n$ with respect to (k, n). Lemma 3 is a consequence of its definition.

Lemma 3. *Let $(a_1, \ldots, a_n)$ be an n-tuple in $TER(x_1, \ldots, x_k)^n$ and $(b_1, \ldots, b_n)$ be an n-tuple of n-polynoms. Let W be a nonempty set. If $(b_1, \ldots, b_n)$ is properly obtained from $(a_1, \ldots, a_n)$ with respect to (k, n) then for all valuations $\mathcal{V}$ on W, there exists a valuation $\mathcal{V}'$ on W such that for all positive integers i, if $i \leq n$, then $\bar{\mathcal{V}}(a_i) = \bar{\mathcal{V}}'(b_i)$ and for all valuations $\mathcal{V}$ on W, there exists a valuation $\mathcal{V}'$ on W such that for all positive integers i, if $i \leq n$, then $\bar{\mathcal{V}}(b_i) = \bar{\mathcal{V}}'(a_i)$.*

For all $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$, let $f^{-1}(\beta_1, \ldots, \beta_n)$ be as in Sect. 6. For all positive integers i, if $i \leq n$ then let Δ_i and c_i be as in Sect. 6. Let υ be the substitution such that

– for all positive integers i, if $i \leq n$ then $\upsilon(x_i) = c_i$,
– for all variables y, if $y \not\in \{x_1, \ldots, x_n\}$ then $\upsilon(y) = y$.

Proposition 5. *For all positive integers i, if $i \leq n$ then considered as formulas in CPL, the terms a_i and $\bar{\upsilon}(b_i)$ are equivalent.*

Proof. Let i be a positive integer such that $i \leq n$. Considered as formulas in CPL, the following terms are equivalent:

1. $\bar{\upsilon}(b_i)$.
2. $\bigcup\{c_1^{\beta_1} \cap \ldots \cap c_n^{\beta_n} : x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon(k, a_i)$ and $f(\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_n)\}$.

3. $\bigcup\{\bigcup\{x_1^{\alpha'_1} \cap \ldots \cap x_k^{\alpha'_k} : (\alpha'_1, \ldots, \alpha'_k) \in f^{-1}(\beta_1, \ldots, \beta_n)\} : x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon(k, a_i) \text{ and } f(\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_n)\}$.
4. $\bigcup\{x_1^{\alpha'_1} \cap \ldots \cap x_k^{\alpha'_k} : (\alpha'_1, \ldots, \alpha'_k) \in f^{-1}(\beta_1, \ldots, \beta_n), x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon(k, a_i) \text{ and } f(\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_n)\}$.
5. $\bigcup\{x_1^{\alpha'_1} \cap \ldots \cap x_k^{\alpha'_k} : x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon(k, a_i) \text{ and } f(\alpha'_1, \ldots, \alpha'_k) = f(\alpha_1, \ldots, \alpha_k)\}$.
6. $\bigcup mon(k, a_i)$.
7. a_i.

The equivalence between 1 and 2 is a consequence of the definition of v; the equivalence between 2 and 3 is a consequence of Proposition 4; the equivalences between 3, 4 and 5 are consequences of simple set-theoretic properties; the equivalence between 5 and 6 is a consequence of the definition of $\sim^k_{(a_1,\ldots,a_n)}$ and the fact that for all $(\alpha_1, \ldots, \alpha_k), (\alpha'_1, \ldots, \alpha'_k) \in \{0, 1\}^k$, if $f(\alpha_1, \ldots, \alpha_k) = f(\alpha'_1, \ldots, \alpha'_k)$ then $(\alpha_1, \ldots, \alpha_k) \sim^k_{(a_1,\ldots,a_n)} (\alpha'_1, \ldots, \alpha'_k)$; the equivalence between 6 and 7 is a consequence of Lemma 1.

We define on $FOR(x_1, \ldots, x_n)$ the equivalence relation $\equiv^n_{\mathcal{C}}$ as follows:

– $\varphi \equiv^n_{\mathcal{C}} \psi$ iff $\varphi \leftrightarrow \psi$ is $\mathcal{C}$-valid.

Proposition 6. *$\equiv^n_{\mathcal{C}}$ has finitely many equivalence classes on $FOR(x_1, \ldots, x_n)$.*

Proof. Each formula φ in $FOR(x_1, \ldots, x_n)$ is a combination of formulas of the form $P(a, b)$ where a and b are terms in $TER(x_1, \ldots, x_n)$ and P is one of the 8 binary predicates of $RCC8$. Hence, $\equiv^n_{\mathcal{C}}$ has finitely many equivalence classes on $FOR(x_1, \ldots, x_n)$.

Let A_n be the set of all n-tuples of terms. Note that n-tuples of terms in A_n may contain occurrences of variables outside $\{x_1, \ldots, x_n\}$. Given a model $(W, R, \mathcal{V})$ on a frame in $\mathcal{C}$ and $(a_1, \ldots, a_n) \in A_n$, let $\Phi^{(W,R,\mathcal{V})}_{(a_1,\ldots,a_n)}$ be the set of all equational formulas $\varphi(x_1, \ldots, x_n)$ in $FOR(x_1, \ldots, x_n)$ such that $(W, R, \mathcal{V}) \models \varphi(a_1, \ldots, a_n)$. Consider a complete list of representatives for each equivalence class on $\Phi^{(W,R,\mathcal{V})}_{(a_1,\ldots,a_n)}$ modulo $\equiv^n_{\mathcal{C}}$ and let $\varphi^{(W,R,\mathcal{V})}_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n)$ be their conjunction.

We define on A_n the equivalence relation $\cong^n_{\mathcal{C}}$ as follows:

– $(a_1, \ldots, a_n) \cong^n_{\mathcal{C}} (b_1, \ldots, b_n)$ iff for all formulas $\varphi(x_1, \ldots, x_n)$ in $FOR(x_1, \ldots, x_n)$, $\varphi(a_1, \ldots, a_n)$ is $\mathcal{C}$-valid iff $\varphi(b_1, \ldots, b_n)$ is $\mathcal{C}$-valid.

Now, we define on A_n the equivalence relation $\simeq^n_{\mathcal{C}}$ as follows:

– $(a_1, \ldots, a_n) \simeq^n_{\mathcal{C}} (b_1, \ldots, b_n)$ iff for all equational formulas $\varphi(x_1, \ldots, x_n)$ in $FOR(x_1, \ldots, x_n)$, $\varphi(a_1, \ldots, a_n)$ is $\mathcal{C}$-valid iff $\varphi(b_1, \ldots, b_n)$ is $\mathcal{C}$-valid.

Obviously, $\cong^n_{\mathcal{C}}$ is finer than $\simeq^n_{\mathcal{C}}$. Lemma 4 is a consequence of Proposition 6; Lemma 5 is a consequence of Lemma 4; Lemma 6 is a consequence of the definition of $\simeq^n_{\mathcal{C}}$ and Lemma 3; Lemma 7 is a consequence of Lemma 6; Lemma 8 is a consequence of the definition of $\varphi^{(W,R,\mathcal{V})}_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n)$.

Lemma 4. *$\cong_{\mathcal{C}}^{n}$ has finitely many equivalence classes on A_n.*

Lemma 5. *$\simeq_{\mathcal{C}}^{n}$ has finitely many equivalence classes on A_n.*

Lemma 6. *Let $(a_1, \ldots, a_n)$ be an n-tuple in $TER(x_1, \ldots, x_k)^n$ and $(b_1, \ldots, b_n)$ be an n-tuple of n-polynoms. If $(b_1, \ldots, b_n)$ is properly obtained from $(a_1, \ldots, a_n)$ with respect to (k, n) then $(a_1, \ldots, a_n) \simeq_{\mathcal{C}}^{n} (b_1, \ldots, b_n)$.*

Lemma 7. *$TER(x_1, \ldots, x_n)^n$ constitutes a complete set of representatives for each equivalence class on A_n modulo $\simeq_{\mathcal{C}}^{n}$.*

Lemma 8. *Let $(W, R, \mathcal{V})$ be a model on a frame in $\mathcal{C}$ and $(a_1, \ldots, a_n) \in A_n$. $(W, R, \mathcal{V}) \models \varphi_{(a_1,\ldots,a_n)}^{(W,R,\mathcal{V})}(a_1, \ldots, a_n)$.*

Proposition 7. *If $\mathcal{C}$ is balanced then for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in A_n$, if $(a_1, \ldots, a_n) \simeq_{\mathcal{C}}^{n} (b_1, \ldots, b_n)$ then $(a_1, \ldots, a_n) \cong_{\mathcal{C}}^{n} (b_1, \ldots, b_n)$.*

Proof. Suppose $\mathcal{C}$ is balanced. Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in A_n$ be such that $(a_1, \ldots, a_n) \simeq_{\mathcal{C}}^{n} (b_1, \ldots, b_n)$ and $(a_1, \ldots, a_n) \not\cong_{\mathcal{C}}^{n} (b_1, \ldots, b_n)$. Let $\varphi(x_1, \ldots, x_n)$ be a formula in $FOR(x_1, \ldots, x_n)$ such that $\varphi(a_1, \ldots, a_n)$ is $\mathcal{C}$-valid not-iff $\varphi(b_1, \ldots, b_n)$ is $\mathcal{C}$-valid. Without loss of generality, let us assume that $\varphi(a_1, \ldots, a_n)$ is $\mathcal{C}$-valid and $\varphi(b_1, \ldots, b_n)$ is not $\mathcal{C}$-valid. Since $\mathcal{C}$ is balanced, therefore let $(W, R, \mathcal{V})$ be a balanced model on a countable frame in $\mathcal{C}$ such that $(W, R, \mathcal{V}) \not\models \varphi(b_1, \ldots, b_n)$. By Lemma 8, $(W, R, \mathcal{V}) \models \varphi_{(b_1,\ldots,b_n)}^{(W,R,\mathcal{V})}(b_1, \ldots, b_n)$. Hence, $\neg\varphi_{(b_1,\ldots,b_n)}^{(W,R,\mathcal{V})}(b_1, \ldots, b_n)$ is not $\mathcal{C}$-valid. Since $(a_1, \ldots, a_n) \simeq_{\mathcal{C}}^{n} (b_1, \ldots, b_n)$, therefore $\neg\varphi_{(b_1,\ldots,b_n)}^{(W,R,\mathcal{V})}(a_1, \ldots, a_n)$ is not $\mathcal{C}$-valid. Since $\mathcal{C}$ is balanced, therefore let $(W', R', \mathcal{V}')$ be a balanced model on a countable frame in $\mathcal{C}$ such that $(W', R', \mathcal{V}') \models \varphi_{(b_1,\ldots,b_n)}^{(W,R,\mathcal{V})}(a_1, \ldots, a_n)$. Now, consider $(\beta_1, \ldots, \beta_n) \in \{0, 1\}^n$. If $\bar{\mathcal{V}}(b_1^{\beta_1} \cap \ldots \cap b_n^{\beta_n}) = \emptyset$ then $(W, R, \mathcal{V}) \models EQ(b_1^{\beta_1} \cap \ldots \cap b_n^{\beta_n}, 0)$. Thus, $\varphi_{(b_1,\ldots,b_n)}^{(W,R,\mathcal{V})}(x_1, \ldots, x_n) \to EQ(x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n}, 0)$ is $\mathcal{C}$-valid. Since $(W', R', \mathcal{V}') \models \varphi_{(b_1,\ldots,b_n)}^{(W,R,\mathcal{V})}(a_1, \ldots, a_n)$, therefore $(W', R', \mathcal{V}') \models EQ(a_1^{\beta_1} \cap \ldots \cap a_n^{\beta_n}, 0)$. Consequently, $\bar{\mathcal{V}}'(a_1^{\beta_1} \cap \ldots \cap a_n^{\beta_n}) = \emptyset$. Similarly, the reader may easily verify that if $\bar{\mathcal{V}}(b_1^{\beta_1} \cap \ldots \cap b_n^{\beta_n}) = W$ then $\bar{\mathcal{V}}'(a_1^{\beta_1} \cap \ldots \cap a_n^{\beta_n}) = W'$ and if $\bar{\mathcal{V}}(b_1^{\beta_1} \cap \ldots \cap b_n^{\beta_n})$ is infinite and coinfinite then $\bar{\mathcal{V}}'(a_1^{\beta_1} \cap \ldots \cap a_n^{\beta_n})$ is infinite and coinfinite. In all cases, there exists a bijection $g_{(\beta_1,\ldots,\beta_n)}$ from $\bar{\mathcal{V}}(b_1^{\beta_1} \cap \ldots \cap b_n^{\beta_n})$ to $\bar{\mathcal{V}}'(a_1^{\beta_1} \cap \ldots \cap a_n^{\beta_n})$. Let g be the union of all $g_{(\beta_1,\ldots,\beta_n)}$ when $(\beta_1, \ldots, \beta_n)$ describes $\{0, 1\}^n$. The reader may easily verify that g is a bijection from W to W' such that for all $u \in W$ and for all $(\beta_1, \ldots, \beta_n) \in \{0, 1\}^n$, $u \in \bar{\mathcal{V}}(b_1^{\beta_1} \cap \ldots \cap b_n^{\beta_n})$ iff $g(u) \in \bar{\mathcal{V}}'(a_1^{\beta_1} \cap \ldots \cap a_n^{\beta_n})$. Let R'_g be the binary relation on W' defined by $u' R'_g v'$ iff $g^{-1}(u') R g^{-1}(v')$. Obviously, g is an isomorphism from (W, R) to (W', R'_g). Since $\varphi(a_1, \ldots, a_n)$ is $\mathcal{C}$-valid, therefore $(W', R'_g, \mathcal{V}') \models \varphi(a_1, \ldots, a_n)$. Hence, $(W, R, \mathcal{V}) \models \varphi(b_1, \ldots, b_n)$: a contradiction.

8 Computability of Unifiability

Let $\mathcal{C}$ be a class of frames. Lemma 9 is a consequence of the definitions in Sect. 4.

Lemma 9

1. *For all $a \in TER(\emptyset)$, either $EQ(a, 0)$ is $\mathcal{C}$-valid, or $EQ(a, 1)$ is $\mathcal{C}$-valid. Moreover, the formula in $\{EQ(a, 0), EQ(a, 1)\}$ that is $\mathcal{C}$-valid can be computed in linear time.*
2. *For all $a, b \in TER(\emptyset)$, either $DC(a, b)$ is $\mathcal{C}$-valid, or $EQ(a, b)$ is $\mathcal{C}$-valid. Moreover, the formula in $\{DC(a, b), EQ(a, b)\}$ that is $\mathcal{C}$-valid can be computed in linear time.*

Lemma 10 is a consequence of the definition of unifiability.

Lemma 10. *For all formulas $\varphi(x_1, \ldots, x_n)$, φ is $\mathcal{C}$-unifiable iff there exists $a_1, \ldots, a_n \in TER(\emptyset)$ such that $\varphi(a_1, \ldots, a_n)$ is $\mathcal{C}$-valid.*

Proposition 8. *The elementary $\mathcal{C}$-unifiability problem is in NP.*

Proof. By Lemmas 9 and 10, for all formulas $\varphi(x_1, \ldots, x_n)$, $\varphi(x_1, \ldots, x_n)$ is $\mathcal{C}$-unifiable iff there exists $a_1, \ldots, a_n \in \{0, 1\}$ such that $\varphi(a_1, \ldots, a_n)$ is $\mathcal{C}$-valid. Obviously, this can be decided in polynomial time.

Proposition 9. *Let $a(x_1, \ldots, x_n)$ be a term. $EQ(a(x_1, \ldots, x_n), 1)$ is $\mathcal{C}$-unifiable iff considered as a formula in CPL, $a(x_1, \ldots, x_n)$ is satisfiable.*

Proof. Suppose considered as a formula in CPL, $a(x_1, \ldots, x_n)$ is satisfiable. Let $b_1, \ldots, b_n$ in $\{0, 1\}$ be such that $a(b_1, \ldots, b_n)$ is a tautology. Hence, $EQ(a(b_1, \ldots, b_n), 1)$ is $\mathcal{C}$-valid. Thus, $EQ(a(x_1, \ldots, x_n), 1)$ is $\mathcal{C}$-unifiable.

Suppose $EQ(a(x_1, \ldots, x_n), 1)$ is $\mathcal{C}$-unifiable. By Lemmas 9 and 10, let $b_1, \ldots, b_n$ in $\{0, 1\}$ be such that $EQ(a(b_1, \ldots, b_n), 1)$ is $\mathcal{C}$-valid. Consequently, $a(b_1, \ldots, b_n)$ is a tautology. Hence, considered as a formula in CPL, $a(x_1, \ldots, x_n)$ is satisfiable.

Proposition 10. *The elementary $\mathcal{C}$-unifiability problem is NP-hard.*

Proof. By Proposition 9 and the NP-hardness of the satisfiability problem of formulas in CPL.

It follows from Propositions 8 and 10 that

Proposition 11. *The elementary unifiability problem in $\mathcal{C}_{all}$, $\mathcal{C}_{ind}$ and $\mathcal{C}_{con}$ is NP-complete.*

In other respect,

Proposition 12. *Let A be a complexity class. If $\mathcal{C}$ is balanced and the $\mathcal{C}$-satisfiability problem is in A then the elementary $\mathcal{C}$-admissibility problem is in $coNEXP^A$.*

Proof. Suppose $\mathcal{C}$ is balanced and the $\mathcal{C}$-satisfiability problem is in A. By Lemma 7 and Proposition 7, for all inference rules $\frac{\varphi(x_1,\ldots,x_n)}{\psi(x_1,\ldots,x_n)}$, $\frac{\varphi(x_1,\ldots,x_n)}{\psi(x_1,\ldots,x_n)}$ is not $\mathcal{C}$-admissible iff there exists $(b_1,\ldots,b_n) \in TER(x_1,\ldots,x_n)^n$ such that $\neg\varphi(b_1,\ldots,b_n)$ is not $\mathcal{C}$-satisfiable and $\neg\psi(b_1,\ldots,b_n)$ is $\mathcal{C}$-satisfiable. Obviously, this can be decided in exponential time with oracle in A.

Since the satisfiability problem in $\mathcal{C}_{all}$ and $\mathcal{C}_{ind}$ is in NP and the satisfiability problem in $\mathcal{C}_{con}$ is in $PSPACE$, it follows from Propositions 3 and 12 that

Proposition 13. *The elementary admissibility problem in $\mathcal{C}_{all}$ and $\mathcal{C}_{ind}$ is in co$NEXP^{NP}$ and the elementary admissibility problem in $\mathcal{C}_{con}$ is in co$NEXP^{PSPACE}$.*

Still, we do not know whether the elementary $\mathcal{C}$-admissibility problem is in co$NEXP$. We conjecture that in $\mathcal{C}_{all}$, $\mathcal{C}_{ind}$ and $\mathcal{C}_{con}$, it is co$NEXP$-complete

9 Unification Type

Let $\mathcal{C}$ be a class of frames.

Proposition 14. *If $\mathcal{C}$ agrees with unions then $EQ(0,x) \vee EQ(1,x)$ is not $\mathcal{C}$-unitary.*

Proof. Suppose $\mathcal{C}$ agrees with unions and $EQ(0,x) \vee EQ(1,x)$ is $\mathcal{C}$-unitary. Let σ_0 and σ_1 be substitutions such that $\sigma_0(x) = 0$ and $\sigma_1(x) = 1$. Obviously, σ_0 and σ_1 are $\mathcal{C}$-unifiers of $EQ(0,x) \vee EQ(1,x)$. Since $EQ(0,x) \vee EQ(1,x)$ is $\mathcal{C}$-unitary, therefore let τ be a $\mathcal{C}$-unifier of $EQ(0,x) \vee EQ(1,x)$ such that $\tau \preceq_{\mathcal{C}} \sigma_0$ and $\tau \preceq_{\mathcal{C}} \sigma_1$. Let μ, ν be substitutions such that $\tau \circ \mu \simeq_{\mathcal{C}} \sigma_0$ and $\tau \circ \nu \simeq_{\mathcal{C}} \sigma_1$. Hence, $EQ(\bar{\mu}(\tau(x)), 0)$ is $\mathcal{C}$-valid and $EQ(\bar{\nu}(\tau(x)), 1)$ is $\mathcal{C}$-valid. Thus, neither $EQ(0, \tau(x))$ is $\mathcal{C}$-valid, nor $EQ(1, \tau(x))$ is $\mathcal{C}$-valid. Let (W, R) and (W', R') be disjoint frames in $\mathcal{C}$, $\mathcal{V}$ be a valuation on W and $\mathcal{V}'$ be a valuation on W' such that neither $\bar{\mathcal{V}}(\tau(x)) = \emptyset$, nor $\bar{\mathcal{V}}'(\tau(x)) = W'$. Since $\mathcal{C}$ agrees with unions, therefore let (W'', R'') be a frame in $\mathcal{C}$ such that $W \cup W' = W''$. Let $\mathcal{V}''$ be the valuation on W'' such that for all variables z, $\mathcal{V}''(z) = \mathcal{V}(z) \cup \mathcal{V}'(z)$. Obviously, for all terms a, $\bar{\mathcal{V}}''(a) = \bar{\mathcal{V}}(a) \cup \bar{\mathcal{V}}'(a)$. Since neither $\bar{\mathcal{V}}(\tau(x)) = \emptyset$, nor $\bar{\mathcal{V}}'(\tau(x)) = W'$, therefore neither $\bar{\mathcal{V}}''(\tau(x)) = \emptyset$, nor $\bar{\mathcal{V}}''(\tau(x)) = W''$. Consequently, τ is not a $\mathcal{C}$-unifier of $EQ(0,x) \vee EQ(1,x)$: a contradiction.

Proposition 15. *If $\mathcal{C}$ agrees with unions then the substitutions σ_0 and σ_1 such that $\sigma_0(x) = 0$, $\sigma_1(x) = 1$ and for all variables y, if $x \neq y$ then $\sigma_0(y) = y$ and $\sigma_1(y) = y$ constitute a complete set of $\mathcal{C}$-unifiers of $EQ(0,x) \vee EQ(1,x)$. Moreover, $EQ(0,x) \vee EQ(1,x)$ is $\mathcal{C}$-finitary.*

Proof. Suppose $\mathcal{C}$ agrees with unions. Hence, by Proposition 14, $EQ(0,x) \vee EQ(1,x)$ is not $\mathcal{C}$-unitary. Obviously, σ_0 and σ_1 are $\mathcal{C}$-unifiers of $EQ(0,x) \vee EQ(1,x)$. Let τ be an arbitrary $\mathcal{C}$-unifier of $EQ(0,x) \vee EQ(1,x)$ such that neither $\sigma_0 \preceq_{\mathcal{C}} \tau$, nor $\sigma_1 \preceq_{\mathcal{C}} \tau$. Thus, neither $EQ(0, \tau(x))$ is $\mathcal{C}$-valid, nor $EQ(1, \tau(x))$

is $\mathcal{C}$-valid. Following the same line of reasoning as in the proof of Proposition 14, we conclude τ is not a $\mathcal{C}$-unifier of $EQ(0,x) \vee EQ(1,x)$: a contradiction. Since τ was arbitrary, therefore σ_0 and σ_1 constitute a complete set of $\mathcal{C}$-unifiers of $EQ(0,x) \vee EQ(1,x)$. Consequently, $EQ(0,x) \vee EQ(1,x)$ is either $\mathcal{C}$-unitary, or $\mathcal{C}$-finitary. Since $EQ(0,x) \vee EQ(1,x)$ is not $\mathcal{C}$-unitary, therefore $EQ(0,x) \vee EQ(1,x)$ is $\mathcal{C}$-finitary.

Proposition 16. *If $\mathcal{C}$ is balanced then elementary unification in $\mathcal{C}$ is either finitary, or unitary. Moreover, if $\mathcal{C}$ agrees with unions then elementary unification in $\mathcal{C}$ is finitary.*

Proof. Suppose $\mathcal{C}$ is balanced. Let $\varphi(x_1,\ldots,x_n)$ be an arbitrary $\mathcal{C}$-unifiable formula. Let σ be an arbitrary substitution such that $\bar{\sigma}(\varphi)$ is $\mathcal{C}$-valid. Without loss of generality, we can assume that for all variables y, if $y \notin \{x_1,\ldots,x_n\}$ then $\sigma(y) = y$. Let k be a nonnegative integer and $(a_1,\ldots,a_n) \in TER(x_1,\ldots,x_k)^n$ be such that for all positive integers i, if $i \leq n$ then $\sigma(x_i) = a_i$. Since $\bar{\sigma}(\varphi)$ is $\mathcal{C}$-valid, therefore $\varphi(a_1,\ldots,a_n)$ is $\mathcal{C}$-valid. Let $\sim^k_{(a_1,\ldots,a_n)}$, $f:\ \{0,1\}^k \longrightarrow \{0,1\}^n$ and $(b_1,\ldots,b_n)$ be as in Sect. 7. By Lemma 6, $(a_1,\ldots,a_n) \simeq^n_{\mathcal{C}} (b_1,\ldots,b_n)$. Since $\mathcal{C}$ is balanced, therefore by Proposition 7, $(a_1,\ldots,a_n) \cong^n_{\mathcal{C}} (b_1,\ldots,b_n)$. Let τ be the substitution such that for all positive integers i, if $i \leq n$ then $\tau(x_i) = b_i$ and for all variables y, if $y \notin \{x_1,\ldots,x_n\}$ then $\tau(y) = y$. Note that $(\tau(x_1),\ldots,\tau(x_n)) \in TER(x_1,\ldots,x_n)^n$. Moreover, since $\varphi(a_1,\ldots,a_n)$ is $\mathcal{C}$-valid and $(a_1,\ldots,a_n) \cong^n_{\mathcal{C}} (b_1,\ldots,b_n)$, therefore $\varphi(b_1,\ldots,b_n)$ is $\mathcal{C}$-valid. Hence, τ is a $\mathcal{C}$-unifier of φ. For all positive integers i, if $i \leq n$ then let Δ_i and c_i be as in Sect. 6. Let υ be as in Sect. 7. By Proposition 5, for all positive integers i, if $i \leq n$ then considered as formulas in CPL, the terms a_i and $\bar{\upsilon}(b_i)$ are equivalent. Thus, for all positive integers i, if $i \leq n$ then $EQ(\bar{\upsilon}(\tau(x_i)),\sigma(x_i))$ is $\mathcal{C}$-valid. Consequently, $\tau \circ \upsilon \simeq_{\mathcal{C}} \sigma$. Hence, $\tau \preceq_{\mathcal{C}} \sigma$. Since σ was arbitrary and $(\tau(x_1),\ldots,\tau(x_n)) \in TER(x_1,\ldots,x_n)^n$, therefore φ is either $\mathcal{C}$-finitary, or $\mathcal{C}$-unitary. Since φ was arbitrary, therefore elementary unification in $\mathcal{C}$ is either finitary, or unitary. Now, suppose $\mathcal{C}$ agrees with unions. By Proposition 15, elementary unification in $\mathcal{C}$ is not unitary. Since elementary unification in $\mathcal{C}$ is either finitary, or unitary, therefore elementary unification in $\mathcal{C}$ is finitary.

It follows from the above discussion that elementary unification in $\mathcal{C}_{all}$, $\mathcal{C}_{ind}$ and $\mathcal{C}_{con}$ is finitary.

10 Conclusion

Much remains to be done. For example, what becomes of the computability of unifiability and admissibility when the language is extended by the connectedness predicate considered in [14,16]? What becomes of the unification type? And when the language is interpreted in different Euclidean spaces as in [15,16]? In other respect, it remains to see how decision procedures for unifiability and admissibility can be used to improve the performance of algorithms that handle the satisfiability problem. Finally, one may as well consider these questions when the language is extended by a set of propositional constants (denoted p, q, etc.). In this case: **(i)** the unifiability problem is to determine, given a formula $\varphi(p_1, \ldots, p_m, x_1, \ldots, x_n)$, whether there exists terms $a_1, \ldots, a_n$ such that $\varphi(p_1, \ldots, p_m, a_1, \ldots, a_n)$ is valid; **(ii)** the admissibility problem is to determine, given an inference rule $\frac{\varphi(p_1,\ldots,p_m,x_1,\ldots,x_n)}{\psi(p_1,\ldots,p_m,x_1,\ldots,x_n)}$, whether for all terms $a_1, \ldots, a_n$, if $\varphi(p_1, \ldots, p_m, a_1, \ldots, a_n)$ is valid then $\psi(p_1, \ldots, p_m, a_1, \ldots, a_n)$ is valid. We conjecture that in $\mathcal{C}_{all}$, $\mathcal{C}_{ind}$ and $\mathcal{C}_{con}$, unification with constants is $NEXP$-complete but still finitary.

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