Sturmian images of non Sturmian words and standard morphisms [§]

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November 17, 2017

Abstract

We prove that if a Sturmian word is the image by a morphism of a word which is a fixed point of another morphism, then this latter word is mostly a Sturmian word, and the involved morphisms are Sturmian. This gives a characterization of Sturmian words that are generated by HD0L systems.

We also characterize the non Sturmian words which can be sent on Sturmian words by morphism, and the involved morphisms. We prove that the same Sturmian images can be obtained by using the standard morphisms of which the above morphisms are the conjugates, and we show how to obtain these Sturmian morphisms from their standard representatives.

Keywords: Sturmian words, Sturmian morphisms, standard morphisms, D0L systems, HD0L systems, Fibonacci word

1 Introduction

Sturmian words are infinite binary words of minimal complexity and Sturmian morphisms are those binary morphisms which preserve Sturmian words. These words and morphisms have been widely studied (see [8] for a general overview) and the Sturmian morphisms are very well known, especially those which generate Sturmian words (see, *e.g.*, [17]). However words generated by morphisms are only a particular case of words generated by D0L systems and, even more generally, by HD0L systems which are special sub-families of Lindenmayer systems, a large class of mechanisms to produce infinite words which has been the object of hundreds of papers.

Surprisingly, it seems that Sturmian words and morphisms were not studied in the framework of Lindenmayer systems.

It is one of the aim of this paper to complete the knowledge of the links between Sturmian words and Sturmian morphisms when D0L systems are considered. First we shall characterize Sturmian words that are fixed points of morphisms (see Theorem 3.2 and Corollary 3.3). Except in the case of the identity morphism, these words are exactly the Sturmian words generated by morphisms (D0L systems where the starting word is a single letter). This complements Parvaix results [11] about the action of morphisms on mechanical words.

The family of HD0L systems strictly contains that of D0L systems (roughly, an HD0L system needs two morphisms when a D0L system uses only one). One of the oldest examples is the *Arshon word* that Berstel [2] proved to be generated by an HD0L system but not by a D0L system. However, as in the case

[§]A preliminary version of this work was presented at “Journées annuelles SDA2” CNRS, Marseille, 2017

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of the Arshon word, almost all of the interesting known examples use a very particular case of HD0L systems, namely the *tag-systems* introduced by Cobham [5] (see also [1], Theorem 6.3.2).

Here, Theorem 4.3 and its Corollary 4.4 give an example of a wide family of words generated by HD0L systems which are not tag-systems (because a Sturmian morphism is never uniform), some of these words being not generated by D0L-systems (Proposition 4.5).

The other goal of this paper is to provide a sharp characterizion of the morphisms which transform non Sturmian words into Sturmian words. This is done with Theorems 5.2, 6.6 and 6.7.

After some definitions and basic results (Section 2) we study Sturmian fixed points of morphisms (Section 3). Then, in Section 4, we characterize the words whose images by a morphism are Sturmian (Section 4.1) and those which are an image by a morphism of a fixed point of another morphism (Section 4.2). Section 4 ends with an example of a Sturmian word generated by an HD0L system but not by a D0L system (Section 4.3). In Section 5 we detail the case of non Sturmian words whose images by morphisms are Sturmian words and we give a characterization of the morphisms which realize such a transformation. The last part of the paper (Section 6) is dedicated to evidence the link between these morphisms and the standard morphisms of which they are the conjugates, and to show how to directly obtain the former from the latter.

2 Preliminaries

We assume that the reader is familiar with basics of combinatorics on words (if necessary, see [8]).

Let $\mathcal{A} = \{a, b\}$. We will often consider a letter $c \in \mathcal{A}$: the letter $\bar{c}$ is then defined by $\{c, \bar{c}\} = \mathcal{A}$.

If u is a word over $\mathcal{A}$ and n a non-negative integer, u^n is the power n of u, *i.e.*, the word $u \cdots u$ made of n occurrences of u. We also use, as usual, the notation u^* to denote the set $\{u^n / n \in \mathbb{N}\}$ of all the powers of u.

A word x over $\mathcal{A}$ is *balanced* if, for all u, v factors of x such that $|u| = |v|$, $||u|_a - |v|_a| \leq 1$ (since $\mathcal{A}$ is a two-letter alphabet, this implies $||u|_b - |v|_b| \leq 1$).

An infinite word x over $\mathcal{A}$ is *ultimately periodic* if there exist words $y \in \mathcal{A}^*$ and $z \in \mathcal{A}^+$ such that $x = yz^\omega$. Otherwise x is *aperiodic*.

Sturmian words are infinite words over $\mathcal{A}$ that are aperiodic and balanced.

We recall here a property of Sturmian words which is useful in the sequel.

Property 2.1 ([8], Proposition 2.1.3)**.** *If an aperiodic infinite word over $\mathcal{A}$ is not Sturmian, then it contains as factors both awa and bwb for some $w \in \mathcal{A}^*$.*

A *(endo)morphism on* $\mathcal{A}$ is an application f from $\mathcal{A}^*$ onto $\mathcal{A}^*$ such that $f(uv) = f(u)f(v)$ for every words u, v over $\mathcal{A}$. The identity morphism on $\mathcal{A}$ is written $Id_{\mathcal{A}}$ and in the whole paper, since there is no ambiguity, the composition of morphisms $f \circ g$ will be denoted fg.

A morphism f on $\mathcal{A}$ is *prolongable on the word* $u \in \mathcal{A}^+$ if $f(u)$ begins with u and $|f^{p+1}(u)| > |f^p(u)|$ for every $p \in \mathbb{N}$. In this case, $f^\omega(u) = \lim_{n\to\infty} f^n(u)$ is a well defined infinite word which is called the *infinite word generated by* f. Here we remark that if f is not prolongable on u then $|f(u)|_u \leq 1$.

2.1 L systems

Words generated by morphisms are a particular case of a more general notion: *words generated by Lindenmayer systems* (see, *e.g.*, [15]). In the present paper, we only deal with two sorts of such systems (we use the definitions given in [6]).

- A *D0L system* is a triple $G = (\mathcal{A}, \sigma, u)$ where $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ is a morphism and $u \in \mathcal{A}^+$ is a word. If σ is prolongable on u then G generates the infinite word $\sigma^\omega(u)$.
- An *HD0L system* is a 4-tuple $H = (\mathcal{A}, \sigma, \tau, u)$ where $(\mathcal{A}, \sigma, u)$ is a *D0L* system and τ is a morphism over $\mathcal{A}$. If σ is prolongable on u then H generates the infinite word $\tau(\sigma^\omega(u))$.

2.2 Sturmian morphisms

A morphism f on $\mathcal{A}$ is *Sturmian* if $f(x)$ is a Sturmian word whenever x is a Sturmian word. For details about Sturmian words and morphisms, see [8], Chapter 2.

In particular, it is a well-known fact that each Sturmian morphism can be obtained by composition from the three morphisms

$$E: \begin{array}{lcl} a & \mapsto & b \\ b & \mapsto & a \end{array} \qquad \varphi: \begin{array}{lcl} a & \mapsto & ab \\ b & \mapsto & a \end{array} \qquad \tilde{\varphi}: \begin{array}{lcl} a & \mapsto & ba \\ b & \mapsto & a \end{array}$$

Thus f is a Sturmian morphism if and only if $f \in St = \{E, \varphi, \tilde{\varphi}\}^*$.

Writing convention. Let f be a Sturmian morphism. To a given decomposition of $f = f_1 f_2 \cdots f_n$, $f_i \in \{E, \varphi, \tilde{\varphi}\}$, is associated the word $f_1 f_2 \cdots f_n$ over the alphabet $\{E, \varphi, \tilde{\varphi}\}$ (here $E, \varphi, \tilde{\varphi}$ are considered as letters). We also denote this word f ($f \in \{E, \varphi, \tilde{\varphi}\}^*$) and we will use such an f either as the morphism or as the associated word, without noticing it when there is no ambiguity.

Since $E^2 = Id_{\mathcal{A}}$, Sturmian morphisms can be obtained by an infinite number of compositions of E, φ and $\tilde{\varphi}$ and are thus associated with an infinite number of words over the alphabet $\{E, \varphi, \tilde{\varphi}\}$. We sometimes use the terminology *writing* of f to indicate one particular word associated with the morphism f. For example, the morphism $\tilde{\varphi} E \varphi$ has two writings containing no occurrence of E^2: the words $\tilde{\varphi} E \varphi$ and $\varphi E \tilde{\varphi}$ (because, as a morphism, $\tilde{\varphi} E \varphi = \varphi E \tilde{\varphi}$).

We call *length* of a Sturmian morphism f the length of (one of) the shortest word associated with f. In particular, two Sturmian morphisms can be compared by their lengths.

To end, in Sections 4 and 6, we will use the notation $|f|_{\varphi, \tilde{\varphi}}$ to indicate the sum of the number of occurrences of φ and $\tilde{\varphi}$ in a writing of a given morphism f (we will see that this value is the same for all the writings of a given f). For example, $|\varphi \tilde{\varphi} E \varphi E \tilde{\varphi}|_{\varphi, \tilde{\varphi}} = 4$.

A characterization of non-trivial elements of St is the following (see [9]):

$$St \setminus \{Id_{\mathcal{A}}, E\} = \{\varphi, E\varphi, \varphi E, E\varphi E, \tilde{\varphi}, E\tilde{\varphi}, \tilde{\varphi}E, E\tilde{\varphi}E\}^+. \tag{1}$$

It is aspecially clear with this decomposition that if f is a Sturmian morphism such that $f(a) = a$ (resp. $f(b) = b$) then

$$f \in \{\varphi E, \tilde{\varphi} E\}^* (\text{resp. } f \in \{E\varphi, E\tilde{\varphi}\}^*) \tag{2}$$

and we have the following property.

Lemma 2.2 ([9]). *The word generated by a Sturmian morphism f is a Sturmian word if and only if $f \notin \{\varphi E, \tilde{\varphi} E\}^* \cup \{E\varphi, E\tilde{\varphi}\}^*$* (i.e., *if $f(a) \neq a$ and $f(b) \neq b$*). ■

We give here two useful properties of the morphisms E, φ, and $\tilde{\varphi}$ (the first one is an observation, originally made by Berstel [3]).

Lemma 2.3. *For every $u \in \mathcal{A}^*$, $a\tilde{\varphi}(u) = \varphi(u)a$. For every infinite word x over $\mathcal{A}$, $a\tilde{\varphi}(x) = \varphi(x)$.* ■

Lemma 2.4. *Let x be an infinite word over $\mathcal{A}$.*

(i) If $E(x)$ is Sturmian then x is Sturmian.

(ii) If $\varphi(x)$ is Sturmian then x is Sturmian.

(iii) If $\tilde{\varphi}(x)$ is Sturmian and x starts with the letter a, then x is Sturmian.

(iv) If $\tilde{\varphi}(x)$ is Sturmian and x starts with the letter b, then $x = bx'$ and x' is Sturmian.

Proof. Item (i) is a direct consequence of the definition of Sturmian words, and items (ii) and (iii) are Proposition 2.3.2 of [8].

We prove item (iv). Let x be an infinite word over $\mathcal{A}$ such that $\tilde{\varphi}(x)$ is Sturmian. If x starts with the letter b, then $x = bx'$ and $\tilde{\varphi}(x) = a\tilde{\varphi}(x')$. Since $a\tilde{\varphi}(x') = \varphi(x')$ (Lemma 2.3), x' is Sturmian from (ii). ■

3 Sturmian fixed points of morphisms

We start our study by considering the case of a Sturmian word which is a fixed point of a morphism: we show that the morphism is Sturmian, and if it is different from $Id_{\mathcal{A}}$ then the word is generated by this morphism.

The first step is given separately because it is interesting in itself.

Proposition 3.1. *Let x be an infinite word over $\mathcal{A}$ and f a morphism on $\mathcal{A}$. If $f(x)$ is Sturmian then $f \in St$.*

It is worth noticing that this proposition is an improvement of Theorem 2.3.7 of [8]. Indeed Theorem 2.3.7 indicates that, to prove that a morphism is in St it is enough to verify that the image of one Sturmian word is a Sturmian word. The present proposition proves that the condition according to which the starting word has to be Sturmian is not even necessary: it is sufficient that the image is a Sturmian word.

Proof. Let x be an infinite word over $\mathcal{A}$ and f a morphism on $\mathcal{A}$ such that $f(x)$ is Sturmian.

Before starting, we note that if $f(x)$ is Sturmian then it is aperiodic and thus contains at least one occurrence of both $f(ab)$ and $f(ba)$, and $f(a) \neq \varepsilon$, $f(b) \neq \varepsilon$.

From [8], Theorem 2.3.7, if x is a Sturmian word then $f \in St$. Consequently, we suppose in the following that x is not a Sturmian word (but x is aperiodic, otherwise $f(x)$ is ultimately periodic thus non Sturmian). From Property 2.1, there exists a word $w \in \mathcal{A}^*$ such that both awa and bwb are factors of x.

If $|f(a)| = |f(b)|$ then we first observe that $|f(a)|_a = |f(b)|_a$. For if not, $||f(awa)|_a - |f(bwb)|_a| \geq 2$; a contradiction with $f(x)$ Sturmian.

- Now, if $f(a) = f(b)$ then $f(x)$ is periodic; a contradiction with $f(x)$ Sturmian.
- But, if $f(a) \neq f(b)$ then there exist words r, t, t' over $\mathcal{A}$ such that $f(a) = rat$ and $f(b) = rbt'$. Moreover, since $|f(a)|_a = |f(b)|_a$, one has $|t'|_a = |t|_a + 1$ and, since $|f(a)| = |f(b)|$, $|t'| = |t|$. But in this case, $f(ab) = ratrbt'$ and $f(ba) = rbt'rat$ thus $f(x)$ contains the two factors trb and $t'ra$ which are such that $|trb| = |t'ra|$ and $|trb|_a = |t'ra|_a - 2$; again a contradiction with $f(x)$ Sturmian.

Thus if $f(x)$ is Sturmian then $|f(a)| \neq |f(b)|$.

Now, suppose that $f(a)$ (resp. $f(b)$) is neither a prefix nor a suffix of $f(b)$ (resp. $f(a)$).

This means that there exist words r, r_1, r_2, t, t_1, t_2 such that $f(a) = ryt_1 = r_1y't$ and $f(b) = rzt_2 = r_2z't$ for $\{y, z\} = \{y', z'\} = \{a, b\}$.

- If $y = y'$ then $z = z'$ $(\neq y)$. In this case $f(awa)$ contains the factor $y'tf(w)ry$ and $f(bwb)$ contains the factor $z'tf(w)rz$; a contradiction with $f(x)$ Sturmian because $||y'try|_a - |z'trz|_a| = 2$.
- If $y = z'$ then $z = y'$ $(\neq y)$. In this case $f(ab) = r_1y'trzt_2$ and $f(ba) = r_2z'tryt_1$; again a contradiction with $f(x)$ Sturmian because $||y'trz|_a - |z'try|_a| = 2$.

Thus, if x is not a Sturmian word but $f(x)$ is Sturmian then $f(a)$ (resp. $f(b)$) is a proper prefix or a proper suffix of $f(b)$ (resp. $f(a)$).

We prove that $f \in St$ by induction on $\|f\|$.

Since $f(a)$ is a proper prefix of $f(b)$ one has $\|f\| \geq 3$.

If $\|f\| = 3$, since $f(b)$ cannot be equal to aa or bb because $f(x)$ is Sturmian, $f \in \{\varphi E, E\varphi E\} \subset St$.

Now, suppose that $\|f\| > 3$ and, by induction, if g is such that $\|g\| < \|f\|$ and $g(y)$ is Sturmian for some infinite word y on $\mathcal{A}$ then $g \in St$.

Here we suppose that $f(a)$ is a prefix of $f(b)$ (the other cases are similar).

Since $f(a)$ is a proper prefix of $f(b)$, there exists $s \in \mathcal{A}^+$ such that $f(b) = f(a)s$. Let g be defined by $g(a) = f(a)$ and $g(b) = s$. Since $s \neq \varepsilon$, $\|g\| < \|f\|$. Moreover $f = g\varphi E$ thus $g(y)$ is Sturmian for $y = \varphi E(x)$. Consequently $g \in St$ by induction and, since St is closed under composition, $f \in St$. ■

From this we are able to show that if a Sturmian word is a fixed point of a morphism different from $Id_{\mathcal{A}}$ then it is generated by this morphism which is Sturmian.

Theorem 3.2. *Let x be a Sturmian word over $\mathcal{A}$. If $x = g(x)$ for some morphism g on $\mathcal{A}$ then $g \in St$, and if $g \neq Id_{\mathcal{A}}$ then there exists $c \in \mathcal{A}$ such that g is prolongable on c and $x = g^{\omega}(c)$.*

Remark. In [11], B. Parvaix gives the value of g depending on x. His characterization is based on the notion of *mechanical words* which is out of the scope of the present paper.

Proof. Let x be a Sturmian word, $c \in \mathcal{A}$ the letter such that x begins with c, and g a morphism such that $x = g(x)$.

Since x is Sturmian then, by Proposition 3.1, $g \in St$ thus the first part of the result is proved.

Now, suppose that $g \neq Id_{\mathcal{A}}$. Since $x = g(x)$ and x begins with the letter c, $g(c)$ starts with c thus either $g(c) = c$ or g is prolongable on c. Indeed, if $g(c) \neq c$ and g is not prolongable on c then $|g(c)|_c \leq 1$ implies $g(c) = c\bar{c}^k$ for some $k \in \mathbb{N}$, $k \geq 1$, and $g(\bar{c}) = \varepsilon$; a contradiction with $g \in St$.

- If $g(c) = c$ then, since $g \neq Id_{\mathcal{A}}$, $g(\bar{c}) \neq \bar{c}$.

  Let $p \neq 0$ such that x begins with $c^p\bar{c}$ (such a p does exist because x is Sturmian thus $x \neq c^{\omega}$). If $g(\bar{c})$ begins with c then $g(x)$ starts with c^{p+1}; a contradiction.

  Consequently $g(\bar{c})$ begins with $\bar{c}$ and, since $g(\bar{c}) \neq \bar{c}$, g is prolongable on $\bar{c}$: g generates the Sturmian word x' such that $x = c^p x'$.

  Since $g \in St$ and $g(c) = c$, $g \in \{\varphi E, \tilde{\varphi} E\}^+ \cup \{E\varphi, E\tilde{\varphi}\}^+$ (cf (2), page 3).

  But in this case, from Lemma 2.2, g does not generate any Sturmian infinite word; a contradiction.

- If g is prolongable on c then, for all $n \in \mathbb{N}$, $|g^{n+1}(c)| > |g^n(c)|$ and, since x begins with c, the word $g^n(c)$ is a prefix of x. Consequently $x = g^{\omega}(c)$. ■

This result admits three interesting corollaries.

Corollary 3.3. *Let x be a Sturmian word over $\mathcal{A}$. Are equivalent:*

1. *x is a fixed point of a morphism $g \neq Id_{\mathcal{A}}$*
2. *x is generated by a D0L system $G = (\mathcal{A}, g, c)$ where $g \in St$ is prolongable on the letter $c \in \mathcal{A}$.*
3. *x is generated by a D0L system $G = (\mathcal{A}, g, u)$ where $g \in St$ is prolongable on the word $u \in \mathcal{A}^+$.*

Proof. Equivalence between 1. and 2. is Theorem 3.2, and the implication 2. $\Rightarrow$ 3. is straightforward.

We prove 3. $\Rightarrow$ 2.

Let us suppose that a Sturmian word x is generated by a D0L system $(\mathcal{A}, g, u)$ where $g \in St$ is prolongable on the word $u \in \mathcal{A}^+$, $|u| \geq 2$.
Let c be the first letter of u. Since $f(u)$ begins with u, $f(c)$ begins with c.

- If $f(c) = c$ then, from Lemma 2.2, f does not generate a Sturmian word; a contradiction.
- Otherwise f is prolongable on c and x is generated by the D0L system $(\mathcal{A}, g, c)$, proving 2. ■

Corollary 3.4. *If a Sturmian word x over $\mathcal{A}$ is a fixed point of a morphism $g \neq Id_{\mathcal{A}}$ then $g \in \{\varphi, E\varphi, \varphi E, E\varphi E, \tilde{\varphi}, E\tilde{\varphi}, \tilde{\varphi}E, E\tilde{\varphi}E\}^+ \setminus (\{\varphi E, \tilde{\varphi}E\}^+ \cup \{E\varphi, E\tilde{\varphi}\}^+)$ and there exists $c \in \mathcal{A}$ such that $x = g^{\omega}(c)$.*

Proof.
From Theorem 3.2 $g \in St$ and, since $g \neq Id_{\mathcal{A}}$, one has $g \in \{\varphi, E\varphi, \varphi E, E\varphi E, \tilde{\varphi}, E\tilde{\varphi}, \tilde{\varphi}E, E\tilde{\varphi}E\}^+$ (cf (1), page 3).
Now, from Corollary 3.3, x is generated by g hence the result follows from Lemma 2.2. ■

Corollary 3.5. *If a Sturmian word x over $\mathcal{A}$ is a fixed point of some morphism $g \neq Id_{\mathcal{A}}$ then there exists a morphism $h \in St$ such that g is a power of h (maybe $g = h$), and each morphism of which x is a fixed point is also a power of h. In other words, x is* rigid.

Proof. Since x is generated by g, this is a direct consequence of [17], Theorem 7 which states that all the Sturmian words generated by morphisms are rigid (see also [14] for a complete proof of this theorem). ■

4 Sturmian images by morphisms

In this second part, we consider Sturmian words which are image by a morphism of an infinite fixed point of another morphism. Initially, this study needs to know the infinite words which can be transformed into Sturmian words by a morphism. After providing the main result in a second part, we end by proving that some words given here do not fulfill the conditions of Theorem 3.2.

4.1 Sturmian images by morphisms of infinite words

We start by giving a characterization of infinite words whose images under morphisms are Sturmian. It was originally proved by Parvaix [12]. The proof, given here in order to be self-contained, is based on the following important lemma.

Let $c \in \mathcal{A}$ and $NS = \{x \text{ infinite word over } \mathcal{A} \;/ x = c^k\bar{c}x' \text{ with } c \in \mathcal{A}, k \geq 1, \bar{c}x' \text{ non Sturmian}\}$.

Lemma 4.1. *Let $x \in NS$ and $f \in St$. Then $f(x) \in NS$.*

Proof. Since every Sturmian morphism is a composition of E, φ and $\tilde{\varphi}$ it is enough to prove the result for $f = E, \varphi$ or $\tilde{\varphi}$. The case $f = E$ is straightforward.

If $c = a$ then $\varphi(x) = (ab)^k a\varphi(x')$ and $\tilde{\varphi}(x) = (ba)^k a\tilde{\varphi}(x') = b(ab)^{k-1}a\varphi(x')$ (Lemma 2.3). Moreover, since bx' is not Sturmian then $a\varphi(x') = \varphi(bx')$ is not Sturmian from Lemma 2.4(ii). Thus $(ba)^k\varphi(x')$ and $(ab)^{k-1}a\varphi(x')$ are not Sturmian

If $c = b$ then $\varphi(x) = a^k ab\varphi(x')$ and $\tilde{\varphi}(x) = a^k ba\tilde{\varphi}(x') = a^k b\varphi(x')$ (Lemma 2.3).

But, since ax' is not Sturmian, from Property 2.1, there exists a word v such that ava and bvb are factors of ax'. This implies that $b\varphi(x')$, which contains as factors both $b\varphi(v)ab$ and $a\varphi(v)aa$, is not Sturmian.

In all the cases the words $\varphi(x)$ and $\tilde{\varphi}(x)$ are in NS. ∎

Proposition 4.2. *Let x be an infinite word over $\mathcal{A}$. Are equivalent:*

1. *there exists a morphism f on $\mathcal{A}$ such that $f(x)$ is Sturmian;*
2. *there exist $n \in \mathbb{N}$, a letter $c \in \mathcal{A}$, and a Sturmian word x' such that $x = c^n x'$.*

Proof. Assume that $f(x)$ is Sturmian for a morphism f on $\mathcal{A}$. From Proposition 3.1, $f \in St$.

If x is ultimately periodic then $f(x)$ is ultimately periodic (maybe empty) for any morphism f and consequently $f(x)$ cannot be Sturmian. So x is aperiodic and there exists an infinite word x' and an integer $n \geq 1$ such that $x = c^n\bar{c}x'$ for some $c \in \mathcal{A}$.

If $\bar{c}x'$ is not Sturmian then $x \in NS$ and, from Lemma 4.1, $f(x) \in NS$ thus is not Sturmian; a contradiction. Consequently $\bar{c}x'$ is Sturmian and the first implication is proved.

To prove the converse it is enough to observe that, from Lemma 2.3, for every infinite word x over $\mathcal{A}$, $(\tilde{\varphi}E)^n\tilde{\varphi}(b^{n+1}x) = a^{n+1}(\tilde{\varphi}E)^n\tilde{\varphi}(x) = (\varphi E)^n\varphi(x)$ for every $n \in \mathbb{N}$.

Thus, since from Lemma 2.4(i) and (ii) $(\varphi E)^n\varphi(x)$ is Sturmian whenever x is Sturmian, if $x = b^{n+1}x'$, $n \in \mathbb{N}$, with x' Sturmian then $(\tilde{\varphi}E)^n\tilde{\varphi}(b^{n+1}x')$ is also Sturmian. And so is $(\tilde{\varphi}E)^n\tilde{\varphi}E(a^{n+1}E(x'))$. ∎

Here it is worth noticing that, in the above proposition, n can be as large as needed because if x is any Sturmian word containing a^2 as a factor, the word $b^n x$ is not Sturmian for every $n \geq 2$ but $(\tilde{\varphi}E)^n\tilde{\varphi}(b^{n+1}x)$ is Sturmian because $(\varphi E)^n\varphi \in St$. This also indicates that the two propositions 3.1 and 4.2 are not empty ($i.e.$, apply for words that are not Sturmian) since there exist words x which are not Sturmian but for which $f(x)$ is Sturmian for a morphism f on $\mathcal{A}$. An example of such a word is given in [8], page 84.

4.2 Sturmian images by morphisms of fixed points of morphisms

Now, we synthesize the results of Sections 3 and 4.1 by studying the case of a Sturmian word which is the image of a fixed point of some morphism different from $Id_{\mathcal{A}}$.

Theorem 4.3. *Let y be a Sturmian word over $\mathcal{A}$. If $y = f(x)$ and $x = g(x)$ for some morphisms f, g on $\mathcal{A}$ and a word x over $\mathcal{A}$ then:*

- *$f \in St$ and $g \in St$;*
- *there exist $n \in \mathbb{N}$ and a Sturmian word x' such that $x = a^n x'$ or $x = b^n x'$;*
- *if $g \neq Id_{\mathcal{A}}$ then x is Sturmian and x is generated by g.* ■

Proof. Let x, y be two infinite words over $\mathcal{A}$ and f, g two morphisms on $\mathcal{A}$ such that $y = f(x)$ and $x = g(x)$.

If y is Sturmian then, from Proposition 3.1, $f \in St$.

Moreover, since $x = g(x)$, one has $y = fg(x)$ thus, again from Proposition 3.1, $fg \in St$. Now, since St is left unitary ([4] Corollary 2.3.9), $f \in St$ and $fg \in St$ imply $g \in St$.

This proves the first part of the result.

The second part comes from Proposition 4.2: there exist $n \in \mathbb{N}$, $c \in \mathcal{A}$, and a Sturmian word x' such that $x = c^n x'$.

If $n = 0$ then $x = x'$ is a Sturmian word and the third part is proved thanks to Theorem 3.2.

So, in the following, we will suppose that $n \geq 1$. We also assume, by way of contradiction, that x is not Sturmian (but x is aperiodic because x' is Sturmian, thus Property 2.1 applies).

The morphism g is not prolongable on c. Indeed if g is prolongable on c then, since x begins with c, $g^k(c)$ is a prefix of $x = g^k(x) = g^k(c^n x')$ for every $k \in \mathbb{N}$. Now, if x is not Sturmian then, from Property 2.1, it contains the two factors cwc and $\bar{c}w\bar{c}$ for some $w \in \mathcal{A}^*$. For a sufficiently large k, these two words are factors of the prefix $g^k(c)$ of x, consequently $g^k(c)$ is not balanced. But, since x' is aperiodic it contains infinitely many occurrences of c thus of $g^k(c)$, which implies that $g^k(c)$ is balanced; a contradiction.

Since $g(c)$ begins with c (because $g(x) = g(c^n x') = x = c^n x'$ with $n \geq 1$), the fact that g is not prolongable on c implies that $g(c) = c$. For if not, $|g(c)|_c \leq 1$ implies $g(c) = c\bar{c}^k$ for some $k \in \mathbb{N}$, $k \geq 1$, and g not being prolongable on c means $g(\bar{c}) = \varepsilon$; a contradiction with $g \in St$.

Moreover $g(\bar{c})$ begins with $\bar{c}$. Indeed let $m \in \mathbb{N}$ be such that x starts with $c^m\bar{c}$: if $g(\bar{c})$ begins with c then $g(c^m\bar{c})$ begins with c^{m+1}; a contradiction with $x = g(x)$.

Let $p < n$ be the integer such that $c^p x'$ is Sturmian and $c^{p+1}x'$ is not Sturmian. Then, from Property 2.1, $c^{p+1}x'$ contains the two factors cwc and $\bar{c}w\bar{c}$ for some $w \in \mathcal{A}^*$. Moreover, since $c^p x'$ is Sturmian, the unique occurrence of cwc in x is as a prefix of $c^{p+1}x'$ thus wc is a prefix of $c^p x'$. But, since every Sturmian word is recurrent ([10], Proposition 6.1.2), wc appears as a factor of x' in another position, which means that $\bar{c}wc$ is also a factor of x.

Now, since $g \neq Id_{\mathcal{A}}$, $g \in St$ implies that one of the two words $g(c)$ or $g(\bar{c})$ is a proper prefix or a proper suffix of the other one ([17], Lemma 3). But $g(c) = c$ and $g(\bar{c})$ begins with $\bar{c}$, thus $g(c)$ is a proper suffix of $g(\bar{c})$ which means that $g(\bar{c})$ ends with c.

This implies that $g(\bar{c}wc)$ ends with $cg(wc)$. But $c^{p+1}x'$ begins with cwc thus $x = c^{n-p-1}c^{p+1}x'$ starts with $c^{n-p-1}awa$ and $g(x) = g(c^{n-p-1}c^{p+1}x') = c^{n-p-1}c^{p+1}g(x') = c^{n-p-1}cg(c^p x')$ starts with $c^{n-p-1}cg(wc)$.

Thus, since $x = g(x)$, one has that $g(wc)$ begins with wc which implies that $g(\bar{c}wc)$ contains cwc. This is, in x, an occurrence of cwc in a different position than the prefix of $c^{p+1}x'$; a contradiction.

Consequently, if $g \neq Id_{\mathcal{A}}$ then x is Sturmian and it is generated by g from Theorem 3.2. ■

Another formulation of this result, joined with Corollary 3.3, is the following.

Corollary 4.4. *Let y be a Sturmian word over $\mathcal{A}$. Are equivalent:*

1. *y is the image by a morphism f on $\mathcal{A}$ of a fixed point of a morphism g on $\mathcal{A}$, $g \neq Id_{\mathcal{A}}$*

2. *y is generated by an HD0L system $H = (\mathcal{A}, g, f, c)$ where the morphisms f and g are Sturmian, $g^\omega(c)$ is the Sturmian word generated by the D0L system $(\mathcal{A}, g, c)$ where g is prolongable on the letter $c \in \mathcal{A}$, and $y = f(g^\omega(c))$.*

3. *y is generated by an HD0L system $H = (\mathcal{A}, g, f, u)$ where the morphisms f and g are Sturmian, $g^\omega(u)$ is the Sturmian word generated by the D0L system $(\mathcal{A}, g, u)$ where g is prolongable on the word $u \in \mathcal{A}^+$, and $y = f(g^\omega(u))$.* ■

4.3 An example

We end this section with an example showing that the two families described in Theorems 3.2 and 4.3 are not equal, the first one being strictly included in the second.

Proposition 4.5. *Let F be the Fibonacci word. The word $\tilde{\varphi}(F)$ fulfills the conditions of Theorem 4.3 but not those of Theorem 3.2 (with $g \neq Id_{\mathcal{A}}$ in both cases).*

Before proving this result, we recall from Séébold, 1985 [16] some properties of the elements of the monoid $\{\varphi, \tilde{\varphi}\}^*$.

We recall that $|h|_{\varphi,\tilde{\varphi}}$ is the number of elements of $\{\varphi, \tilde{\varphi}\}$ implied in any decomposition of a morphism h on $\mathcal{A}$ (from *2.* of Proposition 4.6 below, a given h may have several decompositions but all of them contain the same number of elements of $\{\varphi, \tilde{\varphi}\}$).

Property 4.6 ([16]). *Let h be a morphism on $\mathcal{A}$.*

1. *If h generates an infinite word having the same set of factors than F then $h \in \{\varphi, \tilde{\varphi}\}^+$.*

2. *The only non trivial relations between φ and $\tilde{\varphi}$ are $\varphi^2\tilde{\varphi} = \tilde{\varphi}^2\varphi$ and its derivatives.*

3. *If h_1, h_2 are two morphisms in $\{\varphi, \tilde{\varphi}\}^*$ such that $|h_1|_{\varphi,\tilde{\varphi}} = |h_2|_{\varphi,\tilde{\varphi}}$ then $|h_1(a)| = |h_2(a)|$ and $|h_1(b)| = |h_2(b)|$.* ■

Proof of Proposition 4.5. The Sturmian word F is defined as $F = \varphi^\omega(a)$ thus F is generated by the D0L system $(\mathcal{A}, \varphi, a)$ and, consequently, $\tilde{\varphi}(F) = \tilde{\varphi}(\varphi^\omega(a))$ is generated by the HD0L system $(\mathcal{A}, \varphi, \tilde{\varphi}, a)$.

Now, we prove that $\tilde{\varphi}(F)$ is not generated by a morphism.

Assume, contrary to what we want to prove, that there exists a morphism f on $\mathcal{A}$ such that $\tilde{\varphi}(F) = f^\omega(b)$ (the starting letter of $\tilde{\varphi}(F)$ is b).

From Lemma 2.3, $a\tilde{\varphi}(F) = \varphi(F) = F$ thus $\tilde{\varphi}(F)$ and F have the same set of factors, which implies $f \in \{\varphi, \tilde{\varphi}\}^+$ from *1.* of Property 4.6.

Let $p = |f|_{\varphi,\tilde{\varphi}}$. Of course, since $\tilde{\varphi}(F) = f^\omega(b)$, $f \neq Id_{\mathcal{A}}$ thus $p \neq 0$.

Since $\varphi^p(F) = F$, one has $\tilde{\varphi}\varphi^p(F) = \tilde{\varphi}(F)$. On the other hand, since $\tilde{\varphi}(F)$ is generated by f, $f\tilde{\varphi}(F) = \tilde{\varphi}(F)$.

Consequently $\tilde{\varphi}\varphi^p(F) = f\tilde{\varphi}(F)$ and, since $|\tilde{\varphi}\varphi^p|_{\varphi,\tilde{\varphi}} = |f\tilde{\varphi}|_{\varphi,\tilde{\varphi}}$, one has $\tilde{\varphi}\varphi^p = f\tilde{\varphi}$ from *3.* of Property 4.6; a contradiction with *2.* of Property 4.6 because $p \neq 0$. ■

5 Morphisms sending non Sturmian words on Sturmian words

From Lemma 4.1 and Proposition 4.2, we deduce that the whole family of binary infinite words can be partitioned in three sets:

- the set of Sturmian words;
- the set of non Sturmian words which can be transformed into Sturmian words by applying a morphism, *i.e.*, the words $c^n x$ with $c \in \mathcal{A}$, $n \in \mathbb{N} \setminus \{0\}$, x Sturmian and cx non Sturmian;
- the set NS of non Sturmian words which cannot be transformed into Sturmian words by applying a morphism.

In the following, we are interested in the second of these sets. The aim is to give a characterization of the morphisms which change non Sturmian words into Sturmian words, *i.e.*, those morphisms involved in Proposition 4.2. This characterization needs an intermediate construction.

Let $c \in \mathcal{A}$. For every $n \in \mathbb{N}$ we denote S_{c^n} the set of words

$$S_{c^n} = \{c^n x, x \text{ Sturmian and } cx \text{ non Sturmian}\}.$$

Of course, $S_{a^0} \subset S$ and $S_{b^0} \subset S$ where S is the set of Sturmian words, and for every $n \in \mathbb{N}$, if $y \in S_{c^n}$ then y is aperiodic.

The sets S_{c^n}, $n \in \mathbb{N}$, have the following properties.

Lemma 5.1.

1. *For every $f \in St$, $f(S) \subset S$ and $f(NS) \subset NS$.*
2. *For every $n \in \mathbb{N}$ and $c \in \mathcal{A}$, $E(S_{c^n}) = S_{\bar{c}^n}$.*
3. *For every $n \in \mathbb{N} \setminus \{0\}$, $\varphi(S_{b^n}) \subset S_{a^n}$ and $\tilde{\varphi}(S_{b^n}) \subset S_{a^{n-1}}$.*
4. *For every $n \in \mathbb{N} \setminus \{0\}$, $\varphi(S_{a^n}) \subset NS$.*
5. *For every $n \in \mathbb{N} \setminus \{0, 1\}$, $\tilde{\varphi}(S_{a^n}) \subset NS$.*
6. *$\tilde{\varphi}(S_a) \subset S_b$.*

Proof. The first part of relation 1. is the definition of St and the second part is Lemma 4.1.

2. This second relation is straightforward. We prove the others.
3. Let $n \in \mathbb{N}$, $n \geq 1$, and $y \in S_{b^n}$: $y = b^n x$ with x Sturmian and bx non Sturmian.

   Since x is Sturmian, so is $\varphi(x)$. Moreover, since bx is not Sturmian, $a\varphi(x) = \varphi(bx)$ is not Sturmian (Lemma 2.4(ii)). From this we deduce

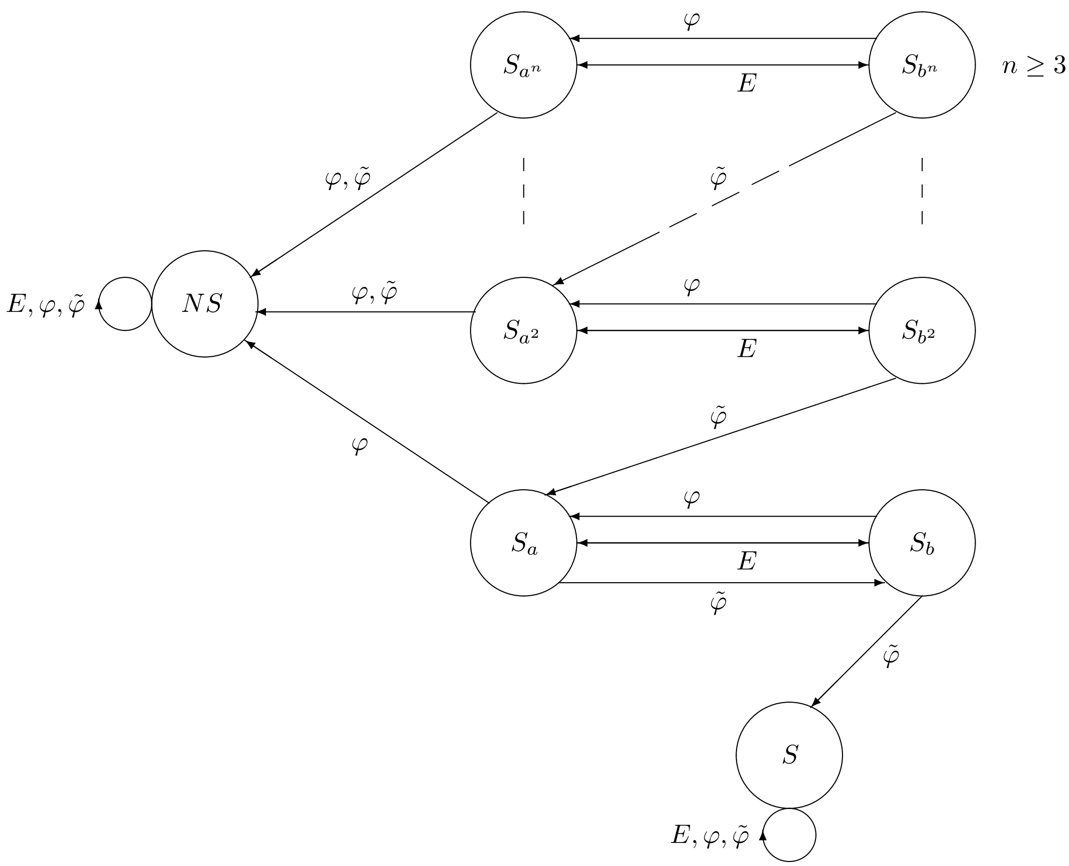
   - $\varphi(y) = \varphi(b^n x) = a^n \varphi(x) \in S_{a^n}$;
   - $\tilde{\varphi}(y) = \tilde{\varphi}(b^n x) = a\tilde{\varphi}(b^{n-1}x) =$ (by Lemma 2.3) $\varphi(b^{n-1}x) = a^{n-1}\varphi(x) \in S_{a^{n-1}}$.
4. Let $n \in \mathbb{N}$, $n \geq 1$, and $y \in S_{a^n}$: $y = a^n x$ with x Sturmian and ax non Sturmian.

   From Lemma 2.3, $b\varphi(a^{n-1}x) = ba\tilde{\varphi}(a^{n-1}x) = \tilde{\varphi}(a^n x)$ which is not Sturmian from Lemma 2.4(iii). Hence $\varphi(y) = \varphi(a^n x) = ab\varphi(a^{n-1}x) \in NS$ from Lemma 4.1.
5. Let $n \in \mathbb{N}$, $n \geq 2$, and $y \in S_{a^n}$: $y = aa^{n-1}x$ with x Sturmian and ax non Sturmian.

   Since $n-1 \geq 1$, ax non Sturmian implies $a^{n-1}x$ non Sturmian thus $a\tilde{\varphi}(a^{n-1}x) = \tilde{\varphi}(ba^{n-1}x)$ is not Sturmian from Lemma 2.4(iv). Hence $\tilde{\varphi}(y) = \tilde{\varphi}(aa^{n-1}x) = ba\tilde{\varphi}(a^{n-1}x) \in NS$ from Lemma 4.1.
6. Let $y \in S_a$: $y = ax$ with x Sturmian and $y = ax$ non Sturmian.

   From Lemma 2.4(iii), $\tilde{\varphi}(y)$ is not Sturmian. But $\tilde{\varphi}(y) = \tilde{\varphi}(ax) = ba\tilde{\varphi}(x) = b\varphi(x)$ (Lemma 2.3) and, since x is Sturmian, $\varphi(x)$ is Sturmian from Lemma 2.4(ii), thus $\tilde{\varphi}(y) \in S_b$. ■

These results are summarized in the following automaton



From Lemma 5.1, we deduce that if f is a morphism on $\mathcal{A}$ and if an infinite word y on $\mathcal{A}$ is such that there exist $c \in \mathcal{A}$ and $n \in \mathbb{N}$ with $y = c^n x$, x Sturmian and cx non Sturmian, then $f(y)$ is Sturmian if and only if $f = f'g$ where f' is any Sturmian morphism and g is the mirror image of the label, in the previous automaton, of a path starting at the state S_{c^n} and ending at the state S. Note that the if part can more or less be deduced from Chapter 5 of Parvaix, 1998 [12].

This (and Proposition 4.2) leads to the following characterization of morphisms which transform non Sturmian words into Sturmian words.

Theorem 5.2. *Let x be a Sturmian word and let f be a morphism on $\mathcal{A}$.*

- *$f(x)$ is Sturmian if and only if $f \in St$;*
- *if bx is not Sturmian, $f(bx)$ is Sturmian if and only if $f = f'\tilde{\varphi}$, $f' \in St$;*
- *if ax is not Sturmian, $f(ax)$ is Sturmian if and only if $f = f'g$, $f' \in St$ and $g \in \tilde{\varphi}\{E, \tilde{\varphi}\}$;*
- *if bx is not Sturmian, $f(b^n x)$ is Sturmian for some integer $n \geq 2$ if and only if $f = f'g$, $f' \in St$ and $g \in \tilde{\varphi}\{E, \tilde{\varphi}\}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^* \tilde{\varphi} \left[(E\varphi)^* E\tilde{\varphi}\right]^{n-2} (E\varphi)^*$;*
- *if ax is not Sturmian, $f(a^n x)$ is Sturmian for some integer $n \geq 2$ if and only if $f = f'g$, $f' \in St$ and $g \in \tilde{\varphi}\{E, \tilde{\varphi}\}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^* \tilde{\varphi} \left[(E\varphi)^* E\tilde{\varphi}\right]^{n-2} (E\varphi)^* E$.* ■

6 A characterization with standard morphisms

In Séébold, 1998 [17] it is proved that each Sturmian morphism is a conjugate of a unique standard morphism obtained by replacing, in any writing of the morphism, all the occurrences of $\tilde{\varphi}$ by φ. This construction is important because many properties of the morphism can be deduced from that of its standard representative. In this section we produce a new interesting property of the standard representatives of the morphisms described in Theorem 5.2.

More precisely, we will prove that the morphisms g of Theorem 5.2 are conjugates of standard morphisms and we will indicate how to directly obtain such a morphism from its standard representative.

This requires more information regarding Sturmian morphisms.

6.1 Standard morphisms and conjugacy

Standard morphisms are those Sturmian morphisms obtained by composition from E and φ only. They are representatives of the whole Sturmian morphisms thanks to the following conjugacy operation (see [8] and [17]).

Let γ be the *permutation* over $\mathcal{A}^+$ defined by $\gamma(cu) = uc$, $c \in \mathcal{A}$, $u \in \mathcal{A}^*$. Let f be a standard morphism and let $||f|| = |f(a)| + |f(b)|$. For $0 \leq i \leq ||f|| - 2$, the *i-th conjugate* of f is the morphism f_i defined by $f_i(ab) = \gamma^i(f(ab))$ and $|f_i(a)| = |f(a)|$. Of course, $f_0 = f$. We call $f_{||f||-2}$ the *last conjugate* of f.

In the following proposition we put together several results from [17].

Proposition 6.1. *Let f be a standard morphism.*

- *The $||f|| - 1$ conjugates of f are Sturmian and are pairwise different.*
- *For $0 \leq i \leq ||f|| - 3$, $f_i(a)$ and $f_i(b)$ start with the same letter.*
- *$f_{||f||-2}(a)$ and $f_{||f||-2}(b)$ start with different letters.*
- *For $0 \leq i, j \leq ||f|| - 2$, $|f_i|_{\varphi,\tilde{\varphi}} = |f_j|_{\varphi,\tilde{\varphi}}$.*

Let f be a Sturmian morphism. There exists a unique standard morphism g of which f is a conjugate and

- *$f = f_1 f_2 \cdots f_n$, $f_i \in \{E, \varphi, \tilde{\varphi}\}$, $1 \leq i \leq n$;*
- *$g = g_1 g_2 \cdots g_n$, $g_i \in \{E, \varphi\}$, $1 \leq i \leq n$;*
- *if $f_i = E$ or $f_i = \varphi$ then $g_i = f_i$; if $f_i = \tilde{\varphi}$ then $g_i = \varphi$.* ■

Let $R : St \to St$, the transformation defined by $R(E) = E$, $R(\tilde{\varphi}) = R(\varphi) = \varphi$. For every $f \in St$, $R(f)$ is called the *standard representative* of f: $R(f)$ is the standard morphism of which f is a conjugate.

Since $\tilde{\varphi} = \tilde{\varphi}EE$, it is straightforward that, with each writing over $\{E, \varphi, \tilde{\varphi}\}$ of any Sturmian morphism f, can be associated a writing of f over $\{E, \varphi, \tilde{\varphi}E\}$: $St = \{E, \varphi, \tilde{\varphi}\}^* = \{E, \varphi, \tilde{\varphi}E\}^*$. In Richomme, 2003 [13] it is proved that the number, as a conjugate of its standard representative, of a Sturmian morphism f is obtained by searching the occurrences of $\tilde{\varphi}E$ in any writing of f over $\{E, \varphi, \tilde{\varphi}E\}$. We give here a binary version of this result (see [7]).

Let us call $NC(f)$ the *conjugate number* of a Sturmian morphism f: $NC(f)$ is such that f is the $NC(f)$-th conjugate of $R(f)$.

Lemma 6.2 ([13], Proposition 6.2)**.** *Let f be a Sturmian morphism and $n \in \mathbb{N}$ such that $f = f_0 \dots f_n$ with $f_i \in \{E, \varphi, \tilde{\varphi}E\}$, $0 \leq i \leq n$. Then $NC(f) = \sum_{0 \leq i \leq n | f_i = \tilde{\varphi}E} |f_0 \dots f_{i-1}(a)|$ (with $|f_0 \dots f_{i-1}(a)| = 1$ if $i = 0$).* ■

This means that if $f = g_{NC(f)}$ where g is a standard morphism, then $NC(f)$ is obtained from any decomposition of f over $\{E, \varphi, \tilde{\varphi}E\}$, $f = f_0 \dots f_n$, by summing the lengths of the words $f_0 \dots f_{i-1}(a)$ for each i, $0 \le i \le n$, such that $f_i = \tilde{\varphi}E$.

Example. Let $f = \tilde{\varphi}\varphi E\tilde{\varphi}\tilde{\varphi} = \tilde{\varphi}EE\varphi E\tilde{\varphi}EE\tilde{\varphi}EE$. Then, $NC(f) = 1+|\tilde{\varphi}EE\varphi E(a)|+|\tilde{\varphi}EE\varphi E\tilde{\varphi}EE(a)| = 1+2+5 = 8$. Indeed, $f = g_8$ with $g = \varphi\varphi E\varphi\varphi$.

Let f, f' be morphisms on $\mathcal{A}$ such that $f = f'cd(E\tilde{\varphi})^t\tilde{\varphi}(E\tilde{\varphi})^s e$ for some $c \in \{\varphi, \tilde{\varphi}\}$, $d \in \{E, \tilde{\varphi}\}$, $e \in \{Id_{\mathcal{A}}, E\}$, and $t, s \in \mathbb{N}$. We call *seed* of f the morphism $s_f = \tilde{\varphi}d(E\tilde{\varphi})^t\tilde{\varphi}(E\tilde{\varphi})^s e$.

It must be noticed that the value of the seed depends only on the letter d, *i.e.*, the last letter of $f'cd$. Moreover, from Theorem 5.2, if $t = 0$ and $s = n-2$ then s_f is the shortest morphism such that $s_f(b^n x)$ (or $s_f(a^n x)$) is Sturmian for a Sturmian word x with bx (or ax) non Sturmian.

We now prove that if a morphism f is as above, its seed is the last conjugate of $R(f)$ and we can give the value of $NC(f)$.

Proposition 6.3. *Let $f = \tilde{\varphi}d(E\tilde{\varphi})^t\tilde{\varphi}(E\tilde{\varphi})^s e$ for some $d \in \{E, \tilde{\varphi}\}$, $e \in \{Id_{\mathcal{A}}, E\}$, and $t, s \in \mathbb{N}$. Then f is the last conjugate of $R(f)$ and $NC(f) = (2t + 1 + 2|d|_{\tilde{\varphi}})(s + 2)$.*

Proof. Let $n, p \in \mathbb{N}$. Since $\tilde{\varphi}EE = \tilde{\varphi}$, and $\tilde{\varphi}E(a) = a$, $\tilde{\varphi}E(b) = ba$, one has:

1. $(\tilde{\varphi}E)^n(a) = a$, $(\tilde{\varphi}E)^n(b) = ba^n$: $|(\tilde{\varphi}E)^n(a)| = 1$
2. $\tilde{\varphi}EE(a) = ba$, $\tilde{\varphi}EE(b) = a$: $|\tilde{\varphi}EE(a)| = 2$
3. $\tilde{\varphi}EE(\tilde{\varphi}E)^n(a) = ba$, $\tilde{\varphi}EE(\tilde{\varphi}E)^n(b) = a(ba)^n$: $|\tilde{\varphi}EE(\tilde{\varphi}E)^n(a)| = 2$
4. $\tilde{\varphi}EE(\tilde{\varphi}E)^n\tilde{\varphi}EE(a) = a(ba)^{n+1}$, $\tilde{\varphi}EE(\tilde{\varphi}E)^n\tilde{\varphi}EE(b) = ba$: $|\tilde{\varphi}EE(\tilde{\varphi}E)^n\tilde{\varphi}EE(a)| = 2n+3$
5. $\tilde{\varphi}EE(\tilde{\varphi}E)^n\tilde{\varphi}EE(\tilde{\varphi}E)^p(a) = a(ba)^{n+1}$, $\tilde{\varphi}EE(\tilde{\varphi}E)^n\tilde{\varphi}EE(\tilde{\varphi}E)^p(b) = ba(a(ba)^{n+1})^p$: $|\tilde{\varphi}EE(\tilde{\varphi}E)^n\tilde{\varphi}EE(\tilde{\varphi}E)^p(a)| = 2n+3$.

Now, let $f = \tilde{\varphi}d(E\tilde{\varphi})^t\tilde{\varphi}(E\tilde{\varphi})^s e$ for some $d \in \{E, \tilde{\varphi}\}$, $e \in \{Id_{\mathcal{A}}, E\}$, and $t, s \in \mathbb{N}$.

- If $d = E$ and $t = 0$, then $f = (\tilde{\varphi}E)^{s+1}\tilde{\varphi}e$. From 1. above, $|\tilde{\varphi}E(a)| = \dots = |(\tilde{\varphi}E)^{s+1}(a)| = 1$ thus $NC(f) = 1 + (s+1) = s+2$.
- If $d = \tilde{\varphi}$ and $t = 0$, then $f = \tilde{\varphi}EE\tilde{\varphi}EE(\tilde{\varphi}E)^s\tilde{\varphi}e$. From 2. and 5. above, $|\tilde{\varphi}EE(a)| = 2$ and $|\tilde{\varphi}EE\tilde{\varphi}EE(a)| = |\tilde{\varphi}EE\tilde{\varphi}EE(\tilde{\varphi}E)(a)| = \dots = |\tilde{\varphi}EE\tilde{\varphi}EE(\tilde{\varphi}E)^s(a)| = 3$.
  Consequently $NC(f) = 1 + 2 + 3(s+1) = 3(s+2)$.
- If $d = E$ and $t \ge 1$, then $f = \tilde{\varphi}EE(\tilde{\varphi}E)^{t-1}\tilde{\varphi}EE(\tilde{\varphi}E)^s\tilde{\varphi}e$. From 2. to 5. above, $NC(f) = 1 + 2 + 2(t-1) + 2(t-1) + 3 + s(2(t-1)+3) = (2t+1)(s+2)$.
- If $d = \tilde{\varphi}$ and $t \ge 1$, then $f = \tilde{\varphi}EE(\tilde{\varphi}E)^t\tilde{\varphi}EE(\tilde{\varphi}E)^s\tilde{\varphi}e$. From 2. to 5. above, $NC(f) = 1 + 2 + 2t + 2t + 3 + s(2t+3) = (2t+3)(s+2)$.

In the four cases, $NC(f) = (2t + 1 + 2|d|_{\tilde{\varphi}})(s+2)$ and, from 1. to 5. above, $f(a)$ and $f(b)$ start with different letters which implies, from Proposition 6.1, that f is the last conjugate of $R(f)$. ∎

6.2 More about Sturmian morphisms

It is known (see, *e.g.*, [8]) that the set St of Sturmian morphisms has the following presentation

$$E^2 = Id_{\mathcal{A}} \tag{3}$$

$$\varphi(\varphi E)^k E\tilde{\varphi} = \tilde{\varphi}(\tilde{\varphi}E)^k E\varphi. \tag{4}$$

In particular, when $k = 0$ and $k = 1$, relation (4) gives

$$\varphi E\tilde{\varphi} = \tilde{\varphi}E\varphi \text{ and } \varphi\varphi\tilde{\varphi} = \tilde{\varphi}\tilde{\varphi}\varphi. \tag{5}$$

Relations (3) and (4) indicate that a given Sturmian morphism may have several different writings. However it is important for what follows to notice that, from Proposition 6.1, if $g_1g_2\cdots g_n$ and $g'_1g'_2\cdots g'_n$ are two writings of a given Sturmian morphism g (with $g_i, g'_i \in \{E, \varphi, \tilde{\varphi}\}$, $1 \le i \le n$) then $|g_1g_2\cdots g_n|_{\varphi,\tilde{\varphi}} = |g'_1g'_2\cdots g'_n|_{\varphi,\tilde{\varphi}}$.

The following lemma gives a set of relations induced by the presentation of St.

Lemma 6.4. *Let $u \in \tilde{\varphi}\{E, \tilde{\varphi}\}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^*$. Then $u\varphi = R(u)\tilde{\varphi}$.*

Proof. Let $n \in \mathbb{N}$ be such that $u \in \tilde{\varphi}\{E, \tilde{\varphi}\}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^n$. The proof is by induction on n.

By (5), $\tilde{\varphi}E\varphi = \varphi E\tilde{\varphi}$ and $\tilde{\varphi}\tilde{\varphi}\varphi = \varphi\varphi\tilde{\varphi}$ thus the property is true if $n = 0$.

Now we suppose that, for every $n \ge 0$, the property is true for every non negative integer less than n, and we prove that it is also true for $n+1$.

Let $u' \in \{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^n$. By induction, $\tilde{\varphi}Eu'\varphi = \varphi ER(u')\tilde{\varphi}$ and $\tilde{\varphi}\tilde{\varphi}u'\varphi = \varphi\varphi R(u')\tilde{\varphi}$.

$$\begin{array}{rcl} \text{Moreover, } \{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^{n+1} & = & \{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^n \\ & = & E\tilde{\varphi}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^n \cup \varphi E\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^n \cup \varphi\tilde{\varphi}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^n. \end{array}$$

- $$\begin{array}{rcl} \text{If } u = \tilde{\varphi}EE\tilde{\varphi}u' \text{ then } u\varphi & = & \tilde{\varphi}EE\tilde{\varphi}u'\varphi \\ & = & \tilde{\varphi}\tilde{\varphi}u'\varphi \\ & = & \varphi\varphi R(u')\tilde{\varphi} \text{ (by induction)} \\ & = & \varphi EE\varphi R(u')\tilde{\varphi}. \end{array}$$
- $$\begin{array}{rcl} \text{If } u = \tilde{\varphi}E\varphi Eu' \text{ then } u\varphi & = & \tilde{\varphi}E\varphi Eu'\varphi \\ & = & \varphi E\tilde{\varphi}Eu'\varphi \\ & = & \varphi E\varphi ER(u')\tilde{\varphi} \text{ (by induction).} \end{array}$$
- $$\begin{array}{rcl} \text{If } u = \tilde{\varphi}E\varphi\tilde{\varphi}u' \text{ then } u\varphi & = & \tilde{\varphi}E\varphi\tilde{\varphi}u'\varphi \\ & = & \varphi E\tilde{\varphi}\tilde{\varphi}u'\varphi \\ & = & \varphi E\varphi\varphi R(u')\tilde{\varphi} \text{ (by induction).} \end{array}$$
- $$\begin{array}{rcl} \text{If } u = \tilde{\varphi}\tilde{\varphi}\varphi Eu' \text{ then } u\varphi & = & \tilde{\varphi}\tilde{\varphi}\varphi Eu'\varphi \\ & = & \varphi\varphi\tilde{\varphi}Eu'\varphi \\ & = & \varphi\varphi\varphi ER(u')\tilde{\varphi} \text{ (by induction).} \end{array}$$
- $$\begin{array}{rcl} \text{If } u = \tilde{\varphi}\tilde{\varphi}\varphi\tilde{\varphi}u' \text{ then } u\varphi & = & \tilde{\varphi}\tilde{\varphi}\varphi\tilde{\varphi}u'\varphi \\ & = & \varphi\varphi\tilde{\varphi}\tilde{\varphi}u'\varphi \\ & = & \varphi\varphi\varphi\varphi R(u')\tilde{\varphi} \text{ (by induction).} \end{array}$$
- If $u = \tilde{\varphi}\tilde{\varphi}E\tilde{\varphi}u'$ and u' does not contain any occurrence of φE or $\varphi\tilde{\varphi}$ then $u = \tilde{\varphi}\tilde{\varphi}E\tilde{\varphi}(E\tilde{\varphi})^n$ (this includes the case $n = 0$).
  $$\begin{array}{rcl} \text{Thus } u\varphi & = & \tilde{\varphi}\tilde{\varphi}E\tilde{\varphi}(E\tilde{\varphi})^n\varphi \\ & = & \tilde{\varphi}(\tilde{\varphi}E)^{n+2}E\varphi \\ & = & \varphi(\varphi E)^{n+2}E\tilde{\varphi} \text{ (by (4))} \\ & = & \varphi\varphi E\varphi(E\varphi)^n\tilde{\varphi}. \end{array}$$
- Otherwise $u = \tilde{\varphi}\tilde{\varphi}E\tilde{\varphi}u'$ and u' contains at least one occurrence of φE or $\varphi\tilde{\varphi}$ (this means in particular that $n \ge 1$). There are two subcases
  $\star$ If $u' = (E\tilde{\varphi})^p\varphi Eu''$, with $u'' \in \{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^{n-p-1}$ for some $p \in \mathbb{N}$ then $u = \tilde{\varphi}\tilde{\varphi}(E\tilde{\varphi})^{p+1}\varphi Eu''$.
  $$\begin{array}{rcl} \text{Thus } u\varphi & = & \tilde{\varphi}\tilde{\varphi}(E\tilde{\varphi})^{p+1}\varphi Eu''\varphi \\ & = & \tilde{\varphi}(\tilde{\varphi}E)^{p+2}E\varphi Eu''\varphi \\ & = & \varphi(\varphi E)^{p+2}E\tilde{\varphi}Eu''\varphi \text{ (by (4))} \\ & = & \varphi\varphi(E\varphi)^{p+1}\tilde{\varphi}Eu''\varphi \\ & = & \varphi\varphi(E\varphi)^{p+1}\varphi ER(u'')\tilde{\varphi} \text{ (by induction since } n-p-1 < n). \end{array}$$
  $\star$ If $u' = (E\tilde{\varphi})^p\varphi\tilde{\varphi}u''$, with $u'' \in \{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^{n-p-1}$ for some $p \in \mathbb{N}$ then $u = \tilde{\varphi}\tilde{\varphi}(E\tilde{\varphi})^{p+1}\varphi\tilde{\varphi}u''$.
  $$\begin{array}{rcl} \text{Thus } u\varphi & = & \tilde{\varphi}\tilde{\varphi}(E\tilde{\varphi})^{p+1}\varphi\tilde{\varphi}u''\varphi \\ & = & \tilde{\varphi}(\tilde{\varphi}E)^{p+2}E\varphi\tilde{\varphi}u''\varphi \\ & = & \varphi(\varphi E)^{p+2}E\tilde{\varphi}\tilde{\varphi}u''\varphi \text{ (by (4))} \\ & = & \varphi\varphi(E\varphi)^{p+1}\tilde{\varphi}\tilde{\varphi}u''\varphi \\ & = & \varphi\varphi(E\varphi)^{p+1}\varphi\varphi R(u'')\tilde{\varphi} \text{ (by induction since } n-p-1 < n). \end{array}$$

In all the cases, $u\varphi = R(u)\tilde{\varphi}$. ■

6.3 Two characterizations

As seen above, standard morphisms are canonical representatives of Sturmian morphisms. We will show that this canonical representation extends to the computation of the words $g(c^n x)$, $c \in \mathcal{A}$, described in Theorem 5.2 since, for each g, the application of its standard representative to the word x gives the same word as $g(c^n x)$.

In order to prove this, we need the following lemma which is a trivial generalization of Lemma 2.3.

Lemma 6.5. *Let x be an infinite word over $\mathcal{A}$.*

1. *For any $n \in \mathbb{N}$, $E\tilde{\varphi}(b^{n+1}x) = b^n E\varphi(x)$.*
2. *For any $n \in \mathbb{N}$, $E\varphi(b^n x) = b^n E\varphi(x)$.*
3. *$\tilde{\varphi}(b^2 x) = a\varphi(x)$; $\varphi E(ax) = a\varphi E(x)$; $\varphi\tilde{\varphi}(ax) = a\varphi^2(x)$; $E\tilde{\varphi}(ax) = aE\varphi(x)$.*
4. *$\tilde{\varphi}E(ax) = \varphi E(x)$; $\tilde{\varphi}^2(ax) = \varphi^2(x)$.* ■

Theorem 6.6. *Let x be an infinite word over $\mathcal{A}$ and $n \in \mathbb{N}$, $n \geq 2$.*
Let $g_1 \in \tilde{\varphi}\{E, \tilde{\varphi}\}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^*\tilde{\varphi}\left[(E\varphi)^* E\tilde{\varphi}\right]^{n-2}(E\varphi)^*$
and $g_2 \in \tilde{\varphi}\{E, \tilde{\varphi}\}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^*\tilde{\varphi}\left[(E\varphi)^* E\tilde{\varphi}\right]^{n-2}(E\varphi)^* E$.
Then $g_1(b^n x) = [R(g_1)](x)$ and $g_2(a^n x) = [R(g_2)](x)$.

We remark that this provides an alternative proof of the if parts of Theorem 5.2. Indeed if $f \in St$ and x is a Sturmian word, then $R(f) \in \{E, \varphi\}^*$ thus $\big[R(f)\big](x)$ is Sturmian by Lemma 2.4(i) and (ii).

Proof. Let x be an infinite word over $\mathcal{A}$.

- Let $n \in \mathbb{N}$, $n \geq 2$, and let $f_1 \in \left[(E\varphi)^* E\tilde{\varphi}\right]^{n-2}(E\varphi)^*$: there exist non-negative integers $p_1, p_2, \cdots p_{n-2}, p_{n-1}$ such that $f_1 = (E\varphi)^{p_1}E\tilde{\varphi}(E\varphi)^{p_2}E\tilde{\varphi}\cdots(E\varphi)^{p_{n-2}}E\tilde{\varphi}(E\varphi)^{p_{n-1}}$ and, from 1. and 2. of Lemma 6.5, $f_1(b^n x) = b^2(E\varphi)^r(x)$ with $r = n - 2 + p_1 + p_2 + \cdots + p_{n-2} + p_{n-1}$.

- Let $f_2 \in \{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^*$: there exist non-negative integers $k, p_1, q_1, r_1, p_2, q_2, r_2, \cdots p_k, q_k, r_k$ such that $f_2 = (E\tilde{\varphi})^{p_1}(\varphi E)^{q_1}(\varphi\tilde{\varphi})^{r_1}(E\tilde{\varphi})^{p_2}(\varphi E)^{q_2}(\varphi\tilde{\varphi})^{r_2}\cdots(E\tilde{\varphi})^{p_k}(\varphi E)^{q_k}(\varphi\tilde{\varphi})^{r_k}$. From 3. of Lemma 6.5, $f_2\tilde{\varphi}(b^2 x) = f_2\left[a\varphi(x)\right] = a\big[(E\varphi)^{p_1}(\varphi E)^{q_1}(\varphi^2)^{r_1}(E\varphi)^{p_2}(\varphi E)^{q_2}(\varphi^2)^{r_2}\cdots(E\varphi)^{p_k}(\varphi E)^{q_k}(\varphi^2)^{r_k}\varphi\big](x)$.

Applying 4. of Lemma 6.5, we calculate that if $g \in \tilde{\varphi}\{E, \tilde{\varphi}\}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^*\tilde{\varphi}\left[(E\varphi)^* E\tilde{\varphi}\right]^{n-2}(E\varphi)^*$ then $g(b^n x) = [R(g)](x)$ and $gE(a^n x) = g(b^n E(x)) = [R(g)](E(x)) = [R(gE)](x)$. ■

To end, we prove that there is a canonical writing (called normalized writing) of morphisms of Theorem 6.6 which can be obtained directly from the standard representatives by their conjugate number.

Let us call *normalizable* a Sturmian morphism f for which there exist $f_1 \in \{E, \varphi\}^*$, $f_2 \in \{E, \tilde{\varphi}\}^*$ such that $f = f_1 f_2$. We call this decomposition the *normalized writing* of f. Of course, there exist Sturmian morphisms which are not normalizable, as $\tilde{\varphi}\varphi$ or $\varphi\tilde{\varphi}\varphi E\tilde{\varphi}$ for example. From Proposition 6.1 the normalized writing of a normalizable morphism is unique, up to occurrences of E^2. In the following, we denote $N(f)$ the unique normalized writing without E^2 of a normalizable morphism f.

In Theorem 6.7 below we prove that the morphisms g of Theorem 5.2 are all normalizable and we show how to obtain, from its standard representative, the normalized writing of such a morphism.

Notation. In what follows, we will consider words of the set $\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^*$. Such a word should be written $(E\tilde{\varphi})^{p_1}(\varphi E)^{q_1}(\varphi\tilde{\varphi})^{r_1}(E\tilde{\varphi})^{p_2}(\varphi E)^{q_2}(\varphi\tilde{\varphi})^{r_2}\cdots(E\tilde{\varphi})^{p_k}(\varphi E)^{q_k}(\varphi\tilde{\varphi})^{r_k}$ for non-negative integers $k, p_1, q_1, r_1, p_2, q_2, r_2, \cdots p_k, q_k, r_k$. But, since the precise values of $k, p_1, q_1, r_1, p_2, q_2, r_2, \cdots p_k, q_k, r_k$ do not matter we will simplify, writing $(E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi})^{p,q,r}$ to indicate that, globally, there are p occurrences of $E\tilde{\varphi}$, q occurrences of φE, and r occurrences of $\varphi\tilde{\varphi}$.

Theorem 6.7. *Let $n \in \mathbb{N}$, $n \geq 2$, and $f \in \tilde{\varphi}\{E, \tilde{\varphi}\}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^*\tilde{\varphi}\left[(E\varphi)^*E\tilde{\varphi}\right]^{n-2}(E\varphi)^*\{Id_{\mathcal{A}}, E\}$. There exist $p, q, r, t \in \mathbb{N}$, $c \in \{\varphi, \tilde{\varphi}\}$, $d, d' \in \{E, \tilde{\varphi}\}$, a morphism f' with $f'cd \in \{\tilde{\varphi}d, \tilde{\varphi}d'(E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi})^{p,q,r}\varphi d\}$, and $e \in \{Id_{\mathcal{A}}, E\}$ such that*

- $f = f'cd(E\tilde{\varphi})^t\tilde{\varphi}(E\tilde{\varphi})^{n-2}e$ *(f is normalizable);*
- $f = N(f) = R(f')s_f$*;*
- *f is the $[(2t+1+2|d|_{\tilde{\varphi}})n]$-th conjugate of $R(f)$.*

Proof. Let $f \in \tilde{\varphi}\{E, \tilde{\varphi}\}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^*\tilde{\varphi}\left[(E\varphi)^*E\tilde{\varphi}\right]^{n-2}(E\varphi)^*\{Id_{\mathcal{A}}, E\}$, $n \geq 2$.

From (5), $\tilde{\varphi}\left[(E\varphi)^*E\tilde{\varphi}\right]^{n-2}(E\varphi)^*\{Id_{\mathcal{A}}, E\} = (\varphi E)^*\tilde{\varphi}(E\tilde{\varphi})^{n-2}\{Id_{\mathcal{A}}, E\}$.

This implies that

$$\begin{aligned} & \tilde{\varphi}\{E, \tilde{\varphi}\}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^*\tilde{\varphi}\left[(E\varphi)^*E\tilde{\varphi}\right]^{n-2}(E\varphi)^*\{Id_{\mathcal{A}}, E\} \\ = \; & \tilde{\varphi}\{E, \tilde{\varphi}\}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^*(\varphi E)^*\tilde{\varphi}(E\tilde{\varphi})^{n-2}\{Id_{\mathcal{A}}, E\} \\ = \; & \tilde{\varphi}\{E, \tilde{\varphi}\}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^*\tilde{\varphi}(E\tilde{\varphi})^{n-2}\{Id_{\mathcal{A}}, E\}. \end{aligned}$$

Consequently, $f \in \tilde{\varphi}\{E, \tilde{\varphi}\}\{E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi}\}^*\tilde{\varphi}(E\tilde{\varphi})^{n-2}\{Id_{\mathcal{A}}, E\}$ and there exist $p, q, r, t \in \mathbb{N}$ and $c \in \{\varphi, \tilde{\varphi}\}$, $d, d' \in \{E, \tilde{\varphi}\}$, $e \in \{Id_{\mathcal{A}}, E\}$ such that $f = f'cd(E\tilde{\varphi})^t\tilde{\varphi}(E\tilde{\varphi})^{n-2}e$ for a morphism f' with $f'cd \in \{\tilde{\varphi}d, \tilde{\varphi}d'(E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi})^{p,q,r}\varphi d\}$. This proves the first point.

Now, if $f'cd = \tilde{\varphi}d$ then $f' = \varepsilon$ thus $R(f') = \varepsilon$ and $f = cd(E\tilde{\varphi})^t\tilde{\varphi}(E\tilde{\varphi})^{n-2}e = \tilde{\varphi}d(E\tilde{\varphi})^t\tilde{\varphi}(E\tilde{\varphi})^{n-2}e = s_f = R(f')s_f$.

Otherwise, from Lemma 6.4, $f'c = \tilde{\varphi}d'(E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi})^{p,q,r}\varphi = R(\tilde{\varphi}d'(E\tilde{\varphi}, \varphi E, \varphi\tilde{\varphi})^{p,q,r})\tilde{\varphi} = R(f')\tilde{\varphi}$. Consequently $f = f'cd(E\tilde{\varphi})^t\tilde{\varphi}(E\tilde{\varphi})^{n-2}e = R(f')\tilde{\varphi}d(E\tilde{\varphi})^t\tilde{\varphi}(E\tilde{\varphi})^{n-2}e = R(f')s_f$.

Since $R(f') \in \{E, \varphi\}^*$ and $s_f \in \{E, \tilde{\varphi}\}^*$, the second point is established.

From Proposition 6.3, the conjugate number of s_f is $NC(s_f) = (2t+1+2|d|_{\tilde{\varphi}})n$. But since $R(f') \in \{E, \varphi\}^*$, from Lemma 6.2, $NC(R(f')s_f) = NC(s_f)$ thus $NC(s_f)$ is the conjugate number of $N(f)$, which completes the proof of the theorem. ■

As a conclusion we remark that Theorem 6.7 is, as we could hope, a generalization of the construction given at the end of the proof of Proposition 4.2. If a Sturmian word x is such that bx is not Sturmian, then a way to obtain a morphism f such that $f(b^nx)$ is Sturmian for some $n \in \mathbb{N}$, $n \geq 2$, is the following. Take any standard morphism g ending with $(\varphi E)^{n-1}\varphi$: $f = g_n$.

Acknowledgments

I would like to thank Christophe Reutenauer for suggesting the study of these questions to me and for his helpful comments on the first version of this paper. I am greatly indebted to Gwenaël Richomme for correcting many mistakes and proposing several improvements.

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