Exponential decay of scattering coefficients

Irène Waldspurger

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Irène Waldspurger
CNRS and CEREMADE, Université Paris Dauphine
INRIA, équipe Mokaplan

Abstract

The scattering transform is a deep representation, defined as a cascade of wavelet transforms followed by the application of a complex modulus. In her PhD, the author showed that, under some conditions on the wavelets, the norm of the scattering coefficients at a given layer only depends on the values of the signal outside a frequency band whose size is exponential in the depth of the layer. We give here a succinct presentation of this result, and slightly generalize it by removing one of the assumptions on the wavelets (namely the weak analyticity condition).

I. Introduction

The scattering transform, introduced in [1], is a deep representation, defined as a cascade of wavelet transform modulus. Because it is invariant to translations, and stable to small deformations, it has proved an efficient tool for various data analysis tasks that require invariance or stability to these operations [2], [3], [4].

On a more theoretical level, the scattering transform has the interest of modelling in a simple, yet realistic manner, the deep learned convolutional representations that have allowed, in the past five years, impressive progresses on a wide range of machine learning tasks, or at least the first layers of these representations. Indeed, it has the architecture of a deep convolutional representation. On various tasks, it performs essentially as well as learned convolutional representations [5], [6]. On other tasks, it can replace the first layers of a deep representation while retaining or improving the classification accuracy [7]. But compared to learned representations, the scattering transform has an entirely explicit expression. It is thus more amenable to mathematical analysis, and offers some insight into the behavior of deep learned representations, notably in terms of their invariance properties.

To refine our understanding of the scattering transform, the main question that must be answered is, informally: which properties of a signal does its scattering transform characterize, and, on the contrary, which properties is the scattering transform insensitive to? We already know that, as previously said, the scattering transform is invariant to translations, and stable to small deformations; this is shown in [1] under a so-called *admissibility* condition on the wavelets, in [8] for more general filters, but with a different definition of stability. On the other hand, the transform uniquely characterizes some structural properties of stationary processes [9], [10].

In this vein, in her PhD [11], the author showed that, under some conditions on the wavelets, the decay of the (1D) scattering coefficients as a function of their order is controlled by the frequential decay of the signal: the norm of the n-th layer scattering coefficients is essentially bounded by the energy contained in the signal outside the frequency band $[-a_0c^n, a_0c^n]$, for some $a_0 > 0, c > 1$. As a byproduct, it allows to generalize the results of [1] by replacing the admissibility condition with weaker assumptions. It also implies that, for band-limited signals, the scattering coefficients decay exponentially as a function of their order, which suggests that further analyses could focus on understanding better the low-order coefficients, while neglecting the high-order ones. A similar result has since been proved for an other family of frames than wavelets [12].

The goal of the present article is to succintly present the result from [11], and to slightly generalize it by removing the need for the wavelets to satisfy a weak analyticity condition (which in particular implies that the result now holds for real-valued wavelets).

In Section II, we give the formal definition of the scattering transform. Section III contains the statements of the main results, and Section IV provides an overview of their proof.

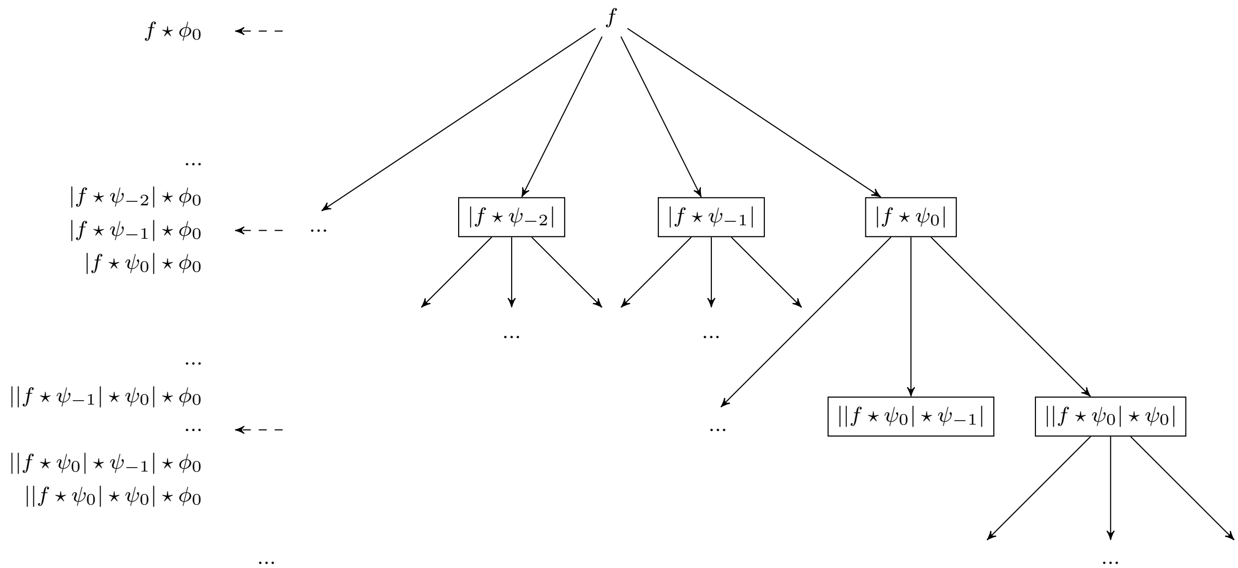

Fig. 1. Schematic illustration of the scattering transform: the tree on the right represents the cascade of modulus of wavelet transforms; the output scattering coefficients are on the left.

II. Definition of the scattering transform

Let $\psi \in L^1 \cap L^2(\mathbb{R})$ be a function such that

$$\int_{\mathbb{R}} \psi = 0.$$

We define a family of wavelets $(\psi_j)_{j\in\mathbb{Z}}$ by

$$\begin{aligned} &\forall j \in \mathbb{Z}, t \in \mathbb{R}, \qquad \psi_j(t) = 2^{-j}\psi(2^{-j}t); \\ \iff\quad &\forall j \in \mathbb{Z}, \omega \in \mathbb{R}, \qquad \hat{\psi}_j(\omega) = \hat{\psi}(2^j\omega). \end{aligned}$$

We also fix a real-valued positive function $\phi_0 \in L^1 \cap L^2(\mathbb{R})$, such that $\hat{\phi}_0(0) = 1$.

The scattering transform is composed of successive layers. At each layer, the wavelet transform of the functions in the previous layer is computed. The low-frequency component is output. A non-linearity (a complex modulus) is applied to the high-frequency components, which increases their invariance to deformations, and the resulting functions are fed as input to the next layer. The structure of the scattering transform is illustrated by Figure 1.

Formally, for any function $f \in L^2(\mathbb{R})$, we set:

$$U[\varnothing]f = f$$

and iteratively define, for any n-uplet $(j_1, ..., j_n) \in \mathbb{Z}^n$, with $n \geq 1$,

$$U[(j_1, ..., j_n)]f = \big|U[(j_1, ..., j_{n-1})]f \star \psi_{j_n}\big|.$$

We set $\mathcal{P}_0 = \{(j_1, \dots, j_n), n \in \mathbb{N}, j_1, \dots, j_n \in \mathbb{Z}, j_1, \dots, j_n \leq 0\}$. We refer to the elements of $\mathcal{P}_0$ as paths, and denote the length (that is, the number of elements) of a path p by $|p|$.

For any $p \in \mathcal{P}_0$, we define:

$$S_0[p]f = U[p]f \star \phi_0$$

The scattering coefficients associated to f are the set $\{S_0[p]f\}_{p\in\mathcal{P}_0}$.

III. Main statement

We first introduce a nonnegative function χ that we will use as a cutoff.

Lemme III.1 (Proof in Appendix A). *Let $\chi \in L^2(\mathbb{R})$ be the function whose Fourier transform is*

$$\hat{\chi} : \omega \in \mathbb{R} \to \frac{1}{(1+\sqrt{|\omega|})^4}.$$

The function χ is nonnegative.

For any $a > 0$, we define

$$\chi_a : t \in \mathbb{R} \to a\chi(at),$$

whose Fourier transform is

$$\hat{\chi}_a : \omega \in \mathbb{R} \to \hat{\chi}\left(\frac{\omega}{a}\right).$$

We can now state the main theorem: the norm of the n-th layer scattering coefficients is upper bounded by the norm of $\hat{f}$, multiplied by a high-pass filter with cutoff frequency proportional to $c^n a_0$. The only conditions required on the wavelet family is that it must form a frame, with upper frame bound at most 1, and that it must have slightly more than one vanishing moment.

Théorème III.2. *We assume that there exists a constant $c_0 > 0$ such that, for any $\omega \in \mathbb{R}$,*

$$c_0 \leq |\hat{\phi}_0(\omega)|^2 + \frac{1}{2}\sum_{j\leq 0}(|\hat{\psi}_j(\omega)|^2 + |\hat{\psi}_j(-\omega)|^2) \leq 1. \tag{1}$$

and that there exists $\epsilon > 0$ such that

$$|\hat{\psi}(\omega)| = O(|\omega|^{1+\epsilon}) \text{ when } \omega \to 0. \tag{2}$$

There exist $a_0 > 0, c > 1$ such that, for any $f \in L^2(\mathbb{R},\mathbb{R})$, and any $n \geq 2$,

$$\sum_{|p|\geq n} ||S_0[p]f||_2^2 \leq ||f||_2^2 - ||f \star \chi_{c^n a_0}||_2^2$$
$$= \int_{\mathbb{R}} |\hat{f}(\omega)|^2 (1 - |\hat{\chi}_{c^n a_0}(\omega)|^2) d\omega.$$

As a corollary, we get that the energy of scattering coefficients decays exponentially as a function of the order n when the input signal is band-limited. More precisely, if $M > 0$ is the size of the bandwith, there is a low-order regime, up to $n = O(\log(M))$, during which the energy may not significantly decay, then a high-order regime, for $n \geq O(\log(M))$, where the exponential decay begins, with a rate that does not depend on M.

Corollaire III.3 (Proof in Appendix D). *There exists $\eta \in]0;1[$, and $A > 0$ such that the following property is true: for any $M > 0$ and any $f \in L^2(\mathbb{R})$ such that*

$$\hat{f}(\omega) = 0, \qquad \forall \omega \notin [-M; M],$$

for any $n \in \mathbb{N}$,

$$\sum_{|p|\geq n} ||S_0[p]f||_2^2 \leq \eta^{\max(0,n-A\log M)} ||f||_2^2.$$

IV. Proof of Theorem III.2

Proof. Because, from Condition (1), the wavelet transform has norm at most 1,

$$\sum_{|p|\geq n} ||S_0[p]f||_2^2 \leq \sum_{j_1,\dots,j_n\leq 0} ||\,|\dots||f\star\psi_{j_1}|\star\psi_{j_2}|\dots|\star\psi_{j_n}||_2^2,$$

so it suffices to show

$$\sum_{j_1,\dots,j_n\leq 0} ||\,|\dots||f\star\psi_{j_1}|\star\psi_{j_2}|\dots|\star\psi_{j_n}||_2^2 \leq ||f||_2^2 - ||f\star\chi_{c^n a_0}||_2^2.$$

Up to a change of variables, it actually suffices to show the existence of $a_0>0, c>1$ such that, for any $n\geq 2$,

$$\sum_{j_1,\dots,j_n\leq 0} ||\,|\dots||f\star\psi_{j_1}|\star\psi_{j_2}|\dots|\star\psi_{j_n}||_2^2 \leq ||f||_2^2 - ||f\star\chi_{c^{n-2} a_0}||_2^2.$$

We proceed by recursion over n. The initialization step ($n=2$) is a separate lemma.

Lemme IV.1 (Proof in Appendix B)**.** *There exists $a_0>0$ such that, for any $f\in L^2(\mathbb{R},\mathbb{R})$,*

$$\sum_{j_1,j_2\leq 0} ||\,|f\star\psi_{j_1}|\star\psi_{j_2}||_2^2 \leq ||f||_2^2 - ||f\star\chi_{a_0}||_2^2.$$

We define $c>1$ as in the technical lemma IV.3. We assume that the result has been proved for some $n\geq 2$, and prove it for $n+1$.

We have, from the inductive hypothesis,

$$\begin{aligned}\sum_{j_1,\dots,j_{n+1}\leq 0} & ||\,|\dots||f\star\psi_{j_1}|\star\psi_{j_2}|\dots|\star\psi_{j_{n+1}}||_2^2 \\ &= \sum_{j\leq 0}\left(\sum_{j_1,\dots,j_n\leq 0} ||\,|\dots||f\star\psi_j|\star\psi_{j_1}|\dots|\star\psi_{j_n}||_2^2\right) \\ &\leq \sum_{j\leq 0} ||f\star\psi_j||_2^2 - ||\,|f\star\psi_j|\star\chi_{c^{n-2}a_0}||_2^2. \end{aligned} \tag{3}$$

The next part is to lower bound $||\,|f\star\psi_j|\star\chi_{c^{n-2}a_0}||_2^2$. The key tool for that is a lemma proved by Mallat in [1].

Lemme IV.2 ([1, Lemma 2.7])**.** *For any functions $g\in L^2(\mathbb{R}), h\in L^1(\mathbb{R})$, for any real number r, if $h(t)\in\mathbb{R}^+$ for all $t\in\mathbb{R}$, then*

$$||g|\star h|(t) \geq |g\star\tilde{h}|(t) \quad \textit{for almost every } t\in\mathbb{R},$$

where $\tilde{h}$ is defined as

$$\tilde{h}: t\in\mathbb{R}\to h(t)e^{irt}.$$

We apply this lemma, and use the fact that f is real-valued to "symmetrize" the Fourier transform:

$$\begin{aligned} ||f\star\psi_j||_2^2 &- ||\,|f\star\psi_j|\star\chi_{c^{n-2}a_0}||_2^2 \\ &= \frac{||f\star\psi_j||_2^2 + ||f\star\overline{\psi_j}||_2^2 - ||\,|f\star\psi_j|\star\chi_{c^{n-2}a_0}||_2^2 - ||\,|f\star\overline{\psi_j}|\star\chi_{c^{n-2}a_0}||_2^2}{2} \\ &\leq \frac{||f\star\psi_j||_2^2 + ||f\star\overline{\psi_j}||_2^2}{2} \\ &\quad - \frac{||f\star\psi_j\star\chi^{(2\pi\delta 2^{-j})}_{c^{n-2}a_0}||_2^2 + ||f\star\psi_j\star\chi^{(-2\pi\delta 2^{-j})}_{c^{n-2}a_0}||_2^2 + ||f\star\overline{\psi_j}\star\chi^{(2\pi\delta 2^{-j})}_{c^{n-2}a_0}||_2^2 + ||f\star\overline{\psi_j}\star\chi^{(-2\pi\delta 2^{-j})}_{c^{n-2}a_0}||_2^2}{4}, \end{aligned}$$

where $\delta > 0$ is the number associated to c by Lemma IV.3, and we have defined

$$\chi_{c^{n-2}a_0}^{(2\pi\delta 2^{-j})} : t \in \mathbb{R} \to \chi_{c^{n-2}a_0}(t) e^{2\pi i \delta 2^{-j} t};$$
$$\chi_{c^{n-2}a_0}^{(-2\pi\delta 2^{-j})} : t \in \mathbb{R} \to \chi_{c^{n-2}a_0}(t) e^{-2\pi i \delta 2^{-j} t}.$$

As a consequence,

$$\begin{aligned} &||f \star \psi_j||_2^2 - ||\,|f \star \psi_j| \star \chi_{c^{n-2}a_0}||_2^2 \\ &\qquad \le \int_{\mathbb{R}} |\hat{f}(\omega)|^2 F_j(\omega) \left(1 - \left(\frac{|\hat{\chi}_{c^{n-2}a_0}(\omega - \delta 2^{-j})|^2 + |\hat{\chi}_{c^{n-2}a_0}(\omega + \delta 2^{-j})|^2}{2} \right) \right) d\omega, \end{aligned}$$

where, by definition, for all $\omega \in \mathbb{R}$, $F_j(\omega) = \frac{|\hat{\psi}_j(\omega)|^2 + |\hat{\psi}_j(-\omega)|^2}{2}$.

We plus this inequality into Equation (3), and rearrange the terms.

$$\begin{aligned} &\sum_{j_1,\dots,j_{n+1} \le 0} ||\,|\dots||f \star \psi_{j_1}| \star \psi_{j_2}| \dots | \star \psi_{j_{n+1}}||_2^2 \\ &\quad \le \int_{\mathbb{R}} |\hat{f}(\omega)|^2 \left(\sum_{j \le 0} F_j(\omega) \left(1 - \left(\frac{|\hat{\chi}_{c^{n-2}a_0}(\omega - \delta 2^{-j})|^2 + |\hat{\chi}_{c^{n-2}a_0}(\omega + \delta 2^{-j})|^2}{2} \right) \right) \right) d\omega \\ &\quad = ||f||_2^2 - \int_{\mathbb{R}} |\hat{f}(\omega)|^2 \left(\left(1 - \sum_{j \le 0} F_j(\omega) \right) + \sum_{j \le 0} F_j(\omega) \left(\frac{|\hat{\chi}_{c^{n-2}a_0}(\omega - \delta 2^{-j})|^2 + |\hat{\chi}_{c^{n-2}a_0}(\omega + \delta 2^{-j})|^2}{2} \right) \right) d\omega \end{aligned}$$

The expression inside the parenthesis can be lower bounded with a final lemma.

Lemme IV.3 (Proof in Appendix C)**.** *Let $a_0 > 0$ be fixed. For any $\delta > 0$ small enough, there exists $c > 1$ such that the following inequality holds for any $a \ge a_0$ and $\omega \in \mathbb{R}$:*

$$\left(1 - \sum_{j \le 0} F_j(\omega) \right) + \sum_{j \le 0} F_j(\omega) \left(\frac{|\hat{\chi}_a|^2 \left(\omega - \delta 2^{-j}\right) + |\hat{\chi}_a|^2 \left(\omega + \delta 2^{-j}\right)}{2} \right) \ge |\hat{\chi}_{ca}\left(\omega\right)|^2.$$

So we get

$$\begin{aligned} \sum_{j_1,\dots,j_{n+1} \le 0} &||\,|\dots||f \star \psi_{j_1}| \star \psi_{j_2}| \dots | \star \psi_{j_{n+1}}||_2^2 \\ &\le ||f||_2^2 - \int_{\mathbb{R}} |\hat{f}(\omega)|^2 |\hat{\chi}_{c^{(n+1)-2}a_0}(\omega)|^2 d\omega \\ &= ||f||_2^2 - ||f \star \chi_{c^{(n+1)-2}a_0}||_2^2, \end{aligned}$$

which is what we wanted to prove. $\square$

Appendix A
Proof of Lemma III.1

Lemme (Lemma III.1)**.** *Let $\chi \in L^2(\mathbb{R})$ be the function whose Fourier transform is*

$$\hat{\chi} : \omega \in \mathbb{R} \to \frac{1}{(1 + \sqrt{|\omega|})^4}.$$

The function χ is nonnegative.

Proof. For all $t \geq 0$,

$$\begin{aligned}
\chi(t) &= \int_{-\infty}^{+\infty} \hat{\chi}(\omega) e^{2\pi i \omega t} d\omega \\
&= 2 \int_{0}^{+\infty} \hat{\chi}(\omega) \cos(2\pi\omega t) d\omega \\
&= 2 \sum_{k=0}^{+\infty} \int_{k/t}^{(k+1)/t} \hat{\chi}(\omega) \cos(2\pi\omega t) d\omega \\
&= 2 \sum_{k=0}^{+\infty} \left(\int_{k/t}^{(k+1/4)/t} \hat{\chi}(\omega) \cos(2\pi\omega t) d\omega + \int_{(k+1/4)/t}^{(k+1/2)/t} \hat{\chi}(\omega) \cos(2\pi\omega t) d\omega \right. \\
&\qquad\qquad \left. + \int_{(k+1/2)/t}^{(k+3/4)/t} \hat{\chi}(\omega) \cos(2\pi\omega t) d\omega + \int_{(k+3/4)/t}^{(k+1)/t} \hat{\chi}(\omega) \cos(2\pi\omega t) d\omega \right) \\
&= 2 \sum_{k=0}^{+\infty} \left(\int_{k/t}^{(k+1/4)/t} \hat{\chi}(\omega) \cos(2\pi\omega t) d\omega - \int_{k/t}^{(k+1/4)/t} \hat{\chi}((2k+1/2)/t - \omega) \cos(2\pi\omega t) d\omega \right. \\
&\qquad\qquad \left. - \int_{k/t}^{(k+1/4)/t} \hat{\chi}(\omega + 1/(2t)) \cos(2\pi\omega t) d\omega + \int_{k/t}^{(k+1/4)/t} \hat{\chi}((2k+1)/t - \omega) \cos(2\pi\omega t) d\omega \right) \\
&= 2 \sum_{k=0}^{+\infty} \int_{k/t}^{(k+1/4)/t} \Big(\hat{\chi}(\omega) - \hat{\chi}((2k+1/2)t - \omega) - \hat{\chi}(\omega + 1/(2t)) + \hat{\chi}((2k+1)/t - \omega) \Big) \cos(2\pi\omega t) d\omega.
\end{aligned}$$

For any $k \in \mathbb{N}$, $\cos(2\pi\omega t)$ takes only positive values on $[k/t; (k+1/4)/t]$. Moreover, as $\hat{\chi}$ is convex on $\mathbb{R}^+$, for any $\omega \in [k/t; (k+1/4)/t]$,

$$\begin{aligned}
\hat{\chi}(\omega) &- \hat{\chi}((2k+1/2)/t - \omega) - \hat{\chi}(\omega + 1/(2t)) + \hat{\chi}((2k+1)/t - \omega) \\
&\geq -\left((2k+1/2)/t - 2\omega\right) (\hat{\chi})'((2k+1/2)t - \omega) \\
&\quad + \left((2k+1/2)/t - 2\omega\right) (\hat{\chi})'(\omega + 1/(2t)) \\
&= \left((2k+1/2)/t - 2\omega\right) \Big((\hat{\chi})'(\omega + 1/(2t)) - (\hat{\chi})'((2k+1/2)/t - \omega) \Big) \\
&\geq 0.
\end{aligned}$$

The last inequality comes from the fact that $(\hat{\chi})'$ is a non-decreasing function, and $\omega + 1/(2t) \geq (2k+1/2)/t - \omega$.

We plug this into the previous integral, and get that $\chi(t) \in \mathbb{R}^+$ for any $t \geq 0$. As $\hat{\chi}$ is real and even, χ is an even function, so the result also holds for $t < 0$. □

Appendix B
Proof of Lemma IV.1

Lemme (Lemma IV.1). *There exists $a_0 > 0$ such that, for any $f \in L^2(\mathbb{R}, \mathbb{R})$,*

$$\sum_{j_1, j_2 \leq 0} || \, |f \star \psi_{j_1}| \star \psi_{j_2} ||_2^2 \leq ||f||_2^2 - ||f \star \chi_{a_0}||_2^2.$$

Proof. For any $j \in \mathbb{Z}$, we define

$$F_j : \omega \in \mathbb{R} \to \frac{|\hat{\psi}_j(\omega)|^2 + |\hat{\psi}_j(-\omega)|^2}{2}.$$

For any j_1, if we define $g_{j_1} = |f \star \psi_{j_1}|$,

$$\sum_{j_2 \leq 0} ||\,|f \star \psi_{j_1}| \star \psi_{j_2}||_2^2 = \int_{\mathbb{R}} |\hat{g}_{j_1}(\omega)|^2 \left(\sum_{j_2 \leq 0} F_{j_2}(\omega) \right) d\omega.$$

We introduce a nonnegative function Φ as in the next lemma.

Lemme B.1 (Proof in Paragraph B-A)**.** *There exists $\Phi \in L^1 \cap L^2(\mathbb{R}, \mathbb{R})$, with nonnegative values, such that, for any $\omega \in \mathbb{R}$,*

$$|\hat{\Phi}(\omega)|^2 \leq 1 - \sum_{j \leq 0} F_j(\omega),$$

and $|\hat{\Phi}|^2 \geq \frac{1}{2}$ on some neighborhood of 0.

With this definition,

$$\begin{aligned}\sum_{j_2 \leq 0} ||\,|f \star \psi_{j_1}| \star \psi_{j_2}||_2^2 &\leq \int_{\mathbb{R}} |\hat{g}_{j_1}(\omega)|^2 (1 - |\hat{\Phi}(\omega)|^2) d\omega \\ &= ||g_{j_1}||_2^2 - ||g_{j_1} \star \Phi||_2^2 \\ &= ||f \star \psi_{j_1}||_2^2 - ||\,|f \star \psi_{j_1}| \star \Phi||_2^2. \end{aligned} \tag{4}$$

For any $m \in \mathbb{R}$, we define

$$\Phi^{(m)} : t \in \mathbb{R} \to \Phi(t) e^{2\pi i m t}.$$

From Lemma IV.2, it holds for any $j_1 \in \mathbb{Z}$ and $m \in \mathbb{R}$ that

$$||\,|f \star \psi_{j_1}| \star \Phi||_2^2 \geq ||f \star \psi_{j_1} \star \Phi^{(m)}||_2^2.$$

We therefore also have, for any $M > 0$,

$$\begin{aligned}||\,|f \star \psi_{j_1}| \star \Phi||_2^2 &= \frac{2^{j_1 - 1}}{M} \int_{-M2^{-j_1}}^{M2^{-j_1}} ||\,|f \star \psi_{j_1}| \star \Phi||_2^2 dm \\ &\geq \frac{2^{j_1 - 1}}{M} \int_{-M2^{-j_1}}^{M2^{-j_1}} ||f \star \psi_{j_1} \star \Phi^{(m)}||_2^2 dm \\ &= \frac{2^{j_1 - 1}}{M} \int_{-M2^{-j_1}}^{M2^{-j_1}} \int_{\mathbb{R}} |\hat{f}(\omega)|^2 |\hat{\psi}_{j_1}(\omega)|^2 |\hat{\Phi}(\omega - m)|^2 dm d\omega \\ &= \int_{\mathbb{R}} |\hat{f}(\omega)|^2 |\hat{\psi}_{j_1}(\omega)|^2 \left(|\hat{\Phi}|^2 \star \frac{1_{[-M2^{-j_1}; M2^{-j_1}]}}{2^{-(j_1 - 1)} M} \right) (\omega) d\omega \\ &= \int_{\mathbb{R}} |\hat{f}(\omega)|^2 F_{j_1}(\omega) \left(|\hat{\Phi}|^2 \star \frac{1_{[-M2^{-j_1}; M2^{-j_1}]}}{2^{-(j_1 - 1)} M} \right) (\omega) d\omega \end{aligned}$$

The last inequality relies on the fact that $|\hat{f}|$ and $|\hat{\Phi}|^2 \star \frac{1_{[-M2^{-j_1}; M2^{-j_1}]}}{2^{-(j_1 - 1)} M}$ are even, because f and Φ are real-valued.

We combine this result with Equation (4):

$$\sum_{j_2 \leq 0} ||\,|f \star \psi_{j_1}| \star \psi_{j_2}||_2^2 \leq \int_{\mathbb{R}} |\hat{f}(\omega)|^2 F_{j_1}(\omega) \left(1 - |\hat{\Phi}|^2 \star \frac{1_{[-M2^{-j_1}; M2^{-j_1}]}}{2^{-(j_1 - 1)} M} \right) (\omega) d\omega$$

We sum over j_1:

$$\sum_{j_1,j_2\leq 0} ||\,|f\star\psi_{j_1}|\star\psi_{j_2}||_2^2 \leq \int_{\mathbb{R}} |\hat{f}(\omega)|^2 \sum_{j\leq 0}\left(F_j(\omega)\left(1-|\hat{\Phi}|^2\star\frac{1_{[-M2^{-j};M2^{-j}]}}{2^{-(j-1)}M}\right)(\omega)\right)d\omega.$$

The next lemma allows us to derive an upper bound for this expression.

Lemme B.2 (Proof in Paragraph B-B)**.** *There exist $M, a_0>0$ such that, for any $\omega\in\mathbb{R}$,*

$$\left(1-\sum_{j\leq 0}F_j(\omega)\right)+\sum_{j\leq 0}F_j(\omega)\left(|\hat{\Phi}|^2\star\frac{1_{[-M2^{-j};M2^{-j}]}}{2^{-(j-1)}M}\right)(\omega)\geq|\hat{\chi}_{a_0}(\omega)|^2,$$

where, for any $j\leq 0$, $F_j(\omega)=\frac{|\hat{\psi}_j(\omega)|^2+|\hat{\psi}_j(-\omega)|^2}{2}$.

If we define $M, a_0>0$ as in the lemma,

$$\begin{aligned}\sum_{j_1,j_2\leq 0} ||\,|f\star\psi_{j_1}|\star\psi_{j_2}||_2^2 &\leq \int_{\mathbb{R}}|\hat{f}(\omega)|^2(1-|\hat{\chi}_{a_0}(\omega)|^2)d\omega\\ &=||f||_2^2-||f\star\chi_{a_0}||_2^2.\end{aligned}$$

□

A. *Proof of Lemma B.1*

Lemme (Lemma B.1)**.** *There exists $\Phi\in L^1\cap L^2(\mathbb{R},\mathbb{R})$, with nonnegative values, such that, for any $\omega\in\mathbb{R}$,*

$$|\hat{\Phi}(\omega)|^2\leq 1-\sum_{j\leq 0}F_j(\omega), \tag{5}$$

and $|\hat{\Phi}|^2\geq\frac{1}{2}$ on some neighborhood of 0.

Proof. Let $H:\mathbb{R}\to\mathbb{R}$ be a compactly-supported, nonnegative, smooth and even function, such that $H(0)=1$ and H is non-increasing over $\mathbb{R}^+$. Then $H\star H$ is also compactly supported, smooth and even. It is also non-increasing over $\mathbb{R}^+$ (its derivative $H'\star H$ can be shown to be nonnegative), which implies

$$(H\star H)(0)=\max_{\omega\in\mathbb{R}}(H\star H)(\omega). \tag{6}$$

From Condition (2),

$$1-\sum_{j\leq 0}F_j(\omega)\to 1 \text{ when } \omega\to 0.$$

So there exists $\eta>0$ such that

$$1-\sum_{j\leq 0}F_j(\omega)\geq\frac{3}{4},\qquad \forall\omega\in[-\eta;\eta].$$

We define

$$\tilde{H}:\omega\in\mathbb{R}\to\sqrt{\frac{3}{4}}\frac{(H\star H)(m\omega)}{(H\star H)(0)},$$

where $m>0$ is large enough so that the support of $\omega\to(H\star H)(m\omega)$ is included in $[-\eta;\eta]$. From the definition of m,

$$|\tilde{H}(\omega)|^2=0\leq 1-\sum_{j\leq 0}F_j(\omega),\qquad \text{when } \omega\notin[-\eta;\eta].$$

And from Equality (6),

$$|\tilde{H}(\omega)|^2 \leq \frac{3}{4} \leq 1 - \sum_{j\leq 0} F_j(\omega), \qquad \text{when } \omega \in [-\eta;\eta].$$

The two previous equations, together, say that $|\tilde{H}|^2 \leq 1 - \sum_{j\leq 0} F_j$ on $\mathbb{R}$.

Let Φ be the inverse Fourier transform of $\tilde{H}$. We have just seen that it satisfies Conditions (5). Its Fourier transform is smooth and compactly-supported, so it belongs to $L^1 \cap L^2(\mathbb{R})$. It is also real-valued (the Fourier transform is even). From the definition of $\tilde{H}$, $|\hat{\Phi}|^2 = \frac{3}{4}$ in 0, and $\hat{\Phi}$ is continuous, so $|\hat{\Phi}|^2 \geq \frac{1}{2}$ on some neighborhood of 0.

The last thing to show is that Φ is positive, but this is immediate, because Φ can be written as

$$\Phi : t \in \mathbb{R} \to \sqrt{\frac{3}{4}} \frac{(h(t/m))^2}{m(H \star H)(0)},$$

where h is the (real-valued) inverse Fourier transform of H. □

B. *Proof of Lemma B.2*

Lemme (Lemma B.2)**.** *There exist $M, a_0 > 0$ such that, for any $\omega \in \mathbb{R}$,*

$$\left(1 - \sum_{j\leq 0} F_j(\omega)\right) + \sum_{j\leq 0} F_j(\omega)\left(|\hat{\Phi}|^2 \star \frac{1_{[-M2^{-j};M2^{-j}]}}{2^{-(j-1)}M}\right)(\omega) \geq |\hat{\chi}_{a_0}(\omega)|^2,$$

where, for any $j \leq 0$, $F_j(\omega) = \frac{|\hat{\psi}_j(\omega)|^2 + |\hat{\psi}_j(-\omega)|^2}{2}$.

Proof. Let $\eta > 0$ be such that $|\hat{\Phi}|^2 \geq \frac{1}{2}$ on $[-\eta;\eta]$. Such an η exists, from the definition of Φ.

We can write, for any j,

$$\begin{aligned} |\hat{\Phi}|^2 \star 1_{[-M2^{-j};M2^{-j}]} &\geq \frac{1_{[-\eta;\eta]} \star 1_{[-M2^{-j};M2^{-j}]}}{2} \\ &\geq \eta 1_{[-(M2^{-j}-\eta);M2^{-j}-\eta]}, \end{aligned}$$

We assume that M is larger than 2η, which implies $[-(M2^{-j}-\eta); M2^{-j}-\eta] \supset [-M2^{-(j+1)}; M2^{-(j+1)}]$.

For any $\omega \in \mathbb{R}$,

$$\begin{aligned} &\left(1 - \sum_{j\leq 0} F_j(\omega)\right) + \sum_{j\leq 0} F_j(\omega)\left(|\hat{\Phi}|^2 \star \frac{1_{[-M2^{-j};M2^{-j}]}}{2^{-(j-1)}M}\right)(\omega) \\ &\qquad \geq \left(1 - \sum_{j\leq 0} F_j(\omega)\right) + \frac{\eta}{2M} \sum_{j\leq 0} 2^j F_j(\omega) 1_{[-M2^{-(j+1)};M2^{-(j+1)}]}(\omega). \end{aligned} \tag{7}$$

Let us show that, when M is large enough, this last function is larger than $|\hat{\chi}_{a_0}|^2$ for some $a_0 > 0$.

First, from Condition (2), there exists $\mu > 0$ such that

$$F_j(\omega) \leq \mu |\omega|^{2(1+\epsilon)},$$

which implies

$$\sum_{j\leq 0} F_j(\omega) \leq \mu' |\omega|^{2(1+\epsilon)},$$

for $\mu' = \mu/(1 - 2^{-2(1+\epsilon)})$.

So for any $a_0 \in]0;1]$, and for any ω such that $|\omega| \leq \min\left(1, \frac{1}{(2\mu')^{\frac{2}{3+4\epsilon}}}\right)$,

$$\begin{aligned}\left(1 - \sum_{j\leq 0} F_j(\omega)\right) + &\frac{\eta}{2M}\sum_{j\leq 0} 2^j F_j(\omega) 1_{[-M2^{-(j+1)};M2^{-(j+1)}]}(\omega)\\ &\geq 1 - \mu'|\omega|^{2(1+\epsilon)}\\ &\geq 1 - \frac{\sqrt{|\omega|}}{2}\\ &\geq |\hat{\chi}_1(\omega)|^2\\ &\geq |\hat{\chi}_{a_0}(\omega)|^2. \end{aligned} \tag{8}$$

We define $\omega_0 = \min\left(1, \frac{1}{(2\mu')^{\frac{2}{3+4\epsilon}}}\right)$. We now have to consider the case where $|\omega| \geq \omega_0$.

From the Littlewood-Paley inequality (1), we have that, for any $\omega \in \mathbb{R} - \{0\}$,

$$c_0 \leq \sum_{j\in\mathbb{Z}} F_j(\omega) \leq 1.$$

(Indeed, for any $k \in \mathbb{N}, \omega \in \mathbb{R}$, Equation (1) implies $c_0 \leq |\hat{\phi}(2^k\omega)|^2 + \sum_{j\leq k} F_j(\omega) \leq 1$, and $|\hat{\phi}(2^k\omega)|^2 \to 0$ when $k \to +\infty$, if $\omega \neq 0$, because $\hat{\phi}$ is the Fourier transform of a function in $L^1(\mathbb{R})$, and hence goes to 0 in $\pm\infty$.)

From this, using the continuity of the F_j and a compacity argument, we see that there exists $j_0 \in \mathbb{N}$ such that, for any $\omega \in \mathbb{R} - \{0\}$,

$$\sum_{2^{-j_0}|\omega| \leq 2^{-j} \leq 2^{j_0}|\omega|} F_j(\omega) \geq \frac{c_0}{2}.$$

We then have, for any $\omega \in \mathbb{R} - \{0\}$, if we assume that M is large enough so that $M > \max(2\eta, 2^{j_0+1})$,

$$\begin{aligned}\left(1 - \sum_{j\leq 0} F_j(\omega)\right) + &\frac{\eta}{2M}\sum_{j\leq 0} 2^j F_j(\omega) 1_{[-M2^{-(j+1)};M2^{-(j+1)}]}(\omega)\\ &\geq \sum_{j>0} F_j(\omega) + \frac{\eta}{2M}\sum_{j\leq 0} 2^j F_j(\omega) 1_{[-M2^{-(j+1)};M2^{-(j+1)}]}(\omega)\\ &\geq \frac{\eta}{2M}\sum_{j\in\mathbb{Z}} 2^{\min(0,j)} F_j(\omega) 1_{[-M2^{-(j+1)};M2^{-(j+1)}]}(\omega)\\ &= \frac{\eta}{2M}\sum_{\frac{2|\omega|}{M} \leq 2^{-j}} 2^{\min(0,j)} F_j(\omega)\\ &\geq \frac{\eta}{2M}\sum_{2^{-j_0}|\omega| \leq 2^{-j} \leq 2^{j_0}|\omega|} 2^{\min(0,j)} F_j(\omega). \end{aligned}$$

If we moreover assume that $|\omega| \geq \omega_0$, we have, for any j such that $2^{-j_0}|\omega| \leq 2^{-j}$,

$$\begin{aligned} & 2^j \leq \frac{2^{j_0}}{\omega_0},\\ \Rightarrow \qquad & 2^{\min(0,j)} = \min(1, 2^j) \geq \omega_0 2^{j-j_0}, \end{aligned}$$

so

$$\left(1-\sum_{j\le 0}F_j(\omega)\right)+\frac{\eta}{2M}\sum_{j\le 0}2^jF_j(\omega)1_{[-M2^{-(j+1)};M2^{-(j+1)}]}(\omega)$$
$$\ge \frac{\eta\omega_0}{2^{j_0+1}M}\sum_{2^{-j_0}|\omega|\le 2^{-j}\le 2^{j_0}|\omega|}2^jF_j(\omega)$$
$$\ge \frac{\eta\omega_0}{2^{j_0+1}M}\sum_{2^{-j_0}|\omega|\le 2^{-j}\le 2^{j_0}|\omega|}\frac{2^{-j_0}}{|\omega|}F_j(\omega)$$
$$\ge \frac{\eta\omega_0c_0}{2^{2(j_0+1)}M}\frac{1}{|\omega|}.$$

For any $a>0$, when $|\omega|\ge a$,

$$|\hat{\chi}_a(\omega)|^2=\frac{1}{(1+\sqrt{|\omega|/a})^8}\le\left(\frac{a}{|\omega|}\right)^8\le\frac{a}{|\omega|}.$$

So if we choose $a_0>0$ such that $a_0<\min\left(1,\omega_0,\frac{\eta\omega_0c_0}{2^{2(j_0+1)}M}\right)$, we have, for any ω such that $|\omega|\ge\omega_0$,

$$\left(1-\sum_{j\le 0}F_j(\omega)\right)+\frac{\eta}{2M}\sum_{j\le 0}2^jF_j(\omega)1_{[-M2^{-(j+1)};M2^{-(j+1)}]}(\omega)$$
$$\ge\frac{a_0}{|\omega|}$$
$$\ge|\hat{\chi}_{a_0}(\omega)|^2. \tag{9}$$

We combine Equations (7), (8) and (9) together, and they prove the lemma.

□

APPENDIX C
PROOF OF LEMMA IV.3

Lemme (Lemma IV.3)**.** *Let $a_0>0$ be fixed. For any $\delta>0$ small enough, there exists $c>1$ such that the following inequality holds for any $a\ge a_0$ and $\omega\in\mathbb{R}$:*

$$\left(1-\sum_{j\le 0}F_j(\omega)\right)+\sum_{j\le 0}F_j(\omega)\left(\frac{|\hat{\chi}_a|^2\left(\omega-\delta 2^{-j}\right)+|\hat{\chi}_a|^2\left(\omega+\delta 2^{-j}\right)}{2}\right)\ge|\hat{\chi}_{ca}(\omega)|^2.$$

Proof. As the left and right hand side of the inequality are even functions, we can restrict ourselves to nonnegative values of ω.

We divide the proof in two cases: when ω is larger than $O(\delta)$, and when ω is at most $O(\delta)$. For the first case, we use the following lemma.

Lemme C.1 (Proof in Paragraph C-A)**.** *There exists $C,C'>0$ such that, when $\delta>0$ is small enough, for any $a\ge a_0$ and any $\omega\ge C\delta$,*

$$\sum_{j\le 0}F_j(\omega)\left(\frac{|\hat{\chi}_a(\omega-\delta 2^{-j})|^2-2|\hat{\chi}_a(\omega)|^2+|\hat{\chi}_a(\omega+\delta 2^{-j})|^2}{2}\right)\ge C'\delta^2\omega^2(|\hat{\chi}_a|^2)''(\omega).$$

The constants C,C' do not depend on δ.

We define C, C' as in the lemma. From this lemma, when δ is small enough and $\omega \geq C\delta$,

$$\begin{aligned}\left(1-\sum_{j\leq 0}F_j(\omega)\right) + \sum_{j\leq 0}F_j(\omega)\left(\frac{|\hat{\chi}_a|^2\left(\omega-\delta 2^{-j}\right)+|\hat{\chi}_a|^2\left(\omega+\delta 2^{-j}\right)}{2}\right)&\\ \geq \left(1-\sum_{j\leq 0}F_j(\omega)\right) + |\hat{\chi}_a(\omega)|^2\left(\sum_{j\leq 0}F_j(\omega)\right) + C'\delta^2\omega^2(|\hat{\chi}_a|^2)''(\omega)&\\ \geq |\hat{\chi}_a(\omega)|^2\left(1-\sum_{j\leq 0}F_j(\omega)\right) + |\hat{\chi}_a(\omega)|^2\left(\sum_{j\leq 0}F_j(\omega)\right) + C'\delta^2\omega^2(|\hat{\chi}_a|^2)''(\omega)&\\ = |\hat{\chi}_a(\omega)|^2 + C'\delta^2\omega^2(|\hat{\chi}_a|^2)''(\omega)&\\ = |\hat{\chi}_1(\omega/a)|^2 + C'\delta^2\left(\frac{\omega}{a}\right)^2(|\hat{\chi}_1|^2)''(\omega/a).&\end{aligned}$$

We use another lemma to conclude.

Lemme C.2 (Proof in Paragraph C-B)**.** *For any $\eta > 0$, there exists $c > 1$ such that*

$$|\hat{\chi}_1(x)|^2 + \eta x^2(|\hat{\chi}_1|^2)''(x) \geq |\hat{\chi}_c(x)|^2, \qquad \forall x \in \mathbb{R}-\{0\}.$$

If we combine the lemma and the equation immediately before, we get that, when δ is small enough, there exists $c > 1$ (depending on δ) such that, when $\omega \geq C\delta$,

$$\begin{aligned}\left(1-\sum_{j\leq 0}F_j(\omega)\right) + \sum_{j\leq 0}F_j(\omega)\left(\frac{|\hat{\chi}_a|^2\left(\omega-\delta 2^{-j}\right)+|\hat{\chi}_a|^2\left(\omega+\delta 2^{-j}\right)}{2}\right)&\\ \geq |\hat{\chi}_c(\omega/a)|^2&\\ = |\hat{\chi}_{ca}(\omega)|^2.&\end{aligned}$$

This concludes the case where $\omega \geq C\delta$. Let us now consider the case where $\omega < C\delta$. We use another lemma.

Lemme C.3 (Proof in Paragraph C-C)**.** *There exists $C' > 0$ such that, for any $a \geq a_0, \omega > 0$,*

$$\sum_{j\leq 0}F_j(\omega)\left(\frac{|\hat{\chi}_a(\omega-\delta 2^{-j})|^2 - 2|\hat{\chi}_a(\omega)|^2 + |\hat{\chi}_a(\omega+\delta 2^{-j})|^2}{2}\right) \geq -C'\omega^{2(1+\epsilon)}\sqrt{\frac{\delta}{a}}.$$

This lemma implies that

$$\begin{aligned}\left(1-\sum_{j\leq 0}F_j(\omega)\right) + \sum_{j\leq 0}F_j(\omega)\left(\frac{|\hat{\chi}_a|^2\left(\omega-\delta 2^{-j}\right)+|\hat{\chi}_a|^2\left(\omega+\delta 2^{-j}\right)}{2}\right)&\\ \geq \left(1-\sum_{j\leq 0}F_j(\omega)\right) + |\hat{\chi}_a(\omega)|^2\left(\sum_{j\leq 0}F_j(\omega)\right) - C'\omega^{2(1+\epsilon)}\sqrt{\frac{\delta}{a}}&\\ = 1-(1-|\hat{\chi}_a(\omega)|^2)\left(\sum_{j\leq 0}F_j(\omega)\right) - C'\omega^{2(1+\epsilon)}\sqrt{\frac{\delta}{a}}.&\end{aligned}$$

The function $\omega \to \sum_{j\leq 0} F_j(\omega)$ is upper bounded by $\frac{1}{4}$ on $[0;C\delta]$, provided that $\delta > 0$ is small enough. So

$$\begin{aligned}\left(1-\sum_{j\leq 0}F_j(\omega)\right)+\sum_{j\leq 0}F_j(\omega)\left(\frac{|\hat\chi_a|^2\left(\omega-\delta 2^{-j}\right)+|\hat\chi_a|^2\left(\omega+\delta 2^{-j}\right)}{2}\right)&\\ \geq 1-\frac{1}{4}(1-|\hat\chi_a(\omega)|^2)-C'\omega^{2(1+\epsilon)}\sqrt{\frac{\delta}{a}}&\\ \overset{(a)}{\geq} 1-2\sqrt{\frac{\omega}{a}}-C'\omega^{2(1+\epsilon)}\sqrt{\frac{\delta}{a}}&\\ \overset{(b)}{\geq} 1-(2+C'C^{3/2+2\epsilon}\delta^{2(1+\epsilon)})\sqrt{\frac{\omega}{a}}.&\end{aligned}$$

For (a), we have used the inequality $|\hat\chi_a(\omega)|^2 \geq 1-8\sqrt{\frac{\omega}{a}}$, valid for any $\omega \geq 0$. For (b), we have used the hypothesis $0 \leq \omega \leq C\delta$.

In particular, when δ is small enough,

$$\begin{aligned}\left(1-\sum_{j\leq 0}F_j(\omega)\right)+\sum_{j\leq 0}F_j(\omega)\left(\frac{|\hat\chi_a|^2\left(\omega-\delta 2^{-j}\right)+|\hat\chi_a|^2\left(\omega+\delta 2^{-j}\right)}{2}\right)&\\ \geq 1-3\sqrt{\frac{\omega}{a}}.&\end{aligned}\tag{10}$$

On the other hand, if we compute the Taylor expansion of $|\hat\chi_1|^2$ in 0, we see that there exists $\eta > 0$ such that

$$\begin{aligned}|\hat\chi_1(\omega)|^2 \leq 1-6\sqrt{\omega},\quad \forall\omega\in[0;\eta],\\ \Rightarrow |\hat\chi_4(\omega)|^2 \leq 1-3\sqrt{\omega},\quad \forall\omega\in[0;4\eta].\end{aligned}$$

Combining this and Equation (10), we get that, when $\delta \leq \frac{4\eta}{C}$,

$$\left(1-\sum_{j\leq 0}F_j(\omega)\right)+\sum_{j\leq 0}F_j(\omega)\left(\frac{|\hat\chi_a|^2\left(\omega-\delta 2^{-j}\right)+|\hat\chi_a|^2\left(\omega+\delta 2^{-j}\right)}{2}\right)\geq |\hat\chi_4(\omega)|^2,$$

for any $\omega \in [0;C\delta]$. □

A. *Proof of Lemma C.1*

Lemme (Lemma C.1)**.** *There exists $C, C' > 0$ such that, when $\delta > 0$ is small enough, for any $a \geq a_0$ and any $\omega \geq C\delta$,*

$$\sum_{j\leq 0}F_j(\omega)\left(\frac{|\hat\chi_a(\omega-\delta 2^{-j})|^2-2|\hat\chi_a(\omega)|^2+|\hat\chi_a(\omega+\delta 2^{-j})|^2}{2}\right)\geq C'\delta^2\omega^2(|\hat\chi_a|^2)''(\omega).$$

The constants C, C' do not depend on δ.

Proof. We first compute various lower bounds for the term inside the parenthesis, depending on the respectives values of ω, a, δ and j.

For any $a, \omega, \delta > 0$, $j \leq 0$ such that $\delta 2^{-j} \leq \frac{\omega}{2}$, from Taylor's theorem,

$$\frac{|\hat\chi_a(\omega-\delta 2^{-j})|^2-2|\hat\chi_a(\omega)|^2+|\hat\chi_a(\omega+\delta 2^{-j})|^2}{2}=\frac{(\delta 2^{-j})^2}{2}(|\hat\chi_a|^2)''(\omega_0),\tag{11}$$

for some $\omega_0 \in [\omega - \delta 2^{-j}; \omega + \delta 2^{-j}] \subset \left[\frac{\omega}{2}; \frac{3\omega}{2}\right]$.

Explicitely computing the second derivative of $|\hat{\chi}_a|^2$ shows that it is a decreasing function on $\mathbb{R}^+$, and that, for some $\alpha > 0$,

$$(|\hat{\chi}_a|^2)''\left(\frac{3z}{2}\right) \geq \alpha(|\hat{\chi}_a|^2)''(z), \qquad \forall z > 0.$$

So if we combine these remarks with Equation (11),

$$\begin{aligned}\frac{|\hat{\chi}_a(\omega-\delta 2^{-j})|^2 - 2|\hat{\chi}_a(\omega)|^2 + |\hat{\chi}_a(\omega+\delta 2^{-j})|^2}{2} &= \frac{(\delta 2^{-j})^2}{2}(|\hat{\chi}_a|^2)''(\omega_0)\\ &\geq \frac{(\delta 2^{-j})^2}{2}(|\hat{\chi}_a|^2)''\left(\frac{3\omega}{2}\right)\\ &\geq \frac{\alpha}{2}(\delta 2^{-j})^2(|\hat{\chi}_a|^2)''(\omega). \end{aligned} \tag{12}$$

When $\frac{\omega}{2} < \delta 2^{-j} \leq \omega$, this inequality may not be true but, because $|\hat{\chi}_a|^2$ is convex on $\mathbb{R}^+$, we at least have

$$\frac{|\hat{\chi}_a(\omega-\delta 2^{-j})|^2 - 2|\hat{\chi}_a(\omega)|^2 + |\hat{\chi}_a(\omega+\delta 2^{-j})|^2}{2} \geq 0. \tag{13}$$

When $\delta 2^{-j} > \omega$, we use the inequalities

$$1 - 8\sqrt{\frac{|z|}{a}} \leq |\hat{\chi}_a(z)|^2 \leq 1, \qquad \forall z \in \mathbb{R},$$

and we deduce from them

$$\begin{aligned}\frac{|\hat{\chi}_a(\omega-\delta 2^{-j})|^2 - 2|\hat{\chi}_a(\omega)|^2 + |\hat{\chi}_a(\omega+\delta 2^{-j})|^2}{2} &\geq -4\left(\sqrt{\frac{\delta 2^{-j}-\omega}{a}} + \sqrt{\frac{\delta 2^{j}+\omega}{a}}\right)\\ &\geq -8\sqrt{\frac{\delta 2^{-j}}{a}}. \end{aligned} \tag{14}$$

Additionally, when $\delta 2^{-j} > \max(a, \omega)$,

$$\frac{|\hat{\chi}_a(\omega-\delta 2^{-j})|^2 - 2|\hat{\chi}_a(\omega)|^2 + |\hat{\chi}_a(\omega+\delta 2^{-j})|^2}{2} \geq -|\hat{\chi}_a(\omega)|^2. \tag{15}$$

We first consider the case where $\omega \geq \max(C\delta, a)$. Then, we combine Equations (12), (13) and (14), to show that

$$\begin{aligned}&\sum_{j\leq 0} F_j(\omega)\left(\frac{|\hat{\chi}_a(\omega-\delta 2^{-j})|^2 - 2|\hat{\chi}_a(\omega)|^2 + |\hat{\chi}_a(\omega+\delta 2^{-j})|^2}{2}\right)\\ &\qquad\geq \frac{\alpha}{2}\delta^2(|\hat{\chi}_a|^2)''(\omega)\left(\sum_{\delta\leq\delta 2^{-j}\leq\omega/2} 2^{-2j}F_j(\omega)\right) - |\hat{\chi}_a(\omega)|^2\left(\sum_{\omega\leq\delta 2^{-j}} F_j(\omega)\right). \end{aligned} \tag{16}$$

From the Littlewood-Paley condition (Equation (1)), as explained in the proof of Lemma B.2, we can show that there exists $j_0 \in \mathbb{N}$ such that, for any $\omega > 0$,

$$\sum_{2^{-j_0}\omega\leq 2^{-j}\leq 2^{j_0}\omega} F_j(\omega) \geq \frac{c_0}{2}.$$

So if $\delta \leq 2^{-(j_0+1)}$ and $C \geq 2^{j_0}$ (which implies $\omega \geq 2^{j_0}\delta$, as $\omega \geq C\delta$),

$$\sum_{\delta\leq\delta 2^{-j}\leq\omega/2} 2^{-2j}F_j(\omega) \geq \sum_{2^{-j_0}\omega\leq 2^{-j}\leq 2^{j_0}\omega} 2^{-2j}F_j(\omega) \geq (2^{-j_0}\omega)^2\frac{c_0}{2}. \tag{17}$$

From Condition (2), there exists $M > 0$ such that

$$F_j(\omega) \leq M|\omega|^{2(1+\epsilon)}, \qquad \forall \omega \in \mathbb{R}$$

so

$$\begin{aligned}
\sum_{\omega \leq \delta 2^{-j}} F_j(\omega) &\leq M\omega^{2(1+\epsilon)} \sum_{\omega/\delta \leq 2^{-j}} 2^{2j(1+\epsilon)} \\
&\leq M\omega^{2(1+\epsilon)} \frac{1}{1-2^{-2(1+\epsilon)}} \left(\frac{\delta}{\omega}\right)^{2(1+\epsilon)} \\
&\leq \frac{4M}{3}\delta^{2(1+\epsilon)}.
\end{aligned} \tag{18}$$

From Equations (16), (17) and (18),

$$\begin{aligned}
&\sum_{j\leq 0} F_j(\omega) \left(\frac{|\hat{\chi}_a(\omega-\delta 2^{-j})|^2 - 2|\hat{\chi}_a(\omega)|^2 + |\hat{\chi}_a(\omega+\delta 2^{-j})|^2}{2}\right) \\
&\qquad \geq 2^{-2(j_0+1)}\alpha c_0 (\delta\omega)^2 (|\hat{\chi}_a|^2)''(\omega) - \frac{4M}{3}\delta^{2(1+\epsilon)}|\hat{\chi}_a(\omega)|^2 \\
&\qquad = \frac{4M}{3}\delta^2 (|\hat{\chi}_1|^2)''(\omega/a)\left(\frac{\omega}{a}\right)^2 \left(\alpha' - \delta^{2\epsilon}\frac{|\hat{\chi}_1(\omega/a)|^2}{\left(\frac{\omega}{a}\right)^2 (|\hat{\chi}_1|^2)''(\omega/a)}\right),
\end{aligned}$$

where $\alpha' > 0$ is an absolue constant.

It is easy to check that the function $z \to \frac{|\hat{\chi}_1(z)|^2}{z^2(|\hat{\chi}_1|^2)''(z)}$ is uniformly bounded on $[1;+\infty[$, so when δ has been chosen small enough,

$$\begin{aligned}
&\sum_{j\leq 0} F_j(\omega) \left(\frac{|\hat{\chi}_a(\omega-\delta 2^{-j})|^2 - 2|\hat{\chi}_a(\omega)|^2 + |\hat{\chi}_a(\omega+\delta 2^{-j})|^2}{2}\right) \\
&\qquad \geq \frac{2M\alpha'}{3}\delta^2 (|\hat{\chi}_1|^2)''(\omega/a)\left(\frac{\omega}{a}\right)^2 \\
&\qquad = \frac{2M\alpha'}{3}\delta^2\omega^2 (|\hat{\chi}_a|^2)''(\omega).
\end{aligned}$$

This establishes the lemma for the values of ω larger than a. Let us now assume that $C\delta \leq \omega < a$, and prove the same thing.

We combine Equation (12), (13), (14) and (15):

$$\begin{aligned}
&\sum_{j\leq 0} F_j(\omega) \left(\frac{|\hat{\chi}_a(\omega-\delta 2^{-j})|^2 - 2|\hat{\chi}_a(\omega)|^2 + |\hat{\chi}_a(\omega+\delta 2^{-j})|^2}{2}\right) \\
&\qquad \geq \frac{\alpha}{2}\delta^2 (|\hat{\chi}_a|^2)''(\omega)\left(\sum_{\delta \leq \delta 2^{-j} \leq \omega/2} 2^{-2j}F_j(\omega)\right) - 8\sqrt{\frac{\delta}{a}}\left(\sum_{\omega < \delta 2^{-j} \leq a} 2^{-j/2}F_j(\omega)\right) \\
&\qquad\quad - |\hat{\chi}_a(\omega)|^2 \left(\sum_{a < \delta 2^{-j}} F_j(\omega)\right).
\end{aligned} \tag{19}$$

With the same proof as Equation (18), we show that,

$$\sum_{\omega < \delta 2^{-j} \leq a} 2^{-j/2}F_j(\omega) \leq 2M\delta^{3/2+2\epsilon}\sqrt{\omega} \qquad \text{and} \qquad \sum_{a<\delta 2^{-j}} F_j(\omega) \leq \frac{4M}{3}\delta^{2(1+\epsilon)}\left(\frac{\omega}{a}\right)^{2(1+\epsilon)}.$$

Plugging this and Equation (17) in Equation (19), we get

$$\begin{aligned}
&\sum_{j\leq 0} F_j(\omega)\left(\frac{|\hat{\chi}_a(\omega-\delta 2^{-j})|^2-2|\hat{\chi}_a(\omega)|^2+|\hat{\chi}_a(\omega+\delta 2^{-j})|^2}{2}\right)\\
&\qquad\geq 16M\delta^2\left(\alpha'\omega^2(|\hat{\chi}_a|^2)''(\omega)-\delta^{2\epsilon}\left(\sqrt{\frac{\omega}{a}}+\frac{1}{12}|\hat{\chi}_a(\omega)|^2\left(\frac{\omega}{a}\right)^{2(1+\epsilon)}\right)\right)\\
&\qquad= 16M\delta^2\left(\frac{\omega}{a}\right)^2(|\hat{\chi}_1|^2)''(\omega/a)\left(\alpha'-\delta^{2\epsilon}\left(\frac{\sqrt{\frac{\omega}{a}}+\frac{1}{12}|\hat{\chi}_1(\omega/a)|^2\left(\frac{\omega}{a}\right)^{2(1+\epsilon)}}{\left(\frac{\omega}{a}\right)^2(|\hat{\chi}_1|^2)''(\omega/a)}\right)\right).
\end{aligned}$$

With an explicit computation, it is easy to check that the function

$$x\to\frac{\sqrt{x}+\frac{1}{12}|\hat{\chi}_1(x)|^2x^{2(1+\epsilon)}}{x^2(|\hat{\chi}_1|^2)''(x)}$$

is uniformly bounded on $]0;1]$. Consequently, if $\delta>0$ has been chosen small enough, when $\omega<a$,

$$\begin{aligned}
&\sum_{j\leq 0} F_j(\omega)\left(\frac{|\hat{\chi}_a(\omega-\delta 2^{-j})|^2-2|\hat{\chi}_a(\omega)|^2+|\hat{\chi}_a(\omega+\delta 2^{-j})|^2}{2}\right)\\
&\qquad\geq 8\alpha' M\delta^2\left(\frac{\omega}{a}\right)^2(|\hat{\chi}_1|^2)''(\omega/a)\\
&\qquad= 8\alpha' M\delta^2\omega^2(|\hat{\chi}_a|^2)''(\omega),
\end{aligned}$$

which ends the proof of the lemma. □

B. Proof of Lemma C.2

Lemme (Lemma C.2)**.** *For any $\eta>0$, there exists $c>1$ such that*

$$|\hat{\chi}_1(x)|^2+\eta x^2(|\hat{\chi}_1|^2)''(x)\geq|\hat{\chi}_c(x)|^2,\qquad\forall x\in\mathbb{R}-\{0\}.$$

Proof. When $x\to 0$,

$$|\hat{\chi}_1(x)|^2+\eta x^2(|\hat{\chi}_1|^2)''(x)=1-8\sqrt{|x|}+2\eta\sqrt{|x|}+O(x),$$

and

$$|\hat{\chi}_c(x)|^2=1-8\sqrt{\frac{|x|}{c}}+O(x).$$

So the inequality is satisfied in a neighborhood of 0, provided that

$$1<c<\frac{1}{(1-\eta/4)^2},$$

and the size of the neighborhood can be chosen independent of c.

When $|x|\to+\infty$,

$$|\hat{\chi}_1(x)|^2+\eta x^2(|\hat{\chi}_1|^2)''(x)=\frac{1+20\eta}{x^4},$$

and

$$|\hat{\chi}_c(x)|^2=\frac{c^4}{x^4}.$$

So the inequality is also satisfied for $|x|$ large enough (larger than some constant independent from c), provided that

$$c<\sqrt[4]{1+20\eta}.$$

To conclude, we simply need to consider the values of x contained in a compact set that does not include 0. On this compact set, when $c \to 1^+$,

$$|\hat{\chi}_1(x)|^2 + \eta x^2 (|\hat{\chi}_1|^2)''(x) - |\hat{\chi}_c(x)|^2 \to \eta x^2 (|\hat{\chi}_1|^2)''(x),$$

and the convergence is uniform on the compact set. As $x \to \eta x^2 (|\hat{\chi}_1|^2)''(x)$ is lower bounded by a strictly positive constant (on the compact set), we have that

$$|\hat{\chi}_1(x)|^2 + \eta x^2 (|\hat{\chi}_1|^2)''(x) - |\hat{\chi}_c(x)|^2 > 0$$

for any x in the compact set, as soon as $c > 1$ is small enough.

□

C. Proof of Lemma C.3

Lemme (Lemma C.3)**.** *There exists $C' > 0$ such that, for any $a \geq a_0, \omega > 0$,*

$$\sum_{j \leq 0} F_j(\omega) \left(\frac{|\hat{\chi}_a(\omega - \delta 2^{-j})|^2 - 2|\hat{\chi}_a(\omega)|^2 + |\hat{\chi}_a(\omega + \delta 2^{-j})|^2}{2} \right) \geq -C' \omega^{2(1+\epsilon)} \sqrt{\frac{\delta}{a}}.$$

Proof. We reuse Inequality 14 from the proof of Lemma C.1:

$$\frac{|\hat{\chi}_a(\omega - \delta 2^{-j})|^2 - 2|\hat{\chi}_a(\omega)|^2 + |\hat{\chi}_a(\omega + \delta 2^{-j})|^2}{2} \geq -8\sqrt{\frac{\delta 2^{-j}}{a}}.$$

This is true when $\delta 2^{-j} > \omega$, and also when $\delta 2^{-j} \leq \omega$ because, by the convexity of $|\hat{\chi}_a|^2$ on $\mathbb{R}^+$, the left-hand side term is then positive, while the right-hand side one is always negative.

So

$$\sum_{j \leq 0} F_j(\omega) \left(\frac{|\hat{\chi}_a(\omega - \delta 2^{-j})|^2 - 2|\hat{\chi}_a(\omega)|^2 + |\hat{\chi}_a(\omega + \delta 2^{-j})|^2}{2} \right) \geq -8\sqrt{\frac{\delta}{a}} \sum_{j \leq 0} F_j(\omega) 2^{-j/2}.$$

With the same reasoning as for Equation (18), we show that

$$\sum_{j \leq 0} F_j(\omega) 2^{-j/2} \leq 2M\omega^{2(1+\epsilon)},$$

for some constant $M > 0$.

Plugging this into the previous equation yields

$$\sum_{j \leq 0} F_j(\omega) \left(\frac{|\hat{\chi}_a(\omega - \delta 2^{-j})|^2 - 2|\hat{\chi}_a(\omega)|^2 + |\hat{\chi}_a(\omega + \delta 2^{-j})|^2}{2} \right) \geq -16M\omega^{2(1+\epsilon)} \sqrt{\frac{\delta}{a}}.$$

□

APPENDIX D
PROOF OF COROLLARY III.3

Corollaire (Corollary III.3)**.** *There exists $\eta \in]0;1[$, and $A > 0$ such that the following property is true: for any $M > 0$ and any $f \in L^2(\mathbb{R})$ such that*

$$\hat{f}(\omega) = 0, \qquad \forall \omega \notin [-M; M],$$

for any $n \in \mathbb{N}$,

$$\sum_{|p|=n} ||S_0[p]f||_2^2 \leq \eta^{\max(0, n - A\log M)} ||f||_2^2.$$

Proof. We define $a_0 > 0, c > 1$ as in Theorem III.2. Let us define

$$g : k \in \mathbb{R}_+^* \to 1 - \frac{1}{\left|1 + \sqrt{\frac{1}{a_0 c^k}}\right|^8}.$$

This function is upper bounded by 1 on $\mathbb{R}$, and decays exponentially when $k \to +\infty$, because

$$g(k) \sim 8\sqrt{\frac{1}{a_0 c^k}} \quad \text{when } k \to +\infty.$$

So there exists $\eta \in]0;1[$ such that, for any $k \in \mathbb{R}_+^*$,

$$g(k) \leq \eta^k.$$

We set $A = \frac{1}{\log c}$. Then for any f such that the support of $\hat{f}$ is included in $[-M; M]$, for any $n \geq A \log M$, from Theorem III.2,

$$\begin{aligned}
\sum_{|p|\geq n} ||S_0[p]f||_2^2 &\leq \int_{\mathbb{R}} |\hat{f}(\omega)|^2 (1 - |\hat{\chi}_{c^n a_0}(\omega)|^2) d\omega \\
&= \int_{-M}^{M} |\hat{f}(\omega)|^2 (1 - |\hat{\chi}_{c^n a_0}(\omega)|^2) d\omega \\
&\leq (1 - |\hat{\chi}_{c^n a_0}(M)|^2) \left(\int_{-M}^{M} |\hat{f}(\omega)|^2 d\omega \right) \\
&= (1 - |\hat{\chi}_{a_0 c^{n - A\log M}}(M / c^{A \log M})|^2) ||f||_2^2 \\
&= (1 - |\hat{\chi}_{a_0 c^{n - A\log M}}(1)|^2) ||f||_2^2 \\
&= g(n - A \log M) ||f||_2^2 \\
&\leq \eta^{n - A\log M} ||f||_2^2.
\end{aligned}$$

For $n < A \log M$, we simply have

$$\sum_{|p|\geq n} ||S_0[p]f||_2^2 \leq ||f||_2^2 = \eta^{\max(0, n - A\log M)} ||f||_2^2,$$

so the desired inequality holds.

$\square$

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