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# Design of integral controllers for nonlinear systems governed by scalar hyperbolic partial differential equations

Ngoc-Tu Trinh, Vincent Andrieu, and Cheng-Zhong Xu\*

**Abstract**—The paper deals with the control and regulation by integral controllers for the nonlinear systems governed by scalar quasi-linear hyperbolic partial differential equations. Both the control input and the measured output are located on the boundary. The closed-loop stabilization of the linearized model with the designed integral controller is proved first by using the method of spectral analysis and then by the Lyapunov direct method. Based on the elaborated Lyapunov function we prove local exponential stability of the nonlinear closed-loop system with the same controller. The output regulation to the set-point with zero static error by the integral controller is shown upon the nonlinear system. Numerical simulations by the Preissmann scheme are carried out to validate the robustness performance of the designed controller to face unknown constant disturbances.

**Index Terms**—Boundary control, PI controller, hyperbolic system, Lyapunov function, partial differential equations, exponential stability, numerical simulation.

## I. INTRODUCTION

The paper is concerned with the control of nonlinear systems governed by scalar quasi-linear hyperbolic partial differential equations. This type of systems appear in many industrial applications and in study of traffic flow. For instance, quasi-linear hyperbolic equations include Burgers equations [8] which are employed in modeling turbulent fluid motion. Another example is given by the equation employed by Lighthill-Whitham in [21] to describe traffic flow on long crowded roads. Finally scalar conservation laws can also be regarded as a particular simpler case of quasi-linear hyperbolic systems under some regularity assumptions (see for instance [2], [6]).

The problem of controlling systems governed by hyperbolic partial differential equations (PDE) with both inputs and outputs on the boundary has attracted a considerable amount of studies [14], [5], [13]. Interested readers can find a nice literature review on the fields in the section 2 of [10] and in [2]. Many available results have established appropriate boundary conditions to ensure asymptotic or exponential stability of the equilibrium state as in [20], [15], [9], [10], [18] and [31] or

[32] (most of them in  $C^1$  topology but also in the  $H^2$  topology, see [9]).

In these papers the boundary conditions are given as a function of the output. In other words, it is the static control law that has been investigated. As it is well-known, one of the drawbacks of this type of controllers is the lack of regulation effect in the presence of constant perturbations. This motivates the introduction of integral actions in the control law by using a dynamic output feedback.

In this paper, our objective is to design an integral controller to guarantee asymptotic stability of the equilibrium of the nonlinear closed-loop system and the output regulation to a given set-point. The idea of using dynamic output feedback control for infinite dimensional systems is inspired by the works of [25], [26] and [30], with implementation of integral action for infinite-dimensional linear systems. These results have been further developed in recent publications for linearized hyperbolic systems, by using Lyapunov techniques in [33] and [12], by using Laplace transformation in frequency domain in [1] and by using a semi-group approach in [33] and [29]. The backstepping method has been exploited in [19] to elaborate PI (proportional and integral) controllers in the same context. For nonlinear systems, the work of Tamasoiu [27] considers a PI controller with damping for a scalar hyperbolic system. However it loses the regulation effect because of the damping required in the PI controller. It is also interesting to note that asymptotic stabilization of entropy solutions to scalar conservation laws has been recently studied by Perrollaz in [24] and by Balandin *et al.* in [3]. In particular they have considered the stabilization problem of weak entropy solutions by boundary control and internal control around a constant equilibrium state for a scalar 1D conservation law with strictly convex flux. In [24] an internal state feedback control law has been designed to asymptotically stabilize the entropy solutions around a constant equilibrium in the topologies  $L^1$  and  $L^\infty$ . A stabilizing nonlocal boundary control law (depending on time and the whole initial data) has been proposed in [3] to get asymptotic stability of the constant equilibrium in the  $L^2$  topology.

In our work, we consider a 1D scalar conservation law with strictly increasing flux. We are interested in the classical solutions of the system around an equilibrium state. We propose a boundary integral output feedback controller to asymptotically stabilize the system. The local stability and the regulation effect for the nonlinear closed-loop system are proven in the  $H^2$  topology by using Lyapunov techniques. The

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contribution of our paper is threefold: (i) inspired from [9] we construct a new Lyapunov functional to prove exponential stabilization of the closed-loop system with the designed integral controller; (ii) we provide a mathematical proof of the regulation effect with zero static error offered by the integral action of the dynamical control law; (iii) the Preissmann scheme is implemented to realize numerical simulations for the nonlinear closed-loop PDE system.

The paper is organized as follows. Section II is devoted to the statement of the regulation problem and the announcement of the main result. Section III presents the proof of the main result. More precisely, in Section III-A asymptotic stabilization of the linearized system is considered by proposing a Lyapunov functional; In Section III-B the stabilization problem is solved for the nonlinear system by extending the proposed Lyapunov functional. Numerical simulations to validating the theoretical results are implemented in Section IV by applying the Preissmann numerical scheme. Finally Section V is devoted to our conclusions.

## II. STATEMENT OF THE PROBLEM AND MAIN RESULT

In this paper, we consider a 1D quasi-linear hyperbolic system of the form :

$$\frac{\partial \psi}{\partial t}(x, t) + \tilde{F}(\psi(x, t)) \frac{\partial \psi}{\partial x}(x, t) = 0, \quad x \in (0, L), \quad t \in \mathbb{R}_+, \quad (1)$$

where  $L$  is a positive constant,  $\psi : (0, L) \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is the state in  $C([0, \infty), H^2(0, L))$ , and  $\tilde{F} : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^2$  function such that  $\tilde{F}(\sigma) > 0 \forall \sigma \in \mathbb{R}$ . The initial condition is given by  $\psi(\cdot, 0) = \psi_0 \in H^2(0, L)$ . Notice that  $H^2(0, L)$  is the usual Sobolev space defined by

$$H^2(0, L) = \{f \in L^2(0, L) \mid f', f'' \in L^2(0, L)\}$$

where  $L^2(0, L)$  denotes the usual Hilbert space of square summable functions on the open set  $(0, L)$ . The Sobolev space  $H^2(0, L)$  is normed by

$$\|f\|_{H^2}^2 = \int_0^L (|f(x)|^2 + |f'(x)|^2 + |f''(x)|^2) dx$$

$$\forall f \in H^2(0, L).$$

We consider the control  $u$  on the boundary  $x = 0$ , i.e.,

$$\psi(0, t) = u(t), \quad t \in \mathbb{R}_+.$$

The output we wish to regulate is also located on the boundary and eventually corrupted by an additive unknown disturbance, i.e.

$$y(t) = \psi(L, t) + w_o, \quad t \in \mathbb{R}_+,$$

where  $w_o \in \mathbb{R}$  is an unknown constant. Our control objective is to design a dynamic output feedback control law in order to achieve asymptotic stabilization of the closed loop system and to ensure that the output  $y(t)$  converges to a desired set-point  $y_r \in \mathbb{R}$ , as  $t \rightarrow \infty$ .

In our study, the control action  $u(t)$  has the structure of an integral controller. We assume that an unknown constant

disturbance may corrupt the control. Hence we write the control law as follows

$$u(t) = -k_I \zeta(t) + w_c, \quad \dot{\zeta}(t) = y(t) - y_r$$

where  $w_c \in \mathbb{R}$  is an unknown constant and  $k_I$  is a positive constant called tuning parameter.

To summarize, the closed-loop system with disturbances is governed by the following PDE :

$$\begin{cases} \frac{\partial \psi}{\partial t}(x, t) = -\tilde{F}(\psi(x, t)) \frac{\partial \psi}{\partial x}(x, t) \\ \dot{\zeta}(t) = \psi(L, t) - y_r + w_o \\ \psi(0, t) = -k_I \zeta(t) + w_c \\ \psi(x, 0) = \psi_0(x), \quad \zeta(0) = \zeta_0. \end{cases} \quad (2)$$

We are studying a nonlinear infinite-dimensional system controlled by an integral controller faced with unknown constant disturbances on the control and the output. The purpose of the paper is to find sufficient conditions on the control parameter  $k_I > 0$  such that the three objectives are realized : (a) the closed-loop system (2) is well posed; (b) asymptotic stability of the closed-loop system is guaranteed; and (c) the regulation property holds

$$\lim_{t \rightarrow \infty} |y(t) - y_r| = 0. \quad (3)$$

As only the classical solutions are considered, in the following we restrict ourselves to study the solutions from the initial data  $(\psi_0, \zeta_0)$  in  $H^2(0, L) \times \mathbb{R}$  which satisfy the  $C^0$  and  $C^1$  compatibility conditions :

$$\begin{cases} \psi_0(0) = -k_I \zeta_0 + w_c \\ \tilde{F}(\psi_0(0)) \psi_0'(0) = k_I (\psi_0(L) - y_r + w_o). \end{cases} \quad (4)$$

To be simple the initial data with the compatibility condition satisfied up to the required order are called compatible initial data throughout the paper. From now on, the state space  $X$  for (2) is the Hilbert space  $X = H^2(0, L) \times \mathbb{R}$  equipped with the norm  $\|(f, z)\|_X^2 = \|f\|_{H^2}^2 + z^2$ . Note that due to the constants  $w_0$ ,  $w_c$  and  $y_r$ ,  $(\psi, \zeta) = (0, 0)$  is not a steady state of the closed-loop system. In fact the equilibrium denoted  $(\psi_\infty, \zeta_\infty)$  is defined as follows

$$\psi_\infty = y_r - w_o, \quad \zeta_\infty = k_I^{-1}(w_o + w_c - y_r). \quad (5)$$

Let  $B_X((f, z), \delta)$  denote the open ball in  $X$  centered at  $(f, z)$  with radius  $\delta > 0$ , i.e.,

$$B_X((f, z), \delta) = \{(\psi, \zeta) \in X \mid \|(\psi, \zeta) - (f, z)\|_X < \delta\}.$$

Then the main result of the paper is stated as follows.

*Theorem 1:* There exist positive real constants  $k_I^*$  and  $\delta$  such that, for each  $k_I \in (0, k_I^*)$ , and for every  $(y_r, w_o, w_c) \in \mathbb{R}^3$  and every compatible  $(\psi_0, \zeta_0) \in B_X((\psi_\infty, \zeta_\infty), \delta)$ , the following assertions hold true :

- 1) The closed-loop system (2) has a unique solution  $(\psi, \zeta) \in C([0, \infty), X)$ ;
- 2) The solution of the closed-loop system (2) converges exponentially to the equilibrium state  $(\psi_\infty, \zeta_\infty)$  in the state space  $X$  as  $t \rightarrow \infty$ , and the disturbed output is regulated to the desired set-point  $y_r$ , i.e.,

$$\lim_{t \rightarrow \infty} |y(t) - y_r| = 0.$$

3) There exist real constants  $M > 0$  and  $\omega > 0$  such that

$$\begin{aligned} & \|(\psi(\cdot, t) - \psi_\infty, \zeta(t) - \zeta_\infty)\|_X \leq \\ & Me^{-\omega t} \|(\psi_0 - \psi_\infty, \zeta_0 - \zeta_\infty)\|_X \quad \forall t \geq 0. \end{aligned}$$

**Remark :**

The equilibrium state is some constant state determined by  $y_r$ ,  $w_o$  and  $w_c$ . Though it is unknown *a priori*, the state of the closed-loop system is bounded because of the asymptotic stability property of the equilibrium. Moreover the output is always regulated to the set-point independently of the unknown disturbances. It is the virtue of the integral controller that allows to suppress the static error and hence achieves output regulation. A more general situation is explained in the paper [33]. ◀

**Remark :**

The solution considered in Theorem 1 is a classical solution in the sense of Li and Yu [20]. However the topology used here is the topology induced by the Hilbert  $H^2$  norm instead of the  $C^1$  norm. Moreover, we have only local exponential stability of the equilibrium of the closed-loop system. For initial compatible data outside some neighborhood of the equilibrium, the classical solution to the Cauchy problem (2) may not be extended on the whole positive time axis. ◀

The proof of Theorem 1 is given in the next section. Our proof is based on the construction of an appropriate Lyapunov functional. The direct Lyapunov approach allows us to consider the tuning parameter  $k_I$  relatively bigger than the classical method (see [11], [25] and [30]). The proposed upper bound on  $k_I$  is computed directly from the given system. This may be the advantage of our approach with respect to that of the literature [25], [30], [33].

### III. PROOF OF THE MAIN RESULT

To prove Theorem 1 we consider the following transformation :

$$\phi(x, t) = \psi(x, t) - \psi_\infty, \quad \xi(t) = \zeta(t) - \zeta_\infty \quad (6)$$

where  $\psi_\infty$  and  $\zeta_\infty$  are defined in (5). Then we obtain a perturbation free nonlinear closed-loop system as follows :

$$\phi_t(x, t) = -F(\phi(x, t))\phi_x(x, t) \quad (7)$$

$$\dot{\xi}(t) = \phi(L, t) \quad (8)$$

$$\phi(0, t) = -k_I \xi(t) \quad (9)$$

$$\phi(x, 0) = \phi_0(x) = \psi_0(x) - \psi_\infty \quad (10)$$

$$\xi(0) = \xi_0 = \zeta_0 - \zeta_\infty \quad (11)$$

where  $\phi_t(x, t)$  denotes the time partial derivative of  $\phi(x, t)$ , and we have defined  $F(\phi) = \tilde{F}(\phi + \psi_\infty)$ . In the new coordinates, the output is written as

$$y(t) = \phi(L, t) + y_r.$$

Hence the output regulation to  $y_r$  is achieved if

$$\lim_{t \rightarrow \infty} |\phi(L, t)| = 0.$$

To guarantee the output regulation of the disturbed nonlinear system (2), we design the integral controller so as to ensure local asymptotic stabilization to the origin of the equivalent system (7)-(11).

In the following, the integral stabilization problem of the equivalent system is considered first for the linearized case in Section III-A and then for the nonlinear case in Section III-B. Finally the complete proof of Theorem 1 is presented in Section III-C.

#### A. Linear hyperbolic system

The purpose of this Section is to study stability property of the origin for the nonlinear hyperbolic system with an integral controller on the boundary as described in (7)-(11). To begin with, we consider the particular case where the system is linear, i.e.,  $F$  does not depend on  $\phi$ . This is the case if for instance the considered system is obtained by the tangent linearization of the nonlinear system around the equilibrium state. In this subsection, we consider the following linear system :

$$\phi_t = -r\phi_x, \quad r > 0 \quad (12)$$

$$\dot{\xi} = \phi(L, t), \quad \phi(0, t) = -k_I \xi(t) \quad (13)$$

$$\phi(x, 0) = \phi_0(x), \quad \xi(0) = \xi_0. \quad (14)$$

To the system (12)-(14) is associated the state space  $Z$  which is the Hilbert space  $Z = L^2(0, L) \times \mathbb{R}$  equipped with the scalar product

$$\begin{aligned} \langle (\phi_a, \xi_a), (\phi_b, \xi_b) \rangle_Z &= \int_0^L \phi_a(x)\phi_b(x)dx + \xi_a\xi_b \\ &\quad \forall ((\phi_a, \xi_a), (\phi_b, \xi_b)) \in Z^2. \end{aligned}$$

and we denote by  $\|\cdot\|_Z$  its associated norm. The first stability result is obtained by employing the Laplace transform approach.

*Proposition 1:* The closed-loop linear system (12)-(14) is exponentially stable in  $Z$  w.r.t.  $L^2$  norm if and only if  $k_I \in \left(0, \frac{r\pi}{2L}\right)$ .

The proof of this result can be obtained from [4, p.444, Chapter 13] or from [16, Appendix Theorem A.5]. For the reader's convenience a simple proof is given in Appendix A.

By frequency-domain analysis it is possible to establish some necessary and sufficient conditions on the parameter  $k_I$  for asymptotic stability of the equilibrium to the linear closed-loop system (12)-(14). However the approach is no longer applicable when dealing with a general nonlinear system. This is the reason why we introduce a Lyapunov functional for the linear system which allows us to tackle the nonlinear hyperbolic system in the following section.

The Lyapunov functional candidate  $V : Z \rightarrow \mathbb{R}$  has the following form :

$$V(\phi, \xi) = \int_0^L \left[ \phi^2(x)e^{-\mu x} + q_1 \xi \phi(x)e^{-\frac{\mu x}{2}} \right] dx + q_2 \xi^2 \quad (15)$$

where  $\mu > 0$  and  $q_i > 0 \quad \forall i = 1, 2$ . Consider the function  $\Pi : [0, 2] \rightarrow \mathbb{R}$  such that  $\Pi(z) = \sqrt{z(2-z)}e^{-z/2}$ . We have

$\Pi(2 - \sqrt{2}) \simeq 0.3395$  that is the maximum value of  $\Pi(z)$  in  $[0, 2]$ .

Given  $T > 0$  and a function  $\phi : (0, L) \times (0, T) \rightarrow \mathbb{R}$ , we use the notation  $\phi(t) := \phi(\cdot, t)$  when there is no ambiguity. Assume that the initial condition is smooth enough so that the solution of (12)-(14) is continuously differentiable with respect to time  $t$  and space  $x$ . Then, by differentiating  $V(\phi(t), \xi(t))$  with time along the solution and by using integration by parts we get

$$\begin{aligned} \dot{V}(\phi(t), \xi(t)) &= -re^{-\mu L} \phi^2(L, t) - k_I r(q_1 - k_I) \xi^2(t) \\ &\quad - \mu r \int_0^L e^{-\mu x} \phi^2(x, t) dx \\ &\quad + \left(2q_2 - q_1 r e^{-\frac{\mu L}{2}}\right) \xi(t) \phi(L, t) \\ &\quad - \frac{\mu q_1 r}{2} \xi(t) \int_0^L e^{-\frac{\mu x}{2}} \phi(x, t) dx \\ &\quad + q_1 \phi(L, t) \int_0^L e^{-\frac{\mu x}{2}} \phi(x, t) dx. \end{aligned} \quad (16)$$

*Lemma 1:* Let  $k_I^* = \left(\frac{r}{2L}\right) \Pi(2 - \sqrt{2})$ . Take  $k_I \in (0, k_I^*)$  and  $\mu \in (0, (2 - \sqrt{2})/L]$  such that  $\left(\frac{r}{2L}\right) \Pi(\mu L) > k_I$ . Let  $q_1 = 2k_I$  and let  $q_2 = rk_I e^{-\mu L/2}$ . Then there exist positive constants  $M \geq 1$  and  $\alpha > 0$  such that

$$M^{-1} \|(\phi, \xi)\|_Z^2 \leq V(\phi, \xi) \leq M \|(\phi, \xi)\|_Z^2 \quad \forall (\phi, \xi) \in Z, \quad (17)$$

and for every smooth compatible  $(\phi_0, \xi_0) \in Z$

$$\dot{V}(\phi(t), \xi(t)) \leq -\alpha V(\phi(t), \xi(t)) - \left(\frac{re^{-\mu L}}{2}\right) \phi^2(L, t). \quad (18)$$

**Proof :** Rewrite  $V(\phi, \xi)$  as follows

$$V(\phi, \xi) = \int_0^L \begin{bmatrix} \phi(x) e^{-\mu x/2} \\ \frac{\xi}{\sqrt{L}} \end{bmatrix}^\top P \begin{bmatrix} \phi(x) e^{-\mu x/2} \\ \frac{\xi}{\sqrt{L}} \end{bmatrix} dx$$

where

$$P = \begin{bmatrix} 1 & \frac{\sqrt{L}q_1}{2} \\ \frac{\sqrt{L}q_1}{2} & q_2 \end{bmatrix}.$$

We claim that the matrix  $P$  is positive definite. Indeed, we have

$\det(P) =$

$$Lk_I \left\{ \frac{r\Pi(\mu L)}{2L} - k_I + \frac{r}{2L} e^{-\mu L/2} \left(2 - \sqrt{\mu L(2 - \mu L)}\right) \right\}.$$

Since  $\left(\frac{r}{2L}\right) \Pi(\mu L) > k_I$  and  $\mu L < 2$ , it is easy to see that  $\det(P) \geq \frac{rk_I}{2} e^{-\frac{\mu L}{2}}$ . Hence there is some real constant  $M \geq 1$  such that the inequality (17) holds.

By substituting the given  $q_1$  and  $q_2$  into (16) we have the following

$$\begin{aligned} \dot{V}(\phi(t), \xi(t)) &= -re^{-\mu L} \phi^2(L, t) - \mu r \int_0^L e^{-\mu x} \phi^2(x, t) dx \\ &\quad - k_I^2 r \xi^2(t) - \mu r k_I \xi(t) \int_0^L e^{-\frac{\mu x}{2}} \phi(x, t) dx \\ &\quad + 2k_I \phi(L, t) \int_0^L e^{-\frac{\mu x}{2}} \phi(x, t) dx. \end{aligned} \quad (19)$$

By using the Young and Cauchy-Schwarz inequalities we get

$$\begin{aligned} &2k_I \phi(L, t) \int_0^L e^{-\frac{\mu x}{2}} \phi(x, t) dx \\ &\leq \left(\frac{re^{-\mu L}}{2}\right) \phi^2(L, t) + \left(\frac{2Lk_I^2 e^{\mu L}}{r}\right) \int_0^L e^{-\mu x} \phi^2(x, t) dx \end{aligned} \quad (20)$$

and

$$\begin{aligned} &\left| \mu r k_I \xi(t) \int_0^L e^{-\frac{\mu x}{2}} \phi(x, t) dx \right| \\ &\leq \left(\frac{rk_I^2}{2}\right) \xi^2(t) + \left(\frac{r\mu^2 L}{2}\right) \int_0^L e^{-\mu x} \phi^2(x, t) dx. \end{aligned} \quad (21)$$

Substituting (20) and (21) into (19) leads us to the following inequality

$$\begin{aligned} \dot{V}(\phi(t), \xi(t)) &\leq -\left(\frac{re^{-\mu L}}{2}\right) \phi^2(L, t) - \left(\frac{k_I^2 r}{2}\right) \xi^2(t) \\ &\quad - \left(\frac{r}{2L} \Pi(\mu L) + k_I\right) \left(\frac{r}{2L} \Pi(\mu L) - k_I\right) J_{\phi, \xi} \end{aligned} \quad (22)$$

where

$$J_{\phi, \xi} = \left(\frac{2L}{r}\right) e^{\mu L} \int_0^L e^{-\mu x} \phi^2(x, t) dx.$$

By the choice of  $\mu$ , we have  $\frac{r}{2L} \Pi(\mu L) - k_I > 0$ . It follows from (22) that there exists a positive real number  $M_1$  such that

$$\begin{aligned} \dot{V}(\phi(t), \xi(t)) &\leq -M_1 \left(\xi^2(t) + \int_0^L e^{-\mu x} \phi^2(x, t) dx\right) \\ &\quad - \left(\frac{re^{-\mu L}}{2}\right) \phi^2(L, t). \end{aligned} \quad (23)$$

The required inequality (18) is true by (23) and (17).  $\square$

**Remark :**

It can be noticed that the set of parameter  $k_I$  which makes the Lyapunov functional decreasing along solutions is smaller than the set of parameter obtained from Proposition 1. Hence, in the linear context our Lyapunov approach is conservative. However the Lyapunov functional allows us to deal with nonlinear systems as it will be shown in the next Section.  $\blacktriangleleft$

## B. Nonlinear system

In this section, we consider the problem for the nonlinear system (7)-(11) with  $F(0) = r > 0$ . By the designed integral controller the nonlinear closed-loop system (7)-(11) is written as follows

$$\begin{aligned} \phi_t + F(\phi) \phi_x &= 0 \\ \dot{\xi} &= \phi(L, t) \\ \phi(0, t) &= -k_I \xi \\ \phi(x, 0) &= \phi_0(x), \quad \xi(0) = \xi_0. \end{aligned} \quad (24)$$

Let us set :

$$s(x, t) = \phi_x(x, t), \quad p(x, t) = \phi_{xx}(x, t).$$

By successive derivatives and compatibility conditions we find that the dynamics of  $s(x, t)$  and  $p(x, t)$  are governed by the following PDE, respectively,

$$\begin{aligned} s_t + F(\phi)s_x &= -F'(\phi)s^2 \\ F(\phi(0, t))s(0, t) &= k_I\phi(L, t) \\ s(x, 0) &= \phi'_0(x) \end{aligned} \quad (25)$$

and

$$\begin{aligned} p_t + F(\phi)p_x &= -3F'(\phi)sp - F''(\phi)s^3 \\ F^2(\phi(0, t))p(0, t) &= k_I F(\phi(L, t))s(L, t) \\ &\quad - 2k_I F'(\phi(0, t))\phi(L, t)s(0, t) \\ p(x, 0) &= \phi''_0(x). \end{aligned} \quad (26)$$

Now we use the idea presented in [9] to extend the Lyapunov functional from the linear system (in  $L^2$  norm) to the nonlinear system (in  $H^2$  norm). Therefore local asymptotic stability of the equilibrium state and the set-point output regulation will be proved for the nonlinear closed-loop system (24). To do that, we consider the Lyapunov functional candidate  $S : X \rightarrow \mathbb{R}_+$  such that

$$S(\phi, \xi) = V(\phi, \xi) + q_3 V_1(\phi_x) + q_4 V_1(\phi_{xx}) \quad (27)$$

where  $V(\phi, \xi)$  is defined in (15) with  $q_1$  and  $q_2$  given in Lemma 1 and

$$V_1(\phi_x) = \int_0^L e^{-\mu x} \phi_x^2(x) dx \quad (28)$$

with the real positive constants  $q_3$  and  $q_4$  to be determined later.

For the moment we assume that all the required regularity is satisfied and carry out formal computations.

**Lemma 2:** The time derivative of  $V(\phi(t), \xi(t))$  along each regular trajectory of the nonlinear system (24) is written as follows

$$\begin{aligned} \dot{V}(\phi(t), \xi(t)) &= -re^{-\mu L} \phi^2(L, t) - k_I^2 r \xi^2(t) - \mu r \int_0^L e^{-\mu x} \phi^2(x, t) dx \\ &\quad - \mu r k_I \xi(t) \int_0^L e^{-\frac{\mu x}{2}} \phi(x, t) dx + 2k_I \phi(L, t) \int_0^L e^{-\frac{\mu x}{2}} \phi(x, t) dx \\ &\quad - \phi^3(L, t) F_1(\phi(L, t)) e^{-\mu L} + \phi^3(0, t) F_1(\phi(0, t)) \\ &\quad + \int_0^L e^{-\mu x} [F'(0) + \phi(x, t) F_2(\phi(x, t))] \phi_x(x, t) \phi^2(x, t) dx \\ &\quad - \mu \int_0^L e^{-\mu x} F_1(\phi(x, t)) \phi^3(x, t) dx \\ &\quad - 2k_I \xi(t) \int_0^L e^{-\mu x/2} F_1(\phi(x, t)) \phi(x, t) \phi_x(x, t) dx \end{aligned} \quad (29)$$

where

$$\begin{aligned} F(z) &= F(0) + F_1(z)z \\ F'(z) &= F'(0) + F_2(z)z \end{aligned} \quad (30)$$

with  $F_1(z) = \int_0^1 F'(\lambda z) d\lambda$  and  $F_2(z) = \int_0^1 F''(\lambda z) d\lambda$ .

**Proof :** By differentiating  $V(\phi(t), \xi(t))$  along each regular trajectory of (24) the following identity holds true

$$\dot{V}(\phi(t), \xi(t)) = - \int_0^L 2e^{-\mu x} \phi(x, t) F(\phi(x, t)) \phi_x(x, t) dx$$

$$\begin{aligned} &- q_1 \int_0^L e^{-\mu x/2} F(\phi(x, t)) \phi_x(x, t) \xi(t) dx \\ &+ q_1 \int_0^L e^{-\mu x/2} \phi_x(x, t) dx \phi(L, t) + 2q_2 \xi(t) \phi(L, t). \end{aligned}$$

By integration by parts and by using the boundary condition (24) and the parameters  $q_1$  and  $q_2$  given in Lemma 1 as well as the relations (30) we prove the required identity (29).  $\square$

Similarly we may prove the following lemmas.

**Lemma 3:** With the same notations as in Lemma 2, the time derivative of  $V_1(\phi_x(t))$  along every regular trajectory of the nonlinear system (24)-(25) is written as follows

$$\begin{aligned} \dot{V}_1(\phi_x(t)) &= -re^{-\mu L} s^2(L, t) + r^{-1} k_I^2 \phi^2(L, t) \\ &\quad - r\mu \int_0^L e^{-\mu x} s^2(x, t) dx - k_I^2 F_3(\phi(0, t)) \phi(0, t) \phi^2(L, t) \\ &\quad - e^{-\mu L} F_1(\phi(L, t)) \phi(L, t) s^2(L, t) \\ &\quad - \int_0^L [(F(\phi(x, t)))_x + \mu F_1(\phi(x, t)) \phi(x, t)] e^{-\mu x} s^2(x, t) dx \end{aligned} \quad (31)$$

where  $F_3(z) = \int_0^1 \frac{F'(\lambda z)}{F^2(\lambda z)} d\lambda$ .

**Lemma 4:** With the same notations as in Lemma 2, the time derivative of  $V_1(\phi_{xx}(t))$  along each regular trajectory of the nonlinear system (24)-(26) is written as follows

$$\begin{aligned} \dot{V}_1(\phi_{xx}(t)) &= -e^{-\mu L} F(\phi(L, t)) p^2(L, t) + \frac{k_I^2 F^2(\phi(L, t))}{F^3(\phi(0, t))} s^2(L, t) \\ &\quad - \mu \int_0^L e^{-\mu x} F(\phi(x, t)) p^2(x, t) dx \\ &\quad + \frac{4k_I^2 (F'(\phi(0, t)))^2}{F^3(\phi(0, t))} \phi^2(L, t) s^2(0, t) \\ &\quad - \frac{4k_I^3 F(\phi(L, t)) F'(\phi(0, t))}{F^4(\phi(0, t))} s(L, t) \phi^2(L, t) \\ &\quad - 5 \int_0^L e^{-\mu x} F'(\phi(x, t)) s(x, t) p^2(x, t) dx \\ &\quad - 2 \int_0^L e^{-\mu x} F''(\phi(x, t)) s^3(x, t) p(x, t) dx. \end{aligned} \quad (32)$$

Let  $T > 0$ . For each function  $(\phi, \xi) \in C([0, T]; C^1[0, L] \times \mathbb{R})$  we define

$$\begin{aligned} \|(\phi, \phi_x, \xi)\|_{T, \infty} &= \sup_{t \in [0, T]} |\xi(t)| + \\ &+ \sup_{\substack{x \in [0, L] \\ t \in [0, T]}} |\phi(x, t)| + \sup_{x \in [0, L]} |\phi_x(x, t)|. \end{aligned}$$

By combining results of Lemma 1-4 the following theorem is obtained.

**Theorem 2:** Let the parameters  $k_I$ ,  $\mu$ ,  $q_1$  and  $q_2$  be determined as in Lemma 1. Then there are positive real constants  $q_3$ ,  $q_4$ ,  $\delta$  and  $\beta$  such that, for each function  $(\phi, \xi) \in C([0, T]; C^3[0, L] \times \mathbb{R}) \cap C^1([0, T]; C^2[0, L] \times \mathbb{R})$  satisfying

the PDE (24)-(26) and the condition  $\|(\phi, \phi_x, \xi)\|_{T, \infty} < \delta$ , the following differential inequality holds true

$$\dot{S}(\phi(t), \xi(t)) \leq -\beta S(\phi(t), \xi(t)) \quad \forall t \in [0, T]. \quad (33)$$

Moreover there exists a positive constant  $K \geq 1$  such that

$$K^{-1}\|(\phi, \xi)\|_X^2 \leq S(\phi, \xi) \leq K\|(\phi, \xi)\|_X^2 \quad \forall (\phi, \xi) \in X. \quad (34)$$

**Proof :** Without loss of generality we assume that  $\delta \leq 1$ . For the sake of simplicity we write  $C_T = \|(\phi, \phi_x, \xi)\|_{T, \infty}$ . By Lemma 2, Lemma 1 and the Cauchy-Schwarz inequality there exists a positive constant  $K_1 > 0$  such that

$$\begin{aligned} \dot{V}(\phi(t), \xi(t)) &\leq -\alpha V(\phi(t), \xi(t)) - (r/2)e^{-\mu L} \phi^2(L, t) + \\ &+ K_1 C_T \left[ \int_0^L e^{-\mu x} \phi^2(x, t) dx + \xi^2(t) + \phi^2(L, t) \right]. \end{aligned} \quad (35)$$

Similarly, by Lemma 3 there exists a positive constant  $K_2 > 0$  such that

$$\begin{aligned} \dot{V}_1(\phi_x(t)) &\leq -(re^{-\mu L} - K_2 C_T) s^2(L, t) \\ &+ r^{-1} k_I^2 \phi^2(L, t) - r\mu \int_0^L e^{-\mu x} s^2(x, t) dx \\ &+ K_2 C_T \left( \phi^2(L, t) + \int_0^L e^{-\mu x} s^2(x, t) dx \right). \end{aligned} \quad (36)$$

Similarly, by Lemma 4 there exists a positive constant  $K_3 > 0$  such that

$$\begin{aligned} \dot{V}_1(\phi_{xx}(t)) &\leq \\ &- (re^{-\mu L} - K_3 C_T) p^2(L, t) + (r^{-2} k_I^2 + K_3 C_T) s^2(L, t) \\ &+ K_3 C_T \phi^2(L, t) + K_3 C_T \int_0^L e^{-\mu x} s^2(x, t) dx \\ &- (r\mu - K_3 C_T) \int_0^L e^{-\mu x} p^2(x, t) dx. \end{aligned} \quad (37)$$

As  $C_T$  can be made as small as we like with  $\delta$ , adding the inequalities (35)-(37) and taking  $\delta$ ,  $q_3$  and  $q_4$  sufficiently small lead us directly to the following differential relation

$$\begin{aligned} \dot{S}(\phi(t), \xi(t)) &\leq -\frac{\alpha}{2} V(\phi(t), \xi(t)) \\ &- \frac{q_3 r \mu}{2} \int_0^L e^{-\mu x} s^2(x, t) dx \\ &- \frac{q_4 r \mu}{2} \int_0^L e^{-\mu x} p^2(x, t) dx. \end{aligned} \quad (38)$$

Therefore the theorem is proved by using (17), (27) and (38).  $\square$

### C. Proof of Theorem 1

With Theorem 2, we are now ready to prove the main result of the paper.

**Proof of Theorem 1 :** We first prove the local existence of a unique solution to the closed-loop system (7)-(11) for each

compatible initial state  $(\phi_0, \xi_0)$  in  $H^2(0, L) \times \mathbb{R}$ . The closed-loop control system (7)-(11) is governed by the following PDE coupled with an ODE through the boundary as follows:

$$\begin{cases} \phi_t = -F(\phi)\phi_x, & \dot{\xi} = \phi(L, t) \\ \phi(0, t) = -k_I \xi, & (\phi(x, 0), \xi(0)) = (\phi_0(x), \xi_0). \end{cases} \quad (39)$$

Recall that  $X = H^2(0, L) \times \mathbb{R}$  is equipped with the norm  $\|(f, z)\|_X^2 = \|f\|_{H^2}^2 + z^2$ . Assume that the initial condition  $(\phi_0, \xi_0)$  is in  $B_X(0, \delta)$ ,  $\delta > 0$  and satisfies the  $C^0$  and  $C^1$  compatibility conditions as in (24) and (25).

By using the Theorem 1.2 and the Propositions 1.3-1.5 in [28, pp.362-365], or [17, Theorem II] we deduce the existence of a unique solution to (39) for some  $\delta > 0$  and  $T > 0$  :

$$(\phi, \xi) \in C([0, T]; H^2(0, L) \times \mathbb{R}) \cap C^1([0, T]; H^1(0, L) \times \mathbb{R}).$$

The reader is referred to [9] and [2, Appendix B] for a rigorous proof to the initial boundary case.

Now we prove local exponential stability of the null state to (39). Notice that each compatible  $w_0 = (\phi_0, \xi_0) \in X$  admits a sequence of  $w_{0,n} = (\phi_{0,n}, \xi_{0,n}) \in H^4(0, L) \times \mathbb{R}$  satisfying the  $C^k$  compatibility condition,  $k = 0, 1, 2, 3$ , such that  $\lim_{n \rightarrow \infty} \|w_{0,n} - w_0\|_X = 0$  (cf. [7, p.130]). Hence it is sufficient for us to prove the exponential stability for  $w_0 \in H^4(0, L) \times \mathbb{R}$ . As the solution depends continuously on the initial condition (see [17, Theorem III]), then the exponential decay of solution from compatible  $w_0 \in X$  is proved by taking the limit.

Indeed, take a compatible  $w_0 \in (H^4(0, L) \times \mathbb{R}) \cap B_X(0, \delta)$ . As stated above the system (39) has a unique solution  $w(t)$  in  $H^4(0, L) \times \mathbb{R}$  (cf. [28]) such that

$$w \in C([0, T]; H^4(0, L) \times \mathbb{R}) \cap C^1([0, T]; H^3(0, L) \times \mathbb{R})$$

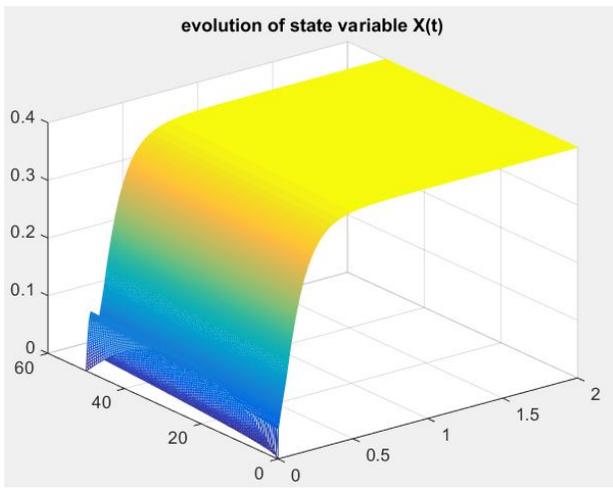
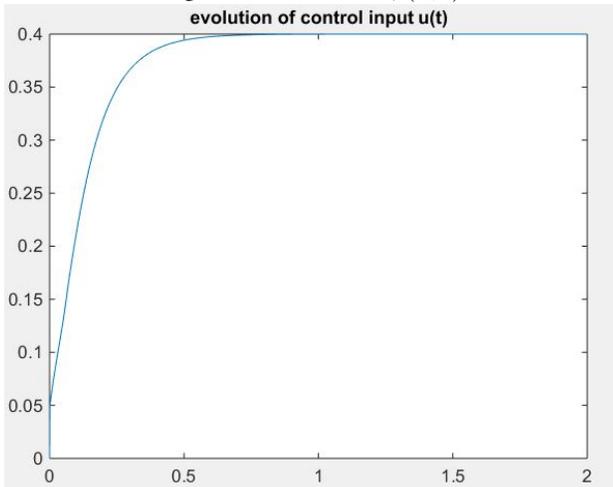
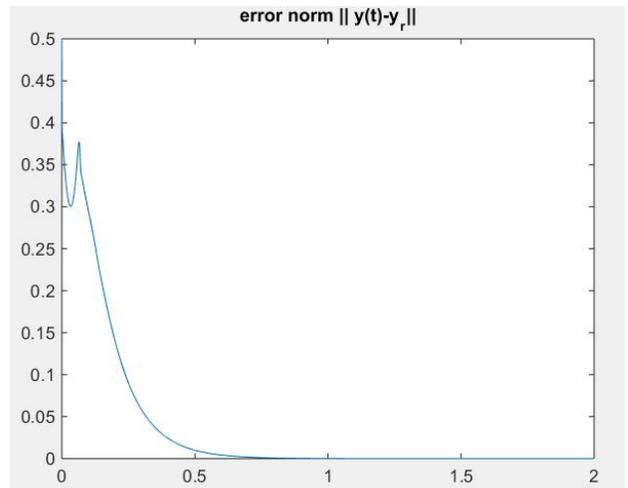
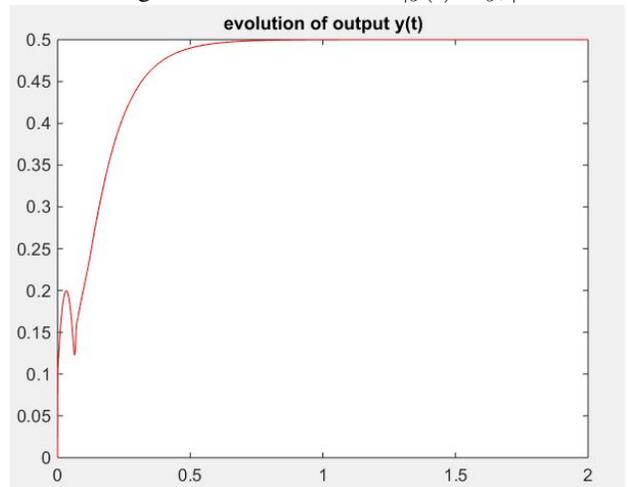
where  $w(t) = (\phi(t), \xi(t))$ . By the continuous embedding (cf. [7, p.167])  $H^n(0, L) \hookrightarrow C^{n-1}[0, L] \forall n \geq 1$  integer, we have the solution

$$w \in C([0, T]; C^3[0, L] \times \mathbb{R}) \cap C^1([0, T]; C^2[0, L] \times \mathbb{R}).$$

Let  $\|w_0\|_X < \delta_1$  for some  $\delta_1 > 0$ . We choose  $\delta_1 > 0$  sufficiently small such that  $\|w_0\|_X < K\delta_1$  implies  $\|(\phi, \phi_x, \xi)\|_{T, \infty} < \delta$  with smaller  $T$  if necessary (cf. [23, Theorem 2.2, p.46]). Notice that  $K$  and  $\delta$  are defined in our Theorem 2. Then direct application of Theorem 2 allows us to get  $\|w(T)\|_X < K\delta_1$ . Since the system is autonomous, the same argument can be used on the time interval  $[T, 2T]$ . By successive iterations we obtain the differential inequality (33) satisfied for all  $t \geq 0$ . By (33)-(34) we find positive constants  $M$  and  $\omega$  such that

$$\|w(t)\|_X \leq M e^{-\omega t} \|w_0\|_X \quad \forall w_0 \in B_X(0, \delta_1).$$

The regulation effect is automatically guaranteed, since  $w \in C([0, \infty); H^2(0, L) \times \mathbb{R})$ . Hence the proof of Theorem 1 is complete.  $\square$

Fig. 1: Evolution of  $\psi(x, t)$ Fig. 2: Evolution of input  $u(t)$ Fig. 3: Evolution of error  $|y(t) - y_r|$ Fig. 4: Evolution of output  $y(t)$ 

#### IV. NUMERICAL SIMULATIONS

The performance of the integral controller on the linearized system has been studied through numerical simulations and discussed in the paper [29]. Here the simulations are done on the nonlinear system to validate the theoretical results of Theorem 1. These simulations are realized by using the discretization method with the Preissmann scheme [22]. The details of the method have been presented in our paper [29] with the following parameters :  $N$ ,  $\theta$ ,  $\Delta t$  and  $\Delta x$ . Note that  $N$  is the number of discretized space intervals,  $\Delta t$  and  $\Delta x$  are the time discretization step and the space discretization step, respectively, and the weight parameter  $\theta \in [0.5, 1]$  is made use of to compute the value of a function from its neighbor values. In this section, the parameters take the following values:  $L = 50m$ ,  $N = 100$ ,  $\theta = 0.55$ , and  $\frac{\Delta t}{\Delta x} = 0.5$ . Moreover, to simulate the nonlinear closed-loop system (2), the following flux function and numerical values are applied:  $F(\psi) = \psi^2 + 3$ ,  $k_I = 0.05$ , and  $y_r = 0.5$ . The constant perturbations on the output and on the control are given by  $w_o = 0.1$  and  $w_c = 0.05$ , respectively.

Figure 1 shows asymptotic stability of the nonlinear closed-loop system and illustrates the evolution of the state  $\psi(x, t)$ .

Moreover the regulation of the output  $y(t)$  to the desired set-point  $y_r$  is illustrated by Figure 3 and Figure 4. Finally Figure 2 shows the evolution of the control input  $u(t)$  perturbed by  $w_c$ . As clearly indicated by the simulations, by virtue of the integral action the output converges to the set-point as  $t \rightarrow \infty$  independently of the constant perturbations.

#### V. CONCLUSIONS

We have considered the design of stabilizing integral controllers for the nonlinear systems described by scalar hyperbolic PDE. First we have proposed an interval of integral gain for stabilization and then proved exponential stability of the linearized system controlled by the designed controller. Moreover, for the linearized system we have been able to establish a necessary and sufficient condition for the integral gain to get exponential stability of the controlled system in the  $L^2$  norm. Then we have proved local exponential stability of the nonlinear controlled system by the same integral controller in the  $H^2$  norm. Both of two main proofs have used Lyapunov techniques with the Lyapunov functions in the quadratic form. The regulation of the output to the set-point is automatically

guaranteed from the local exponential stability of the closed-loop system in  $H^2$  norm. Numerical simulations for the nonlinear closed-loop system have been carried out to validate the performance of the controlled system. In the future, the work is to extend the design of stabilizing PI controllers for networks of scalar systems governed by nonlinear hyperbolic PDE.

## APPENDIX

### A. Proof of Proposition 1

A necessary and sufficient condition for exponential stability of the system (12)-(14) is that all the poles of the transfer function have negative real part (see [34], [4, Chapter 13], or [16, Appendix Theorem 3.5]). To formulate the transfer function, we set  $v(t)$  as the new control input with  $y(t)$  as the output :

$$\phi(0, t) = -k_I \xi(t) + v(t), \quad y(t) = \phi(L, t). \quad (40)$$

By taking the Laplace transform in (12)-(14), we obtain:

$$s\hat{\phi} + r\hat{\phi}_x = 0 \quad (41)$$

$$s\hat{\xi} = \hat{\phi}(L, s), \quad (42)$$

$$\hat{\phi}(0, s) = -k_I \hat{\xi}(s) + \hat{v}(s), \quad \hat{y}(s) = \hat{\phi}(L, s) \quad (43)$$

From (41) we have the solution  $\hat{\phi}(x, s) = \hat{\phi}(0, s)e^{-sr^{-1}x}$ . Combining it with (42) and (43), we obtain:

$$\begin{aligned} \hat{y}(s) &= \hat{\phi}(L, s) = \hat{\phi}(0, s)e^{-sLr^{-1}} = \\ &e^{-sLr^{-1}}(-k_I \hat{\xi}(s) + \hat{v}(s)) = e^{-sLr^{-1}} \left( -k_I \frac{\hat{y}(s)}{s} + \hat{v}(s) \right) \end{aligned}$$

Hence,

$$\left( 1 + \frac{k_I}{s} e^{-sLr^{-1}} \right) \hat{y}(s) = e^{-sLr^{-1}} \hat{v}(s)$$

Therefore we get the transfer function as follows:

$$G(s) = \frac{\hat{y}(s)}{\hat{v}(s)} = \frac{s}{k_I + se^{sLr^{-1}}}$$

The poles of transfer function are solutions of the following equation :

$$k_I + se^{sLr^{-1}} = 0 \quad (44)$$

We set

$$\mu = sLr^{-1} \quad \text{and} \quad \alpha = k_I Lr^{-1}. \quad (45)$$

Note that  $\alpha > 0$ . The characteristic equation now becomes

$$\alpha + \mu e^\mu = 0 \quad (46)$$

The proposition is proved if we show that the equation (46) has all the solutions  $\mu$  in the left-half complex plane  $\Re(\mu) < 0$  if and only if  $\alpha \in (0, \frac{\pi}{2})$ .

Let set  $\mu = \sigma + i\eta$ , where  $\sigma, \eta \in \mathbb{R}$ . Then (46) is rewritten as follows :

$$(\sigma + i\eta)e^{\sigma+i\eta} + \alpha = 0$$

By separating the real part and the imaginary part, we obtain:

$$-e^\sigma(\sigma \cos(\eta) - \eta \sin(\eta)) = \alpha \quad (47)$$

$$\eta \cos(\eta) + \sigma \sin(\eta) = 0 \quad (48)$$

We consider the following two cases.

- If  $\sin(\eta) = 0$ , by (48),  $\eta \cos(\eta) = 0$  implies  $\eta = 0$ . From (47), we have  $\alpha = -\sigma e^\sigma$ . The last equation has no solution  $\sigma \geq 0$  whatever is  $\alpha > 0$ . Hence each solution  $\sigma$  is negative if and only if  $\alpha \in (0, \pi/2)$ .
- If  $\sin(\eta) \neq 0$ , from (48),

$$\sigma = -\frac{\eta \cos(\eta)}{\sin(\eta)} \quad (49)$$

Thus  $\alpha = H(\eta)$  where

$$H(\eta) = \frac{\eta}{\sin(\eta)} \exp\left(\frac{-\eta \cos(\eta)}{\sin(\eta)}\right)$$

Because  $H(\eta)$  is a pair function, we only need to consider the case where  $\eta > 0$ . Thus  $\alpha > 0$  if and only if  $\sin(\eta) > 0$ . As  $\eta > 0$  and  $\sin(\eta) > 0$ , we set  $\eta = \gamma + 2k\pi$ , where  $\gamma \in (0, \pi)$  and  $k \in \mathbb{N}$ .

Now considering the function  $H(\eta)$ , we have :

$$\frac{\partial H(\eta)}{\partial \eta} = e^{-\frac{\eta \cos(\eta)}{\sin(\eta)}} \frac{\sin^2(\eta) - \eta \sin(2\eta) + \eta^2}{\sin^3(\eta)}$$

One can easily check that  $\frac{\sin^2(\eta) - \eta \sin(2\eta) + \eta^2}{\sin^3(\eta)} \geq 0$  for all  $\eta > 0$ . Therefore,  $\frac{\partial H(\eta)}{\partial \eta} \geq 0$ . Hence, on each interval  $(2k\pi, 2k\pi + \pi)$ , the function  $H(\eta)$  is continuous and monotonic increasing. Moreover we have

$$\lim_{\eta \rightarrow (2k+1)\pi^-} H(\eta) = +\infty, \quad \lim_{\eta \rightarrow 2k\pi + \frac{\pi}{2}} H(\eta) = 2k\pi + \frac{\pi}{2}.$$

In addition,  $\lim_{\eta \rightarrow 2k\pi^+} H(\eta) = 0 \quad \forall k \in \mathbb{N}$  and  $k > 0$ , and  $\lim_{\eta \rightarrow 0} H(\eta) = e^{-1}$ . By (49) and  $\eta = \gamma + 2k\pi$ , we have  $\sigma < 0$  if and only if  $\gamma \in (0, \frac{\pi}{2})$ . Obviously  $\sigma \geq 0$  if  $\gamma \in [\frac{\pi}{2}, \pi)$ . Therefore  $\sigma < 0$  if and only if the equation  $H(\eta) - \alpha = 0$  has all its solutions in  $\cup_{k=0}^{\infty} (2k\pi, 2k\pi + \frac{\pi}{2})$ . Since  $\alpha$  needs to be in one interval including 0, we have  $\alpha \in (0, \frac{\pi}{2})$ .

From the two cases, the proposition is proved.

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