Multiple Hopf bifurcations in coupled networks of planar systems

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MULTIPLE HOPF BIFURCATIONS IN COUPLED NETWORKS OF PLANAR SYSTEMS

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Abstract. In this communication, we study coupled networks built with non-identical instances of a dynamical system exhibiting a Hopf bifurcation. We first show how the coupling generates the birth of multiple limit cycles. Next, we project those coupled networks in the real plane, and construct a polynomial Hamiltonian system of degree n, admitting $O(n^2)$ non-degenerate centers. We explore various perturbations of that Hamiltonian system and implement an algorithm for the symbolic computation of the Melnikov coefficients.

Keywords. Limit cycle, Hopf bifurcation, Hamiltonian system, Melnikov coefficients, coupled network.

1 Sequence of Hopf bifurcations in a directed chain of oscillators

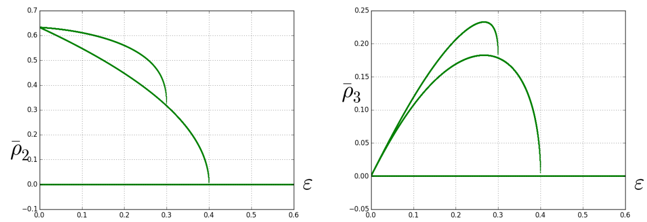
Let us consider a directed chain of oscillators, built with the normal form of a Hopf bifurcation

$$\dot{\rho} = \rho(\lambda - \rho^2), \quad \dot{\theta} = 1, \tag{1}$$

where $(\rho,\ \theta)$ denote the polar coordinates of a generic point $(x,\ y)$ in $\mathbb{R}^2$. The whole coupled network is given by

$$\begin{cases} \dot{\rho}_1 = \rho_1(\lambda_1 - \rho_1^2) - \varepsilon\rho_1 \\ \dot{\rho}_k = \rho_k(\lambda_k - \rho_k^2) + \varepsilon\rho_{k-1} - \varepsilon\rho_k, 2 \leq k \leq n-1 \\ \dot{\rho}_n = \rho_n(\lambda_n - \rho_n^2) + \varepsilon\rho_{n-1}. \end{cases} \tag{2}$$

We consider that the cells have the same angular velocity, that is $\dot{\theta} = 1$, so we omit it in the equations of the network. For $n = 3$ and $\lambda_3 < 0 < \lambda_1 < \lambda_2$, in absence of coupling ($\varepsilon = 0$), cells 1 and 2 each admit one unstable equilibrium point, and one stable limit cycle whose respective radius $\bar{\rho}_1$, $\bar{\rho}_2$ satisfy $\bar{\rho}_1 < \bar{\rho}_2$. Cell 3 admits a unique stable equilibrium point. If the coupling strength ε weakly increases, we observe the appearance of a second cycle in cell 2, and simultaneously the appearance of two different cycles in cell 3. If ε keeps increasing, the cycle of greater radius first vanishes, and finally the other cycle vanishes too. Finally, the three cells present the same dynamic only for quite large values of ε. The bifurcation diagrams for cells 2 and 3 are depicted in Figure 1. This shows that a very simple network can also generate new limit cycles.



(a) (b)

Figure 1: Bifurcation diagrams for a three non identical cells directed chain.

In order to prove that the chain exhibits this sequence of Hopf bifurcations, we can make a local analysis based on the calculation of the first Lyapunov coefficient in the Poincaré normal form of the Hopf bifurcation [2, 4]. After basic computations, the first Lyapunov coefficient is given by

$$\ell_1(\mu) = -\frac{(2\delta + 3\mu)^2}{(\lambda_1 - \mu)^2}, \tag{3}$$

where $\delta = \lambda_2 - \lambda_1$ measures the difference of the dynamics of each node, and $\mu = \lambda_1 - \varepsilon$ is a translation of the coupling strength introduced in order to lighten the computations. In particular, we have $\ell_1(0) = \frac{-4\delta^2}{\lambda_1^2}$, thus the Hopf bifurcation that occurs in the network is supercritical, and degenerates into a Bautin bifurcation point if $\lambda_1 = \lambda_2$.

A longer chain can present a greater number of Hopf bifurcations. Such coupled networks have been applied to the study of neural networks or electric circuits networks [5]. It is remarkable that the number of limit cycles increases, but the degree of the polynomial involved in the system (2) is constant. In the following, we investigate the effect of projecting such a coupled network in the real plane.

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2 Near-hamiltonian planar polynomial systems of degree n admitting n^3 limit cycles

In this section, we consider a Hamiltonian system admitting n^2 centers, and build a perturbation of that system. We show some examples for which the perturbed system admits n^3 limit cycles. To that aim, we introduce the following Hamiltonian system:

$$(\Sigma_{n,m}) \begin{cases} \dot{x} = E_y \\ \dot{y} = -E_x, \end{cases} \tag{4}$$

with $n, m \in \mathbb{N}^*$,

$$E = \sum_{k=1}^{m} E_k \prod_{\substack{l=1 \\ l \neq k}}^{m} (x^2 + y^2 - \rho_l^2)^2, \tag{5}$$

where $\rho_1 < \rho_2 < \cdots < \rho_m$,

$$E_k = \frac{1}{2} \prod_{i=1}^{n} \left[(x - x_{i,k})^2 + (y - y_{i,k})^2 \right], \tag{6}$$

with $x_{i,k} = \rho_k \cos \frac{2i\pi}{n}$ and $y_{i,k} = \rho_k \sin \frac{2i\pi}{n}$.

Proposition 2.1. *For any $n, m \in \mathbb{N}^*$, the system $(\Sigma_{n,m})$ is Hamiltonian, invariant under rotation of center $(0,\ 0)$ and angle $\frac{2\pi}{n}$. Furthermore, it is polynomial of degree $O(n+m)$, and it admits $n \times m$ non-degenerate centers at $(x_{i,k},\ y_{i,k})$, $1 \leq i \leq n$, $1 \leq k \leq m$.*

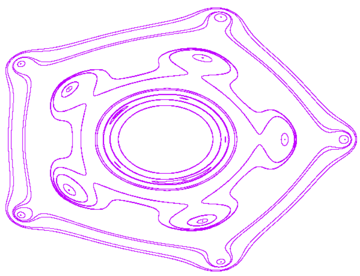

Figure 2: Energy levels of $(\Sigma_{5,3})$.

Now the difficult question is to find a polynomial perturbation of $(\Sigma_{n,m})$, of low degree, so that each center bifurcates into a given number a limit cycles. Thus we introduce for $\varepsilon > 0$ and $r \in \mathbb{N}^*$ the near-Hamiltonian system:

$$(\Sigma^{\varepsilon}_{n,m,r}) \begin{cases} \dot{x} = E_y + \varepsilon P(x, y, \delta) \\ \dot{y} = -E_x + \varepsilon Q(x, y, \delta), \end{cases} \tag{7}$$

with

$$P(x, y, \delta) = \sum_{k=1}^{m} P_k(x, y, \delta) \prod_{\substack{l=1 \\ l \neq k}}^{m} (x^2 + y^2 - \rho_l^2), \tag{8}$$

where

$$P_k(x, y, \delta) = \sum_{i=1}^{n} - \sin \tfrac{2i\pi}{n} p_k \left(-\sin \tfrac{2i\pi}{n} x + \cos \tfrac{2i\pi}{n} y \right) \times \left[\prod_{\substack{j=1 \\ j \neq i}}^{n} \left(-\sin \tfrac{2j\pi}{n} x + \cos \tfrac{2j\pi}{n} y \right) \right],$$

and $p_k(u) = \sum_{s=0}^{r} \delta_{k,s} u^{2s+3}$ for $1 \leq k \leq m$.

Proposition 2.2. *For any $n, m, r \in \mathbb{N}^*$, the near-Hamiltonian system $(\Sigma^{\varepsilon}_{n,m,r})$ is polynomial of degree $O(n + m + r)$.*

Finally, we present some examples for which we have computed the Melnikov coefficients [1] at each non-degenerate center, in order to show that $(\Sigma^{\varepsilon}_{n,m,r})$ can admit $n \times m \times r$ limit cycles (see Table 1). It is a work in progress to prove it in the general case. Such a general theorem would constitute a new lower bound for Hilbert's number, that is $H(n) \geq O(n^3)$ (see [3] and the references therein).

Table 1: Some examples of system $(\Sigma^{\varepsilon}_{n,m,r})$ admitting $n \times m \times r$ limit cycles.

n	m	r	Number of limit cycles
3	2	2	12
3	3	3	27
3	3	4	36
3	4	3	36
3	4	4	48
3	5	3	45
5	2	2	20
5	3	2	30

References

[1] M. Han, J. Yang, and P. Yu. Hopf bifurcations for near-hamiltonian systems. *International Journal of Bifurcation and Chaos*, 19(12):4117–4130, 2009.

[2] B. Hassard and Y. Wan. Bifurcation formulae derived from center manifold theory. *Journal of Mathematical Analysis and Applications*, 63(1):297–312, 1978.

[3] Y. Ilyashenko. Centennial history of hilbert's 16th problem. *Bulletin of the American Mathematical Society*, 39(3):301–354, 2002.

[4] Y. Kuznetsov. *Elements of Applied Bifurcation Theory*. Applied Mathematical Sciences. Springer New York, 2004.

[5] C. Zhang, B. Zheng, and L. Wang. Multiple hopf bifurcations of three coupled van der pol oscillators with delay. *Applied Mathematics and Computation*, 217(17):7155–7166, 2011.