An Average Study of the Signalized Cell Transmission Model

Carlos Canudas de Wit

An Average Study of the Signalized Cell Transmission Model

Carlos Canudas-de-Wit*

***Abstract*— In this paper we present an average study between the continuous-time signalized T-periodic Link Transmission Model (LTM), and its averaged version. Those are macroscopic models capturing the time-evolution of the vehicles densities in urban traffic networks controlled by periodic traffic light of period T. In this paper, we formalize the mathematic sense in which the solutions of the periodic signalized LTM model are approximated by the solutions of its averaged version. In particular, we shown that the error norm between the solutions of the signalized and the averaged models is bounded, in both a finite and infinite time-intervals, by constant proportional to the the ratio, T/L, between the traffic light time-cycle T, and the considered road segment (link) length, L. This result confirm the intuition that the precision of the averaged models improves with the increase of traffic light frequencies and link road lengths.**

I. Introduction

One of the most popular flow traffic model is the LWR-model, proposed by Lighthill and Whitham [13] and Richards [15] in the middle of the 50's. The model originally derives from the conservation law of vehicles. It is described by an scalar hyperbolic Partial Differential Equation (PDE) in term of macroscopic distributed variables such as the density, flow and average speed of the vehicles.

For convenience of their use for simulations and for control design, the LWR model can be approximated by finite-dimensional ordinary differential/difference equations. A popular method to do that is the Godunov scheme [11]. The discrete version of LWR model with triangular fundamental diagram, is then formulated as an iterated coupled map with time and space discretized into time and space (cells) steps, and supplemented by a special "supply-demand" update rule to describe interactions between adjacent freeway cells as well as shock and advection waves. This discretized version of the LWR model, is known as the Cell Transmission Model (CTM) [3], and has become one of the most popular macroscopic traffic models used for control design. A complete treatment of stability analysis, control properties and the use of the CTM for ramp metering can be found in [10].

The CTM naturally addresses models of highways, but it can be also used to modeling traffic in urban networks by including diverging and merging cell, but also signalized intersections [4]. In case of a single cell between intersection the CTM is then named "Link Transmission Model (LTM)" in relation to the node-link structure of urban networks where a single link is used to describe the road between two intersections (nodes). In these models, the traffic signal (named here u) could be constant (describing the turning "average" priority of the flow), time-dependent (alternating between 0 and 1 after a suitable number of clock ticks), or depending on the traffic load when used as a feedback control. In many cases, traffic signal profiles are computed in advance and deployed based in some optimized periodic schedule.

When u is time-dependent and periodic, this model is named as the *signalized-LTM* (S-LTM), and leads to a set of nonlinear periodic ODEs[1] of the form $\dot{x} = f(x, u(t))$, where $u(t) = u(t+T) \in \{0, 1\}$ describes the integer (binary) control variable indicating the stop (red $= 0$) or the release (green $= 1$) of flows at the traffic light. It is possible to use this model for control design at expenses of an increase in the design complexity due to the mixed integer/real nature of the control/state variables. To simplify the complexity of the control design, it is convenient to replace the periodic input $u(t)$ by its average over a period, $\bar{u} \in [0, 1]$, and to use this averaged input as the new *continuous* control variable. The control design can now be performed on the basis of the resulting "averaged" model $\dot{y} = f(y, \bar{u})$.

An averaged version of the *signalized-LTM* has been recently used for designing one-step ahead optimal control policies for the traffic lights [8], where time-dependent traffic

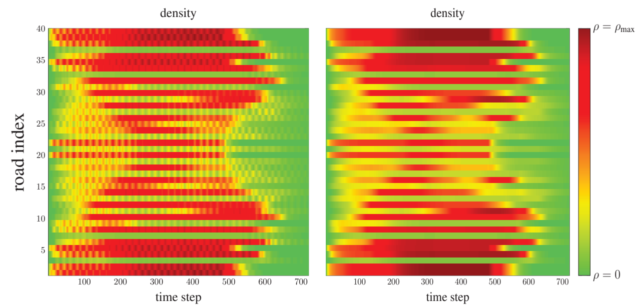

Fig. 1. Density time-evolution per row of a planar square network of 40 roads connected by standard 4-ways intersections. Left Figure show the real system ($\dot{x} = f(x, u(t))$), and the right figure the average one ($\dot{y} = f(y, \bar{u})$). Figure from [8].

This work has been partly funded by the EU FP7 project SPEEDD (619435).

*Director of research at the CNRS, GIPSA-Lab. NeCS team, Grenoble, France. carlos.canudas-de-wit(at)gipsa-lab.fr

[1] For simplicity in the presentation and for convenience of the averaged analysis, in this note we will use the the continuous-time version of the LTM rather than the discrete-time version. We will refer then to *differential* equation, rather than *difference* equations as originally stated in the LTM model.

signals are then replaced by it time-average (or duty-cycle) over a traffic light cycle. The problem has been efficiently solved via linear programming, because the optimization time-horizon has been limited to one step (one traffic-light cycle), but also because the optimization variable $u(t)$ was replaced by his average $\bar{u}$ over the considered optimization cycle. The paper present some numerical comparison between the signalized time-dependent model versus the averaged one, as shown in Fig.1. The comparison reports average differences of 10%, with maximum picks of 20%, see [8] for details.

This kind of averaged representation has also been used in other models to describe the network traffic flow process in a simplified way, so as to circumvent the inclusion of discrete variables. An example is the ”store-and-forward” discrete-time model [6] proposed by Gazis and Potts (1963), and more recently used for designing LQ-optimal controllers [1]. Other studies formulate the ”store-and-forward” discrete-time model in a stochastic setting, where vehicle inflows and outflows are assumed to be random [18]. The model is used as a basis for the design of the Max-pressure control that maximizes throughput.

It is worth to notice that the ”averaged” models allow for highly efficient control methods to be deployed in large scale networks by mitigating complexity of the associated optimization problems. At the other hand, these modeling simplifications allows only for split optimization, while cycle time and offsets must be delivered by other control algorithms [7]. A main reason being that averaging is invariant with respect to offsets (delays).

In spite of their popularity and its widely spread use for control design, the theoretical foundations and the analytic properties of these averaged models have been disregarded. In particular, and to the best of the the author knowledge, theoretical questions concerning how and in which sense the solutions of the “averaged” system $y(t)$, approximated the solutions of the true system $x(t)$, remain unanswered to date. This contribution provides a formal answer to this question for the continuous-time version of the LTM.

For simplicity of exposition, we assume constant demand and supply at the boundaries, and we first shown that the open-loop system is globally bounded, and continuously Lipschitz. Those properties are used to shown that the difference between the true and the averaged systems is bounded by $c(T)\epsilon$, in some finite time-interval depending of ϵ, i.e.

$$||x(t) - y(t)|| \leq c(T)\epsilon, \quad 0 \leq t \leq T_F/\epsilon$$

where $\epsilon = 1/L > 0$ is the inverse of the cell length, and $c(T)$ depends on the physical parameters of the model, including in particular the traffic light time-period T. The paper also analyze other properties (equilibria, stability of the equilibria, contractivity), which are useful to extend the result to all the positive time-axis. In particular, this extension is mainly due to the contractivity property of the averaged model around its equilibria.

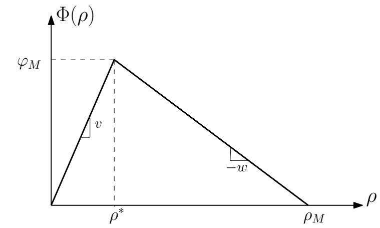

Fig. 2. Schematic representation of the triangular piece-wise fundamental diagram, where φ_M is the the maximum flow, ρ_M is the maximum density, ρ^* is the critical density, v is the free-flow velocity, and w is the congested velocity.

II. Preliminaries

The LWR macroscopic model is based on the vehicle conservation’s principle, and on the assumption that the traffic can be described by the empiric relation between the flow, φ, and density, ρ, as: $\varphi = \Phi(\rho)$, where the function $\Phi(\cdot)$ is called Fundamental Diagram. The constitutive assumption of this model, motivated by experimental data, is that the vehicles tend to travel at an equilibrium speed and that $\varphi = V(\rho)\rho$, where $V(\rho)$ is the flow speed depending on the density. $V(\rho)$ varies in the range $[0, v]$, where v is the maximum velocity at free-flow. As shown in Fig. 2, the fundamental diagram can be defined, in its simplest form, as a triangle with its maximum at $\varphi_M = \Phi(\rho^*)$ describing the maximum capacity of the road. The critical density ρ^* defines the boundary between the free-flow and the congested modes, while ρ_M is the maximum density that the road can withstand. The slope, w, defines the speed at which congestion travels upstream.

The evolution of the number of vehicles, N, within any spatial section $(0, L)$ is given by the following vehicles conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}N = \varphi_{in} - \varphi_{out}, \qquad N = \int_0^L \rho(x,t)\mathrm{d}x \tag{1}$$

where φ_{in} and φ_{out} are the input (at $x = L$) and output (at $x = 0$) flows at the boundaries of the road section of length L. Equation (1) can be rewritten (see [13]) as a hyperbolic equation involving only the density

$$\partial_t\rho + \partial_x\Phi(\rho) = \partial_t\rho + \partial_\rho\Phi \cdot \partial_x\rho = 0 \tag{2}$$

The macroscopic continuous density dynamics is then given by the LWR Cauchy problem described by (2) with the initial condition $\rho(x, 0) = \rho^0(x)$. The model has been shown to be consistent with hydrodynamic theory [3]. Validation tests with real data have been reported in [14].

A. Signalized Link-Transmission Model

The analysis presented in this note, can be extended to multi-road networks and to CTM models with arbitrarily numbers of cells. For the sake of clarity in the exposition of the technical results, we limited this analysis to a single link road, described by a signalized one-cell CTM, named before as the Link Transmission Model (LTM).

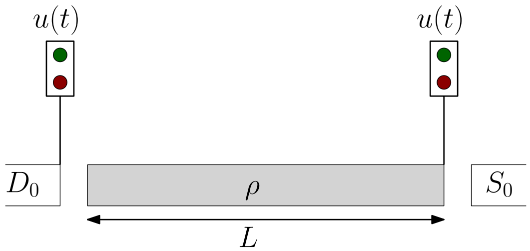

Fig. 3. Schematic description of the considered scenario for this study.

Consider the scenario shown in Fig. 3 of a single link road of length L with T-periodic traffic light $u(t) = u(t+T)$, and density $\rho(t)$. Let D_0 and S_0 be the boundary (external) demand and supply, respectively. For the sake of simplicity, assume they are constant although the presented analysis also holds if they are considered time, or state-dependent. Also, we do not consider the light phases as they have no effect in the average approximation. Under those assumptions, the continuous-time version of the Signalized LTM, can be written as:

$$\dot{\rho}(t) \quad = \quad \frac{1}{L}u(t)\left[\min\{D_0, S(\rho)\} - \min\{D(\rho), S_0\}\right] \quad (3)$$

with,

$$\begin{aligned} D(\rho) &= \min\{v\rho_i, \varphi_M\}, & (4)\\ S(\rho) &= \min\{\varphi_M, w(\rho_M - \rho)\} & (5)\end{aligned}$$

Note that, the standard discrete-time S-LTM form can be recovered by replacing the time-derivative, $\dot{\rho}(t)$, by its Euler approximation, i.e. $\dot{\rho}(t) \approx (\rho(k+1) - \rho(k))/\Delta t$, where Δt is the time-discretization step. It is worth to recall that, as the conservation laws like (2) generate irregular flows, they cannot be integrated numerically using standard methods (see [4], [12]), but efficient methods that reproduces correctly the propagation of the shock waves, like the Godunov scheme [9].

B. Averaged Link-Transmission Model

In the context of this study, the averaged model is now defined using the "averaged" density $\bar{\rho}$, and the averaged input $\bar{u}$ over the traffic light cycle T-period, where the dynamics of $\bar{\rho}$ is defined by:

$$\dot{\bar{\rho}}(t) = \frac{1}{L}\bar{u}\left[\min\{D_0, S(\bar{\rho})\} - \min\{D(\bar{\rho}), S_0\}\right] \quad (6)$$

Since $u(t)$ is assumed to be periodic, and being of the form:

$$u(t) = \begin{cases} 1 & if \quad nT < t \le \bar{u}(n+1)T \\ 0 & if \quad \bar{u}(n+1)T < t \le (n+1)T \end{cases}$$

for all $n \in Z^+$, and $\forall t \ge 0$. Then the duty cycle $\bar{u}$ is indeed the average of the control input, i.e.

$$\bar{u} = \frac{1}{T}\int_0^T u(t)dt, \quad \bar{u} \in [0,1]$$

III. Problem definition and model properties

Let's write *original model* (3) in a general form by letting $x = \rho$ be the state, and $\epsilon = 1/L > 0$ be the "small" parameter:

$$\dot{x}(t) = \epsilon u(t) g(x) = \epsilon f(x, u(t)) = \epsilon f(x,t) \quad (7)$$

with $g(x) = g_{in}(x) - g_{out}(x)$, where $g_{in}(x)$ is the inflow, and $g_{out}(x)$ is the outflow:

$$\begin{aligned} g_{in}(x) &= \min\{D_0, S(x)\} & (8)\\ g_{out}(x) &= \min\{D(x), S_0\} & (9)\end{aligned}$$

Let *average model*, with $y = \bar{\rho}$, be described as

$$\dot{y}(t) = \epsilon f_{av}(y) \quad (10)$$

with

$$\begin{aligned} f_{av}(y) &= \frac{1}{T}\int_0^T f(y,t)dt \\ &= g(y)\frac{1}{T}\int_0^T u(t)dt \\ &= \bar{u}g(y)\end{aligned}$$

The problem is to study in which sense the averaged system (10) approximates the original system (7). Or, in other words, how the norm $|x(t) - y(t)|$ can be upperbounded by a function depending on ϵ, with $x(0) = y(0)$.

A. Model properties

The following properties holds for every possible value of the boundary demand and supply below its maximal capacity; $|D_0(t)| \le \varphi_M$, $|S_0(t)| \le \varphi_M$, and for any $u(t), \bar{u} \in [0,1]$.

Lemma 1 (Boundednees) *Let $\Omega = [0, \rho_M]$ be the compact (closed and bounded) set defining the solution space for system* (7)*. The set Ω is indeed an positive invariant set for all the solutions of* (7),

$$x(0) \in \Omega, \Rightarrow x(t) \in \Omega, \forall t \ge 0.$$

Proof. This property comes from the vehicle conservation property of the CTM, and can be easily verified by noting that: if $x(0) = 0$, then $\dot{x}(t) \ge 0$, while if $x(0) = \rho_M$, then $\dot{x}(t) \le 0$. This property also shown the system "open-loop" boundedness for any $x(0) \in \Omega$.

Lemma 2 *The functions $g_{in}(x)$, and $g_{out}(x)$ are continuous Lipschitz in Ω, i.e. There exist two Lipschitz constants, $\lambda_{in} = w$, and $\lambda_{out} = v$, such that:*

i) $|g_{in}(x_1) - g_{in}(x_2)| \le \lambda_{in}|x_1 - x_2|, \quad \forall x_1, x_2 \in \Omega$
ii) $|g_{out}(x_1) - g_{out}(x_2)| \le \lambda_{out}|x_1 - x_2|, \quad \forall x_1, x_2 \in \Omega$

Proof. The proof of this statement can be easily verified by observing that the slope of the secant line joining $(x_1, g_{in}(x_1))$ and $(x_2, g_{in}(x_2))$ is always bounded above by w, and by v for the secant line joining $(x_1, g_{out}(x_1))$ and $(x_2, g_{out}(x_2))$. Note that Lemma 2 holds for every possible value of the boundary demand and supply below the maximal capacity.

Lemma 3 (Lipschitz continuity) *As a consequence of Lemma 2 the function $f(x,t)$ is continuous Lipschitz in Ω, with the Lipschitz constant $\lambda = w + v$, i.e.*

$$|f(x_1,t) - f(x_2,t)| \leq \lambda |x_1 - x_2|, \quad \forall x_1, x_2 \in \Omega, \forall t \geq 0.$$

Proof. The proof follows from the property that a linear combination of arbitrary Lipschitz continuous functions is Lipschitz continuous:

$$\begin{aligned}
|f(x_1,t) - f(x_2,t)| &= |u(t)g(x_1) - u(t)g(x_2)| \\
&\leq |u(t)||g(x_2) - g(x_2)| \\
&\leq |g(x_1) - g(x_2)| \\
&\leq |g_{in}(x_1) - g_{in}(x_2)| + \\
&\quad\ |g_{out}(x_1) - g_{out}(x_2)| \\
&\leq \lambda_{in}|x_1 - x_2| + \lambda_{out}|x_1 - x_2| \\
&= (\lambda_{in} + \lambda_{out})|x_1 - x_2| \\
&= \lambda |x_1 - x_2|
\end{aligned}$$

where we have used the fact that $|u(t)| \leq 1$, and the previous bounds from Lemma 2.

Lemma 4 (Equilibria) *Let $x(0) \in \Omega$, then system (7) has the following equilibria:*

- *If $D_0 < S_0$, there exists an unique free-flow equilibrium: $x_f^* = \frac{D_0}{v}$,*
- *If $D_0 > S_0$, there exists an unique congested equilibrium given by: $x_c^* = \rho_M - \frac{S_0}{w}$,*
- *If $D_0 = S_0 = \Phi_{DS}$, there exist an equilibrium manifold $\Omega_e \subseteq \Omega$,*

$$\Omega_e = \left\{ x^* : \frac{\Phi_{DS}}{v} \leq x^* \leq \rho_M - \frac{\Phi_{DS}}{w} \right\}.$$

Proof. By direct inspection of the values for x^* satisfying:

$$g(x^*) = \min\{D_0, S(x^*)\} - \min\{D(x^*), S_0\} = 0$$

for all considered cases.

Remark 1 *Properties from Lemma 1 to Lemma 4 of the original system (7), also hold for the average system (10).*

Lemma 5 *Consider the average system (10) with $y(0) \in \Omega$, and $\tilde{y}(t) = y(t) - y^*$, then:*

a) If $D_0 < S_0$, $\exists$ positive constants c_f, μ_f, such that:

$$|\tilde{y}(t)| \leq c_f e^{-\mu_f t}, \forall t \geq 0, \quad y^* = y_f^* = D_0/v,$$

b) If $D_0 > S_0$, $\exists$ positive constants c_c, μ_c, such that:

$$|\tilde{y}(t)| \leq c_c e^{-\mu_c t}, \forall t \geq 0, \quad y^* = y_c^* = \rho_M - S_0/w,$$

c) If $D_0 = S_0 = \Phi_{DS}$, then:

c1) $|\tilde{y}(t)| \leq |\tilde{y}(0)| e^{-\mu_f t}, \forall t \geq 0, y^* = y_1^*,\ y(0) \in \Omega_1$,
c2) $|\tilde{y}(t)| \leq |\tilde{y}(0)| e^{-\mu_c t}, \forall t \geq 0, y^* = y_2^*,\ y(0) \in \Omega_2$,
c3) $|\tilde{y}(t)| = 0, \forall t \geq 0, \quad y^* = y(0),\ y(0) \in \Omega_e$.

with $y_1^* = \Phi_{DS}/v$, $y_2^* = \rho_M - \Phi_{DS}/w$.

Proof. Note first that the analysis is conducted around the equilibrium points as specified by Lemma 4, which is also valid for the averaged system (10). The three stability cases and the position of the corresponding equilibria are sketched in Fig. 4.

Case a). From Fig. 4 and equations (10), we can see that,

$$\dot{y} = \epsilon \bar{u} \begin{cases} D_0 - D(y) & if \quad y \in \Omega_1 \\ D_0 - S_0 < 0 & if \quad y \in \Omega_2 \end{cases}$$

from which we get

$$\dot{\tilde{y}} = \epsilon \bar{u} \begin{cases} -v\tilde{y} & if \quad y \in \Omega_1 \\ D_0 - S_0 < 0 & if \quad y \in \Omega_2 \end{cases}$$

If initial conditions are taking as $y(0) \in \Omega_1$, then $\tilde{y}(t) = \tilde{y}(0)e^{-\mu_f(t)}$, with $\mu_f = \epsilon \bar{u} v$. If $y(0) \in \Omega_2$, then $\tilde{y}(T_1) = \tilde{y}(0) + (D_0 - S_0)T_1$ decreases linearly in time ($(D_0 - S_0) < 0$), enters in the the region Ω_1 in finite-time T_1, and thereafter converges exponentially to its equilibrium with a rate given by μ_f. These two cases can be combined to get the upper bound,

$$|\tilde{y}(t)| \leq |\tilde{y}(T_1)| e^{\mu_f T_1} e^{-\mu_f t} = c_f e^{-\mu_f t}, \forall t \geq 0.$$

Case b). Following similar steps we get

$$\dot{\tilde{y}} = \epsilon \bar{u} \begin{cases} D_0 - S_0 > 0 & if \quad y \in \Omega_1 \\ -w\tilde{y} & if \quad y \in \Omega_2 \end{cases}$$

which leads to

$$|\tilde{y}(t)| \leq |\tilde{y}(T_2)| e^{\mu_c T_2} e^{-\mu_c t} = c_c e^{-\mu_c t}, \forall t \geq 0.$$

with $\mu_c = \epsilon \bar{u} w$, and T_2 being the time to reach Ω_2 from $\tilde{y}(0)$.

In the case c), the error system dynamics writes as:

$$\dot{\tilde{y}} = \epsilon \bar{u} \begin{cases} -v\tilde{y} & if \quad y \in \Omega_1 \\ -w\tilde{y} & if \quad y \in \Omega_2 \\ 0 & if \quad y \in \Omega_e \end{cases}$$

For cases $c1$) and $c2$), it is easy to see that the error system converges exponentially to the respective left and right boundaries of Ω_e with rates: μ_f and μ_c. In the case $c2$), both the signalized and the averaged model have the same trivial solutions ($x(t) = y(t) = x(0) = y(0)$), and needs not be analyzed.

Aggregating all the cases in Lemma 5, we get the following result.

Lemma 6 (Exponential stability) *The average system (10) with $y(0) \in \Omega$, $(D_0, S_0) \leq \varphi_M$, and $\tilde{y}(t) = y(t) - y^*$, is globally exponentially stable, i.e.*

$$|\tilde{y}(t)| \leq c e^{-\mu t}$$

where $c = \max\{|\tilde{y}(0)|, c_f, c_c\}$, and $\mu = \min\{\mu_f, \mu_c\} = \mu_c$.

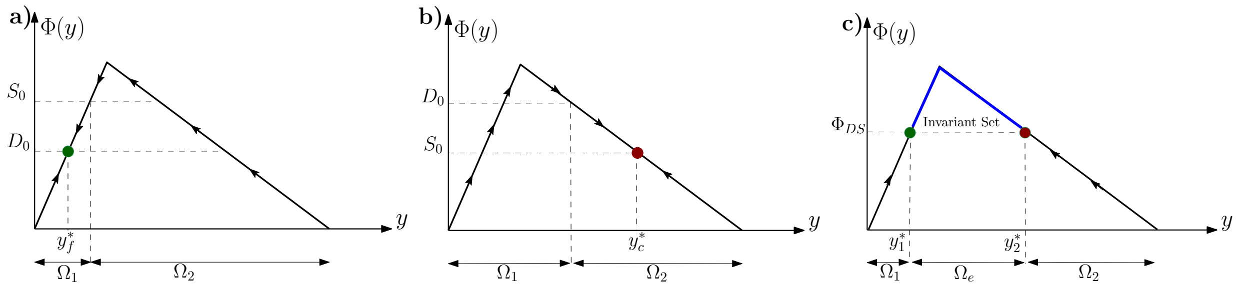

Fig. 4. Equilibria points and stability profiles for: $a)$ $S_0 > D_0$, $b)$ $D_0 > S_0$, and $c)$ $D_0 = S_0 = \Phi_{DS}$.

Lemma 7 (Contractivity) *Consider two solutions $y_1(t)$, and $y_2(t)$ of the average system* (10) *with $(y_1(t_0), y_1(t_0)) \in \Omega$, then there exists a large enough time, T_3 (independent of ϵ), and a constant $0 < k < 1$, such that*

$$|y_1(t) - y_2(t)| \leq k|y_1(t_0) - y_2(t_0)|, \quad \forall t > T_3 > 0.$$

with $T_3 = \max\{T_1, T_2\}$.

Proof. Introduce $\bar{y} = y_1(t) - y_2(t) = \tilde{y}_1(t) - \tilde{y}_2(t)$, from Lemma 5 we can see that $\bar{y}$ decreases first linearly in time up to $t = T_3$; and then it follows an μ exponential decay, i.e:

$$\bar{y}(t) = (c(D_0, S_0)T_3 + \bar{y}(t_0))\, e^{-\mu(t-T_3)}, \forall t \geq T_3$$

where $c(D_0, S_0)$ is a constant with sign such that $c(D_0, S_0)t + \bar{y}(t_0) \leq \bar{y}(t_0)$. Then the following upperbound applies:

$$|\bar{y}(t)| \leq |\bar{y}(t_0)|e^{-\mu(t-T_3)}$$

Therefore, $e^{\mu(t-T_3)} < 1, \forall t > T_3$, and the contractivity property holds.

Remark 2 *The contractivity property of the averaged model is an essential property to show the time-horizon validity of the error upperbound can be extended to the infinite.*

IV. Main Result

Let introduce the error system dynamics, $e(t) = x(t) - y(t)$, given by:

$$\dot{e}(t) = \epsilon\left(f(x,t) - f_{av}(y)\right), \quad e(0) = 0. \tag{11}$$

The main results below states that given some time-interval $0 \leq t \leq T_F/\epsilon$, the error variable is bounded by the product of ϵ and a constant depending on the period T.

Theorem 1 (Finite-time horizon) *For all $\epsilon > 0$, and any arbitrarily $T_F > 0$, there exist a constant $c(T)$ such that*

$$|x(t) - y(t)| \leq c(T)\epsilon$$

for $0 \leq t \leq T_F/\epsilon$.

Before proceeding the the proof, note that due to the boundedness property of Lemma 1, Theorem 1 holds for any ϵ, and any T_F. This is in opposition to the standard first-order averaging results where T_F may be chosen arbitrarily, but then ϵ and Ω are chosen in response. Here the system solutions are bounded in the set Ω independently to the value of ϵ, and T_F.

Proof. The proof follows general ideas for analyzing first-order averaging systems from [16] by establishing suitable bounds on the error system (11) in the time interval of interest $0 \leq t \leq T_F/\epsilon$.

Let first rewrite the error system (11) as:

$$\dot{e}(t) \;=\; \epsilon\left[f(x,t) - f(y,t)\right] + \epsilon\left[f(y,t) - f_{av}(y)\right]$$

with $e(0) = 0$. From this equation we have that

$$\begin{aligned} e(t) \;=\; & \epsilon \int_0^t \left[f(x,\tau) - f(y,\tau)\right] d\tau \,+ \\ & \epsilon \int_0^t \left[f(y,\tau) - f_{av}(y)\right] d\tau \end{aligned} \tag{12}$$

The next step is to find upperbounds for each of the two integral terms. For the first term in the RHS of (12), we can use the Lipsichtz property in Lemma 3 to get the following upperbound:

$$\epsilon \int_0^t |f(x,\tau) - f(y,\tau)|\, d\tau \leq \epsilon\lambda \int_0^t |e(\tau)| d\tau \tag{13}$$

Derivation of an upperbound for the second term in the RHS of (12) is a bit more involved, but it can be obtained with the help of the Besjes's Lemma [2], as indicated in [16]. Before proceed, let introduce some useful properties on the integrant of the second term,

Lemma 1 *Let $\tilde{f}(y,\tau) = f(y,\tau) - f_{av}(y)$, the $\tilde{f}(y,\tau)$ has the following properties:*

- $i)$ *$\tilde{f}(y,\tau)$ is periodic in τ, with a period T,*
- $ii)$ *$\tilde{f}(y,\tau)$ has zero mean for fixed y,*
- $iii)$ *$\tilde{f}(y,\tau)$ is bounded for all t, and for all $y \in \Omega$,*
- $iv)$ *$\tilde{f}(y,\tau)$ is continuous Lipschitz in Ω, and has the Lipschitz constant 2λ.*

$i)$ follows directly from the periodicity of $f(y,\tau)$. $ii)$ is also easy to derive by noticing that

$$\int_0^T \tilde{f}(y,\tau) = g(y) \int_0^T (u(\tau) - \bar{u}) d\tau = 0$$

$iii)$ follows from Lemma 1, and $iv)$ follows from Lemma 3, i.e.

$$\begin{aligned}|\tilde{f}(y_1,\tau)-\tilde{f}(y_2,\tau)| &\leq |f(y_1,\tau)-f(y_2,\tau)|+\\ &\quad |f_{av}(y_1)-f_{av}(y_2)|\\ &\leq \lambda|y_1-y_2|+\lambda|y_1-y_2|\\ &= 2\lambda|y_1-y_2|\end{aligned}$$

Now for the second term and following Besjes's Lemma [2], we can partition the integral in the interval $[0,t]$, in m intervals of duration T, and one fractional; $[0,T],[T,2T],\dots[(m-1)T,mT],[mT,t]$,

$$\begin{aligned}\epsilon\left|\int_0^t\tilde{f}(y,\tau)d\tau\right| &\leq \epsilon\sum_{i=1}^{m}\left|\int_{(i-1)T}^{iT}\tilde{f}(y,\tau)d\tau\right|+\\ &+\ \epsilon\left|\int_{mT}^{t}\tilde{f}(y,\tau)d\tau\right|\end{aligned}$$

For the $[0,mT]$-time interval, we have

$$\begin{aligned}\left|\int_{(i-1)T}^{iT}\tilde{f}(y,\tau)d\tau\right| &= \left|\int_{(i-1)T}^{iT}\tilde{f}(y,\tau)d\tau-\right.\\ &\quad\left.\int_{(i-1)T}^{iT}\tilde{f}(y((i-1)T),\tau)d\tau\right|\\ &\leq \int_{(i-1)T}^{iT}\left|(\tilde{f}(y,\tau)-\tilde{f}(y((i-1)T),\tau)\right|d\tau\\ &\leq 2\lambda\int_{(i-1)T}^{iT}|y(\tau)-y((i-1)T)|d\tau\end{aligned}$$

where the equality holds because $\tilde{f}(y((i-1)T),\tau)$ has zero mean for fixed y (Property ii in Lemma 1), and the upperbound results from the continuous Liptchitz property of $\tilde{f}$ (property iv in Lemma 1).

Using (10), the integrant of the last inequality can be rewritten as

$$y(\tau)-y((i-1)T)=\int_{(i-1)T}^{\tau}\dot{y}(s)ds=\epsilon\bar{u}\int_{(i-1)T}^{\tau}g(y(s))ds$$

$\forall\tau$ in the interval $(i-1)T\leq\tau\leq iT$. Now, since $|g(y)|\leq\varphi_M,\forall y$, and $|\tau-(i-1)T|\leq T$ the following bound holds;

$$\begin{aligned}|y(\tau)-y((i-1)T)| &\leq \epsilon\bar{u}\int_{(i-1)T}^{\tau}|g(y(s))|ds\\ &\leq \epsilon\bar{u}|\tau-(i-1)T|\cdot\max\{|g(y)|\}\\ &\leq \epsilon\bar{u}T\varphi_M\\ &= \epsilon T\varphi_M \qquad (14)\end{aligned}$$

from which we get

$$\begin{aligned}\left|\int_{(i-1)T}^{iT}\tilde{f}(y,\tau)d\tau\right| &\leq 2\lambda\int_{(i-1)T}^{iT}|y(\tau)-y((i-1)T)|d\tau\\ &\leq 2\lambda\epsilon T^2\varphi_M \qquad (15)\end{aligned}$$

For the $[mT,t]$ interval, and from the boundedness property in Lemma 1 (i.e, $|\tilde{f}(y)|\leq 2\varphi_M$), we have

$$\left|\int_{mT}^{t}\tilde{f}(y,\tau)d\tau\right| \leq 2\varphi_M T \qquad (16)$$

Using (15) and (16), we have that,

$$\begin{aligned}\epsilon\left|\int_0^t\tilde{f}(y,\tau)d\tau\right| &\leq 2\lambda\epsilon^2 mT^2\varphi_M+\epsilon 2\varphi_M T \qquad (17)\\ &\leq 2\epsilon\lambda T_F T\varphi_M+2\epsilon\varphi_M T \qquad (18)\\ &\leq \epsilon(c_1(T)+c_2(T))=\epsilon c_3(T) \qquad (19)\end{aligned}$$

with $c_1(T)=2\lambda T_F T\varphi_M$, $c_2(T)=2\varphi_M T$, and $c_3(T)=c_1(T)+c_2(T)$. The second inequality comes from the fact that the analysis is performed in the time interval $0\leq t\leq T_F/\epsilon$, and hence, $mT\leq t\leq T_F/\epsilon$.

Finally using (13) and (19) in (12), we get:

$$|e(t)|\leq\epsilon\lambda\int_0^t|e(\tau)|d\tau+\epsilon c_3(T)$$

and by the use of the Gronwall-Bellman lemma, we get

$$|e(t)|\leq\epsilon c_3(T)e^{\epsilon\lambda t}\leq\epsilon c_3(T)e^{\lambda T_F}=\epsilon c(T)$$

which conclude the proof, with $c(T)=c_3(T)e^{\lambda T_F}$.

Theorem 1 holds for a finite-time horizon. Extension of such a result to infinite time horizon will typically require that the average system (10) be contractive as already demonstrated in Lemma 5.

Theorem 2 (Infinite-time horizon) *For all $\epsilon>0$, there exist a constant $C(T)$ such that*

$$|x(t)-y(t)|\leq C(T)\epsilon,\quad\forall t\geq 0.$$

Proof. The proof is based on the time-partition trick proposed by [17] to extend the error grown to the interval $[0,\infty)$. For other similar analysis see Theorem 5.5.1 in, [16].

Let split the positive time axis t, in time intervals equally spaced by large enough length T_I/ϵ, such that both the contractivity property (Lemma 7) and the finite time boundedness (Theorem 1) hold for each considered time interval $I_m=[\frac{mT_I}{\epsilon},\frac{(m+1)T_I}{\epsilon}]$.

In addition to the exact solution $x(t)$, and the averaged solution $y(t)$, let consider the solution, $z(t)$, of the *"reset-averaged"* (or switched) system, with initial conditions reset at the beginning of each interval I_m, such that they coincide with the real solution, see Fig. 5,

$$\dot{z}(t)=\epsilon f_{av}(z),\quad z(\frac{mT_I}{\epsilon})=x(\frac{mT_I}{\epsilon}),\forall m\in\mathbb{N}$$

In virtue of Theorem 1, we have

$$|x(t)-z(t)|\leq c(T)\epsilon=\delta(\epsilon),\quad\forall t\in I_m$$

From the triangle inequality, we get $\forall t\in I_m$,

$$\begin{aligned}|x(t)-y(t)| &\leq |x(t)-z(t)|+|y(t)-z(t)|\\ &\leq \delta(\epsilon)+|y(t)-z(t)| \qquad (20)\end{aligned}$$

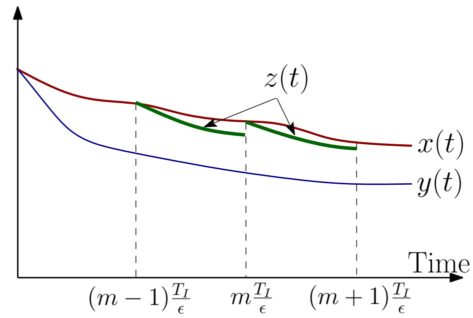

Fig. 5. Schematization of the 3 solutions: the real $x(t)$, the averaged $y(t)$, and the reset-averaged $z(t)$ under: $x(0) = y(0)$, and $z(\frac{mT_I}{\epsilon}) = x(\frac{mT_I}{\epsilon})$.

From Lemma 7, and considering large enough intervals I_m, the term $|y(t) - z(t)|$ can easily be shown to be contractive at the boundaries of I_m, i.e.

$$|y_{m+1} - z_{m+1}| \leq k|y_m - z_m|, \quad 0 < k < 1$$

where for simplicity of notation we use $z_m = z(\frac{mT_I}{\epsilon})$, $y_m = y(\frac{mT_I}{\epsilon})$.

Evaluating inequality (20) at $t = \frac{(m+1)T_I}{\epsilon}$, we get

$$\begin{aligned} |x_{m+1} - y_{m+1}| &\leq \delta(\epsilon) + |y_{m+1} - z_{m+1}| \\ &\leq \delta(\epsilon) + k|y_m - z_m| \\ &\leq \delta(\epsilon) + k|y_m - x_m| \end{aligned}$$

The second line is obtained using the contractivity property, and the last line comes from the fact that $z_m = x_m$.

By recursion, and using the limit of a power law, we get

$$\begin{aligned} |x_{m+1} - y_{m+1}| &\leq \delta(\epsilon) \sum_{j=0}^{m} k^j + k^m |y_0 - x_0| \\ &\leq \delta(\epsilon) \frac{1}{1-k} + k^m |y_0 - x_0| \end{aligned}$$

Finally, by taking the limit $m \to \infty$,

$$|x_{m+1} - y_{m+1}| \leq \delta(\epsilon) \frac{1}{1-k}$$

the result is proved, with $C(T) = c(T)\frac{1}{1-k}$.

V. Conclusions

In this paper we have presented an average study between the signalized T-periodic Link Transmission Model (LTM), and its averaged version. We have formalized the mathematic sense in which the solutions of the periodic signalized LTM model approximate the solutions of its averaged version. We have shown that the error norm between the solutions of the signalized and the averaged models can be bounded, in both finite and infinite time-intervals, by a constant proportional to the ratio between the light cycle T and the road segment length L.

For the sake of simplicity, and the clarity of the analysis and exposition of the main results, the analysis has been carried out under some simplified hypothesis. Those assumptions may be relaxed at the price of a more involved analysis. For instance, it may be possible to extend the result to: different control inputs at each intersection, time-varying boundary demands and supply, and networks with multiple cells. However, we feel that the main formal features confirming the natural intuition that the precision of the averaged models improves with small traffic light periods and large link road lengths, are captured by the proposed analysis even under some simplified hypothesis.

References

[1] K. Aboudolas, M. Papageorgiou, and E. Kosmatopoulos. *Store-and-forward based methods for the signal control problem in large-scale congested urban road networks.* Transportation Research Part-C, 17 (2009) 163–174.

[2] J.G. Besjes. *On the asymptotic methods for non-linear differential equations.* Journal de Mécanique, 8:357-373, 1969.

[3] C.F. Daganzo, *The cell transmission model: a dynamic representation of highway traffic consistent with the hydrodynamic theory*,Transport. Res. B-Meth., Vol.4, No.28, pp.269–287, 1994.

[4] C.F. Daganzo, *The cell transmission model,part II:Network traffic*,Transport. Res. B-Meth., Vol.2, No.29, pp.79–93, 1995.

[5] M. Garavello, B. Piccoli, *Traffic Flow on Networks: Conservation Laws Models*, AIMS Series on Applied Mathematics, 2006.

[6] D.C., Gazis, and R.B. Potts, *The oversaturated intersection.* In: Proceedings of the Second International Symposium on Traffic Theory, London, UK, pp. 221–237, 1963.

[7] P. Grandinetti, F. Garin and C. Canudas-de-Wit, *Towards scalable optimal traffic control.*, 54th IEEE Conference on Decision and Control, Osaka, Japan, 2015.

[8] P. Grandinetti, C. Canudas-de-Wit and F. Garin, *An efficient one-step-ahead optimal control for urban signalized traffic networks based on an averaged Cell-Transmission model*, 14th IEEE European Control Conference, Jul 2015, Linz, Austria.

[9] S. Godunov: *A difference scheme for numerical solution of discontinuous solution of hydrodynamic equations*, Matematicheskii Sbornik, Vol. 47, pp. 271-306, 1959.

[10] G. Gomes, R. Horowitz, A. A. Kurzhanskiy, J. Kwon,and P. Varaiya: *Behavior of the Cell Transmission Model and Effectiveness of Ramp Metering*, Transportation Research C, Vol. 16, pp. 485-513, 2008.

[11] R.J. LeVeque, *Finite Volume Methods for Hyperbolic Problems*, Cambridge University Press, 2004.

[12] R.J. LeVeque: *Numerical methods for conservation laws*, Birkhäuser, 1992.

[13] M. J. Lighthill and G. B. Whitham: *On kinematic waves II: A theory of traffic flow on long crowded roads*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Vol. 229, pp. 317-345, 1955.

[14] W.H. Lin and D. Ahanotu: *Validating the Basic Cell Transmission Model on a Single Freeway Link*, PATH Technical Note 95-3, Institute of Transportation Studies, University of California at Berkeley, 1994.

[15] P. I. Richards: *Shock Waves on the Highway*, Operations Research, Vol. 4, pp. 42-51, 1956.

[16] J.-A. Sanders, F. Verhulst, and J. Murdock: *Averaging Methods in Nonlinear Dynamical Systems*. Applied Mathematical Sciences Volume 59. Springer.

[17] E. Sanchez-Palencia: *Methode de Contrage-Estimation de l'erreur et comportement des trajectoire dans l'espace des phases*. Int. J. Non-Linear Mechanics, Vol.11, pp.251-263, Pergamon Press, 1976.

[18] P. Varaiya. *Max pressure control of a network of signalized intersections*. Transportation Research Part C Vol.36, (2013), 177-195.