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Abstract

In this chapter, we present the empirical estimation of some reliability measures, such as the rate of occurrence of failures and the steady-state availability, for a discrete-time semi-Markov system. The probability of first occurred failure is introduced and estimated. A numerical application is given to illustrate the strong consistency of these estimators.

Keywords: Semi-Markov chain, empirical estimation, rate of occurrence of failures, steady-state availability, first occurred failure

1 Introduction

Semi-Markov chains constitute a generalization of Markov chains and renewal chains. For a Markov chain, the sojourn time in each state is geometrically distributed, whereas for the semi-Markov case, the sojourn time distribution can be any distribution on $\mathbb{N}$. An introduction to homogeneous semi-Markov chains is given by Howard (1971) as well as by Mode and Pickens (1998) and Mode and Sleeman (2000). For non-homogeneous discrete-time semi-Markov systems see Vassiliou and Papadopoulou (1992, 1994), whereas for ergodic theory of semi-Markov chains see Anselone (1960). Moreover, for semi-Markov replacement chains see Gerontidis (1994). Barbu and Limnios (2006) study an empirical estimator of the discrete-time semi-Markov kernel and its asymptotic properties, with application to reliability. Chryssaphinou et al. (2011) establish a discrete-time reliability system with multiple components under semi-Markov hypothesis. An overview in the theory on semi-Markov chains oriented toward applications in modeling and estimation is presented in Barbu and Limnios (2008).

In this work, the investigation of the rate of occurrence of failures (ROCOF) is addressed for the first time for semi-Markov chains. The ROCOF may be interpreted as the expected number of transitions of a semi-Markov chain to a subset of its state space at a specific moment. Firstly, a simple formula for evaluating the ROCOF is derived. As a consequence of this result, a statistical estimator of this function is proposed. The continuous-time version of the ROCOF is calculated in a wide range of scientific fields including reliability (Lam, 1997; Ouhbi and Limnios, 2002) and seismology (Votsi et al., 2012).

Afterwards, we examine the steady-state (or asymptotic) availability by two different aspects. At first, we consider the pointwise availability and we take its limit as the time tends to infinity. Substituting with the empirical estimator for the pointwise availability, we obtain an estimator for the asymptotic measure. Alternatively, the steady-state availability is written as the sum of the stationary distribution of the semi-Markov chain in the working states of the system (see

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Barbu and Limnios, 2008). Two different proposed nonparametric estimators of the stationary distribution are presented and the estimator of the availability is expressed in terms of them. As a sequence, we derive the empirical estimation of the steady-state availability through three different approaches.

Finally, we introduce the measure of the probability of the first occurred failure. Consider a sequence of disjoint subsets of the failure states of a semi-Markov system. Given the initial state of the system, the probability of first occurred failure expresses the probability to entry for first time into a failure subset before any other one. An empirical estimator for the probability of first occurred failure is then proposed. The asymptotic property of the strong consistency is presented for the estimators of ROCOF, steady-state availability and probability of first occurred failure.

In Section 2, the necessary preliminaries of a semi-Markov model are introduced and, in the next one, the definitions of the reliability measures studied in this chapter are given. Section 4 depicts their empirical estimation and their strong consistency. Finally, we apply these results to a numerical example in Section 5.

2 Discrete-time semi-Markov model

Let us firstly introduce some preliminaries from the theory of semi-Markov chains absolutely necessary for our purposes. Let $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. Consider a stochastic system with finite state space E. We suppose that the evolution in time of the system is described by the following chains :

1. The chain $\boldsymbol{J} := (J_n)_{n\in\mathbb{N}}$ with state space E, where J_n is the system state at the n-th jump time;

2. The chain $\boldsymbol{S} := (S_n)_{n\in\mathbb{N}}$ with state space $\mathbb{N}$, where S_n is the n-th jump time. We suppose that $S_0 = 0$ and $0 < S_1 < S_2 < \ldots < S_n < S_{n+1} < \ldots$;

3. The chain $\boldsymbol{X} := (X_n)_{n\in\mathbb{N}^*}$ with state space $\mathbb{N}$, with $X_n := S_n - S_{n-1}$ for all $n \in \mathbb{N}^*$. Thus, for all $n \in \mathbb{N}^*$, X_n is the sojourn time in state J_{n-1}, before the n-th jump.

The chain $(\boldsymbol{J}, \boldsymbol{S}) := (J_n, S_n)_{n\in\mathbb{N}}$ is said to be a Markov renewal chain (MRC) with state space $E \times \mathbb{N}$, if for all $n \in \mathbb{N}$, $j \in E$ and $k \in \mathbb{N}$, it satisfies almost surely (a.s.)

$$\mathbb{P}(J_{n+1} = j, S_{n+1} - S_n = k|J_0, \ldots, J_n; S_0, \ldots, S_n) = \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n = k|J_n).$$

If the above equation is independent of n, then $(\boldsymbol{J}, \boldsymbol{S})$ is said to be homogeneous.

The process $\boldsymbol{J}$ is the embedded Markov chain (EMC) of the MRC $(\boldsymbol{J}, \boldsymbol{S})$ with initial distribution $\boldsymbol{\alpha} := (\alpha_i;\ i \in E)$, where $\alpha_i := \mathbb{P}(J_0 = i)$, and stationary distribution $\boldsymbol{\nu} = (\nu_i;\ i \in E)$. The transition matrix $\boldsymbol{P} := (P(i,j);\ i,j \in E)$ of $\boldsymbol{J}$ is given by

$$P(i,j) := \mathbb{P}(J_{n+1} = j|J_n = i), \quad n \in \mathbb{N}. \tag{1}$$

Moreover, for all $k \in \mathbb{N}$, we define the discrete-time counting process of the number of jumps in $[1,k] \subset \mathbb{N}$ by $N(k) := \max\{n \in \mathbb{N} : S_n \leqslant k\}$. The semi-Markov chain (SMC) $\boldsymbol{Z} := (Z_k)_{k\in\mathbb{N}}$ is defined as $Z_k = J_{N(k)}$, $k \in \mathbb{N}$, and it gives the system's state at time k. We have also $J_n = Z_{S_n}$ and $S_n = \min\{k > S_{n-1} : Z_k \neq Z_{k-1}\}$, $n \in \mathbb{N}$. We note that the initial distribution of the SMC $\boldsymbol{Z}$ coincides with that of $\boldsymbol{J}$.

The evolution of the discrete-time semi-Markov system is governed by the semi-Markov kernel $\boldsymbol{q}(k) := (q_{ij}(k);\ i,j \in E)$, $k \in \mathbb{N}$, defined by

$$q_{ij}(k) := \mathbb{P}(J_{n+1} = j, X_{n+1} = k|J_n = i), \quad n \in \mathbb{N}. \tag{2}$$

For all $k \in \mathbb{N}^*$, the entries of the semi-Markov kernel $\boldsymbol{q}(k)$ are written as

$$q_{ij}(k) = p_{ij}\, f_{ij}(k), \quad i, j \in E,$$

where $f_{ij}(k) := \mathbb{P}(X_{n+1} = k | J_n = i, J_{n+1} = j)$ $i, j \in E$, $k \in \mathbb{N}$, is the conditional distribution of the sojourn time in state i given that the next visited state is j. In this case the sojourn times are attached to transitions and when a sojourn time in a state i expires, we can determine the next visited state j by using the probability of the EMC as well as the duration of this time.

Let us denote by $H_i(k)$, $k \in \mathbb{N}$, the sojourn time cumulative distribution function in any state $i \in E$, i.e.

$$H_i(k) := \mathbb{P}(X_{n+1} \leqslant k | J_n = i) = \sum_{j \in E} \sum_{l=0}^{k} q_{ij}(l),$$

and by $\overline{H}_i(k)$, $k \in \mathbb{N}$, the survival function in any state i, i.e.

$$\overline{H}_i(k) := \mathbb{P}(X_{n+1} > k | J_n = i) = 1 - H_i(k) = 1 - \sum_{j \in E} \sum_{l=0}^{k} q_{ij}(l).$$

Also, consider the matrices $\boldsymbol{H}(k) := diag(H_i(k);\, i \in E)$ and $\overline{\boldsymbol{H}}(k) := diag(\overline{H}_i(k);\, i \in E)$, $k \in \mathbb{N}$.

The transition function $\boldsymbol{P}(k) := (P_{ij}(k);\, i, j \in E)$, $k \in \mathbb{N}$, of the SMC $\boldsymbol{Z}$ is defined by

$$P_{ij}(k) := \mathbb{P}(Z_k = j | Z_0 = i).$$

Let $\boldsymbol{\psi}(k) := (\psi_{ij}(k);\, i, j \in E)$ be the Markov renewal function given by

$$\psi_{ij}(k) := \sum_{n=0}^{k} q_{ij}^{(n)}(k),$$

where $q_{ij}^{(n)}(k) := \mathbb{P}(J_n = j, S_n = k | J_0 = i)$ is the n-fold discrete-time convolution of $q_{ij}(k)$ (see Barbu et al., 2004). Then, the transition function $\boldsymbol{P}(k)$ of the SMC $\boldsymbol{Z}$ can be written as

$$\boldsymbol{P}(k) = \boldsymbol{\psi} * (\boldsymbol{I} - \boldsymbol{H})(k).$$

Also, the stationary distribution $\boldsymbol{\pi} := (\pi_i; i \in E)$ $k \in \mathbb{N}$, of the SMC $\boldsymbol{Z}$ is defined, when it exists, by

$$\pi_i := \lim_{k \to \infty} P_{li}(k), \quad l \in E.$$

In addition, we denote by $\boldsymbol{m} := (m_i; i \in E)^\top$ the mean sojourn times of the SMC $\boldsymbol{Z}$ in any state i defined by

$$m_i := \mathbb{E}[S_1 | J_0 = i] = \sum_{n \in \mathbb{N}} \overline{H}_i(n).$$

and, then, the stationary distribution of the SMC is expressed in terms of the stationary distribution of the EMC :

$$\pi_i = \frac{\nu_i m_i}{\sum_{k \in E} \nu_k m_k}.$$

We assume that the MRC $(\boldsymbol{J}, \boldsymbol{S})$ is irreducible and aperiodic, with finite mean sojourn times.

Let $\boldsymbol{U} := (U_k)_{k \in \mathbb{N}}$ be the sequence of the backward recurrence times for the SMC $\boldsymbol{Z}$ defined as follows :

$$U_k := \begin{cases} k, & \text{if } k < S_1, \\ k - S_{N(k)}, & \text{if } k \geqslant S_1, \end{cases}$$

where, since $S_0 = 0$, we have that $U_0 = 0$. We note that, for all $k \in \mathbb{N}$, $U_k \leqslant k$. The stochastic process $(\boldsymbol{Z}, \boldsymbol{U}) := (Z_k, U_k)_{k \in \mathbb{N}}$ is a Markov chain with values in $E \times \mathbb{N}$ (Limnios and Oprişan, 2001). A study of this process is given by Chryssaphinou et al. (2008). It is worth noticing that the Markov chain $(\boldsymbol{Z}, \boldsymbol{U})$ is time-homogeneous.

We denote by $\tilde{\boldsymbol{\alpha}}$ the initial distribution of the Markov chain $(\boldsymbol{Z}, \boldsymbol{U})$, whereas its transition matrix $\widetilde{\boldsymbol{P}} := (\widetilde{P}\big((i, t_1), (j, t_2)\big);\ (i, t_1), (j, t_2) \in E \times \mathbb{N})$ can be written as :

$$\begin{aligned} \widetilde{P}\big((i, t_1), (j, t_2)\big) &:= \mathbb{P}(Z_{k+1} = j, U_{k+1} = t_2 | Z_k = i, U_k = t_1) \\ &= \begin{cases} q_{ij}(t_1 + 1)/\overline{H}_i(t_1), & \text{if } i \neq j, t_2 = 0, \\ \overline{H}_i(t_1 + 1)/\overline{H}_i(t_1), & \text{if } i = j, t_2 - t_1 = 1, \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \tag{3}$$

for every $k \in \mathbb{N}$ such that $\mathbb{P}(Z_k = i, U_k = t_1) > 0$. Let us further denote by $(\tilde{\boldsymbol{\alpha}}\widetilde{\boldsymbol{P}}^{k-1})(i, m)$ the (i, m) element of the vector $\tilde{\boldsymbol{\alpha}}\widetilde{\boldsymbol{P}}^{k-1}$.

After introducing the basic notation and giving a brief description of the model under study, we proceed to the calculation of the ROCOF along with steady-state availability and the probability of first occurred failure for the semi-Markov case.

3 Reliability and probability of first occurred failure

Let us consider that the possible states of the semi-Markov chain $\boldsymbol{Z}$ belong to the set $E = \{1, \ldots, s\}$, $s \in \mathbb{N}^*$. Let $U := \{1, \ldots, r\}$ and $D := \{r + 1, \ldots, s\}$ be the subsets of the state space E of the working states and the down states of the system, respectively, with $U \cup D = E$ and $U \cap D = \varnothing$. We present now the reliability measures under consideration in this work.

3.1 Rate of occurrence of failures

We further consider an arbitrary subset B of the state space E, with $B \neq \varnothing$ and $B \neq E$. Initially, we are concentrated on the study of an important parameter in semi-Markov chains, the ROCOF. Before providing a formula for the evaluation of the ROCOF, let us clarify its meaning. For all $k \in \mathbb{N}$, we denote by $N_B(k)$ the counting process, up to time k, of the transitions of the SMC $\boldsymbol{Z}$ from $B^{\complement}$ to B, namely

$$N_B(k) := \sum_{l=1}^{k} \mathbf{1}_{\{Z_{l-1} \in B^{\complement}, Z_l \in B\}}.$$

The ROCOF is interpreted as the expected number of transitions of the SMC to the set B at time k, i.e.

$$\tilde{r}_B(k) := \mathbb{E}[N_B(k) - N_B(k-1)].$$

The following proposition gives a simple formula of the ROCOF for semi-Markov chains.

Proposition 1. *The ROCOF of the SMC $\boldsymbol{Z}$ at time k is given by*

$$\tilde{r}_B(k) = \sum_{i \in B^{\complement}} \sum_{j \in B} \sum_{m=0}^{k-1} [(\tilde{\alpha}\widetilde{P}^{k-1})(i, m)]\widetilde{P}\left((i, m), (j, 0)\right). \tag{4}$$

3.2 Steady-state availability

Another interesting measure concerning the asymptotic reliability theory is the steady-state availability of a system. Firstly, let us define the pointwise availability A of a system at time $k \in \mathbb{N}$ as the probability that the system is operational at time k, independently of the fact that the system has failed or not in $[0, k)$, i.e.

$$A(k) := \mathbb{P}(Z_k \in U).$$

Proposition 2. *Given a stochastic system described by a SMC $\boldsymbol{Z}$, the pointwise availability is given by*

$$A(k) = \boldsymbol{\alpha P}(k)\mathbf{1}_{s,r}, \tag{5}$$

where $\mathbf{1}_{s,r}$ is a s-column vector whose the r first elements are 1's and the last $s-r$ ones are 0's.

The steady-state availability A_∞ of a system is defined as the limit of the pointwise availability, when the limit exists, as the time tends to infinity, i.e.

$$A_\infty := \lim_{k \to \infty} A(k).$$

Proposition 3. *For a semi-Markov system, the steady-state availability is given by*

$$A_\infty = \sum_{i \in U} \pi_i. \tag{6}$$

3.3 Probability of first occurred failure

The measure of the probability of first occurred failure is introduced here. We consider the subset U of the working states, as defined in the previous subsection. We further consider the partition of the subset D to k subsets of failure states $(C_k)_{k\in\mathbb{N}}$ with $D = \cup_k C_k$ and $C_k \cap C_l = \varnothing$, $k \neq l$.

Let $T_D := \min\{n \in \mathbb{N} : \ Z_n \in D\}$ be the first hitting time of the subset D. We consider the process $\boldsymbol{Z}' := (Z'_k)_{k\in\mathbb{N}}$ defined by

$$Z'_k := \begin{cases} Z_k, & t < T_D, \\ \Delta, & t \geqslant T_D, \end{cases}$$

where Δ is an additional absorbing state .

Denote by $T_{C_k} := \min\{n \in \mathbb{N} : \ Z'_n \in C_k\}$ the first hitting time of the subset C_k and $\boldsymbol{\rho}_{C_k} := (\rho_{C_k}(i);\ i \in U)^\top$ the column-vector of probabilities defined below. If, $j \in \mathbb{N}^*$, $T_{C_j} < \infty$, then $T_{C_k} = +\infty$ for any $k \neq j$. Under this notation, for any $i \in U$, we define the probability of first occurred failure and get

$$\begin{aligned} \rho_{C_k}(i) &:= \mathbb{P}(T_{C_k} < \infty | J_0 = i) \\ &= \mathbb{P}_i(T_{C_k} < \infty) \\ &= \mathbb{P}_i(J_1 \in C_k, T_{C_k} < \infty) + \mathbb{P}_i(J_1 \in U, T_{C_k} < \infty) \\ &= \mathbb{P}_i(J_1 \in C_k) + \sum_{j \in U} P(i,j)\rho_{C_k}(j). \end{aligned}$$

Let $\boldsymbol{P}_{00}$ be the restriction on the subset $U \times U$ of the transition matrix $\boldsymbol{P}$ of the EMC $\boldsymbol{J}$ and $\boldsymbol{P}_{0k}$ the column-vector defined as $\boldsymbol{P}_{0k} := (P(i, C_k);\ i \in U)^\top$ with $P(i, C_k) := \sum_{j \in C_k} P(i,j)$. Then, we have

$$\boldsymbol{\rho}_{C_k} = \boldsymbol{P}_{0k} + \boldsymbol{P}_{00}\boldsymbol{\rho}_{C_k}.$$

Consequently, we obtain the following proposition :

Proposition 4. *For a stochastic system governed by the modified process $\boldsymbol{Z}'$ and given a failure class C_k, if the matrix $\boldsymbol{I}-\boldsymbol{P}_{00}$ is nonsingular, the probability of first occurred failure is given by*

$$\boldsymbol{\rho}_{C_k} = (\boldsymbol{I}-\boldsymbol{P}_{00})^{-1}\boldsymbol{P}_{0k}. \tag{7}$$

Note that

$$\sum_k \boldsymbol{\rho}_{C_k} = \mathbf{1},$$

where $\mathbf{1}$ is a column-vector with all entries equal to 1.

4 Nonparametric estimation of reliability measures

In this section, we follow the next observational procedure concerning the statistical inference of stochastic processes : a single realization of the process is observed over the fixed time interval $[0, M]$, $M \in \mathbb{N}^*$. The asymptotic property of strong consistency is obtained as the censoring time M tends to infinity (practically, when it becomes large). We consider an observation $\mathcal{H}_M$ of the ergodic MRC $(\boldsymbol{J}, \boldsymbol{S})$, censored at a fixed arbitrary time $M \in \mathbb{N}^*$, defined by

$$\mathcal{H}_M := \begin{cases} \{J_0, X_1, J_1, \ldots, X_{N(M)}, J_{N(M)}, U_M\}, & \text{if } N(M) > 0, \\ \{J_0, U_M = M\}, & \text{if } N(M) = 0, \end{cases}$$

where $N(M)$ is the discrete-time counting process of the number of jumps in $[1, M]$ and $U_M := M - S_{N(M)}$ is the censored sojourn time in the last visited state $J_{N(M)}$. Let us consider the following set : $T_M := \{0, \ldots, M\}$.

Additionally, for all states $i, j \in E$, let us define the counting processes :

1. $N_i(M) := \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i\}}$ is the number of visits to state i of the EMC, up to time M;
2. $N_{ij}(M) := \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n=j\}}$ is the number of transitions of the EMC from i to j, up to time M.

The proposed empirical estimator $\widehat{\boldsymbol{q}}(k, M) := (\widehat{q}_{ij}(k, M);\ i, j \in E)$, $k \in T_M$, $M \in \mathbb{N}^*$, of the semi-Markov kernel (2) is defined by the following equation

$$\widehat{q}_{ij}(k, M) := \frac{1}{N_i(M)} \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n=j, X_n=k\}}.$$

For further study of the asymptotic properties of the proposed empirical estimator, see Barbu and Limnios (2006). Once the estimator of the semi-Markov kernel is obtained, any measure conserning the SMC can be estimated, after having been expressed as a function of the semi-Markov kernel.

4.1 Estimation of ROCOF

The estimators $\widehat{H}_i(k, M)$ and $\widehat{\overline{H}}_i(k, M)$, $k \in T_M$, $M \in \mathbb{N}^*$, $i \in E$, for the sojourn time cumulative distribution functions $H_i(k, M)$ and the survival function $\overline{H}_i(k)$, respectively, are given by

$$\widehat{H}_i(k, M) := \sum_{j\in E}\sum_{l=0}^{k} \widehat{q}_{ij}(l, M) \quad \text{and} \quad \widehat{\overline{H}}_i(k, M) := 1 - \sum_{j\in E}\sum_{l=0}^{k} \widehat{q}_{ij}(l, M).$$

Consider also the estimators $\widehat{\boldsymbol{H}}(k, M) := diag(\widehat{H}_i(k, M); i \in E)$ and $\widehat{\overline{\boldsymbol{H}}}(k, M) := diag(\widehat{\overline{H}}_i(k, M); i \in E)$.

The empirical estimator $\widehat{\widetilde{\boldsymbol{P}}}_M := \left(\widehat{\widetilde{P}}\big((i,t_1),(j,t_2)\big);\ (i,t_1),(j,t_2) \in E \times \mathbb{N}\right)$ of the transition matrix (3) of the Markov chain $(\boldsymbol{Z},\boldsymbol{U})$ has entries defined by

$$\widehat{\widetilde{P}}_M\big((i,t_1),(j,t_2)\big) = \begin{cases} \widehat{q}_{ij}(t_1+1,M)/\widehat{\overline{H}}_i(t_1,M), & \text{if } i \neq j, t_2 = 0, \\ \widehat{\overline{H}}_i(t_1+1,M)/\widehat{\overline{H}}_i(t_1,M), & \text{if } i = j, t_2 - t_1 = 1, \\ 0, & \text{otherwise.} \end{cases}$$

On the basis of Proposition 1, we propose the following estimator $\widehat{\widetilde{r}}_B(k,M)$, $k \in T_M$, for the ROCOF (4) of the semi-Markov system,

$$\widehat{\widetilde{r}}_B(k,M) = \sum_{i \in B^{\complement}} \sum_{j \in B} \sum_{m=0}^{k-1} [(\widehat{\widetilde{\boldsymbol{\alpha}}} \widehat{\widetilde{\boldsymbol{P}}}_M^{k-1})(i,m)] \widehat{\widetilde{P}}_M\big((i,m),(j,0)\big), \tag{8}$$

$(\widehat{\widetilde{\boldsymbol{\alpha}}} \widehat{\widetilde{\boldsymbol{P}}}_M^{k-1})(i,m)$ is the (i,m)-th element of the vector $\widehat{\widetilde{\boldsymbol{\alpha}}} \widehat{\widetilde{\boldsymbol{P}}}_M^{k-1}$. The following proposition gives the uniform strong consistency of the ROCOF's estimator.

Proposition 5. *For any arbitrary $k \in \mathbb{N}$ fixed, the estimator (8) of the ROCOF (4) at instant k is strongly consistent in the sense that*

$$\widehat{\widetilde{r}}_B(k,M) \xrightarrow{a.s.} \widetilde{r}_B(k), \quad \text{as } M \to \infty.$$

4.2 Estimation of the steady-state availability

Concerning the stationary distribution $\boldsymbol{\pi}$ of the SMC, we consider two empirical estimators $\widehat{\boldsymbol{\pi}}(M) := (\widehat{\pi}_i(M);\ i \in E)$ and $\widetilde{\boldsymbol{\pi}}(M) := (\widetilde{\pi}_i(M);\ i \in E)$, defined as follows :

$$\widetilde{\pi}_i(M) := \frac{1}{M} \sum_{k=1}^{M} \mathbf{1}_{\{Z_{k-1}=i\}}, \tag{9}$$

$$\widehat{\pi}_i(M) := \frac{\widehat{\nu}_i(M)\widehat{m}_i(M)}{\sum_{k \in E} \widehat{\nu}_k(M)\widehat{m}_k(M)}, \tag{10}$$

where $\widehat{\nu}_i(M)$ and $\widehat{m}_i(M)$ are the empirical estimators of the stationary distribution ν_i of the EMC $\boldsymbol{J}$ and the mean sojourn time m_i, respectively, given by

$$\widehat{\nu}_i(M) := \frac{N_i(M)}{N(M)} \quad \text{and} \quad \widehat{m}_i(M) := \sum_{n \geqslant 0} \widehat{\overline{H}}_i(n,M).$$

For any censoring time $M \in \mathbb{N}^*$, we consider the estimators $\widehat{\boldsymbol{\psi}}(k,M) := (\widehat{\psi}_{ij}(k,M);\ i,j \in E)$ and $\widehat{\boldsymbol{P}}(k,M) := (\widehat{P}_{ij}(k,M),\ i,j \in E)$, $k \in T_M$, of the Markov renewal function $\boldsymbol{\psi}(k)$ and the transition function $\boldsymbol{P}(k)$ of the SMC $\boldsymbol{Z}$, respectively, taking the form

$$\widehat{\boldsymbol{\psi}}(k,M) := \sum_{n=0}^{k} \widehat{\boldsymbol{q}}^{(n)}(k,M) \quad \text{and} \quad \widehat{\boldsymbol{P}}(k,M) = \widehat{\boldsymbol{\psi}} * (\boldsymbol{I} - \widehat{\boldsymbol{H}})(k,M).$$

Then, the empirical estimator $\widehat{A}(k,M)$ of the steady-state availability $A(k)$, $k \in T_M$, can be written as

$$\widehat{A}(k,M) = \widehat{\boldsymbol{\alpha}} \widehat{\boldsymbol{P}}(k,M) \mathbf{1}_{s,r}. \tag{11}$$

Remark 1. *Formally, the estimator of the initial distribution* α *is defined as* $\widehat{\alpha} := \delta_{Z_0}$. *In our case, as one trajectory is taken into account, the estimation of the initial distributions* $\boldsymbol{\alpha}$ *and* $\widetilde{\boldsymbol{\alpha}}$ *is trivial.*

From equation (6), we may obtain two different estimators for the steady-state availability :

$$\widetilde{A}_\infty(M) = \sum_{i \in U} \widetilde{\pi}_i(M), \tag{12}$$

$$\widehat{A}_\infty(M) = \sum_{i \in U} \widehat{\pi}_i(M), \tag{13}$$

where $\widehat{A}(k, M)$, $\widetilde{\pi}_i(M)$ and $\widehat{\pi}_i(M)$ are given in (11), (9) and (10), respectively.

Proposition 6. *For any fixed* $k \in \mathbb{N}$, *the estimator (11) of the pointwise availability* $A(k)$, *and the estimators (**??**), (12) and (13) of the steady-state availability* A_∞, *are strongly consistent, i.e.*

$$\widehat{A}(k, M) \xrightarrow{a.s.} A(k),$$
$$\widetilde{A}_\infty \xrightarrow{a.s.} A_\infty,$$
$$\widehat{A}_\infty \xrightarrow{a.s.} A_\infty,$$

as $M \to \infty$.

4.3 Estimation of the probability of first occurred failure

The empirical estimator $\widehat{\boldsymbol{P}}_M := (\widehat{P}_M(i,j);\ i,j \in E)$, $M \in \mathbb{N}^*$, of the transition matrix (1) of the EMC $\boldsymbol{J}$ is given by

$$\widehat{P}_M(i,j) := \frac{N_{ij}(M)}{N_i(M)}.$$

Based on the estimation $\widehat{\boldsymbol{P}}$ of the transition matrix (1), we obtain the estimators $\widehat{\boldsymbol{P}}_{00}(M)$ and $\widehat{\boldsymbol{P}}_{0k}(M)$ of $\boldsymbol{P}_{00}$ and $\boldsymbol{P}_{0k}$, respectively. For any failure subset C_k, the estimator $\widehat{\boldsymbol{\rho}}_{C_k}(M) := (\widehat{\rho}_{C_k}(i, M);\ i \in U)^\top$ of the probability of first occurred failure (7) takes the form

$$\widehat{\boldsymbol{\rho}}_{C_k}(M) = (\boldsymbol{I} - \widehat{\boldsymbol{P}}_{00}(M))^{-1} \widehat{\boldsymbol{P}}_{0k}(M). \tag{14}$$

Proposition 7. *For any* $M \in \mathbb{N}^*$, *the estimator (14) of the probability of first occurred failure (7) of the failure subset* C_k, *is strongly consistent, i.e.*

$$\widehat{\boldsymbol{\rho}}_{C_k}(M) \xrightarrow{a.s.} \boldsymbol{\rho}_{C_k}, \quad \text{as } M \to \infty.$$

5 Numerical application

In this section, we apply the previous results to a four-state semi-Markov system, as described in Figure 1.

The state space of the system $E = \{1, 2, 3, 4\}$ is partitioned into the up-state set $U = \{1, 2\}$ and the down-state set $D = \{3, 4\}$. Since B is arbitrary chosen, for sake of simplicity, we set $B \equiv D$ and $B^\complement \equiv U$. The initial law $\boldsymbol{\alpha}$ and the transition matrix $\boldsymbol{P}$ of the EMC are given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{P} = \begin{pmatrix} 0 & 0.8 & 0.2 & 0 \\ 0.9 & 0 & 0 & 0.1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

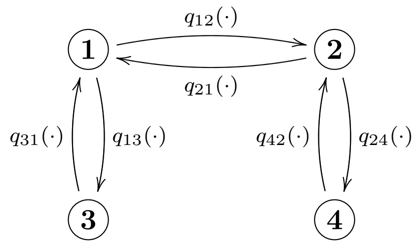

Figure 1: Four-state discrete-time semi-Markov system.

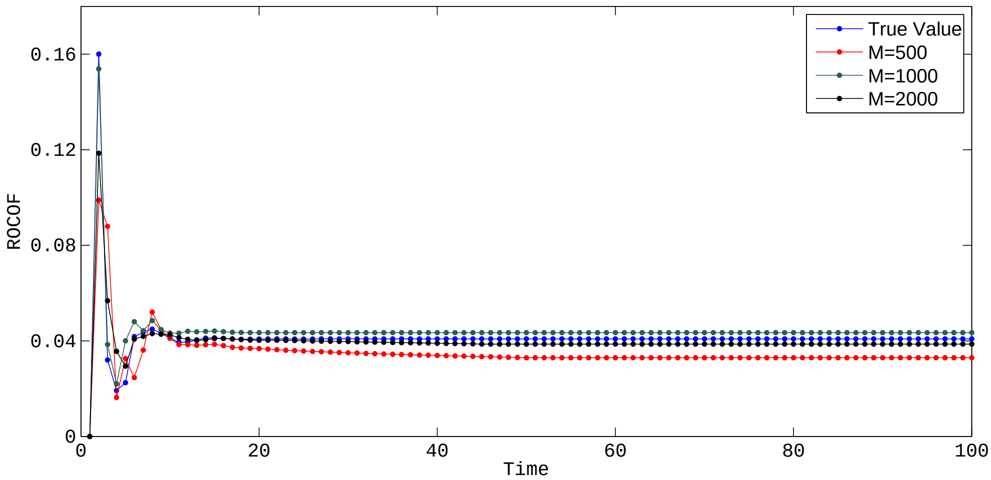

Figure 2: ROCOF plot.

We suppose that the conditional sojourn-time distributions $f_{13}(\cdot)$ and $f_{24}(\cdot)$ are geometric ones defined by

$$f(k) := \begin{cases} p(1-p)^{k-1}, & \text{if } k \geqslant 1, \\ 0, & \text{if } k = 0, \end{cases}$$

with $p = 0.8$ and $p = 0.8$, respectively. The distributions $f_{12}(\cdot)$, $f_{21}(\cdot)$, $f_{31}(\cdot)$ and $f_{42}(\cdot)$ are discrete-time Weibull ones with

$$f(k) := \begin{cases} q^{(k-1)^b} - q^{k^b}, & \text{if } k \geqslant 1, \\ 0, & \text{if } k = 0, \end{cases}$$

where $(q, b) = (0.8, 1.6)$ for the transition $1 \to 2$, $(q, b) = (0.7, 1.6)$ for the transition $2 \to 1$, $(q, b) = (0.4, 0.7)$ for the transition $3 \to 1$ and $(q, b) = (0.3, 0.7)$ for the transition $4 \to 2$.

Three independent trajectories of the SMC up to fixed censoring times $M = 500$, $M = 1000$ and $M = 2000$ are obtained by means of a Monte Carlo method. In the following figures, we present the plots of the theoretical and estimation values for the ROCOF and the availability of the system for the first 100 time units. In Table 1, the estimation of the steady-state availability is depicted.

Remark 2. *Note that, since $M \to \infty$ and $k \to \infty$, the estimation of the point-wise availability converges to the steady-state availability. Thus, we may derive an alternative estimation for the steady-state availability through the availability function.*

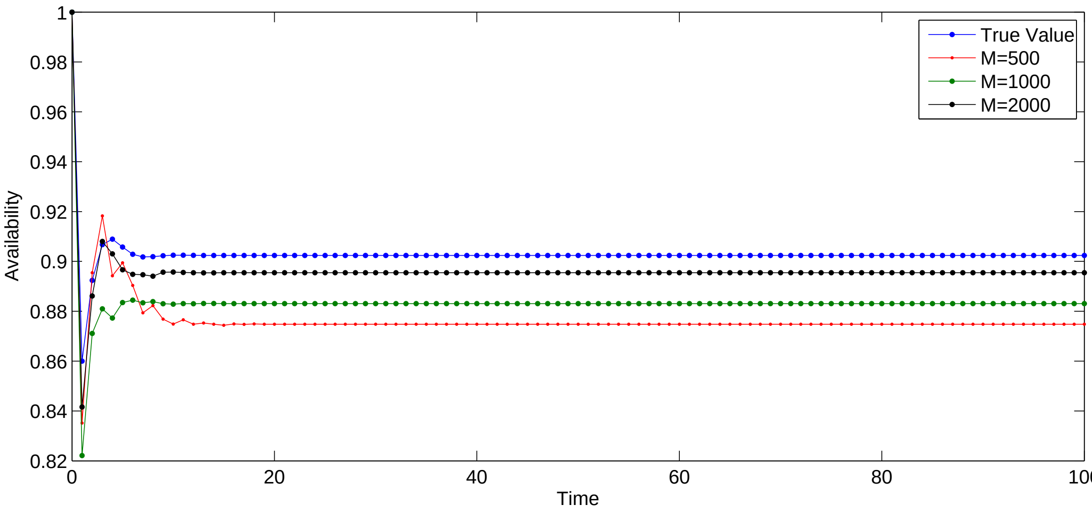

Figure 3: Availability plot.

	$M = 500$	$M = 3000$	$M = 5000$
A_∞	0.9024		
$\tilde{A}_\infty(M)$	0.8760	0.8830	0.8955
$\hat{A}_\infty(M)$	0.8795	0.8829	0.8954

Table 1: Estimation values of the steady-state availability.

To investigate the estimation of the probability of first occurred failure, we consider two failure subsets $C_1 = \{3\}$ and $C_2 = \{4\}$ with $C_1 \cup C_2 = D$. The probability of first occurred failure and its estimation, given the subsets C_1 and C_2, is presented in Table 2.

	$M = 500$	$M = 1000$	$M = 2000$
$\boldsymbol{\rho}_{C_1}$	$(0.7143, 0.6429)^\top$		
$\boldsymbol{\rho}_{C_2}$	$(0.2857, 0.3571)^\top$		
$\widehat{\boldsymbol{\rho}}_{C_1}(M)$	$(0.6813, 0.5772)^\top$	$(0.8227, 0.7635)^\top$	$(0.7267, 0.6528)^\top$
$\widehat{\boldsymbol{\rho}}_{C_2}(M)$	$(0.3187, 0.4028)^\top$	$(0.1773, 0.2365)^\top$	$(0.2733, 0.3472)^\top$

Table 2: Estimation values of the probability of first occurred failure.

The consistency of the estimators of the ROCOF, the steady-state availability and the probability of first occurred failure seems to be verified by the results of the preceding plots and tables and, the estimation values approach the true ones as the length of the trajectory becomes larger. Also, we see that two proposed estimators of A_∞ tend to coincide, with a slightly better estimator to be $\tilde{A}_\infty(M)$.

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