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CROSS-FLOW-INDUCED CHAOTIC VIBRATIONS OF HEAT-EXCHANGER TUBES IMPACTING ON LOOSE SUPPORTS

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The chaotic dynamics of heat exchanger tubes impacting on the generally loose baffle plates is studied using an analytic model that involves delay differential equations. The critical flow velocity for the local instability of the flexible cylinder near the static equilibrium position is obtained by assuming a harmonic solution in the discretized linearized model and solving the resulting algebraic equations. Numerical solutions show that, with increasing flow beyond the critical, the amplitude of motion grows until impacting with the loose support occurs; more complex motions then arise, leading to chaos for a sufficiently high flow velocity. A Lyapunov exponent technique is developed for delay differential equations and is applied to the present model, showing definitely and unequivocally—for the first time for this system—that chaotic motions do occur. To understand the system behaviour better, a one-degree-of-freedom, forced, negatively damped impact oscillator is studied. The response is quite complex and results in chaos in certain parameter regions. The analysis is performed by finding periodic solutions and determining their stability and bifurcations with the Poincaré map technique.

1. INTRODUCTION

Arrays of cylinders subject to cross-flow are known to be subject to fluid elastic instabilities at sufficiently high flow velocities. In tube-in-shell type heat exchangers, steam generators, boilers, condensers and similar equipment, these instabilities can cause severe damage to the tubes: the resultant high-amplitude vibration may sometimes cause inter-tube clashing and always wear with the tube-supporting baffle plates [1], eventually leading to breakdowns.

Two mechanisms are generally accepted as being responsible for fluid elastic instability [2–5]: a velocity controlled, negative damping mechanism, and a stiffness controlled one. The first mechanism gives rise to single-degree-of-freedom flutter; hence array stability may be analyzed, approximately, by considering all cylinders of the array except one to be rigid, and considering motions of just that one cylinder [6; 4, 5]. The second mechanism involves at least two degrees of freedom and gives rise to coupled-mode flutter [5]. To clarify matters further by reference to well known aeroelastic phenomena, the first mechanism is similar to Den Hartog's for the galloping of iced transmission lines; the second is akin to wake flutter of aeroplane wings.

For cylinder arrays, both mechanisms are generally at work, although the contribution of one of the two may in some cases be so small as to be negligible; thus, for instance,

allowing in some circumstances the analysis of a fully flexible array of cylinders by considering only one to be flexible, as mentioned above. For some array geometries,[†] it has been found that for low values of the mass-damping parameter,[‡] $m\delta/\rho D^2$, say for $m\delta/\rho D^2 < 300$ approximately, the negative damping instability is dominant. This is the case, for instance, for rotated triangular arrays and square arrays. However, this is not always true. For *rotated* square arrays, even for low values of $m\delta/\rho D^2$, the instability is of the two-degree-of-freedom variety; moreover, this type of array is prone to be subject to a recently discovered static instability, i.e. fluid-elastic divergence [7].

There are three main analytic or quasi-analytic (semi-empirical) models for predicting oscillatory fluid-elastic instability [2, 4, 6, 8, 9], as well as others. The interested reader is referred to Chen's [10] and Paidoussis' [11] discussions of alternative classifications of these models, as well as to Paidoussis and Price's [5] discussion of the mechanisms giving rise to the instability in simple terms. All these models are linearized, both in terms of the fluid-dynamic forces acting on the cylinders and in terms of structural motions. Of course, in reality, non-linear fluid and structural forces come into play when the oscillation amplitude becomes substantial, limiting motion into a stable limit cycle. The effect of fluid-force non-linearities on the dynamics of arrays of cylinders in cross-flow has been studied by Price and Valerio [12]. However, in the particular case of heat-exchanger type cylinder arrays, the amplitude limitation may be controlled by another, physical, motion-limiting constraint, as will be explained below.

In heat exchangers, the tubular cylinders are threaded through the baffle plates (the purpose of which is to direct the flow and to support the cylinders) through holes, which are oversize to permit easy threading and to allow for thermal expansion. After some use, crud may be deposited on the baffle plates, rendering some, but not all, loose-fitting holes into positive supports; also, the fact that the tubes are not perfectly straight originally helps explain how in a real heat exchanger some baffle plates provide positive support to any given cylinder, while others do not. Now, after the onset of fluid elastic instability, impacting will occur with the baffle plate at the loose supports, which is known to give rise to structural non-linear forces [13, 14]. This non-linearity is considered to perhaps be more important for heat exchanger arrays than the fluid-force non-linearities, since impacting occurs even at relatively small amplitude (the gap to the baffle plate being generally small), when fluid non-linearities are still relatively unimportant.

The present paper is concerned with the dynamics of heat exchanger type cylinders in cross-flow, of the kind (in terms of geometry and values of δ and $m/\rho D^2$) where the single-degree-of-freedom, negative damping instability is dominant. Hence, the analysis is carried out by considering a single flexible cylinder, with all the other cylinders in the array rigid and immobile; the analytic model used is that of Price and Paidoussis [4]. For the reasons given in the previous paragraph, the only non-linearities considered here are those associated with impacting on the baffle plates.

Although the proposed model is in many ways close to an actual heat-exchanger type array, some simplifications and idealizations have been introduced in order to enable the analysis to bring out the essential dynamics, without the encumbrances that more realistic modelling would entail. Thus, although tubular cylinders in most heat exchangers are multi-span, here a two-span model will be considered, with the extremes of the cylinder being positively supported (clamped) and the middle support being loose; see Figure 1(a).

[†] Here "geometry" is characterized by the layout of the cylinders (e.g., into square or triangular patterns) and by their orientation *vis-à-vis* the upstream flow velocity vector (thus, "normal triangular", or "rotated square" arrays).

[‡] Here m is the cylinder mass per unit length, δ is the logarithmic decrement of damping, the ρ is the fluid density and D is the cylinder diameter.

Figure 1. (a) The system under consideration, showing the flexible cylinder positively supported at its extremities and with a loose support at x_b , subjected to cross-flow. (b) A cross-sectional view of the idealized system, in which the cylinder vibrates only in the cross-stream direction and the loose support (gap $d \ll D$) is modelled as shown.

Also, although in a real heat exchanger the baffle plates are circular and hence post-impact sliding could involve quasi-orbital motion of the flexible cylinders,[†] it will here be assumed for simplicity that motions will be planar and in the direction of the single degree of freedom considered, in the *cross-stream* direction.

The main motivation for this work is to show definitively, in quantitative terms, whether chaotic oscillations are possible in such systems. Axisa *et al.* [13] obtained chaotic-looking oscillations with their model of an impacting heat exchanger tube; however, since they did not conduct calculations which would provide a quantitative measure of whether the oscillations were truly chaotic, the question of the existence of chaos in such systems was not unequivocally answered in that work. Nevertheless, a most significant finding of that work [13] was that increased wear rates are obtained with the higher frequencies associated with such oscillations. Thus, this problem is of both fundamental and practical interest.

The dynamics of the system are studied through simulation, obtaining bifurcation and phase flow diagrams and also Lyapunov exponents. In order to understand the dynamics better and explain some of the phenomena observed in the numerical results, an even more simplified model is examined in the last part of the paper, involving a single-degree-of-freedom, negatively damped impact oscillator under the excitation of harmonic forcing. Such a model allows the performance of some non-linear analysis, which helps toward understanding the underlying mechanism for the observed chaotic behaviour.

2. THE EQUATION OF MOTION

The equation of motion of the cylindrical beam subject to cross-flow may be written in the form

$$EI \frac{\partial^4 w}{\partial x^4} + c \frac{\partial w}{\partial t} + m \frac{\partial^2 w}{\partial t^2} + \delta(x - x_b) f(w) = F(w, \dot{w}, \ddot{w}), \quad (1)$$

[†] The dynamics of generalized impacting of a circular cylinder in a circular hole can be very complex, involving both Coulomb and sliding frictional effects.

where w is the cross-stream lateral displacement of the cylinder, El its flexural rigidity, m its mass per unit length, and c is the damping coefficient; f is the non-linear force due to the constraint, with $\delta(x - x_b)$ being the Dirac delta function and x_b the constraint location (see Figure 1(a)), and F is the flow-induced force. The cylinder is supposed to be clamped at its two extreme supports (Figure 1(a)) rather than simply supported, to disallow natural frequencies which could be integral multiples of the first.† The flow-induced force F may generally be expressed as

$$F(w, \dot{w}, \ddot{w}) = M \frac{\partial^2 w}{\partial t^2}(x, t) + B \frac{\partial w}{\partial t}(x, t) + Cw(x, t - \Delta t), \quad (2)$$

where, in accordance with the Price and Païdoussis [4, 8] model,

$$M = -\frac{\pi}{4} \rho D^2 C_{ma}, \quad B = -\frac{1}{2} \rho U D C_D, \quad C = \frac{1}{2} \rho U^2 D \frac{\partial C_L}{\partial w}, \quad \Delta t = \mu \frac{D}{U},$$

and D is the diameter of the cylinder, ρ and U are the fluid density and velocity, respectively, C_D and C_L are the drag and lift coefficients, respectively, based on the flow velocity in the gap between the cylinders, C_{ma} is the virtual or “added” mass coefficient of the fluid around the cylinder, Δt is the time delay, and $\mu \sim \mathcal{O}(1)$; see also Païdoussis and Price [5]. The origin of the delay Δt is in the assumption made in this quasi-steady model that there is a delay (phase lag) between cylinder motions and the fluid-dynamic forces generated thereby, associated with re-adjustment of the viscous wake to the altered position of the cylinder *vis-à-vis* its neighbours.

It should be mentioned here that if another model leading to a negative-damping type of fluid-elastic instability had been utilized instead of that of references [4, 8], qualitatively similar results would have been obtained; thus, the conclusions of this study are not specific to the fluid-elastic instability model utilized. None of the available mathematical models are “perfect”; for a discussion of strengths and weaknesses of the available models, the reader is referred to references [10, 11].

By introducing the following non-dimensional quantities, in which the cylinder length is denoted by L ,

$$\begin{aligned} \eta &= \frac{w}{D}, & \xi &= \frac{x}{L}, & \tilde{t} &= \lambda_1^2 \sqrt{\frac{El}{mL^4}} t = \Omega_1 t, & \zeta &= \frac{1}{m\Omega_1}, \\ \tilde{m} &= \frac{m}{\rho D^2}, & \tilde{U} &= \frac{2\pi U}{D\Omega_1}, & \tilde{f} &= \frac{f}{m\Omega_1^2}, & \beta &= 1 \left/ \left(1 + \frac{4\tilde{m}}{\pi C_{ma}} \right) \right., \end{aligned} \quad (3)$$

and then dropping the tilde over time t , equation (1), combined with (2), may be written in the following dimensionless form:

$$\begin{aligned} \frac{1}{1-\beta} \frac{\partial^2 \eta}{\partial \tau^2}(\xi, \tau) + \left(\zeta + \frac{\tilde{U} C_D}{4\pi \tilde{m}} \right) \frac{\partial \eta}{\partial \tau}(\xi, \tau) + \frac{1}{\lambda_1^4} \frac{\partial^4 \eta}{\partial \xi^4}(\xi, \tau) \\ - \frac{\tilde{U}^2}{8\pi^2 \tilde{m}} \frac{\partial C_L}{\partial \eta} \eta(\xi, \tau - \tau) + \delta(\xi - \xi_b) \tilde{f}(\eta) = 0, \end{aligned} \quad (4)$$

where λ_1 is the dimensionless eigenvalue of the first mode for a clamped beam, and $\tau = 2\pi/\tilde{U}$ for $\mu = 1$.

† Supports in real heat exchangers are somewhere between a simple support and a clamped one, and exact integral relationships between frequencies are rather rare.

The system is now discretized by utilizing the standard Galerkin expansion

$$\eta(\xi, t) = \sum_{i=1}^N \phi_i(\xi) q_i(t), \quad (5)$$

where the $\phi_i(\xi)$ are the dimensionless orthonormal set of eigenfunctions for the clamped-clamped beam and have the properties:

$$\int_0^1 \phi_i(\xi) \phi_j(\xi) d\xi = \begin{cases} 1 & \text{for } i=j, \\ 0 & \text{for } i \neq j; \end{cases} \quad \text{and} \quad \int_0^1 \phi_i(\xi) \frac{d^4 \phi_j(\xi)}{d\xi^4} d\xi = \begin{cases} \lambda_i^4 & \text{for } i=j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Upon substituting equation (5) into (4), multiplying by $\phi_j(\xi)$ and integrating over the domain $[0, 1]$, one obtains

$$\frac{1}{1-\beta} \ddot{q}_i(t) + \left(\frac{\delta_i v_i}{\pi} + \frac{\tilde{U} C_D}{4\pi \tilde{m}} \right) \dot{q}_i(t) + v_i^2 q_i(t) - \frac{\tilde{U}^2}{8\pi^2 \tilde{m}} \frac{\partial C_L}{\partial \eta} q_i(t-\tau) + \tilde{f}(\eta_b) \phi_i(\xi_b) = 0, \quad (6)$$

where $v_i = (\lambda_i/\lambda_1)^2$ are the ratios of the non-dimensional natural frequencies to the first one, η_b is the displacement at $\xi = \xi_b$, and the viscous damping term has been replaced by the modal damping $\delta_i v_i/\pi$, in which it has been assumed that damping coefficients are small.

Notice that the only coupling term in the above equation is the force $f(\eta_b)$ due to the constraint at the intermediate loose support. If there were no such constraint, then the resulting equations would be decoupled.

In what follows, equation (6) will be studied by modelling the constraint force by a cubic spring, $f(\eta) = \kappa \eta^3$, and by a trilinear spring, $f(\eta) = \kappa[\eta - \frac{1}{2}(|\eta + d| - |\eta - d|)]$, respectively, where $2d$ represents the gap at the constraint, non-dimensionalized with respect to D . For the cubic spring representation of the constraint force, symmetric and asymmetric cases are studied separately ($\xi_b = 0.50$ and 0.45 , respectively, in sections 4 and 6). The cubic spring representation is less physically realistic than the trilinear one; the reason for utilizing it at all is that it is analytic and so permits calculations (e.g., of the Lyapunov exponents, section 5) which are not possible otherwise.

Some of the calculations are time-consuming and hence they were all conducted for a single set of physical parameters. The system chosen was a square array with pitch-to-diameter ratio $p/D = 1.5$, for which $C_D = 0.26$ and $\partial C_L/\partial \eta = -8.1$ have been measured [4].† To avoid complications arising from multiple stability boundaries [4], a not-too-low value of $\tilde{m}$ and δ were chosen, namely $\tilde{m} = 3$ and $\delta_i = 0.06$ for all i . Note that $\tilde{m} = 3$ signifies that, realistically, the fluid is a liquid (e.g., water flowing around brass tubes); with $C_{ma} = 1.2$ this leads to $\beta = 0.24$.

Concerning the impact stiffness, Antunes [15] calculated a value of 6.6×10^6 N/m, while Yetisir and Weaver [16] give values of 10^6 – 10^7 N/m for impact with anti-vibration bars (AVB). Adopting the range of 10^6 – 10^7 N/m, this gives a non-dimensional stiffness $\kappa = 10^3$ – 10^4 . Since numerical simulation showed that the dynamics of the system are qualitatively insensitive to κ in this range, $\kappa = 10^3$ is used throughout.

In summary, unless otherwise stated, the fixed parameters in this study are $C_D = 0.26$, $\partial C_L/\partial \eta = -8.1$, $\tilde{m} = 3$, $\delta_i = 0.06$, $\beta = 0.24$ and $\kappa = 10^3$. The main parameter to be varied is the dimensionless flow velocity, $\tilde{U}$.

In what follows, all quantities are understood to be dimensionless, so the tildes on $\tilde{U}$, $\tilde{f}$ and $\tilde{m}$ are henceforth dropped.

† Note that the values of drag, lift and their derivatives given by Price and Paidoussis [4] have been converted here from those corresponding to far-upstream flow velocity to values associated with the so-called reference gap velocity, by dividing by $[p/(p-D)]^2$, equal to 9 in this case.

3. THE HOPF BIFURCATION

To find the critical flow velocity for the onset of instability, all system parameters are fixed as defined in the previous section, and the dimensionless flow velocity U is gradually increased. Eventually, for high enough U , there is a point, U_H , at which the stable equilibrium (at the undeformed position) of the cylinder is lost. Thereafter, for any $U > U_H$, the motion of the cylinder is amplified, until the non-linear elements of the system counter-balance the growth, bringing the system to a steady oscillatory state. A Hopf bifurcation is defined as the loss of the stable equilibrium and onset of amplified oscillation.

The critical value, U_H , may be found by solving the linearized system of equations (6) by letting $f(\eta_b) = 0$. Upon substituting the parameter values given in section 2, one obtains

$$\ddot{q}_i(t) + (\alpha_1 + \alpha_2 U) \dot{q}_i(t) + \alpha_3 q_i(t) + \alpha_4 U^2 q_i(t - \tau) = 0, \quad (7)$$

where $\alpha_1 = 0.0145\nu_i$, $\alpha_2 = 0.00524$, $\alpha_3 = 0.76\nu_i^2$, $\alpha_4 = 0.026$; and ν_i and thus α_i are known for each mode. The above equations are decoupled and thus can be treated independently. For the i th mode, assuming a solution of the form

$$q_i = q_0 e^{j\omega_i t},$$

in which j is the imaginary unit and ω_i is the i th dimensionless eigenfrequency, and substituting it into equation (7), gives

$$-\omega_i^2 + (\alpha_1 + \alpha_2 U)j\omega_i + (\alpha_3 + \alpha_4 U^2 e^{-j\omega_i \tau}) = 0.$$

At the threshold of the Hopf bifurcation, $U = U_H$, ω_i is purely real, and this equation leads to

$$-\omega_i^2 + \alpha_3 + \alpha_4 U_H^2 \cos \omega_i \tau = 0, \quad (\alpha_1 + \alpha_2 U_H)\omega_i - \alpha_4 U_H^2 \sin \omega_i \tau = 0, \quad (8)$$

where $\tau = 2\pi/U_H$. For fixed parameter values, U_H and ω_i are the only unknowns for each mode, and are solved for numerically; their values are listed in Table 1.

Of all the critical flow velocities, the first one, $U_{H1} = 1.785$, is the most important, since thereafter an oscillatory state prevails for the double-span system of Figure 1. However, higher velocities are also of interest ($U > U_{H1}$), since in the design of such systems the value of U_H is normally computed assuming a *single* span (i.e., of length $L/2$ in Figure 1). Referring to the definition of the dimensionless U in equation (3), it is seen that, based on $L/2$, the "design" dimensional critical flow velocity is higher than even that of the second mode Hopf bifurcation found here, because the middle support is loose. If three or more spans had been considered, and they happened to be loose also, one can see that flow velocities many times larger than U_{H1} are of interest.

TABLE 1
Critical flow velocity values, U_H , and the corresponding non-dimensional natural frequencies

Mode	U_H	ω_i
1	1.785	0.824
2	4.920	2.272
3	9.645	4.455
4	19.944	7.364
5	23.818	11.000

Of course, linear analysis, yielding the Hopf bifurcation, cannot give any information about limit-cycle motion; moreover, in the linear system the modes are uncoupled. Nevertheless, it is instructive to discuss here how limit-cycle motion can develop with the model in this paper. Once U exceeds U_H , amplified motions lead to impact with the loose support, whereupon the modes do become coupled. Thereafter, the energy flows into the system (from the flowing fluid) in the mode or modes for which $U > U_H$, while the higher modes that are still stable provide damping. Thus, a balance between the negatively damped unstable modes, dumping energy into the system, and the positively damped stable ones, dissipating it, leads to a state of dynamic equilibrium, i.e., to a limit cycle.

4. BIFURCATION DIAGRAM FOR THE SYMMETRIC CASE

Numerical simulations of the dynamics of certain types of cross-flow systems have been reported by many researchers (see Axisa *et al.* [13] and Fricker [14], for example). But all of them were presented in the form of either phase flows or time histories for certain individual values of the flow velocity, with other parameters fixed. With the fast computers now available, it is easy and helpful to display the system response for some continuous ranges of the flow velocity in order to obtain a full understanding of system dynamics; this is also helpful in deciding more definitely if the system becomes chaotic or not. Such a diagram also allows one to observe and pinpoint any sudden changes (bifurcations) in the system behaviour. The specific choice of the output signal to be processed in this way depends on the individual system, and usually the one which can provide most information is used.

For the present system, the simulation was carried out by integrating equation (6) using the standard fourth order Runge-Kutta algorithm. The parameter varied is U , and the output utilized to display bifurcations is the approximate dimensionless displacement at the middle point of the cylinder, $\eta(0.5, t)$, obtained through equation (5). The triggering signal is $\partial\eta(0.5, t)/\partial t$: when it is equal to zero and $\eta(0.5, t) > 0$, then the value of $\eta(0.5, t)$ is recorded. In the bifurcation diagrams to be presented, solutions for the first 200 time units were discarded, so as to achieve steady state solutions, and then $\eta(0.5, t)$ was recorded for the next 50 time units.

Before proceeding with the computation, it is important first to decide on the number of modes necessary in the simulation to generate the essential dynamics of the physical system. For the cubic constraint model (symmetric when $\xi_b = 0.5$ and asymmetric when $\xi_b = 0.45$) and the trilinear one, phase plots were generated for values of U before and after the critical value for instability. Study of these phase flows indicates that simulation results from models with five modes and higher are qualitatively the same both in magnitude and in shape when the integration step size is sufficiently small, and thus five-mode models ($N=5$) are used in all the simulations to follow.

The bifurcation diagram for the symmetric cubic model is displayed in Figure 2. It is clearly seen that the Hopf bifurcation occurs at $U_H = 1.78$, as predicted earlier, and thereupon stability is lost, leading to the birth of a limit cycle; for $U < U_H$, all oscillations, if any, die out as time becomes large. It can also be seen that the first post-Hopf bifurcation occurs at $U_b = 3.24$, followed by another one at $U_{ch} = 4.39$. In order to have a clear picture of the system dynamics in the neighbourhood of these bifurcations, phase flows were computed for some specially chosen U , and are shown in Figure 3.

A regular limit cycle for $U = 3.0$ ($U_H < U < U_b$) is shown in Figure 3(a). After the first post-Hopf bifurcation, the cylinder vibrates slightly once, before it "leaves" the spring†

† Strictly speaking, the cylinder is always in contact with the spring for the cubic model of the constraint. However, the spring force *versus* deflection characteristic has the following feature: the spring force is very small for small deflections and then abruptly very high for large deflections; hence, one may use such words "leaving" and "hitting" the spring.

Figure 2. The bifurcation diagram for the system representing an in-line square array with $p/D=1.5$ ($C_D=0.26$, $\partial C_L/\partial \eta=-8.1$), $\dot{m}=3$, $\delta_i=0.06$, $\beta=0.24$ and $\kappa=10^3$, showing the dimensionless mid-point displacement amplitude in terms of the dimensionless fluid velocity, U . The loose support, positioned at the cylinder mid-point ("symmetric" system, $x_b/L=0.5$), is modelled here as a cubic spring. U_H denotes the critical U for the Hopf bifurcation, and U_b is the first post-Hopf bifurcation; U_{ch} denotes the onset of chaos.

and moves to the other side, as seen in Figure 3(b) for $U=3.6$ where the motion is symmetric. A further increase in U drives the cylinder to hit the springs harder, and the corresponding phase flow, as shown in Figure 3(c) for $U=4.0$, looks still periodic but may have higher period. In Figure 3(d) is shown the phase flow for $U=4.3$ before the occurrence of chaos; it is seen that sudden changes of the direction of the phase flow appear. As U increases further, chaotic response develops, as indicated by the orderless phase flow shown in Figure 3(e). Regardless of how long the simulation was carried out, no pattern could be observed. Qualitatively speaking, this phase flow is similar to the flow for a two-well potential oscillator, where homoclinic orbits exist and the homoclinic bifurcations lead the system to chaos, as seen in Figure 4 of reference [17].[†]

A common feature of chaotic systems is that, within the chaotic regimes, there exist windows in which the system response is periodic. In Figure 3(f) is shown such an example for the present system ($U=4.82$). This periodic window can also be seen fairly clearly in Figure 2, buried in the chaotic region. It should be commented that chaotic response in physical systems may become periodic (transient chaos), and periodic motion surrounded by chaos may also become chaotic with the increase in time. The phase flow in Figure 3(f) was recorded in a time period of 500–520 non-dimensional units.

5. LYAPUNOV EXPONENTS

There are many ways in which to characterize chaotic motions. Phase flows, Poincaré maps and power spectra, to name but a few, are common techniques utilized for this purpose, but are only useful for each individual signal. To study the system dynamics over a range of a given parameter, for example the flow velocity, the appropriate tool is the Lyapunov exponent technique, especially useful in numerical simulation. Unlike some of the other aforementioned techniques, this one gives a quantitative measure for the onset of chaos. For each given parameter value, one can compute a Lyapunov exponent for the corresponding motion. The motion is attractive (a stable fixed point), periodic or chaotic, if the exponent is negative, zero or positive, respectively. This simple procedure can easily

[†] The underlying mechanism which makes the phase flows here similar to those in Moon's non-autonomous problem [17] could not be pinpointed. Nevertheless, similar structures can be observed in other systems which, on the surface at least, are quite different: see, for example, Ueda's [18] and Plaut and Hsieh's [19] work.

Figure 3. Phase flow diagrams associated with the bifurcation diagram of Figure 2, for the following values of U : (a) $U=3.0$; (b) $U=3.6$; (c) $U=4.0$; (d) $U=4.3$; (e) $U=4.61$; (f) $U=4.82$.

be repeated for different values of the parameter over the range of interest, and the results saved and looked at later.

Lyapunov exponents for a set of *delay* differential equations, such as those for the system under study, can be derived in the same spirit as that utilized for ordinary differential equations. To proceed with this derivation, the delay differential equation in the form

$$\dot{x}(t) = f(x(t)) + g(x(t - \tau)) \quad (9)$$

is considered, where x represents an n -dimensional vector function, and f and g are analytic functions of x at t and $t - \tau$, respectively. Denoting by $\phi(t)$ a solution of equation (9), and introducing a variational vector function $u(t)$ such that $x(t) = \phi(t) + u(t)$, then substituting $x(t)$ into equation (9) and Taylor expanding yields

$$\dot{u}(t) = Df(\phi(t))u(t) + Dg(\phi(t - \tau))u(t - \tau), \quad (10)$$

where Df and Dg are the Jacobian matrices of f and g along the trajectory $\phi(t)$, and the non-linear terms have been discarded. With $\phi(t)$ known, by numerically integrating equation (9) for given initial conditions, the solution of (10) gives information concerning the stability of $\phi(t)$. For a bounded solution $\phi(t)$, its Lyapunov exponents are defined by

$$\sigma \rightarrow \frac{1}{T} \ln \frac{|u(T)|}{|u(0)|} \quad \text{as } T \rightarrow +\infty. \quad (11)$$

There are n values of σ for each ϕ , since n linearly independent $u(0)$ may be chosen in $\mathbb{R}^n$. However, only the maximum exponent, denoted simply by σ , is important for detecting chaos, and this σ may be computed by assigning an arbitrary $u(0)$ [20].

The computation of Lyapunov exponents for the cubic model was carried out by first writing equation (6) in its first order form and then deriving the variationals:

$$\begin{aligned} \dot{q}_i &= p_i, & \dot{p}_i &= -(a_1 + a_2 U)p_i - a_3 q_i - a_4 U^2 q_i(t - \tau) - \kappa \left(\sum_j \phi_j q_j \right)^3 \phi_i, \\ \dot{u}_i &= v_i, & \dot{v}_i &= -(a_1 + a_2 U)v_i - a_3 u_i - a_4 U^2 u_i(t - \tau) - \kappa \left(\sum_j \phi_j q_j \right)^2 \left(\sum_j \phi_j u_j \right) \phi_i, \end{aligned} \quad (12)$$

where the only delay terms are those associated with the coefficients a_4 ; the eigenfunctions $\phi_j(\xi)$ are evaluated at $\xi = \xi_b$. When an N -mode model is employed in the computation, then $i, j = 1, 2, \dots, N$.

The exponents for the symmetric case of section 4 are shown in Figure 4(a), where $\xi_b = 0.5$. It is clearly shown that for $U < U_H = 1.78$, the exponents are indeed negative, and are nearly zero for periodic motions, until the point at which the motion becomes chaotic, at U_{ch} . Direct comparison of the results in Figure 4(a) with the bifurcation diagram in Figure 2 shows that the system indeed undergoes a Hopf bifurcation at U_H and the transition to chaos occurs at U_{ch} . This result also confirms the validity of the application of the Lyapunov exponent technique to delay differential systems. However, one may have noticed that the periodic window in the neighbourhood of $U = 4.8$, shown in Figure 4(a), is considerably larger than the one shown by the bifurcation diagram in Figure 2; an alternative possibility is that this region contains numerous thin periodic regions interspersed between equally thin chaotic ones. Individual checks of some points in this neighbourhood showed that some seemingly chaotic motions (observed from the phase flows) do have near-zero Lyapunov exponents. Such motions may well be quasi-periodic, as will be suggested by an analysis for a simplified model in section 8.

Notice that the original definition of the Lyapunov exponent, equation (11), is over an infinite time interval. This, of course, is not practical in numerical simulation and was never achieved. In practice, the result obtained from calculations over a finite time period is perfectly acceptable if convergence is reached. This averaging process is shown in Figure 4(b) over a time interval of 800 non-dimensional units (that over the first 200 was discarded) for $U = 1.5, 3.0$ and 4.61 , respectively; these three cases correspond to attractive, periodic and chaotic motions.

There are many technical subtleties in the implementation of the computation of Lyapunov exponents for delay differential equations. The most important one is the process of renormalization during the computation. Notice that when $\sigma > 0$ (in the chaotic region), the solution to the variational equation (10), which is linear, will blow up as time approaches infinity. Numerically, one corrects this problem by renormalizing $u(t)$ within some finite time intervals. When doing so, one must also renormalize the history in $(t - \tau, t)$, since this information is needed for the computation later.

Figure 4. (a) The largest Lyapunov exponent for the system of Figure 2 as a function of the dimensionless flow velocity, U ; (b) the rate of convergence of the calculation for three specific values of U as a function of dimensionless time.

With the results discussed in this section it is considered that the main question raised in this study has been answered: chaotic oscillations are indeed possible for sufficiently high flow velocities.

6. BIFURCATION DIAGRAM FOR THE ASYMMETRIC CASE

So far, the necessary tools have been developed for characterizing chaotic responses and they have successfully been applied to the symmetric model ($\xi_b = 0.5$). Now it is of interest to investigate what changes would result if the stop constraint was placed off the mid-cylinder point. It is noticed that, when $\xi_b = 0.5$, due to symmetry the even modes do not contribute, and thus do not affect the system behaviour; they can be entirely ignored. When $\xi_b \neq 0.5$, however, full participation of all modes is expected in the simulated dynamics.

In Figure 5 are displayed the bifurcation diagram and the corresponding Lyapunov exponents in the velocity range of 3.0–4.5 for $\xi_b = 0.45$. The Hopf bifurcation remains unaffected, at $U_H = 1.78$. Then, at $U_b = 3.39$, another bifurcation occurs, and the cylinder now begins to oscillate one more time “locally” before it moves to the other side of the spring, this being analogous with the symmetric case (cf. Figure 3(b)). At $U_{ch} = 4.12$, chaotic motion begins to appear, and the solution blows up for U near 4.5. A major difference between the two bifurcation diagrams, Figures 2 and 5(a), and the corresponding

Figure 5. (a) The bifurcation diagram for the asymmetric version ($x_b/L=0.45$) of the system of Figure 2; (b) the corresponding largest Lyapunov exponents.

Figure 6. A typical phase flow diagram for the system of Figure 5 (asymmetric location of loose support, $x_b/L=0.45$); $U=4.3$.

Lyapunov exponent diagrams, Figures 4(a) and 5(b), is that the periodic window just beyond the onset of chaos, observed in the symmetric case, did not materialize in the asymmetric case. A typical chaotic phase flow diagram for the asymmetric case is shown in Figure 6.

7. NUMERICAL RESULTS FOR THE TRILINEAR MODEL

Up to this point, the constraint force has been modelled by a cubic spring, $f(\eta) = \kappa\eta^3$, which represents a compromise: although this form of f is mathematically convenient, it

does not model the physical situation perfectly, since it does not recognize that in the free gap, the constraint force is zero, rather than just small. Therefore, in this section, the constraint is modelled by a trilinear spring, $f = \kappa[\eta - \frac{1}{2}(|\eta + d| - |\eta - d|)]$, which is obviously closer to physical reality when κ and d are chosen properly. It is noticed that the representation of the trilinear stiffness is continuous but not analytic at $|x| = d$ (i.e., at the contact points), and thus no Lyapunov exponents can be computed from the governing differential equations.

The phase trajectory for $U = 2.5$ ($> U_H$) is shown in Figure 7. It was simulated by setting a small initial displacement ($\eta < d$) and zero initial velocity. With increasing time, the phase trajectory begins to grow until it impacts with the stop constraints. Thereafter, the cylinder vibrates more irregularly back and forth within the gap, repeatedly striking the two sides of the loose support. In an attempt to see any possible convergence to a limit cycle, the simulation was performed for the rather long time period of 1000 non-dimensional time units; however, no periodic pattern was found.

In Figure 8(a) and (b) are shown the phase flows for $U = 1.9$ (just after the Hopf bifurcation) and $U = 3$, with the transient trajectories of the first 500 time units discarded. For the case of $U = 1.9$, the response appears to have a small chaotic component superimposed on a larger periodic component indicated by the main peak of its power spectrum, shown in Figure 8(c). For the case of larger flow velocity, $U = 3.0$, the phase plot looks more chaotic (Figure 8(b)), and the corresponding power spectrum is broad-banded (Figure 8(d)). The same trends continue for higher flow velocities. The increase in response frequencies with flow is of particular importance, since that would result in increased wear of the tubes in heat exchangers.

Although the constraint force is modelled by a trilinear stiffness with $\kappa = 1000$, the system essentially behaves as if it were impacting on two rigid stops. This may be seen from the phase flows in Figures 7 and 8: when the cylinder hits one side of the constraint, it immediately bounces back. The resulting motion appears to be chaotic, without any pattern. This behaviour is similar to that of a zero-stiffness impact oscillator where chaotic motions are possible even for very small forcing [21]. A similar phenomenon was also observed for a pipe conveying fluid when the motion-limiting constraint was modelled by a trilinear spring [22]. In general, for a multiple-degree-of-freedom non-linear system with delay terms, the theoretical analysis is difficult, and may be practically impossible, and one is usually limited to numerical simulation. For the present system with a trilinear spring model, it may be useful to simplify the system further, to a point which will allow some

Figure 7. Phase flow diagram for the system with trilinear modelling of the constraint stiffness at the loose support (at $x_b/L = 0.5$), for $U = 2.5$, showing the growth of the oscillation in response to impact with the loose supports, and the subsequent motion.

Figure 8. (a, b) Phase flow and (c, d) power spectral density diagrams for the system with trilinear modelling of constraint stiffness: (a, c) for $U=1.9$, just beyond the Hopf bifurcation; (b, d) for $U=3.0$ showing strongly chaotic motion.

meaningful analysis and could provide some insight regarding the underlying mechanism leading to chaos.

Considering all the results obtained in sections 4–7, the situation may be summarized as follows. With both models of the impact stiffness and for the symmetric and asymmetric loose-baffle location, it has been shown that, for sufficiently high cross-flow, chaotic oscillations are possible. Hence, perhaps the erratic, chaotic-looking oscillations reported by Axisa *et al.* [13] were chaotic also.

Nevertheless, there remain several unanswered questions. First, the route to chaos for the present models is still unclear, although qualitative changes in the bifurcation diagrams can be clearly observed. The second question is the significance of the apparently chaotic behaviour obtained for some parameter ranges with the symmetric, cubic-spring model (Figure 2 for $U \approx 4.7$ – 5.0), which are shown to be periodic by the Lyapunov exponent calculations (Figure 4(a)). The third question is related to the result obtained with the trilinear spring, where it was found that chaotic motions appear to occur at any U large enough to cause impacting. It will be attempted to provide possible answers to these remaining questions in the next section, through an even further simplified, lower-dimensional model.

8. ANALYSIS WITH A SIMPLIFIED MODEL

In its simplest form, the analytic model utilized in the foregoing has only two modes: one which for high enough U develops negative damping, and the other with positive damping; the two are coupled via the impact-related non-linear stiffness terms. The modes concerned are the lowest two interactive modes: for the symmetric case, the first and the

third; for the asymmetric case, the first and the second. Here, the simplest possible model is proposed, namely a one-degree-of-freedom model involving a negatively damped, forced impact oscillator. The negative damping feature is carried over directly from the previous model, and is associated with the lowest mode behaviour therein. The positively damped higher modes are represented in the model by the forcing function, where it is understood that the forcing frequency ought to be substantially higher than that of the oscillator; finally, the damping associated with the higher mode(s) is represented by a restitution coefficient less than unity. This forced impact oscillator cannot pretend to model the physics of the real system; nevertheless, it does possess some similar key features, and it is in this sense that it is pursued.

The mathematical model for the impact oscillator described above is a second order piecewise linear differential equation for which there exist explicit solutions for the time intervals between any two consecutive impacts. The non-dimensional equation of the oscillator takes the form

$$\begin{aligned}\ddot{x} + 2\zeta\dot{x} + x &= \gamma \cos \omega t, & \text{for } |x| < 1, \\ \dot{x} &\rightarrow -r\dot{x}, & \text{for } |x| = 1,\end{aligned}\tag{13}$$

where ζ (< 0) is the negative damping coefficient, and r (< 1) is the restitution coefficient, representing the energy dissipation in the system, ω (> 1) and γ represent the forcing frequency and amplitude, respectively; the system is non-dimensionalized such that impact occurs at $|x| = 1$ rather than at the real physical stops.

It should be remembered that there have been many studies on piecewise linear impact oscillators, notably by Shaw [23] and Li *et al.* [21]. The system (13) looks rather simple, but its dynamics are extremely rich and complicated, and could themselves be the subject of a separate study (the case of $\zeta = 0$ is reported by Shaw [23]). Here only some key results are presented which may be relevant to this problem, and which may provide possible explanations to some of the unanswered questions referred to at the end of the previous section. Before proceeding, the Poincaré map technique for system (13) is introduced, as it is needed in later analysis.

8.1. THE POINCARÉ MAP

Equation (13) is three-dimensional, and may be written into a set of first order equations by introducing a phase angle ϕ :

$$\begin{aligned}\dot{x} &= y, & \dot{y} &= -2\zeta y - x + \gamma \cos \omega \phi, & \text{for } |x| < 1, \\ \dot{\phi} &= 1, & y &\rightarrow -ry, & \text{for } |x| = 1,\end{aligned}\tag{14}$$

where $\phi = t \pmod{2\pi/\omega}$ is simply the time t , except that it only takes values from 0 to $2\pi/\omega$. The central idea of the Poincaré map technique is to choose a cross-section and convert a continuous system into a discrete one on this surface, reducing the system dimension by one. Equation (14) is discontinuous at $|x| = 1$, and thus it is convenient to choose such a cross-section at $x = 1$ (or -1):

$$\Sigma = \{(x, y, \phi) \in \mathbb{I} \times \mathbb{R}^+ \times \mathbb{S} : x = +1, y > 0, \phi \in [0, 2\pi/\omega]\},$$

where $\mathbb{I} = [-1, 1]$, and $\mathbb{S}$ is the circle of $2\pi/\omega$. The variables in the Poincaré section are ϕ and y . The Poincaré map is a mapping from Σ onto Σ , defined symbolically by

$$(\phi, y) = P(\phi_0, y_0).\tag{15}$$

In words, P takes the point (ϕ_0, y_0) in Σ back to another point (ϕ, y) still in Σ via equations (14). If the orbit is periodic, then $(\phi, y) = (\phi_0, y_0)$.

The advantage of employing this technique for system (13) is that thereby one is effectively dealing with a two-dimensional discrete map rather than a three-dimensional discontinuous flow, and the stability information of a continuous periodic solution of (13) may be obtained by computing eigenvalues of P at the corresponding fixed point. The solution is stable if all $|\lambda_i| < 1$. The computation of P is made through obtaining solutions (14) in intervals between two consecutive impacts and using the rebound condition at the two boundaries.

8.2. DETERMINATION OF PERIODIC SOLUTIONS

Equation (14) contains many different types of periodic solutions of order n , depending on the values of the system parameters and the initial conditions. The simplest type of such solutions is the one in which the system impacts only once with each of the two boundaries, and spends a period of $2n\pi/\omega$ to complete a whole cycle. This is the type of solution considered in the following.

To find such a periodic solution, it is assumed that the oscillator starts at a point p_0 on the Poincaré section $p_0 = (x_0 = 1, y_0 > 0, \phi_0) \in \Sigma$. A complete cycle is as follows: from p_0 the solution jumps to point $p_1 = (x_1 = 1, y_1 < 0, \phi_1) \notin \Sigma$ via the impact rule, and then it leaves p_1 and flows toward the left until it hits the left boundary at point $p_2 = (x_2 = -1, y_2 < 0, \phi_2)$; at that moment, the oscillator once again undergoes a jump from p_2 to another point $p_3 = (x_3 = -1, y_3 > 0, \phi_3)$, and after that it continues its journey toward the right until it returns to the Poincaré section at $p_4 = (x_4 = 1, y_4 > 0, \phi_4) \in \Sigma$. The motion is periodic when $\phi_4 = \phi_0$ and $y_4 = y_0$.

In the following analysis, all parameters will be fixed except for the forcing amplitude γ , which will be varied to investigate possible bifurcations. To avoid confusion, t is used still, instead of ϕ , while keeping in mind that $\phi = t \pmod{2\pi/\omega}$. A general solution of equation (14) between two consecutive impacts has the form

$$\begin{aligned} x &= e^{-\zeta(t-t_0)}[a \cos \omega_d(t-t_0) + b \sin \omega_d(t-t_0)] + \beta \cos(\omega t - \psi) \equiv f(t; t_0, x_0, y_0), \\ y &= \dot{x} \equiv g(t; t_0, x_0, y_0), \end{aligned} \quad (16)$$

where

$$\begin{aligned} \omega_d &= \sqrt{1 - \zeta^2}, \quad \beta = \frac{\gamma}{\sqrt{(1 - \omega^2)^2 + (2\zeta\omega)^2}}, \\ \psi &= \begin{cases} \arctan\left(\frac{2\zeta\omega}{1 - \omega^2}\right) & \text{for } \omega < 1, \\ -\pi + \arctan\left(\frac{2\zeta\omega}{1 - \omega^2}\right) & \text{for } \omega > 1, \end{cases} \end{aligned}$$

and a, b are integration constants depending on the initial conditions t_0, x_0 and y_0 .

In order to determine the Poincaré map, it is necessary to find a mathematical solution from p_0 to $y_4 \in \Sigma$. This can be done as follows:

$$\begin{aligned} \text{from } p_0 \text{ to } p_1: & y_1 = -ry_0, \quad t_1 = t_0; \\ \text{from } p_1 \text{ to } p_2: & -1 = f(t_2; +1, y_1), \quad y_2 = g(t_2; t_1, +1, y_1); \\ \text{from } p_2 \text{ to } p_3: & y_3 = -ry_2, \quad t_3 = t_2; \\ \text{from } p_3 \text{ to } p_4: & +1 = f(t_4; t_3, -1, y_3), \quad y_4 = g(t_4; t_3, -1, y_3). \end{aligned} \quad (17)$$

From the above equations, one can in principle solve for t_4 and y_4 in terms of t_0 and y_0 ,

step by step from p_0 to p_4 . It is immediately recognized that, because of the involvement of trigonometric and transcendental functions, such an implicit solution is impossible. It is fortunate that to fulfill the purpose of determining stability and bifurcations, an implicit form of the Poincaré map P is sufficient. After all, the values of γ and initial conditions (t_0, x_0, y_0) for periodic solutions can be computed numerically through solving the above non-linear algebraic equations.

In order to determine the stability of a periodic solution emanating from (t_0, y_0) , the Jacobian matrix of the Poincaré map P at (t_0, y_0) must be computed:

$$DP(t_0, y_0) = \frac{\partial(t, y)}{\partial(t_0, y_0)}. \quad (18)$$

Because of the discontinuity in y , the computation of the derivative (18) must be divided into four parts:

$$DP(t_0, y_0) = \frac{\partial(t_4, y_4)}{\partial(t_3, y_3)} \frac{\partial(t_3, y_3)}{\partial(t_2, y_2)} \frac{\partial(t_2, y_2)}{\partial(t_1, y_1)} \frac{\partial(t_1, y_1)}{\partial(t_0, y_0)}. \quad (19)$$

All the individual derivative terms can easily be determined using the impact rule and the functions f and g defined in equations (16):

$$\frac{\partial(t_1, y_1)}{\partial(t_0, y_0)} = \frac{\partial(t_3, y_3)}{\partial(t_2, y_2)} \begin{bmatrix} 1 & 0 \\ 0 & -r \end{bmatrix}, \quad (20a)$$

$$\frac{\partial(t_2, y_2)}{\partial(t_1, y_1)} = \begin{bmatrix} -\frac{\partial f}{\partial t_1} / \frac{\partial f}{\partial t_2} & -\frac{\partial f}{\partial y_1} / \frac{\partial f}{\partial t_2} \\ \frac{\partial g}{\partial t_2} \frac{\partial t_2}{\partial t_1} + \frac{\partial g}{\partial t_1} & \frac{\partial g}{\partial t_2} \frac{\partial t_2}{\partial y_1} + \frac{\partial g}{\partial y_1} \end{bmatrix}, \quad (20b)$$

and the derivative $\partial(t_4, y_4)/\partial(t_3, y_3)$ can be obtained from equation (20b) by shifting the numbers from 1 to 3 and 2 to 4. Multiplying the four matrices, in order, yields $DP(t_0, y_0)$ and because of the assumption that (t_0, y_0) is periodic, y_4 should be replaced by y_0 in the final result. The eigenvalues of DP can be determined from

$$\lambda^2 - \text{tr } \lambda + \det = 0, \quad (21)$$

where

$$\det = e^{-2\xi\Delta_{21}} e^{-2\xi\Delta_{43}} r^4 \quad (22)$$

is the determinant of DP , with Δ_{21} and Δ_{43} being the time periods spent from p_1 to p_2 and p_3 to p_4 , and tr is the trace of the matrix, which is too long to be given here.

For a given γ , the initial conditions for a periodic solution can be computed numerically from equations (17). In turn, the characteristic equation (21), involving the initial conditions (t_0, x_0, y_0) and γ , may be solved to give the eigenvalues λ .

8.3. RESULTS

All calculations in this section have been conducted with $\omega = 3.5$. $\zeta = -0.05$ and $r = 0.75$.

The main objective here was to find whether a period-doubling sequence exists, which is one of the common paths leading to chaos for many physical systems. For a discrete map, such a bifurcation sequence is often predicted by a (-1) eigenvalue of the

Figure 9. Phase flow diagrams for the simplified model of the negatively damped, forced impact oscillator ($\zeta = -0.05$, $\omega = 3.5$, $r = 0.75$): (a) a periodic solution for a forcing amplitude of $\gamma = 0.9010$; (b) a period-two solution for $\gamma = 0.9323$.

map. To test whether this phenomenon was occurring, λ was first set at $\lambda = -0.75$ in equation (21) and then the resulting equation was solved, yielding $\gamma = 0.9010$. The initial conditions corresponding to this periodic solution were $t_0 = 0.8697$, $x_0 = +1$ and $y_0 = 1.0336$. The phase flow for such a periodic solution is shown in Figure 9(a). Since $|\lambda| < 1$, the solution was stable and, regardless of how long one observed the simulation, it never changed. By then setting $\lambda = -1.25$ and repeating the same procedure, $\gamma = 0.9329$, $t_0 = 0.8728$, $x_0 = +1$ and $y_0 = 1.0378$ were obtained, corresponding to an unstable solution. In this case, the phase flows over a short time period looked the same as that of the previous case, but as time was increased the flow gradually diverged to a new steady state. The result is shown in Figure 9(b) after discarding the transient solution. It is clear that a period-doubling bifurcation has indeed occurred at $\gamma_c = 0.9133$, obtained directly by setting $\lambda = -1$ in equation (21) and solving it.

In order to determine whether this single period-doubling bifurcation event is the first in a doubling sequence, a bifurcation diagram was systematically simulated, displaying the steady state positive velocities ($y > 0$) at $x = 0$ versus the parameter γ , over a fairly large range, $0.05 < \gamma < 1.2$. The result, plotted in Figure 10(a), is in remarkably accurate agreement with the prediction of γ_c . However, as was commented at the beginning of this section, the periodic solutions being investigated in the foregoing analysis are but a small portion of the whole group of periodic solutions. The whole picture of the system dynamics is far more complex. This is clearly seen in Figure 10(a): even before the predicted bifurcation at γ_c , the system response has already become chaotic. In Figure 10(b) is shown an enlarged portion in the neighbourhood of $\gamma = 0.67$, and one sees that there exist some higher order period-doubling sequences involving impacting more than once with each

Figure 10. (a) Bifurcation diagram for the simplified, three-dimensional model of the negatively damped, forced impact oscillator ($\zeta = -0.05$, $\omega = 3.5$, $r = 0.75$), with the forcing amplitude, γ , as the bifurcation parameter; (b) a portion of the bifurcation diagram, $0.64 < \gamma < 0.70$, in enlarged form.

boundary within a period near $\gamma = 0.683$ before the system becomes apparently chaotic for $U > 0.69$.

Another interesting phenomenon observed in Figure 10(a) for low values of γ is the characterization of chaotic motions. For $\gamma < 0.375$, the response has a clear pattern and is undoubtedly periodic. However, over most of the range $0.375 < \gamma < 0.8$, one may think that the motion is mostly chaotic by direct observation of this figure. It is nevertheless difficult to arrive at a clear conclusion as to which motions are chaotic and which are not simply by observing the phase flows, such as those shown in Figure 11(a-c) for $\gamma = 0.5$, 0.7 and 1.0 . Of course, the Lyapunov exponent technique is inapplicable here for deciding this question because of the discontinuity of the model. However, the Poincaré map turns out to be the right tool in this case for doing the same job. The results for the three phase flows in Figure 11 are shown in (d), (e) and (f) respectively. The conclusion is that the response for $\gamma = 0.5$ is quasi-periodic, that for $\gamma = 0.7$ is chaotic, and for $\gamma = 1.0$ the motion is very chaotic.

The importance of these results is in their application, by analogy, to the original problem. The last set of results (Figure 11) shows clearly that very-long-period motions can arise which can easily be mistaken for chaos (cf. results of section 4). The other point is that despite the apparent disorder in Figure 10, the underlying mechanism leading to chaos is through a cascade of period-doubling bifurcations. The qualitative similarity of Figures 10, 2 and 5(a) is evident, and hence the same conclusion with regard to the route to chaos may apply to the earlier model as well. Finally, the results obtained by the trilinear model, giving chaos immediately after impacting begins, may now be seen as

Figure 11. (a-c) Phase flow diagrams and (d-f) the corresponding Poincaré maps for the simplified three-dimensional model of the negatively damped, forced impact oscillator ($\zeta = -0.05$, $\omega = 3.5$, $r = 0.75$): (a, d) for $\gamma = 0.5$; (b, e) for $\gamma = 0.7$; (c, f) for $\gamma = 1.0$.

being associated with the immediacy of multiple impacting and quasi-periodic motions. In principle, there may be a period-doubling sequence leading to this, but the interval of the varied parameter (U) over which this occurs may be too small to be pinpointed.

9. CONCLUSIONS

The purpose of this paper was to decide whether chaotic oscillations are possible in multi-span heat exchanger tubes subjected to cross-flow, when there exists at least one loose support, a common occurrence in practice. This question was addressed with the aid of a simple model involving one two-span flexible tube in the midst of an array of rigid cylinders, with the middle support on the flexible tube being loose, i.e., with a gap between the tube and the supporting baffle plate. To make matters simple, motion was supposed to occur exclusively in the cross-stream direction, and impact dynamics was modelled via cubic and trilinear spring representations.

The study involved numerical computation of bifurcation and phase flow diagrams and Lyapunov exponents. It was shown that, as the flow velocity was increased to beyond the first Hopf bifurcation (negative-damping fluid elastic instability), limit-cycle motion

resulted; beyond that, impact of the cylinder on the supports gives birth to chaos when the flow velocity becomes sufficiently high. With the trilinear spring model of the impact constraint, chaotic-looking results are obtained, immediately after (in terms of increasing flow velocity) the occurrence of impacting on the constraint.

The first and foremost contribution of this paper is to show, for the first time and in quantitative terms, that chaotic oscillations are indeed possible for this system; i.e., for loosely supported heat-exchanger tubes subjected to cross-flow. This result is of practical importance also, since the higher frequencies that are associated with such motions have been known to cause accelerated wear at the loose support, and hence to signal premature breakdown in heat exchangers.

The second contribution made by this paper is that, to the authors' best knowledge, this is the first application of the Lyapunov exponent technique to delay differential equations; it was through the results obtained thereby that the existence of chaos could be decided upon unequivocally. Of course, the importance of this technique has long been recognized; its implementation to delay differential equations broadens its applicability even further. In the present problem, the computation of the Lyapunov exponents for the case of symmetric location of the constraint clarified an additional point: whereas the system response in phase flow and bifurcation diagrams over a small range of U appeared to be aperiodic and chaotic-looking, Lyapunov exponent calculations showed this to be in fact quasi-periodic motion.

A number of small points remained unanswered, and the 10-dimensional (five-degree-of-freedom) model utilized in the calculations did not lend itself to analytic or semi-analytic methods for clarifying these points. Among these points were (i) the determination of the route to chaos and (ii) the finding that for the trilinear spring case chaos apparently occurred at any flow velocity at which impact could take place.[†] For that purpose, a highly simplified one-degree-of-freedom mathematical model was implemented in Section 8, involving a forced impact oscillator with negative damping. This being a three-dimensional model only, it was possible to determine semi-analytically the mechanism underlying the onset of chaotic oscillations: a period-doubling cascade was predicted and confirmed numerically. The results suggest that even though the actual route to chaos for the physical system was not clearly demonstrated, period-doubling cascades may well be the underlying structure for chaos. Also, the results obtained with this model suggest that the chaotic-looking response obtained with the trilinear model of the impact constraint beyond, but close to, the Hopf bifurcation point may in fact be quasi-periodic.

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[†] Since the trilinear model is not analytic, the computation of Lyapunov exponents, which would have decided the issue, is not possible.

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