# Comments on observability and stabilization of magnetic Schrödinger equations

Kaïs Ammari, Mourad Choulli, Luc Robbiano

# COMMENTS ON OBSERVABILITY AND STABILIZATION OF MAGNETIC SCHRÖDINGER EQUATIONS

KAÏS AMMARI, MOURAD CHOULLI, AND LUC ROBBIANO

Abstract. We are mainly interested in extending the known results on observability inequalities and stabilization for the Schrödinger equation to the magnetic Schrödinger equation. That is in presence of a magnetic potential. We establish observability inequalities and exponential stabilization by extending the usual multiplier method, under the same geometric condition that needed for the Schrödinger equation. We also prove, with the help of elliptic Carleman inequalities, logarithmic stabilization results through a resolvent estimate. Although the approach is classical, these results on logarithmic stabilization seem to be new even for the Schrödinger equation.

Contents



## 1. Introduction

Prior to give the precise statement of our main results we need to consider IBVP's for magnetic Schödinger equation that we are interested in. For this, we firstly give the main notations and the preliminary results that we will use frequently in this text

*Date*: February 6, 2019.

1.1. **Notations and preliminaries.** Denote by $dx$ the Lebesgue measure on $\mathbb{R}^d$, $d \geq 1$, and $d\sigma$ the Lebesgue measure on a submanifold $S$ of $\mathbb{R}^k$ of dimension $k-1$. Let $X$ be an open subset of $\mathbb{R}^d$ and $Y = (X, d\mu)$, $Y = (S, d\mu)$ or $Y = (X \times S, d\mu)$, where $d\mu = dx$ if $Y = X$, $d\mu = d\sigma$ if $Y = S$ and $d\mu = dx \otimes d\sigma$ if $Y = X \times S$.

For $f, g \in L^2(Y) = L^2(Y, \mathbb{C})$ and $E \subset Y$ is measurable, we set

$$(f|g)_{0,E} = \int_E f\overline{g}d\mu,$$

$$\|f\|_{0,E} = \left(\int_E |f|^2 d\mu\right)^{1/2}$$

and, if in addition $Y = (X, d\mu)$ and $f \in H^1(Y) = H^1(Y, \mathbb{C})$, let

$$\|f\|_{1,E} = \left(\|f\|_{0,E}^2 + \sum_{j=1}^d \|\partial_j f\|_{0,E}^2\right)^{1/2}.$$

Similarly, for $F, G \in L^2(Y, \mathbb{C}^\ell)$, $\ell \geq 1$, we define

$$(F|G)_{0,E} = \int_E F \cdot \overline{G} d\mu,$$

$$\|F\|_{0,E} = \left(\int_E |F|^2 d\mu\right)^{1/2}.$$

Finally, for $f \in L^\infty(X, \mathbb{R}^\ell)$, $\ell \geq 1$, we set

$$\|f\|_\infty = \||f|\|_{L^\infty(X,\mathbb{R})}.$$

Throughout this text, $\Omega$ is a $C^\infty$ bounded domain of $\mathbb{R}^n$, $n \geq 1$, with boundary $\Gamma$. Let $\nu$ denotes the outward unit normal vector field on $\Gamma$.

Henceforth $\mathbf{a} = (a_1, \ldots, a_n) \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ is a fixed vector field. We define the magnetic Laplacian and the magnetic gradient respectively by

$$\Delta_{\mathbf{a}} = \sum_{j=1}^n (\partial_j + i\, a_j)^2 = \Delta + 2i\mathbf{a} \cdot \nabla + i\mathrm{div}(\mathbf{a}) - |\mathbf{a}|^2$$

and

$$\nabla_{\mathbf{a}} = \nabla + i\mathbf{a}.$$

We shall also need the notation

$$\partial_{\nu_{\mathbf{a}}} = \nabla_{\mathbf{a}} \cdot \nu = \partial_\nu + i\mathbf{a} \cdot \nu.$$

The following identities will be useful in the sequel. There are obtained by making integrations by parts

$$(\Delta_{\mathbf{a}} f|g)_{0,\Omega} = -(\nabla_{\mathbf{a}} f|\nabla_{\mathbf{a}} g)_{0,\Omega} + (\partial_{\nu_{\mathbf{a}}} f|g)_{0,\Gamma}, \quad f \in H^2(\Omega),\ g \in H^1(\Omega), \tag{1.1}$$

$$(\Delta_{\mathbf{a}} f|g)_{0,\Omega} = (f|\Delta_{\mathbf{a}} g)_{0,\Omega}, \quad f, g \in H^2(\Omega) \cap H_0^1(\Omega). \tag{1.2}$$

Note that we can take $f \in H_\Delta(\Omega)$ in (1.1) instead of $f \in H^2(\Omega)$. In that case $(\partial_{\nu_{\mathbf{a}}} f|g)_{0,\Gamma}$ has to be interpreted as a duality pairing between $\partial_{\nu_{\mathbf{a}}} f \in H^{-1/2}(\Gamma)$ and $g \in H^{1/2}(\Gamma)$.

Let $\Lambda$ be a nonempty open subset of $\Gamma$ and

$$\mathcal{H} = \{u \in H^1(\Omega);\ u = 0 \text{ on } \Lambda\}. \tag{1.3}$$

The Poincaré constant of $\mathcal{H}$ will be denoted by $\varkappa(\mathcal{H})$. That is $\varkappa(\mathcal{H})$ is the best constant so that

$$\|u\|_{0,\Omega} \le C\|\nabla u\|_{0,\Omega}, \;\; u \in \mathcal{H}.$$

We have in particular

$$\|u\|_{0,\Omega} \le \varkappa(\mathcal{H})\|\nabla u\|_{0,\Omega}, \;\; u \in \mathcal{H}.$$

**Magnetic gradient semi-norm**. Consider on $H_0^1(\Omega) = H_0^1(\Omega, \mathbb{C})$ the semi-norm

$$f \in H_0^1(\Omega) \mapsto \|\nabla_{\mathbf{a}} f\|_{0,\Omega}.$$

As it is shown by Esteban and Lions [14, page 406], we have

$$|\nabla|f|| \le |\nabla_{\mathbf{a}} f| \quad \text{a.e. in } \Omega.$$

Indeed, bearing in mind that $\mathbf{a}$ takes its values in $\mathbb{R}^n$, we have

$$|\nabla|f|| = \left|\Re\left(\nabla f \frac{\overline{f}}{|f|}\right)\right| = \left|\Re\left((\nabla f + i\mathbf{a} f)\frac{\overline{f}}{|f|}\right)\right| \quad \text{a.e. in } \Omega.$$

As a consequence of this relation, we deduce that $\|\nabla_{\mathbf{a}} \cdot\|_{0,\Omega}$ defines a norm on $H_0^1(\Omega)$. This norm is not in general equivalent to the natural norm $\|\nabla \cdot\|_{0,\Omega}$ on $H_0^1(\Omega)$. For simplicity sake's, even it is not always necessary, we assume that $\mathbf{a}$ is chosen is such a way that $\|\nabla_{\mathbf{a}} \cdot\|_{0,\Omega}$ is equivalent to $\|\nabla \cdot\|_{0,\Omega}$. This is achieved for instance if 0 is not an eigenvalue of the $\Delta_{\mathbf{a}}$, under Dirichlet boundary condition. We refer to [8, Proposition 3.1] for a proof and other equivalent conditions.

Note that if $\|\mathbf{a}\|_\infty$ is sufficiently small then $\|\nabla_{\mathbf{a}} \cdot\|_{0,\Omega}$ and $\|\nabla \cdot\|_{0,\Omega}$ are equivalent on $H_0^1(\Omega)$. This follows in a straightforward manner by observing that if $\varkappa = \varkappa\left(H_0^1(\Omega)\right)$, then

$$(1 - \|\mathbf{a}\|_\infty\varkappa)\|\nabla u\|_{0,\Omega} \le \|\nabla_{\mathbf{a}} u\|_{0,\Omega} \le (1 + \|\mathbf{a}\|_\infty\varkappa)\|\nabla u\|_{0,\Omega}. \tag{1.4}$$

Whence, under the smallness condition

$$\|\mathbf{a}\|_\infty < \frac{1}{\varkappa}, \tag{1.5}$$

$\|\nabla_{\mathbf{a}} \cdot\|_{0,\Omega}$ and $\|\nabla \cdot\|_{0,\Omega}$ are equivalent on $H_0^1(\Omega)$.

More generally, if $\mathcal{H}$ is of the form (1.3) and $\|\mathbf{a}\|_\infty < \frac{1}{\varkappa(\mathcal{H})}$, then $\|\nabla_{\mathbf{a}} \cdot\|_{0,\Omega}$ and $\|\nabla \cdot\|_{0,\Omega}$ are equivalent on $\mathcal{H}$.

1.2. **IBVP's for the magnetic Schrödinger operator.** Consider $\Gamma_0$ and $\Gamma_1$ two disjoint nonempty open subsets of $\Gamma$ so that $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$.

We consider henceforward the following assumptions on the damping coefficients.

$(A_c)$ $0 \le c \in L^\infty(\Omega)$ and there exist $\omega$, an open subset of $\Omega$, and $c_0 > 0$ so that $c \ge c_0$ a.e. in $\omega$.

$(A_d)$ $0 \le d \in L^\infty(\Gamma_0)$ and there exist $\gamma_0$, an open subset of $\Gamma_0$, and $d_0 > 0$ so that $d \ge d_0$ a.e. on $\gamma_0$.

We deal with systems governed by IBVP's for the magnetic Schrödinger operator with different types of dampings. The first system we consider is given by the IBVP

$$\begin{cases} iu_t + \Delta_{\mathbf{a}} u + ic(x)u = 0 & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \Gamma \times (0, +\infty), \\ u(\cdot, 0) = u_0. \end{cases} \tag{1.6}$$

As a consequence of (1.2) we obtain that the unbounded operator $A : L^2(\Omega) \to L^2(\Omega)$ given by $Au = \Delta_{\mathbf{a}} u$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ is self-adjoint.

From inequality (1.1)

$$\Re(Au|u)_0 = -\|\nabla_{\mathbf{a}} u\|_0^2 \le 0, \quad u \in D(A).$$

As a non negative self-adjoint densely defined operator, $A$ is m-dissipative. Then so is $A_0 = iA, D(A_0) = D(A)$, and, consequently, $A_0$ generates a strongly continuous group $e^{tA_0}$.

Assume that $c$ obeys to assumption $(A_c)$ and let $A_1 = i\Delta_{\mathbf{a}} - c$ with domain $D(A_1) = D(A_0)$. As a bounded perturbation of $A_0$, $A_1$ generates a strongly continuous semigroup $e^{tA_1}$.

For $u_0 \in L^2(\Omega)$, define the energy for the system (1.6) by

$$\mathcal{E}^1_{u_0}(t) = \frac{1}{2}\left\|e^{tA_1}u_0\right\|^2_{0,\Omega}.$$

If $u(t) = e^{tA_1}u_0$, we get by using identity (1.1)

$$\frac{d}{dt}\|u(t)\|^2_{0,\Omega} = 2\Re(u'(t), u(t))_{0,\Omega} = 2\Re\left[i\|\nabla_{\mathbf{a}} u(t)\|^2_{0,\Omega} - \|\sqrt{c}u(t)\|^2_{0,\Omega}\right], \;\; t > 0.$$

Hence

$$\frac{d}{dt}\mathcal{E}^1_{u_0}(t) = -\|\sqrt{c}u(t)\|^2_{0,\Omega}, \;\; t > 0.$$

Therefore $t \mapsto \mathcal{E}^1_{u_0}(t)$ is decreasing when $u_0 \neq 0$. We can then address the question to know how fast this energy decay. This issue will be one of our objectives in the coming sections.

The second system is associated with an IBVP with boundary damping.

$$\left\{\begin{array}{ll} iu_t + \Delta_{\mathbf{a}} u = 0 & \text{in } \Omega \times (0, +\infty), \\ \partial_{\nu_{\mathbf{a}}} u + du_t = 0 & \text{on } \Gamma_0 \times (0, +\infty), \\ u = 0 & \text{on } \Gamma_1 \times (0, +\infty), \\ u(\cdot, 0) = u_0. & \end{array}\right. \tag{1.7}$$

Introduce

$$V = \{u \in H^1(\Omega);\; u_{|\Gamma_1} = 0\}.$$

Then, as we have seen before, under the smallness condition

$$\|\mathbf{a}\|_\infty < \frac{1}{\varkappa(V)}, \tag{1.8}$$

$\|\nabla_{\mathbf{a}} \cdot\|_{0,\Omega}$ and $\|\nabla \cdot\|_{0,\Omega}$ are equivalent on $V$. In particular, $V$ endowed with the norm $\|\nabla_{\mathbf{a}} \cdot\|_{0,\Omega}$ is a Hilbert space.

Let $d$ satisfies assumption $(A_d)$ and consider the unbounded operator $A_2 : V \to V$ given by

$$A_2 = i\Delta_{\mathbf{a}} \;\text{ and }\; D(A_2) = \{u \in V;\; \Delta_{\mathbf{a}} u \in V \text{ and } \partial_{\nu_{\mathbf{a}}} u + id\Delta_{\mathbf{a}} u = 0 \text{ on } \Gamma_0\}.$$

Let $W = \{u \in V;\; \Delta_a u \in V \text{ and } \partial_{\nu_{\mathbf{a}}} u \in L^2(\Gamma_0)\}$. Apply then twice (1.1) in order to derive, for $u, v \in W$,

$$(\nabla_{\mathbf{a}}(i\Delta_{\mathbf{a}} u)|\nabla_{\mathbf{a}} v)_{0,\Omega} = -i(\Delta_{\mathbf{a}} u|\Delta_{\mathbf{a}} v)_{0,\Omega} + i(\Delta_{\mathbf{a}} u|\partial_{\nu_{\mathbf{a}}} v)_{0,\Gamma_0}, \tag{1.9}$$

$$(\nabla_{\mathbf{a}} u|\nabla_{\mathbf{a}}(-i\Delta_{\mathbf{a}} v))_{0,\Omega} = -i(\Delta_{\mathbf{a}} u|\Delta_{\mathbf{a}} v)_{0,\Omega} + i(\partial_{\nu_{\mathbf{a}}} u|\Delta_{\mathbf{a}} v)_{0,\Gamma_0}. \tag{1.10}$$

Take in (1.9) and (1.10) $u \in D(A_2)$ and $v \in W$, we find

$$(\nabla_{\mathbf{a}}(i\Delta_{\mathbf{a}} u)|\nabla_{\mathbf{a}} v)_{0,\Omega} = (\nabla_{\mathbf{a}} u|\nabla_{\mathbf{a}}(-i\Delta_{\mathbf{a}} v))_{0,\Omega} + i(\Delta_{\mathbf{a}} u|\partial_{\nu_{\mathbf{a}}} v - id\Delta_a v)_{0,\Gamma_0}. \tag{1.11}$$

Pick $\varphi \in C^\infty(\overline{\Omega}\setminus\Gamma_1)$ and let $u \in V$ be the variational solution of the BVP

$$\begin{cases} -\Delta_a u = \varphi & \text{in } \Omega, \\ \partial_{\nu_{\mathbf{a}}} u = id\varphi & \text{in } \Gamma_0. \end{cases}$$

It is then not hard to check that $u \in D(A_2)$. Hence $(\Delta_{\mathbf{a}} u|\partial_{\nu_{\mathbf{a}}} v - id\Delta_a v)_{0,\Gamma_0} = 0$ for any $u \in D(A_2)$ implies in particular $(\varphi|\partial_{\nu_{\mathbf{a}}} v - id\Delta_a v)_{0,\Gamma_0} = 0$ for any $\varphi \in C^\infty(\overline{\Omega}\setminus\Gamma_1)$. Hence $\partial_{\nu_{\mathbf{a}}} v - id\Delta_a v = 0$ on $\Gamma_0$. From this and (1.11) we obtain that

$$A_2^* = -i\Delta_{\mathbf{a}} \text{ and } D(A_2) = \{u \in V;\ \Delta_{\mathbf{a}} u \in V \text{ and } \partial_{\nu_{\mathbf{a}}} u - id\Delta_{\mathbf{a}} u = 0 \text{ on } \Gamma_0\}.$$

Here we identified the Hilbert space $V$ with its dual space.

Now $u = v$, with $u \in D(A_2)$, in (1.9) yields

$$\Re(\nabla_{\mathbf{a}}(A_2 u)|\nabla_{\mathbf{a}} u)_{0,\Omega} = -\|\sqrt{d}\Delta_{\mathbf{a}} u\|^2_{0,\Gamma_0} \le 0. \tag{1.12}$$

We get similarly from (1.10), where $u \in D(A_2^*)$,

$$\Re(\nabla_{\mathbf{a}}(A_2^* u)|\nabla_{\mathbf{a}} u)_{0,\Omega} = -\|\sqrt{d}\Delta_{\mathbf{a}} u\|^2_{0,\Gamma_0} \le 0.$$

In other words, $A_2$ and $A_2^*$ are dissipative. On the other hand, as for the Laplace operator, one can prove that $A_2$ is closed graph. Therefore, according to [37, Proposition 3.1.11, page 73], $A_2$ is m-dissipative. Whence $A_2$ is the generator of strongly continuous semigroup $e^{tA_2}$.

The energy associated to the system (1.7) is given by

$$\mathcal{E}^2_{u_0}(t) = \frac{1}{2}\left\|\nabla_{\mathbf{a}} e^{tA_2} u_0\right\|^2_{0,\Omega}, \quad u_0 \in V.$$

In light of (1.12), we have

$$\frac{d}{dt}\mathcal{E}^2_{u_0}(t) = \Re(\nabla_{\mathbf{a}} u(t)|\nabla_{\mathbf{a}} u'(t))_{0,\Omega} = \Re(\nabla_{\mathbf{a}}(A_2 u(t))|\nabla_{\mathbf{a}} u(t))_{0,\Omega} = -\|\sqrt{d}\Delta_{\mathbf{a}} u(t)\|_{0,\Gamma_0}, \quad t > 0.$$

Here again, we see that $t \mapsto \mathcal{E}^2_{u_0}(t)$ is decreasing whenever $u_0 \neq 0$.

The third system is again an IBVP with a boundary damping

$$\begin{cases} iu_t + \Delta_{\mathbf{a}} u = 0 & \text{in } \Omega \times (0,+\infty), \\ \partial_{\nu_{\mathbf{a}}} u - idu = 0 & \text{on } \Gamma_0 \times (0,+\infty), \\ u = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ u(\cdot,0) = u_0. \end{cases} \tag{1.13}$$

Define the unbounded operator $A_3 : L^2(\Omega) \to L^2(\Omega)$ given by

$$A_3 = i\Delta_{\mathbf{a}} \text{ and } D(A_3) = \{u \in V;\ \Delta_{\mathbf{a}} u \in L^2(\Omega) \text{ and } \partial_{\nu_{\mathbf{a}}} u - idu = 0 \text{ on } \Gamma_0\},$$

where $d$ obeys to assumption $(A_d)$.

We repeat the same argument that we used to prove that $A_2$ is dissipative in order to derive that $A_3$ is also $m$-dissipative. Therefore $A_3$ generates a strongly continuous semigroup $e^{tA_3}$.

The energy corresponding to the system (1.13) is

$$\mathcal{E}^3_{u_0}(t) = \frac{1}{2}\left\|e^{tA_3} u_0\right\|^2_{0,\Omega}, \quad u_0 \in L^2(\Omega).$$

In light of identity (1.1), for $u, v \in D(A_3)$, we have

$$(i\Delta_{\mathbf{a}} u|v)_{0,\Omega} = -i(\nabla_{\mathbf{a}} u|\nabla_{\mathbf{a}} v)_{0,\Omega} - (du|v)_{0,\Gamma_0}.$$

Whence

$$\frac{d}{dt}\mathcal{E}^3_{u_0}(t) = -\|\sqrt{d}u(t)\|^2_{0,\Gamma_0}, \quad t > 0,$$

where $u(t)=e^{tA_3}u_0$, $u_0\in L^2(\Omega)$.

One more time, we observe that, if $u_0\neq 0$ then $t\mapsto \mathcal{E}^3_{u_0}(t)$ is decreasing.

If $\overline{\Gamma_0}\cap\overline{\Gamma_1}\neq\emptyset$, we do not have necessarily $D(A_j)\subset H^2(\Omega)$. In order to avoid this case, we assume in the rest of this text, even if it is not always necessary, that $\overline{\Gamma_0}\cap\overline{\Gamma_1}=\emptyset$. In other words, $\Gamma$ has at least two connected components.

Prior to give sufficient condition guaranteeing that $D(A_j)\subset H^2(\Omega)$, $j=2,3$, we introduce, for $s\in\mathbb{R}$ and $1\le r\le\infty$,

$$B_{s,r}(\mathbb{R}^{n-1}):=\{w\in\mathscr{S}'(\mathbb{R}^{n-1});\ (1+|\xi|^2)^{s/2}\widehat{w}\in L^r(\mathbb{R}^{n-1})\},$$

where $\mathscr{S}'(\mathbb{R}^{n-1})$ is the space of temperated distributions on $\mathbb{R}^{n-1}$ and $\widehat{w}$ is the Fourier transform of $w$. Endowed with its natural norm

$$\|w\|_{B_{s,r}(\mathbb{R}^{n-1})}:=\|(1+|\xi|^2)^{s/2}\widehat{w}\|_{L^r(\mathbb{R}^{n-1})},$$

$B_{s,r}(\mathbb{R}^{n-1})$ is a Banach space (it is noted that $B_{s,2}(\mathbb{R}^{n-1})$ is merely the usual Sobolev space $H^s(\mathbb{R}^{n-1})$). By using local charts and a partition of unity, we construct $B_{s,r}(\Gamma)$ from $B_{s,r}(\mathbb{R}^{n-1})$ similarly as $H^s(\Gamma)$ is built from $H^s(\mathbb{R}^{n-1})$.

The main interest in these spaces is that the multiplication by a function from $B_{s,1}(\Gamma_0)$, $s\ge 0$, defines a bounded operator on $H^s(\Gamma_0)$ (see [12, Theorem 2.1]).

Additionally to the previous conditions on $\mathbf{a}$ and $d$, we assume in the rest of this text that $\mathbf{a}\cdot\nu\in B_{1/2,1}(\Gamma_0)$ and $d\in B_{1/2,1}(\Gamma_0)$.

Under these supplementary assumptions, for $u\in D(A_j)$, $j=2,3$, $\partial_\nu u\in H^{1/2}(\Gamma_1)$ and, since

$$\left[2i\mathbf{a}\cdot\nabla+i\mathrm{div}(\mathbf{a})-|\mathbf{a}|^2\right]u\in L^2(\Omega),$$

the usual $H^2$-regularity for the Laplacian with mixed boundary conditions entail $u\in H^2(\Omega)$. Whence, $D(A_j)\subset H^2(\Omega)$, $j=1,2,3$.

*Remark* 1.1. 1) Let $\psi\in W^{2,\infty}(\Omega,\mathbb{R})$ and denote by $A_j^\psi$, $j=1,2,3$, the operator $A_j$ in which we substituted $\mathbf{a}$ by $\mathbf{a}+\nabla\psi$. Straightforward computations give

$$e^{-i\psi}\nabla_{\mathbf{a}}e^{i\psi}=\nabla_{\mathbf{a}+\nabla\psi},\quad e^{-i\psi}\Delta_{\mathbf{a}}e^{i\psi}=\Delta_{\mathbf{a}+\nabla\psi}$$

and then

$$e^{tA_j^\psi}=e^{-i\psi}e^{tA_j}e^{i\psi},\ \ j=1,2,3. \tag{1.14}$$

In particular,

$$\|e^{tA_j^\psi}\|_{\mathscr{B}(H)}=\|e^{tA_j}\|_{\mathscr{B}(H)},\ \ H=L^2(\Omega)\text{ for } j=1,3 \text{ and } H=V \text{ if } j=2.$$

Let $\mathcal{E}^{j,\psi}_{u_0}$ the energy corresponding to $A_j^\psi$, with $u_0\in L^2(\Omega)$, $j=1,3$ and $u_0\in V$ for $j=2$. In light of (1.14), we have

$$\mathcal{E}^{j,\psi}_{u_0}=\mathcal{E}^{j}_{e^{i\psi}u_0},\ \ j=1,2,3.$$

2) Assume $n=1$ and let $\Omega=(0,1)$. Denote by $A_j^0$ the operator $A_j$ when $\mathbf{a}=0$, $j=1,2,3$. Using that $\psi(x)=\int_0^x\mathbf{a}(t)dt$ satisfies $\partial_x\psi=\mathbf{a}$, we get from 1)

$$e^{tA_j}=e^{-i\psi}e^{tA_j^0}e^{i\psi}\ \text{ and }\ \mathcal{E}^j_{u_0}=\mathcal{E}^{0,j}_{e^{i\psi}u_0},\ \ j=1,2,3.$$

Here $\mathcal{E}^{0,j}_{u_0}$ is the energy corresponding to $A_j^0$, $j=1,2,3$. Therefore, all the results existing in the literature without the presence of magnetic potential can be transferred to the magnetic case.

1.3. **Main results.** Let $\mathcal{H}_\ell = L^2(\Omega)$ if $\ell = 1, 3$ and $\mathcal{H}_2 = V$. The first results we are going to prove concern logarithmic stabilization in both cases of interior or boundary damping. We will prove in each case of $\ell = 1$, $\ell = 2$ or $\ell = 3$,

**Theorem 1.1.** *Assume that assumptions $(A_c)$ and $(A_d)$ hold. For every $\mu \in \mathbb{R}$, $A_\ell - i\mu$ is invertible and*
(i) $\|(A_\ell - i\mu)^{-1}\|_{\mathscr{B}(\mathcal{H}_\ell)} \le Ce^{K\sqrt{|\mu|}}$, *$\mu \in \mathbb{R}$, for some constants $C > 0$ and $K > 0$,*
(ii) *there exists a constant $C_1 > 0$ such that*

$$\|e^{tA_\ell}u_0\|_{\mathcal{H}_\ell} \le \frac{C_1}{\ln^{2k}(2+t)}\|u_0\|_{D(A_\ell^k)}, \quad u_0 \in D(A_\ell^k).$$

Next, we establish observability inequalities for the magnetic Schrödinger operator. To this end, fix $x_0 \in \mathbb{R}^n$, let $m = m(x) = x - x_0$, $x \in \mathbb{R}^n$ and assume that

$$\Gamma_0 = \{x \in \Gamma;\ m(x) \cdot \nu(x) > 0\}. \tag{1.15}$$

Observe that in the present case the condition $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ is satisfied for instance if $\Omega = \Omega_0 \backslash \Omega_1$, with $\Omega_1 \Subset \Omega_0$, $\Omega_j$ star-shaped with respect to $x_0 \in \Omega_1$ and $\Gamma_j = \partial\Omega_j$, $j = 0, 1$.

**Proposition 1.1.** *Assume that $\Gamma_0$ is of the form* (1.15). *Then there exists a constant $C > 0$, only depending on $\Omega$ and $T$, so that, for any $u_0 \in D(A_0)$ and $u(t) = e^{tA_0}u_0$, we have*

$$\|\nabla_{\mathbf{a}} u_0\|_{0,\Omega} \le C\|\partial_{\nu_{\mathbf{a}}} u\|_{0,\Sigma_0}.$$

Consider the following assumption: (A) $\omega$ is a neighborhood of $\Gamma_0$ in $\Omega$ so that $\overline{\omega} \cap \Gamma_1 = \emptyset$.

**Proposition 1.2.** *Under assumption (A), there exists a constant $C > 0$, only depending on $\Omega$, $T$, $\Omega$ and $\Gamma_0$, so that, for any $u_0 \in D(A_0)$ and $u(t) = e^{tA_0}u_0$, we have*

$$\|u_0\|_{0,\Omega} \le C\|u\|_{0,Q_\omega}.$$

*Here $Q_\omega = \omega \times (0, T)$.*

Finally, we use these observability inequalities to obtain the following exponential stabilization results.

**Theorem 1.2.** *If the assumption (A) holds, then there exists a constant $\varrho > 0$, depending only on $\Omega$, $T$, $\Omega$ and $\Gamma_0$, so that*

$$\mathcal{E}^1_{u_0}(t) \le e^{-\varrho t}\mathcal{E}^1_{u_0}(0), \quad u_0 \in L^2(\Omega).$$

**Theorem 1.3.** *Let $\Gamma_0$ be of the form* (1.15) *for some $x_0$. Then there exists $0 < \varsigma \le \frac{1}{2\varkappa(V)}$, depending on $x_0$ and $\Omega$, with the property that, if $\|\mathbf{a}\|_\infty \le \varsigma$ and $\mathbf{a} = 0$ on $\Gamma_0$, then there exists two constants $C > 0$ and $\varrho > 0$, depending only on $x_0$ and $\Omega$, so that*

$$\mathcal{E}^2_{u_0}(t) \le Ce^{-\varrho t}\mathcal{E}^2_{u_0}(0), \quad u_0 \in V.$$

1.4. **State of art.** Observability inequalities for the Schrödinger equation were established by Machtyngier [25] by the multiplier method. The corresponding exponential stabilization results are due to Machtyngier and Zuazua [26]. Our observability inequalities together with exponential stabilisation extend those in [25, 26].

Under the so-called geometric control condition, Lebeau [22] showed that the Schrödinger equation is exactly controlable (or equivalently exactly observable) for an arbitrary fixed time (see also Phung [34], Laurent [21] and Dehman, Gérard and Lebeau [13] for the nonlinear case). In the case of a square, Ramdani, Takahashi, Tenenbaum and Tucsnak [35] obtained an observability inequality by a spectral method which is build on the fact that observability is equivalent to an observality resolvent estimate, known also as Hautus test. This equivalence was first proved by Burq and Zworski [10] (see also Miller [33]).

Early observability estimates for the Schrödinger equation on torus were established by Haraux [17] and Jaffard [19] in two dimensions and without potentials. The case of Schrödinger equation on spheres and Zoll manifolds was studied in Macià [27], Marcià and Rivière [28, 29]. The observability inequalities for the Schrödinger equation on the torus and the disk was also considered by Anantharaman, Fermanian-Kammerer and Macià [1], Anantharaman and M. Léautaud [4], Anantharaman, M. Léautaud and Macià [2], Anantharaman and Macià [3]. Its is worth mentioning that the results in [1, 2, 3] allow time-dependent potentials and these results hold without any geometric condition on the observation set.

Exact observability inequalities for the (magnetic) wave equation can transferred to observability inequalities for the (magnetic) Schrödinger equation and vice versa via a transmutation method (see Miller [33] and references therein) or by an abstract framework consisting in transforming a second order evolution equation into a first order evolution equation (see [37, Theorem 6.7.5 and Proposition 6.8.2] for more details).

There is wide literature on control, observability and stabilization for the wave equation. We only quote the following few reference [6, 11, 15, 16, 20, 36].

1.5. **Outline.** The rest of this text is organized as follows. Section 2 is devoted to establish logarithmic decay of each of the energies $\mathcal{E}_{u_0}^j$, $j = 1, 2, 3$. The main step consists in proving a resolvent estimate via elliptic Carleman inequalities. Logarithmic energy decay is obtained by using an abstract theorem guaranteeing such decay when the resolvent satisfies some estimates. We note that the logarithmic stability results we establish in Section 2 hold without any geometric condition. We revisit in Section 3 the multiplier method with the objective to extend the existing results for the Schördinger equation to the magnetic Schördinger equation, provided that the magnetic potential satisfies certain conditions. In Section 3, we need the usual geometric conditions on the control subregion. Namely, the boundary control region must contain a part of the boundary enlightened by a point in the space. For the internal control region, its boundary must contain again a part of the boundary enlightened by a point in the space. In the last section, we added supplementary comments. Precisely, we give an exponential stabilization estimate based on a direct application of a Carleman inequality and an observability inequality in a product space.

## 2. Logarithmic stabilization

We firstly recall some interior Carleman estimates as well as boundary Carleman estimates. For this last case we have several estimates depending on the a priori knowledge we have on traces. Next, we apply these inequalities in order to get resolvent estimates on imaginary axis, yielded to obtain energy decay of logarithmic type.

### 2.1. Carleman estimates.

Carleman estimates can be viewed as weighted energy estimates with a large parameter. The crucial assumption is the sub-ellipticity condition introduced in this context by Hörmander [18].

Henceforth

$$X = (-2,2)\times\Omega \ \text{ and } \ L = (-2,2)\times\Gamma.$$

Let $P$ be equal to the Laplace operator plus an operator of order 1 with bounded coefficients. The principal symbol of $P$ is then $p(y,\eta) = |\eta|^2$.

Set, for $\varphi \in \mathscr{C}^\infty(\mathbb{R}^{n+1},\mathbb{R})$,

$$p_\varphi(y,\eta,\tau) = p(y,\eta + i\tau\nabla\varphi(y)).$$

**Definition 2.1.** *Let $\mathcal{O}$ be a bounded open set in $\mathbb{R}^{n+1}$ and $\varphi \in \mathscr{C}^\infty(\mathbb{R}^{n+1},\mathbb{R})$. We say that $\varphi$ satisfies the* sub-ellipticity *condition in $\overline{\mathcal{O}}$ if $|\nabla\varphi| > 0$ in $\overline{\mathcal{O}}$ and*

$$p_\varphi(y,\eta,\tau) = 0,\ (y,\eta)\in\overline{\mathcal{O}}\times\mathbb{R}^{n+1},\ \tau > 0 \ \Rightarrow\ \{\Im p_\varphi, \Re p_\varphi\}(y,\eta,\tau) > 0, \tag{2.1}$$

*where $\{\cdot,\cdot\}$ is the usual Poisson bracket.*

*Remark* 2.1. Note that the sub-ellipticity condition is not really too restrictive. To see that, pick $\psi \in \mathscr{C}^\infty(\mathbb{R}^{n+1},\mathbb{R})$ such that $\nabla\psi(y) \neq 0$ for every $y \in \overline{\mathcal{O}}$. Then $\varphi(y) = e^{\lambda\psi(y)}$ satisfies obviously the sub-ellipticity property in $\overline{\mathcal{O}}$ if $\lambda$ is chosen sufficiently large. This gives a method to construct a weight function having the sub-ellipticity property in $\overline{\mathcal{O}}$ but other choices could be possible.

#### 2.1.1. *Interior Carleman estimate.*

The following Carleman estimate is classical and we can find a proof in Hörmander [18, Theorem 8.3.1].

**Theorem 2.1.** *Let $U$ be an open subset of $X$ and assume that $\varphi$ obeys to the sub-ellipticity condition in $\overline{U}$. Then there exist $C > 0$ and $\tau_0 > 0$, such that*

$$\tau^3\|e^{\tau\varphi}f\|_{0,X}^2 + \tau\|e^{\tau\varphi}\nabla f\|_{0,X}^2 \leq C\|e^{\tau\varphi}Pf\|_{0,X}^2, \tag{2.2}$$

*for all $\tau \geq \tau_0$, $f \in \mathscr{C}_0^\infty(U)$.*

#### 2.1.2. *Boundary Carleman estimates.*

For simplicity sake's, we use in the sequel the notation

$$Y = (-2,2)\times\overline{\Omega}.$$

Let $y_0 \in L$ and $\mathcal{O}$ be a neighborhood of $y_0$ in $(-2,2)\times\mathbb{R}^n$. We say that $f \in \overline{\mathscr{C}_0^\infty}(\mathcal{O}_{|X})$ if there exists $g \in \mathscr{C}_0^\infty(\mathcal{O})$ such that $f = g_{|Y}$. In particular $f \in \mathscr{C}^\infty(Y)$. This definition allows functions with non null traces on $\partial X$ but with null traces on $\partial(\mathcal{O}\cap X)\setminus L$. The following theorem is proved in [24, Proposition 1].

**Theorem 2.2.** *Let $y_0 \in \partial X$ and $\mathcal{O}$ a neighborhood of $y_0$ in $(-2,2)\times\mathbb{R}^n$ and assume that $\varphi$ satisfies the sub-ellipticity condition in $\overline{\mathcal{O}\cap X}$. We also assume that*

$\partial_\nu\varphi(y) \neq 0$ *in* $\overline{\partial\mathcal{O}\cap\partial X}$. *Then there exist* $C>0$ *and* $\tau_0>0$, *such that*

$$\tau^3\|e^{\tau\varphi}f\|^2_{0,X}+\tau\|e^{\tau\varphi}\nabla f\|^2_{0,X}+\tau\|e^{\tau\varphi}\nabla f\|^2_{0,L} \leq C\left(\|e^{\tau\varphi}Pf\|^2_{0,X}+\tau^3\|e^{\tau\varphi}f\|^2_{0,L}+\tau\|e^{\tau\varphi}\partial_\nu f\|^2_{0,L}\right),$$

*for all* $\tau\geq\tau_0$, $f\in\overline{\mathscr{C}_0^\infty}(\mathcal{O}_{|X})$.

This Carleman estimate is useful when we know Dirichlet and Neumann traces of $f$ on a part of the boundary. It allows to estimate the function $f$ in an interior domain by its Dirichlet and Neumann traces on a part of the boundary and $Pf$.

The two next theorems only assume that the knowledge of the Dirichlet trace or Neumann trace. They permit to estimate the function $f$ up to the boundary by $Pf$ and a priori knowledge of $f$ in a small domain contained in $X$.

Henceforth, $\nabla_T$ denotes the tangential gradient on $\Sigma$. The following theorem is proved in [23, Proposition 1]

**Theorem 2.3.** *Let* $y_0\in\partial X$ *and* $\mathcal{O}$ *a neighborhood of* $y_0$ *in* $(-2,2)\times\mathbb{R}^n$, *assume that* $\varphi$ *satisfies the sub-ellipticity condition in* $\overline{\mathcal{O}\cap X}$ *and* $\partial_\nu\varphi(y)<0$ *on* $\overline{\partial\mathcal{O}\cap\partial X}$. *Then there exist* $C>0$ *and* $\tau_0>0$, *such that*

$$\tau^3\|e^{\tau\varphi}f\|^2_{0,X}+\tau\|e^{\tau\varphi}\nabla f\|^2_{0,X}+\tau\|e^{\tau\varphi}\partial_\nu f\|^2_{0,L} \leq C\left(\|e^{\tau\varphi}Pf\|^2_{0,X}+\tau^3\|e^{\tau\varphi}f\|^2_{0,L}+\tau\|e^{\tau\varphi}\nabla_T f\|^2_{0,L}\right),$$

*for all* $\tau\geq\tau_0$, $f\in\overline{\mathscr{C}_0^\infty}(\mathcal{O}_{|X})$.

The following theorem is a consequence of [24, Lemma 4].

**Theorem 2.4.** *Let* $y_0\in\partial X$ *and* $\mathcal{O}$ *a neighborhood of* $y_0$ *in* $(-2,2)\times\mathbb{R}^n$, *assume that* $\varphi$ *satisfies the sub-ellipticity condition in* $\overline{\mathcal{O}\cap X}$ *and* $\partial_\nu\varphi(y)<0$ *on* $\overline{\partial\mathcal{O}\cap\partial X}$. *Then there exist* $C>0$ *and* $\tau_0>0$, *such that*

$$\tau^3\|e^{\tau\varphi}f\|^2_{0,X}+\tau\|e^{\tau\varphi}\nabla f\|^2_{0,X}+\tau^3\|e^{\tau\varphi}f\|^2_{0,L}+\tau\|e^{\tau\varphi}\nabla f\|^2_{0,L} \leq C\left(\|e^{\tau\varphi}Pf\|^2_{0,X}+\tau\|e^{\tau\varphi}\partial_\nu f\|^2_{0,L}\right),$$

*for all* $\tau\geq\tau_0$, $f\in\overline{\mathscr{C}_0^\infty}(\mathcal{O}_{|X})$.

2.1.3. *Global Carleman estimates.* We can patch together the interior and boundary Carleman estimates to obtain a global one. The global Carleman estimate we obtain will be very useful to tackle the stabilization issue for system (1.6).

**Theorem 2.5.** *Let* $Z$ *be a open subset of* $X$ *and assume that* $\varphi$ *satisfies the sub-ellipticity condition in* $Y\setminus Z$. *Assume moreover that* $\partial_\nu\varphi(y)<0$ *in* $L$. *Then there exist* $C>0$ *and* $\tau_0>0$, *such that*

$$C\left(\tau^3\|e^{\tau\varphi}f\|^2_{0,X}+\tau\|e^{\tau\varphi}\nabla f\|^2_{0,X}+\tau\|e^{\tau\varphi}\partial_\nu f\|^2_{0,L}\right) \leq \|e^{\tau\varphi}Pf\|^2_{0,X}+\tau^3\|e^{\tau\varphi}f\|^2_{0,L}+\tau\|e^{\tau\varphi}\nabla_T f\|^2_{0,L} + \|e^{\tau\varphi}f\|^2_{0,Z}+\|e^{\tau\varphi}\nabla f\|^2_{0,Z},$$

*for all* $\tau\geq\tau_0$, $f\in\mathscr{C}^\infty(\overline{X})$.

We now state a theorem that we will use to deal with stabilization issue for systems (1.7) and (1.13)

Set

$$L_j=(-2,2)\times\Gamma_j,\quad j=0,1.$$

**Theorem 2.6.** *Let $\Lambda$ an open subset of $L_0$. Assume that $\varphi$ satisfies the sub-ellipticity condition in $Y$ and $\partial_\nu\varphi(y) < 0$ in $L \setminus \Lambda$. Then there exist $C > 0$ and $\tau_0 > 0$, such that*

$$\begin{aligned} C\left(\tau^3\|e^{\tau\varphi}f\|_{0,X}^2 + \tau\|e^{\tau\varphi}\nabla f\|_{0,X}^2 + \tau^3\|e^{\tau\varphi}f\|_{0,L}^2 + \tau\|e^{\tau\varphi}\nabla f\|_{0,L}^2\right) \\ \le \|e^{\tau\varphi}Pf\|_{0,X}^2 + \tau^3\|e^{\tau\varphi}f|_{0,L_1}^2 + \tau\|e^{\tau\varphi}\nabla_T f\|_{0,L_1}^2 \\ + \tau\|e^{\tau\varphi}\partial_\nu f\|_{0,L_0\setminus\Lambda}^2 + \tau^3\|e^{\tau\varphi}f\|_{0,\Lambda}^2 + \tau\|e^{\tau\varphi}\partial_\nu f\|_{0,\Lambda}^2, \end{aligned}$$

*for all $\tau \ge \tau_0$, $f \in \mathscr{C}^\infty(\overline{X})$.*

To prove Theorems 2.5 and 2.6 we can proceed similarly to the proof of [18, Lemma 8.3.1]. A rough idea of the proof is the following. Assume that we have a Carleman estimate in a neighborhood $\mathcal{U}$ of each point of $\overline{X}$. If $(U_j)$ is a finite covering of $\overline{X}$ of such neighborhoods, we pick $(\chi_j)$ a partition of unity subordinate to this covering. In each $U_j$, we apply the corresponding Carleman estimate to $\chi_j u$, i.e. Theorem 2.1 if $U_j \subset X$ or one of Theorem 2.2, 2.3 and 2.4 if $U_j \cap \partial X \neq \emptyset$, depending on assumptions we have on boundary terms. Putting together all these estimates in order to get, in a classical way, Theorems 2.5 and 2.6.

*Remark* 2.2. All the previous theorems still hold if we substitute $P$ by $P$ plus a first order operator $Q$ having bounded coefficients. For that it is enough to observe that

$$\|e^{\tau\varphi}Pf\|_{0,X} \le \|e^{\tau\varphi}(P+Q)f\|_{0,X} + \|e^{\tau\varphi}Qf\|_{0,X}$$

and that the term $\|e^{\tau\varphi}Qf\|_{0,X}$ can be absorbed by the left hand side of (2.2), by modifying $\tau_0$ if necessary.

The assumptions on the weight function may impose some constraints on the topology of $\Omega$. In Theorem 2.5, if $\varphi$ satisfies $\partial_\nu\varphi(y) < 0$ in $L$, $\varphi$ has a maximum in $X$, thus we have to impose that this maximum belongs to $Z$. In Theorem 2.6, we need $\nabla\varphi \neq 0$ in $Y$. This is always possible as long as we do not assume that $\partial_\nu\varphi$ is of constant sign on $Z$. However one can construct weight functions $\varphi$ obeying to the assumptions of the preceding theorems.

**Proposition 2.1.** *Let $Z$ an open subset of $X$. There exists $\psi \in \mathscr{C}^\infty(\overline{X})$ such that $\partial_\nu\psi < 0$ on $L$ and $\nabla\psi \neq 0$ in $Y \setminus Z$.*

**Proposition 2.2.** *Let $\Lambda$ be an open subset of $L_0$. There exists $\psi \in \mathscr{C}^\infty(\overline{X})$ such that $\partial_\nu\psi < 0$ on $L \setminus \Lambda$ and $\nabla\psi \neq 0$ in $Y$.*

To prove the existence of such functions $\psi$, we first construct $\psi$ in a neighborhood of $L$ (resp. $L\setminus\Lambda$). Next, we extend this function to $Y$ and approximate the extended function by a Morse function. Finally, we *push* the singularities in $Z$ along paths to singularities in a point in $Z$ (resp. in the exterior of $X$ along paths passing through $\Lambda$). We refer for instance [37, Section 14.2, page 437] for a proof. One can then check that $\varphi = e^{\lambda\psi}$ possesses the assumptions of Theorem 2.5 (resp. Theorem 2.6) for $\psi$ constructed in Proposition 2.1 (resp. Proposition 2.2).

2.2. **Stabilization by a resolvent estimate.** The resolvent set of an operator $B$ will denoted by $\rho(B)$.

The following abstract theorem is the key tool in establishing the logarithmic stabilization for each of the three systems we are interested in.

**Theorem 2.7.** *Let $B$ the generator of a continuous semigroup $e^{tB}$ on a Hilbert space $H$. Assume that*
(i) $\sup_{t\geq 0}\|e^{tB}\|_{\mathscr{B}(H)}<\infty$,
(ii) $i\mathbb{R}\subset\rho(B)$,
(iii) $\|(B-i\mu)^{-1}\|_{\mathscr{B}(H)}\leq Ce^{K\sqrt{|\mu|}}$, $\mu\in\mathbb{R}$, *for some constants $C>0$ and $K>0$.*
*Then there exists a constant $C_1>0$, such that*

$$\|e^{tB}f\|_H\leq\frac{C_1}{\ln^{2k}(2+t)}\|f\|_{D(B^k)},\quad f\in D(B^k)$$

*or equivalently*

$$\|e^{tB}B^{-k}\|_{\mathscr{B}(H)}\leq\frac{C_1}{\ln^{2k}(2+t)}.$$

This result is a particular case of [7, Theorem 1.5].

2.2.1. *Interior damping.* We deal in this subsection with the system (1.6). Specifically we are going to apply Theorem 2.7 with $B=A_1$ and $H=L^2(\Omega)$. That is we will prove Theorem 1.1 when $\ell=1$. We restate here for convenience this result.

**Theorem 2.8.** *Assume that assumption $(A_c)$ is satisfied. For every $\mu\in\mathbb{R}$, $A_1-i\mu$ is invertible and*
(i) $\|(A_1-i\mu)^{-1}\|_{\mathscr{B}(L^2(\Omega))}\leq Ce^{K\sqrt{|\mu|}}$, $\mu\in\mathbb{R}$, *for some constants $C>0$ and $K>0$,*
(ii) *there exists a constant $C_1>0$, such that*

$$\|e^{tA_1}u_0\|_{L^2(\Omega)}\leq\frac{C_1}{\ln^{2k}(2+t)}\|u_0\|_{D(A_1^k)},\quad u_0\in D(A_1^k).$$

*Proof.* Let us first consider the resolvent equation $(A_1-i\mu)u=g$, $g\in L^2(\Omega)$. Changing $g$ by $-ig$, we are lead to solve

$$\Delta_{\mathbf{a}}u+icu-\mu u=g. \tag{2.3}$$

Multiplying this equation by $\overline{u}$ and integrating on $\Omega$, we have

$$(\Delta_{\mathbf{a}}u|u)_{0,\Omega}+i(cu|u)_{0,\Omega}-\mu(u|u)_0=(g|u)_{0,\Omega},$$

We obtain by applying (1.1)

$$-\|\nabla_{\mathbf{a}}u\|_{0,\Omega}^2+i(cu|u)_{0,\Omega}-\mu\|u\|_{0,\Omega}^2=(g|u)_{0,\Omega}. \tag{2.4}$$

Taking the real part of this equation, we obtain

$$-\|\nabla_{\mathbf{a}}u\|_{0,\Omega}^2-\mu\|u\|_{0,\Omega}^2=\Re(g|u)_{0,\Omega}.$$

If $\mu\geq 0$, this estimate entails

$$\|\nabla_{\mathbf{a}}u\|_{0,\Omega}^2\leq\|g\|_{0,\Omega}\|u\|_{0,\Omega}$$

and hence

$$\|u\|_{0,\Omega}\leq\varkappa^2k^2\|g\|_{0,\Omega},\quad\mu\geq 0.$$

Here $\varkappa$ is the Poincaré constant of $H_0^1(\Omega)$ and $k$ is a constant so that $\|\nabla w\|_{0,\Omega}\leq k\|\nabla_{\mathbf{a}}w\|_{0,\Omega}$, for each $w\in H_0^1(\Omega)$. In other words, we proved the resolvent estimate when $\mu\geq 0$.

Next, simple computations show that $(iA_1)^*=iA_1+2ic$. Whence

$$\operatorname{ind}(iA_1+\mu)=-\operatorname{ind}(iA_1+2ic+\mu)=-\operatorname{ind}(iA_1+\mu)$$

and then $\operatorname{ind}(iA_1+\mu)=0$. Therefore, $A_1-i\mu$ is invertible if and only if it is injective.

To prove that $A_1-i\mu$ is injective, take, for $g=0$, the imaginary part of equation (2.4) in order to obtain that $u=0$ in $\omega$. Hence $\Delta_{\mathbf{a}}u+icu-\mu u=0$ in $\Omega$ and $u=0$ in $\omega$. Then, by the unique continuation property, $u=0$ in $\Omega$.

We complete the proof by establishing the resolvent estimate when $\mu<0$. By continuity argument, we are reduced to prove the resolvent estimate for large $|\mu|$. To do that, we obtain, by taking again the imaginary part of equation (2.4),

$$c_0\|u\|_{0,\omega}^2\le \|u\|_{0,\Omega}\|g\|_{0,\Omega}. \tag{2.5}$$

Now are now going to apply a Carleman inequality to estimate $\|u\|_{0,\Omega}$ in terms of $\|u\|_{0,\omega}$. To this end, define $f(s,x)=e^{\alpha s}u(x)$, where $\alpha=\sqrt{-\mu}$. Since $u$ is the solution of (2.3), we easily get that $f$ satisfies

$$\partial_s^2 f+\Delta_{\mathbf{a}}f+icf=e^{s\alpha}g. \tag{2.6}$$

Fix $\omega'\Subset\omega$ and set

$$X_1=(-1,1)\times\Omega \text{ and } X_2=(-1/2,1/2)\times\omega'.$$

Pick $\chi\in\mathscr{C}_0^\infty(\mathbb{R})$, such that $\chi(s)=1$ for $|s|\le 3/4$ and $\chi(s)=0$ for $|s|\ge 1$. We put

$$\varphi(s,x)=e^{\lambda(-\beta s^2+\psi(x))},$$

where $\psi$ satisfies Proposition 2.1 with $Z=X_2$ and $\beta>0$ is fixed in what follows. The critical points of $-\beta s^2+\psi(x)$ are located in $X_2$. Then for $\lambda$ sufficiently large (but fixed from now on) $\varphi$ satisfies the sub-ellipticity condition according to Remark 2.1. We can apply Theorem 2.5, with $\chi f$ instead of $f$. We obtain as $\chi f$ satisfies the Dirichlet boundary condition

$$\begin{aligned}\tau^3\|e^{\tau\varphi}\chi f\|_{0,X}^2+\tau\|e^{\tau\varphi}\nabla(\chi f)\|_{0,X}^2 \lesssim \|e^{\tau\varphi}\big(\partial_s^2(\chi f)+\Delta_{\mathbf{a}}(\chi f)+ic\chi f\big)\|_{0,X}^2\\ +\|e^{\tau\varphi}f\|_{0,X_2}^2+\|e^{\tau\varphi}\nabla f\|_{0,X_2}^2.\end{aligned} \tag{2.7}$$

Here and until the end of this proof, $Q_1\lesssim Q_2$ means that $Q_1\le CQ_2$, for some generic constant $C$, only depending on $\Omega$, $\psi$, $\mathbf{a}$ and $c$.

We have

$$\partial_s^2(\chi f)+\Delta_{\mathbf{a}}(\chi f)+ia\chi f=e^{s\alpha}\chi g+2\partial_s\chi\partial_s f+f\partial_s^2\chi.$$

As $\partial_s\chi$ is supported in the set $\{s\in\mathbb{R}, 3/4\le|s|\le 1\}$, we get

$$\|e^{\tau\varphi}(2\partial_s\chi\partial_s f+f\partial_s^2\chi)\|_{0,X}^2\lesssim \alpha e^{C_1\tau+2\alpha}\|u\|_{0,\Omega}^2, \tag{2.8}$$

with $C_1=e^{\lambda(-9\beta/16+\max_\Omega\psi)}$.

On the other hand

$$\|e^{\tau\varphi}f\|_{0,X_2}^2+\|e^{\tau\varphi}\nabla f\|_{0,X_2}^2\lesssim \alpha e^{\tau C_3+2\alpha}\|u\|_{1,\omega'}^2, \tag{2.9}$$

$$\|e^{\tau\varphi}e^{s\alpha}\chi g\|_{0,X}^2\lesssim e^{\tau C_3+2\alpha}\|g\|_{0,X}^2. \tag{2.10}$$

where $C_3=2e^{\lambda\max_\Omega\psi}$.

Inequalities (2.8), (2.9) and (2.10) in (2.7) yield

$$\tau^3\|e^{\tau\varphi}\chi f\|_{0,X}^2+\tau\|e^{\tau\varphi}\nabla(\chi f)\|_{0,X}^2\lesssim \alpha e^{\tau C_3+2\alpha}\left(\|u\|_{1,\omega'}^2+\|g\|_{0,\Omega}^2\right)+\alpha e^{C_1\tau+2\alpha}\|u\|_{0,\Omega}^2. \tag{2.11}$$

Let $\chi_0^2 \in \mathscr{C}_0^\infty(\omega)$ where $\chi_0 = 1$ on $\omega'$. We multiply (2.3) by $\chi_0^2\overline{u}$ and we make an integration by parts. We obtain

$$\|\nabla u\|_{0,\omega'}^2 \lesssim \alpha\|u\|_{0,\omega}^2 + \|g\|_{0,\Omega}^2$$

for which we deduce

$$\tau^3\|e^{\tau\varphi}\chi f\|_{0,X}^2 + \tau\|e^{\tau\varphi}\nabla(\chi f)\|_{0,X}^2 \lesssim \alpha^2 e^{\tau C_3 + 2\alpha}\left(\|u\|_{0,\omega}^2 + \|g\|_{0,\Omega}^2\right) + \alpha e^{C_1\tau + 2\alpha}\|u\|_{0,\Omega}^2. \tag{2.12}$$

In the set $X \cap \{(s,x);\ |s| \le 1/2\}$, $\chi = 1$ and

$$\varphi \ge e^{\lambda(-\beta/4 + \min_\Omega \psi)}.$$

Then $e^{2\tau\varphi} \ge e^{\tau C_2}$, where

$$C_2 = 2e^{\lambda(-\beta/4 + \min_\Omega \psi)}.$$

Fix then $\beta$ sufficiently large in such a way that $C_1 < C_2 < C_3$. From (2.12) we thus obtain

$$e^{\tau C_2 + \alpha}\|u\|_{0,\Omega}^2 \lesssim \alpha^2 e^{\tau C_3 + 2\alpha}\left(\|u\|_{0,\omega}^2 + \|g\|_{0,\Omega}^2\right) + \alpha e^{C_1\tau + 2\alpha}\|u\|_{0,\Omega}^2.$$

Taking $\tau = \gamma\alpha = \gamma\sqrt{|\mu|}$ with $\gamma$ sufficiently large, there exist $C_4, C_5 > 0$ such that

$$\|u\|_{0,\Omega}^2 \lesssim e^{C_4\alpha}(\|u\|_{0,\omega}^2 + \|g\|_{0,\Omega}^2) + e^{-C_5\alpha}\|u\|_{0,\Omega}^2.$$

For $\alpha$ sufficiently large, we have

$$\|u\|_{0,\Omega}^2 \lesssim e^{C_4\alpha}(\|u\|_{0,\omega}^2 + \|g\|_{0,\Omega}^2).$$

From (2.5) we have

$$\|u\|_{0,\Omega}^2 \le Ke^{C_4\alpha}\left(\|u\|_{0,\Omega}\|g\|_{0,\Omega} + \|g\|_{0,\Omega}^2\right).$$

As

$$Ke^{C_4\alpha}\|u\|_{0,\Omega}\|g\|_{0,\Omega} \le \|u\|_{0,\Omega}^2/2 + (K^2/2)e^{2C_4\alpha}\|g\|_{0,\Omega}^2,$$

we obtain

$$\|u\|_{0,\Omega}^2 \lesssim e^{2C_4\alpha}\|g\|_{0,\Omega}^2,$$

which is exactly the expected resolvent estimate. □

2.2.2. *Boundary damping.* This subsection is devoted to the proof of Theorem 1.1 when $\ell = 2$ and $\ell = 3$. We first restate for conveniance the result for $\ell = 2$.

**Theorem 2.9.** *Let assumption $(A_d)$ holds. For every $\mu \in \mathbb{R}$, $A_2 - i\mu$ is invertible and*
(i) $\|(A_2 - i\mu)^{-1}\|_{\mathscr{B}(V)} \le Ce^{K\sqrt{|\mu|}}$, *$\mu \in \mathbb{R}$, for some constants $C > 0$ and $K > 0$,*
(ii) *there exists a constant $C_1 > 0$, such that*

$$\|e^{tA_2}u_0\|_V \le \frac{C_1}{\ln^{2k}(2+t)}\|u_0\|_{D(A_2^k)},\quad u_0 \in D(A_2^k).$$

*Proof.* We are going to prove that $B = A_2$ obeys to the conditions of Theorem 2.7 when $H = V$. As in the preceding proof, we solve the resolvent equation: for $g \in V$, find $u \in D(A_2)$ satisfying

$$(A_2 - i\mu)u = g.$$

Substituting $g$ by $-ig$, we are reduced the following equation: find $u \in D(A_2)$ so that

$$\Delta_{\mathbf{a}}u - \mu u = g. \tag{2.13}$$

Multiply this equation by $\overline{u}$ and integrate over $\Omega$ in order to get

$$(\Delta_{\mathbf{a}} u|u)_{0,\Omega} - \mu(u|u)_{0,\Omega} = (g|u)_{0,\Omega},$$

In combination with (1.1), this identity yields

$$-\|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2 - \mu\|u\|_{0,\Omega}^2 + (\partial_{\nu_A} u|u)_{0,\Gamma_0} = (g|u)_{0,\Omega}.$$

As $\partial_{\nu_A} u = -id\Delta_{\mathbf{a}} u$ and $\Delta_{\mathbf{a}} u = \mu u + g$ on $\Gamma_0$, we have

$$-\|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2 - \mu\|u\|_{0,\Omega}^2 - i\mu\|\sqrt{d}u\|_{0,\Gamma_0} = (g|u)_{0,\Omega} + i(dg|u)_{0,\Gamma_0}. \tag{2.14}$$

Taking the real part, we get

$$-\|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2 - \mu\|u\|_{0,\Omega}^2 = \Re(g|u)_{0,\Omega} + \Re(idg|u)_{0,\Gamma_0}. \tag{2.15}$$

For $\mu \geq 0$, we have

$$\|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2 + \mu\|u\|_{0,\Omega}^2 \leq \|g\|_{0,\Omega}\|u\|_{0,\Omega} + \|d\|_\infty \|g\|_{0,\Gamma_0}\|u\|_{0,\Gamma_0}.$$

We know that $\|\nabla_{\mathbf{a}} \cdot \|_{0,\Omega}$ is equivalent to the natural norm of $V$ induced by that on $H_0^1(\Omega)$. Therefore, the trace operator $\mathbf{tr} : V \to L^2(\Gamma_0)$ is bounded when $V$ is endowed with norm $\|\nabla_{\mathbf{a}} \cdot \|_{0,\Omega}$.

Thus, we have

$$\begin{aligned}\|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2 + \mu\|u\|_0^2 &\leq \varkappa_1^2 \|\nabla_{\mathbf{a}} g|\|_{0,\Omega}\|\nabla_{\mathbf{a}} u\|_{0,\Omega} + \|d\|_\infty \|\mathbf{tr}\| \||\nabla_{\mathbf{a}} g|\|_{0,\Omega}\||\nabla_{\mathbf{a}} u|\|_{0,\Omega} \\ &\leq \left(\varkappa_1^2 + \|d\|_\infty \|\mathbf{tr}\|^2\right) \|\nabla_{\mathbf{a}} g\|_{0,\Omega}\|\nabla_{\mathbf{a}} u\|_{0,\Omega},\end{aligned}$$

where $\|\mathbf{tr}\|$ denotes the norm of $\mathbf{tr}$ in $\mathscr{B}(V, L^2(\Gamma_0))$ and $\varkappa_1$ is the Poincaré constant of $V$. In particular

$$\|\nabla_{\mathbf{a}} u\|_{0,\Omega} \leq \left(\varkappa_1^2 + \|d\|_\infty \|\mathbf{tr}\|^2\right) \|\nabla_{\mathbf{a}} g\|_{0,\Omega}. \tag{2.16}$$

This is nothing but the resolvent estimate for $\mu \geq 0$.

Let us now consider the case $\mu < 0$. To this end, we firstly observe that $A_2 - i\mu$ is injective. Indeed, take $g = 0$ and then the imaginary part in (2.14) to get $u = 0$ on $\gamma_0$ yielding $\partial_\nu u = 0$ on $\gamma_0$. Whence $u = 0$ by the unique continuation property. Obviously, $g = 0$ and $\mu = 0$ entail $\nabla_{\mathbf{a}} u = 0$ and then $u = 0$. Next, as $A_2$ is invertible by the preceding step and $D(A_2)$ is compactly embedded in $V$ according to the elliptic regularity, $A_2^{-1} : V \to V$ is compact. Therefore $B_2 = I - i\mu A_2^{-1}$, injective, is onto by Fredholm's alternative and hence $A_2 - i\mu = A_2 B_2$ is also onto.

To complete the proof, it remains to prove the resolvent estimate for $\mu < 0$. As in the preceding proof it is enough to establish such an estimate for $|\mu|$ large. To this end, taking one more time the imaginary part of (2.14), we obtain

$$-\mu\|\sqrt{d}u\|_{0,\Gamma_0}^2 = -\Re(ig|u)_{0,\Omega} + \Re(dg|u)_{0,\Gamma_0}. \tag{2.17}$$

From (2.17), we get by using the continuity of the trace operator $\mathbf{tr}$ and $|\mu| \geq 1$,

$$d_0\|u\|_{0,\gamma_0}^2 \leq \|d\|_\infty \|\mathbf{tr}\|^2 \|\nabla_{\mathbf{a}} g\|_{0,\Omega}\|\nabla_{\mathbf{a}} u\|_{0,\Omega}. \tag{2.18}$$

Next we proceed as in the preceding theorem. We first use a Carleman inequality to estimate $\|\nabla_{\mathbf{a}} u\|_{0,\Omega}$ by $\|u\|_{0,\Gamma_0}$. Set $f(s,x) = e^{\alpha s}u(x)$, where $s \in (-2,2)$ and $\alpha = \sqrt{-\mu}$. Then it is straightforward to check that $f$ satisfies

$$Pf = \partial_s^2 f + \Delta_{\mathbf{a}} f = e^{s\alpha} g. \tag{2.19}$$

Recall that

$$X = (-2,2) \times \Omega, \quad X_1 = (-1,1) \times \Omega$$

and define

$$\ell_0 = (-1/2, 1/2) \times \gamma_0.$$

Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$, such that $\chi(s) = 1$ for $|s| \le 3/4$ and $\chi(s) = 0$ for $|s| \ge 1$. We set

$$\varphi(s, x) = e^{\lambda(-\beta s^2 + \psi(x))},$$

where $\psi$ is given by Proposition 2.2 with $\Lambda = \ell_0$ and $\beta > 0$ is fixed in what follows. The function $-\beta s^2 + \psi(x)$ has no critical point in $X$. Then for $\lambda$ sufficiently large (but fixed from now on) $\varphi$ satisfies the sub-ellipticity condition of Remark 2.1.

In the rest of this proof $Q_1 \lesssim Q_2$ means $Q_1 \le C Q_2$, for some generic constant $C$, only depending on $n$, $\Omega$, $\mathbf{a}$ and $d$.

We can apply Theorem 2.6, with $\chi f$ instead of $f$. As $\chi f$ satisfies Dirichlet boundary condition on $\Gamma_1$ and as $\partial_{\nu_{\mathbf{a}}} = \partial_\nu + i\mathbf{a} \cdot \nu$, we get $\tau^3 \|e^{\tau\varphi} f\|_{0,\ell_0}^2 + \tau \|e^{\tau\varphi} \partial_\nu f\|_{0,\ell_0}^2$ is equivalent to $\tau^3 \|e^{\tau\varphi} f\|_{0,\ell_0}^2 + \tau \|e^{\tau\varphi} \partial_{\nu_{\mathbf{a}}} f\|_{0,\ell_0}^2$. Then

$$\begin{aligned}\tau^3 \|e^{\tau\varphi} \chi f\|_{0,X}^2 + \tau \|e^{\tau\varphi} \nabla(\chi f)\|_{0,X}^2 &\lesssim \|e^{\tau\varphi} P(\chi f)\|_{0,X}^2 \\ &\quad + \tau \|e^{\tau\varphi} \partial_{\nu_{\mathbf{a}}} f\|_{0,L_0 \setminus \ell_0}^2 + \tau^3 \|e^{\tau\varphi} f\|_{0,\ell_0}^2 + \tau \|e^{\tau\varphi} \partial_{\nu_{\mathbf{a}}} f\|_{0,\ell_0}^2. \end{aligned} \tag{2.20}$$

We recall the constants defined in the previous section,

$$\begin{aligned} C_1 &= 2e^{\lambda(-9\beta/16 + \sup_\Omega \psi)}, \\ C_2 &= 2e^{\lambda(-\beta/4 + \min_\Omega \psi)}, \\ C_3 &= 2e^{\lambda \sup_\Omega \psi(x)}. \end{aligned} \tag{2.21}$$

Similarly to the previous section, we have

$$\begin{aligned} &\|e^{\tau\varphi} P(\chi f)\|_{0,X}^2 \lesssim e^{2\alpha + C_3\tau} \|\nabla_{\mathbf{a}} g\|_{0,\Omega}^2 + e^{2\alpha + C_1\tau} \|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2, \\ &\|e^{\tau\varphi} f|_{0,\ell_0}^2 \lesssim e^{2\alpha + C_3\tau} \|u\|_{0,\gamma_0}^2 \lesssim e^{2\alpha + C_3\tau} \|\nabla_{\mathbf{a}} g\|_{0,\Omega} \|\nabla_{\mathbf{a}} u\|_{0,\Omega}, \end{aligned} \tag{2.22}$$

from (2.18). The two other terms of the right hand side of (2.20) may be estimated by $\|e^{\tau\varphi} \partial_{\nu_{\mathbf{a}}} f\|_{0,L_0}^2$. We have

$$(\partial_{\nu_A} f)_{|L_0} = e^{\alpha s} (\partial_{\nu_A} u)_{|\Gamma_0} = -ide^{\alpha s} (\Delta_{\mathbf{a}} u)_{|\Gamma_0} = -ide^{\alpha s} (g + \mu u)_{|\Gamma_0}.$$

Whence

$$\begin{aligned} \|e^{\tau\varphi} \partial_{\nu_{\mathbf{a}}} f\|_{0,L_0}^2 &\lesssim e^{2\alpha + C_3\tau} (\|g\|_{0,\Gamma_0}^2 + |\mu|^2 \|d^{1/2} u\|_{0,\Gamma_0}^2) \\ &\lesssim e^{2\alpha + C_3\tau} \left( \|\nabla_{\mathbf{a}} g\|_{0,\Omega}^2 + \alpha^4 \|\nabla_{\mathbf{a}} g\|_{0,\Omega} \|\nabla_{\mathbf{a}} u\|_{0,\Omega} \right). \end{aligned} \tag{2.23}$$

On the other hand, it is straightforward to check

$$\begin{aligned} \tau^3 \|e^{\tau\varphi} \chi f\|_{0,X}^2 + \tau \|e^{\tau\varphi} \nabla(\chi f)\|_{0,X}^2 &\gtrsim \tau^3 \|e^{\tau\varphi} f\|_{0,(-1/2,1/2)\times\Omega}^2 \\ &\qquad + \tau \|e^{\tau\varphi} \nabla f\|_{0,(-1/2,1/2)\times\Omega}^2 \\ &\gtrsim e^{\alpha + \tau C_2} (\|u\|_{0,\Omega}^2 + \|\nabla u\|_{0,\Omega}^2). \end{aligned} \tag{2.24}$$

Inequalities (2.20) and (2.22) to (2.24) yield

$$\begin{aligned} &e^{\alpha + \tau C_2} \|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2 \\ &\qquad \lesssim e^{2\alpha + C_3\tau} (\|\nabla_{\mathbf{a}} g\|_{0,\Omega}^2 + \alpha^4 \|\nabla_{\mathbf{a}} g\|_{0,\Omega} \|\nabla_{\mathbf{a}} u\|_{0,\Omega}) + \alpha e^{2\alpha + C_1\tau} \|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2. \end{aligned}$$

As we have done in the preceding proof, taking $\beta$ sufficiently large, we have $C_1 < C_2 < C_3$ and, for $\tau = \gamma\alpha$ with $\gamma$ sufficiently large, we find $C_4 > 0$ and $C_5 > 0$ so that

$$\|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2 \le Ce^{C_4\alpha}\left(\|\nabla_{\mathbf{a}} g\|_{0,\Omega}^2 + \alpha^4\|\nabla_{\mathbf{a}} g\|_{0,\Omega}\|\nabla_{\mathbf{a}} u\|_{0,\Omega}\right) + Ce^{-C_5\alpha}\|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2.$$

Choose $\alpha$ sufficiently large in such a way that $Ce^{-C_5\alpha} \le 1/4$. Then

$$C\alpha^4 e^{C_4\alpha}\|\nabla_{\mathbf{a}} g\|_{0,\Omega}\|\nabla_{\mathbf{a}} u\|_{0,\Omega} \le C^2\alpha^8 e^{2C_4\alpha}\|\nabla_{\mathbf{a}} g\|_{0,\Omega}^2 + \|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2/2.$$

The last two estimates entail

$$\|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2 \le Ce^{C\alpha}\|\nabla_{\mathbf{a}} g\|_{0,\Omega}^2. \tag{2.25}$$

The proof is then complete. □

We move now to the system (1.13) for which we aim to prove Theorem 1.1 for $\ell = 3$. We restate here for convenience this result.

**Theorem 2.10.** *Under assumption $(A_d)$, for every $\mu \in \mathbb{R}$, $A_3 - i\mu$ is invertible and*
(i) $\|(A_3 - i\mu)^{-1}\|_{\mathscr{B}(L^2(\Omega))} \le Ce^{K\sqrt{|\mu|}}$, $\mu \in \mathbb{R}$, *for some constants $C > 0$ and $K$,*
(ii) *there exists a constant $C_1 > 0$, such that*

$$\|e^{tA_3}u_0\|_{0,\Omega} \le \frac{C_1}{\ln^{2k}(2+t)}\|u_0\|_{D(A_3^k)}, \quad u_0 \in D(A_3^k).$$

*Proof.* As in the preceding two proofs, we first solve the resolvent equation: for $g \in L^2(\Omega)$, find $u \in D(A_3)$ so that

$$\Delta_{\mathbf{a}} u - \mu u = g. \tag{2.26}$$

With the help of identity (1.1), we get

$$-\|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2 - \mu\|u\|_{0,\Omega}^2 + (\partial_{\nu_A} u|u)_{0,\Gamma_0} = (g|u)_{0,\Omega}.$$

As $\partial_{\nu_A} u = idu$, we have

$$-\|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2 - \mu\|u\|_{0,\Omega}^2 + i\|\sqrt{d}u\|_{0,\Gamma_0}^2 = (g|u)_{0,\Omega}. \tag{2.27}$$

Take the real part of each side in order to derive

$$-\|\nabla_{\mathbf{a}} u\|_{0,\Omega}^2 - \mu\|u\|_{0,\Omega}^2 = \Re(g|u)_{0,\Omega}. \tag{2.28}$$

When $\mu \ge 0$, we obtain

$$\|\nabla_{\mathbf{a}} u\|_{0,\Omega} \le \|g\|_{0,\Omega}. \tag{2.29}$$

This and Poincaré inequality on $V$ imply the resolvent estimate when $\mu \ge 0$.

When $\mu < 0$, we can repeat the argument we used for $A_2$. That is $A_3 - i\mu$ will be invertible if it is injective. Here again the fact that $A_3 - i\mu$ is injective follows from a unique continuation property.

Next assume that $\mu < 0$. We get by taking the imaginary part of each side of (2.27)

$$\|\sqrt{d}u\|_{0,\Gamma_0}^2 = -\Re(ig|u)_{0,\Omega}.$$

Hence

$$d_0\|u\|_{0,\gamma_0}^2 \le \|g\|_{0,\Omega}\|u\|_{0,\Omega}. \tag{2.30}$$

In this proof $\lesssim$ has the same meaning as in the proof of Theorem 2.9.

With the notations of the preceding proof, we have

$$\begin{aligned}\|e^{\tau\varphi}P(\chi f)\|_{0,X}^2 &\lesssim e^{2\alpha+C_3\tau}\|g\|_{0,\Omega}^2+\alpha e^{2\alpha+C_1\tau}\|u\|_{1,\Omega}^2\\ \|e^{\tau\varphi}f\|_{0,L_0}^2 &\lesssim e^{2\alpha+C_3\tau}\|u\|_{L^2(\gamma_0)}^2\lesssim e^{2\alpha+C_3\tau}\|g\|_{0,\Omega}\|u\|_{0,\Omega},\end{aligned} \tag{2.31}$$

where we used (2.30).

The two other terms of the right hand side of (2.20) are estimated by $\|e^{\tau\varphi}\partial_{\nu_{\mathbf{a}}}f\|_{0,L_0}^2$. We have

$$(\partial_{\nu_{\mathbf{a}}}f)_{|L_0}=e^{\alpha s}(\partial_{\nu_{\mathbf{a}}}u)_{|\Gamma_0}=ide^{\alpha s}u_{|\Gamma_0}.$$

Whence, using (2.30), we get

$$\begin{aligned}\|e^{\tau\varphi}\partial_{\nu_{\mathbf{a}}}f\|_{0,L_0}^2 &\lesssim e^{2\alpha+C_3\tau}\|\sqrt{d}u\|_{0,\Gamma_0}^2\\ &\lesssim e^{2\alpha+C_3\tau}\|g\|_{0,\Omega}\|u\|_{0,\Omega}.\end{aligned} \tag{2.32}$$

On the other hand,

$$\begin{aligned}\tau^3\|e^{\tau\varphi}\chi f\|_{0,X}^2+\tau\|e^{\tau\varphi}\nabla(\chi f)\|_{0,X}^2 &\gtrsim \tau^3\|e^{\tau\varphi}f\|_{0,(-1/2,1/2)\times\Omega}^2\\ &\qquad+\tau\|e^{\tau\varphi}\nabla f\|_{0,(-1/2,1/2)\times\Omega}^2\\ &\gtrsim e^{\alpha+\tau C_2}(\|u\|_{0,\Omega}^2+|\nabla u\|_{0,\Omega}^2).\end{aligned} \tag{2.33}$$

Estimates (2.20) and (2.31) to (2.33), imply

$$e^{\alpha+\tau C_2}\|u\|_{1,\Omega}^2\lesssim e^{2\alpha+C_3\tau}(\|g\|_{0,\Omega}^2+\|g\|_{0,\Omega}\|u\|_{0,\Omega})+e^{2\alpha+C_1\tau}\|u\|_{1,\Omega}^2. \tag{2.34}$$

Similarly to the proof of the preceding theorem, we can take $\beta$ large enough in order to ensure that $C_1<C_2<C_3$ and, for $\tau=\gamma\alpha$ with $\gamma$ sufficiently large, there exist $C_4>0$ and $C_5>0$ so that

$$\|u\|_{1,\Omega}^2\leq Ce^{C_4\alpha}(\|g\|_{0,\Omega}^2+\|g\|_{0,\Omega}\|u\|_{0,\Omega})+Ce^{-C_5\alpha}\|u\|_{1,\Omega}^2.$$

Pick $\alpha$ large enough in such a way that $Ce^{-C_5\alpha}\leq 1/4$. Then

$$Ce^{C_4\alpha}\|g\|_{0,\Omega}\|u\|_{0,\Omega}\leq C^2e^{2C_4\alpha}\|g\|_{0,\Omega}^2+\|u\|_{0,\Omega}^2/2.$$

The two last estimates yield

$$\|u\|_{0,\Omega}^2\leq\|u\|_{1,\Omega}^2\leq Ce^{C\alpha}\|g\|_{0,\Omega}^2, \tag{2.35}$$

That is we proved the resolvent estimate for $\mu<0$. □

## 3. Exponential stabilization

**3.1. Observability inequalities.** In this section, we use the following notation

$$Q=\Omega\times(0,T),\ \ \Sigma=\Gamma\times(0,T)\ \text{ and }\ \Sigma_j=\Gamma_j\times(0,T),\ j=0,1.$$

Following Lions and Magenes notation, the anisotropic Sobolev space $H^{2,1}(Q)$ is given by

$$H^{2,1}(Q)=L^2((0,T),H^2(\Omega))\cap H^1((0,T),L^2(\Omega)).$$

We use frequently in the sequel the following Green's formula

$$((\partial_j+ia_j)u|v)_{0,\Omega}=-(u|(\partial_j+ia_j)u)_{0,\Omega}+(u|v\nu_j)_{0,\Gamma}. \tag{3.1}$$

The following proposition is a key tool in the multiplier method.

**Proposition 3.1.** *Let* $\aleph \in C^2(\overline{Q}, \mathbb{R}^n)$, $u \in H^{2,1}(Q)$ *and set*

$$f = i\partial_t u + \Delta_{\mathbf{a}} u.$$

*Then*

$$\begin{aligned}\langle \nabla_{\mathbf{a}} u \cdot \nu | \aleph \cdot \nabla_{\mathbf{a}} u\rangle_{0,\Sigma} - \frac{1}{2}(|\nabla_{\mathbf{a}} u|^2 | \aleph \cdot \nu)_{0,\Sigma} \\ + \frac{1}{2}(div(\aleph) u | \nabla_{\mathbf{a}} u \cdot \nu)_{0,\Sigma} - \frac{i}{2}(u(\aleph\cdot\nu)|\partial_t u)_{0,\Sigma} \\ = \langle D\aleph \nabla_{\mathbf{a}} u | \nabla_{\mathbf{a}} u\rangle_{0,Q} + \frac{1}{2}(u\nabla div(\aleph)|\nabla_{\mathbf{a}} u)_{0,Q} \\ + \frac{i}{2}(u\partial_t \aleph | \nabla_{\mathbf{a}} u)_{0,Q} - \frac{i}{2}\left[(u\aleph|\nabla_{\mathbf{a}} u)_{0,\Omega}\right]_0^T dx \\ + \langle f\aleph|\nabla_{\mathbf{a}} u\rangle_{0,Q} + \frac{1}{2}(div(\aleph) u | f)_{0,Q}.\end{aligned}$$

*Here* $D\aleph = (\partial_k \aleph_\ell)$ *is the Jacobian matrix of* $\aleph$.

*Proof.* For simplicity sake's, we use in this proof the following temporary notation

$$d_j = \partial_j + ia_j \ \text{ and } \ \overline{d_j} = \partial_j - ia_j.$$

**First step.** We prove

(3.2)

$$\begin{aligned}\langle \Delta_{\mathbf{a}} u | \aleph \cdot \nabla_{\mathbf{a}} u\rangle_{0,Q} = -\langle D\aleph\nabla_{\mathbf{a}} u|\nabla_{\mathbf{a}} u\rangle_{0,Q} \\ + \frac{1}{2}(|\nabla_{\mathbf{a}} u|^2|\mathrm{div}(\aleph))_{0,Q} - \frac{1}{2}(|\nabla_{\mathbf{a}} u|^2|\aleph\cdot\nu)_{0,\Sigma} + \langle \nabla_{\mathbf{a}} u\cdot\nu|\aleph\cdot\nabla_{\mathbf{a}} u\rangle_{0,\Sigma}.\end{aligned}$$

From Green's formula (3.1), we have

$$\begin{aligned}(\Delta_{\mathbf{a}} u|\aleph\cdot\nabla_{\mathbf{a}} u)_{0,\Omega} &= \sum_{j,k=1}^n (d_j^2 u \aleph_k | d_k u)_{0,\Omega} \\ &= -\sum_{j,k=1}^n (d_j u | d_j(\aleph_k d_k u))_{0,\Omega} + \sum_{j,k=1}^n (d_j u \nu_j | \aleph_k d_k u)_{0,\Gamma} \\ &= -\sum_{j,k=1}^n (d_j u | d_j(\aleph_k d_k u))_{0,\Omega} + (\nabla_{\mathbf{a}} u\cdot\nu|\aleph\cdot\nabla_{\mathbf{a}} u)_{0,\Gamma}.\end{aligned} \tag{3.3}$$

Elementary calculations show

$$d_j(\aleph_k d_k u) = \partial_j \aleph_k d_k u + \aleph_k d_j d_k u.$$

Therefore

$$(d_j u|d_j(\aleph_k d_k u))_{0,\Omega} = (\partial_j\aleph_k d_j u|d_k u)_{0,\Omega} + (d_j u \aleph_k | d_j d_k u)_{0,\Omega}.$$

Hence

$$\sum_{i,k=1}^n (d_j u|d_j(\aleph_k d_k u))_{0,\Omega} = (D\aleph\nabla_{\mathbf{a}} u|\nabla_{\mathbf{a}} u)_{0,\Omega} + \sum_{i,k=1}^n (d_j u\aleph_k|d_j d_k u)_{0,\Omega}. \tag{3.4}$$

Introduce the auxiliary function $v_j = d_j u$. Then

$$d_j u \overline{d_j}\, \overline{d_k u} = v_j \overline{d_k} \overline{v_j} = v_j \partial_k \overline{v_j} - ia_j |v_j|^2$$

and then

$$\Re[d_j u\overline{d_j}\,\overline{d_k u}] = \Re(v_j\partial_k\overline{v_j}) = \frac{1}{2}(v_j\partial_k\overline{v_j} + \overline{v_j}\partial_k v_j) = \frac{1}{2}\partial_k|v_j|^2 = \frac{1}{2}\partial_k|d_j u|^2.$$

Whence

$$\begin{aligned}\sum_{i,j=1}^{n}\langle\aleph_k d_j u|d_j d_k u\rangle_{0,\Omega} &= \frac{1}{2}(\nabla|\nabla_{\mathbf{a}}u|^2|\aleph)_{0,\Omega}\\ &= -\frac{1}{2}(|\nabla_{\mathbf{a}}u|^2|\mathrm{div}(\aleph))_{0,\Omega} + \frac{1}{2}(|\nabla_{\mathbf{a}}u|^2|\aleph\cdot\nu)_{0,\Gamma}.\end{aligned}$$

This and (3.4) lead

$$\begin{aligned}\sum_{j,k=1}^{n}\langle\aleph_k d_j u|d_j(\aleph_k d_k u)\rangle_{0,\Omega} &= \langle D\aleph\nabla_{\mathbf{a}}u|\nabla_{\mathbf{a}}u\rangle_{0,\Omega}\\ &\quad -\frac{1}{2}(|\nabla_{\mathbf{a}}u|^2|\mathrm{div}(\aleph))_{0,\Omega} + \frac{1}{2}(|\nabla_{\mathbf{a}}u|^2|\aleph\cdot\nu)_{0,\Gamma}.\end{aligned}$$

Combine this identity with the real part of (3.3) and integrate with respect to $t$ in order to get the expected identity.

**Second step.** We have

$$\begin{aligned}2\Re\left[i\partial_t u\left(\aleph\cdot\overline{\nabla_{\mathbf{a}}u}\right)\right] &= i\partial_t u\left(\aleph\cdot\overline{\nabla_{\mathbf{a}}u}\right) - i\partial_t\overline{u}\left(\aleph\cdot\nabla_{\mathbf{a}}u\right)\\ &= i\left[\partial_t u\left(\aleph\cdot\overline{\nabla u}\right) - \partial_t\overline{u}\left(\aleph\cdot\nabla u\right)\right] + (\aleph\cdot\mathbf{a})(\partial_t u\overline{u} + u\partial_t\overline{u})\\ &= i\left[\partial_t u\left(\aleph\cdot\overline{\nabla u}\right) - \partial_t\overline{u}\left(\aleph\cdot\nabla u\right)\right] + (\aleph\cdot\mathbf{a})\partial_t|u|^2.\end{aligned}$$

An integration by parts with respect to $t$ gives

$$(\aleph\cdot\mathbf{a}|\partial_t|u|^2)_{0,(0,T)} = -(\partial_t\aleph\cdot\mathbf{a}||u|^2)_{0,(0,T)} + \left[(\aleph\cdot\mathbf{a})|u|^2\right]_0^T.$$

Therefore

$$\begin{aligned}\langle i\partial_t u\aleph|\nabla_{\mathbf{a}}u\rangle_{0,Q} &= \frac{i}{2}\left[(\partial_t u\aleph|\nabla u)_{0,Q} - (\nabla u|\partial_t u\aleph)_{0,Q}\right]\\ &\quad -\frac{1}{2}(\partial_t\aleph\cdot\mathbf{a}||u|^2)_{0,Q} + \frac{1}{2}\left[(\aleph\cdot\mathbf{a}||u|^2)_{0,\Omega}\right]_0^T.\end{aligned}\tag{3.5}$$

Next we calculate the first term in the right hand side of the identity above. Integrating with respect to $t$, we find

$$(\partial_t u|\aleph\cdot\nabla u)_{0,(0,T)} = -(u|\partial_t\aleph\cdot\nabla u)_{0,(0,T)} - (u|\aleph\cdot\partial_t\nabla u)_{0,(0,T)} + \left[u(\aleph\cdot\overline{\nabla u})\right]_0^T.$$

On the other hand, Green's formula yields

$$(u\aleph|\partial_t\nabla u)_{0,Q} = -(\mathrm{div}(\aleph)u|\partial_t u)_{0,Q} - (\aleph\cdot\nabla u|\partial_t u)_{0,Q} + ((\aleph\cdot\nu)u|\partial_t u)_{0,\Sigma}.$$

Hence

$$\begin{aligned}\frac{i}{2}\left[(\partial_t u\aleph|\nabla u)_{0,Q} - (\nabla u|\partial_t u\aleph)_{0,Q}\right] &= -\frac{i}{2}(u\partial_t\aleph|\nabla u)_{0,Q} + \frac{i}{2}(\mathrm{div}(\aleph)u|\partial_t u)_{0,Q}\\ &\quad + \frac{i}{2}\left[u\aleph|\nabla u)_{0,\Omega}\right]_0^T dx - \frac{i}{2}(u(\aleph\cdot\nu)|\partial_t u)_{0,\Sigma}.\end{aligned}\tag{3.6}$$

**Step three.** We calculate the term $(\mathrm{div}(\aleph)u|\partial_t u)_{0,Q}$ in (3.6). Using $i\partial_t u = -\Delta_{\mathbf{a}}u + f$, we find

$$i(\mathrm{div}(\aleph)u|\partial_t u)_{0,Q} = (\mathrm{div}(\aleph)u|\Delta_{\mathbf{a}}u)_{0,Q} - (\mathrm{div}(\aleph)u|f)_{0,Q}.\tag{3.7}$$

But

$$
\begin{aligned}
(\mathrm{div}(\aleph)u|\Delta_{\mathbf{a}}u)_{0,Q} &= \sum_{j=1}^{n}(\mathrm{div}(\aleph)u|d_j d_j u)_{0,Q} \\
&= -\sum_{j=1}^{n}(d_j(\mathrm{div}(\aleph)u)|d_j u)_{0,Q} + \sum_{j=1}^{n}(\mathrm{div}(\aleph)u\nu_j|d_j u)_{0,\Sigma} \\
&= -\sum_{j=1}^{n}(\mathrm{div}(\aleph)d_j u|d_j u)_{0,Q} - \sum_{j=1}^{n}(\partial_j \mathrm{div}(\aleph)u|d_j u)_{0,Q} \\
&\qquad + \sum_{j=1}^{n}(\mathrm{div}(\aleph)u\nu_j|d_j u)_{0,\Sigma} \\
&= -(\mathrm{div}(\aleph)||\nabla_{\mathbf{a}}u|^2)_{0,Q} - (u\nabla(\mathrm{div}(\aleph))|\nabla_{\mathbf{a}}u)_{0,Q} \\
&\qquad + (\mathrm{div}(\aleph)u|\nabla_{\mathbf{a}}u\cdot\nu)_{0,\Sigma}.
\end{aligned}
\tag{3.8}
$$

A combination of (3.5) to (3.8) entails

$$
\begin{aligned}
\langle i\partial_t u\aleph|\nabla_{\mathbf{a}}u)_{0,Q} = &-\frac{i}{2}(\partial_t\aleph|\nabla u)_{0,Q} - \frac{1}{2}(\mathrm{div}(\aleph)||\nabla_{\mathbf{a}}u|^2)_{0,Q} \\
&-\frac{1}{2}(u\nabla(\mathrm{div}(\aleph))|\nabla_A u)_{0,Q} - \frac{1}{2}(\mathbf{a}\cdot\partial_t\aleph||u|^2)_{0,Q} \\
&+\frac{1}{2}(\mathrm{div}(\aleph)u|f)_{0,Q} \\
&+\frac{i}{2}[(u\aleph|\nabla u)_{0,\Omega}]_0^T dx + \frac{1}{2}[(u\aleph|u\mathbf{a})_{0,\Omega}]_0^T dx \\
&+\frac{1}{2}(\mathrm{div}(\aleph)u|\nabla_{\mathbf{a}}u\cdot\nu)_{0,\Sigma} - \frac{i}{2}(u(\aleph\cdot\nu)|\partial_t u)_{0,\Sigma}.
\end{aligned}
\tag{3.9}
$$

We put together the first and the fourth terms of the right hand side of this inequality. We obtain

$$
-\frac{i}{2}(u\partial_t\aleph|\nabla u)_{0,Q} - \frac{1}{2}(\partial_t\aleph u|u\mathbf{a})_{0,Q} = -\frac{i}{2}(u\partial_t\aleph|\nabla_{\mathbf{a}}u)_{0,Q}.
$$

Similarly, we put together the sixtieth and the ninetieth terms for the right hand of (3.9). We get

$$
\frac{i}{2}\,[(u\aleph|\nabla u)_{0,\Omega}]_0^T\,dx + \frac{1}{2}([(u\aleph|u\mathbf{a})_{0,\Omega}]_0^T\,dx = \frac{i}{2}\,[(u\aleph|\nabla_{\mathbf{a}}u)_{0,\Omega}]_0^T\,dx.
$$

Then (3.9) becomes

$$
\begin{aligned}
\langle i\partial_t u\aleph|\nabla_{\mathbf{a}}u)_{0,Q} = &-\frac{i}{2}(\partial_t\aleph|\nabla_{\mathbf{a}}u)_{0,Q} - \frac{1}{2}(\mathrm{div}(\aleph)||\nabla_{\mathbf{a}}u|^2)_{0,Q} \\
&-\frac{1}{2}(u\nabla(\mathrm{div}(\aleph))|\nabla_{\mathbf{a}}u)_{0,Q} \\
&+\frac{1}{2}(\mathrm{div}(\aleph)u|f)_{0,Q} \\
&+\frac{i}{2}[(u\aleph|\nabla_{\mathbf{a}}u)_{0,\Omega}]_0^T dx \\
&+\frac{1}{2}(\mathrm{div}(\aleph)u|\nabla_{\mathbf{a}}u\cdot\nu)_{0,\Sigma} - \frac{i}{2}(u(\aleph\cdot\nu)|\partial_t u)_{0,\Sigma}.
\end{aligned}
\tag{3.10}
$$

**Step four.** We complete the proof by noting the expected identity follows from (3.2), (3.10) and

$$\Re\left[i\partial_t u\aleph\cdot\overline{\nabla_{\mathbf{a}}u}\right]+\Re\left[\Delta_{\mathbf{a}}u\aleph\cdot\overline{\nabla_A u}\right]=\Re\left[f\aleph\cdot\overline{\nabla_{\mathbf{a}}u}\right].$$

□

Bearing in mind that $D(A_0)=H_0^1(\Omega)\cap H^2(\Omega)$, we derive that, for $u_0\in D(A_0)$,

$$u(t)=e^{tA_0}u_0\in C([0,T],D(A_0))\cap C^1([0,T],L^2(\Omega))\subset H^{2,1}(Q).$$

**Corollary 3.1.** *There exists a constant $C=C_1+C_2\sqrt{T}>0$, the constants $C_1$ and $C_2$ only depend on $\Omega$, so that, for any $u_0\in D(A_0)$ and $u(t)=e^{tA_0}u_0$, we have*

$$\|\nabla_{\mathbf{a}}u\|_{0,\Sigma}\leq C\|\nabla_{\mathbf{a}}u_0\|_{0,\Omega}. \tag{3.11}$$

*Proof.* We firstly note that according to [8, Lemma 3.2],

$$\|u(\cdot,t)\|_{0,\Omega}=\|u_0\|_{0,\Omega}\ \text{ and }\ \|\nabla_{\mathbf{a}}u(\cdot,t)\|_{0,\Omega}=\|\nabla_{\mathbf{a}}u_0\|_{0,\Omega},\ \ 0\leq t\leq T. \tag{3.12}$$

Let us choose $\aleph\in C^\infty(\overline{\Omega},\mathbb{R}^n)$ as an extension of $\nu$. In that case the left hand side of the identity in Proposition 3.1 is equal to the square of the left hand side of (3.11). While the right hand of the identity in Proposition 3.1 is bounded by the square of the right hand side of (3.11). This a consequence of Cauchy-Schwarz's inequality and (3.12). □

In the rest of this section, $x_0\in\mathbb{R}^n$ is fixed, $m=m(x)=x-x_0$, $x\in\mathbb{R}^n$ and

$$\Gamma_0=\{x\in\Gamma;\ m(x)\cdot\nu(x)>0\}.$$

Observe that in the present case the condition $\overline{\Gamma_0}\cap\overline{\Gamma_1}=\emptyset$ is satisfied for instance if $\Omega=\Omega_0\backslash\Omega_1$, with $\Omega_1\Subset\Omega_0$, $\Omega_j$ star-shaped with respect to $x_0\in\Omega_1$ and $\Gamma_j=\partial\Omega_j$, $j=0,1$.

We now sketch the proof of the following observability inequality announced in the introduction.

**Proposition 3.2.** *There exists a constant $C>0$, only depending on $\Omega$ and $T$, so that, for any $u_0\in D(A_0)$ and $u(t)=e^{tA_0}u_0$, we have*

$$\|\nabla_{\mathbf{a}}u_0\|_{0,\Omega}\leq C\|\nabla_{\mathbf{a}}u\|_{0,\Sigma_0}=C\|\partial_{\nu_{\mathbf{a}}}u\|_{0,\Sigma_0}. \tag{3.13}$$

*Sketch of the proof.* Take $\aleph=m$ in the identity of Proposition 3.1. We get

$$(m\cdot\nu||\partial_{\nu_{\mathbf{a}}}u|^2)_{0,\Sigma}=\|\nabla_{\mathbf{a}}u\|_{0,Q}^2-\frac{i}{2}\left[(um|\nabla_{\mathbf{a}}u)_{0,\Omega}\right]_0^T.$$

Whence, in light of (3.12),

$$T\|\nabla_{\mathbf{a}}u_0\|_{0,\Omega}^2\leq(m\cdot\nu||\partial_{\nu_{\mathbf{a}}}u|^2)_{0,\Sigma_0}+\frac{1}{2}\left|\left[(um|\nabla_{\mathbf{a}}u)_{0,\Omega}\right]_0^T\right|.$$

But, for $0<\epsilon<T$, there exists a constant $C_\epsilon>0$, independent on $T$, so that

$$\frac{1}{2}\left|\left[(um|\nabla_{\mathbf{a}}u)_{0,\Omega}\right]_0^T\right|\leq\epsilon\|\nabla_{\mathbf{a}}u_0\|_{0,\Omega}^2+C_\epsilon\|u_0\|_{0,\Omega}^2,$$

where we used again (3.12). Hence

$$(T-\epsilon)\|\nabla_{\mathbf{a}}u_0\|_{0,\Omega}^2\leq\|m\|_\infty\|\partial_{\nu_{\mathbf{a}}}u\|_{0,\Sigma_0}^2+C_\epsilon\|u_0\|_{0,\Omega}^2.$$

As $\|\nabla_{\mathbf{a}}\cdot\|_{0,\Omega}$ and $\|\nabla\cdot\|_{0,\Omega}$ are equivalent on $H_0^1(\Omega)$, we can repeat the compactness argument in [25, Proposition 2.1] to complete the proof. □

We also sketch the proof of the observability inequality with interior control. We restate here for convenience this result.

Recall that for this result $\omega$ is a neighborhood of $\Gamma_0$ in $\Omega$ so that $\overline{\omega} \cap \Gamma_1 = \emptyset$.

**Proposition 3.3.** *There exists a constant $C > 0$, only depending on $\Omega$, $T$, $\Omega$ and $\Gamma_0$, so that, for any $u_0 \in D(A_0)$ and $u(t) = e^{tA_0}u_0$, we have*

$$\|u_0\|_{0,\Omega} \le C\|u\|_{0,Q_\omega}. \tag{3.14}$$

*Here $Q_\omega = \omega \times (0,T)$.*

*Sketch of the proof.* Fix $0 < \delta < T$. Let $\nu_e \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$ be an extension of $\nu$, $0 \le \phi \in C_0^\infty(0,T)$ satisfying $\phi = 1$ in $[\delta, T-\delta]$, and $\psi \in C_0^\infty(\mathbb{R}^n)$ so that $\mathrm{supp}(\psi) \cap \Omega \subset \widetilde{\omega}$ and $\psi = 1$ on $\Gamma_0$.

We have from Proposition 3.2 with $\aleph = \nu_e\phi\psi$, in which $(0,T)$ is substituted by $(\delta, T-\delta)$,

$$\|\nabla_{\mathbf{a}} u_0\|_{0,\Omega} = \|\nabla_{\mathbf{a}} u(\cdot,\delta)\|_{0,\Omega} \le C\|\partial_{\nu_{\mathbf{a}}} u\|_{0,\Gamma_0\times(\delta,T-\delta)} \le C\|(\aleph\cdot\nu)\partial_{\nu_{\mathbf{a}}} u\|_{0,\Sigma}. \tag{3.15}$$

Let $\tilde{\omega}$ be a neighborhood of $\omega$ in $\Omega$ satisfying $\overline{\tilde{\omega}} \cap \Gamma_1 = \emptyset$. As in the proof of Corollary 3.1, we obtain by applying Proposition 3.1, where $Q_{\tilde{\omega}} = \widetilde{\omega} \times (0,T)$,

$$C\|(\aleph\cdot\nu)\partial_{\nu_{\mathbf{a}}} u\|_{0,\Sigma} \le \|\nabla_{\mathbf{a}} u\|_{0,Q_{\tilde{\omega}}} + \|u\|_{0,Q_{\tilde{\omega}}}. \tag{3.16}$$

On the other hand, using $\Delta_{\mathbf{a}} u(\cdot,t) = -i\partial_t u(\cdot,t)$ in $\Omega$ and Caccioppoli's inequality in order to obtain

$$C\|\nabla_{\mathbf{a}} u\|_{0,Q_{\tilde{\omega}}} \le \|u\|_{0,Q_\omega} + \|\partial_t u\|_{L^2((0,T),H^{-1}(\omega))}. \tag{3.17}$$

Inequalities (3.16) and (3.17) at hand, we can mimic the interpolation argument in the end of the proof of [25, Proposition 3.1] to complete the proof. □

3.2. **Stabilization by an internal damping.** The following result was announced in the introduction. In this subsection we aim to prove it.

**Theorem 3.1.** *There exists a constant $\varrho > 0$, depending only on $\Omega$, $T$, $\Omega$ and $\Gamma_0$, so that*

$$\mathcal{E}^1_{u_0}(t) \le e^{-\varrho t}\mathcal{E}^1_{u_0}(0), \quad u_0 \in L^2(\Omega).$$

*Proof.* By density it is enough to give the proof when $u_0 \in D(A_0)$. Fix then $u_0 \in L^2(\Omega)$ and let $u(t) = e^{tA_1}u_0$. We decompose $u$ into two terms, $u = v + w$, with

$$v(t) = e^{tA_0}u_0 \ \text{ and } \ w(t) = -i\int_0^t e^{(t-s)A_0}cu(s)ds.$$

As $\mathcal{E}^1_{u_0}$ is non increasing, we have

$$\mathcal{E}^1_{u_0}(t) \le \mathcal{E}^1_{u_0}(0) = \frac{1}{2}\|u_0\|^2_{0,\Omega}.$$

Hence

$$\mathcal{E}^1_{u_0}(t) \le C\|v\|^2_{0,Q_\omega} \le C\|\sqrt{c}u\|^2_{0,Q_\omega} + C\|w\|^2_{0,Q_\omega}$$

by Proposition 3.3. Whence, using that $c \ge c_0 > 0$ a.e. in $\omega$,

$$\mathcal{E}^1_{u_0}(t) \le C\|\sqrt{c}v\|^2_{0,Q} \le C\|\sqrt{c}u\|^2_{0,Q} + C\|w\|^2_{0,Q}. \tag{3.18}$$

On the other hand, it is straightforward to check that

$$\|w\|^2_{0,Q} \le \|cu\|^2_{0,Q} \le \|c\|_\infty\|\sqrt{c}u\|^2_{0,Q}.$$

This and (3.18) entail

$$\mathcal{E}^1_{u_0}(t) \le C\|\sqrt{c}u\|^2_{0,Q} = -C\frac{d}{dt}\mathcal{E}^1_{u_0}(t).$$

Or equivalently

$$\frac{d}{dt}\mathcal{E}^1_{u_0}(t) \le -C^{-1}\mathcal{E}^1_{u_0}(t).$$

This yields the expected inequality in a straightforward manner. □

3.3. **Stabilization by a boundary damping.** In this subsection we take $d(x) = m(x)\cdot\nu(x)$, $x\in\Gamma_0$, which satisfies obviously the assumption required in Section 1.

Let $u_0 \in V$ and recall the $\mathcal{E}^2_{u_0}(t) = \frac{1}{2}\left\|\nabla_{\mathbf{a}} e^{tA_2}u_0\right\|^2_{0,\Omega}$ satisfies

$$\frac{d}{dt}\mathcal{E}^2_{u_0}(t) = -\|\sqrt{m\cdot\nu}\,u'(t)\|_{0,\Gamma_0} = -\|\sqrt{m\cdot\nu}\,\Delta_{\mathbf{a}}(t)\|_{0,\Gamma_0},\quad t>0.$$

Here $u(t) = e^{tA_2}u_0$.

Introduce,

$$\mathscr{E}^2_{u_0}(t) = \Im(u(t)|m\cdot\nabla u(t))_{0,\Omega}.$$

**Lemma 3.1.** *For any $u_0\in V$ and $u(t) = e^{tA_2}u_0$, we have, where $t>0$,*

(3.19)

$$\frac{d}{dt}\mathscr{E}^2_{u_0}(t) = 2\Re(\Delta_{\mathbf{a}}u|m\cdot\nabla u(t))_{0,\Omega} - n\|\nabla_{\mathbf{a}}u(t)\|^2_{0,\Omega} - \Re((n+i)(m\cdot\nu)u(t)|u'(t))_{0,\Gamma_0}.$$

*Proof.* By density it is sufficient to give the proof when $u_0\in D(A_2)$. In that case, we have

$$\frac{d}{dt}\mathscr{E}^2_{u_0}(t) = \Im\left[(u'(t)|m\cdot\nabla u(t))_{0,\Omega} + (u(t)|m\cdot\nabla u'(t))_{0,\Omega}\right].$$

An integration by parts yields

$$\begin{aligned}(u(t)|m\cdot\nabla u'(t))_{0,\Omega} &= -(\mathrm{div}(u(t)m)|u'(t))_{0,\Omega} + (u(t)(m\cdot\nu)|u'(t))_{0,\Gamma}\\ &= -n(u(t)|u'(t))_{0,\Omega} - (m\cdot\nabla u(t)|u'(t))_{0,\Omega} + (u(t)(m\cdot\nu)|u'(t))_{0,\Gamma}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{dt}\mathscr{E}^2_{u_0}(t) = \Im&\left[(u'(t)|m\cdot\nabla u(t))_{0,\Omega} - n(u(t)|u'(t))_{0,\Omega}\right]\\ &- \Im\left[(\nabla u(t)\cdot m|u'(t))_{0,\Omega} + (u(t)(m\cdot\nu)|u'(t))_{0,\Gamma}\right].\end{aligned}$$

Since

$$(u'(t)|m\cdot\nabla u(t))_{0,\Omega} - (\nabla u(t)\cdot m|u'(t))_{0,\Omega} = 2i\Im(u'(t)|m\cdot\nabla u(t))_{0,\Omega},$$

we obtain

$$\frac{d}{dt}\mathscr{E}^2_{u_0}(t) = 2\Im(u'(t)|m\cdot\nabla u(t))_{0,\Omega} - n\Im(u(t),u'(t))_{0,\Omega} + \Im(u(t)(m\cdot\nu)|u'(t))_{0,\Gamma}.$$

But $u'(t) = i\Delta_{\mathbf{a}}u(t)$. Therefore

$$\frac{d}{dt}\mathscr{E}^2_{u_0}(t) = 2\Re(\Delta_{\mathbf{a}}u|m\cdot\nabla u(t))_{0,\Omega} - n\Re(u(t),\Delta_{\mathbf{a}}(t))_{0,\Omega} + \Im(u(t)(m\cdot\nu)|u'(t))_{0,\Gamma}.$$

This and

$$(\Delta_{\mathbf{a}}u(t),u(t))_{0,\Omega} = -\|\nabla_{\mathbf{a}}u(t)\|^2_{0,\Omega} + (\partial_{\nu_{\mathbf{a}}}u(t)|u(t))_{0,\Gamma}$$

entail

$$\frac{d}{dt}\mathscr{E}^2_{u_0}(t) = 2\Re(\Delta_{\mathbf{a}} u|m\cdot\nabla u(t))_{0,\Omega} - n\|\nabla_{\mathbf{a}} u(t)\|^2_{0,\Omega} + n\Re(\partial_{\nu_{\mathbf{a}}} u(t)|u(t))_{0,\Gamma} + \Im(u(t)(m\cdot\nu)|u'(t))_{0,\Gamma}. \tag{3.20}$$

Using that $\partial_{\nu_{\mathbf{a}}} u = -(m\cdot\nu)u'(t)$ on $\Gamma_0$ and $u = 0$ on $\Gamma_1$, we get

$$n\Re(\partial_{\nu_{\mathbf{a}}} u(t)|u(t))_{0,\Gamma} + \Im(u(t)(m\cdot\nu)|u'(t))_{0,\Gamma} = -\Re((n+i)(m\cdot\nu)u(t)|u'(t))_{0,\Gamma_0}.$$

In (3.20), this identity yields

$$\frac{d}{dt}\mathscr{E}^2_{u_0}(t) = 2\Re(\Delta_{\mathbf{a}} u(t)|m\cdot\nabla u(t))_{0,\Omega} - n\|\nabla_{\mathbf{a}} u(t)\|^2_{0,\Omega} - \Re((n+i)(m\cdot\nu)u(t)|u'(t))_{0,\Gamma_0},$$

which is the expected inequality. □

Henceforward, $\varkappa_1$ is the Poincaré constant of $V$.

**Lemma 3.2.** *Assume that $\|\mathbf{a}\|_\infty \le \frac{1}{2\varkappa_1}$. Then, for any $u \in D(A_2)$, we have*

$$\Re(\Delta_{\mathbf{a}} u, m\cdot\nabla u)_{0,\Omega} \le \frac{n-2}{2}(1+\delta(\|\mathbf{a}\|_\infty))\|\nabla_{\mathbf{a}} u\|^2_{0,\Omega} + \Re(\partial_\nu u|m\cdot\nabla u)_{0,\Gamma_0} - \frac{1}{2}(|\nabla u|^2|m\cdot\nu)_{0,\Gamma_0}, \tag{3.21}$$

*where the function $\delta$, depending only on $\Omega$ and $\Gamma_0$, satisfies $\delta(\rho)\to 0$ as $\rho \to 0$.*

*Proof.* By simple integration by parts, we have

$$\Re(\nabla u|\nabla(m\cdot\nabla u))_{0,\Omega} = -\frac{n-2}{2}\|\nabla u\|^2_{0,\Omega} + \frac{1}{2}(|\nabla u|^2|m\cdot\nu)_{0,\Gamma}. \tag{3.22}$$

But

$$\Re(\Delta_{\mathbf{a}} u, m\cdot\nabla u)_{0,\Omega} = \Re(\Delta u|m\cdot\nabla u)_{0,\Omega} - 2\Im(\mathbf{a}\cdot\nabla u|m\cdot\nabla u)_{0,\Omega} + \Re([i\mathrm{div}(\mathbf{a}) - |\mathbf{a}|^2]u|m\cdot\nabla u)_{0,\Omega}. \tag{3.23}$$

Integrating by parts, the first term in the right hand side of inequality (3.23) in order to get

$$\Re(\Delta u|m\cdot\nabla u)_{0,\Omega} = -\Re(\nabla u|\nabla(m\cdot\nabla u))_{0,\Omega} + \Re(\partial_\nu u|m\cdot\nabla u)_{0,\Gamma}.$$

This identity combined with (3.22) yields

(3.24)

$$\begin{aligned}\Re(\Delta u|m\cdot\nabla u)_{0,\Omega} &= \frac{n-2}{2}\|\nabla u\|^2_{0,\Omega} + \Re(\partial_\nu u|m\cdot\nabla u)_{0,\Gamma} - \frac{1}{2}(|\nabla u|^2|m\cdot\nu)_{0,\Gamma}.\\ &= \frac{n-2}{2}\left(\|\nabla_{\mathbf{a}} u\|^2_{0,\Omega} + 2\Im(u|\mathbf{a}\cdot\nabla u)_{0,\Omega} - \||\mathbf{a}|u\|^2_{0,\Omega}\right)\\ &\quad + \Re(\partial_\nu u|m\cdot\nabla u)_{0,\Gamma} - \frac{1}{2}(|\nabla u|^2|m\cdot\nu)_{0,\Gamma}.\end{aligned}$$

Under the assumption on $\mathbf{a}$, straightforward computations show

$$\|\nabla u\|_{0,\Omega} \le 2\|\nabla_{\mathbf{a}} u\|_{0,\Omega}$$

and

$$\|u\|_{0,\Omega} \le 2\varkappa_1\|\nabla_{\mathbf{a}} u\|_{0,\Omega}.$$

These inequalities enable us to derive from (3.24)

$$\Re(\Delta u|m\cdot\nabla u)_{0,\Omega}\le\frac{n-2}{2}(1+\delta_0)\|\nabla_{\mathbf{a}}u\|_{0,\Omega}^2+\Re(\partial_\nu u|m\cdot\nabla u)_{0,\Gamma}-\frac{1}{2}(|\nabla u|^2|m\cdot\nu)_{0,\Gamma},\tag{3.25}$$

where

$$\delta_0=4(2\varkappa_1+\varkappa_1^2)\|\mathbf{a}\|_\infty.$$

Similarly, we have

$$\left|-2\Im(\mathbf{a}\cdot\nabla u|m\cdot\nabla u)_{0,\Omega}+\Re([i\mathrm{div}(\mathbf{a})-|\mathbf{a}|^2]u|m\cdot\nabla u)_{0,\Omega}\right|\le\frac{n-2}{2}\delta_1\|\nabla_{\mathbf{a}}u\|_{0,\Omega}^2,\tag{3.26}$$

the constant $\delta_1=\delta_1(\|\mathbf{a}\|_\infty)$ is so that $\delta_1(\rho)\to 0$ as $\rho\to 0$.

In light of (3.25) and (3.26), we get from (3.23)

$$\Re(\Delta_{\mathbf{a}}u,m\cdot\nabla u)_{0,\Omega}\le\frac{n-2}{2}(1+\delta)\|\nabla_{\mathbf{a}}u\|_{0,\Omega}^2+\Re(\partial_\nu u|m\cdot\nabla u)_{0,\Gamma}-\frac{1}{2}(|\nabla u|^2|m\cdot\nu)_{0,\Gamma}.\tag{3.27}$$

Here $\delta=\delta_0+\delta_1$.

On the other hand,

$$\begin{aligned}\Re(\partial_\nu u|m\cdot\nabla u)_{0,\Gamma_1}&-\frac{1}{2}(|\nabla u|^2|m\cdot\nu)_{0,\Gamma_1}\\&=(|\partial_\nu u|^2|m\cdot\nu)_{0,\Gamma_1}-\frac{1}{2}(|\partial_\nu u|^2|m\cdot\nu)_{0,\Gamma_1}\\&=\frac{1}{2}(|\partial_\nu u|^2|m\cdot\nu)_{0,\Gamma_1}\le 0.\end{aligned}\tag{3.28}$$

A combination of (3.27) and (3.28) yields

$$\Re(\Delta_{\mathbf{a}}u,m\cdot\nabla u)_{0,\Omega}\le\frac{n-2}{2}(1+\delta)\|\nabla_{\mathbf{a}}u\|_{0,\Omega}^2+\Re(\partial_\nu u|m\cdot\nabla u)_{0,\Gamma_0}-\frac{1}{2}(|\nabla u|^2|m\cdot\nu)_{0,\Gamma_0}.$$

The proof is then complete. □

We are now able to prove the second exponential stabilization result. We recall that this result is the following

**Theorem 3.2.** *There exists $0<\varsigma\le\frac{1}{2\varkappa_1}$, depending on $x_0$ and $\Omega$, with the property that, if $\|\mathbf{a}\|_\infty\le\varsigma$ and $\mathbf{a}=0$ on $\Gamma_0$, then there exists two constants $C>0$ and $\varrho>0$, depending only on $x_0$ and $\Omega$, so that*

$$\mathcal{E}_{u_0}^2(t)\le Ce^{-\varrho t}\mathcal{E}_{u_0}^2(0),\quad u_0\in V.$$

*Proof.* Let $u_0\in V$ and set $u(t)=e^{tA_2}u_0$. Since $\mathbf{a}=0$ on $\Gamma_0$, we have

$$\Re(\partial_\nu u(t)|m\cdot\nabla u(t))_{0,\Gamma_0}=-\Re((m\cdot\nu)u'(t)|m\cdot\nabla u(t))_{0,\Gamma_0}.$$

This inequality and (3.21) entail

$$\begin{aligned}2\Re(\Delta_{\mathbf{a}}u,m\cdot\nabla u)_{0,\Omega}&\le(n-2)(1+\delta(\|\mathbf{a}\|_\infty))\|\nabla_{\mathbf{a}}u\|_{0,\Omega}^2\\&\quad-2\Re((m\cdot\nu)u'(t)|m\cdot\nabla u(t))_{0,\Gamma_0}-\Re(m\cdot\nu||\nabla u|^2)_{0,\Gamma_0}.\end{aligned}$$

Using this inequality in (3.19), we get

$$\frac{d}{dt}\mathscr{E}^2_{u_0}(t) \le -\|\nabla_{\mathbf{a}} u(t)\|^2_{0,\Omega} - 2\Re((m\cdot\nu)u'(t)|m\cdot\nabla u(t))_{0,\Gamma_0} - \Re(m\cdot\nu||\nabla u|^2)_{0,\Gamma_0} - \Re((n+i)u(t)|u'(t))_{0,\Gamma_0}, \tag{3.29}$$

provided that $\delta \le \frac{1}{n-2}$. This last condition is satisfied whenever $\|\mathbf{a}\|_\infty \le \varsigma$, for some $0<\varsigma\le \frac{1}{2\varkappa_1}$.

Define, for $\epsilon>0$,

$$\mathcal{E}^{2,\epsilon}_{u_0} = \mathcal{E}^2_{u_0} + \epsilon\mathscr{E}^2_{u_0}.$$

From inequality (3.29), we have

$$\frac{d}{dt}\mathcal{E}^{2,\epsilon}_{u_0}(t) \le -\epsilon\mathcal{E}^2_{u_0}(t) - (m\cdot\nu||u'(t)|^2)_{0,\Gamma_0} - \epsilon\left[2\Re((m\cdot\nu)u'(t)|m\cdot\nabla u(t))_{0,\Gamma_0} + \Re(m\cdot\nu||\nabla u|^2)_{0,\Gamma_0} + \Re((n+i)u(t)|u'(t))_{0,\Gamma_0}\right]. \tag{3.30}$$

Let $\|\mathbf{tr}\|$ be the norm of the trace operator

$$u\in V \to \sqrt{m\cdot\nu}\, u_{|\Gamma_0} \in L^2(\Gamma_0),$$

when $V$ is endowed with the norm $\|\nabla_{\mathbf{a}}\cdot\|_{0,\Omega}$. Then

$$\begin{aligned}|((m\cdot\nu)(n+i)u(t)|u'(t))_{0,\Gamma_0}| &\le \frac{\|\mathbf{tr}\|^2}{2}(n^2+1)\|\sqrt{m\cdot\nu}u'(t)\|^2_{0,\Gamma_0} \\ &\quad + \frac{1}{2\|\mathbf{tr}\|^2}\|\sqrt{m\cdot\nu}u(t)\|^2_{0,\Gamma_0} \\ &\le \frac{\|\mathbf{tr}\|}{2}(n^2+1)\|\sqrt{m\cdot\nu}u'(t)\|^2_{0,\Gamma_0} + \frac{1}{2}\|\nabla_{\mathbf{a}} u(t)\|^2_{0,\Omega}.\end{aligned} \tag{3.31}$$

Also,

$$2\left|u'(t)(m\cdot\nabla\overline{u(t)})\right| \le \|m\|^2_\infty|u'(t)|^2 + |\nabla u(t)|^2. \tag{3.32}$$

If $\vartheta = \frac{\|\mathbf{tr}\|}{2}(n^2+1) + \|m\|^2_\infty$, then inequalities (3.31) and (3.32) in (3.30) entail

$$\frac{d}{dt}\mathcal{E}^{2,\epsilon}_{u_0}(t) \le -\frac{\epsilon}{2}\mathcal{E}^2_{u_0}(t) - (1-\epsilon\vartheta)\|\sqrt{m\cdot\nu}u'(t)\|^2_{0,\Gamma_0}.$$

That is

$$\frac{d}{dt}\mathcal{E}^{2,\epsilon}_{u_0}(t) \le -\frac{\epsilon}{2}\mathcal{E}^2_{u_0}(t) \text{ if } 1-\epsilon\vartheta\ge 0. \tag{3.33}$$

On the other hand, as

$$\mathscr{E}^2_{u_0}(t) \le 2\varkappa_1\|m\|_\infty\|\nabla_{\mathbf{a}} u(t)\|^2_{0,\Omega} = 2\varkappa\|m\|_\infty\mathcal{E}^2_{u_0}(t),$$

we have

$$\mathcal{E}^{2,\epsilon}_{u_0}(t) \le (1+2\varkappa_1\|m\|_\infty)\mathcal{E}^2_{u_0}(t).$$

This in (3.33) yields

$$\frac{d}{dt}\mathcal{E}^{2,\epsilon}_{u_0}(t) \le -\epsilon\mu\mathcal{E}^{2,\epsilon}_{u_0}(t),\quad 0<\epsilon\le\epsilon_0=\frac{1}{\vartheta}.$$

Here $\mu = \frac{1}{2+4\varkappa_1\|m\|_\infty}$. Hence

$$\mathcal{E}^{2,\epsilon}_{u_0}(t) \le e^{-\epsilon\mu t}\mathcal{E}^{2,\epsilon}_{u_0}(0)$$

But

$$(1-2\varkappa_1\|m\|_\infty)\mathcal{E}_{u_0}^2(t) \le \mathcal{E}_{u_0}^{2,\epsilon}(t) \le (1+2\varkappa_1\|m\|_\infty)\mathcal{E}_{u_0}^2(t).$$

Therefore

$$\mathcal{E}_{u_0}^2(t) \le \frac{1+2\varkappa_1\|m\|_\infty}{2}e^{-\epsilon\mu t}\mathcal{E}_{u_0}^2(0), \quad 0<\epsilon\le \min\left(\epsilon_0, \frac{1}{4\varkappa_1\|m\|_\infty}\right).$$

The proof is then complete. □

## 4. **Additional comments**

4.1. **Exponential stabilization via a Carleman inequality.** In this subsection we show that we can retrieve the exponential stabilization result in Theorem 3.1 by using an argument based on a Carleman inequality.

Assume that $\omega$ can be chosen in such a way that there exists $\psi \in C^4(\overline{\Omega})$ satisfying

$$\psi > 0 \text{ in } \overline{\Omega}, \quad \nabla\psi \neq 0 \text{ in } \overline{\Omega\setminus\omega}, \quad \partial_\nu\psi \le 0 \text{ on } \Gamma$$

and the following pseudo-convexity condition: there exists $\varpi > 0$ so that

$$|\nabla\psi(x)\cdot\xi|^2 + \nabla^2\psi(x)\xi\cdot\overline{\xi} \ge \varpi|\xi|^2, \;\; x\in\overline{\Omega\setminus\omega},\; \xi\in\mathbb{C}^n.$$

Here where $\nabla^2\psi = (\partial_{ij}\psi)$.

Note that since $\nabla^2\psi$ is symmetric, $\nabla^2\psi\xi\cdot\overline{\xi}$ is real.

We call this condition on $\omega$ by $(\mathcal{G})$.

Let us provide a domain $\omega$ obeying to condition $(\mathcal{G})$. In fact, any neighborhood $\omega$ of $\Gamma$ in $\Omega$ possesses this property. To see that, pick $\omega$ a neighborhood of $\Gamma$ in $\Omega$, $x_0$ an arbitrary point in $\mathbb{R}^n\setminus\overline{\Omega}$ and $0\le\chi\in C_0^\infty(\Omega)$ satisfying $\chi=1$ in a neighborhood of $\overline{\Omega\setminus\omega}$. Then it is obvious to check that $\psi(x)=1+\chi(x)|x-x_0|^2$ satisfies all the conditions listed in $(\mathcal{G})$. This construction can be improved to include domains satisfying the condition for the exponential stabilization discussed in the multiplier method. To this end, fix again $x_0$ an arbitrary point in $\mathbb{R}^n\setminus\overline{\Omega}$ and set

$$\Gamma_0 = \{x\in\Gamma;\; \nu(x)\cdot(x-x_0)>0\}.$$

Pick $\omega$ a neighborhood of $\Gamma_0$ in $\Omega$ so that $\overline{\omega}\cap\Gamma_1=\emptyset$ and let $0\le\chi\in C_0^\infty(\Omega)$ with $\chi=1$ in a neighborhood of $\overline{\Omega\setminus\omega}$ and $\mathrm{supp}(\chi)\cap\overline{\Gamma_0}=\emptyset$. A straightforward computations show that $\psi(x)=1+\chi(x)|x-x_0|^2$ fulfills condition $(\mathcal{G})$.

Substituting, if necessary, $\psi$ by $\psi+C$, for some large constant $C$, we can assume that

$$\psi > \frac{2}{3}\|\psi\|_\infty \;\text{ in }\; \overline{\Omega}.$$

In the sequel, the two functions $\theta$ and $\varphi$, defined on $Q$, are given by

$$\theta(x,t)=\frac{e^{\lambda\psi(x)}}{t(T-t)}, \quad \varphi(x,t)=\frac{e^{2\lambda\|\psi\|_\infty}-e^{\lambda\psi(x)}}{t(T-t)}.$$

Here $\lambda$ is a parameter to be specified later.

Let

$$\mathscr{H}=\{w\in L^2((0,T),H_0^1(\Omega));\; i\partial_t+\Delta_{\mathbf{a}}\in L^2(Q)\}.$$

A straightforward modification of the proof [31, Corollary 3.2] gives

**Lemma 4.1.** *There are three constants $\lambda_0 \geq 1$, $s_0 \geq 1$ and $C_0 > 0$ such that for all $\lambda \geq \lambda_0$, $s \geq s_0$ and $w \in \mathscr{H}$, it holds*

$$\|\sqrt{\lambda s\theta}e^{-s\varphi}\nabla_{\mathbf{a}}w\|_{0,Q} + \|\lambda^2 s\theta\sqrt{s\theta}e^{-s\varphi}w\|_{0,Q}$$
$$\leq C_0\left(\|e^{-s\varphi}(i\partial_t + \Delta_{\mathbf{a}})w\|_{0,Q} + \|\sqrt{\lambda s\theta}e^{-s\varphi}\nabla_{\mathbf{a}}w\|_{0,Q_\omega} + \|\lambda^2 s\theta\sqrt{s\theta}e^{-s\varphi}w\|_{0,Q_\omega}\right).$$

Pick $u_0 \in D(A_0)$ and let $u(t) = e^{tA_0}u_0$. Taking into account that

$$\|u(t)\|_0 = \|u_0\|_{0,\Omega} \text{ and } \|\nabla_{\mathbf{a}}u(t)\|_{0,\Omega} = \|\nabla_{\mathbf{a}}u_0\|_{0,\Omega},$$

we obtain by applying Lemma 4.1 the following observability inequality

**Corollary 4.1.** *There exists a constant $C > 0$, depending on $\Omega$, $\mathbf{a}$, $\omega$ and $T$, so that for any $u_0 \in D(A_0)$ and $u(t) = e^{tA_0}u_0$, we have*

$$\|\nabla_{\mathbf{a}}u_0\|_{0,\Omega} \leq C\left(\|\nabla_{\mathbf{a}}u\|_{0,Q_\omega} + \|u\|_{0,Q_\omega}\right). \tag{4.1}$$

This observability inequality at hand, we can proceed similarly to the proof of Theorem 3.1 in order to get the following theorem.

**Theorem 4.1.** *Let $\omega$ be a neighborhood of $\omega_0$ in $\Omega$, where $\omega_0$ obeys to the condition $(\mathcal{G})$. Then there exists a constant $\varrho > 0$, depending only on $\Omega$, $T$, $\omega$, so that*

$$\mathcal{E}^1_{u_0}(t) \leq e^{-\varrho t}\mathcal{E}^1_{u_0}(0), \quad u_0 \in L^2(\Omega).$$

Clearly, from the previous discussion, Theorem 4.1 improve Theorem 3.1. However we do not know whether we can construct a domain $\omega$ obeying to condition $(\mathcal{G})$ but doesn't satisfy the assumption in Theorem 3.1.

*Remark* 4.1. As in Theorem 3.1, one step in the proof consists in establishing the following observability inequality

$$\|u_0\|_{0,\Omega} \leq C\|e^{tA_0}u_0\|_{0,Q_\omega}, \quad u_0 \in L^2(\Omega).$$

According to [32, Theorem 5.1], under the assumption of Theorem 4.1, this inequality is equivalent to the following the so-called observability resolvent estimate: there exists two constants $\aleph_0$ and $\aleph_1$, depending on $\Omega$, $\omega$ and $\mathbf{a}$ so that, for any $\mu \in \mathbb{R}$ and $u \in D(A_0)$, we have

$$\|u\|^2_{0,\Omega} \leq \aleph_0\|(A_0 - i\mu)u\|^2_0 + \aleph_1\|u\|^2_{0,\omega}.$$

4.2. **Observability inequality in a product space.** We consider the case in which $\Omega = \Omega_1 \times \Omega_2$, with $\Omega_j$ a $C^\infty$ bounded domain of $\mathbb{R}^{n_j}$, $j = 1, 2$ and $n_1 + n_2 = n$. Assume that

$$\mathbf{a}(x_1, x_2) = (\mathbf{a_1}(x_1), \mathbf{a}_2(x_2)) \in \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}, \quad (x_1, x_2) \in \Omega.$$

where $\mathbf{a}_j$ satisfies the same assumptions as $\mathbf{a}$ when $\Omega$ is substituted by $\Omega_j$, $j = 1, 2$. Denote by $A_{0,j}$ the operator $A_0$ when $\Omega = \Omega_j$ and $\mathbf{a}$ is substituted by $\mathbf{a}_j$, $j = 1, 2$.

For $u_{0,j} \in L^2(\Omega_j)$, $j = 1, 2$, it is not hard to check that

$$e^{tA_0}(u_{0,1} \otimes u_{0,2}) = e^{tA_{0,1}}u_{0,1} \otimes e^{tA_{0,2}}u_{0,2}. \tag{4.2}$$

Let $\omega_1$ be an open subset of $\Omega_1$, $Q_{\omega_1} = \omega_1 \times (0, T)$, $\omega = \omega_1 \times \Omega_2$ and $Q_\omega = \omega \times (0, T)$.

Following a simple idea in [10], we have

**Theorem 4.2.** *Assume that there exists a constant $C > 0$ so that the following observability inequality holds*

$$\|u_{0,1}\|^2_{0,\Omega_1} \le C\|e^{tA_{0,1}}u_{0,1}\|_{0,Q_{\omega_1}}, \quad u_{0,1} \in L^2(\Omega_1). \tag{4.3}$$

*Then*

$$\|u_0\|^2_{0,\Omega} \le C\|e^{tA_0}u_0\|_{0,Q_\omega}, \quad u_0 \in L^2(\Omega).$$

*Proof.* Denote by $(\phi_k)_{k\ge 1}$ an orthonormal basis consisting of eigenfunctions of $A_{0,2}$. For $u_0 \in L^2(\Omega)$, we have

$$u_0(x_1, x_2) = \sum_{k\ge 1} \psi_k(x_1)\phi_k(x_2).$$

Here

$$\psi_k(x_1) = (u(x_1, \cdot)|\phi_k)_{0,\Omega_2} \in L^2(\Omega_1), \quad k \ge 1$$

In light of (4.2) we have, where $(i\lambda_k) \subset i\mathbb{R}$ is the sequence of eigenvalues of $A_{0,2}$,

$$e^{tA_0}u_0(x_1, x_2) = \sum_{k\ge 1} e^{i\lambda_k t}e^{tA_{0,1}}\psi_k(x_1)\phi_k(x_2).$$

We get by applying Parseval's identity

$$\|e^{tA_0}u_0\|^2_{0,\Omega} = \sum_{k\ge 1}\|e^{tA_{0,1}}\psi_k\|^2_{0,\Omega_1} = \sum_{k\ge 1}\|\psi_k\|^2_{0,\Omega_1} = \|u_0\|^2_{0,\Omega}$$

and

$$\|e^{tA_0}u_0\|^2_{0,\omega} = \sum_{k\ge 1}\|e^{tA_{0,1}}\psi_k\|^2_{0,\omega_1}. \tag{4.4}$$

On other hand, apply observability inequality (4.3) in order to obtain

$$\|u_0\|^2_{0,\Omega} = \sum_{k\ge 1}\|\psi_k\|^2_{0,\Omega_1} \le C\sum_{k\ge 1}\|e^{tA_{0,1}}\psi_k\|^2_{0,Q_{\omega_1}}.$$

This and (4.4) entail

$$\|u_0\|^2_{0,\Omega} \le C\|e^{tA_0}u_0\|^2_{0,Q_\omega}.$$

This is the expected inequality. □

**Acknowledgements** We would like to thank the referees for their valuable comments which enabled us to improve the paper.

## References

[1] N. Anantharaman, C. Fermanian-Kammerer and F. Macià, *Semiclassical completely integrable systems : long-time dynamics and observability via two-microlocal Wigner measures*, Amer. J. Math. 137 (3) (2015), 577-638.

[2] N. Anantharaman, M. Léautaud and F. Macià. *Wigner measures and observability for the Schroödinger equation on the disk*, Inventiones Mathematicae. 206 (2) (2016), 485-599.

[3] N. Anantharaman and F. Macià, *Semiclassical measures for the Schrödinger equation on the torus*, J. European Math. Soc. 16 (6) (2014), 1253-1288.

[4] N. Anantharaman and M. Léautaud, *Sharp polynomial decay rates for the damped wave equation on the torus. With an appendix by Stéphane Nonnenmacher*, Anal. PDE. 7 (2014), 159-214.

[5] N. Anantharaman and G. Rivière, *Dispersion and controllability for the Schrödinger equation on negatively curved manifolds*, Anal. PDE. 5 (2012), 313-338.

[6] C. Bardos, G. Lebeau and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim. 30 (1992), 1024-1065.
[7] C. J. K. Batty and T. Duyckaerts, *Non-uniform stability for bounded semi-groups on Banach spaces*, J. Evol. Equ. 8 (2008), 765-780.
[8] M. Bellassoued and M. Choulli, *Stability estimate for an inverse problem for the magnetic Schrödinger equation from the Dirichlet-to-Neumann map*, J. Funct. Anal. 258 (2010) 161-195.
[9] N. Burq and M. Zworski, *Control for Schrödinger operators on tori*, Math. Res. Lett. 19 (2012), 309-324.
[10] N. Burq and M. Zworski, *Geometric control in the presence of a black box*, J. Amer. Math. Soc. 17 (2004), 443-471.
[11] N. Burq, *Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel*, Acta Math. 180 (1998), 1-29.
[12] M. Choulli, *Stability estimates for an inverse elliptic problem*, J. Inverse Ill-Posed Probl. 10 (6) (2002), 601-610.
[13] B. Dehman, P. Gérard and G. Lebeau, *Stabilization and control for the nonlinear Schrödinger equation on a compact surface*, Math. Z. 254 (4) (2006), 729-749.
[14] M. J. Esteban and P. L. Lions, *Stationary solutions of nonlinear Schrödinger equations with an external magnetic field*, Partial differential equations and the calculus of variations, Vol. I, 401-449, Progr. Nonlinear Differential Equations Appl., 1, Birkhäuser Boston, Boston, MA, 1989.
[15] A. Fursikov and O. Yu. Imanuvilov, *Controllability of evolution equations,* Lecture notes, 34, Seoul National University, Korea, 1996.
[16] P. Grisvard, *Contrôlabilité exacte des solutions de l'équation des ondes en présence de singularités*, J. Math. Pures Appl. 68 (1989), 215-259.
[17] A. Haraux, *Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire*, J. de Math. Pures Appl. 68 (4), (1990) 457-465.
[18] L. Hörmander, *Linear partial differential operators,* Springer-Verlag, Berlin, 1963.
[19] S. Jaffard, *Contrôle interne exact des vibrations d'une plaque rectangulaire*, Portugal. Math. 47(4) (1990), 423-429.
[20] V. Komornik and E. Zuazua, *A direct method for the boundary stabilization of the wave equation*, J. Math. Pures et Appl. 69 (1990), 33-54.
[21] C. Laurent, *Internal control of the Schrödinger equation*, Math. Control Relat. Fields. 4 (2) (2014), 161-186.
[22] G. Lebeau, *Contrôle de l'équation de Schrödinger*, J. Math. Pures Appl. 71 (1992), 267-291.
[23] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur*, Commun. Partial Differ. Equat. 20 (1995), 335–356.
[24] G. Lebeau and L. Robbiano, *Stabilisation de l'équation des ondes par le bord*, Duke Math. J. 86 (1997), 465-491.
[25] E. Machtyngier, *Exact controllability for the Schrödinger equation*, SIAM. J. Control Optim. 32 (1994), 24–34.
[26] E. Machtyngier and E. Zuazua, *Stabilization of the Schrödinger equation*, Portugal. Math. 51 (1994), 244-256.
[27] F. Macià, *Semiclassical measures and the Schrödinger flow on Riemannian manifolds* Nonlinearity 22 (2009) , 1003-1020.
[28] F. Macià and G. Rivière, *Concentration and non-concentration for the Schrödinger evolution on Zoll manifolds*, Commun. Math. Phy. 345 (3) (2016), 1019-1054.
[29] F. Macià and G. Rivière, *Observability and Quantum Limits for the Schrödinger equation on* $\mathbb{S}^d$. Preprint (2016).
[30] F. Macià, *High-frequency propagation for the Schrödinger equation on the torus*, J. Funct. Anal. 258 (2010), 933-955.
[31] A. Mercado, A. Osses and L. Rosier, *Inverse problems for the Schödinger equation via Carleman inequalities with degenerate weights*, Inverse Problems. 24 (2008) 015017 (18pp).
[32] L. Miller, *Controllability cost of conservative systems: resolvent condition and transmutation*, J. Funct. Anal. 218 (2005), 425-444.
[33] L. Miller, *Non-structural controllability of linear elastic systems with structural damping*, J. Funct. Anal. 236 (2006), 592–608.

[34] K.-D. Phung, *Observability and control of Schrödinger equations*, SIAM J. Control Optim. 40 (2001), 211-230.
[35] K. Ramdani, T. Takahashi, G. Tenenbaum and M. Tucsnak, *A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator*, J. Funct. Anal. 226 (1) (2005),193-229.
[36] L. Robbiano, *Fonction de coût et contrôle des solutions des équations hyperboliques*, Asymp. Anal. 10 (1995), 95-115.
[37] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups.* Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, 2009.

UR Analysis and Control of Pde, UR 13ES64, Department of Mathematics, Faculty of Sciences of Monastir, University of Monastir, 5019 Monastir, Tunisia and LMV-UVSQ/Université de Paris-Saclay, France
*E-mail address*: kais.ammari@fsm.rnu.tn

Université de Lorraine, 34 cours Léopold, 54052 Nancy cedex, France
*E-mail address*: mourad.choulli@univ-lorraine.fr

Laboratoire de Mathématiques, Université de Versailles Saint-Quentin en Yvelines, 78035 Versailles, France
*E-mail address*: luc.robbiano@uvsq.fr