Weighted functional inequalities: Constructive approach

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WEIGHTED FUNCTIONAL INEQUALITIES: CONSTRUCTIVE APPROACH

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Abstract. Consider an ergodic stationary random field A on the ambient space $\mathbb{R}^d$. In a companion article we introduced the notion of *weighted* functional inequalities, which extend standard functional inequalities like spectral gap, covariance, and logarithmic Sobolev inequalities, and we studied the associated concentration properties for nonlinear functions $X(A)$ of the field. In the present contribution we develop a constructive approach to produce random fields that satisfy such weighted functional inequalities. The construction typically relies on devising approximate chain rules for nonlinear and random changes of variables for random fields. This approach allows us to cover Gaussian fields with non-necessarily integrable covariance function, Poisson random inclusions with (unbounded) random radii, random parking and Matérn-type processes, as well as Poisson random tessellations (Voronoi or Delaunay). These weighted functional inequalities, which we primarily develop here in view of their application to quantitative stochastic homogenization, are of independent interest.

Contents

1. Introduction

In the companion article [7] we introduced the notion of weighted functional inequalities, which are generalizations of standard functional inequalities like spectral gap, covariance, or logarithmic Sobolev inequalities, and imply strong concentration properties. The aim of the present contribution is to complete this work by developing a constructive approach that generates random fields that do satisfy weighted functional inequalities.

In many fields of mathematical analysis, complex objects in a low-dimensional space can be described as the projection of simpler objects of a higher-dimensional space. A prototypical example is given by quasi-periodic structures. Conversely, suitable projections can be a powerful way to generate many (possibly complex) lower-dimensional objects from simpler higher-dimensional objects while preserving some essential properties, which is a useful point of view for modeling. For quasi-periodic functions, the simple high-dimensional objects are periodic functions (on a high-dimensional torus), the projection corresponds to the composition with a winding matrix, and the preserved essential property is some quantitative averaging property. In this contribution, we apply this idea to functional inequalities.

Consider a random field $A = \Phi(A_0)$ on $\mathbb{R}^d$ obtained as the image by some "projection" Φ of some higher-dimensional random field A_0 on $\mathbb{R}^d \times \mathbb{R}^l$. In this article we argue that standard functional inequalities satisfied by A_0 can be transferred to A in the form of the weighted functional inequalities introduced in the companion article [7]. To this aim, we develop in Section 2 an abstract yet constructive approach to such inequalities, which amounts to making suitable assumptions on the "projection operator" Φ. In Section 3, we make use of this constructive approach to prove the validity of weighted functional inequalities for various examples of random fields considered in the literature.

To conclude this introduction, we describe three classes of random fields A that satisfy weighted functional inequalities.

(I) *Gaussian-like fields*: A is (possibly the image by a Lipschitz function of) the convolution of some white noise with some deterministic kernel, which leads to Gaussian fields with arbitrary covariance function.

(II) *Independent coloring of random geometric patterns*: A is characterized by a random geometric pattern completed by an independent product structure. The random geometric pattern is typically constructed starting from a point process (e.g. Poisson, random parking, or Matérn-type processes) by considering inclusions centered at the points, or (Voronoi or Delaunay) tessellations. The associated product structure then determines the values of A on the cells of the random pattern, or even completes the description of the random pattern (e.g. conferring random sizes and shapes to the inclusions). This leads to possibly long-range correlations of the geometric pattern.

(III) *Dependent coloring of random geometric patterns*: This corresponds to (II) for a coloring that does not come from a product structure but from a field that is itself correlated (e.g. of the class (I)). This leads to possibly long-range correlations of the colors of the inclusions (in the sense of e.g. value of A, size, or orientation of the inclusions), on top of the correlations of the geometric pattern.

Details are provided in Section 3. The above three classes of random fields encompass all the examples considered in [26], a reference textbook on random heterogeneous structures for materials sciences, which brings the use of functional inequalities (in their weighted versions) in stochastic homogenization to the state-of-the-art of materials science.

Notation.

- d is the dimension of the ambient space $\mathbb{R}^d$;
- C denotes various positive constants that only depend on the dimension d and possibly on other controlled quantities; we write $\lesssim$ and $\gtrsim$ for $\le$ and $\ge$ up to such multiplicative constants C; we use the notation $\simeq$ if both relations $\lesssim$ and $\gtrsim$ hold; we add a subscript in order to indicate the dependence of the multiplicative constants on other parameters;
- the notation $a \ll b$ (or equivalently $b \gg a$) stands for $a \le \frac{1}{C} b$ for some large enough constant $C \simeq 1$;
- $Q^k := [-1/2, 1/2)^k$ denotes the unit cube centered at 0 in dimension k, and for all $x \in \mathbb{R}^d$ and $r > 0$ we set $Q^k(x) := x + Q^k$, $Q^k_r := rQ^k$ and $Q^k_r(x) := x + rQ^k$; when $k = d$ or when there is no confusion possible on the meant dimension, we drop the superscript k;
- we use similar notation for balls, replacing Q^k by B^k (the unit ball in dimension k);
- the Euclidean distance between subsets of $\mathbb{R}^d$ is denoted by $d(\cdot, \cdot)$;
- $\mathcal{B}(\mathbb{R}^k)$ denotes the Borel σ-algebra on $\mathbb{R}^k$;
- $\mathbb{E}[\cdot]$ denotes the expectation, $\mathrm{Var}[\cdot]$ the variance, and $\mathrm{Cov}[\cdot;\cdot]$ the covariance in the underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and the notation $\mathbb{E}[\cdot \| \cdot]$ stands for the conditional expectation;
- for all subsets A of a reference set B, we let $A^c := B \setminus A$ denote the complement of A in B;
- for all $a, b \in \mathbb{R}$, we set $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, and $a_+ := a \vee 0$;
- for all matrices F, we denote by F^t its transpose matrix;
- $\lceil a \rceil$ denotes the smallest integer larger or equal to a;
- $\mathcal{F}$ denotes Fourier transformation.

2. Constructive approach to weighted functional inequalities

In this section we consider random fields that can be constructed as transformations of product structures. Under suitable assumptions we describe how the standard spectral gaps, covariance inequalities, and logarithmic Sobolev inequalities satisfied by "hidden product structures" are deformed into weighted functional inequalities for the random fields of interest. The analysis of the examples mentioned in the introduction is postponed to Section 3.

2.1. **Weighted functional inequalities.** We start by recalling the definition of weighted functional inequalities introduced in the companion article [7]. Let $A : \mathbb{R}^d \times \Omega \to \mathbb{R}$ be a jointly measurable random field on $\mathbb{R}^d$, constructed on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A spectral gap in probability for A is a functional inequality that allows one to control the variance of any function $X(A)$ in terms of its local dependence on A, that is, in terms of some "derivative" of $X(A)$ with respect to local restrictions of A. In the continuum setting that we consider here, there is no canonical choice of such a (wide-sense) derivative with respect to the field A, and we recall below three such possible notions.

- The so-called *Glauber derivative* ∂^{G} is defined as follows, letting A' denote an i.i.d. copy of A, and denoting by $\mathbb{E}'[\cdot]$ the expectation wrt A' only,

$$\partial^{\mathrm{G}}_{A,S} X(A) = \mathbb{E}'\big[\big(X(A) - X(A')\big)^2 \,\big\|\, A'|_{\mathbb{R}^d \setminus S} = A|_{\mathbb{R}^d \setminus S}\big]^{\frac{1}{2}}.$$

- The *oscillation* ∂^{osc} is formally defined by

$$\begin{aligned}\partial^{\text{osc}}_{A,S}\, X(A) \;&:= \sup_{A,S}\text{ess}\, X(A) - \inf_{A,S}\text{ess}\, X(A)\\ &\text{“=”}\ \sup\text{ess}\Big\{X(A') : A' \in \text{Mes}(\mathbb{R}^d;\mathbb{R}),\ A'|_{\mathbb{R}^d\setminus S} = A|_{\mathbb{R}^d\setminus S}\Big\}\\ &\qquad - \inf\text{ess}\Big\{X(A') : A' \in \text{Mes}(\mathbb{R}^d;\mathbb{R}),\ A'|_{\mathbb{R}^d\setminus S} = A|_{\mathbb{R}^d\setminus S}\Big\}, \quad (2.1)\end{aligned}$$

  where the essential supremum and infimum are taken with respect to the measure induced by the field A on the space $\text{Mes}(\mathbb{R}^d;\mathbb{R})$ (endowed with the cylindrical σ-algebra). This definition (2.1) of $\partial^{\text{osc}}_{A,S}X(A)$ is not measurable in general, and we rather define

$$\partial^{\text{osc}}_{A,S}\, X(A) := \mathcal{M}[X\|A|_{\mathbb{R}^d\setminus S}] + \mathcal{M}[-X\|A|_{\mathbb{R}^d\setminus S}]$$

  in terms of the conditional essential supremum $\mathcal{M}[\cdot\|A_{\mathbb{R}^d\setminus S}]$ given $\sigma(A|_{\mathbb{R}^d\setminus S})$, as introduced in [1]. Alternatively, we may simply define $\partial^{\text{osc}}_{A,S}X(A)$ as the measurable envelope of (2.1).
- The (integrated) *functional (or Malliavin) derivative* ∂^{fct} is the closest generalization of the usual partial derivatives commonly used in the discrete setting. Let us denote by $M \subset \mathrm{L}^\infty(\mathbb{R}^d)$ some open set such that the random field A takes its values in M. Given a $\sigma(A)$-measurable random variable $X(A)$, and given an extension $\tilde X : M \to \mathbb{R}$, its Fréchet derivative $\partial\tilde X(A)/\partial A \in \mathrm{L}^1_{\text{loc}}(\mathbb{R}^d)$ is defined for any compactly supported perturbation $\delta A \in \mathrm{L}^\infty(\mathbb{R}^d)$ by

$$\lim_{t\to 0}\frac{\tilde X(A+t\delta A)-\tilde X(A)}{t} = \int_{\mathbb{R}^d}\delta A(x)\frac{\partial\tilde X(A)}{\partial A}(x)\,dx,$$

  if the limit exists. Since we are interested in the local averages of this derivative, we rather define for all bounded Borel subset $S \subset \mathbb{R}^d$,

$$\partial^{\text{fct}}_{A,S}X(A) = \int_S\Big|\frac{\partial\tilde X(A)}{\partial A}(x)\Big|dx.$$

  This derivative is additive with respect to the set S: for all disjoint Borel subsets $S_1, S_2 \subset \mathbb{R}^d$ we have $\partial^{\text{fct}}_{A,S_1\cup S_2}X(A) = \partial^{\text{fct}}_{A,S_1}X(A) + \partial^{\text{fct}}_{A,S_2}X(A)$.

It is clear by definition that the oscillation dominates the Glauber derivative. Henceforth we use the notation $\tilde\partial$ for any of the above-defined (wide-sense) derivatives with respect to the random field A. We define weighted functional inequalities as follows.

Definition 2.1. Given an integrable function $\pi : \mathbb{R}_+ \to \mathbb{R}_+$, we say that A satisfies the *weighted spectral gap* ($\tilde\partial$-WSG) *with weight* π if for all $\sigma(A)$-measurable random variable $X(A)$ we have

$$\text{Var}\,[X(A)] \le \mathbb{E}\left[\int_0^\infty\int_{\mathbb{R}^d}\Big(\tilde\partial_{A,B_{\ell+1}(x)}X(A)\Big)^2 dx\,(\ell+1)^{-d}\pi(\ell)\,d\ell\right];$$

it satisfies the *weighted covariance inequality* ($\tilde\partial$-WCI) *with weight* π if for all $\sigma(A)$-measurable random variables $X(A)$ and $Y(A)$ we have

$$\mathrm{Cov}\left[X(A);Y(A)\right] \\ \le \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\tilde\partial_{A,B_{\ell+1}(x)}X(A)\right)^2\right]^{\frac12} \mathbb{E}\left[\left(\tilde\partial_{A,B_{\ell+1}(x)}Y(A)\right)^2\right]^{\frac12} dx\,(\ell+1)^{-d}\pi(\ell)\,d\ell;$$

it satisfies the *weighted logarithmic Sobolev inequality* ($\tilde\partial$-WLSI) *with weight* π if for all $\sigma(A)$-measurable random variable $Z(A)$ we have

$$\mathrm{Ent}\left[Z(A)^2\right] := \mathbb{E}\left[Z(A)^2\log Z(A)^2\right] - \mathbb{E}\left[Z(A)^2\right]\log\mathbb{E}\left[Z(A)^2\right] \\ \le \mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d}\left(\tilde\partial_{A,B_{\ell+1}(x)}Z(A)\right)^2 dx\,(\ell+1)^{-d}\pi(\ell)\,d\ell\right]. \quad \square$$

Standard functional inequalities (spectral gap (SG), covariance (CI), and logarithmic Sobolev inequality (LSI)) are recovered by taking a compactly supported weight π (or equivalently, skipping the integral over ℓ).

Remark 2.2. In each of the examples considered in the sequel, if the functional inequality ($\tilde\partial$-WSG), ($\tilde\partial$-WCI), or ($\tilde\partial$-WLSI) is proved to hold with some weight π, then for all $L \ge 1$ the rescaled field $A_L := A(L\cdot)$ satisfies the same functional inequality with the same weight π. See Remarks 2.10 and B.3 for detail. $\square$

Classical arguments yield the following sufficient criterion for standard functional inequalities. A standard proof is included for completeness in Appendix A and will be referred to at several places in this contribution. (Note that the logarithmic Sobolev inequality (LSI) is only established here with the oscillation ∂^{osc}, while the version with the Glauber derivative ∂^{G} is well-known to be much more restrictive, crucially depending on the law of the underlying product structure.)

Proposition 2.3. *Let A_0 be a random field on $\mathbb{R}^d$ with values in some measurable space such that restrictions $A_0|_S$ and $A_0|_T$ are independent for all disjoint Borel subsets $S,T \subset \mathbb{R}^d$. Let A be a random field on $\mathbb{R}^d$ that is an R-local transformation of A_0, in the sense that for all $S \subset \mathbb{R}^d$ the restriction $A|_S$ is $\sigma(A_0|_{S+B_R})$-measurable. Then, the field A satisfies* (∂^{G}-CI) *and* (∂^{osc}-LSI) *with radius $R+\varepsilon$ for all $\varepsilon > 0$.* $\square$

Note that any field satisfying the assumption in the above criterion has finite range of dependence. Conversely any field that satisfies (CI) has necessarily finite range of dependence (cf. [7, Proposition 2.3]). One does not expect, however, finite range of dependence to be a sufficient condition for the validity of (SG) in general (compare indeed with the constructions in [5, 3]).

Although the Glauber derivative ∂^{G} and the functional derivative ∂^{fct} are particularly convenient measures of sensitivity of a random variable $X(A)$ with respect to local restrictions of A, they are essentially only adapted to product structures and to Gaussian-like random fields, respectively. On the other hand, the oscillation ∂^{osc} is adapted to a much larger variety of random fields (cf. Section 2.3), but it involves taking (essential) suprema, which might be difficult to control in various applications (and in particular in stochastic homogenization, cf. [8]).

In the course of the article, we consider various classes of random fields on $\mathbb{R}^d$ that can be constructed as (possibly random) projections of random fields having a product structure in a higher-dimensional space $\mathbb{R}^d \times \mathbb{R}^l$. Such projections naturally allow one to "deform" the underlying Glauber derivative in a way that cannot be strictly speaking written as a Glauber derivative, but which shares important properties (and in particular avoids taking suprema). The following definition (which can be skipped at the first reading) gives such a proxy for the Glauber derivative, which can typically be used in functional inequalities with loss of integrability.

Definition 2.4. Given $l \geq 0$, let $\mathcal{X}$ be some random field on $\mathbb{R}^d \times \mathbb{R}^l$ with values in some measure space, and assume that the random field A under consideration is $\sigma(\mathcal{X})$-measurable, $A = A(\mathcal{X})$. Choose $\mathcal{X}'$ an i.i.d. copy of the field $\mathcal{X}$, and for all x, t let the perturbed field $\mathcal{X}^{x,t}$ be defined by $\mathcal{X}^{x,t}|_{(\mathbb{R}^d\times\mathbb{R}^l)\setminus(Q^d(x)\times Q^l(t))} = \mathcal{X}|_{(\mathbb{R}^d\times\mathbb{R}^l)\setminus(Q^d(x)\times Q^l(t))}$ and $\mathcal{X}^{x,t}|_{Q^d(x)\times Q^l(t)} = \mathcal{X}'|_{Q^d(x)\times Q^l(t)}$. We use the short-hand notation

$$\partial^{\text{dis}}_{\ell,x,t} X(A) := (X(A) - X(A(\mathcal{X}^{x,t})))\mathbb{1}_{A|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)} = A(\mathcal{X}^{x,t})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}}, \tag{2.2}$$

which we abusively call a *discrete derivative*, and we define a spectral gap with loss (∂^{dis}-WSG') as follows: given a family $(\pi_\lambda)_\lambda$ of integrable functions $\pi_\lambda : \mathbb{R}^l \times \mathbb{R}_+ \to \mathbb{R}_+$, the spectral gap with loss (∂^{dis}-WSG') with weights $(\pi_\lambda)_\lambda$ is said to hold if for all $\sigma(A)$-measurable random variables $X(A)$ and all $\lambda \in (0,1)$ we have

$$\mathrm{Var}\left[X(A)\right] \leq \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^l} \mathbb{E}\left[\left(\partial^{\text{dis}}_{\ell,x,t} X(A)\right)^{\frac{2}{1-\lambda}}\right]^{1-\lambda} \pi_\lambda(t,\ell)\, dt\, dx\, d\ell. \qquad \square$$

2.2. **Transformation of product structures.** Let the random field A on $\mathbb{R}^d$ be $\sigma(\mathcal{X})$-measurable for some random field $\mathcal{X}$ defined on some measure space X and with values in some measurable space M. Assume that we have a partition $X = \biguplus_{x\in\mathbb{Z}^d, t\in\mathbb{Z}^l} X_{x,t}$, on which $\mathcal{X}$ is *completely independent*, that is, the family of restrictions $(\mathcal{X}|_{X_{x,t}})_{x\in\mathbb{Z}^d,t\in\mathbb{Z}^l}$ are all independent.

In the sequel, the case $l = 0$ (that is, the case when there is no variable t) is referred to as the non-projective case, while the case $l \geq 1$ is referred to as the projective case. Note however that the non-projective case is a particular case of the projective one, simply defining $X_{x,0} = X_x$ and $X_{x,t} = \varnothing$ for all $t \neq 0$. The random field $\mathcal{X}$ can be e.g. a random field on $\mathbb{R}^d \times \mathbb{R}^l$ with values in some measure space (choosing $X = \mathbb{R}^d \times \mathbb{R}^l$, $X_{x,t} = Q^d(x) \times Q^l(t)$, and M the space of values), or a random point process (or more generally a random measure) on $\mathbb{R}^d \times \mathbb{R}^l \times X'$ for some measure space X' (choosing $X = \mathbb{Z}^d \times \mathbb{Z}^l \times X'$, $X_{x,t} = \{x\} \times \{t\} \times X'$, and M the space of measures on $Q^d \times Q^l \times X'$).

Let $\mathcal{X}'$ be some given i.i.d. copy of $\mathcal{X}$. For all x, t, we define a perturbed random field $\mathcal{X}^{x,t}$ by setting $\mathcal{X}^{x,t}|_{X\setminus X_{x,t}} = \mathcal{X}|_{X\setminus X_{x,t}}$ and $\mathcal{X}^{x,t}|_{X_{x,t}} = \mathcal{X}'|_{X_{x,t}}$. By complete independence, the random fields $\mathcal{X}$ and $\mathcal{X}^{x,t}$ (resp. $A = A(\mathcal{X})$ and $A(\mathcal{X}^{x,t})$) have the same law. Arguing as in the proof of Proposition 2.3 (cf. (A.3) and (A.4) in Appendix A), the complete independence assumption ensures that $\mathcal{X}$ satisfies the following standard functional inequalities.

Proposition 2.5. *For all $\sigma(\mathcal{X})$-measurable random variables $Y(\mathcal{X})$ and $Z(\mathcal{X})$, we have*

$$\mathrm{Var}\left[Y(\mathcal{X})\right] \le \frac{1}{2} \sum_{x\in\mathbb{Z}^d} \sum_{t\in\mathbb{Z}^l} \mathbb{E}\left[\left(Y(\mathcal{X}) - Y(\mathcal{X}^{x,t})\right)^2\right], \tag{2.3}$$

$$\mathrm{Ent}[Y(\mathcal{X})] \le 2 \sum_{x\in\mathbb{Z}^d} \sum_{t\in\mathbb{Z}^l} \mathbb{E}\left[\sup_{\mathcal{X}'}\mathrm{ess}\,\left(Y(\mathcal{X}) - Y(\mathcal{X}^{x,t})\right)^2\right], \tag{2.4}$$

$$\mathrm{Cov}\left[Y(\mathcal{X}); Z(\mathcal{X})\right] \le \frac{1}{2} \sum_{x\in\mathbb{Z}^d} \sum_{t\in\mathbb{Z}^l} \mathbb{E}\left[\left(Y(\mathcal{X}) - Y(\mathcal{X}^{x,t})\right)^2\right]^{\frac{1}{2}} \mathbb{E}\left[\left(Z(\mathcal{X}) - Z(\mathcal{X}^{x,t})\right)^2\right]^{\frac{1}{2}}. \tag{2.5}$$

□

2.3. **Abstract criteria and action radius.** We now describe general situations for which the functional inequalities for the hidden product structure $\mathcal{X}$ are deformed into weighted inequalities for the random field A. We distinguish the following two cases:

- *deterministic localization*, that is, when the random field A is a deterministic convolution of some product structure, so that the dependence pattern is prescribed deterministically a priori; it leads to weighted functional inequalities with the functional derivative ∂^{fct};
- *random localization*, that is, when the dependence pattern is encoded by the underlying product structure $\mathcal{X}$ itself (and therefore may depend on the realization, whence the terminology "random"); the localization of the dependence pattern is then measured in terms of what we call the *action radius*; it leads to weighted inequalities with the derivatives ∂^{osc} and ∂^{dis}, and generalizes the idea of local transformations of Proposition 2.3.

The case of deterministic localization essentially concerns Gaussian fields, which have been thoroughly studied in the literature. Weighted functional inequalities for such random fields then follow from standard functional inequalities (typically formulated in terms of Malliavin calculus on Wiener space, see e.g. [12, 13, 19]) combined with a *deterministic* radial change of variables to reformulate the RHS (extracting a 1D weight from Hilbert norms encoding the covariance structure, see the proof of Theorem B.2 below). The RHS of weighted functional inequalities is indeed more explicit (and flexible when it turns to estimates — see e.g. bounds by duality in [8]). A self-contained approach to deterministic localization is included in Appendix B.

In the rest of this section we focus on the more original setting of *random* localization (which involves a *random* change of variable, due to the randomness of the dependence pattern). More precisely, we introduce the notion of *action radius* as a probabilistic measure of the localization of the dependence pattern. General criteria for weighted spectral gaps are then obtained in terms of the properties of this action radius. Various examples that are included in this framework are described in Section 3 below.

We use the notation of Section 2.2: A is a $\sigma(\mathcal{X})$-measurable random field on $\mathbb{R}^d$, where $\mathcal{X}$ is a completely independent random field on some measure space $X = \biguplus_{x\in\mathbb{Z}^d, t\in\mathbb{Z}^l} X_{x,t}$ with values in some measurable space M. The following definition is inspired by the notion of stabilization radius first introduced by Lee [15, 16] and crucially used in the works by Penrose, Schreiber, and Yukich on random sequential adsorption processes [22, 21, 23, 25] (see also [14]).

Definition 2.6. Given an i.i.d. copy $\mathcal{X}'$ of the field $\mathcal{X}$, an *action radius for A with respect to $\mathcal{X}$ on $X_{x,t}$* (with reference perturbation $\mathcal{X}'$), if it exists, is defined as a nonnegative $\sigma(\mathcal{X},\mathcal{X}')$-measurable random variable ρ such that we have a.s.,

$$A(\mathcal{X}^{x,t})\big|_{\mathbb{R}^d\setminus(Q(x)+B_\rho)} = A(\mathcal{X})|_{\mathbb{R}^d\setminus(Q(x)+B_\rho)},$$

where as before the perturbed random field $\mathcal{X}^{x,t}$ is defined by $\mathcal{X}^{x,t}|_{X\setminus X_{x,t}} := \mathcal{X}|_{X\setminus X_{x,t}}$ and $\mathcal{X}^{x,t}|_{X_{x,t}} := \mathcal{X}'|_{X_{x,t}}$. □

Note that if $\mathcal{X} = A_0$ is a random field on $\mathbb{R}^d$, and if for some $R > 0$ the random field A is an R-local transformation of A_0 in the sense of Proposition 2.3, then the constant $\rho = R$ is an action radius for A with respect to A_0 on any set. Reinterpreted in the case when $\mathcal{X} = \mathcal{P}$ is a random point process on $\mathbb{R}^d \times \mathbb{R}^l \times X'$ for some measure space X', the above definition takes on the following guise: given a subset $E \times F \subset \mathbb{R}^d \times \mathbb{R}^l$ and given an i.i.d. copy $\mathcal{P}'$ of $\mathcal{P}$, an *action radius for A with respect to $\mathcal{P}$ on $E \times F$*, if it exists, is a nonnegative random variable ρ such that we have a.s.,

$$A\Big((\mathcal{P}\setminus(E\times F\times X'))\bigcup(\mathcal{P}'\cap(E\times F\times X'))\Big)\Big|_{\mathbb{R}^d\setminus(E+B_\rho)} = A(\mathcal{P})|_{\mathbb{R}^d\setminus(E+B_\rho)}.$$

We display two general results, Theorems 2.7 and 2.9 below. The first result is a general criterion for the validity of weighted spectral gaps in terms of the properties of an action radius, whereas the second result is based on more elaborate properties of action radii and is useful to avoid loss of integrability in some situations. Note that the condition for the validity of the weighted logarithmic Sobolev inequality below is rather stringent (see Section 3 for examples).

Theorem 2.7. *Let the fields $A, \mathcal{X}$ be as above. Given an i.i.d. copy $\mathcal{X}'$ of the field $\mathcal{X}$, assume that:*

(a) For all x, t, there exists an action radius $\rho_{x,t}$ for A with respect to $\mathcal{X}$ in $X_{x,t}$.
(b) The transformation A of $\mathcal{X}$ is stationary, that is, the random fields $A(\mathcal{X}(\cdot+z,\cdot))$ and $A(\mathcal{X})(\cdot+z)$ have the same law for all $z \in \mathbb{Z}^d$. Moreover, the law of the action radius $\rho_{x,t}$ is independent of x.

Then the following holds.

(i) Setting

$$\pi(t,\ell) := \mathbb{P}\left[\ell-1 \le \rho_{0,t} < \ell,\ A(\mathcal{X}^{0,t}) \neq A(\mathcal{X})\right],$$

we have for all $\sigma(A)$-measurable random variable $Z(A)$ and all $\lambda \in (0,1)$,

$$\operatorname{Var}[Z(A)] \le \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{\ell=1}^{\infty}\sum_{t\in\mathbb{Z}^l}\pi(t,\ell)^\lambda\,\mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell,x,t}Z(A)\Big)^{\frac{2}{1-\lambda}}\right]^{1-\lambda} \tag{2.6}$$

and

$$\operatorname{Cov}[Y(A);Z(A)] \le \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{t\in\mathbb{Z}^l}\bigg(\sum_{\ell=1}^{\infty}\pi(t,\ell)^\lambda\,\mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell,x,t}Y(A)\Big)^{\frac{2}{1-\lambda}}\right]^{1-\lambda}\bigg)^{\frac12}$$
$$\times\bigg(\sum_{\ell'=1}^{\infty}\pi(t,\ell')^\lambda\,\mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell',x,t}Z(A)\Big)^{\frac{2}{1-\lambda}}\right]^{1-\lambda}\bigg)^{\frac12}, \tag{2.7}$$

where $\partial^{\mathrm{dis}}_{\ell,x,t} Z(A)$ *is the notation defined in* (2.2)*, that is,*

$$\partial^{\mathrm{dis}}_{\ell,x,t} Z(A) := \big(Z(A) - Z(A(\mathcal{X}^{x,t}))\big)\mathbb{1}_{A|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}=A(\mathcal{X}^{x,t})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}}.$$

In particular, for all $\lambda \in (0,1)$*, if we set*

$$\pi_\lambda(\ell) := (\ell+1)^d \sum_{t\in\mathbb{Z}^l} \mathbb{P}\left[\ell - 1 \le \rho_{0,t} < \ell,\, A(\mathcal{X}^{0,t}) \neq A(\mathcal{X})\right]^\lambda,$$

we obtain for all $\sigma(A)$*-measurable random variables* $Z(A)$*,*

$$\mathrm{Var}\left[Z(A)\right] \le \frac{1}{2}\sum_{\ell=1}^\infty (\ell+1)^{-d}\pi_\lambda(\ell) \sum_{x\in\mathbb{Z}^d} \mathbb{E}\left[\Big(\ \partial^{\mathrm{osc}}_{A,Q_{2\ell+1}(x)}\ Z(A)\Big)^{\frac{2}{1-\lambda}}\right]^{1-\lambda}. \tag{2.8}$$

If in addition the random variable $\rho_{x,t}$ *is* $\sigma(\mathcal{X})$*-measurable for all* x,t*, then we have*

$$\mathrm{Ent}[Z(A)] \le 2\sum_{\ell=1}^\infty (\ell+1)^{-d}\pi_\lambda(\ell) \sum_{x\in\mathbb{Z}^d} \mathbb{E}\left[\Big(\ \partial^{\mathrm{osc}}_{A,Q_{2\ell+1}(x)}\ Z(A)\Big)^{\frac{2}{1-\lambda}}\right]^{1-\lambda}. \tag{2.9}$$

(ii) *Assume that for all* x,t *the action radius* $\rho_{x,t}$ *is independent of* $A|_{\mathbb{R}^d\setminus(Q(x)+B_{f(\rho_{x,t})})}$ *for some* influence function $f:\mathbb{R}_+\to\mathbb{R}_+$ *with* $f(u)\ge u$ *for all* u*. Then, with the convention* $0/0=0$*, if we set*

$$\tilde\pi(t,\ell) := \mathbb{P}\left[\mathcal{X}^{0,t}\neq\mathcal{X}\right]\frac{\mathbb{P}\left[\,\ell-1\le\rho_{0,t}<\ell \,\|\, \mathcal{X}^{0,t}\neq\mathcal{X}\right]}{\mathbb{P}\left[\rho_{0,t}<\ell\right]},$$
$$\pi(\ell) := (\ell+1)^d\sum_{t\in\mathbb{Z}^l}\tilde\pi(t,\ell),$$

we have for all $\sigma(A)$*-measurable random variables* $Z(A)$*,*

$$\mathrm{Var}\left[Z(A)\right] \le \frac{1}{2}\sum_{\ell=1}^\infty (\ell+1)^{-d}\pi(\ell)\sum_{x\in\mathbb{Z}^d}\mathbb{E}\left[\Big(\ \partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\ Z(A)\Big)^2\right] \tag{2.10}$$

and

$$\mathrm{Cov}\left[Y(A);Z(A)\right] \le \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{t\in\mathbb{Z}^l}\bigg(\sum_{\ell=1}^\infty\tilde\pi(t,\ell)\,\mathbb{E}\left[\Big(\ \partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\ Y(A)\Big)^2\right]\bigg)^{\frac12}$$
$$\times\bigg(\sum_{\ell'=1}^\infty\tilde\pi(t,\ell')\,\mathbb{E}\left[\Big(\ \partial^{\mathrm{osc}}_{A,Q_{2f(\ell')+1}(x)}\ Z(A)\Big)^2\right]\bigg)^{\frac12}. \tag{2.11}$$

If in addition the random variable $\rho_{x,t}$ *is* $\sigma(\mathcal{X})$*-measurable for all* x,t*, then we have*

$$\mathrm{Ent}[Z(A)] \le 2\sum_{\ell=1}^\infty(\ell+1)^{-d}\pi(\ell)\sum_{x\in\mathbb{Z}^d}\mathbb{E}\left[\Big(\ \partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\ Z(A)\Big)^2\right]. \tag{2.12}$$

□

Remark 2.8. The covariance inequalities (2.7) and (2.11) are not in the canonical form of Definition 2.1. However note that if $\tilde\pi(t,\ell)$ is non-increasing with respect to ℓ then the inequality (2.11) (and likewise for (2.7)) easily leads to

$$\operatorname{Cov}\left[Y(A);Z(A)\right] \le \sum_{x\in\mathbb{Z}^d}\sum_{\ell=1}^{\infty}(\ell+1)^{-d}\Big(\sum_{\ell'=1}^{\ell}\pi(\ell')\Big)\,\mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\,Y(A)\Big)^2\right]^{\frac12} \times \mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\,Z(A)\Big)^2\right]^{\frac12},$$

which is now in the correct form, although the weight $\sum_{\ell'=1}^{\ell}\pi(\ell')$ seems to be suboptimal whenever π has algebraic decay. □

We now turn to a more complex situation when the dependence pattern is intricate but sufficiently well controlled in terms of a family of action radii. The aim of the following is to avoid the loss of integrability which would follow from Theorem 2.7(i) in the case of the random parking process and of Poisson tessellations.

Theorem 2.9. *Let $A=A(\mathcal{X})$ be a $\sigma(\mathcal{X})$-measurable random field on $\mathbb{R}^d$, where $\mathcal{X}$ is a completely independent random field on some measure space $X=\biguplus_{x\in\mathbb{Z}^d}X_x$ with values in some measurable space M. For all $x\in\mathbb{Z}^d$, $\ell\in\mathbb{N}$, set $X_x^\ell:=\bigcup_{y\in\mathbb{Z}^d:|x-y|_\infty\le\ell}X_y$. Given an i.i.d. copy $\mathcal{X}'$ of the field $\mathcal{X}$, let the perturbed field $\mathcal{X}^{x,\ell}$ be defined by*

$$\mathcal{X}^{x,\ell}|_{X\setminus X_x^\ell}=\mathcal{X}|_{X\setminus X_x^\ell},\qquad\text{and}\qquad \mathcal{X}^{x,\ell}|_{X_x^\ell}=\mathcal{X}'|_{X_x^\ell},$$

and assume that:

(a) For all x,ℓ, there exists an action radius ρ_x^ℓ for A with respect to $\mathcal{X}$ in X_x^ℓ, that is, a nonnegative random variable ρ_x^ℓ such that we have a.s.,

$$A(\mathcal{X}^{x,\ell})|_{\mathbb{R}^d\setminus(Q_{2\ell+1}(x)+B_{\rho_x^\ell})}=A(\mathcal{X})|_{\mathbb{R}^d\setminus(Q_{2\ell+1}(x)+B_{\rho_x^\ell})}.$$

(b) The transformation A of $\mathcal{X}$ is stationary, that is, the random fields $A(\mathcal{X}(\cdot+z,\cdot))$ and $A(\mathcal{X})(\cdot+z)$ have the same law for all $z\in\mathbb{Z}^d$. Moreover, the law of the action radius ρ_x^ℓ is independent of x.

Further assume that

(c) For all x,ℓ, the random variable ρ_x^ℓ is $\sigma\Big(\mathcal{X}|_{X_x^{\ell+\rho_x^\ell}\setminus X_x^\ell}\Big)$-measurable.
(In particular, for all x,ℓ,R, given the event $\rho_x^\ell\le R$, the random variables ρ_x^ℓ and $\rho_x^{\ell+R}$ are independent.)

Let $R\ge1$ be chosen large enough so that

$$\sup_{\ell\ge R}\mathbb{P}\big[\rho_x^\ell\ge\ell\big]\le\frac14, \tag{2.13}$$

let $\pi_0:\mathbb{R}_+\to\mathbb{R}_+$ be a non-increasing function such that $\mathbb{P}\left[\ell/4\le\rho_x^{\ell_0}<\ell\right]\le\pi_0(\ell)$ holds for all $0\le\ell_0\le\ell/4$, and define the weight

$$\pi(\ell):=(\ell+1)^d\begin{cases}1, & \text{if } \ell\le 4R;\\ 8\ell^{-1}\pi_0(\ell/4), & \text{if } \ell>4R.\end{cases}$$

Then for all $\sigma(A)$-measurable random variables $Y(A), Z(A)$, we have

$$\mathrm{Var}\left[Z(A)\right] \le \frac{1}{2}\int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,B_{\sqrt{d}(2\ell+3)}(x)}\, Z(A)\Big)^2\right] dx\,(\ell+1)^{-d}\pi(\ell)d\ell, \tag{2.14}$$

$$\mathrm{Ent}[Z(A)] \le 2\int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,B_{\sqrt{d}(2\ell+3)}(x)}\, Z(A)\Big)^2\right] dx\,(\ell+1)^{-d}\pi(\ell)d\ell, \tag{2.15}$$

$$\begin{aligned}\mathrm{Cov}\left[Y(A); Z(A)\right] \le \frac{1}{2}\int_{\mathbb{R}^d}& \bigg(\int_0^\infty \mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,B_{\sqrt{d}(2\ell+3)}(x)}\, Y(A)\Big)^2\right](\ell+1)^{-d}\pi(\ell)d\ell\bigg)^{\frac12}\\ &\times\bigg(\int_0^\infty \mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,B_{\sqrt{d}(2\ell+3)}(x)}\, Z(A)\Big)^2\right](\ell+1)^{-d}\pi(\ell)d\ell\bigg)^{\frac12} dx. \end{aligned}\tag{2.16}$$

□

Remark 2.10. We briefly address the claim contained in Remark 2.2 in the context of examples of random fields with random localization. By definition, for all $L \ge 1$, an action radius for A with respect to $\mathcal{X}$ on $X_{0,t}$ is always also an action radius for the rescaled field $A_L := A(L\cdot)$ with respect to $\mathcal{X}$ on $X_{0,t}$. This proves that in Theorems 2.7 and 2.9 any result stated for the field A also holds in the very same form (with the same constants and weights) for A_L with $L \ge 1$. □

We start with the proof of Theorem 2.7, and then turn to the proof of Theorem 2.9.

Proof of Theorem 2.7. Recall that for all x, t the perturbed random field $\mathcal{X}^{x,t}$ is defined by $\mathcal{X}^{x,t}|_{X\setminus X_{x,t}} = \mathcal{X}|_{X\setminus X_{x,t}}$ and $\mathcal{X}^{x,t}|_{X_{x,t}} = \mathcal{X}'|_{X_{x,t}}$. By complete independence of $\mathcal{X}$, the fields $\mathcal{X}$ and $\mathcal{X}^{x,t}$ (hence $A = A(\mathcal{X})$ and $A(\mathcal{X}^{x,t})$) have the same law. The strategy of the proof consists in deforming the functional inequalities of Proposition 2.5 with respect to the transformation $A(\mathcal{X})$ in terms of the action radius. We split the proof into four steps.

Step 1. Proof of the spectral gap (2.6).
Conditioning the RHS of (2.3) with respect to the values of the action radius $\rho_{x,t}$, applying the Hölder inequality, and using the stationarity assumption (b) to recognize the weight $\pi(t,\ell)$, we obtain for all $0 < \lambda < 1$,

$$\begin{aligned}\mathrm{Var}\left[Z(A)\right] &\overset{(2.3)}{\le} \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{t\in\mathbb{Z}^l}\mathbb{E}\left[\big(Z(A)-Z(A(\mathcal{X}^{x,t}))\big)^2\right]\\ &= \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{\ell=1}^{\infty}\sum_{t\in\mathbb{Z}^l}\mathbb{E}\left[\big(Z(A)-Z(A(\mathcal{X}^{x,t}))\big)^2\mathbb{1}_{\ell-1\le\rho_{x,t}<\ell}\right] \qquad (2.17)\\ &\le \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{\ell=1}^{\infty}\sum_{t\in\mathbb{Z}^l}\pi(t,\ell)^{\lambda}\,\mathbb{E}\left[\big(Z(A)-Z(A(\mathcal{X}^{x,t}))\big)^{\frac{2}{1-\lambda}}\mathbb{1}_{\rho_{x,t}<\ell}\right]^{1-\lambda}.\end{aligned}$$

Noting that the event $\rho_{x,t} < \ell$ entails that $A|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)} = A(\mathcal{X}^{x,t})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}$, the above can be rewritten as follows,

$$\mathrm{Var}\left[Z(A)\right] \le \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{\ell=1}^{\infty}\sum_{t\in\mathbb{Z}^l}\pi(t,\ell)^{\lambda}\,\mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell,x,t}Z\Big)^{\frac{2}{1-\lambda}}\right]^{1-\lambda},$$

that is, (2.6).

Step 2. Proof of the spectral gap (2.10).
For all x, t, conditioning with respect to the values of $\rho_{x,t}$, we may decompose

$$\mathbb{E}\left[\big(Z(A)-Z(A(\mathcal{X}^{x,t}))\big)^2\right] = g_x^1(t)+g_x^2(t), \tag{2.18}$$
$$g_x^1(t) := \sum_{\ell=2}^{\infty}\mathbb{E}\left[\big(Z(A)-Z(A(\mathcal{X}^{x,t}))\big)^2\mathbb{1}_{\ell-1\le\rho_{x,t}<\ell}\right],$$
$$g_x^2(t) := \mathbb{E}\left[\big(Z(A)-Z(A(\mathcal{X}^{x,t}))\big)^2\mathbb{1}_{\rho_{x,t}<1}\right].$$

We first estimate the term $g_x^1(t)$. Recalling that the influence function f satisfies $f(u)\ge u$ for all u, we obtain

$$\begin{aligned}
g_x^1(t) &= \sum_{\ell=2}^{\infty}\mathbb{E}\left[\big(Z(A)-Z(A(\mathcal{X}^{x,t}))\big)^2\mathbb{1}_{\mathcal{X}|_{X_{x,t}}\neq\mathcal{X}'|_{X_{x,t}}}\mathbb{1}_{\ell-1\le\rho_{x,t}<\ell}\right]\\
&\le \sum_{\ell=2}^{\infty}\mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\,Z(A)\Big)^2\mathbb{1}_{\mathcal{X}|_{X_{x,t}}\neq\mathcal{X}'|_{X_{x,t}}}\mathbb{1}_{\ell-1\le\rho_{x,t}<\ell}\right]\\
&= \sum_{\ell=2}^{\infty}\mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\,Z(A)\Big)^2\mathbb{1}_{\mathcal{X}|_{X_{x,t}}\neq\mathcal{X}'|_{X_{x,t}}}\,\Big\|\,\ell-1\le\rho_{x,t}<\ell\right]\mathbb{P}\left[\ell-1\le\rho_{x,t}<\ell\right].
\end{aligned}$$

By definition, given $\rho_{x,t}<\ell$, the restriction $A|_{\mathbb{R}^d\setminus Q_{2f(\ell)+1}(x)}$ is independent of $\mathcal{X}|_{X_{x,t}}$ and $\mathcal{X}'|_{X_{x,t}}$. The above thus yields

$$\begin{aligned}
g_x^1(t) \;\le\; &\sum_{\ell=2}^{\infty}\mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\,Z(A)\Big)^2\,\Big\|\,\ell-1\le\rho_{x,t}<\ell\right]\\
&\qquad\times\mathbb{P}\left[\ell-1\le\rho_{x,t}<\ell,\,\mathcal{X}|_{X_{x,t}}\neq\mathcal{X}'|_{X_{x,t}}\right].
\end{aligned}$$

By assumption in item (ii), the restriction $A|_{\mathbb{R}^d\setminus Q_{2f(\rho_{x,t})+1}(x)}$ is independent of $\rho_{x,t}$, so that we may deduce

$$g_x^1(t)\le\sum_{\ell=2}^{\infty}\mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\,Z(A)\Big)^2\,\Big\|\,\rho_{x,t}<\ell\right]\mathbb{P}\left[\ell-1\le\rho_{x,t}<\ell,\,\mathcal{X}|_{X_{x,t}}\neq\mathcal{X}'|_{X_{x,t}}\right].$$

To simplify notation, we set for all $\ell\ge1$,

$$Y_\ell := \Big(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\,Z(A)\Big)^2.$$

Estimating

$$\mathbb{E}\left[Y_\ell\,\|\,\rho_{x,t}<\ell\right]\le\frac{\mathbb{E}\left[Y_\ell\right]}{\mathbb{P}\left[\rho_{x,t}<\ell\right]},$$

and using the stationarity assumption (b) for the action radius, we may conclude

$$\begin{aligned} g_x^1(t) &\le \sum_{\ell=2}^{\infty} \frac{\mathbb{E}\left[Y_\ell\right]}{\mathbb{P}\left[\rho_{x,t}<\ell\right]} \mathbb{P}\left[\ell-1\le\rho_{x,t}<\ell,\, \mathcal{X}|_{X_{x,t}} \ne \mathcal{X}'|_{X_{x,t}}\right] \\ &= \sum_{\ell=2}^{\infty} \frac{\mathbb{E}\left[Y_\ell\right]}{\mathbb{P}\left[\rho_{0,t}<\ell\right]} \mathbb{P}\left[\ell-1\le\rho_{0,t}<\ell,\, \mathcal{X}|_{X_{0,t}} \ne \mathcal{X}'|_{X_{0,t}}\right] \\ &= \sum_{\ell=2}^{\infty} \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\, Z(A)\Big)^2\right] \mathbb{P}\left[\mathcal{X}|_{X_{0,t}} \ne \mathcal{X}'|_{X_{0,t}}\right] \\ &\qquad\qquad \times \frac{\mathbb{P}\left[\ell-1\le\rho_{0,t}<\ell \,\|\, \mathcal{X}|_{X_{0,t}} \ne \mathcal{X}'|_{X_{0,t}}\right]}{\mathbb{P}\left[\rho_{0,t}<\ell\right]}. \end{aligned} \tag{2.19}$$

We now turn to the estimate of the term $g_x^2(t)$. Since the influence function f satisfies $f(u)\ge u$ for all u, we find

$$\begin{aligned} g_x^2(t) &= \mathbb{E}\left[\Big(Z(A)-Z(A(\mathcal{X}^{x,t}))\Big)^2 \mathbb{1}_{\mathcal{X}|_{X_{x,t}}\ne\mathcal{X}'|_{X_{x,t}}} \mathbb{1}_{\rho_{x,t}<1}\right] \\ &\le \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2f(1)+1}(x)}\, Z(A)\Big)^2 \mathbb{1}_{\mathcal{X}|_{X_{x,t}}\ne\mathcal{X}'|_{X_{x,t}}} \,\Big\|\, \rho_{x,t}<1\right] \mathbb{P}\left[\rho_{x,t}<1\right]. \end{aligned}$$

By definition, given $\rho_{x,t}<1$, the restriction $A|_{\mathbb{R}^d\setminus Q_{2f(1)+1}(x)}$ is independent of $\mathcal{X}|_{X_{x,t}}$ and $\mathcal{X}'|_{X_{x,t}}$. The above thus yields

$$\begin{aligned} g_x^2(t) &\le \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2f(1)+1}(x)}\, Z(A)\Big)^2\right] \mathbb{P}\left[\mathcal{X}|_{X_{x,t}} \ne \mathcal{X}'|_{X_{x,t}} \,\|\, \rho_{x,t}<1\right] \\ &= \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2f(1)+1}(x)}\, Z(A)\Big)^2\right] \mathbb{P}\left[\mathcal{X}|_{X_{x,t}} \ne \mathcal{X}'|_{X_{x,t}}\right] \frac{\mathbb{P}\left[\rho_{x,t}<1 \,\|\, \mathcal{X}|_{X_{x,t}} \ne \mathcal{X}'|_{X_{x,t}}\right]}{\mathbb{P}\left[\rho_{x,t}<1\right]}. \end{aligned}$$

Using the stationarity assumption (b) again, and combining this with (2.18) and (2.19), the conclusion (2.10) follows.

Step 3. Proof of the logarithmic Sobolev inequalities (2.9) and (2.12).
Conditioning the RHS of (2.4) with respect to the values of the action radius $\rho_{x,t}$, we obtain

$$\begin{aligned} \mathrm{Ent}[Z(A)] &\le 2\sum_{x\in\mathbb{Z}^d}\sum_{\ell=1}^{\infty}\sum_{t\in\mathbb{Z}^l} \mathbb{E}\left[\sup_{\mathcal{X}'}\mathrm{ess}\, \Big(\big(Z(A(\mathcal{X}))-Z(A(\mathcal{X}^{x,t}))\big)^2 \mathbb{1}_{\ell-1\le\rho_{x,t}<\ell}\Big)\right] \\ &\le 2\sum_{x\in\mathbb{Z}^d}\sum_{\ell=1}^{\infty}\sum_{t\in\mathbb{Z}^l} \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2\ell+1}(x)}\, Z(A)\Big)^2 \sup_{\mathcal{X}'}\mathrm{ess}\, \big(\mathbb{1}_{\ell-1\le\rho_{x,t}<\ell}\big)\right]. \end{aligned}$$

Hence, if for all x,t the random variable $\rho_{x,t}$ is $\sigma(\mathcal{X})$-measurable, we may deduce

$$\mathrm{Ent}[Z(A)] \le 2\sum_{x\in\mathbb{Z}^d}\sum_{\ell=1}^{\infty}\sum_{t\in\mathbb{Z}^l} \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2\ell+1}(x)}\, Z(A)\Big)^2 \mathbb{1}_{\ell-1\le\rho_{x,t}<\ell}\right].$$

The result (2.9) follows from the Hölder inequality, while the result (2.12) follows as in Step 2.

Step 4. Proof of the covariance inequalities (2.7) and (2.11).
Conditioning the RHS of (2.5) with respect to the values of the action radius $\rho_{x,t}$, we obtain

$$\operatorname{Cov}[Y(A);Z(A)] \le \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{t\in\mathbb{Z}^l}\bigg(\sum_{\ell=1}^\infty \mathbb{E}\Big[\big(Y(A)-Y(A(\mathcal{X}^{x,t}))\big)^2\mathbb{1}_{\ell-1\le\rho_{x,t}<\ell}\Big]\bigg)^{\frac12} \times\bigg(\sum_{\ell'=1}^\infty \mathbb{E}\Big[\big(Z(A)-Z(A(\mathcal{X}^{x,t}))\big)^2\mathbb{1}_{\ell'-1\le\rho_{x,t}<\ell'}\Big]\bigg)^{\frac12}.$$

Now the sums over ℓ,ℓ' are estimated exactly as in Steps 1 and 2, and the results (2.7) and (2.11) follow. □

We now prove Theorem 2.9.

Proof of Theorem 2.9. We only prove the spectral gap (2.14). The proof of the logarithmic Sobolev inequality (2.15) and of the covariance inequality (2.16) is similar, based on (2.4) and (2.5), respectively. For all x, let the field $\mathcal{X}^x$ be defined by $\mathcal{X}^x|_{X\setminus X_x}=\mathcal{X}|_{X\setminus X_x}$ and $\mathcal{X}^x|_{X_x}=\mathcal{X}'|_{X_x}$, and recall that the spectral gap (2.3) for $\mathcal{X}$ takes the form

$$\operatorname{Var}[Z(A)] \le \frac12\sum_{x\in\mathbb{Z}^d}\mathbb{E}\Big[\big(Z(A)-Z(A(\mathcal{X}^x))\big)^2\Big].$$

The conclusion (2.14) then follows provided we prove that for all $x\in\mathbb{Z}^d$,

$$\mathbb{E}\Big[\big(Z(A)-Z(A(\mathcal{X}^x))\big)^2\Big] \le \int_0^\infty \mathbb{E}\bigg[\Big(\partial^{\mathrm{osc}}_{A,Q_{2\ell+1}(x)}Z(A)\Big)^2\bigg](\ell+1)^{-d}\pi(\ell)d\ell. \tag{2.20}$$

Without loss of generality, it suffices to consider the case $x=0$. Moreover, by an approximation argument, we may assume that the random variable $Z(A)$ is bounded. For simplicity, we set $\rho(r):=r+\rho_0^r$ and $\partial_r^{\mathrm{osc}}:=\partial^{\mathrm{osc}}_{A,Q_{2r+1}}$. Note that the choice (2.13) of R then takes the form

$$\sup_{\ell\ge R}\mathbb{P}\big[\rho(\ell)\ge 2\ell\big]\le\frac14. \tag{2.21}$$

We split the proof into two steps.

Step 1. Conditioning argument.
In this step, we prove for all $r_2\ge 2r_1\ge 2R$,

$$\mathbb{E}\bigg[\Big(\partial^{\mathrm{osc}}_{r_2}Z(A)\Big)^2\mathbb{1}_{\frac12 r_2\le\rho(r_1)<r_2}\bigg]\le 2\,\mathbb{P}\big[\tfrac12 r_2\le\rho(r_1)<r_2\big] \times\bigg(\mathbb{E}\bigg[\Big(\partial^{\mathrm{osc}}_{2r_2}Z(A)\Big)^2\bigg]+\sum_{\ell=2}^\infty\mathbb{E}\bigg[\Big(\partial^{\mathrm{osc}}_{2^\ell r_2}Z(A)\Big)^2\mathbb{1}_{2^{\ell-1}r_2\le\rho(r_2)<2^\ell r_2}\bigg]\bigg). \tag{2.22}$$

Conditioning the LHS with respect to the value of $\rho(r_2)$, we decompose

$$\mathbb{E}\bigg[\Big(\partial^{\mathrm{osc}}_{r_2}Z(A)\Big)^2\mathbb{1}_{\frac12 r_2\le\rho(r_1)<r_2}\bigg]\le\mathbb{E}\bigg[\Big(\partial^{\mathrm{osc}}_{r_2}Z(A)\Big)^2\mathbb{1}_{\frac12 r_2\le\rho(r_1)<r_2}\mathbb{1}_{\rho(r_2)<2r_2}\bigg] +\sum_{\ell=2}^\infty\mathbb{E}\bigg[\Big(\partial^{\mathrm{osc}}_{r_2}Z(A)\Big)^2\mathbb{1}_{\frac12 r_2\le\rho(r_1)<r_2}\mathbb{1}_{2^{\ell-1}r_2\le\rho(r_2)<2^\ell r_2}\bigg]. \tag{2.23}$$

We estimate each of the RHS terms separately. For that purpose, note that the definition of ρ and assumption (c) ensure that, given $\rho(\ell_1)\le\ell_2$ and $\rho(\ell_2)\le\ell_3$, the random variable $\rho(\ell_1)$ is independent of $\partial^{\mathrm{osc}}_{\ell_3} Z(A)$. This observation directly yields

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{r_2} Z(A)\Big)^2\mathbb{1}_{\frac12 r_2\le\rho(r_1)<r_2}\mathbb{1}_{\rho(r_2)<2r_2}\Big]\\
&\quad\le\ \mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{2r_2} Z(A)\Big)^2\mathbb{1}_{\rho(r_1)\ge\frac12 r_2}\,\Big\|\,\rho(r_1)<r_2,\,\rho(r_2)<2r_2\Big]\,\mathbb{P}\big[\rho(r_1)<r_2,\,\rho(r_2)<2r_2\big]\\
&\quad\le\ \mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{2r_2} Z(A)\Big)^2\Big]\,\frac{\mathbb{P}\big[\frac12 r_2\le\rho(r_1)<r_2\big]}{\mathbb{P}\left[\rho(r_1)<r_2,\,\rho(r_2)<2r_2\right]}\\
&\quad\le\ \mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{2r_2} Z(A)\Big)^2\Big]\,\frac{\mathbb{P}\big[\frac12 r_2\le\rho(r_1)<r_2\big]}{1-\mathbb{P}\left[\rho(r_1)\ge r_2\right]-\mathbb{P}\left[\rho(r_2)\ge 2r_2\right]}.
\end{aligned}$$

For $r_2\ge 2r_1\ge 2R$, the choice (2.21) of R yields

$$\mathbb{P}\left[\rho(r_1)\ge r_2\right]+\mathbb{P}\left[\rho(r_2)\ge 2r_2\right]\le\mathbb{P}\left[\rho(r_1)\ge 2r_1\right]+\mathbb{P}\left[\rho(r_2)\ge 2r_2\right]\le\frac12,$$

so that the above takes the simpler form

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{r_2} Z(A)\Big)^2\mathbb{1}_{\frac12 r_2\le\rho(r_1)<r_2}\mathbb{1}_{\rho(r_2)<2r_2}\Big]\\
&\qquad\qquad\le 2\,\mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{2r_2} Z(A)\Big)^2\Big]\,\mathbb{P}\big[\tfrac12 r_2\le\rho(r_1)<r_2\big].
\end{aligned}\tag{2.24}$$

On the other hand, further recalling that assumption (c) ensures that given $\rho(\ell_1)\le\ell_2$ the random variables $\rho(\ell_1)$ and $\rho(\ell_2)$ are independent, we similarly obtain

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{r_2} Z(A)\Big)^2\mathbb{1}_{\frac12 r_2\le\rho(r_1)<r_2}\mathbb{1}_{2^{\ell-1}r_2\le\rho(r_2)<2^\ell r_2}\Big]\\
&\quad\le\ \mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{2^\ell r_2} Z(A)\Big)^2\mathbb{1}_{\rho(r_2)\ge 2^{\ell-1}r_2}\,\Big\|\,\rho(r_1)<r_2,\,\rho(r_2)<2^\ell r_2\Big]\\
&\qquad\qquad\times\mathbb{P}\big[\rho(r_1)\ge\tfrac12 r_2\,\big\|\,\rho(r_1)<r_2,\,\rho(r_2)<2^\ell r_2\big]\,\mathbb{P}\big[\rho(r_1)<r_2,\,\rho(r_2)<2^\ell r_2\big]\\
&\quad\le\ \mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{2^\ell r_2} Z(A)\Big)^2\mathbb{1}_{2^{\ell-1}r_2\le\rho(r_2)<2^\ell r_2}\Big]\,\frac{\mathbb{P}\left[\frac12 r_2\le\rho(r_1)<r_2\right]}{\mathbb{P}\left[\rho(r_1)<r_2,\,\rho(r_2)<2^\ell r_2\right]}\\
&\quad\le\ \mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{2^\ell r_2} Z(A)\Big)^2\mathbb{1}_{2^{\ell-1}r_2\le\rho(r_2)<2^\ell r_2}\Big]\,\frac{\mathbb{P}\left[\frac12 r_2\le\rho(r_1)<r_2\right]}{1-\mathbb{P}\left[\rho(r_1)\ge r_2\right]-\mathbb{P}\left[\rho(r_2)\ge 2^\ell r_2\right]}.
\end{aligned}$$

With the choice (2.21) of R, for $r_2\ge 2r_1\ge 2R$ and $\ell\ge1$, this turns into

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{r_2} Z(A)\Big)^2\mathbb{1}_{\frac12 r_2\le\rho(r_1)<r_2}\mathbb{1}_{2^{\ell-1}r_2\le\rho(r_2)<2^\ell r_2}\Big]\\
&\qquad\qquad\le 2\,\mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{2^\ell r_2} Z(A)\Big)^2\mathbb{1}_{2^{\ell-1}r_2\le\rho(r_2)<2^\ell r_2}\Big]\,\mathbb{P}\left[\tfrac12 r_2\le\rho(r_1)<r_2\right].
\end{aligned}$$

Combining this with (2.23) and (2.24), the conclusion (2.22) follows.

Step 2. Proof of (2.20).
Conditioning the LHS of (2.20) with respect to the value of the action radius $\rho(0)$, we obtain

$$\mathbb{E}\Big[\big(Z(A)-Z(A(\mathcal{X}^x))\big)^2\Big]\le\mathbb{E}\Big[\Big(\partial^{\rm osc}_{R}\,Z(A)\Big)^2\Big]+\sum_{\ell=1}^\infty\mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^\ell R}\,Z(A)\Big)^2\mathbb{1}_{2^{\ell-1}R\le\rho(0)<2^\ell R}\Big].$$

We now iteratively apply (2.22) to estimate the last RHS terms: with the short-hand notation $\pi(\ell_2;\ell_1):=\mathbb{P}\big[\frac12\ell_2\le\rho(\ell_1)<\ell_2\big]$, we obtain for all $n\ge1$,

$$\begin{aligned}\mathbb{E}\Big[\big(Z(A)-Z(A(\mathcal{X}^x))\big)^2\Big]&\le\mathbb{E}\Big[\Big(\partial^{\rm osc}_{R}\,Z(A)\Big)^2\Big]+2\sum_{\ell_1=1}^\infty\pi(2^{\ell_1}R;0)\,\mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^{\ell_1+1}R}\,Z(A)\Big)^2\Big]\\&\quad+2^2\sum_{\ell_1=1}^\infty\pi(2^{\ell_1}R;0)\sum_{\ell_2=\ell_1+2}^\infty\pi(2^{\ell_2}R;2^{\ell_1}R)\,\mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^{\ell_2+1}R}\,Z(A)\Big)^2\Big]+\ldots\\&+2^n\sum_{\ell_1=1}^\infty\pi(2^{\ell_1}R;0)\sum_{\ell_2=\ell_1+2}^\infty\pi(2^{\ell_2}R;2^{\ell_1}R)\ldots\sum_{\ell_n=\ell_{n-1}+2}^\infty\pi(2^{\ell_n}R;2^{\ell_{n-1}}R)\,\mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^{\ell_n+1}R}\,Z(A)\Big)^2\Big]\\&\quad+2^n\sum_{\ell_1=1}^\infty\pi(2^{\ell_1}R;0)\sum_{\ell_2=\ell_1+2}^\infty\pi(2^{\ell_2}R;2^{\ell_1}R)\ldots\sum_{\ell_n=\ell_{n-1}+2}^\infty\pi(2^{\ell_n}R;2^{\ell_{n-1}}R)\\&\qquad\times\sum_{\ell_{n+1}=\ell_n+2}^\infty\mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^{\ell_{n+1}}R}\,Z(A)\Big)^2\mathbb{1}_{2^{\ell_{n+1}-1}R\le\rho(2^{\ell_n}R)<2^{\ell_{n+1}}R}\Big].\end{aligned}$$

With the choice (2.21) of R in the form

$$\sup_{\ell_0\ge0}\sum_{\ell=\ell_0+2}^\infty\pi(2^\ell R;2^{\ell_0}R)=\sup_{\ell_0\ge0}\mathbb{P}\big[\rho(2^{\ell_0}R)\ge2^{\ell_0+1}R\big]\le\frac14,$$

the definition $\tilde\pi(\ell):=\sup_{\ell_0:0\le\ell_0\le\ell/4}\pi(\ell;\ell_0)$ of the weight, and recalling that the random variable $Z(A)$ is bounded, we deduce

$$\begin{aligned}\mathbb{E}\Big[\big(Z(A)-Z(A(\mathcal{X}^x))\big)^2\Big]&\le\mathbb{E}\Big[\Big(\partial^{\rm osc}_{R}\,Z(A)\Big)^2\Big]\\&\quad+2\Big(\sum_{m=0}^{n-1}2^{-m}\Big)\sum_{\ell=1}^\infty\tilde\pi(2^\ell R)\,\mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^{\ell+1}R}\,Z(A)\Big)^2\Big]+2^{-n-2}\|Z\|_{\mathrm{L}^\infty}.\end{aligned}$$

Letting $n\uparrow\infty$, we thus obtain

$$\mathbb{E}\Big[\big(Z(A)-Z(A(\mathcal{X}^x))\big)^2\Big]\;\le\;\mathbb{E}\Big[\Big(\partial^{\rm osc}_{R}\,Z(A)\Big)^2\Big]+4\sum_{\ell=1}^\infty\tilde\pi(2^\ell R)\,\mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^{\ell+1}R}\,Z(A)\Big)^2\Big].$$

Comparing sums to integrals and using the definition of π, the conclusion (2.20) follows. □

2.4. **Local operations.** In this subsection, we describe two typical operations on random fields that do preserve functional inequalities: local transformations and gluing of independent random fields with respect to an independent pattern. These operations allow one to generate many variations around the examples of Section 3.

2.4.1. *Local transformations.* Given a random field A_0 on $\mathbb{R}^d$, we say that a random field A on $\mathbb{R}^d$ is a R-local transformation of A_0 (as in Proposition 2.3) if $A|_S$ is $\sigma(A|_{S+B_R})$-measurable for all Borel subsets $S \subset \mathbb{R}^d$. Important particular cases are local smoothing (e.g. by convolution with a smooth kernel with bounded support) and truncation (e.g. by applying a Lipschitz function).

Lemma 2.11. *If A_0, A are two random fields on $\mathbb{R}^d$, and if A is a R-local transformation of A_0, then we have for all Borel subsets $S \subset \mathbb{R}^d$ and all $\sigma(A)$-measurable random variables $X(A)$,*

$$\partial^{\text{osc}}_{A_0,S} X(A(A_0)) \le \partial^{\text{osc}}_{A,S+B_R} X(A)$$

and

$$\partial^{\text{fct}}_{A_0,S} X(A(A_0)) \le R^d \left\| \frac{\partial A}{\partial A_0} \right\|_{\mathrm{L}^\infty} \partial^{\text{fct}}_{A,S+B_R} X(A),$$

so that functional inequalities for A_0 with the oscillation or the functional derivative imply the corresponding functional inequalities for A with the oscillation or the functional derivative (provided $\partial A/\partial A_0$ is bounded if the functional derivative is used). ☐

Proof. By assumption, $A|_{\mathbb{R}^d\setminus(S+B_R)}$ is $\sigma(A_0|_{\mathbb{R}^d\setminus S})$-measurable, so that the sub-σ-algebra $\sigma(A|_{\mathbb{R}^d\setminus(S+B_R)})$ is contained in $\sigma(A_0|_{\mathbb{R}^d\setminus B})$, and the inequality follows. ☐

2.4.2. *Independent gluing.* The following result shows how independent localized fields can be glued together. Since it is a direct consequence of the standard tensorization arguments used e.g. in the proof of Proposition 2.3, details are omitted.

Lemma 2.12. *Let A_1, A_2, and A_3 be three independent random fields on $\mathbb{R}^d$. Assume that $|A_1 - A_3| \le C$ a.s. for some deterministic constant $C > 0$, that A_2 has values in $[0, 1]$, and consider the "glued" random field $A := A_2A_1 + (1 - A_2)A_3$. If A_1, A_2, and A_3 satisfy different forms of weighted spectral gaps (resp. covariance inequality, resp. logarithmic Sobolev inequality), then the random field A satisfies the worst of these spectral gaps (resp. covariance inequality, resp. logarithmic Sobolev inequality), that is, with the RHS replaced by the sum of the corresponding RHSs.* ☐

3. Examples

In this section we consider four representative examples: Gaussian fields, tessellations associated with a Poisson point process, random parking bounded inclusions, and Poisson or random parking inclusions with unbounded radii. The main results are summarized in the table below.

Example of field	**Key property**	**Functional inequalities**
Gaussian random field	covariance function $\mathcal{C}$ $\sup_{B(x)} \lvert\mathcal{C}\rvert \le c(\lvert x\rvert)$	(∂^{fct}-WSG), (∂^{fct}-WLSI) weight $\pi(\ell) \simeq (-c'(\ell))_+$
Poisson tessellations (Voronoi/Delaunay)	$\sigma(\mathcal{X})$-measurable action radius	(∂^{osc}-WSG), (∂^{osc}-WLSI) weight $\pi(\ell) \simeq e^{-\frac{1}{C}\ell^d}$
Random parking bounded inclusions	$\sigma(\mathcal{X})$-measurable action radius & exponential stabilization	(∂^{osc}-WSG), (∂^{osc}-WLSI) weight $\pi(\ell) \simeq e^{-\frac{1}{C}\ell}$
Poisson random inclusions with random radii	radius law V $\gamma(\ell) := \mathbb{P}[\ell - 4 \le V < \ell + 2]$	(∂^{osc}-WSG) weight $\pi(\ell) \simeq (\ell + 1)^d\gamma(\ell)$ (and (∂^{osc}-LSI) if V bounded)

3.1. **Gaussian random fields.** Gaussian random fields are the main examples of deterministically localized fields as introduced in Section 2.3 (and studied in Appendix B). Note that this result is a weighted reformulation of the "coarsened" functional inequalities used in the first version of [9] for Gaussian fields.

Corollary 3.1. *Let A be a jointly measurable stationary Gaussian random field on $\mathbb{R}^d$ with covariance function $\mathcal{C}(x) := \mathrm{Cov}\,[A(x); A(0)]$.*

(i) If $x \mapsto \sup_{B(x)} |\mathcal{C}|$ is integrable, then A satisfies (∂^{fct}-SG) and (∂^{fct}-LSI) with any radius $R > 0$.
(ii) If $\sup_{B(x)} |\mathcal{C}| \le c(|x|)$ holds for some Lipschitz function $c : \mathbb{R}_+ \to \mathbb{R}_+$, then A satisfies (∂^{fct}-WSG) and (∂^{fct}-WLSI) with weight $\pi(\ell) \simeq (-c'(\ell))_+$.
If $\mathcal{F}\mathcal{C} \in \mathrm{L}^1(\mathbb{R}^d)$ and if $\sup_{B(x)} |\mathcal{F}^{-1}(\sqrt{\mathcal{F}\mathcal{C}})| \le r(|x|)$ holds for some non-increasing Lipschitz function $r : \mathbb{R}_+ \to \mathbb{R}_+$, then A satisfies (∂^{fct}-WCI) with weight $\pi(\ell) \simeq (\ell + 1)^d\, r(\ell)(-r'(\ell))$. □

As shown in the companion article [7, Proposition 2.3], this result is sharp: each sufficient condition is (essentially) necessary.

Proof. Let W denote a Gaussian white noise with intensity 1, that is, a random noise W on $\mathbb{R}^d$ such that for all bounded Borel subsets $E \subset \mathbb{R}^d$ the random variable $W(E)$ has a centered Gaussian law with variance $\mathbb{E}[W(E)^2] = |E|$. As shown in [6, Section XI.8], a stationary Gaussian random field A on $\mathbb{R}^d$ can be rewritten as a convolution (B.1) with a Gaussian white noise whenever the field A has an absolutely continuous spectral measure, or equivalently, whenever the Fourier transform of the covariance function $\mathcal{C}$ is in $\mathrm{L}^1(\mathbb{R}^d)$. Under such a restriction on $\mathcal{C}$, since Gaussian random variables satisfy the standard spectral gap (B.3) and logarithmic Sobolev inequality (B.7) (cf. [11]), we can directly apply Proposition B.1 and Theorem B.2 to establish the validity of weighted spectral gaps, covariance, and logarithmic Sobolev inequalities.

It remains to show that this restriction on $\mathcal{C}$ can be relaxed in the case of spectral gaps and logarithmic Sobolev inequalities. To this end, it suffices to prove that the conclusion of Proposition B.1 (that is, the validity of Brascamp-Lieb type inequalities) always holds for any jointly measurable Gaussian stationary random field A. This is achieved by an approximation argument. We focus on the Brascamp-Lieb inequality (B.4), while the argument is analogous for (B.8). As an approximation argument shows, it is enough to establish (B.4) for those random variables $X(A)$ that depend on A only via their spatial averages on the partition $\{Q_\varepsilon(z)\}_{z \in B_R \cap \varepsilon\mathbb{Z}^d}$ with $\varepsilon, R > 0$. Let us introduce the following notation for these averages,

$$A_\varepsilon(z) := \fint_{Q_\varepsilon(z)} A, \qquad \text{for } z \in \varepsilon\mathbb{Z}^d. \tag{3.1}$$

In this case, the Fréchet derivative $\{\frac{\partial X}{\partial A}(x)\}_{x \in \mathbb{R}^d}$ and the partial derivatives $\{\frac{\partial X}{\partial A_\varepsilon(z)}\}_{z \in \varepsilon\mathbb{Z}^d}$ of $X(A)$ are related via

$$\varepsilon^d \frac{\partial X}{\partial A}(x) = \frac{\partial X}{\partial A_\varepsilon(z)}, \qquad \text{for } x \in Q_\varepsilon(z),\ z \in \varepsilon\mathbb{Z}^d. \tag{3.2}$$

We infer from (3.1) that $\{A_\varepsilon(z)\}_{z\in\varepsilon\mathbb{Z}^d}$ is a discrete centered Gaussian random field (which is now stationary with respect to the action of $\varepsilon\mathbb{Z}^d$), characterized by its covariance

$$\mathcal{C}_\varepsilon(z-z') := \fint_{Q_\varepsilon(z)} \fint_{Q_\varepsilon(z')} \mathcal{C}(x-x')dx'dx. \tag{3.3}$$

By the discrete result (B.9) obtained in the proof of Proposition B.1 (based on the standard spectral gap for Gaussian random variables [11]), we deduce for all $\varepsilon, R > 0$ and all random variables $X(A)$ that depend on A only via its spatial averages on the partition $\{Q_\varepsilon(z)\}_{z\in B_R\cap\varepsilon\mathbb{Z}^d}$,

$$\mathrm{Var}\,[X] \le C \sum_{z\in B_R\cap\varepsilon\mathbb{Z}^d} \sum_{z'\in B_R\cap\varepsilon\mathbb{Z}^d} |\mathcal{C}_\varepsilon(z-z')|\,\mathbb{E}\left[\Big|\frac{\partial X}{\partial A_\varepsilon(z)}\Big|\Big|\frac{\partial X}{\partial A_\varepsilon(z')}\Big|\right].$$

Injecting (3.2) and (3.3), the conclusion (B.4) follows. ☐

3.2. **Poisson random tessellations.** In this section, we consider random fields that take i.i.d. values on the cells of a tessellation associated with a stationary random point process $\mathcal{P}$ on $\mathbb{R}^d$. Such random fields can be formalized as projections of decorated random point processes. Given a point process $\mathcal{P}$ on $\mathbb{R}^d$ and given a random element G with values in some measurable space X, we call *decorated random point process associated with $\mathcal{P}$ and G* a point process $\hat{\mathcal{P}}$ on $\mathbb{R}^d\times X$ defined as follows: choose a measurable enumeration $\mathcal{P}=\{X_j\}_j$, pick independently a sequence $(G_j)_j$ of i.i.d. copies of the random element G, and set $\hat{\mathcal{P}} := \{X_j, G_j\}_j$ (that is, in measure notation, $\hat{\mathcal{P}} := \sum_j \delta_{(X_j,G_j)}$). Note that by definition $\hat{\mathcal{P}}$ is completely independent whenever $\mathcal{P}$ is completely independent.

We focus here on the case when the underlying point process $\mathcal{P}$ is some Poisson point process $\mathcal{P}=\mathcal{P}_0$ on $\mathbb{R}^d$ with intensity $\mu=1$. Choose a measurable random field V on $\mathbb{R}^d$, corresponding to the values on the cells. We study both Voronoi and Delaunay tessellations.

(1) *Voronoi tessellation:* Let $\hat{\mathcal{P}}_1 := \{X_j, V_j\}_j$ denote a decorated point process associated with the random point process $\mathcal{P}_0 := \{X_j\}_j$ and the random element V (hence $(V_j)_j$ is a sequence of i.i.d. copies of the random field V). We define a $\sigma(\hat{\mathcal{P}}_1)$-measurable random field A_1 as follows,

$$A_1(x) = \sum_j V_j(x)\mathbb{1}_{C_j}(x),$$

where $\{C_j\}_j$ denotes the partition of $\mathbb{R}^d$ into the Voronoi cells associated with the Poisson points $\{X_j\}_j$, that is,

$$C_j := \{x\in\mathbb{R}^d : |x-X_j| < |x-X_k|,\ \forall k\ne j\}.$$

(2) *Delaunay tessellation:* Let $\tilde V := (\tilde V_\zeta)_\zeta$ denote a family of i.i.d. copies of the random element V, indexed by sets ζ of $d+1$ distinct integers. We define a random field A_2 as follows,

$$A_2(x) = \sum_j \tilde V_{\zeta(D_j)}(x)\mathbb{1}_{D_j}(x),$$

where $\{D_j\}_j$ denotes the partition of $\mathbb{R}^d$ into the Delaunay d-simplices associated with the Poisson points $\{X_j\}_j$ (the Delaunay triangulation is indeed almost surely uniquely defined), and where $\zeta(D_j)$ denotes the set of the $d+1$ indices $i_1,\ldots,i_{d+1}$ of the vertices $X_{i_1},\ldots,X_{i_{d+1}}$ of D_j.

Since large holes in the Poisson process have exponentially small probability, large cells in the corresponding Voronoi or Delaunay tessellations also have exponentially small probability. This allows one to prove the following weighted functional inequalities with stretched exponential weights.

Proposition 3.2. *For $s = 1, 2$, the above-defined random field A_s satisfies* (∂^{osc}-WSG), (∂^{osc}-WLSI), *and* (∂^{osc}-WCI) *with weight $\pi(\ell) = e^{-\frac{1}{C}\ell^d}$ for some constant $C > 0$. Moreover for all $\sigma(A_s)$-measurable random variables $Z(A_s)$ and all $\lambda \in (0,1)$ we have*

$$\mathrm{Var}\left[Z(A_s)\right] \le C \sum_{x\in\mathbb{Z}^d} \sum_{\ell=1}^{\infty} e^{-\frac{\lambda}{C}\ell^d}\, \mathbb{E}\left[\left(\partial^{\mathrm{dis}}_{\ell,x} Z(A_s)\right)^{\frac{2}{1-\lambda}}\right]^{1-\lambda},$$

with the notation $\partial^{\mathrm{dis}}_{\ell,x} Z(A_s)$ defined in (2.2) *(with $l = 0$).* □

Proof. We focus on the case of the Voronoi tessellation (the argument for the Delaunay tessellation is similar). We shall appeal to Theorem 2.9, and need to construct and control action radii, which we do in two separate steps. (The weighted spectral gap with loss and discrete derivative follows from Theorem 2.7(i).)

Step 1. Definition and properties of the action radius.
Let $x \in \mathbb{R}^d$, $\ell \in \mathbb{N}$ be fixed. Changing the point configuration of $\hat{\mathcal{P}}_1 = \{X_j, V_j\}_j$ inside $Q_{2\ell+1}(x) \times \mathbb{R}^{\mathbb{R}^d}$ only modifies the Voronoi tessellation (hence the field A_1) inside the set

$$V_{\mathcal{P}_0,\ell}(x) := \big\{y \in \mathbb{R}^d : \exists z \in Q_{2\ell+1}(x) \\ \text{such that } |y-z| \le |y-X| \text{ for all } X \in \mathcal{P}_0 \setminus Q_{2\ell+1}(x)\big\}.$$

An action radius for A_1 with respect to $\hat{\mathcal{P}}_1$ on $Q_{2\ell+1}(x) \times \mathbb{R}^{\mathbb{R}^d}$ is thus given by

$$\rho_x^\ell := 2\,\mathrm{diam}\, V_{\mathcal{P}_0,\ell}(x) + 1 - \ell,$$

and property (a) of Theorem 2.9 is proved. The stationarity property (b) follows by construction, and it remains to prove the measurability property (c). In particular, we need to prove that ρ_x^ℓ is $\sigma(\mathcal{P}_0|_{Q_{2(\ell+\rho_x^\ell)+1}(x)\setminus Q_{2\ell+1}(x)})$-measurable. Since ρ_x^ℓ is $\sigma(\mathcal{P}_0|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)})$-measurable by construction, it remains to prove it is $\sigma(\mathcal{P}_0|_{Q_{2(\ell+\rho_x^\ell)+1}(x)})$-measurable. To this aim, let $\tilde{\mathcal{P}}$ be an arbitrary locally finite point set and consider the compound point set $\tilde{\mathcal{P}}_{0,\ell}(x) = \mathcal{P}_0|_{Q_{2(\ell+\rho_x^\ell)+1}(x)} \cup \tilde{\mathcal{P}}|_{\mathbb{R}^d\setminus Q_{2(\ell+\rho_x^\ell)+1}(x)}$. The claimed measurability then follows from the identity $V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x) = V_{\mathcal{P}_0,\ell}(x)$. We start with the proof that $V_{\mathcal{P}_0,\ell}(x) \subset V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x)$. Let $y \in V_{\mathcal{P}_0,\ell}(x)$. Then for all $X \in \tilde{\mathcal{P}}_{0,\ell}(x)|_{\mathbb{R}^d\setminus Q_{2(\ell+\rho_x^\ell)+1}(x)}$ we have by the triangle inequality

$$|X-y| \ge |X-x| - |x-y| \ge \ell + \rho_x^\ell - \mathrm{diam}\, V_{\mathcal{P}_0,\ell}(x) \;=\; \mathrm{diam}\, V_{\mathcal{P}_0,\ell}(x) + 1 \ge |x-y|,$$

so that $y \in V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x)$. Let us turn to the converse inclusion. By definition, $V_{\mathcal{P}_0,\ell}(x)$ and $V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x)$ are convex, and thus simply connected. Set $\eta = \frac{1}{2}$ and consider $y \in (B_\eta + V_{\mathcal{P}_0,\ell}(x)) \setminus V_{\mathcal{P}_0,\ell}(x)$ (the η-fattened boundary of $V_{\mathcal{P}_0,\ell}(x)$). By definition we have $y \notin V_{\mathcal{P}_0,\ell}(x)$, so that for all $z \in Q_{2\ell+1}(x)$ there exists $X \in \mathcal{P}_0 \setminus Q_{2\ell+1}(x)$ such that $|y-z| > |y-X|$. Let us argue that $X \in Q_{2(\ell+\rho_x^\ell)+1}(x)$. Indeed, by the triangle inequality,

$$|X-x| \le |X-y| + |y-x| < |y-z| + |y-x| \\ \le \mathrm{diam}\, V_{\mathcal{P}_0,\ell}(x) + \eta + \mathrm{diam}\, V_{\mathcal{P}_0,\ell}(x) + \eta \;=\; \rho_x^\ell + \ell.$$

Hence, we deduce $X \in \tilde{\mathcal{P}}_{0,\ell}(x)$, which in turn implies $y \notin V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x)$. This proves the inclusion $V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x) \subset V_{\mathcal{P}_{0},\ell}(x) \cup (\mathbb{R}^d \setminus (B_\eta + V_{\mathcal{P}_0,\ell}(x)))$. Combined with the inclusion $V_{\mathcal{P}_0,\ell}(x) \subset V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x)$ and the fact that both sets are simply connected, this yields the desired identity $V_{\mathcal{P}_0,\ell}(x) = V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x)$ and therefore proves the claimed measurability property (c). We then appeal to Theorem 2.9, and it remains to estimate the weights.

Step 2. Control of the weight.
By scaling and change of intensity, it is enough to consider $\ell = 0$ (we omit the sub- and superscripts ℓ in the notation) and a Poisson point process $\mathcal{P}_0$ of general intensity $\mu > 0$. Denote by $\mathcal{C}_i = \{x \in \mathbb{R}^d : x_i \geq \frac{5}{6}|x|\}$ the d cones in the canonical directions e_i of $\mathbb{R}^d$, and consider the $2d$ cones $\mathcal{C}_i^\pm := \pm(2e_i + C_i)$. By an elementary geometric argument, for some constant $C \simeq 1$ the following implication holds: for all $L > C$,

$$\sharp\big(\mathcal{P}_0 \cap \mathcal{C}_i^\pm \cap \{x : C \leq |x_i| \leq L\}\big) > 0 \text{ for all } i \text{ and } \pm \quad \Longrightarrow \quad \operatorname{diam} V_{\mathcal{P}_0}(0) \leq CL.$$

A union bound then yields for all $L > C$,

$$\begin{aligned} \mathbb{P}\left[\operatorname{diam} V_{\mathcal{P}_0}(0) \geq L\right] &\leq \mathbb{P}\left[\exists 1 \leq i \leq d, \exists \pm : \sharp\big(\mathcal{P}_0 \cap \mathcal{C}_i^\pm \cap \{x : |x_i| \leq \frac{1}{C}L\}\big) = 0\right] \\ &\leq 2d\, e^{-\frac{\mu}{C}L^d}. \end{aligned}$$

Combined with the definition of the action radius in Step 1, this implies the desired estimate. ☐

3.3. **Random parking process.** In this section we let $\mathcal{P}$ be the random parking point process on $\mathbb{R}^d$ with given radius $R > 0$. As shown by Penrose [20] (see also [10, Section 2.1]), the random parking point process $\mathcal{P}$ can be constructed as a transformation $\mathcal{P} = \Phi(\mathcal{P}_0)$ of a Poisson point process $\mathcal{P}_0$ on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity 1. Let us recall the graphical construction of this transformation Φ. We first construct an oriented graph on the points of $\mathcal{P}_0$ in $\mathbb{R}^d \times \mathbb{R}_+$, by putting an oriented edge from (x, t) to (x', t') whenever $B(x, R) \cap B(x', R) \neq \varnothing$ and $t < t'$ (or $t = t'$ and x precedes x' in the lexicographic order, say). We say that (x', t') is an offspring (resp. a descendant) of (x, t), if (x, t) is a direct ancestor (resp. an ancestor) of (x', t'), that is, if there is an edge (resp. a directed path) from (x, t) to (x', t'). The set $\mathcal{P} := \Phi(\mathcal{P}_0)$ is then constructed as follows. Let F_1 be the set of all roots in the oriented graph (that is, the points of $\mathcal{P}_0$ without ancestor), let G_1 be the set of points of $\mathcal{P}_0$ that are offsprings of points of F_1, and let $H_1 := F_1 \cup G_1$. Now consider the oriented graph induced on $\mathcal{P}_0 \setminus H_1$, and define F_2, G_2, H_2 in the same way, and so on. By construction, the sets $(F_j)_j$ and $(G_j)_j$ are all disjoint and constitute a partition of $\mathcal{P}_0$. We finally define $\mathcal{P} := \Phi(\mathcal{P}_0) := \bigcup_{j=1}^\infty F_j$.

In this setting we show that there exists an action radius with exponential moments for $\mathcal{P}$ with respect to $\mathcal{P}_0$. The proof follows from the exponential stabilization results of [25].

Proposition 3.3. *For all $x \in \mathbb{Z}^d$ and $\ell \geq 0$, the random parking point process $\mathcal{P}$ with radius $R > 0$ as constructed above admits an action radius ρ_x^ℓ with respect to $\mathcal{P}_0$ on $Q_{2\ell+1}(x) \times \mathbb{R}_+$, which satisfies for all $L \geq 0$,*

$$\mathbb{P}[\rho_x^\ell \geq L] \leq C_R(\ell + 1)^d e^{-L/C_R},$$

and which is $\sigma\big(\mathcal{P}_0|_{((Q_{2(\ell+\rho_x^\ell)+1}(x)\setminus Q_{2\ell+1}(x))\times\mathbb{R}_+}\big)$*-measurable.*
In particular, the point process $\mathcal{P}$ *satisfies* (∂^{osc}-WSG), (∂^{osc}-WLSI), *and* (∂^{osc}-WCI) *with weight* $\pi(\ell) =: e^{-\ell/C_R}$. ☐

Proof. The proof relies on the notion of *causal chains* defined in the proof of [25, Lemma 3.5] to which we refer the reader. Note that for all consecutive points (x,t) and (y,s) in a causal chain we necessarily have $|x-y|<2R$. By definition, it follows that an action radius for $\mathcal{P}$ given $\mathcal{P}_0$ on $Q_{2\ell+1}(x)\times\mathbb{R}_+$ can be defined by the maximum of the distances $2R+d(y,Q_{2\ell+1}(x))$ on the set of points $(y,s)\in\mathcal{P}_0$ such that there exists a causal chain between a point of $\mathcal{P}_0$ in $((Q_{2\ell+1}(x)+B_{2R})\setminus Q_{2\ell+1}(x))\times\mathbb{R}_+$ and (y,s). We denote by ρ_x^ℓ this maximum. By construction, we note that this random variable ρ_x^ℓ is $\sigma\big(\mathcal{P}_0|_{((Q_{2\ell+1}(x)+B_{\rho_x^\ell})\setminus Q_{2\ell+1}(x))\times\mathbb{R}_+}\big)$-measurable.
It remains to estimate the decay of its probability law. First, note that by definition the event $\rho_x^\ell>L$ entails the existence of some $(y,s)\in\mathcal{P}_0$ with $y\in(Q_{2\ell+1}(x)+B_{L+2R})\setminus(Q_{2\ell+1}(x)+B_L)$ and of a causal chain between a point of $((Q_{2\ell+1}(x)+B_{2R})\setminus Q_{2\ell+1}(x))\times\mathbb{R}_+$ and (y,s). Second, the exponential stabilization result of [25, Lemma 3.5] states that for all $z\in\mathbb{R}^d$ and all $L>0$ the probability that there exists $(y,s)\in Q(z)\times\mathbb{R}_+$ and a causal chain from a point outside $(Q(z)+B_L)\times\mathbb{R}_+$ towards (y,s) is bounded by C_Re^{-L/C_R}. For $L\ge R$, covering $(Q_{2\ell+1}(x)+B_{L+2R})\setminus(Q_{2\ell+1}(x)+B_L)$ with $C(L+\ell)^{d-1}R$ unit cubes and covering $Q_{2\ell+1}(x)+B_{2R}$ with $C(R+\ell)^d$ unit cubes, a union bound then yields

$$\mathbb{P}\big[\rho_x^\ell>L\big]\le C_R\big(L^{d-1}+\ell^d\big)e^{-L/C_R}\le C_R(\ell+1)^de^{-L/C_R}.$$

All the assumptions of Theorem 2.9 are then satisfied with $\pi(\ell)=C_Re^{-\ell/C_R}$, and the conclusion follows. ☐

3.4. **Random inclusions with random radii.** We consider typical examples of random fields on $\mathbb{R}^d$ taking random values on random inclusions centered at the points of some random point process $\mathcal{P}$. The inclusions are allowed to have i.i.d. random shapes (hence in particular i.i.d. random radii). For the random point process $\mathcal{P}$, we consider projections $\Phi(\mathcal{P}_0)$ of some Poisson point process $\mathcal{P}_0$ on $\mathbb{R}^d\times\mathbb{R}^l$ with intensity $\mu>0$, and shall assume that for all $x\in\mathbb{Z}^d$ the process $\mathcal{P}$ admits an action radius ρ_x *with respect to* $\mathcal{P}_0$ *on* $Q(x)\times\mathbb{R}^l$.

We turn to the construction of the random inclusions. Let V be a nonnegative random variable (corresponding to the random radius of the inclusions). In order to define the random shapes, we consider the set Y of all nonempty Borel subsets $E\subset\mathbb{R}^d$ with $\sup_{x\in E}|x|=1$, and endow it with the σ-algebra $\mathcal{Y}$ generated by all subsets of the form $\{E\in Y:x_0\in E\}$ with $x_0\in\mathbb{R}^d$. Let S be a random nonempty Borel subset of $\mathbb{R}^d$ with $\sup_{x\in S}|x|=1$ a.s., that is, a random element in the measurable space Y. (Note that V and S need not be independent.) Let $\hat{\mathcal{P}}_0:=\{X_j,V_j,S_j\}_j$ be a decorated point process associated with the random point process $\mathcal{P}_0=\{X_j\}_j$ and the random element (V,S). The collection of random inclusions is then given by $\{I_j\}_j$ with $I_j:=X_j+V_jS_j$.

It remains to associate random values to the random inclusions. Since inclusions may intersect each other, several constructions can be considered; we focus on the following three typical choices.

(1) Given $\alpha,\beta\in\mathbb{R}$, we set $\hat{\mathcal{P}}_1:=\hat{\mathcal{P}}_0$, and we consider the $\sigma(\hat{\mathcal{P}}_1)$-measurable random field A_1 that is equal to α inside the inclusions, and to β outside. More precisely,

$$A_1:=\beta+(\alpha-\beta)\mathbb{1}_{\bigcup_j I_j}.$$

The simplest example is the random field A_1 obtained for $\mathcal{P}$ a Poisson point process on $\mathbb{R}^d$ with intensity $\mu = 1$, and for S the unit ball centered at the origin in $\mathbb{R}^d$; this is referred to as the *Poisson unbounded spherical inclusion model.*

(2) Let $\beta \in \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function, and let W be a measurable random field on $\mathbb{R}^d$. Let $\hat{\mathcal{P}}_2 := \{X_j, V_j, S_j, W_j\}$ be a decorated point process associated with $\hat{\mathcal{P}}_0$ and W. We then consider the $\sigma(\hat{\mathcal{P}}_2)$-measurable random field A_2 that is equal to $f(\sum_{j:x\in I_j} W_j)$ at any point x of the inclusions, and to β outside. More precisely,

$$A_2(x) := \beta + \bigg(f\Big(\sum_j W_j(x)\mathbb{1}_{I_j}(x)\Big) - \beta\bigg)\mathbb{1}_{\bigcup_j I_j}(x).$$

(Of course, this example can be generalized by considering more general functions than simple sums of the values W_j; the corresponding concentration properties will then remain the same.)

(3) Let $\beta \in \mathbb{R}$, let W be a measurable random field on $\mathbb{R}^d$, and let U denote a uniform random variable on $[0, 1]$. Let $\hat{\mathcal{P}}_3 := \{X_j, V_j, S_j, W_j, U_j\}$ be a decorated point process associated with $\hat{\mathcal{P}}_0$ and (W, U). Given a $\sigma(VS, W)$-measurable random variable $P(VS, W)$, we say that inclusion I_j has the priority on inclusion I_i if $P(V_jS_j, W_j) < P(V_iS_i, W_i)$ or if $P(V_jS_j, W_j) = P(V_iS_i, W_i)$ and $U_j < U_i$. Since the random variables $\{U_j\}_j$ are a.s. all distinct, this defines a priority order on the inclusions on a set of maximal probability. Let us then relabel the inclusions and values $\{(I_j, V_j)\}_j$ into a sequence $(I'_j, V'_j)_j$ in such a way that for all j the inclusion I'_j has the j-th highest priority. We then consider the $\sigma(\hat{\mathcal{P}}_3)$-measurable random field A_3 defined as follows,

$$A_3 := \beta + \sum_j (W'_j - \beta)\mathbb{1}_{I'_j \setminus \bigcup_{i:i<j} I'_i}.$$

(Note that this example includes in particular the case when the priority order is purely random (choosing $P \equiv 0$), as well as the case when the priority is given to inclusions with e.g. larger or smaller radius (choosing $P(VS, W) = V$ or $-V$, respectively).)

In each of these three examples, $s = 1, 2, 3$, the random field A_s is $\sigma(\hat{\mathcal{P}}_s)$-measurable, for some completely independent random point process $\hat{\mathcal{P}}_s$ on $\mathbb{R}^d \times \mathbb{R}^l \times \mathbb{R}_+ \times Y_s$ and some measurable space Y_s (the set $\mathbb{R}^d\times\mathbb{R}^l$ stands for the domain of the point process $\mathcal{P}_0 = \{X_j\}_j$, and the set $\mathbb{R}_+$ stands for the domain of the radius variables $\{V_j\}_j$). In order to recast this into the framework of Section 2.2, we may define $\mathcal{X}_s(x, t, v) := \mathcal{P}_s|_{Q(x)\times Q(t)\times Q(v)\times Y_s}$, so that $\mathcal{X}_s$ is a completely independent measurable random field on the space $X = \mathbb{Z}^d \times \mathbb{Z}^l \times \mathbb{Z}$ with values in the space of (locally finite) measures on $Q^d \times Q^l \times Q^1 \times Y_s$.

Rather than stating a general result, we focus on the representative examples of the Poisson and of the random parking point processes. For the latter, a refined analysis is needed to avoid a loss of integrability. Note that the proof below yields slightly more general statements than contained in the proposition (and can easily be adapted to various other situations).

Proposition 3.4. *Set* $\gamma(v) := \mathbb{P}[v - 1/2 \le V < v + 1/2]$.

(i) Assume that $\mathcal{P} = \mathcal{P}_0$ *is a Poisson point process on* $\mathbb{R}^d$ *with constant intensity* μ *(hence* $l = 0$*). Then, for each* $s = 1, 2, 3$*, the above-defined random field* A_s *satisfies (*∂^{osc}*-WSG) and (*∂^{osc}*-WCI) with weight* $\ell \mapsto \mu\,(\ell + 1)^d \sup_{0\le u\le 2} \gamma(\frac{1}{\sqrt{d}}\ell - u)$*. In addition,*

for all $\lambda \in (0,1)$,

$$\begin{aligned}\mathrm{Cov}\left[Y(A_s);Z(A_s)\right] \\ \le \frac{(2\mu)^\lambda}{2}\sum_{x\in\mathbb{Z}^d}\sum_{v=0}^\infty \gamma(v)^\lambda \mathbb{E}\left[\left(\partial^{\mathrm{dis}}_{v+1,x,v}Y\right)^{\frac{2}{1-\lambda}}\right]^{\frac{1-\lambda}{2}}\mathbb{E}\left[\left(\partial^{\mathrm{dis}}_{v+1,x,v}Z\right)^{\frac{2}{1-\lambda}}\right]^{\frac{1-\lambda}{2}},\end{aligned} \tag{3.4}$$

where $\partial^{\mathrm{dis}}_{v+1,x,v}Z$ *is the notation defined in* (2.2), *that is,*

$$\begin{aligned}\partial^{\mathrm{dis}}_{v+1,x,v}Z &:= \big(Z(A_s)-Z(A_s(\mathcal{X}^{x,v})\big)\mathbb{1}_{A_s|_{\mathbb{R}^d\setminus Q_{2v+3}(x)}=A_s(\mathcal{X}^{x,v})|_{\mathbb{R}^d\setminus Q_{2v+3}(x)}} \\ &= Z(A_s)-Z(A_s(\mathcal{X}^{x,v})).\end{aligned}$$

In the case when the radius law V *is almost surely bounded by a deterministic constant, the standard logarithmic Sobolev inequality* (∂^{osc}-LSI) *holds.*

(ii) Assume that $\mathcal{P}$ *is a random parking point process on* $\mathbb{R}^d$ *with radius* $R>0$ *as constructed in Section 3.3. Then, for each* $s=1,2,3$, *the above-defined random field* A_s *satisfies* (∂^{osc}-WSG) *with weight* $\pi_R(\ell):=C_R\big(e^{-\ell/C_R}+(\ell+1)^d\gamma(\ell)\big)$. *More generally it satisfies the following covariance inequality: for all* $\sigma(A_s)$*-measurable random variables* $Y(A_s), Z(A_s)$ *we have*

$$\begin{aligned}\mathrm{Cov}\left[Y(A_s);Z(A_s)\right] \le \int_{\mathbb{R}^d}\left(\int_0^\infty \mathbb{E}\left[\left(\partial^{\mathrm{osc}}_{A_s,B_{2\ell+1}(x)}\,Y(A_s)\right)^2\right](\ell+1)^{-d}\pi_R(\ell)d\ell\right)^{\frac12} \\ \times\left(\int_0^\infty \mathbb{E}\left[\left(\partial^{\mathrm{osc}}_{A_s,B_{2\ell+1}(x)}\,Z(A_s)\right)^2\right](\ell+1)^{-d}\pi_R(\ell)d\ell\right)^{\frac12}dx.\end{aligned} \tag{3.5}$$

In addition, for all $\lambda\in(0,1)$,

$$\begin{aligned}\mathrm{Cov}\left[Y(A_s);Z(A_s)\right] \le \sum_{x\in\mathbb{Z}^d}\sum_{v=0}^\infty\left(\sum_{\ell=1}^\infty \pi_R(v,\ell)^\lambda\,\mathbb{E}\left[\left(\partial^{\mathrm{dis}}_{\ell,x,v}Y\right)^{\frac{2}{1-\lambda}}\right]^{1-\lambda}\right)^{\frac12} \\ \times\left(\sum_{\ell'=1}^\infty \pi_R(v,\ell')^\lambda\,\mathbb{E}\left[\left(\partial^{\mathrm{dis}}_{\ell',x,v}Z\right)^{\frac{2}{1-\lambda}}\right]^{1-\lambda}\right)^{\frac12},\end{aligned} \tag{3.6}$$

where we have set

$$\pi_R(v,\ell) := C_R\Big(\gamma(v)\mathbb{1}_{\ell-1\le v<\ell}+\gamma(v)\wedge\Big(e^{-\ell/C_R}+\sup_{r\ge\ell/2}\gamma(r)\Big)\Big),$$

and where $\partial^{\mathrm{dis}}_{\ell,x,v}Z$ *is the notation defined in* (2.2), *that is,*

$$\partial^{\mathrm{dis}}_{\ell,x,v}Z := \big(Z(A_s)-Z(A_s(\mathcal{X}^{x,v})\big)\mathbb{1}_{A_s|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}=A_s(\mathcal{X}^{x,v})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}}.$$

In the case when the radius law V *is almost surely bounded by a deterministic constant, the logarithmic Sobolev inequality* (∂^{osc}-WLSI) *holds with weight* $\ell\mapsto C_Re^{-\ell/C_R}$. □

Proof. We split the proof into three steps. We first apply the general results of Theorem 2.7, and then treat more carefully the case of the random parking point process in order to avoid the loss of integrability.

Step 1. Proof of the covariance estimates with loss.
Assume for simplicity that the transformation Φ of $\mathcal{P}_0$ into $\mathcal{P}=\Phi(\mathcal{P}_0)$ does not add points and does not move points in the direction of $\mathbb{R}^d$: more precisely, this means that

for any locally finite sequence $(x_j)_j \subset \mathbb{R}^d \times \mathbb{R}^l$ we have $\Phi((x_j)_j) = (p(x_j))_{j\in I}$ for some subset I of indices (depending on $(x_j)_j$), where $p : \mathbb{R}^d \times \mathbb{R}^l \to \mathbb{R}^d$ denotes the projection onto the first factor. Further assume that for all locally finite $(x_j)_j \subset \mathbb{R}^d \times \mathbb{R}^l$, denoting $\Phi((x_j)_j) = (p(x_j))_{j\in I}$, we have $\Phi((x_j)_{j\in J}) = (p(x_j))_{j\in I}$ for all subset $J \supset I$. In this step, we show that, for each $s = 1, 2, 3$, the random field A_s satisfies for all $\sigma(A^s)$-measurable random variables $Y(A_s), Z(A_s)$ and all $\lambda \in (0, 1)$,

$$\operatorname{Cov}[Y(A_s); Z(A_s)] \le \frac{1}{2} \sum_{x\in\mathbb{Z}^d} \sum_{v=0}^{\infty} \bigg(\sum_{\ell=1}^{\infty} \pi(v,\ell)^{\lambda}\, \mathbb{E}\bigg[\Big(\partial^{\text{dis}}_{\ell,x,v} Y\Big)^{\frac{2}{1-\lambda}} \bigg]^{1-\lambda} \bigg)^{\frac{1}{2}} \times \bigg(\sum_{\ell'=1}^{\infty} \pi(v,\ell')^{\lambda}\, \mathbb{E}\bigg[\Big(\partial^{\text{dis}}_{\ell',x,v} Z\Big)^{\frac{2}{1-\lambda}} \bigg]^{1-\lambda} \bigg)^{\frac{1}{2}}, \quad (3.7)$$

where we have set

$$\pi(v,\ell) := 2\big(\mathbb{E}[\sharp(\mathcal{P}\cap Q)] + 1\big) \times \Big(\gamma(v)\mathbb{1}_{\ell-1\le v<\ell} + \gamma(v) \wedge \mathbb{E}\big[\sharp(\mathcal{P}\cap Q_{2\rho_x+1}(x))\, \mathbb{P}[\ell-1 \le \rho_x + V < \ell \,\|\, \rho_x]\big] \Big),$$

and where $\partial^{\text{dis}}_{\ell,x,v} Z$ is the notation defined in (2.2). Applying this in the case of the random parking process together with Proposition 3.3, the weight becomes

$$\pi(v,\ell) \le C_R \Big(\gamma(v)\mathbb{1}_{\ell-1\le v<\ell} + \gamma(v) \wedge \int_0^{\ell} \gamma(\ell-r)\, e^{-r/C_R}\, dr \Big),$$

and estimating the last integral leads to the desired result (3.6).
Let $\mathcal{X}'_s$ denote an i.i.d. copy of the field $\mathcal{X}_s$, and let $\hat{\mathcal{P}}'_s := \{X'_j, V'_j, Y'_{j,s}\}_j$ denote the corresponding i.i.d. copy of $\hat{\mathcal{P}}_s := \{X_j, V_j, Y_{j,s}\}_j$. For all x, v, let the perturbations $\mathcal{X}^{x,v}_s$ and $\hat{\mathcal{P}}^{x,v}_s$ be then defined as usual, and let $\mathcal{P}^{x,v}_0$ be the corresponding projected point process on $\mathbb{R}^d \times \mathbb{R}^l$. Let us consider $J_v(x, r)$ the set of all indices j such that the projection $p(X_j)$ belongs to $(\Phi(\mathcal{P}_0) \cup \Phi(\mathcal{P}^{x,v}_0)) \cap (Q(x) + B_r) \setminus Q(x)$. Given the assumptions on the transformation Φ, an action radius for A_s with respect to $\mathcal{X}_s$ on $\{x\} \times \{v\}$ (or equivalently, with respect to $\hat{\mathcal{P}}_s$ on $Q(x) \times Q(v) \times Y_s$) is then given by

$$\rho^s_{x,v} := \big(v \vee (\rho_x + \max\{V_j : j \in J_v(x,\rho_x)\})\big)\, \mathbb{1}_{\mathcal{X}_s \ne \mathcal{X}^{x,v}_s}.$$

In order to prove (3.7), by Theorem 2.7(i), it remains to estimate the corresponding weights. First, for all $\ell \ge 0$, a union bound yields

$$\begin{aligned} &\mathbb{P}[\ell-1 \le \rho_x + \max\{V_j : j \in J_v(x,\rho_x)\} < \ell] \\ &\quad\le\ \mathbb{E}[\sharp J_v(x,\rho_x)\, \mathbb{P}[\ell-1 \le \rho_x + V < \ell \,\|\, \rho_x]] \\ &\quad\le\ \mathbb{E}[\sharp((\Phi(\mathcal{P}_0) \cup \Phi(\mathcal{P}^{x,v}_0)) \cap Q_{2\rho_x+1}(x))\, \mathbb{P}[\ell-1 \le \rho_x + V < \ell \,\|\, \rho_x]] \\ &\quad\le\ 2\,\mathbb{E}[\sharp(\mathcal{P} \cap Q_{2\rho_x+1}(x))\, \mathbb{P}[\ell-1 \le \rho_x + V < \ell \,\|\, \rho_x]]\,. \end{aligned}$$

Let us now define $I_v(x)$ as the set of all indices j such that either $p(X_j)$ or $p(X'_j)$ belongs to $(\Phi(\mathcal{P}_0) \cup \Phi(\mathcal{P}^{x,v}_0)) \cap Q(x)$. Given the assumptions on the transformation Φ, we may then compute, in terms of the probability law $\gamma(v) = \mathbb{P}[V \in Q(v)]$,

$$\begin{aligned} \mathbb{P}[A_s(\mathcal{X}^{x,v}_s) \ne A_s(\mathcal{X})] &\le \mathbb{P}[\exists j \in I_v(x) : V_j \in Q(v)] \\ &\le \gamma(v)\, \mathbb{E}[\sharp I_v(x)] \le 2\gamma(v)\, \mathbb{E}[\sharp(\mathcal{P} \cap Q)]\,. \end{aligned}$$

Combining the above estimates, we conclude

$$\begin{aligned}
&\mathbb{P}\left[\ell-1 \le \rho^s_{x,v} < \ell,\ A(\mathcal{X}^{x,v}_s) \neq A(\mathcal{X}_s)\right] \\
&\quad\le \ \big(2\gamma(v)\,\mathbb{E}\left[\sharp(\mathcal{P}\cap Q)\right]\big) \wedge \Big(\mathbb{1}_{\ell-1\le v<\ell} + \mathbb{P}\left[\ell-1 \le \rho_x + \max\{V_j : j \in J_v(x,\rho_x)\} < \ell\right]\Big) \\
&\quad\le \ 2\big(\mathbb{E}\left[\sharp(\mathcal{P}\cap Q)\right]+1\big) \\
&\qquad\qquad \times\Big(\gamma(v)\mathbb{1}_{\ell-1\le v<\ell} + \gamma(v)\wedge \mathbb{E}\left[\sharp(\mathcal{P}\cap Q_{2\rho_x+1}(x))\,\mathbb{P}\left[\ell-1\le \rho_x + V < \ell \,\|\, \rho_x\right]\right]\Big).
\end{aligned}$$

The result (3.7) then follows from Theorem 2.7(i).

Step 2. Proof of (i).
We repeat the analysis of Step 1 in the particular case of a Poisson point process $\mathcal{P}=\mathcal{P}_0$ on $\mathbb{R}^d$ with constant intensity $\mu>0$. In this case, we have $\rho_x=0$, hence $J_v(x,r)=\varnothing$, so that the action radius $\rho^s_{x,v}$ takes the simpler form

$$\rho^s_{x,v} = v\,\mathbb{1}_{\mathcal{X}_s\neq\mathcal{X}^{x,v}_s}.$$

Estimating

$$\begin{aligned}
\mathbb{P}\left[\ell-1\le\rho^s_{x,v}<\ell,\ A_s(\mathcal{X}^{x,v}_s)\neq A_s(\mathcal{X}_s)\right] &\le \mathbb{P}\left[\ell-1\le\rho^s_{x,v}<\ell,\ \mathcal{X}^{x,v}_s\neq\mathcal{X}_s\right] \\
&\le \mathbb{P}\left[\mathcal{X}^{x,v}_s\neq\mathcal{X}_s\right]\mathbb{1}_{\ell-1\le v<\ell} \\
&\le 2\mu\gamma(v)\,\mathbb{1}_{\ell-1\le v<\ell},
\end{aligned}$$

the conclusion (3.4) follows from Theorem 2.7(i). It remains to prove (∂^{osc}-WCI). Since obviously $\mathbb{P}\big[\rho^s_{x,v}<\ell\big]=1$ if $v<\ell$, we compute for all $x\in\mathbb{Z}^d$, $v\ge0$, $\ell\ge1$,

$$\frac{\mathbb{P}\left[\ell-1\le\rho^s_{x,v}<\ell,\ \mathcal{X}^{x,v}_s\neq\mathcal{X}\right]}{\mathbb{P}\left[\rho^s_{x,v}<\ell\right]} \le \frac{2\mu\gamma(v)\,\mathbb{1}_{\ell-1\le v<\ell}}{\mathbb{P}\left[\rho^s_{x,v}<\ell\right]} = 2\mu\gamma(v)\,\mathbb{1}_{\ell-1\le v<\ell},$$

and Theorem 2.7(ii) with influence function $f(u)=u$ then yields

$$\begin{aligned}
&\mathrm{Cov}\left[Y(A_s);Z(A_s)\right] \\
&\qquad\le \mu\sum_{x\in\mathbb{Z}^d}\sum_{v=0}^\infty \gamma(v)\,\mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A_s,Q_{2v+3}(x)}\,Y(A_s)\Big)^2\right]^{\frac12}\mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A_s,Q_{2v+3}(x)}\,Z(A_s)\Big)^2\right]^{\frac12}.
\end{aligned}$$

The desired covariance estimate (∂^{osc}-WCI) follows by taking local averages.

Step 3. Proof of (ii).
In this step, we consider the case when the stationary point process $\mathcal{P}$ satisfies a hard-core condition $\sharp(\mathcal{P}\cap Q)\le C$ a.s. for some deterministic constant $C>0$, and also satisfies the following covariance inequality (resp. the corresponding (∂^{osc}-WSG)) with some integrable weight π_0: for all $\sigma(\mathcal{P})$-measurable random variables $Y(\mathcal{P}), Z(\mathcal{P})$,

$$\begin{aligned}
\mathrm{Cov}\left[Y(\mathcal{P});Z(\mathcal{P})\right] \le \int_{\mathbb{R}^d}&\bigg(\int_0^\infty \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{\mathcal{P},B_{\ell+1}(x)}\,Y(\mathcal{P})\Big)^2\right](\ell+1)^{-d}\pi_0(\ell)d\ell\bigg)^{\frac12} \\
&\times\bigg(\int_0^\infty \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{\mathcal{P},B_{\ell+1}(x)}\,Z(\mathcal{P})\Big)^2\right](\ell+1)^{-d}\pi_0(\ell)d\ell\bigg)^{\frac12}dx,
\end{aligned}$$

We then show that, for each $s = 1, 2, 3$, the random field A_s satisfies the following covariance inequality (resp. the corresponding ($\partial^{\rm osc}$-WSG)): for all $\sigma(A_s)$-measurable random variables $Y(A_s), Z(A_s)$ we have

$$\mathrm{Cov}\left[Y(A_s); Z(A_s)\right] \le C \int_{\mathbb{R}^d} \left(\int_0^\infty \mathbb{E}\left[\left(\partial^{\rm osc}_{A_s, B_{2\ell+1}(x)} Y(A_s)\right)^2\right] (\ell+1)^{-d} \pi(\ell) d\ell\right)^{\frac12} \times \left(\int_0^\infty \mathbb{E}\left[\left(\partial^{\rm osc}_{A_s, B_{2\ell+1}(x)} Z(A_s)\right)^2\right] (\ell+1)^{-d} \pi(\ell) d\ell\right)^{\frac12} dx, \quad (3.8)$$

where we have set $\pi(\ell) := \pi_0(\ell) + (\ell+1)^d\, \mathbb{P}\left[\ell - 1 \le V < \ell\right]$. In particular, combined with Proposition 3.3, this implies the covariance inequality (3.5) in the case of the random parking point process.

To simplify notation, we only treat the case of the spectral gap inequality. Consider a measurable enumeration of the point process $\mathcal{P} = \{Z_j\}_j$, let $\{Z_j, V_j, Y_{s,j}\}$ be a decorated point process associated with $\mathcal{P}$ and the decoration law (V, Y_s), and let $\mathcal{D} := \{V_j, Y_{s,j}\}_j$ denote the decoration sequence. Since $\mathcal{P}$ and $\mathcal{D}$ are independent, the expectation $\mathbb{E}$ splits into $\mathbb{E} = \mathbb{E}_{\mathcal{P}} \mathbb{E}_{\mathcal{D}}$, where $\mathbb{E}_{\mathcal{P}}[\cdot] = \mathbb{E}[\cdot \| \mathcal{D}]$ denotes the expectation with respect to $\mathcal{P}$, and where $\mathbb{E}_{\mathcal{D}}[\cdot] = \mathbb{E}[\cdot \| \mathcal{P}]$ denotes the expectation with respect to $\mathcal{D}$. By tensorization of the variance as in (3.15), the spectral gap assumption for $\mathcal{P}$ and the standard spectral gap (2.3) for the i.i.d. sequence $\mathcal{D}$ then yields for all random variables $Z = Z(A_s)$,

$$\begin{aligned}\mathrm{Var}\left[Z(A_s)\right] &= \mathbb{E}_{\mathcal{P}}\big[\mathrm{Var}_{\mathcal{D}}[Z(A_s)]\big] + \mathrm{Var}_{\mathcal{P}}\left[\mathbb{E}_{\mathcal{D}}[Z(A_s)]\right] \\ &\le \frac12 \sum_k \mathbb{E}\left[\big(Z(A_s) - Z(A_s^k)\big)^2\right] \\ &\qquad + \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\partial^{\rm osc}_{\mathcal{P}, B_{\ell+1}(x)} \mathbb{E}_{\mathcal{D}}[Z(A_s)]\right)^2\right] dx\, (\ell+1)^{-d} \pi_0(\ell) d\ell, \quad (3.9)\end{aligned}$$

where A_s^k corresponds to the field A_s with the decoration $(V_k, Y_{s,k})$ replaced by an i.i.d. copy $(V_k', Y_{s,k}')$. We separately estimate the two RHS terms in (3.9), and we begin with the first. For all $x \in \mathbb{R}^d$, we define the following two random variables,

$$N(x) := \sharp(\mathcal{P} \cap B(x)), \qquad R(x) := \max\{V_j : Z_j \in B(x)\}.$$

Let $R_0 \ge 1$ denote the smallest value such that $\mathbb{P}\left[V < R_0\right] \ge \frac12$, which implies in particular by a union bound and by the hard-core assumption

$$\mathbb{P}\left[R(x) < R_0\right] = \mathbb{E}\left[\mathbb{P}\left[V < R_0\right]^{N(x)}\right] \ge \mathbb{E}\left[2^{-N(x)}\right] \ge 2^{-C}. \quad (3.10)$$

Conditioning with respect to the value of $R(x)$, we obtain

$$\begin{aligned}
&\sum_k \mathbb{E}\left[\big(Z(A_s)-Z(A_s^k)\big)^2\right] \\
&\quad\lesssim \int_{R_0}^\infty \int_{\mathbb{R}^d} \sum_k \mathbb{E}\left[\big(Z(A_s)-Z(A_s^k)\big)^2 \mathbb{1}_{Z_k\in B(x)} \mathbb{1}_{\ell-1\le R(x)<\ell}\right] dx\, d\ell \\
&\qquad\qquad + \int_{\mathbb{R}^d} \sum_k \mathbb{E}\left[\big(Z(A_s)-Z(A_s^k)\big)^2 \mathbb{1}_{Z_k\in B(x)} \mathbb{1}_{R(x)<R_0}\right] dx \\
&\quad\le \int_{R_0}^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\text{osc}}_{A_s,B_{\ell+1}(x)}\, Z(A_s)\Big)^2 N(x)\, \mathbb{1}_{\ell-1\le R(x)<\ell}\right] dx\, d\ell \\
&\qquad\qquad + \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\text{osc}}_{A_s,B_{R_0+1}(x)}\, Z(A_s)\Big)^2 N(x)\right] dx \\
&\quad= \int_{R_0}^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\text{osc}}_{A_s,B_{\ell+1}(x)}\, Z(A_s)\Big)^2 N(x)\, \mathbb{1}_{R(x)\ge\ell-1} \,\Big\|\, R(x)<\ell\right] \mathbb{P}\left[R(x)<\ell\right] dx\, d\ell \\
&\qquad\qquad + \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\text{osc}}_{A_s,B_{R_0+1}(x)}\, Z(A_s)\Big)^2 N(x)\right] dx.
\end{aligned}$$

Using the hard-core assumption in the form $N(x)\le C$ a.s., and noting that given $R(x)<\ell$ the random variable $R(x)$ is independent of $A_s|_{\mathbb{R}^d\setminus B_{\ell+1}(x)}$, we deduce

$$\begin{aligned}
\sum_k \mathbb{E}\left[\big(Z(A_s)-Z(A_s^k)\big)^2\right] \lesssim \int_{R_0}^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\text{osc}}_{A_s,B_{\ell+1}(x)}\, Z(A_s)\Big)^2\right] \frac{\mathbb{P}\left[\ell-1\le R(x)<\ell\right]}{\mathbb{P}\left[R(x)<\ell\right]} dx\, d\ell \\
+ \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\text{osc}}_{A_s,B_{R_0+1}(x)}\, Z(A_s)\Big)^2\right] dx.
\end{aligned}$$

Estimating by a union bound $\mathbb{P}\left[\ell-1\le R(x)<\ell\right]\le C\,\mathbb{P}\left[\ell-1\le V<\ell\right]$, and making use of the property (3.10) of the choice of $R_0\ge 1$, we conclude

$$\begin{aligned}
\sum_k \mathbb{E}\left[\big(Z(A_s)-Z(A_s^k)\big)^2\right] \lesssim \int_{R_0}^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\text{osc}}_{A_s,B_{\ell+1}(x)}\, Z(A_s)\Big)^2\right] \mathbb{P}\left[\ell-1\le V<\ell\right] dx\, d\ell \\
+ \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\text{osc}}_{A_s,B_{R_0+1}(x)}\, Z(A_s)\Big)^2\right] dx. \qquad (3.11)
\end{aligned}$$

It remains to estimate the second RHS term in (3.9). The hard-core assumption for $\mathcal{P}$ yields by stationarity $\sharp(\mathcal{P}\cap B_\ell(x))\le C\ell^d$ a.s. Further noting that a union bound gives

$$\begin{aligned}
\mathbb{P}\left[r-1\le \max_{1\le j\le C\ell^d} V_j<r\right] &\le \sum_{j=1}^{C\ell^d} \mathbb{P}\left[V_j\ge r-1, \text{ and } V_k<r\ \forall 1\le k\le C\ell^d\right] \\
&= C\ell^d\, \mathbb{P}\left[V<r\right]^{C\ell^d-1}\, \mathbb{P}\left[r-1\le V<r\right],
\end{aligned}$$

and hence for all $r\ge R_0$,

$$\frac{\mathbb{P}\left[r-1\le \max_{1\le j\le C\ell^d} V_j<r\right]}{\mathbb{P}\left[\max_{1\le j\le C\ell^d} V_j<r\right]} \le C\ell^d \frac{\mathbb{P}\left[r-1\le V<r\right]}{\mathbb{P}\left[V<r\right]} \le 2C\ell^d\, \mathbb{P}\left[r-1\le V<r\right],$$

we find, arguing similarly as above,

$$\int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{\mathcal{P},B_\ell(x)} \, \mathbb{E}_{\mathcal{D}}[Z(A_s)]\Big)^2\right] dx\, (\ell+1)^{-d}\pi_0(\ell)d\ell$$
$$\lesssim \int_0^\infty \int_{R_0}^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A_s,B_{\ell+r}(x)} \, Z(A_s)\Big)^2\right] dx\, \mathbb{P}\left[r-1\le V<r\right] dr\, \pi_0(\ell)d\ell$$
$$+\int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A_s,B_{\ell+R_0}(x)} \, Z(A_s)\Big)^2\right] dx\, \pi_0(\ell)d\ell. \quad (3.12)$$

Combining this with (3.9) and (3.11), the conclusion (3.8) follows in variance form. □

3.5. **Dependent coloring of random geometric patterns.** Up to here, besides Gaussian random fields, all the examples of random fields that we have been considering corresponded to random geometric patterns (various random point processes constructed from a higher-dimensional Poisson process or random tessellations) endowed with an independent coloring determining e.g. the size and shape of the cells and the value of the field in the cells. In the present subsection, we turn to the examples of type (III) mentioned at the end of the introduction, and consider *dependent* colorings of the random geometric patterns. The random field A is now a function of both a product structure (typically some decorated Poisson point process $\hat{\mathcal{P}}$), and of a random field G (e.g. a Gaussian random field) which typically has long-range correlations but is assumed to satisfy some weighted functional inequality. In other words, this amounts to mixing up all the previous examples. Rather than stating general results in this direction, we only treat a number of typical concrete examples in order to illustrate the robustness of the approach.

(1) The first example A_1 is a random field on $\mathbb{R}^d$ corresponding to random spherical inclusions centered at the points of a Poisson point process $\mathcal{P}$ of intensity $\mu=1$, with i.i.d. random radii of law V, but such that the values on the inclusions are determined by some random field G_1 with long-range correlations.
More precisely, we let $\hat{\mathcal{P}}_1 := \{\tilde X_j, \tilde V_j, \tilde U_j\}_j$ denote a decorated point process associated with $\mathcal{P}$ and (V,U), where U denotes an independent uniform random variable on $[0,1]$. Independently of $\hat{\mathcal{P}}_1$ we choose a jointly measurable stationary bounded random field G_1 on $\mathbb{R}^d$, with typically long-range correlations. The collection of random inclusions is given by $\{\tilde I_1^j\}_j$ with $\tilde I_1^j := \tilde X_j + \tilde V_j B$. As in the third example of Section 3.4, we choose a $\sigma(V,U)$-measurable random variable $P(V,U)$, and we say that the inclusion $\tilde I_1^j$ has the priority on inclusion $\tilde I_1^i$ if $P(\tilde V_j,\tilde U_j) < P(\tilde V_i,\tilde U_i)$ or if $P(\tilde V_j,\tilde U_j) = P(\tilde V_i,\tilde U_i)$ and $\tilde U_j < \tilde U_i$. This defines a priority order on the inclusions on a set of maximal probability, and we then relabel the inclusions and the points of $\hat{\mathcal{P}}_1$ into a sequence $(I_1^j, X_j, V_j, U_j)_j$ such that for all j the inclusion I_1^j has the j-th highest priority. Given $\beta\in\mathbb{R}$, we then consider the $\sigma(\hat{\mathcal{P}}_1, G_1)$-measurable random field A_1 defined as follows,

$$A_1 := \beta + \sum_j \big(G_1(X_j)-\beta\big)\mathbb{1}_{I_1^j\setminus\bigcup_{i:i<j} I_1^i}.$$

(2) The second example A_2 is a random field on $\mathbb{R}^d$ corresponding to random inclusions centered at the points of a Poisson point process $\mathcal{P}$ of intensity $\mu=1$, with i.i.d. random radii of law V, but with orientations determined by some random field G_2 with long-range correlations.
More precisely, we let $\hat{\mathcal{P}}_2 := \{X_j, V_j\}_j$ denote a decorated point process associated with

$\mathcal{P}$ and V, we choose a reference shape $S \in \mathcal{B}(\mathbb{R}^d)$ with $0 \in S$, and independently of $\hat{\mathcal{P}}_2$ we choose a jointly measurable stationary bounded random field G_2 on $\mathbb{R}^d$ with values in the orthogonal group $O(d)$ in dimension d, and with typically long-range correlations. The collection of random inclusions is then given by $\{I_2^j\}_j$ with $I_2^j := X_j + G_2(X_j)S$. Given $\alpha, \beta \in \mathbb{R}$, and given a function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(t) = 1$ for $t \le 1$ and $\phi(t) = 0$ for $t \ge 2$, and with $\|\phi'\|_{\mathrm{L}^\infty} \lesssim 1$, we then consider the $\sigma(\hat{\mathcal{P}}_2, G_2)$-measurable random field A_2 defined as follows,

$$A_2(x) := \beta + (\alpha - \beta)\, \phi\Big(d\big(x\,,\, \cup_j I_2^j\big)\Big).$$

(Note that the smoothness of this interpolation ϕ between the values α and β is crucial for the arguments below.)

(3) The third example A_3 is a random field on $\mathbb{R}^d$ corresponding to the Voronoi tessellation associated with the points of a Poisson point process $\mathcal{P}$ of unit intensity, such that the values on the cells are determined by some random field G_3 with long-range correlations.
More precisely, we let $\hat{\mathcal{P}}_3 := \mathcal{P} = \{X_j\}_j$, and we let $\{C_j\}_j$ denote the partition of $\mathbb{R}^d$ into the Voronoi cells associated with the Poisson points $\{X_j\}_j$. Independently of $\hat{\mathcal{P}}_3$ we choose a jointly measurable stationary bounded random field G_3 on $\mathbb{R}^d$. We then consider the $\sigma(\hat{\mathcal{P}}_3, G_3)$-measurable random field A_3 defined as follows,

$$A_3(x) := \sum_j G_3(X_j) \mathbb{1}_{C_j}.$$

For each of these examples, we show functional inequalities with as derivative the supremum of the functional derivative ∂^{fct}, which we define as

$$\partial^{\sup}_{A,S} X(A) \ := \sup_{A,S} \mathrm{ess} \int_S \Big| \frac{\partial \tilde{X}(A)}{\partial A} \Big|.$$

Note that provided A is bounded we have $\partial^{\mathrm{osc}}, \partial^{\mathrm{fct}} \lesssim \partial^{\sup}$. From the proofs in the companion article [7], it is clear that weighted functional inequalities with $\partial^{\sup}$ imply the same concentration properties as the corresponding functional inequalities with ∂^{osc}.

Proposition 3.5. *For $s = 1, 2, 3$, assume that the random field G_s satisfies* (∂^{fct}-WSG) *for some integrable weight π_s. For $s = 1, 2$, set $\gamma(v) := \mathbb{P}\,[v - 4 \le V < v + 4]$. Then the following holds.*

(i) For $s = 1, 2$, the above-defined random field A_s satisfies the following weighted spectral gap: for all $\sigma(A_s)$-measurable random variable $Z(A_s)$ we have

$$\begin{aligned} &\mathrm{Var}\left[Z(A_s)\right] \\ &\lesssim \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\sup}_{A,B_{\ell+C(v+1)}(x)} Z(A_s)\Big)^2\right] dx \big((\ell+1)^{-d} \wedge \gamma(v)\big) \pi_s(\ell) dv d\ell. \quad (3.13) \end{aligned}$$

In the case when the random variable V is almost surely bounded by a deterministic constant, we rather obtain

$$\operatorname{Var}[Z(A_s)] \lesssim \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\text{osc}}_{A_s,B_C(x)} Z(A_s)\Big)^2\right] dx + \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\text{fct}}_{A_s,B_{\ell+C}(x)} Z(A_s)\Big)^2\right] dx\, (\ell+1)^{-d}\pi_s(\ell)d\ell, \quad (3.14)$$

and if the random field G_s further satisfies (∂^{fct}-WLSI) *with weight π_s, then the corresponding logarithmic Sobolev inequality also holds, that is,*

$$\operatorname{Ent}[Z(A_s)] \lesssim \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\text{osc}}_{A_s,B_C(x)} Z(A_s)\Big)^2\right] dx + \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial^{\text{fct}}_{A_s,B_{\ell+C}(x)} Z(A_s)\Big)^2\right] dx\, (\ell+1)^{-d}\pi_s(\ell)d\ell.$$

(ii) The above-defined random field A_3 satisfies (∂^{sup}-WSG) *with weight $\pi(\ell) := C(\pi_3(\ell)+ e^{-\frac{1}{C}\ell^d})$. If the random field G_3 further satisfies* (∂^{fct}-WLSI) *with weight π_3, then A_3 also satisfies* (∂^{sup}-WLSI) *with weight π.* □

The proof of Proposition 3.5 is quite robust and many variants of the above results could be considered.

Proof. For $s = 1, 2, 3$, since $\hat{\mathcal{P}}_s$ and G_s are independent, the expectation $\mathbb{E}$ splits into $\mathbb{E} = \mathbb{E}_{\hat{\mathcal{P}}_s}\mathbb{E}_{G_s}$, where $\mathbb{E}_{\hat{\mathcal{P}}_s}[\cdot] = \mathbb{E}[\cdot \| G_s]$ denotes the expectation with respect to $\hat{\mathcal{P}}_s$, and where $\mathbb{E}_{G_s}[\cdot] = \mathbb{E}[\cdot\|\hat{\mathcal{P}}_s]$ denotes the expectation with respect to G_s. The variance and the entropy also tensorize: for all $\sigma(A_s)$-measurable random variables $Z(A_s)$,

$$\operatorname{Var}[Z(A_s)] = \operatorname{Var}_{G_s}[\mathbb{E}_{\hat{\mathcal{P}}_s}[Z(A_s)]] + \mathbb{E}_{G_s}[\operatorname{Var}_{\hat{\mathcal{P}}_s}[Z(A_s)]], \quad (3.15)$$
$$\operatorname{Ent}[Z(A_s)] = \operatorname{Ent}_{G_s}[\mathbb{E}_{\hat{\mathcal{P}}_s}[Z(A_s)]] + \mathbb{E}_{G_s}[\operatorname{Ent}_{\hat{\mathcal{P}}_s}[Z(A_s)]].$$

In each of the examples under consideration, the estimate on the terms $\operatorname{Var}_{\hat{\mathcal{P}}_s}[Z(A_s)]$ and $\operatorname{Ent}_{\hat{\mathcal{P}}_s}[Z(A_s)]$ (with G_s "frozen") follows from the same arguments as in the proof of Propositions 3.2 and 3.4(i). We therefore focus on the estimates of $\operatorname{Var}_{G_s}[\mathbb{E}_{\hat{\mathcal{P}}_s}[Z(A_s)]]$ and $\operatorname{Ent}_{G_s}[\mathbb{E}_{\hat{\mathcal{P}}_s}[Z(A_s)]]$, and only treat the case of the variance in the proof.

Since the random field G_s is assumed to satisfy (∂^{fct}-WSG) with weight π_s, we obtain

$$\operatorname{Var}_{G_s}[\mathbb{E}_{\hat{\mathcal{P}}_s}[Z(A_s)]] \le \mathbb{E}_{\hat{\mathcal{P}}_s}[\operatorname{Var}_{G_s}[Z(A_s)]] \le \mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \Big(\partial^{\text{fct}}_{G_s,B_{\ell+1}(x)} Z(A_s)\Big)^2 dx\, (\ell+1)^{-d}\pi_s(\ell)d\ell\right]. \quad (3.16)$$

The chain rule yields

$$\partial^{\text{fct}}_{G_s,B_{\ell+1}(x)} Z(A_s) = \int_{B_{\ell+1}(x)} \Big|\frac{\partial Z(A_s(\hat{\mathcal{P}}_s, G_s))}{\partial G_s}(y)\Big| dy \le \int_{B_{\ell+1}(x)} \int_{\mathbb{R}^d} \Big|\frac{\partial Z(A_s)}{\partial A_s}(z)\Big|\Big|\frac{\partial A_s(\hat{\mathcal{P}}_s, G_s)(z)}{\partial G_s}(y)\Big| dz dy.$$

Since A_s is $\sigma(\hat{\mathcal{P}}_s, \{G_s(X_j)\}_j)$-measurable, we obtain

$$\partial^{\rm fct}_{G_s,B_{\ell+1}(x)} Z(A_s) \le \sum_j \mathbb{1}_{X_j\in B_{\ell+1}(x)} \int_{\mathbb{R}^d} \Big|\frac{\partial Z(A_s)}{\partial A_s}(z)\Big|\Big|\frac{\partial A_s(\hat{\mathcal{P}}_s,G_s)(z)}{\partial G_s(X_j)}\Big| dz \tag{3.17}$$

in terms of the usual partial derivative of $A_s(\hat{\mathcal{P}}_s,G_s)(z)$ with respect to $G_s(X_j)$. We now need to compute this derivative in each of the considered examples. We claim that

$$\Big|\frac{\partial A_s(\hat{\mathcal{P}}_s,G_s)(z)}{\partial G_s(X_j)}\Big| \le C\mathbb{1}_{R_s^j}(z), \tag{3.18}$$

where

$$R_s^j := \begin{cases} I_1^j \setminus \bigcup_{i:i<j} I_1^i, & \text{if } s=1;\\ \big\{x \,:\, 0<d(x,I_2^j)<2\wedge d(x,I_2^k),\, \forall k\ne j\big\}, & \text{if } s=2;\\ C_j, & \text{if } s=3. \end{cases}$$

This claim (3.18) is obvious for $s=1$ and $s=3$. For $s=2$, the properties of ϕ and the definition of R_2^j yield

$$\Big|\frac{\partial A_2(\hat{\mathcal{P}}_2,G_2)(z)}{\partial G_2(X_j)}\Big| \le |\alpha-\beta|\Big|\phi'\Big(d\big(z\,,\,\cup_k I_2^k\big)\Big)\Big|\mathbb{1}_{R_2^j}(z) = |\alpha-\beta|\big|\phi'\big(d(z,I_2^j)\big)\big|\mathbb{1}_{R_2^j}(z),$$

which indeed implies (3.18). Now injecting (3.18) into (3.17), and noting that in each case the sets $\{R_s^j\}_j$ are disjoint, we obtain

$$\begin{aligned} \partial^{\rm fct}_{G_s,B_{\ell+1}(x)} Z(A_s) &\le C\sum_j \mathbb{1}_{X_j\in B_{\ell+1}(x)} \int_{R_s^j} \Big|\frac{\partial Z(A_s)}{\partial A_s}\Big| = C\int_{\bigcup_{j:X_j\in B_{\ell+1}(x)} R_s^j} \Big|\frac{\partial Z(A_s)}{\partial A_s}\Big| \\ &\le C\int_{B_{D_s(\ell,x)}(x)} \Big|\frac{\partial Z(A_s)}{\partial A_s}\Big|, \end{aligned} \tag{3.19}$$

with

$$D_s(\ell,x) := \sup\Big\{d(y,x)\,:\, y\in \bigcup_{j:X_j\in B_{\ell+1}(x)} R_s^j\Big\}.$$

For $s=1,2$ with radius law V almost surely bounded by a deterministic constant $R>0$, we obtain $D_1(\ell,x)\le \ell+R+1$ and $D_2(\ell,x)\le \ell+R+3$, and injecting (3.19) into (3.16) directly yields the result (3.14).

We now consider the cases $s=1,2$ with general unbounded radii. Without loss of generality we only treat $s=1$, in which case

$$D_1(\ell,x)\le \ell+1+\bar{D}_1(\ell,x), \qquad \bar{D}_1(\ell,x) := \max\big\{V_j \,:\, X_j\in B_{\ell+1}(x)\big\}.$$

Noting that the restriction $A_1|_{\mathbb{R}^d\setminus B_{\ell+1+\bar{D}_1(\ell,x)}(x)}$ is by construction independent of $\bar{D}_1(\ell,x)$, we obtain, conditioning on the values of $\bar{D}_1(\ell,x)$ and arguing as in Step 2 of the proof of

Theorem 2.7,

$$\begin{aligned}
&\mathbb{E}\bigg[\Big(\int_{B_{\ell+1+\bar{D}_1(\ell,x)}(x)}\Big|\frac{\partial Z(A_1)}{\partial A_1}\Big|\Big)^2\bigg] \\
&\le \sum_{i=1}^{\infty}\mathbb{P}\big[i-1\le \bar{D}_1(\ell,x)<i\big]\mathbb{E}\bigg[\Big(\int_{B_{\ell+i+1}(x)}\Big|\frac{\partial Z(A_1)}{\partial A_1}\Big|\Big)^2\,\bigg\|\,i-1\le \bar{D}_1(\ell,x)<i\bigg] \\
&\le \sum_{i=1}^{\infty}\mathbb{P}\big[i-1\le \bar{D}_1(\ell,x)<i\big]\mathbb{E}\bigg[\operatorname*{sup\,ess}_{A_1,B_{\ell+i+1}(x)}\Big(\int_{B_{\ell+i+1}(x)}\Big|\frac{\partial Z(A_1)}{\partial A_1}\Big|\Big)^2\,\bigg\|\,\bar{D}_1(\ell,x)<i\bigg] \\
&\le \sum_{i=1}^{\infty}\frac{\mathbb{P}\big[i-1\le \bar{D}_1(\ell,x)<i\big]}{\mathbb{P}\big[\bar{D}_1(\ell,x)<i\big]}\mathbb{E}\bigg[\operatorname*{sup\,ess}_{A_1,B_{\ell+i+1}(x)}\Big(\int_{B_{\ell+i+1}(x)}\Big|\frac{\partial Z(A_1)}{\partial A_1}\Big|\Big)^2\bigg]. \qquad (3.20)
\end{aligned}$$

Now by definition of the decorated Poisson point process $\hat{\mathcal{P}}_1$, we may compute for all $i\ge 1$,

$$\begin{aligned}
\mathbb{P}\big[\bar{D}_1(\ell,x)\ge i-1\big] &= \mathbb{P}\big[\exists j\,:\,V_j\ge i-1 \text{ and } X_j\in B_{\ell+1}(x)\big] \\
&= e^{-|B_{\ell+1}|}\sum_{n=0}^{\infty}\frac{|B_{\ell+1}|^n}{n!}\big(1-(1-\mathbb{P}\left[V\ge i-1\right])^n\big) \\
&= 1-e^{-|B_{\ell+1}|\,\mathbb{P}[V\ge i-1]},
\end{aligned}$$

and hence

$$\frac{\mathbb{P}\big[i-1\le \bar{D}_1(\ell,x)<i\big]}{\mathbb{P}\big[\bar{D}_1(\ell,x)<i\big]} = 1-e^{-|B_{\ell+1}|\mathbb{P}[i-1\le V<i]}\le 1\wedge\big(C(\ell+1)^d\,\mathbb{P}\left[i-1\le V<i\right]\big).$$

Combining this computation with (3.16), (3.19) and (3.20), we obtain, setting $\gamma(v):=\mathbb{P}\left[v-2\le V<v+1\right]$,

$$\begin{aligned}
&\mathrm{Var}_{G_1}[\mathbb{E}_{\hat{\mathcal{P}}_1}[Z(A_1)]] \\
&\lesssim \mathbb{E}\bigg[\int_0^\infty\sum_{i=1}^{\infty}\int_{\mathbb{R}^d}\operatorname*{sup\,ess}_{A_1,B_{\ell+i+1}(x)}\Big(\int_{B_{\ell+i+1}(x)}\Big|\frac{\partial Z(A_1)}{\partial A_1}\Big|\Big)^2 dx \\
&\qquad\qquad \times\big((\ell+1)^{-d}\wedge\mathbb{P}\left[i-1\le V<i\right]\big)\pi_s(\ell)d\ell\bigg] \\
&\le \mathbb{E}\bigg[\int_0^\infty\int_0^\infty\int_{\mathbb{R}^d}\operatorname*{sup\,ess}_{A_1,B_{\ell+v+2}(x)}\Big(\int_{B_{\ell+v+2}(x)}\Big|\frac{\partial Z(A_1)}{\partial A_1}\Big|\Big)^2 dx\,\big((\ell+1)^{-d}\wedge\gamma(v)\big)\pi_s(\ell)dvd\ell\bigg],
\end{aligned}$$

and the conclusion (3.13) follows.

We now turn to the case $s=3$, for which

$$D_3(\ell,x)\le \ell+1+\bar{D}_3(\ell,x),\qquad \bar{D}_3(\ell,x):=\max\big\{\,\mathrm{diam}(C_j)\,:\,X_j\in B_{\ell+1}(x)\big\}.$$

Noting that the restriction $A_3|_{\mathbb{R}^d\setminus B_{\ell+1+2\bar D_3(\ell,x)}(x)}$ is by construction independent of $\bar D_3(\ell,x)$ we obtain, after conditioning on the values of $\bar D_3(\ell,x)$ and arguing as in (3.20),

$$\mathbb{E}\bigg[\Big(\int_{B_{\ell+1+\bar D_3(\ell,x)}(x)}\Big|\frac{\partial Z(A_3)}{\partial A_3}\Big|\Big)^2\bigg]\le\mathbb{E}\bigg[\sup_{A_3,B_{3\ell+1}(x)}\!\!\text{ess}\,\Big(\int_{B_{3\ell+1}(x)}\Big|\frac{\partial Z(A_3)}{\partial A_3}\Big|\Big)^2\bigg]\\+\sum_{i=2\ell}^{\infty}\frac{\mathbb{P}\big[i-1\le\bar D_3(\ell,x)<i\big]}{\mathbb{P}\big[\bar D_3(\ell,x)<i\big]}\mathbb{E}\bigg[\sup_{A_3,B_{\ell+i+1}(x)}\!\!\text{ess}\,\Big(\int_{B_{\ell+i+1}(x)}\Big|\frac{\partial Z(A_3)}{\partial A_3}\Big|\Big)^2\bigg].\qquad(3.21)$$

Similar computations as in Step 2 of the proof of Proposition 3.2 yield

$$\mathbb{P}\big[\bar D_3(\ell,x)\ge i\big]\le Ce^{-\frac1C(i-\ell)_+^d}.$$

Combining this with (3.16), (3.19) and (3.21), we obtain

$$\begin{aligned}&\mathrm{Var}_{G_3}[\mathbb{E}_{\hat{\mathcal{P}}_3}[Z(A_3)]]\\&\lesssim\ \mathbb{E}\bigg[\int_0^\infty\int_{\mathbb{R}^d}\sup_{A_3,B_{3\ell+1}(x)}\!\!\text{ess}\,\Big(\int_{B_{3\ell+1}(x)}\Big|\frac{\partial Z(A_3)}{\partial A_3}\Big|\Big)^2dx\,(\ell+1)^{-d}\pi_3(\ell)d\ell\bigg]\\&\qquad+\mathbb{E}\bigg[\int_0^\infty\sum_{i=2\ell}^\infty e^{-\frac1C i^d}\int_{\mathbb{R}^d}\sup_{A_3,B_{2i+1}(x)}\!\!\text{ess}\,\Big(\int_{B_{2i+1}(x)}\Big|\frac{\partial Z(A_3)}{\partial A_3}\Big|\Big)^2dx\,(\ell+1)^{-d}\pi_3(\ell)d\ell\bigg]\\&\lesssim\ \mathbb{E}\bigg[\int_0^\infty\int_{\mathbb{R}^d}\sup_{A_3,B_{3\ell+1}(x)}\!\!\text{ess}\,\Big(\int_{B_{3\ell+1}(x)}\Big|\frac{\partial Z(A_3)}{\partial A_3}\Big|\Big)^2dx\,\big((\ell+1)^{-d}\pi_3(\ell)+e^{-\frac1C\ell^d}\big)d\ell\bigg],\end{aligned}$$

and the result follows. □

Appendix A. Proof of the criterion for standard functional inequalities

In this appendix, we give a proof of Proposition 2.3.

Proof of Proposition 2.3. Let $\varepsilon>0$ be fixed, and consider the partition $(Q_z)_{z\in\mathbb{Z}^d}$ of $\mathbb{R}^d$ defined by $Q_z=\varepsilon z+\varepsilon Q$. Choose an i.i.d. copy A_0' of the field A_0, and for all z define the random field A_0^z by $A_0^z|_{\mathbb{R}^d\setminus Q_z}:=A_0|_{\mathbb{R}^d\setminus Q_z}$ and $A_0^z|_{Q_z}:=A_0'|_{Q_z}$. We split the proof into three steps.

Step 1. Tensorization argument.
Choose an enumeration $(z_n)_n$ of $\mathbb{Z}^d$, and for all n let Π_n and $\mathbb{E}_n$ denote the linear maps on $\mathrm{L}^2(\Omega)$ defined by

$$\Pi_n[X]:=\mathbb{E}\big[X\,\big\|\,A_0|_{\bigcup_{k=1}^n Q_{z_k}}\big],\qquad\mathbb{E}_n[X]:=\mathbb{E}\big[X\,\big\|\,A_0|_{\mathbb{R}^d\setminus Q_{z_n}}\big].$$

Also define

$$\mathrm{Cov}_n[X;Y]:=\mathbb{E}_n[XY]-\mathbb{E}_n[X]\mathbb{E}_n[Y],\qquad\mathrm{Var}_n[X]:=\mathrm{Cov}_n[X;X],\\\mathrm{Ent}_n[X^2]:=\mathbb{E}_n\big[X^2\log(X^2/\mathbb{E}_n[X^2])\big].$$

In this step, we make use of a martingale argument à la Lu-Yau [17] to show the following tensorization identities for the covariance and for the entropy: for all $\sigma(A_0)$-measurable random variables $X(A_0)$ and $Y(A_0)$, we have

$$|\mathrm{Cov}\left[X(A_0);Y(A_0)\right]| \leq \sum_{k=1}^{\infty}\mathbb{E}\left[\,\left|\,\mathrm{Cov}_k\left[\Pi_k[X(A_0)];\Pi_k[Y(A_0)]\right]\right|\,\right], \tag{A.1}$$

$$\mathrm{Ent}\big[X(A_0)^2\big] \leq \sum_{k=1}^{\infty}\mathbb{E}\left[\mathrm{Ent}_k\left[\Pi_k[X(A_0)^2]\right]\right]. \tag{A.2}$$

First note that for all $\sigma(A_0)$-measurable random variables $X(A_0)\in \mathrm{L}^2(\Omega)$, the properties of conditional expectations ensure that $\Pi_n[X(A_0)]\to X(A_0)$ in $\mathrm{L}^2(\Omega)$ as $n\uparrow\infty$. We then decompose the covariance into the following telescopic sum

$$\begin{aligned}
&\mathrm{Cov}\left[\Pi_n[X(A_0)];\Pi_n[Y(A_0)]\right]\\
&= \sum_{k=1}^{n}\Big(\mathbb{E}\left[\Pi_k[X(A_0)]\Pi_k[Y(A_0)]\right]-\mathbb{E}\left[\Pi_{k-1}[X(A_0)]\Pi_{k-1}[Y(A_0)]\right]\Big)\\
&= \sum_{k=1}^{n}\mathbb{E}\left[\mathrm{Cov}_k\left[\Pi_k[X(A_0)];\Pi_k[Y(A_0)]\right]\right],
\end{aligned}$$

so that the result (A.1) follows by taking the limit $n\uparrow\infty$. Likewise, we decompose the entropy into the following telescopic sum

$$\begin{aligned}
&\mathrm{Ent}\big[\Pi_n[X(A_0)^2]\big]\\
&= \sum_{k=1}^{n}\Big(\mathbb{E}\left[\Pi_k[X(A_0)^2]\log(\Pi_k[X(A_0)^2])\right]-\mathbb{E}\left[\Pi_{k-1}[X(A_0)^2]\log(\Pi_{k-1}[X(A_0)^2])\right]\Big)\\
&= \sum_{k=1}^{n}\mathbb{E}\left[\mathrm{Ent}_k\left[\Pi_k[X(A_0)^2]\right]\right],
\end{aligned}$$

and the result (A.2) follows in the limit $n\uparrow\infty$.

Step 2. Preliminary versions of (CI) and (LSI).
In this step, we prove that for all $\sigma(A_0)$-measurable random variables $X(A_0)$ and $Y(A_0)$ we have

$$\begin{aligned}
&|\mathrm{Cov}\left[X(A_0);Y(A_0)\right]|\\
&\leq \frac{1}{2}\sum_{k=1}^{\infty}\mathbb{E}\left[\left|\Pi_k\left[X(A_0)-X(A_0^{z_k})\right]\right|\left|\Pi_k\left[Y(A_0)-Y(A_0^{z_k})\right]\right|\right]\\
&\leq \frac{1}{2}\sum_{z\in\mathbb{Z}^d}\mathbb{E}\left[\big(X(A_0)-X(A_0^{z})\big)^2\right]^{\frac12}\mathbb{E}\left[\big(Y(A_0)-Y(A_0^{z})\big)^2\right]^{\frac12},
\end{aligned} \tag{A.3}$$

and

$$\mathrm{Ent}[X(A_0)]\leq 2\sum_{z\in\mathbb{Z}^d}\mathbb{E}\left[\operatorname*{sup\,ess}_{A_0'}\big(X(A_0)-X(A_0^{z})\big)^2\right]. \tag{A.4}$$

We first prove (A.3): we appeal to (A.1) in the form

$$\begin{aligned}|\mathrm{Cov}\,[X(A_0);Y(A_0)]\,| &\le \frac12\sum_{k=1}^\infty \mathbb{E}\left[\big|\mathbb{E}_k\big[\Pi_k[X(A_0)-X(A_0^{z_k})]\,\Pi_k[Y(A_0)-Y(A_0^{z_k})]\big]\big|\right]\\ &\le \frac12\sum_{k=1}^\infty \mathbb{E}\left[\big|\Pi_k[X(A_0)-X(A_0^{z_k})]\big|\,\big|\Pi_k[Y(A_0)-Y(A_0^{z_k})]\big|\right],\end{aligned}$$

which directly yields (A.3) by Cauchy-Schwarz' inequality. Likewise, we argue that (A.4) follows from (A.2). To this aim, we have to reformulate the RHS of (A.2): using the inequality $a\log a - a + 1 \le (a-1)^2$ for all $a\ge 0$, we obtain for all $k\ge 0$,

$$\begin{aligned}\mathrm{Ent}_k\left[\Pi_k[X(A_0)^2]\right] &\le \mathbb{E}_k[\Pi_k[X(A_0)^2]]\,\mathbb{E}_k\left[\Big(\frac{\Pi_k[X(A_0)^2]}{\mathbb{E}_k[\Pi_k[X(A_0)^2]]}-1\Big)^2\right]\\ &= \frac{\mathrm{Var}_k\left[\Pi_k[X(A_0)^2]\right]}{\mathbb{E}_k[\Pi_k[X(A_0)^2]]}\\ &= \frac{\mathbb{E}_k\left[(\Pi_k[X(A_0)^2]-\Pi_k[X(A_0^{z_k})^2])^2\right]}{2\,\mathbb{E}_k[\Pi_k[X(A_0)^2]]}\\ &= \frac{\mathbb{E}_k\left[(\Pi_k[(X(A_0)-X(A_0^{z_k}))(X(A_0)+X(A_0^{z_k}))])^2\right]}{2\,\mathbb{E}_k[\Pi_k[X(A_0)^2]]}\\ &\le \frac{\mathbb{E}_k\left[\Pi_k[(X(A_0)-X(A_0^{z_k}))^2]\,\Pi_k[(X(A_0)+X(A_0^{z_k}))^2]\right]}{2\,\mathbb{E}_k[\Pi_k[X(A_0)^2]]}.\end{aligned}$$

Since $(A_0,A_0^{z_k})$ and $(A_0^{z_k},A_0)$ have the same law by complete independence, the above implies, using the inequality $(a+b)^2\le 2(a^2+b^2)$ for all $a,b\in\mathbb{R}$,

$$\begin{aligned}\mathrm{Ent}_k\left[\Pi_k[X(A_0)^2]\right] &\le \frac{2\,\mathbb{E}_k\left[\Pi_k[(X(A_0)-X(A_0^{z_k}))^2]\,\Pi_k[X(A_0^{z_k})^2]\right]}{\mathbb{E}_k[\Pi_k[X(A_0^{z_k})^2]]}\\ &\le 2\,\sup_{A_0'|_{Q_{z_k}}}\mathrm{ess}\,\Pi_k[(X(A_0)-X(A_0^{z_k}))^2]\\ &\le 2\,\Pi_k\Big[\sup_{A_0'|_{Q_{z_k}}}\mathrm{ess}\,(X(A_0)-X(A_0^{z_k}))^2\Big].\end{aligned}$$

Estimate (A.4) now follows from (A.2).

Step 3. Proof of (CI) and (LSI).
We start with the proof of (CI). Since $A=A(A_0)$ is $\sigma(A_0)$-measurable, (A.3) yields for all $\sigma(A)$-measurable random variables $X(A)$ and $Y(A)$,

$$\big|\mathrm{Cov}\,[X(A);Y(A)]\,\big| \le \frac12\sum_{z\in\mathbb{Z}^d}\mathbb{E}\left[\big(X(A)-X(A(A_0^z))\big)^2\right]^{\frac12}\mathbb{E}\left[\big(Y(A)-Y(A(A_0^z))\big)^2\right]^{\frac12}.$$

Using that $\mathbb{E}\big[X(A)\,\|\,A_0|_{\mathbb{R}^d\setminus Q_z}\big]=\mathbb{E}\big[X(A(A_0^z))\,\|\,A_0|_{\mathbb{R}^d\setminus Q_z}\big]$ by complete independence of the field A_0,

$$\mathbb{E}\left[\big(X(A)-X(A(A_0^z))\big)^2\right]=\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A_0,Q_z}X(A(A_0))\Big)^2\right].$$

Since the conditional expectation $\mathbb{E}\big[\cdot \,\big\|\, A_0|_{\mathbb{R}^d\setminus Q_z}\big]$ coincides with the L^2-projection onto the $\sigma(A_0|_{\mathbb{R}^d\setminus Q_z})$-measurable functions, and since $\mathbb{E}\big[X(A) \,\big\|\, A|_{\mathbb{R}^d\setminus(Q_z+B_R)}\big]$ is $\sigma(A|_{\mathbb{R}^d\setminus(Q_z+B_R)})$-measurable and therefore $\sigma(A_0|_{\mathbb{R}^d\setminus Q_z})$-measurable by assumption, we have

$$\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A_0,Q_z}X(A(A_0))\Big)^2\right] \le \mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,Q_z+B_R}X(A)\Big)^2\right].$$

Combining these two observations, we deduce that for all $\sigma(A)$-measurable random variables $X(A)$ and $Y(A)$,

$$\big|\mathrm{Cov}\left[X(A);Y(A)\right]\big| \le \frac{1}{2}\sum_{z\in\mathbb{Z}^d}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,Q_z+B_R}X(A)\Big)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,Q_z+B_R}Y(A)\Big)^2\right]^{\frac{1}{2}}.$$

By taking local averages, this turns into

$$\begin{aligned}
&\big|\mathrm{Cov}\left[X(A);Y(A)\right]\big|\\
&\le \frac{\varepsilon^{-d}}{2}\sum_{z\in\mathbb{Z}^d}\int_{\varepsilon Q}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,y+\varepsilon z+\varepsilon Q+B_R}X(A)\Big)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,y+\varepsilon z+\varepsilon Q+B_R}Y(A)\Big)^2\right]^{\frac{1}{2}}dy\\
&= \frac{\varepsilon^{-d}}{2}\int_{\mathbb{R}^d}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,y+\varepsilon Q+B_R}X(A)\Big)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,y+\varepsilon z+\varepsilon Q+B_R}Y(A)\Big)^2\right]^{\frac{1}{2}}dy\\
&\le \frac{\varepsilon^{-d}}{2}\int_{\mathbb{R}^d}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,B_{R+\varepsilon\sqrt{d}/2}(y)}X(A)\Big)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,B_{R+\varepsilon\sqrt{d}/2}(y)}Y(A)\Big)^2\right]^{\frac{1}{2}}dy,
\end{aligned}$$

that is, (CI) for any radius larger than R.
We then turn to the proof of (LSI). For all $\sigma(A)$-measurable random variables $X(A)$, the estimate (A.4) yields

$$\begin{aligned}
\mathrm{Ent}[X(A)] &\le 2\sum_{z\in\mathbb{Z}^d}\mathbb{E}\left[\sup_{A_0'}\mathrm{ess}\,\big(X(A(A_0))-X(A(A_0^z))\big)^2\right]\\
&\le 2\sum_{z\in\mathbb{Z}^d}\mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,Q_z+B_R}\,X(A)\Big)^2\right].
\end{aligned}$$

The desired result (LSI) then follows from taking local averages. □

Appendix B. Abstract criteria for deterministically localized fields

In this appendix, we discuss general criteria for weighted functional inequalities in the case when the random field A is deterministically localized in the sense of Section 2.3. To be precise we focus on the typical example of a convolution of a random noise. In this case we prove the validity of a Brascamp-Lieb inequality from which the desired weighted functional inequalities follow. Although Gaussian random fields are the most prominent examples of this framework, we develop the general argument in a slightly more abstract setting. (Note that we choose to argue by approximation and reduce to discrete fields, rather than appeal to Malliavin calculus and associated functional analysis.)

Let W be a random noise on $\mathbb{R}^d$, that is, a mean-zero stationary completely independent second-order random Borel measure on $\mathbb{R}^d$ (see e.g. [24, Section 2]). More precisely, W

associates a random variable $W(E)$ to any bounded Borel subset $E \subset \mathbb{R}^d$, in such a way that

(i) $\mathbb{E}[W(E)] = 0$ and $\mathbb{E}\left[|W(E)|^2\right] < \infty$ for all bounded Borel subset $E \subset \mathbb{R}^d$;
(ii) if $(E_n)_n$ is a family of disjoint Borel subsets of $\mathbb{R}^d$, then $W\left(\bigcup_{n=1}^\infty E_n\right) = \sum_{n=1}^\infty W(E_n)$ in the L^2-sense;
(iii) $(W(x+E), W(x+E'))$ has the same law as $(W(E), W(E'))$ for any two bounded Borel subsets $E, E' \subset \mathbb{R}^d$ and any $x \in \mathbb{R}^d$;
(iv) $W(E_1), \ldots, W(E_n)$ are independent for any disjoint Borel sets $E_1, \ldots, E_n \subset \mathbb{R}^d$ and any $n \in \mathbb{N}$.

Stationarity implies in particular that the Borel measure $\mathbb{E}[|dW|^2]$ is proportional to the Lebesgue measure: $\mathbb{E}[|dW|^2] = \lambda dx$, for some constant $\lambda \geq 0$ that is called the intensity of the random noise W.

Given a (deterministic) nonnegative Borel function $F \in \mathrm{L}^2(\mathbb{R}^d)$ and a constant $m \in \mathbb{R}^d$, we now define a measurable random field A on $\mathbb{R}^d$ by the following convolution,

$$A(y) = m + \int_{\mathbb{R}^d} F(y-z) dW(z), \tag{B.1}$$

the covariance function of which is then given by

$$\mathcal{C}(x) := \mathrm{Cov}\left[A(x); A(0)\right] = \lambda \int_{\mathbb{R}^d} F(x-z)F(z)dz. \tag{B.2}$$

The following result (which is rather standard) shows that a Brascamp-Lieb inequality holds for such random fields whenever the random noise W satisfies a standard spectral gap, thus mimicking the well-known situation of Gaussian fields. (For Gaussian fields, a discrete version of the Brascamp-Lieb inequality (B.4) below was first due to [4], while a discrete version of the inequality in covariance form (B.5) and in entropy form (B.8) is due to [18] and to [2, Proposition 3.4], respectively.)

Proposition B.1 (Brascamp-Lieb type inequalities). *Let W be a random noise on $\mathbb{R}^d$ with intensity λ, let the stationary random field A on $\mathbb{R}^d$ be given by* (B.1)*, and let $\mathcal{C}$ denote its covariance function.*

(i) Assume that for all $\eta > 0$ the random variable $W(\eta Q)$ satisfies the following spectral gap: for any smooth function ϕ,

$$\mathrm{Var}\left[\phi(W(\eta Q))\right] \leq C\lambda\eta^d \mathbb{E}\left[\phi'(W(\eta Q))^2\right]. \tag{B.3}$$

Then the random field A satisfies the following Brascamp-Lieb inequality: for all $\sigma(A)$-measurable random variables $X(A)$,

$$\mathrm{Var}\left[X(A)\right] \leq C\mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Big|\frac{\partial X(A)}{\partial A}(z)\Big|\Big|\frac{\partial X(A)}{\partial A}(z')\Big| |\mathcal{C}(z-z')| dz dz'\right]. \tag{B.4}$$

Moreover, the following Brascamp-Lieb inequality in covariance form holds: for all $\sigma(A)$-measurable random variables $X(A), Y(A)$ we have

$$\mathrm{Cov}\left[X(A); Y(A)\right] \leq C \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \Big|\frac{\partial X(A)}{\partial A}(z)\Big|\, |\mathcal{F}^{-1}(\sqrt{\mathcal{F}\mathcal{C}})(x-z)|\, dz\right)^2\right]^{\frac12} \times \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \Big|\frac{\partial Y(A)}{\partial A}(z')\Big|\, |\mathcal{F}^{-1}(\sqrt{\mathcal{F}\mathcal{C}})(x-z')|\, dz'\right)^2\right]^{\frac12} dx, \quad \text{(B.5)}$$

and in particular

$$\mathrm{Cov}\left[X(A); Y(A)\right] \leq C \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{E}\left[\Big|\frac{\partial X(A)}{\partial A}(z)\Big|^2\right]^{\frac12} \times \mathbb{E}\left[\Big|\frac{\partial Y(A)}{\partial A}(z')\Big|^2\right]^{\frac12} \tilde{\mathcal{C}}(z-z')dzdz', \quad \text{(B.6)}$$

in terms of

$$\tilde{\mathcal{C}}(x) := \int |\mathcal{F}^{-1}(\sqrt{\mathcal{F}\mathcal{C}})(x-y)||\mathcal{F}^{-1}(\sqrt{\mathcal{F}\mathcal{C}})(y)|dy.$$

(ii) Assume that for all $\eta > 0$ the random variable $W(\eta Q)$ satisfies the corresponding logarithmic Sobolev inequality: for any smooth function ϕ,

$$\mathrm{Ent}\left[\phi(W(\eta Q))^2\right] \leq C\lambda\eta^d \mathbb{E}\left[\phi'(W(\eta Q))^2\right]. \quad \text{(B.7)}$$

Then the random field A satisfies the corresponding Brascamp-Lieb inequality in logarithmic Sobolev form: for all $\sigma(A)$-measurable random variables $X(A)$,

$$\mathrm{Ent}[X(A)] \leq C\mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Big|\frac{\partial X(A)}{\partial A}(z)\Big|\Big|\frac{\partial X(A)}{\partial A}(z')\Big||\mathcal{C}(z-z')|dzdz'\right]. \quad \text{(B.8)}$$

□

In the following theorem, we show that Brascamp-Lieb inequalities imply weighted functional inequalities, using a suitable radial change of variables. Note that in item (ii), the weights obtained for (∂^{fct}-WSG) and (∂^{fct}-WCI) typically have the same scaling.

Theorem B.2. *Let A be a jointly measurable stationary random field on $\mathbb{R}^d$, let $\mathcal{C}$ denote its covariance function. Assume that A satisfies the Brascamp-Lieb inequality* (B.4) *(resp. in logarithmic Sobolev form* (B.8)*).*

(i) If the map $x \mapsto \sup_{B(x)} |\mathcal{C}|$ is integrable, then the field A satisfies (∂^{fct}-SG) *(resp.* (∂^{fct}-LSI)*) for any radius $R > 0$.*

(ii) If $\sup_{B(x)} |\mathcal{C}| \leq c(|x|)$ holds for some non-increasing Lipschitz function $c : \mathbb{R}_+ \to \mathbb{R}_+$, then the field A satisfies (∂^{fct}-WSG) *(resp.* (∂^{fct}-WLSI)*) with weight $\pi(\ell) \simeq (-c'(\ell))$. If the field A further satisfies the Brascamp-Lieb inequality in covariance form* (B.5)*, and if $\sup_{B(x)} |\mathcal{F}^{-1}(\sqrt{\mathcal{F}\mathcal{C}})| \leq r(|x|)$ holds for some non-increasing Lipschitz function $r : \mathbb{R}_+ \to \mathbb{R}_+$, then A satisfies* (∂^{fct}-WCI) *with weight $\pi(\ell) \simeq (\ell+1)^d\, r(\ell)(-r'(\ell))$.* □

Remark B.3. We briefly address the claim contained in Remark 2.2 in the context of examples of random fields with deterministic localization. More precisely, we consider a random field A as in the statement of Theorem B.2 above, and we assume that $\sup_{B(x)} |\mathcal{C}| \leq$

$c(|x|)$ holds for some non-increasing Lipschitz function $c : \mathbb{R}_+ \to \mathbb{R}_+$. By definition, for all $L \geq 1$, the rescaled field $A_L := A(L\cdot)$ has covariance $\mathcal{C}_L := \mathcal{C}(L\cdot)$ and for $|x| \geq 1$ it satisfies $\sup_{B(x)} |\mathcal{C}_L| = \sup_{B_L(Lx)} |\mathcal{C}| \leq c((L|x| - L + 1)_+) \leq c(|x|)$ since c is non-increasing. This shows that the same conclusions as for A in Theorem B.2 also hold for A_L uniformly with respect to $L \geq 1$. ☐

We start with the proof of Proposition B.1, and then turn to the proof of Theorem B.2.

Proof of Proposition B.1. For all $\varepsilon > 0$, consider the following approximations of the random field A,

$$A_\varepsilon(x) := \sum_{y,z \in \varepsilon\mathbb{Z}^d} \mathbb{1}_{Q_\varepsilon(z)}(x)\, W(Q_\varepsilon(y)) \fint_{Q_\varepsilon(z)} \fint_{Q_\varepsilon(y)} F(z'-y')dz'dy'.$$

By an approximation argument, we may reduce the proof of the proposition to the proof of the following discrete counterpart: given a random vector $W := (W_1, \ldots, W_N)$ with N independent components, and given a linear transformation $F \in \mathbb{R}^{N\times N}$, the transformed random vector $A := (A_1, \ldots, A_N) := FW$ satisfies:

(i') If for all $1 \leq j \leq N$ the random variable W_j satisfies the standard spectral gap

$$\mathrm{Var}\left[\phi(W_j)\right] \leq C\mathbb{E}\left[\phi'(W_j)^2\right]$$

for all smooth functions $\phi : \mathbb{R} \to \mathbb{R}$, then the random vector A satisfies for all smooth functions $X, Y : \mathbb{R}^N \to \mathbb{R}$

$$\mathrm{Var}\left[X(A)\right] \leq C \sum_{i,j=1}^N |(FF^t)_{ij}|\, \mathbb{E}\left[\Big|\frac{\partial X(A)}{\partial A_i}\Big|\Big|\frac{\partial X(A)}{\partial A_j}\Big|\right], \tag{B.9}$$

and also

$$\mathrm{Cov}\left[X(A); Y(A)\right] \leq \sum_{i=1}^N \mathbb{E}\left[\Big(\sum_{j=1}^N \frac{\partial X(A)}{\partial A_j} F_{ji}\Big)^2\right]^{\frac12} \mathbb{E}\left[\Big(\sum_{k=1}^N \frac{\partial Y(A)}{\partial A_k} F_{ki}\Big)^2\right]^{\frac12}. \tag{B.10}$$

(ii') If for all $1 \leq j \leq N$ the random variable W_j satisfies the standard logarithmic Sobolev inequality

$$\mathrm{Ent}\big[\phi(W_j)^2\big] \leq C\mathbb{E}\left[\phi'(W_j)^2\right]$$

for all smooth functions $\phi : \mathbb{R} \to \mathbb{R}$, then the random vector A satisfies for all smooth functions $X : \mathbb{R}^N \to \mathbb{R}$,

$$\mathrm{Ent}\big[X(A)^2\big] \leq C \sum_{i,j=1}^N |(FF^t)_{ij}|\, \mathbb{E}\left[\Big|\frac{\partial X(A)}{\partial A_i}\Big|\Big|\frac{\partial X(A)}{\partial A_j}\Big|\right]. \tag{B.11}$$

We start with the proof of item (i'). Using the tensorization identity (A.1), the spectral gap assumption yields

$$\mathrm{Var}\left[X(A)\right] \leq \sum_{i=1}^N \mathbb{E}\left[\mathrm{Var}\left[X(A) \,\|\, (W_j)_{j:j\neq i}\right]\right] \leq \sum_{i=1}^N \mathbb{E}\left[\Big(\frac{\partial X(A)}{\partial W_i}\Big)^2\right],$$

and hence, by the chain rule,

$$\operatorname{Var}[X(A)] \le \sum_{i=1}^N \mathbb{E}\left[\Big(\sum_{j=1}^N \frac{\partial X(A)}{\partial A_j} F_{ji}\Big)^2\right] = \mathbb{E}\left[\nabla X(A)\cdot (FF^t)\nabla X(A)\right] \qquad \text{(B.12)}$$
$$\le \sum_{i,j=1}^N |(FF^t)_{ij}|\,\mathbb{E}\left[\Big|\frac{\partial X(A)}{\partial A_i}\Big|\Big|\frac{\partial X(A)}{\partial A_j}\Big|\right].$$

In covariance form, using again the tensorization identity (A.1), the spectral gap assumption yields

$$\operatorname{Cov}[X(A);Y(A)] \le \sum_{i=1}^N \mathbb{E}\left[\operatorname{Var}[X(A)\,\|\,(W_j)_{j:j\ne i}]\right]^{\frac12} \mathbb{E}\left[\operatorname{Var}[Y(A)\,\|\,(W_j)_{j:j\ne i}]\right]^{\frac12}$$
$$\le \sum_{i=1}^N \mathbb{E}\left[\Big(\frac{\partial X(A)}{\partial W_i}\Big)^2\right]^{\frac12} \mathbb{E}\left[\Big(\frac{\partial Y(A)}{\partial W_i}\Big)^2\right]^{\frac12},$$

and the result (B.10) follows from the chain rule. We now turn to the proof of item (ii'). Using the tensorization identity (A.2), the logarithmic Sobolev inequality assumption yields

$$\operatorname{Ent}\big[X(A)^2\big] \le \sum_{i=1}^N \mathbb{E}\left[\operatorname{Ent}\left[\mathbb{E}\left[X(A)^2\,\|\,(W_j)_{j:j\le i}\right]\,\|\,(W_j)_{j:j\ne i}\right]\right]$$
$$\le C\sum_{i=1}^N \mathbb{E}\left[\Big|\frac{\partial}{\partial W_i}\mathbb{E}\left[X(A)^2\,\|\,(W_j)_{j:j\le i}\right]^{\frac12}\Big|^2\right]$$
$$= C\sum_{i=1}^N \mathbb{E}\left[\mathbb{E}\left[X(A)^2\,\|\,(W_j)_{j:j\le i}\right]^{-1}\Big|\mathbb{E}\left[X(A)\frac{\partial X(A)}{\partial W_i}\,\Big\|\,(W_j)_{j:j\le i}\right]\Big|^2\right]$$
$$\le C\sum_{i=1}^N \mathbb{E}\left[\Big|\frac{\partial X(A)}{\partial W_i}\Big|^2\right].$$

Now arguing as in (B.12), the result of item (ii') follows. □

We now prove Theorem B.2.

Proof of Theorem B.2. We focus on items (i) and (ii) for the variance and the covariance (the arguments for the entropy are similar). Assume that A satisfies the Brascamp-Lieb inequality (B.4). If $x \mapsto \sup_{B(x)}|\mathcal{C}|$ is integrable, the inequality $|ab| \le (a^2+b^2)/2$ for $a,b\in\mathbb{R}$ directly yields for all $\sigma(A)$-measurable random variables $X(A)$ and all $R>0$ (after taking local averages),

$$\operatorname{Var}[X(A)] \le C\,\mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big|\frac{\partial X(A)}{\partial A}(z)\Big|\Big|\frac{\partial X(A)}{\partial A}(z')\Big||\mathcal{C}(z-z')|dzdz'\right]$$
$$\le 2C\Big\|\sup_{B_{2R}(\cdot)}|\mathcal{C}|\Big\|_{\mathrm{L}^1}\mathbb{E}\left[\int_{\mathbb{R}^d}\Big(\fint_{B_R(z)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2dz\right].$$

Now assume that the covariance function $\mathcal{C}$ is not integrable, and that $\sup_{B(x)}|\mathcal{C}| \le c(|x|)$ for some Lipschitz function $c:\mathbb{R}_+\to\mathbb{R}_+$. Given a $\sigma(A)$-measurable random variable $X(A)$, we consider the projection $X_R(A) := \mathbb{E}[X(A)\|A|_{B_R}]$, for $R>0$. Taking local averages, using polar coordinates, and integrating by parts (note that there is no boundary term since the Fréchet derivative $\partial X_R(A)/\partial A$ is compactly supported in B_R), the Brascamp-Lieb inequality (B.4) yields

$$\begin{aligned}
&\mathrm{Var}\left[X_R(A)\right]\\
&\le C\mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\int_0^\infty\Big|\frac{\partial X_R(A)}{\partial A}(z)\Big|\fint_{B(z+\ell u)}\Big|\frac{\partial X_R(A)}{\partial A}(u')\Big|du'\ell^{d-1}c(\ell)d\ell d\sigma(u)dz\right]\\
&= C\mathbb{E}\left[\int_{\mathbb{R}^d}\Big|\frac{\partial X_R(A)}{\partial A}(z)\Big|\int_{\mathbb{S}^{d-1}}\int_0^\infty\int_0^\ell\fint_{B(z+su)}\Big|\frac{\partial X_R(A)}{\partial A}(u')\Big|du's^{d-1}ds(-c'(\ell))d\ell d\sigma(u)dz\right]\\
&\le C\mathbb{E}\left[\int_{\mathbb{R}^d}\Big|\frac{\partial X_R(A)}{\partial A}(z)\Big|\int_0^\infty\Big(\int_{B_{\ell+1}(z)}\Big|\frac{\partial X_R(A)}{\partial A}\Big|\Big)(-c'(\ell))d\ell dz\right].
\end{aligned}$$

Reorganizing the integrals, and taking local spatial averages, we conclude

$$\begin{aligned}
&\mathrm{Var}\left[X_R(A)\right]\\
&\lesssim \mathbb{E}\left[\int_0^\infty\int_{\mathbb{R}^d}\Big|\frac{\partial X_R(A)}{\partial A}(z)\Big|\Big(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(z)}X_R(A)\Big)dz(-c'(\ell))_+d\ell\right]\\
&\lesssim \mathbb{E}\left[\int_0^\infty\int_{\mathbb{R}^d}\int_{B_{\ell+1}}\Big|\frac{\partial X_R}{\partial A}(z+y)\Big|\Big(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(z+y)}X_R(A)\Big)dydz\,(\ell+1)^{-d}(-c'(\ell))_+d\ell\right]\\
&\lesssim \mathbb{E}\left[\int_0^\infty\int_{\mathbb{R}^d}\Big(\partial^{\mathrm{fct}}_{A,B_{2(\ell+1)}(z)}X_R(A)\Big)^2dz\,(\ell+1)^{-d}(-c'(\ell))_+d\ell\right]\\
&\lesssim \mathbb{E}\left[\int_0^\infty\int_{\mathbb{R}^d}\Big(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(z)}X_R(A)\Big)^2dz\,(\ell+1)^{-d}(-c'(\ell))_+d\ell\right],
\end{aligned}$$

where in the last line we used the (sub)additivity of $S\mapsto\partial^{\mathrm{fct}}_{A,S}$. By Jensen's inequality in the form

$$\mathbb{E}\left[\Big(\partial^{\mathrm{fct}}_{A,S}X_R(A)\Big)^2\right]\le\mathbb{E}\left[\Big(\mathbb{E}\left[\partial^{\mathrm{fct}}_{A,S}X(A)\,\Big\|\,A|_{B_R}\right]\Big)^2\right]\le\mathbb{E}\left[\Big(\partial^{\mathrm{fct}}_{A,S}X(A)\Big)^2\right],$$

and passing to the limit $R\uparrow\infty$, the conclusion (∂^{fct}-WSG) follows. Let us now turn to the case when the field A satisfies the Brascamp-Lieb inequality in covariance form (B.5). Assuming that $\sup_{B(x)}|\mathcal{F}^{-1}(\sqrt{\mathcal{FC}})|\le r(|x|)$ for some Lipschitz function $r:\mathbb{R}_+\to\mathbb{R}_+$, a radial integration by parts similar as above yields

$$\begin{aligned}
\mathrm{Cov}\left[X_R(A);Y_R(A)\right]\lesssim\int_{\mathbb{R}^d}\mathbb{E}\left[\Big(\int_0^\infty\Big(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(x)}X_R(A)\Big)(-r'(\ell))_+\,d\ell\Big)^2\right]^{\frac12}&\\
\times\,\mathbb{E}\left[\Big(\int_0^\infty\Big(\partial^{\mathrm{fct}}_{A,B_{\ell'+1}(x)}Y_R(A)\Big)(-r'(\ell'))_+\,d\ell'\Big)^2\right]^{\frac12}dx.&
\end{aligned}$$

By the triangle inequality, this turns into

$$
\begin{aligned}
\operatorname{Cov}\left[X_R(A); Y_R(A)\right] \lesssim \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(x)} X_R(A)\right)^2\right]^{\frac12} \\
\times \mathbb{E}\left[\left(\partial^{\mathrm{fct}}_{A,B_{\ell'+1}(x)} Y_R(A)\right)^2\right]^{\frac12} dx(-r'(\ell))_+ \, d\ell(-r'(\ell'))_+ \, d\ell' \\
\le 2 \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(x)} X_R(A)\right)^2\right]^{\frac12} \mathbb{E}\left[\left(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(x)} Y_R(A)\right)^2\right]^{\frac12} dx \\
\times \Big(\int_0^\ell (-r'(\ell'))_+ \, d\ell'\Big)(-r'(\ell))_+ \, d\ell,
\end{aligned}
$$

and the conclusion (∂^{fct}-WCI) follows after passing to the limit $R \uparrow \infty$. ☐

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