Proceedings of the 13th International Conference on Technology in Mathematics Teaching

Gilles Aldon, Jana Trgalova

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ICTMT 13

École Normale Supérieure de Lyon / Université Claude Bernard Lyon 1

3 to 6 July, 2017

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Chapter 1

Introduction

The 13th International Conference on Technology in Mathematics Teaching – ICTMT 13 was organized by the Ecole Normale Supérieure de Lyon and the University Lyon 1. It was held in Lyon, France, 3 to 6 July, 2017.

This biennial conference is the thirteenth of a series which began in Birmingham, UK, in 1993, under the influential enterprise of Professor Bert Waits from Ohio State University. The last conference was held in Faro, Portugal, in 2015 and the next conference will be held in Essen (Germany) in July 2019.

The ICTMT conference series is unique in that it aims to bring together lecturers, teachers, educators, curriculum designers, mathematics education researchers, learning technologists and educational software designers, who share an interest in improving the quality of teaching and learning, and eventually research, by effective use of technology. It provides a forum for researchers and practitioners in this field to discuss and share best practices, theoretical know-how, innovation and perspectives on educational technologies and their impact on the teaching and learning of mathematics, as well as on research approaches.

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The general theme of this conference is related to the progress of mathematics education research on the design and integration of technology in educational settings, for learners of all ages rom primary schools to universities.

The ICTMT 13 gave to all participants the opportunity to share research and to report progress regarding technology in the mathematics classroom. The following themes were presented and discussed during the four days of the conference and these proceedings are the result of both the proposals and the discussion made during the presentation slots.

Curriculum

Technology and its use impact the ways that the mathematics curriculum is designed and implemented both in schools and at the university level. What are the new impacts of technology on the content, progression and approach to the mathematics curriculum?

Assessment

Technology offers through its functionalities and affordances new possibilities for assessment in mathematics and particularly for formative assessment. How can teachers support the students' learning that make use of these functionalities and affordances? How can technology support students to gain a better awareness of their own learning?

Students

Does technology still motivate students to learn mathematics? How can technology support students' to learn mathematics? How can technology foster the development of creative mathematical thinking in students? How can students use their day-to-day technological skills/experiences to support their mathematics learning in and out of schools?

Teachers

Technology can provide a means for mathematics teachers' professional development through online professional development initiatives, such as blended courses and more recently "massive Open Online Courses (MOOCs). How can technology best support mathematics teachers' professional development? What are the design principles for technology-mediated professional development courses? How can the impact of such courses on mathematics teachers' professional learning be assessed? Does the use of technology within professional courses for practicing mathematics teachers impact positively on teachers' uses of technology in mathematics lessons?

Innovation

New developments in technology for learning and teaching mathematics come both from the design of new applications and from research and innovation. In what ways can these developments enhance mathematics teaching and learning? How can technology become a bridge between mathematics and other subjects? Does creativity in the design of technology impact the creativity of students in maths classes?

Software and applications

What is new in the design of educational software and applications? How can the recent technological developments, such as robotics, touch technology, virtual reality, be exploited to refreash or enhance mathematics teaching and learning?

The plenaries of this ICTMT 13 are available on : https://ictmt13.sciencesconf.org/resource/page/id/16

Chapter 2

CURRICULUM

THE IMPACT OF TECHNOLOGY USE ON THE CURRICULUM OF THE COURSE “PLANE TRANSFORMATIONS IN GEOMETRY”: A SELF-STUDY

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In the current research, we analysed our own teaching experience of integrating technology in the classroom. We traced the impact of integrating technology on the curriculum of the course “Plane transformations in geometry”. This research is a self-study. The course is taught in the mathematics department of Achva Academic College. The students are mathematics student teachers. While adapting our classroom to a high-tech environment we modified the course curriculum. These changes have been traced across almost a decade and are analysed with respect to basic principles of constructivist teaching. The results indicate significant changes in the curriculum, coherent with the constructivist approach to teaching.

Background

Unkefer, Shinde, & McMaster (2009) propose that the implementation of technology in the educational process induces teachers to look for the appropriate learning environment and pedagogical procedure. According to this conception, the principle of integrating a dynamic environment into the educational process entails continual modification of the classroom and the teaching methodology. Furthermore, educational researchers widely agree that one of the critical factors that can lead to the effective integration of technology into teaching is teachers’ belief that technology can improve learning (e.g., Chen, Looi, & Chen, 2009; Ertmer, 2005; Drijvers et al., 2010; Mittal & Chawla, 2013; So & Kim, 2009). Unfortunately, many mathematics teachers still worry that technology might harm the development of formal thinking in mathematics students, although they accept that the visualization of mathematical objects can facilitate students’ understanding of the learning material (Blum & Kirsch, 1991; Pinto & Tall, 2002). As a result of integrating technology, even the teachers’ way of “doing mathematics” may change - from the belief that mathematics has only correct or incorrect statements to the belief that mathematics may mean the process of solving a particular mathematical problem, while refining the understanding and clarifying the correct mathematical ideas which fits well with constructivist approaches to constructing knowledge (Sachs, 2014).

Motivation of the study

In the current research, we intend to analyze the impact of the usage of dynamic software (GeoGebra) on the curriculum of the course “Plane transformations in geometry”.

As to the plane isometries, which are the main part of the transformations studied within the course, the students’ previous acquaintance with these transformations is usually restricted to a visual level of perception, while rigorous study of geometrical definitions and properties related to these transformations is also more or less new to them. Even on the visual level, they encounter difficulties e.g., in discerning a pure reflection from reflection composed with translation, reflection from central symmetry etc. Such issues, for example, a composition of isometries, is important in the course since it is difficult. Hence, in order to provide a good understanding of transformations, one has to teach them anew.

In view of specific features of recently developed computerized tools, it seems almost obvious to consider applying new methods in geometry teaching based on these tools in teaching transformations. In the modern literature on new approaches in geometry teaching and learning, specific computerized tools such as dynamic geometry environments are being regarded as one of the teaching alternatives (see, for example, Healy & Hoyles, 2001). The rationale behind applying dynamic computerized tools in the course "Plane Transformations in Geometry" is related to some essential features of these tools, such as: convenient and adequate visualization of geometrical argumentation; direct implementation of basic and composed plane isometries; flexibility of dynamic structure, which preserves and accentuates essential transformation-invariant relations between elements of geometric objects. Moreover, our own experience of teaching for more than a decade indicates that integrating digital technology contributes to a better understanding of the subject by the students (Barabash, Gurevich & Yanovski, 2009, Gurevich & Gorev, 2012).

In the current research, as instructors' teachers we were interested in testing whether the changes made contribute to the transformation of the process of students' passive acquisition of knowledge into an active, constructive process of knowledge building.

Course curriculum

The course is taught in the mathematics department of Achva Academic College in Israel. The students are mathematics student teachers. The main topics are basic plane isometries: translations, reflections, rotations, glide reflections, and their compositions. Students learn to define isometry in terms of functions. They also study such transformations as central similarity and inversions that are not isometries. They become acquainted with the invariants of different transformations, for example, invariance of images under composition of two reflections with the same angle between their axes, and invariance of an angle under inversion. In addition, students study the properties of each transformation and develop mathematical arguments about geometric relationships. Students become acquainted with matrix representations of different transformations and their compositions by means of matrices, and finally, they become acquainted with the solutions of construction problems using transformations.

Methodology

Participants

We are two teacher educators in the mathematics department of a college of education. Data were drawn from the course "Plane transformations in geometry" across almost a decade (2007-2016). Our students were mathematics student teachers.

A self-study

We have chosen the genre of self-study since as teacher educators we sought to analyse our own teaching with the purpose of adjusting our teaching to current trends. We felt a tension between the intention of teaching the students to present elaborated formal answers and the belief that the most appropriate way of teaching is based on students' construction of knowledge in a computerized environment. In our research, we concentrated on the following themes: presentation of material, classroom activity and homework assignments. In order to discover how the chosen themes developed throughout the research period we examined the data obtained by ourselves over two academic years, 2007 and 2016. These academic years were chosen since in 2007 the course was taught in traditional classrooms where the digital tools were used episodically, mainly for illustration, while in 2016 the dynamic digital tool GeoGebra was fully integrated into teaching/learning process.

Data collection and analysis

We evaluated the mode of our teaching together with the level of activity of our students' process of acquiring knowledge regarding basic characteristics of the traditional and constructivist classroom defined by Brooks and Brooks (1993) as follows:

Table 1. Basic characteristics of the traditional and constructivist classroom (Cited from Brooks and Brooks, 1993, p.17)

Traditional Classrooms	**Constructivist Classroom**
…	…
Strict adherence to fixed curriculum is highly valued.	Pursuit of student questioning is highly valued
Curricular activities rely heavily on textbooks and workbooks.	Curricular activities rely heavily on primary sources of data and manipulative materials.
Students are viewed as "blank slates" onto which information is etched by the teacher.	Students are viewed as thinkers with emerging theories about the world.
Teachers generally behave in a didactic manner, disseminating information to students.	Teachers generally behave in an interactive manner, mediating the environment for students.
Teacher seeks the correct answer to validate student learning.	Teachers seek the student's point of view in order to understand student's present conceptions for use in subsequent lessons.
Assessment of student learning is viewed as separate from teaching and occurs almost entirely through testing.	Assessment of student learning is interwoven with teaching and occurs through teacher observations of students at work and through student exhibitions ….

The data were collected from the following sources: lessons plans, assignments given to the students, examinations, and comments on course evaluations (taken from the researchers' log and notes on office conversations).

The data from each academic year were analyzed with respect to the above characteristics, that is, based on the data we tried to determine to which kind of classroom (traditional vs constructivist) they fit.

Results

Below we present the examples of presentation of material, classroom activity and homework assignments taken from two academic years (2007 and 2016) that illustrate the changes that occurred in reference to the characteristics described in Table 1 that specify the traditional vs. the constructivist classroom. It is important to emphasize that in all the activities described below the students worked in small groups to construct their knowledge by themselves. Plenary discussion followed each activity.

Episode 1 - Presentation of material.

We proved the theorem that if two points A and B are collinear with the centre of inversion O, and Q is any other point, then the angle ∠AQB is preserved under inversion (see Figure 1).

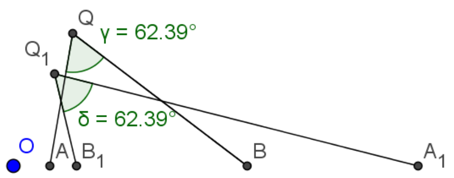

Figure 1. Point O is the center of inversion. Tree points A, B and Q are given where A and B are collinear with the center of inversion. If A_1=Inv(A), B_1=Inv(B), Q_1=Inv(Q) then $\angle AQB = \angle A_1Q_1B_1$.

a) In 2007 the analytical proof of the theorem was presented to the students.
b) In 2016 after the theorem was proven, the students suggested that the theorem holds only when both points (A and B) are either inside or outside the circle of inversion. The students were then asked to explore the given situation using GeoGebra and to test their conjectures. They found that the theorem holds no matter where the points A and B are (while they are on the same ray). Then we analysed the analytical proof to make sure that it does not depend on whether the points are inside or outside the circle of inversion.

Commentary:

In the described situation, digital technology enables the students to explore the theorem by themselves and to make sure that it holds.

Episode 2 - Classroom activity

The students were given the following assignment:

Suppose $T_{(-3,0)}$ is a translation by the vector (3,0). Define $H=S_x\circ T_{(-3,0)}$. What is the transformation represented by H? What is the transformation represented by H∘H?

a) Academic year 2007 – the students had previously learned that the transformation resulting from a composition of translation and reflection when the translation vector is parallel to the reflection line is glide reflection. Moreover, the composition of two glide reflections results in translation by a double vector of the given one. After multiplying the matrices, they confirm this result:

$$H=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 3\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$H\circ H=\begin{pmatrix}1 & 0 & 3\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 3\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 6\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

b) Academic year 2016 – Besides the described above activity the students were asked what would happen if the given vector were not parallel to the reflection line. The students suggested that the result of H∘H, where the H is defined as $H=S_x\circ T_{(a,b)}$ should be a translation by 2(a,b). Given the students' misconception, we decided to experiment with GeoGebra, where the students' conjecture was refuted. Namely, it was observed that the discussed transformation results in translation by a double projection of the given vector on the x-axis (see Figure 3).

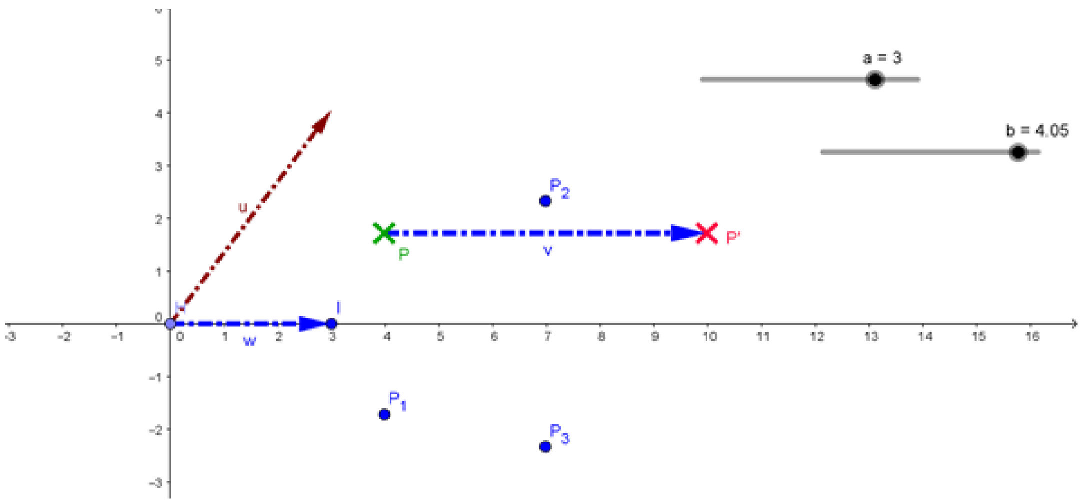

Figure 3. The image of P under H∘H, where H is defined as H=Sx∘ T(a,b) is P': P'= (H∘H)(P). The vector of translation is defined as $\vec{u}=(a,b)$. The given transformation results in translation by the vector $\vec{v}=(2u)_x=(2a,0)$.

After having obtained the above results in GeoGebra, the students derived it analytically using corresponding matrices, as follows:

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & -1 & -b \\ 0 & 0 & 1 \end{pmatrix}$$

$$H \circ H = \begin{pmatrix} 1 & 0 & a \\ 0 & -1 & -b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & a \\ 0 & -1 & -b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Commentary:

The above example shows that GeoGebra enabled the students to explore new situations, test their conjectures, come to a completely unexpected result, and thereby refine their understanding of mathematics.

Episode 3 - Homework assignments

Below we present an example of a homework assignment:

Find the matrix S that represents the transformation obtained by first rotating around the origin (0, 0) by 45˚ and then translating by (-2, 6). What is the transformation that corresponds to S (Find its parameters).

a) In 2007 the students were supposed to find the matrix S by multiplying the corresponding matrix, then they were supposed to identify the resulting transformation as a rotation by 45˚ but with a new origin, and after that to find the image of the given point.

b) In 2016 the students were requested to perform the following steps:

1. Draw the image of the point (4, 3) with respect to S, by using GeoGebra;
2. Find the centre of the rotation found in step 1, using geometrical constructions (see Figure 6);
3. Find the matrix S and then find the centre of the corresponding rotation analytically;
4. Make sure that in both cases (in GeoGebra and analytically) the result is the same.

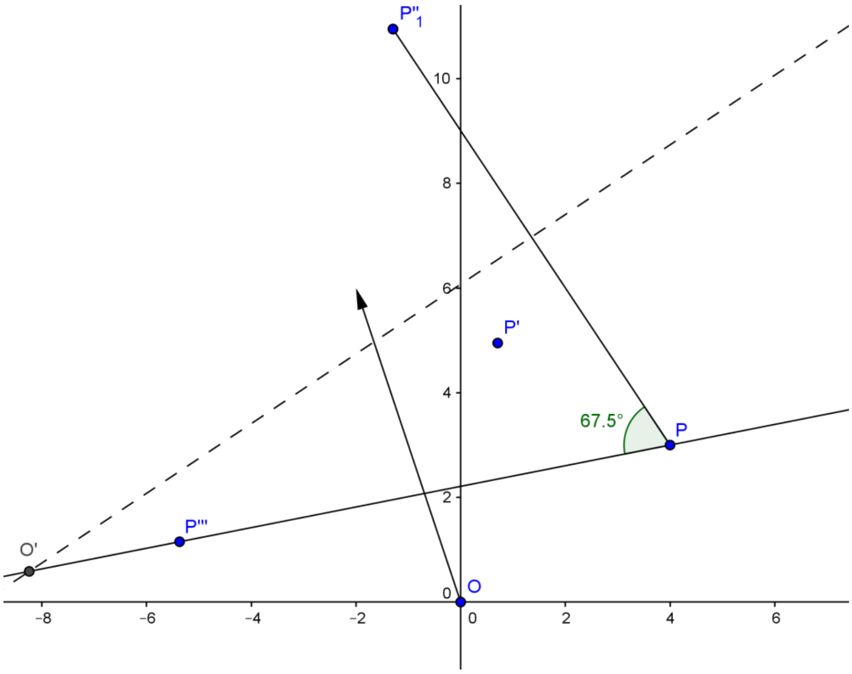

Figure 4. The same point $P_1^{''}$ is obtained as the image of P in two different ways:
1) under composition of rotation around the origin (0, 0) by 45˚ and then translating by (-2, 6);
2) by rotating P around O' by 45˚. A new origin (O') was found as an intersection of a perpendicular bisector of a line sector PP_1'' and a line forming an angle of 67.5˚ with this line sector.

<u>Commentary</u>:

In 2016 the students experiment with the given task: first they perform the given transformations and find the solution by means of GeoGebra, and then solve the same problem analytically. Thus, they can compare the answers received and make sure the solution is correct. Students found evidence that it is possible to get the same results using either geometric or algebraic methods.

Discussion

Dealing with the tension between teaching mathematics courses in a computerized environment and the mandatory requirement for the students to present formal proofs and answers led us to make important changes in the curriculum of the course "Plane transformations in geometry".

The analysis of all the data obtained from two academic years within a period of ten academic years revealed the changes both in our mode of teaching and in the level of involvement of our students in the learning process.

We found that in 2007, as instructors we behaved in a rather didactic manner, disseminating information to our students, while curricular activities relied mainly on textbooks and workbooks. The dynamic geometrical software was used only for visualization.

Referring to the data obtained in 2016, the following changes were found:

- Each new topic was explained and presented both analytically and using the computer, so that the students got the opportunity to explore the topic themselves.
- The students raised conjectures consistent with the given problem, before we presented a formal solution, and thereby created their own knowledge based on their findings.
- The learning activity became interactive, and the students' questions led to additional elaborations of the studied topics.
- The students performed the homework assignments not only analytically but also using GeoGebra.

We have analysed both the mode of our teaching and the level of our students' involvement in the learning process regarding basic characteristics of the traditional vs. the constructivist classroom. It was found that in 2007 they fit the most of characteristics of the traditional classroom, while in 2016 our classroom fits well enough the characteristics of the constructivist classroom. Thus, the obtained results permit us to conclude that during the period described our classroom changed from traditional to constructivist. Moreover, those changes are mainly due to the integration of dynamic mathematical tools into our teaching.

The described changes concur with relevant studies demonstrating the ability of the instructor to take advantage of the dynamic environment when the pedagogy of the course was entirely technology-oriented (Monaghan, 2001; Hollebrands, 2007), and claiming that that the constructivist approach intelligently utilizes a wide variety of computer capabilities to create a computerized learning environment that facilitates constructivist teaching methods (Eshet & Hammer 2006).

Based on our own teaching experience, we believe that teaching mathematics in a computerized environment contributes to understanding the formal subjects taught in mathematical courses. The students understand that although ultimately there must be a formal answer to a mathematical problem, there are various ways to reach the solution. Thus, a computerized environment can improve both learning in class and working at home while preparing assignments.

We as instructors constantly update the curriculum of the course according to our ongoing experience of integrating new technological tools, and we intend to continue using various technological tools that are appropriate to both our teaching goals and the students' levels of ability.

References

Barabash, M., Gurevich, I., & Yanovski, L. (2009). Usage of computerized environment in the course "Plane Transformations and Constructions in Geometry". *The International Journal for Technology in Mathematics Education*, 16(2), 49-62.

Blum, W., & Kirsch, A. (1991). Pre-formal proving: Examples and reflections. *Educational Studies in Mathematics, 22*(2), 183–203.

Brooks, J.G., & Brooks, M.G. (1993). *The case for constructivist classrooms*. Alexandria, Va.: Association for Supervision and Curriculum Development.

Chen, F. H., Looi, C. K., & Chen, W. (2009). Integrating technology in the classroom: A visual conceptualization of teachers' knowledge, goals and beliefs. *Journal of Computer Assisted Learning, 25,* 470–488.

Drijvers, P., Kieran, C., Mariotti, M. A., Ainley, J., Andresen, M., Chan, Y. C., & Meagher ,M. (2010). Integrating technology into mathematics education: Theoretical perspectives. In C. Hoyles & J-B. Lagrange (Eds.), *Mathematics education and technology—Rethinking the terrain; The 17th ICMI Study* (pp. 89–132). Vol. 13, New ICMI Study Series. New York: Springer.

Ertmer, P. A. (2005). Teacher pedagogical beliefs: The final frontier in our quest for technology integration. *Educational Technology Research and Development, 53*(4), 25–39.

Eshet, Y. & Hammer, R. (2006). *Principles of design and analysis of computer-based learning environments*. The Open University of Israel. 31-38. (Hebrew).

Gurevich I., & Gorev D. (2012). Examining the impact of an integrative method of using technology on students' achievement and efficiency of computer usage and on pedagogical procedure in geometry. *The International Journal for Technology in Mathematics Education*, *19*(3), 95-104.

Healy, L., & Hoyles, C. (2001). Software tools for geometrical problem solving: Potentials and pitfalls. *International Journal of Computers for Mathematical Learning*, *6*, 235-256.

Hollebrands, K. F. (2007). The role of a dynamic software program for geometry in the strategies high school mathematics students employ. *Journal for Research in Mathematics Education, 38*(2), 164-192.

Mittal, A. K., & Chawla, N. (2013). Pedagogy of mathematics: Role of technology in teaching-learning mathematics. *Journal of Indian Research, 1*(1), 105-110.

Monaghan, J. (2001). Teachers' classroom interactions in ICT-based mathematics lessons. In M. V. D. Heuvel, (Ed.), *Proceedings of the 25th International Conference for the Psychology of Mathematics Education, Vol. 3*, (pp. 239-257).

Pinto, M., & Tall, D. (2002). Building formal mathematics on visual imagery: A case study and a theory. *For the Learning of Mathematics, 22*(1), 2–10.

Sachs, J. M. (2014). Managing the challenges of technology to support learning: Some lessons from experience. In: M. Gosper & D. Ifenthaler (Eds.): *Curriculum models for 21st century: Using learning technologies in higher education*, 407-422. Springer.

So, H. J., & Kim, B. (2009). Learning about problem based learning: Student teachers integrating technology, pedagogy, and content knowledge. *Australasian Journal of Educational Technology, 25*(1), 101–116.

Unkefer, L., Shinde, S., & McMaster, K. (2009). Integrating advanced technology in teacher education courses. *TechTrends, 53*(5), 80–85.

COMPUTER SCIENCE IN MATHEMATICS' NEW CURRICULA AT PRIMARY SCHOOL: NEW TOOLS, NEW TEACHING PRACTICES?

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Abstract: Based on the observation of a teacher incorporating a programming language for the first time in his teaching, and on previous research centred on the development of teaching practices in mathematics, we highlight here the importance of didactic "landmarks", functioning as references in the dynamics involved along the development of teaching practices with ICT.

Keywords: teaching practices, Scratch, computer science, didactical landmark, instrumental distance

1. INTRODUCTION

In France, since September 2016, new mathematics curricula ask primary schools teachers and secondary mathematics teachers to integrate computer science, algorithmic, programming, using robots or new software such as Scratch. This latter is referred to all along the different school levels, pointing computer science knowledge but also more or less traditional mathematics notions such as the "location in space" (6/7 years-old, MEN 2015, p.86), the "production of simple algorithms" (8 y.o, ibid.), or the "notions of variables and functions" (from the age of 12, ibid. p.378). Yet, the difficuly for ICT to penetrate mathematics classrooms is not new, explained in many research by the "teacher barrier". Will it be different this time? How will practices using these new tools for new curricula develop over time?

We present here a case-study from the on-going ANR research project "DALIE" (Didactics and learning of computer science in primary school), where 24 ordinary teachers (with no training), volunteered to use robots and/ or Scratch software. We focus on the first sessions of René, a primary school teacher, who uses for the first time Scratch. As most primary school teachers and mathematics teachers, René is a beginner in both the functioning of this tool, in the knowledge that it embeds, and *a fortiori* in its didactic uses. How does Scratch become a teaching tool for René and for which aims? What knowledge and practices does he develop? What can be learned from this study for the teacher training to be set up but also the resources to support teachers?

The section 2 details the theoretical tools we use to analyse René's practices, based on our previous researches, and the section 3 our main results of observations. We end by a discussion in section 4.

2. THEORETICAL FRAMES FOR THE STUDY

Inscribed in the field of didactics of mathematics, our analyses are framed by two theoretical frames that we briefly present next: the Double Approach and the Instrumental Approach in didactics.

Components of practices and instrumental approach in didactics

The Double Approach frame (didactic and ergonomic) of Robert & Rogalski (2002) models teaching activity with five components (institutional, social, cognitive, mediative and personal). The institutional and social ones constraint the choices the teacher makes when organizing the students' work: at cognitive level (as choices of contents, of tasks…) and mediative one (space and time organization). Decisions are taken according to the teachers' own person (history, representation of teaching, of education, of mathematics, of learning, etc.). To explain here why teachers act such as they do, we take this personal component as crucial, supposing that daily cognitive/ mediave choices, are imprinted of several didactic knowledge, which pre-exists in this personal component. In other words, we think that the personal component *contains knowledge on the cognitive and mediative ones* themselves. This diversified knowledge acts as didactic landmarks guiding the subsequent activity, which refers to it in order to perform the cognitive and mediative choices.

To specify this general approach to the case of instrumented situations, we turn towards the Instrumental Approach in didactics (Artigue 2002, Guin, Ruthven and Trouche, 2004, Lagrange 1999), which borrows two of the keys ideas from the theory of instrumentation developed in cognitive ergonomic by Vérillon and Rabardel (1995 of): the process of instrumental genesis with its artefact/ instrument distinction, and the fact that this is not a one-way process. Rather there is a dialectic between the subject acting on her personal instrument (*instrumentalization:* the different functionality of the artefact are progressively discovered, eventually transformed in a personal way) and the instrument acting on the subject's mind (*instrumentation*: the progressive constitution of the cognitive schemes of instrumented actions). So, human activity transforms an artefact into an instrument across a long individual process of instrumental genesis, which combines these two interdependent mechanisms. Both points out that instrumentation is not neutral: instruments have impacts on conceptualizations. For example, using a graphic calculator to represent a function can play on student's conceptualizations of the notion of limit. This idea of not neutral "mediation", which exists (and always existed) between mathematics and instruments of mathematical activity, was used in several studies, first on symbolic calculators, then on other software as dynamic geometry or spreadsheets. In what follows, we introduce in more detail the notion that will be used from this frame: the distinction personal/ professional instrumental genesis.

Double instrumental genesis

Applying the notion of instrumental genesis to the teacher entails to divide it into a *professional* genesis and a *personal* genesis. To briefly present here this idea of *double instrumental genesis*, we go back to the research context, which gave birth to it: the study of the spreadsheet integration in mathematics classroom; more recent details can be found in (Haspekian, 2014).

For a person (the students, the teacher), an instrumental genesis (IGpe) can lead the artefact spreadsheet to become a personal instrument of mathematical work. In addition, for the teacher, the same artefact spreadsheet has to progressively become a didactic instrument serving mathematics learning, along a process of a professional instrumental genesis (IGpro). These are, for teachers and students, two different spreadsheet instruments, from the same artefact. In this "splitting in two" instrument, the important point is that they both exist on the teacher's side. The teacher has to organize the students' work, and accompany their instrumental geneses with the spreadsheet, a tool of students' mathematical work. This accompaniment evolves through the teacher's various experiments, along a professional genesis where the spreadsheet becomes an instrument for her professional activity: teaching mathematics. Unlike the students, the teacher thus faces two instruments, one personal (possibly ancient as in the case of pocket calculators for which a IGpe process has generally taken place, former to any teaching context), and a professional one, based on the transforming of the new artefact or already personal instrument (as the pocket calculator) into an instrument to teach mathematics. The example of the pocket calculator as didactic instruments is rather telling if considering the many (and now classic) situations of "broken machines" (in display, in use…) provided in educational resources and developed in this aim of mathematics teaching ("broken key", "defective machine" [1]). This calculator, as a didactic instrument, is quite different from the personal "pocket calculator" instrument, which is ordinarily neither defective, nor with broken keys…

IGpro and IGpe interfere one on each other. Haspekian (2014) shows these interferences in the case of a teacher integrating the spreadsheet while discovering it herself. But even when the IGpe is well advanced, we claim that the process of IGpro is far from being evident. More, it also has to take into account the student's instrumental geneses. Schemes has to be built aiming at organizing the ' work, accompanying their own instrumental geneses with the tool. This piloting role is necessary for Trouche (2004), who speaks about the teacher's *instrumental orchestration* (configurations and mode of exploitation of the tool in class) [2]. The figure 1 shows the relations between this teacher's double instrumental geneses interfering also with those of the students.

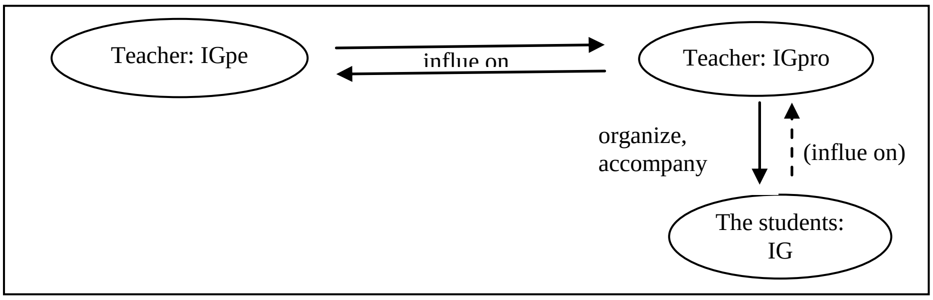

Fig 1 • Instrumental Geneses of the teacher (personal and professional) and the '

Scratch puts René in this complex case. His personal schemes of action with Scratch are evolving simultaneously, non-independently, with its professional schemes that aim students' learning. An additional difficulty comes in his case: knowledge to teach (computer science) is also new…

3. USING SCRATCH WITH FOURTH GRADE (9 YEARS OLD)

Methodology

Collected data consist of videos of the Scratch sessions and pre/ post interviews of the teachers. In these data, we try to understand the activity of the teacher with Scratch, the way instrumental geneses develop, particularly the links between IGpe (teacher and students) and IGpro (teacher). The sessions we focus here are situated at René's very beginning of IGpro: it is his 2nd session with Scratch, the first one consisting of a "free" discovering of Scratch by the students. What did René plan next? What knowledge does he aim at (mathematics? computer science? instrumental only?), through which functionality, in which order and under which modalities? In other words what are René's cognitive, mediative and instrumental choices? Another point makes this second session interesting: the class is divided into two groups with whom René repeats the same 1,5h session on two consecutive slots. We thus directly access to an instant of development of the teacher's IGpro, who's reinvesting with the 2nd group the marks taken with the first. It is interesting to see, in real time conditions, what types of marks he can he reinvest on the spot and why.

Main observed results

A detailed presentation of this session and its repetition is provided in Haspekian & Gélis (to come). We present here a synthesis of the two main results: on the one hand an IGpe too little advanced to efficiently support René's IGpro, on the other hand, despite the difficult situation, an evolution nevertheless of the IGpro, visible in the session repetition.

An IGpe too little advanced: consequences on the IGpro

In the session planed by Rene, the students were to answer two instructions [3] that, considering their own IG advancement with Scratch and their mathematical knowledge at this school level, were rising three foreseeable obstacles: first, the students did not yet meet the coordinates in Scratch, an instrumental knowledge (a) necessary here to both locate, control moves of objects, and give them an initial position. This relates to mathematical knowledge of this school level clearly mentioned in the curricula (location in a plan or space). Then, they did neither meet the necessity (b) to define a starting position with certain movement commands used (with absolute and nonrelative positions, like "Going to…"). This instrumental knowledge is not obvious, insofar as an incompleteness of the program is only visible if run twice (the object does not move anymore). Third, the students do not either know the existence (c) of "scripts of scenario" associated to each object, which is again non-intuitive (only one page of scripts is displayed at once) but necessary to control two or more objects.

Did René's task aim at making this knowledge emerge? The videos and interviews show that Rene did not prepare his session in this approach, having little identified himself these 3 points. Classroom interactions show René having the same interrogations as the students and discovering (a), (b) and (c), more or less realizing their importance on the spot. But René's personal knowledge of Scratch features, even if beginning, far from putting him in discomfort, is on the contrary utilized to show students the importance of seeking solutions, carrying out tests, not discouraging...

This too little advanced GIpe of Rene has two consequences on his GIpro: in the management of the students' GI, and in the definition of the learning objectives with Scratch. Indeed, having not himself anticipated knowledge (a), (b), (c), René could not effectively support the students' difficulties, nor help their IG advance with Scratch. At several moments, in the two sessions, Rene is looking for the origin of the problem. Sometimes he succeeds on the spot (it is the case for the knowledge (b) but in an incomplete way: for the objects moved by translation but not by rotation), but more often he blames Scratch features, saying they do not function well, or dismiss the problem without more explanation, the dysfunction remaining thus not understood by the students. Lastly, Rene does not manage Scratch like a didactic tool of learning mathematical concepts nor informatics concepts, which are not identified at this stage (for example, his vocabulary is unstable: "*coordinates*" is sometimes said "*codes of the character*" or "*codes of movement*"). Yet, René has two other objectives instead. In the interviews, he states aiming at the learning of the French language (reading and understanding of the commands, project of writing a novel, importance of the chronology of a story, of sequencing the actions…) and of transdisciplinary objectives (to seek, to try and adjust, to develop interactions between pairs).

Finding of landmarks and development of the IGpro

Observing Rene at the first stages of his GIpro with Scratch, we see the teacher taking reference points with the first semi-group, and immediately reinvesting part of them with the second.

If the knowledge (c), a bit identified in mid-session 1, is never mentioned again, René clearly evolves on (a): the interactions show that he discovers at the beginning of the group 1 session the display of coordinates on the screen. At the end of this session, he points them directly (yet without seeking the coordinate system that generates them): *"If you don't see it anymore, it means that the x and y coordinates you put are outside of the page. (...) look, there you have the coordinates of the pointer. If you move, the coordinates change"*. Then, with group 2 session, he anticipates and this time mentions (a) during the beginning collective exchange: *"I will save time compared to the previous group: see if we put the pointer here…"* The interview confirms that he discovered knowledge (a) during the session: *"the coordinates of the pointer were displayed on the screen!"*; *"Look here, there, here: x zero! y zero! I had not seen it but in fact when you move you have the exact position!"*

In the same way, but at a later stage (in session 2), René becomes aware of knowledge (b). Once this landmark taken, he immediately identifies the students' difficulties related to uninitialized positions and in a dialogue with a student, he clearly express regrets of not having specified it collectively: *"I see very interesting things but there is a point that you, uh, a point that, besides, we did not specify in common...'*. If the initial position problems in programs with displacements are thus well identified, the similar need to initialize a starting "orientation" in programs with rotations remains unidentified, leaving the students who encounter it blocked.

The table 1 summarizes René's evolution on (a), (b), (c) knowledge, along the consecutive sessions:

Knowledge	Group 1 session	Group 2 session
a : - Coordinates - Existence of a coordinate system	Coord : NO at the beginning, then awareness all along the session System : NO	Coord. : YES and beyond (asking a start and final point different) System : NO
b Initialize (if necessary) the starting position/ orientation	NO	NO at the beginning, then awareness all along the session for the displacements. No for the orientations.
c Scripts per object	NO at the beginning, then YES	YES and NO

Table 1 • Evolution of René's GIpe/pro along the 2 groups

Levers to manage the sessions while finding landmarks in parallel

René is 14 years experienced. His teaching practices are rather stabilized and coherent (Robert & Rogalski, 2002). The irruption of this new tool in the classroom destabilizes these equilibriums until evolving towards a new stability, which maintain the teacher's coherence in his professional activity. What is in the core of this process of evolution? The above analysis shows at least one thing: the teacher is taking landmarks on the utilization of Scratch. Here, this *constructive activity* (Samurçay and Rabardel, 2004) is occurring in the very time of the sessions, then how does Rene manage his sessions for the time duration needed to find landmarks? He reports himself needing time: "*I think it is necessary to redo a week more exercises of uh..., to discover a little because uh...*" But in spite of these difficult conditions (non-specialist, untrained, new tool and with unidentified underlying knowledge, be it algorithmic, mathematics or computer science), René remains at ease in the observed sessions, at no time in difficulties, neither at the macro level of its progression with Scratch, nor at the meso level of each session. What levers does he use?

Our hypothesis is that René has sufficient other landmarks (brought by his experience outside of tools as Scratch) to engage on innovative sessions without being toughly shaken, sessions that will provide him new landmarks. However, his use of Scratch does not lead him to use the tool with a mathematical or computer learning goal; he relates to transversal learning or French language learning. We make the hypothesis that these levers are not fortuitous choices, on the contrary they could be explained, again, in terms of landmarks acquired by the teacher, minimizing the distance that the software introduces to his everyday practices: René knows very well the teaching of French, and choosing transdisciplinary aims (group work, students' socialization, construction of a class project) also provides well-known landmarks, easily transferable because without underlying concepts.

4. DISCUSSION ET PERSPECTIVES FOR RESEARCH AND TEACHER TRAINING

Distance and landmarks

In earlier work, we have encountered two other cases, as René, of teachers minimizing the distance embarked by "newness" in old practices: the introduction of the spreadsheet into algebra teaching, which led to the idea of instrumental distance (Haspekian, 2014), and that of algorithmic into high school, where we observed similar phenomena to those of instrumental distance: tensions and resistances, practices of juxtaposition (homework, not integrated activities) or setting up of situations minimizing the "distance" that we then extended to a "distance to usual mathematical practices"

(Haspekian & Nijimbéré, 2016). The didactic landmarks appear as another hyphen to all these cases, like another face of the distance. If there is distance (to former practices), disturbing the teacher, not simply innovation, added without making waves on the current practices, it is because some didactic landmarks have been already built and the newness moves the teacher away from them, causing a loss feeling. New marks are to be created, either brought by training, resources, or by imagining them oneself or still accepting trying the experiment after all, in a blind manner. A first trial creates new reference marks which can lead to quite a different teaching activity at the second attempt. At a longer scale, several such experiments can supply the teacher with sufficiently robust landmarks enabling him to act in brand-new situations since they're not too distant on what was lived up, or since the teacher manages to bring it closer to what he knows. We saw these on-going processes happening in the case of Rene, but also with the spreadsheets. In other words, an enough experienced teacher will not only have more reference marks but may also be able to transpose, adapt old reference marks to create new ones more quickly and more easily than a beginner. In the same way, when we note phenomena of reduction of the distance, this translates the teacher's attempt to approach a situation in which she finds back didactic references. The distance is problematic when these landmarks are too much disrupted and/or without new ones being considered. For example, the factor making the spreadsheet instrumental distance too large had been analyzed as epistemological. The spreadsheet drops too much references on this dimension; letting the teacher with not enough didactic landmarks particularly in mathematical praxeologies. Thus, speaking of "distance" supposes the existence of an upstream referential to which new practices are compared. Whatever the term to name it, this referential serves the teacher to navigate in her daily practices by carrying a number of preexistent didactic landmarks (which can thus be disturbed, modified, searched, built, rebuilt…). The definition includes this idea of guidance of the teacher's later activity: a didactic landmark is a professional knowing, guiding the teacher in her action. The term "didactic" is taken in a very common sense, to specify that the elements of knowledge in which we are interested are those linked to the teaching-learning (including class management for instance).

The factors identified here and in our former research as contributing to create distance allow a categorization of the didactic landmarks, theoretically structured by the components of the Double Approach where we specifically isolate in the personal one: teachers epistemology and representations:

Even if legitimacy institutional (and social) is given and accepted, the teacher can still feel difficulties on the levels of: - The disciplinary knowledge embarked by the tool: a too long distance to the usual objects of teaching (for example in the case of the spreadsheet in algebra: distance to the discipline and importance of "epistemological" legitimacy). This level where the epistemology of the teacher plays relates to the personal component of the Double Approach (representations on a discipline, on its teaching, its learning) - Mediative knowledge of teaching: too large distance compared to the usual didactic landmarks (example of Scratch here). This level relates to the mediative component of the DA. - Knowledge on the learning of the concepts by the students, on the possible situations, their potentialities, the classic difficulties/ errors, the possible remediation…: the new object must present a cognitive legitimacy but this is not enough. Even if the teacher recognizes it, she can feel its implementation too distant from its current knowledge. This level relates to the cognitive component. - Knowledge on the curricula: the distance can be too large compared to the usual institutional landmarks. This level relates to the institutional and social component of the DA

Table 2 • Factors contributing to distance in general (instrumental in particular) hampering integration of newness (tool, domain or entire discipline)

From this, it comes out the following organization, which opposes legitimacies supporting newness integration to the tensions "landmark-distance" which slows it down:

	Legitimacy of the "newness"	Tension landmarks-distance
I: institutional	- legitimacy given by curricula, inspection, assess-	Require an appropriation on the part of the

S: social	ments, schoolbook; and by societal developments, fully immersed in technology	teacher: new landmarks are to be constructed here, even if curricula give some
Didactic: **- C**: cognitive **- M**: mediative	Research studies, professional training and literature, legitimize the contributions and benefits to cognitive levels (eg dynamic geometry for the notion of geometric figure, spreadsheets for entry into algebra ...) and mediative (saving time in the drawing of geometric constructions, in obtaining a large number of data, in the simulation of random experiments, in automated calculations, curve plots, illustration, etc.)	A priori, for an ordinary teacher: → loss of cognitive marks here → loss of mediative marks here Instrumental professional geneses are to develop in terms of orchestration, particularly to manage students' IG
P: Personal: **- E**: Epistemology of the teacher) **- R**: Representations	Legitimate/ foster or hinder (variable according to teachers): Depend on the person, her very knowledge of the disciplines at stake - Epistemology of the teacher on the impacted disciplines (epistemology of the discipline and of its teaching and learning) **-** Representation, in general, on teaching and learning (not specifically disciplinary)	
		Is function of the distance introduced by the "newness" regarding the disciplines usually taught

Table 3 • Legitimacies, landmarks and distance to ancient: the distance to current school practices is problematic if too few landmarks remain (I, C, M) (negative factors). This loss is counterbalanced on one part by the perceived/ conferred legitimacies at the levels (S, I, C, M) (positive factors), on the other part by the personal component, particularly the teachers' representation and epistemology in the concerned domain (P: R/ E) (factor positive or negative according to the person).

In conclusion, the quantity and the quality of the integration of a new object (in a broad sense) depend on two conditions on each one of the 5 components I, S, C, M, P: a condition on legitimacy and a condition on the didactic landmarks:

1. Legitimacy perceived/conferred by the teacher to this object at the institutional (I, S), didactic (C, M) and personal (E and R) levels
2. This legitimacy alone is not enough, the "newness" should not create (on the level of each components I, C or M) a too big distant situation to the usual practices where the teacher has landmarks (I, C or M), i.e that the integration of new can be done on landmarks close to the already acquired ones. A too large distance (for these components) hinders integration.

Finally, integration/or not, and its qualitative characteristics, depend on balance for each teacher between these various landmark-distance tensions (I, C, M) on one hand and the perceived/ conferred or not legitimacies (I, S, C, M and P) on the other.

Perspectives for rese arch and teacher accompaniment (training and resources)

The study of the case of Rene put in perspective with other research brings elements of comprehension of the practices in cases where the context "moves away" the teachers from their usual practices, either by the introduction of a new artifact, or by the introduction of a new field within mathematics, or by the introduction of a new discipline like informatics at elementary school. That led us to introduce the idea of "didactic landmarks" to speak about these common situations, idea that turns out to be the "counterpart" of that of distance. Defining and studying these are both objects of our current researches (with a theoretical link certainly necessary with the notion of *schemes* (concepts and theorems in acts, here professional; thus related to the Activity theory), but also with that of *beliefs* or Anglo-Saxon research on professional knowledge of the teachers: PCK model of Shulman (1986) and its later developments whose models are not based on the framework of the Double Approach). But if the didactic landmarks prove to be crucial, several interrogations upraise: how to facilitate their acquisition? Are some easier than others? Can some be more easily acquired in autonomy than others? In particular can we reasonably bet on the only experiment to develop didactic landmarks concerning the teaching of computer science concepts? The teachers in DALIE project (with Scratch or with robots) do not appear in a difficulty thanks to strategies of "substitution", why would they turn towards a new knowledge that they did not even identified and what could help them acquire the necessary associated landmarks? These reflections indicate ways for the resources and more generally for needs in teacher training, to work out new didactic landmarks, supporting former and new situations, taking into account various dimensions of these landmarks:

- knowledge disciplinary of the fields, possible praxeologies,
- didactic knowledge in link with these fields (cognitive, mediative, instrumental, including class management in general at mediative level, but also at instrumental one with the orchestrations),

These dimensions should not be separated if one wants changes in practices according the Double Approach frame (Robert & Rogalski, 2002). We assume that if certain didactic landmarks can be more or less quickly acquired through the development of the teacher's GIpro/pe, undoubtedly there is a need to accompany, through training and resources, some conceptual didactic landmarks.

REFERENCES

Artigue M. (2002). Learning Mathematics in a CAS Environment: The Genesis of a Reflection about Instrumentation and the Dialectics between Technical and Conceptual work. *International Journal of Computers for Mathematical Learning* 7 (3): 245-274.

Guin, D., Ruthven, K. & Trouche, L. (eds.) (2004) The Didactical Challenge of Symbolic Calculators: Turning a Computational Device into a athematical Instrument. Springer, New York.

Haspekian, M. et Nijimbéré, C. (2016) Favoriser l'enseignement de l'algorithmique : une question de distance aux mathématiques ? *Education et Didactique,* vol 10(3), 121-135

Haspekian, M. (2014). Teachers' instrumental geneses when integrating spreadsheet software. In Clark-Wilson, Robutti, Sinclair (eds.) *The Mathematics Teacher in the Digital Era*, Springer, pp.241-275

Lagrange J.B. (2000). L'intégration d'instruments informatiques dans l'enseignement : Une approche par les techniques. *Educational Studies in Mathematics 43* (1): 1-30.

Ministère de l'éducation national. (2015). Programme d'enseignement du cycle des apprentissages fondamentaux (cycle 2), du cycle de consolidation (cycle 3) et du cycle des approfondissements (cycle 4). *Bulletin Officiel du 26 novembre 2015*.

Robert, A. & Rogalski, J. (2002). Le système complexe et cohérent des pratiques des enseignants de mathématiques: Une double approche. *Canadian Journal of Science, Mathematics and Technology Education, 2* (4), 505-528

Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations, *International Journal of Computers for Mathematical Learning* 9, 281-307

Samurçay, R., Rabardel, P. (2004). Modèles pour l'analyse de l'activité et des compétences : propositions. In R.Samurçay, P.Pastré (dir.), *Recherches en didactique professionnelle*, Toulouse : Octarès, pp. 163-180.

Vérillon P. & Rabardel P. (1995), Cognition and artifacts: a contribution to the study of though on relation to instrumented activity. *European Journal of Education*. *10* (1). 77-101.

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1 see for instance MEN 2002, or Caron, F. (2007). Au cœur de « la calculatrice défectueuse » : un virus qu'on souhaiterait contagieux ! *Petit x 73*, 71-82, or also online resources, as for example: http://emmanuel.ostenne.free.fr/arras/rallye/rallye8.html or: http://calculatice.ac-lille.fr/calculatice/spip.php?article60

2 regarding Robert & Rogalski frame(2002), they are part of the teacher's personal component

3 with only one initial command, move two characters at the same time, then in a successive way

INTERACTIVE DIAGRAMS USED FOR COLLABORATIVE LEARNING

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The present research focuses on the development of knowledge about motion processes involving collaboration between students and interactive multiple representations diagrams. We designed three settings of interactive diagrams that share an example represented as an animation of multi-process motion but differ in their organizational functions. The 13- and 14-year-old students explored sets of characteristics of the mathematical models in the diagrams to analyze the related phenomena presented as real model and developed meaning of the abstract representations regarding the phenomena. The development of shared knowledge occurred when the students engaged in a reflective activity concerning the other members' reasoning and instruments involved in the collaborative process.

Keywords: Interactive Diagrams; Collaborative Learning; Mathematical Models; Animation

INTRODUCTION

Our research interests are concerned with the development of knowledge about mathematical models of motion processes involving collaboration between students and interactive diagrams. Interactive diagrams (IDs) are relatively small units of interactive text (in e-textbooks or other materials) and are important elements in e-textbooks. The ID`s components include: the given example, its representations (verbal, visual and other) and interactive tools. The difference between an ID and other interactive tools is that an ID is built around a pre-constructed example to carry a specific task. Whereas a static text presents information and a point of view implicitly engaging the viewer in meaningful interpretations, an ID explicitly requires the viewer to take action and change the diagram within given limitations.

Mathematical modeling is defined as the process of constructing a mathematical representation of reality that focuses on selected features of the reality being modeled (Cai, et al. 2014). To help learners construct mathematical representations of reality, the teaching-learning processes need to include the development of tools that will serve them in the practice. There are two approaches to teaching-learning mathematical modeling: (1) to learn by constructing models and (2) to learn by using models (Schwartz, 2007). But the two perspectives should not be in contrast with each other. Students who do not have experience with mathematical models will probably not benefit greatly from constructing their own models, if indeed they can learn to do so at all (*ibid.*). At first, learners tend to explore models by modifying their parameters. Next, they are often asked to modify the models themselves, thus providing them with the original and many similar models with which to work. Finally, students may be asked to devise models of phenomena independently. Pedagogic artistry, or the art of executing the teaching-learning process well, lies in helping students move through this sequence in ways that are appropriate to their current understanding of mathematical modeling.

Using technology to develop interactive curriculum materials, such as interactive textbooks, provides a captivating, engaging tool which encourages learners to explore mathematical models and to devise their own models as suggested by the learning sequence. The material presented in this way attempts to create new avenues for learners to develop knowledge about mathematical modeling. It is especially important that, while students learn about dynamic processes, such as similarly dynamic way by animations and interactive models in order to reinforce their knowledge development (Ainsworth, 2006;Yerushalmy and Naftaliev, 2011; Schwartz, 2007). The animations and models are simplifications that attempt to capture the essential features of the reality they describe. Technological developments are introduced into the range of resources available to students and teachers. In order to guide students to focus their attention on the essential details of the dynamic processes and to analyze the process, interactive curriculum materials should be designed to provide opportunities for exploration of mathematical models.

VISUAL SEMIOTIC ANALYSIS OF ID FUNCTIONS

There are profound differences between the traditional page in math textbooks that appears on paper and the new page that derives its principles of design and organization from the screen and the affordances of technology. Current technology allows for a variety of interactive tools, examples and representations. For example, IDs focused on motion may include the following components: a wide range of representations of motions; a wide repertoire of linking tools, and choices of activation of various representations. The question is, "How do a curriculum designer, a learner, and a teacher decide how and which IDs components of the text to use for different purposes in teaching-learning processes?" To explain some aspects of the design of an ID, we adopted a framework developed by semiotic research of text and visuals and provided a collection of categories that would allow an orderly discussion of the subject. (Naftliev and Yerushalmy, 2017; Yerushalmy, 2005). There are three ID's functions in the framework: the orientational function, the presentational function and the organizational function.

The presentational function focuses on what and how is being illustrated by the diagram. The reader may act within the context of the given example and change it or create other similar examples. Three types of examples are widely used: Random examples, Specific examples and Generic examples. The orientational function relates to the type of relationships that the text design attempts to set between the viewer and the text. IDs can function both as sketches and as diagrams in the sense that they can reveal their details.

The organizational function looks at the system of relations defining wholes and parts and specifically at how the elements of text combine. IDs can be designed to function in three different ways: Illustrating, Elaborating, Guiding. Illustrating IDs are simply operated unsophisticated representations. They are intended to orient the student's thinking to the structure and objectives of the activity by usually offering a single representation and relatively simple actions. For example, an Illustrating ID may have a limited degree of intervention by activation of controls in the animation (Table 1). At any time, users can freeze the positions on the track, continue the run, or initialize the race. Elaborating IDs provide the means that students may need to engage in activities that lead to the formulation of a solution and to operate at a meta-cognitive level. The important components in the design of the Elaborating IDs are rich tools and linked representations that enable various directions in the search for a solution. For example, the same animation that serves as an illustrating ID can be part of an elaborating ID when set within other tools and representations. The ID provides four adjacent, linked representations: a table of values that represents distance and time; a two dimensional graph of distance over time; a one dimensional graph which traces the objects' positions at each time unit; and an animation (Table 1). The variety of linked representations and rich tools in this elaborating ID enables various options in viewing the ID: as a sketch and/or as a neat diagram, as discrete information and/or as a continuous flow of information. We use the term Guiding IDs in relation to guided inquiry. This kind of diagram provides the means for students to explore new ideas. In addition to providing resources that promote inquiry, they also set the boundaries and provide a framework for the process of working with the task. The Guiding IDs are designed to call for action in a specific way that supports the construction of the principal ideas of the activity and may serve to balance constraints and open-ended explorations and support autonomous inquiry. For example, the guiding ID was designed around a known conflict about a time-position graph describing a "motionless" situation over continuously running time (Table 1). The ID consists of two representations of the motion of four cars: an animation and a hot-linked position-time graph. The task is to establish a one-to-one correspondence between the graphs and the cars. The graph and the animation are only partially linked: motion occurs simultaneously on the animation and on the graph but there is no color-match, so the identification process requires extracting data from the animation and the graph in order to link them. The following constraints contribute to making the task an interesting challenge: the small number of animated representations, the partial link between the representations, the absence of representations and controls that could turn the given sketchy nature of the representations into an accurate diagram, and the exceptional example in a list of examples that are aimed at focusing on a motionless situation over time.

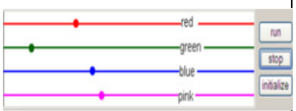
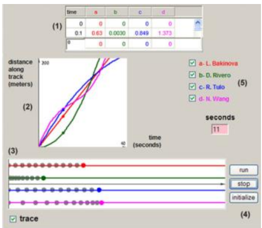
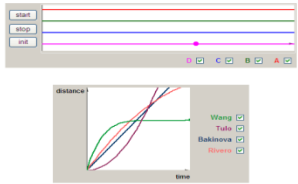
	Illustrating ID 	Elaborating ID 	Guiding ID 
Animation "run", "stop", "timer" choose components	 ✓, ✓, × ✓	 ✓ , ✓, ✓ ✓	 ✓ , ✓, × ✓
1D graph	×	✓	×
2D graph	×	✓ (neat and sketch)	✓ (sketch)
Table of Values	×	✓	×
Links between representations	×	✓	✓ (partial)
Examples	Generic	Generic	Generic (motionless component)

Table 1. Comparative view on the IDs' design

RESEARCH DESIGN

The series of activities included a preliminary activity and three comparable activities that contained different ID's. The IDs shared an example represented as an animation of multi-process motion but they were different in their organizational functions. The activity, which asked the students to describe a motion situation, was first illustrated by a video clip and subsequently as an Illustrating, Elaborating or Guiding IDs, all based on the animation. The three IDs varied by the design choices concerning what was included in the given example and how it was represented and controlled.

Regarding what to include in the example, the animation was designed around simultaneous multi-process motions, to include motion situations known to be challenging, such as non-constant rate-of-change and "no motion" situations, as well as surprising situations such as an ”unexpected win“. Considerations of how to design these choices were driven by the semiotic functions framework. We made comparative decisions about the variety and type of representations, the control features, and the linking features.

The interviews were sorted by students into groups of three. Each interviewee met the interviewer twice. The first meeting included an interview with each student individually. The second meeting was a group interview. Each participant followed a three-step procedure that enabled us to examine and track the role of IDs in the students' knowledge development process concerning mathematical models of motion. At the first stage, the students were given a preliminary task presented as a video clip and designed to evaluate their knowledge and solution techniques. At the second stage, the students were given a task similar to the one they received before, except that it was presented as an ID. The purpose of the interview was to learn how the students constructed their knowledge using the diagram. At the third stage, the three students who had been asked to address similar tasks that included different IDs shared their work and participated in a group discussion. The students were asked to describe the technique they used in their solution, to present their use of the ID, to reflect upon their changes and to be involved in a conversation regarding other students' techniques. The students could use all the diagrams they worked with in the previous stages. The interview time was flexible and varied according to the participants' responses. All the interviews were video recorded.

For the first step of the research, we analyzed the students' emerging, personal, engagement processes as they interacted with one of the three mathematical modeling IDs which were designed to support different functions of inquiry teaching-learning. The findings of our previous research show that similar tasks with different IDs should be considered as different learning settings (Naftaliev and Yerushalmy, 2013, 2017; Yerushalmy and Naftaliev, 2011). In the presented research, we focus on the second step by asking the students who had already been asked to address similar activities that included different IDs to share their work and to participate in a group discussion. The process allowed us to analyze the social construction of knowledge in a new pedagogical setting.

THE SOCIAL CONSTRUCTION OF KNOWLEDGE IN A NEW PEDAGOGICAL SETTING

We are going to analyze one of the groups' engagement processes to present the social construction of knowledge in a new pedagogical setting (Table 2). The group has one student for each of the three types of IDs: Illustrating, Elaborating, and Guiding. We'll begin by looking at their individual work in the two first stages. With video clips, the learners put the emphasis on getting the story right, which required attending to details such as the runners' body motion. The video clip kept learners too close to the situation and prevented them from thinking in the abstract. Elad, the student who worked with the illustrating ID, started by activating the animation. Throughout the process, he stopped the animation several times. During each pause, Elad examined the runners' respective positions and described the changes in speed between each stop. Elad described each runners' changes in speed with reference to their relative positions at specific moments. He mistakenly interpreted continuous change of speed by comparing relative positions. For example, he argued that passing another runner must have meant speeding up; whereas, in reality, the runner maintained a constant speed. To cope with the challenge, Elad resorted to a failed attempt at drawing graphs by himself to complete the diagram. Helena, the student who worked with the elaborating diagram, started by activating the representation and tools in the ID. She learned about the wide variety of options and representations available in the ID, but we didn't have evidence that showed developing knowledge concerning mathematical models of motion processes. Or, the student who worked with the Guiding ID, began his work by identifying a visual and kinematic conflict: while all seven dots moved on the graphs, one of the dots in the animation stopped and remained still. To resolve this conflict, he focused on discrete events much like Elad, using discrete events to match the motions described in the animation and graph extracting discrete motion characteristics such as: average speed, time and distance. He successfully matched the dots yet failed to resolve the conflict.

Knowledge development concerning Characteristics of Motion and the elements of IDs	In stage 2			In stage 3
	Elad with Illustrating ID	Helena with Elaborating ID	Or with Guiding ID	Elad, Helena and Or
Familiarize him/herself with the elements of the IDs	✓	✓	✓	✓
Discrete Characteristics (Animation): average speed, time for distance, distance	✓		✓	✓
Discrete Characteristics (Graph): average speed, time for distance, distance			✓	✓
Continues Characteristics describing the motion process (Animation and Graphs): such as speed, time, and distance as variables in motion processes				✓

Table 2. Knowledge developments in the second and third stage

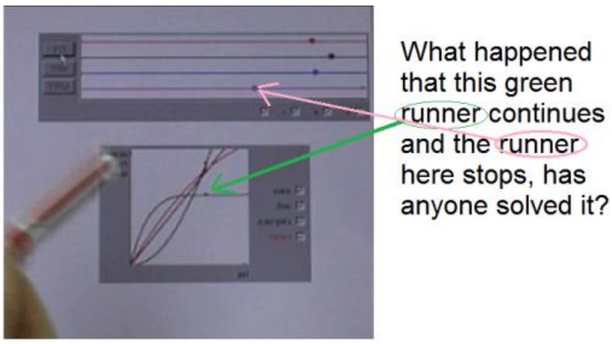


Fig. 1 "…has anyone solved it?"

Progressing to the group discussion, Or decided to open the conversation with the question which remained unsolved in his individual work (Fig. 1). He demonstrated the problem while activating the Guiding ID with which he worked. The two other participants, Elad and Helena, were intrigued by the question and it turned into the goal of their collaborative work. They began by familiarizing themselves with the options of the ID and examples presented in it to resolve the conflict. When they didn't have success resolving the conflict using the Guiding ID and realized their diagrams were different, Helena suggested using representations and tools from her Elaborating ID to accomplish the goal they defined for themselves. Each time she suggested adding only one of the options from the Elaborating ID. They used it firstly to develop meaning regarding the motion presented in the Elaborating ID. Then, they used the ideas which they developed to resolve the conflict using the Guiding ID. The following dialogue presents the process which took place in the last step of their work in which they successfully resolved the conflict.

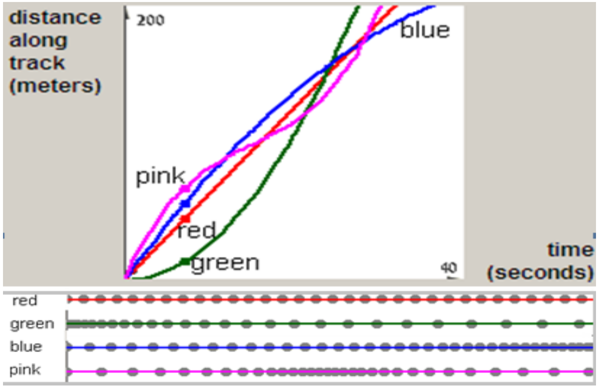


Fig. 2 2D and 1D (traces) graphs recorded while running the animation

Following suggestion of Helena, the students activated the animation with traces, resulting in the generation of a 1D graph of the motion (Fig. 2). While running the animation and generating of a 1D graph, they read the race from the traced motion using the size of the spaces between the traces as a gauge for speed:

Helena: Press on traces. You see! Where they are stopping?

Or: Ahh… Yes, it describes every time point.

Elad: It describes the steps, the distance of the steps.

Helena: Here, you see the black starts [green] to advance more

Elad: Pink starts with greater steps. If the traces describe the steps then here he starts to slow down as the time goes on and here it stays at the same speed

Helena: And the black [green] is really fast

Elad: But in the end he speeds a bit. The black [green] almost doesn't, he starts with slowness, as the time goes on, his steps only enlarge

Helena: The red doesn't change… and the red.. At the same speed

Elad: And the red, like I told you in the beginning, remember? That the red is always at the same distance, at the same speed, the same steps. And the blue at the beginning until the middle at the same speed, same steps and towards the end he starts to slow down.

Following the interpretation of the 1D graph as describing speed, the students check whether this option is available in the Guiding ID. Once they verify it is not, they returned to work with Elaborating ID. They began by interpreting the 2D graph based on the 1D graph in static mode with which they became familiar. At the end they were able to describe the speed by using only the 2D graph.

Helena: Wait, in his [Or] diagram there is it [the traces]? Check

Or: Check

Helena: It's interesting what happened with the pink in his [Or] diagram

Or: No. I think that this [the elaborating ID] is the best.

Helena: The red is running at the same speed. The black in the beginning runs really slow, and then he ups his speed more and more [they closed the 1D graph and continued work only with the 2D graph]. The blue runs really quick and then he starts to slow down. The pink runs fast, in the middle he slows down and then in the end again he runs fast.

Once they have succeeded in interpreting the 2D graph in the Elaborating ID, they were able to resolve the conflict they had about motionless process presented by the Guiding ID:

Or: Yes. So, as the line is steeper, then his speed is... ehh... it is steep and… that's it, I see that in the end it turns into a straight line, plain, something like this. That means that he slowed the speed and even stopped in place.

Elad: If this shows distance, then it means that the distance here does not change.

The episode describes exploration concerning the speed description in four stages: analysis of a dynamic mode of 1D graph which was linked to running animation, analysis of a static representation of 1D graph, analysis of shapes of 2D graphs and analysis of motionless process represented by 2D graph. They examined the graphs, each being composed of curving segments, representing an increase or decrease in speed, and straight segments, describing constant speed.

DISCUSSION

Students do need to have enough experience with abstract models to understand the point of mathematical modeling, its "language" (Schwartz, 2007). Once such representations exist in cognitive "baggage" of learners," it also becomes a tool for mathematical modeling (Wilensky, 1999). In our research we focused on knowledge development concerning mathematical models involving collaboration between students and various IDs.

The students explored sets of characteristics of the mathematical models in the IDs to analyze the related phenomena presented as real model and developed meaning of the abstract representations regarding the phenomena. They looked for ways to bypass the designed constrains of the Guiding ID: they develop meaning regarding the motion represented in the real and mathematical models by using the Elaborating ID, pointing to the speed, time, and distance as continues variables. Then they used the ideas which they had developed to analyze the characteristics of motion presented in the Guiding ID. At the end of the discussion, the mathematical models in static mode prompted them to mentally recreate and describe the motion processes.

The development of shared knowledge occurred when the students engaged in a reflective activity concerning the other members' reasoning and instruments involved in the collaborative process. As a result of the group collaboration, the students generated an interactive text. The participants did the following: posed a new question, decided what component from what ID to bring to discussion, decided on the sequence between the components, defined the role of each component, and, created a representation of the data. All of this was done to accomplish the goal they posed to themselves, thus building meaning concerning mathematical models of motion. Ainsworth (2006), who investigated the use of various technological packages that provide similar pairs of representations for understanding the mathematics of motion, suggested that the second more familiar or concrete representation was intended to bridge understanding of the more complicated and unfamiliar representation. Our analysis clarified that choosing and combining representations from similar tasks, which were designed as different IDs, reflected students' personal choices to anchor their inquiry in the ones they noticed first or with which they were more familiar. The interactive texts became an instrument which supported the development of shared knowledge concerning characteristics of kinematic phenomena and about their mathematical models.

IMPLICATIONS

To educators, who are challenged by the design and the implementation of interactive mathematics instructional materials, this study offers ways and terms to think about the design of interactive texts. Teaching with an interactive textbook should be considered more than a technological change; indeed, it is an attempt to create new paths for the construction of mathematical meaning. Other considerations related to specific affordances of technology should be studied to make the currently offered technological shift in learning and teaching materials an important sustained pedagogical shift.

REFERENCE

Ainsworth, S. (2006). DeFT: A conceptual framework for learning with multiple representations. *Learning and Instruction*, 16(3), 183-198.

Cai, J., Cirillo, M., Pelesko, J. A., Borromeo Ferri, R., Borba, M., Geiger, V., et al. (2014). Mathematical modeling in school education: Mathematical, cognitive, curricular, instructional, and teacher education perspectives. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the 38th PME conference.* Vancouver: PME.

Naftaliev, E., & Yerushalmy, M. (2011). Solving algebra problems with interactive diagrams: Demonstration and construction of examples. *Journal of Mathematical Behavior*, 30(1), 48-61.

Naftaliev, E., & Yerushalmy M. (2013). Guiding explorations: Design principles and functions of interactive diagrams. *Computers in the Schools*, *30*(1-2), 61-75.

Naftaliev, E., & Yerushalmy, M. (2017). Design digital tasks: Interactive Diagrams as resource and constraint. In Leung, A. & Baccaglini-Frank, A. (Eds), *The role and potential of using digital technologies in designing mathematics education tasks*, 153-173. Cham: Springer.

Yerushalmy, M. (2005). Functions of Interactive Visual Representations in Interactive Mathematical Textbooks. *International Journal of Computers for Mathematical learning*, *10*(3), 217-249.

Yerushalmy, M., & Naftaliev, E. (2011). Design of interactive diagrams structured upon generic animations. *Technology, Knowledge and Learning*, *16*(3), 221–245.

Schwartz, J. (2007). Models, simulations, and exploratory environments: A tentative taxonomy. In R. Lesh, E. Hamilton, & J. Kaput (Eds.), *Foundations for the future in mathematics education*, 161–172. Mahwah: Lawrence Erlbaum Associates.

Wilensky, U. (1999). GasLab? An extensible modeling toolkit for exploring micro- and macro-views of gases. In N. Roberts, W. Feurzeig, & B. Hunter (Eds.), *Computer modeling and simulation in science education*, 151-178. Berlin, Germany: Springer-Verlag.

COMPETENCIES AND DIGITAL TECHNOLOGIES – REFLECTIONS ON A COMPLEX RELATIONSHIP

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Advantages and disadvantages of the use of digital technologies (DT) and especially of computer algebra systems (CAS) in mathematics lessons are worldwide discussed controversially. Many empirical studies show the benefit of the use of DT in classrooms. However, despite of inspiring results, classroom suggestions, lesson plans and research reports, the use of DT – and especially CAS – has not succeeded, as many had expected during the last decades. The thesis of this article is that we – the researchers and teachers who are interested in the use of DT since many years or decades – have not been able to convince teachers, lecturers at university and parents of the benefit of DT in the classrooms in a sufficient way. In the following, the working with DT will be related to understanding and classroom activities. The basis of the argumentation is a competence model, which classifies – for a special content – the relation between levels of understanding (of the concept), representations of DT and different kinds of classroom activities.

Keywords: digital technologies, tool-competencies, representations, classroom activities, calculus.

CONCERNING THE USE OF DIGITAL TECHNOLOGIES (DT) IN MATHEMATICS CLASSROOMS

There are many theoretical considerations, empirical investigations and suggestions for the classroom concerning the use of DT in mathematical learning and teaching (e. g. Guin et.al., Zbiek 2007, Drijvers & Weigand 2010, Weigand 2013). In recent times, some empirical studies started integrating DT and especially computer algebra systems (CAS) into regular classroom teaching and covering longer periods of investigation. E. g. the *e-CoLab*[1] (Aldon et al. 2008), *RITEMATHS*[2], *CALIMERO*[3] (Ingelmann and Bruder 2007), *M³-Project*[4] (Weigand 2008, Weigand and Bichler 2010b, Weigand & Bichler 2010c). The main results of these projects and investigations can roughly be summarized as follows: DT (and especially CAS)

- allow a greater variety of strategies in the frame of problem solving processes;
- are a catalyst for individual, partner and group work;
- do not lead to a deficit in paper-and-pencil abilities and mental abilities (if these abilities are regularly supported in the teaching lessons)
- allow more realistic modelling problems in the classroom (but also raise the cognitive level of the understanding of these problems);

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1 e-CoLab = Expérimentation Collaborative de Laboratoires mathématiques. See: http://educmath.inrp.fr/Educmath/dossier-parutions/experimentation-collaborative-de-laboratoires-mathematiques. *Accessed 23 February 2017*

2 RITEMATHS = The project is about the use of real problems (R) and information technology (IT) to enhance (E) students' commitment to, and achievement in, mathematics (MATHS). http://extranet.edfac.unimelb.edu.au/DSME/RITEMATHS. *Accessed 23 February 2017*

3 CALIMERO = Computer Algebra in Mathematics Lessons: Discovering, Calculating, Organizing (translated title).

4 M³ = **M**odel Project New **M**edia in **M**athematics Education

- do not automatically lead to changed or modified test and examination problems (compared to paper-and-pencil tests);
- demand and foster advanced argumentation strategies (e.g. if equations are solved by pressing only one button).

Overall, Drijvers et.al. (2016) concluded from a meta-study-survey of quantitative studies, that there are "significant and positive effects, but with small average effect sizes" (p. 6), if for the benefits of integrating DT in mathematics education is asked. Moreover, there is also a broad consensus, that gainful changes in classroom teaching and learning need didactic and methodic considerations and a thorough thinking about the goals of teaching and learning.

VISIONS AND DISILLUSIONS

The first ICMI study in 1986 "The Influence of Computers and Informatics on Mathematics and its Teaching" (Churchhouse) was affected by a great enthusiasm concerning the perspectives of mathematics education in view of the availability of new technologies. However, in the ICMI Study 17 "Mathematics Education and Technology – Rethinking the terrain" (Hoyles & Lagrange 2010) and the OECD Study (2015), "Students, computers and learning. Making the connection", disappointment is quite often expressed about the fact that – despite the countless ideas, classroom suggestions, lesson plans and research reports – the use of DT has not succeeded, as many had expected at the beginning of the 1990s.

Worldwide, the current situation concerning the use of DT is very versatile. There are countries (like Norway or Denmark) that are intensively using laptops, tablets (with the programs Geogebra or Maple) or symbolic calculators (like the TI-Nspire or the Casio Classpad). These countries even allow using these tools in examinations. There are other countries (like the UK or France) that allow "only" symbolic calculators in examinations, there are countries – especially in Asia – which are very sceptical about the use in examinations, and there are countries (like Germany) where there are a different situations about the use of DT – depending on the state.

Reflecting the developments of the use of DT in the last decade, the results concerning the possibilities of supporting students' learning processes have been started to rethink and then especially raised the question: What is the benefit of using DT in the classroom? (Weigand 2017). More specifically important questions are:

1. In relation to which mathematical contexts does the use of DT make sense and which (mathematical) competencies are supported and developed?
2. Which mathematical and tool competencies are necessary, or at least helpful, when working with DT for specific mathematics content?
3. How can the DT-use be described in a more detailed form?

In the following it will be tried to give answers to these questions, by constructing a model that shows the relation between working with DT, different levels of understanding and specific activities in the classroom. The result is a three-dimensional competence model for DT-use.

COMPETENCE MODELS

Theoretical Foundations

The concepts of *competence* and *competence (level) models* have aroused interest in mathematics education in the past years. Starting with the NCTM Standards (1989) and especially the PISA studies, *competence* and *competencies* are expressions, often used in the context of standards and substituted the "old expression" *goals* which envisaged knowledge and abilities in mathematics education. "Mathematical competence means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role …" (Niss 2004, p. 120). In the PISA studies, competencies are on the one hand related to the *content*, e.g. numbers, space and shape, change, etc., and on the other hand – in a more general way – related to *processes* like problem-solving, modelling and the use of mathematical language. In order to evaluate or operationalize the competencies through the construction of items and tests, it is helpful to organize these competencies in levels, categories or classes. In the PISA studies, each of the possible pairs (content, process) can be divided into three different levels or competence classes (OECD 1999, p. 43): Class 1: reproduction, definitions, and computations; Class 2: connections and integration of problem solving; Class 3: mathematical thinking, generalisation and insight. This leads to a three-dimensional competence-model with the *dimensions* content, basic or process competencies and cognitive activation.[5]

Competence model for symbolic calculators while working with functions

In Weigand and Bichler (2010a) a competence model for the use of symbolic calculators with CAS in mathematics lessons in the frame of working with functions was developed. It can also be extended to the use of DT overall. Different levels of *understanding the function concept* have been seen in relation with the *representations* and – as a third dimension – with *cognitive activation.* These levels of understanding are not strictly hierarchical, because they are intertwined during the developing process, e. g. conceptual aspects have to be seen in relation to intuitive and relational aspects.

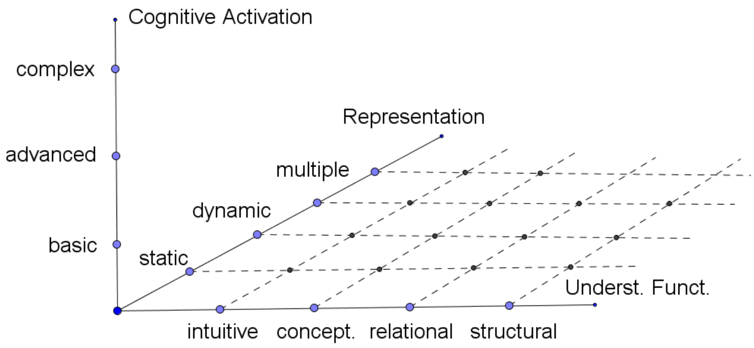

Fig. 1. Competence model for DT while working with functions

The ability or the competence to adequately use the tool requires technical knowledge about the handling of the tool. Moreover, it requires the knowledge of when to use which features and representations and for which problems it might be helpful. Three levels are distinguished, which might also be categorized by using DT as a (simple) function plotter, as a tool for creating dynamic animations and as a multi-representational tool.

Competence model for DT-use in the classroom

The model in Fig. 1 is gainful if tasks and problems have to be classified, e. g. for tests and examinations. It does not adequately fit if activities in the classroom should be integrated and evaluated. In the following model the second dimension "Representation" was changed due to the well-established theory of representation, which emphasizes the reasoning with multiple and dynamic representations (Bauer 2013 or Ainsworth 1999). Moreover, understanding and working with representations are seen in relation to classroom activities. A third dimension with the following activities will be introduced:

5 In PISA, these dimensions are called "Overarching ideas" (content), "Competencies" (process) and "Competence Clusters" (cognitive activation).

- Calculate: DT as a tool for (numeric and symbolic) calculations, and especially CAS are tools, which allow calculation on a symbolic level in notations close to the mathematical language. Example: Seeing parameter dependent functions as functions of several variables allows an efficient working in problem solving processes.
- Consult: DT – especially CAS – as a consultant in the sense of using a formulary. Example: $(a+b)^3=a^3+\ldots$.
- Control: DT as a controller of hand-written solutions, suggestions and ideas on a graphical, numeric or symbolic level.
- Explain: DT are catalysts for the communication between the user (student, learner) and someone who has to interpret or understand the DT-solutions (e.g. a teacher). DT are sources for explanations and argumentations.
- Discover: DT as a tool for evaluating and testing suggestions and strategies in a problem solving process.

This classification may be seen as a hierarchy while moving from a procedural knowledge (Calculate) to a conceptual knowledge (communicate, discover). Of course, also the activity "levels" are intertwined and do not represent a strict hierarchy. This new third dimension is more on the teaching side while the dimension "Cognitive activity" (Fig. 1) is more on the learning side.

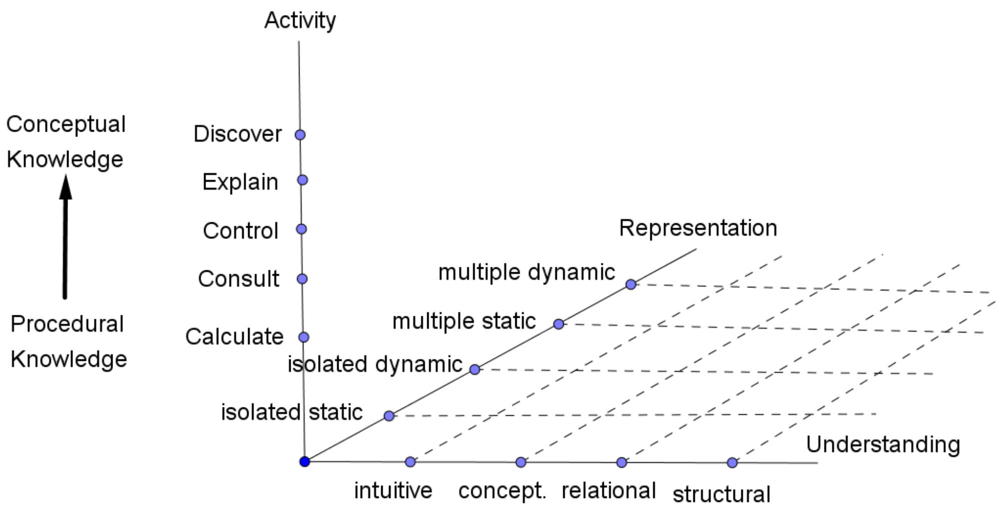

Fig. 2. An extended competence model integrating (classroom) activities

This competence model has three categories and gives us 4 x 4 x 5 = 80 cells. If each cell is again subdivided into three levels of cognitive activation, this makes a total of 240 cells and this is only for a special concept. This already shows that it is very difficult or even impossible to create special examples for each of these cells. This competence model is more for pointing out the directions and goals concerning understanding, the kinds of used representations, and the kinds of activities DT might be used for adequately develop a special concept.

What is meant by *tool competence*? A *tool* as "something you use to do something" (Monaghan et.al. 2016, p. 5) is the quite general definition. *Mathematical tools* allow us to create, to operate with and to change mathematical objects. DT and CAS are *digital tools*. The word *tool* is used instead of *instrument* because the facilities of DT in relation to mathematics aspects in the classroom are in the foreground, and the development of the user-tool-relationship in the frame of an instrumental orchestration, which is the heart of the instrumental genesis (see Artigue 2002, Drijvers et.al. 2010), is left to the user or learner. *Tool-competence* is the ability to refer the competence-model (Fig. 2) to a special concept. Tool-competence describes the development of the understanding of a concept in relation to the tool-representation in the frame of classroom-activities.

While empirical competence models – like the PISA model – help to answer the question *whether* students or learners do benefit from special learning or teaching interventions, the model (Fig. 2) is more process-oriented and should give reasons *why* and *how* this might be the case.

The problem of this model is the specification of the 80 cells with prototypical examples. It is not possible to construct a one-to-one-relationship between a cell and a special example. Problems, situations or examples can mainly or even always be seen under different aspects and never relate only to one level of understanding or to one level of activity. The challenge with regard to an empirical justification of this model is the construction of prototypes of examples which emphasize the triad of one cell of this model.

EXAMPLES

In the following the concentration is on CAS, the fields these tools are mostly used and DT promise the biggest changes compared to traditional courses: algebra and calculus. In this article the restriction on a few spotlights of examples is necessary.

Functions

Working with functions on a structural level means to see functions as objects, to use symbols like f and g on the symbolic level, to e.g. add (f + g) and multiply (f · g) them, and to represent them in different representations. It is expected that learners e. g.

- can work with functions on different levels of understanding;
- can work with functions as objects on a symbolic and a graphical level; they especially interpret changes of variables of a function as geometrical transformations;
- understand the definition of functions of several variables and they can– adequately to the situation–interpret them as functions of one variable with parameters;
- can use functions of several variables to solve mathematical and modelling problems.

Sequences

Working with *recursively-defined sequences* with $a_{k+1}=f(a_k)$, $k \in N$, a first element $a_1 \in R$ and a function $f: R \to R$, CAS allow to calculate the *sequence of iteration*

$a_1, a_2=f(a_1), a_3=f(a_2), \ldots$,

and to represent it numerically and graphically as k-a_k-diagrams or “cobweb-diagrams”. This can be done on a conceptual level of understanding and multiple dynamic representations can be used.

Difference sequences $(\Delta a_k)_N$ with $\Delta a_k := a_{k+1} - a_k$ and a given sequence $(a_k)_N$ are well suitable for a discrete introduction of the difference quotient (see Weigand 2015).

Overall, the meaning of CAS concerning the content *sequences* can be summarized like the following: CAS

- are tools with notations (or a language) quite close to mathematical notations (or the mathematical language);
- allow symbolic calculations and show related numeric and graphic representations;

- allow object-related working with sequences and discrete functions;
- have to be seen or evaluated in relation to other–especially graphical–representations.

Equations

A CAS is a formulary that offers in particular solution formulas for linear and quadratic equations and for systems of linear equations. In relation with a graphic representation, questions concerning the *number* of zeros of a quadratic function can (at first) be answered through experimental exploration. A CAS can be used to calculate the zeros of a function by only pressing one button, but moreover, it serves as visualization. Furthermore, the relation of function and equation is fundamental for the mutual representation in the CAS and the graphic window.

If the CAS is used tool for solving systems of equations with parameters, learners work on a structural level of understanding with multiple dynamic representations.

Example 1: Systems with quadratic equations can be calculated (on a symbolic and graphic level)

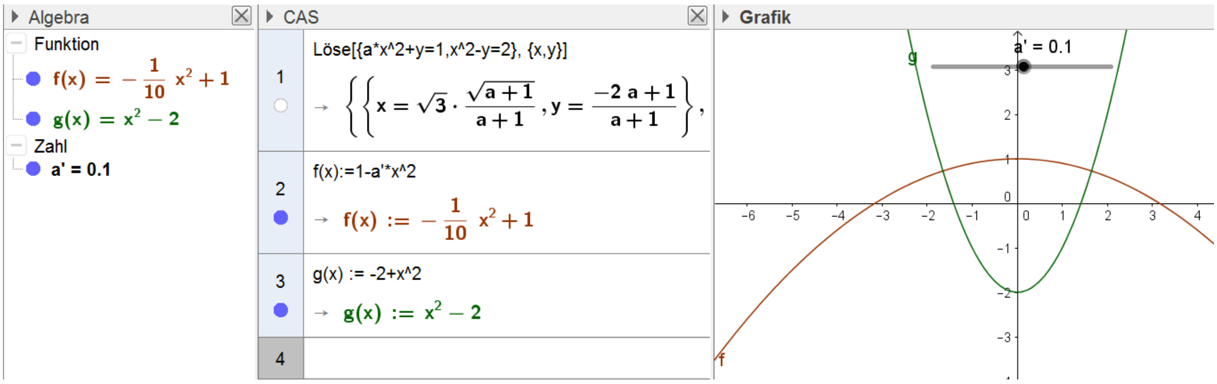

Fig. 3. Solving a system of quadratic equations with one parameter

The CAS provides calculations and solutions on the symbolic level and these have then to be interpreted, especially in relation to the graphical level.

Example 2: The symbolic solutions of more complex equations like $x^3 - x + 1 = 0$, $1 + sin(x) = 2^x$ or $x^7 - 4x^5 + 4x^3 = 0$ depend on the equation. But an efficient use of a CAS is only possible if it is based on mathematical knowledge concerning the solution of equations, the characteristics of the underlying functions of the equations and the possibilities of the solution varieties. For calculations the CAS is used mainly within the static isolated symbolic representation, adding graphic representations for interpreting or explaining symbolic results. The advantage of using CAS is the notation of solutions on a symbolic level, especially while working with equations with parameters. The communication with the tool is possible in a language close to the traditional mathematical language. The CAS is a consultant in the sense of a formulary for symbolic solutions especially for polynomial equations of order 2 or 3.

CONCLUSIONS

The developed competence model is a theoretical or normative model. It applies the understanding of a concept to the working in the classroom and the use of a tool (with different representations). It is a model for the evaluation of the process and development of understanding of a special concept in a tool-supported classroom.

Concerning the empirical justification of the theoretical model and with the aim of constructing an *empirical* competence model, some questions have to be answered.

1. *Task development*: Tasks and appropriate learning environments concerning a special concept for the levels of understanding and kinds of activities have to be developed, which promise a benefit while working with DT compared to traditional paper and pencil working. Will the learners be able to work adequately (in the theoretically expected way) with different kinds of representations and do they develop a normative expected understanding on different levels?

2. *Micro-connectivity of the tool-use*: Working with a CAS on the symbolic level has to be seen in relation with other representations, especially with numerical and graphical representations (the aspect of multiple representations). These additional representations allow interpretations of symbolic results and expressions. Which representations are adequate on which levels of understanding and which kinds of activities?

3. *Dynamic aspects.* The dynamics of representations have to been seen in relation to the dynamic aspects of variables, and in consequence to the dynamics of the concepts of function, equation, derivative, … How is the transition from static to dynamic representations and how is it related to levels of understanding and kinds of activities?

4. *Diagnostic instrument*: The competence model might also be used for diagnostic reasons to evaluate the "tool-competencies" of special learners (or one special learner). Diagnostics are the first step while improving students' understanding; the second step is to establish consequences to *improve* these. How can a learner be supported the best way to attain tool-competencies?

5. From a qualitative to a quantitative model: The PISA studies use a model with a numerical competence scale, which is based on the relative frequency with which students are able to solve a problem. The problem that has been solved successfully is taken as a measure of the difficulty of the exercise. The scale is standardized on a mean value of 500 with a standard deviation of 100 (OECD 2003). This might also be an aim in the context of this competence model. How can the competence model (Fig. 2) be extended or transformed to a quantitative model?

References:

Ainsworth, S. (1999). The functions of multiple representations. *Computers & Education*, 33, 131–152

Aldon, G., Artigue, M., Bardini, C., Trouche, L. (Ed.) (2008). *Une étude sur la conception et les usages didactiques d'une nouvelle plate-forme mathématique, potentialité, complexité.* http://educmath.inrp.fr/Educmath/dossier-parutions/experimentation-collaborative-de-laboratoires-mathematiques. Accessed 29 May 2010.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, *7*, 245–274.

Bauer A. (2013). Reasoning with multiple and dynamic representations. In: Faggione, E., Montone, A.: *Proceedings of the ICTMT 11*, Bari. 327-329 http://www.dm.uniba.it/ictmt11/download/ICTMT11_Proceedings.pdf

Drijvers, P. et.al. (2010). Integrating technology into mathematics education: Theoretical perspectives. In: Hoyles, C., Lagrange, B. (eds.) *Mathematics education and technology-rethinking the terrain. The 17th ICMI study*. Dordrecht: Springer: 89–132

Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y., Machietto, M. (2016). Uses of Technology in Lower Secondary Mathematics Education. ICME-13 Topical Surveys. Springer Open Access

Drijvers, P., Weigand, H.-G. (Hrsg.) (2010). Handheld technology in the mathematics classroom – theory and practice, *ZDM - The International Journal on Mathematics Education*, 7

Guin, D., Ruthven, K. and Trouche L. (Eds.) (2005). *The didactical challenge of symbolic calculators*. New York: Springer.

Hoyles, C. & J.-B. Lagrange (Eds.) (2010). *Mathematics Education and Technology – Rethinking the Terrain. The 17th ICMI Study*, Springer: New York a. o.,

Ingelmann, M., Bruder, R (2007). Appropriate CAS-use in class 7 and 8. In J.-H. Woo et al. (Ed.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*. PME. Seoul: The Korea Society of Educational Studies in Mathematics

Kultusministerkonferenz (KMK) (2004). *Bildungsstandards im Fach Mathematik für den mittleren Schulabschluss*. München: Wolters Kluwer.

Kultusministerkonferenz (KMK) (2012). *Bildungsstandards im Fach Mathematik für die Allgemeine Hochschulreife*. München: Wolters Kluwer.

Niss, M. (2004). Mathematical competencies and the learning of mathematics: The Danish KOM project. In A. Gagtsis, S. Papastavridis: *3rd Mediterranean Conference on mathematical education, 3–5 January 2003, Athens, Greece* (pp. 115–124). Athens: The Hellenic mathematical society, 2003.

OECD: Organisation for Economic Co-operation and Development (Ed.) (1999). *Measuring student knowledge and skills. A new framework for assessment.* Paris: OECD Publication Service.[6] http://www.pisa.oecd.org

OECD (Ed.) (2003). *The PISA 2003 Assessment Frmework: Mathematics, reading, science and problem solving knowledge and skills.* Paris: OECD Publication Service.[7] http://www.pisa.oecd.org

OECD (2015). *Students, computers and learning: making the connection*. Paris: OECD Publishing. http://www.oecd.org/edu/students-computers-and-learning-9789264239555-en.htm

Trouche, L. (2005). Instrumental Genesis, individual and social aspects. In: D. Guin, K. Ruthven, K., L. Trouche (Eds.). *The didactical challenge of symbolic calculators* (pp. 197-230). New York: Springer.

Weigand, H.-G. (2008). Teaching with a symbolic calculator in 10th grade - Evaluation of a one year project. *International Journal for Technology in Mathematics Education, 15*(1), 19–32

Weigand, H.-G. (2013). Looking back and flashing ahead - Didactical implications for the use of digital technologies in the next decade. Faggione, E., Montone, A.: *Proceedings of the ICTMT 11*, Bari. 298-303 http://www.dm.uniba.it/ictmt11/download/ICTMT11_Proceedings.pdf

Weigand, H.-G. (2010). Book Review of Hoyles, C. & J.-B. Lagrange (Eds.) (2010). *Mathematics Education and Technology – Rethinking the Terrain. The 17th ICMI Study*, Springer: New York a. o., ZDM - The International Journal on Mathematics Education 42(7), 801-808

Weigand, H.-G. (2015). From an intuitive-oriented to a content-oriented understanding of the basics of calculus. Amado, N. & Carreira, S. (Eds.). (2015). Proceedings of the 12th International Conference on Technology in Mathematics Teaching. University of Algarve, Faro, Portugal. 95-102. Siehe hier: http://sapientia.ualg.pt/handle/10400.1/6081

Weigand, H.-G. (2017). What is or what might be the benefit of using computer algebra systems in the learning and teaching of calculus. In: Faggiano, E., Ferrera, F., Montone, A. (Ed.). Digital Technologies. Springer. In press.

Weigand, H.-G., Bichler, E. (2010a). Towards a Competence Model for the Use of Symbolic Calculators in Mathematics Lessons – The Case of Functions, ZDM - The International Journal on Mathematics Education 42(7), 697-713

Weigand, H.-G., Bichler, E. (2010b). Symbolic calculators in mathematics lessons – The case of calculus. In *International Journal for Technology in Mathematics Education*, *17*(1), 3–15.

Weigand, H.-G., Bichler, E. (2010c). The long-term Project “Integration of Symbolic Calculator in Mathematics Lessons" – The case of Calculus. In: Durand-Guerrier, V., Soury-Lavergne, S., Arzarello, F. (Eds.). *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education*. 2009, Lyon (France). 1191–1200 http://ife.ens-lyon.fr/publications/edition-electronique/cerme6/wg7-15-weigand-bichler.pdf

Zbiek, R. M., Heid, M. K., Blume G. W., Dick, Th. P. (2007). Research on technology in mathematics education. In F. K. Lester (Ed.). *Second Handbook of Research* (pp. 1169–1206). Charlottte, NC: Information Age Publishing

6 http://www.pisa.oecd.org/document/58/0,2340,en_32252351_32236159_33688954_1_1_1_1,00.html

7 http://www.pisa.oecd.org/document/29/0,3343,en_32252351_32236173_33694301_1_1_1_1,00.html

Workshops / Posters

BLENDING COMPUTATIONAL AND MATHEMATICAL THINKING IN PRIMARY EDUCATION: THE SCRATCHMATHS PROJECT IN ENGLAND

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A 3-year design-based research project in England, ScratchMaths, has developed a set of curriculum materials for the last two years of primary school.These materials use the Scratch programming language to blend computational and mathematical thinking. In this workshop you will have the opportunity to explore some of the curriculum activities as a means to discuss the potential impacts of the underlying pedagogy and curriculum content on pupils' classroom experiences in other countries' contexts.

Keywords: mathematical thinking; computational thinking; Scratch programming language;

INTRODUCTION

Internationally, many countries are seeing a resurgence on the teaching of computer programming or 'coding' within formal and informal educational settings. In England, a revised national curriculum introduced coding within primary school education, which created a sudden need for both curriculum resources and teacher professional development opportunities. Pupils in Year 6 also sit a national examination in mathematics, which is 'high-stakes' in that it both contributes to all primary schools' nationally published accountability data and provides a baseline for pupils' mathematical outcomes as they enter secondary school. A 3-year project funded by the Education Endowment Fund has led to the development of a set of curriculum units that use the *Scratch* programming language (MIT Media Lab, 2013) and are aimed at the final two years of primary school education, Years 5 and 6 (9-10 years and 10-11 years respectively). These materials, which bridge ideas from the computing and mathematics curricula, have the underlying aim to improve pupils' mathematical outcomes. The project team has outlined the theoretical foundations for the ScratchMaths project and reported on aspects of its design and early implementations (Benton, Hoyles, Kalas, & Noss, 2016, 2017). Fundamental to our approach is the foregrounding of the Sc4atchMaths pedagogy, 'the 5 Es', Explore, Envisage, Explain, Exchange and BridgE: all will be explained in the workshop!

The workshop will be 'hands-on', which means that you will be expected to explore some activities from the ScratchMaths curriculum resources by interacting with the Scratch programming environment and the accompanying pupil andteacher materials. As we anticipate that many of the workshop's international participants will have examples of other similar projects, we will aim to address questions such as:

- What are the points of intersection between the primary school computing and mathematics curricula? How does this vary between countries?
- How can these points of intersection be developed?
- Can the language of programming, and the ideas and the approaches represented within it, offer a more open, more accessible and more learnable mathematics without sacrificing what makes mathematics work?

REFERENCES

Benton, L., Hoyles, C., Kalas, I., & Noss, R. (2016). *Building mathematical knowledge with programming: insights from the ScratchMaths project*. Paper presented at the Constructionism Conference 2016, Bangkok, Thailand.

Benton, L., Hoyles, C., Kalas, I., & Noss, R. (2017). Bridging Primary Programming and Mathematics: Some Findings of Design Research in England. *Digital Experiences in Mathematics Education*, 1-24. doi:10.1007/s40751-017-0028-x

MIT Media Lab. (2013). Scratch (Version 2). Massachusetts: Massachusetts Institute of Technology.

Chapter 3

ASSESSMENT

A, B, OR C? EXPLOITING POLLS AS A FORMATIVE ASSESSMENT TOOL FOR MATHEMATICS IN A CONNECTED CLASSROOM ENVIRONMENT

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This contribution addresses the theme of technology for formative assessment in the mathematics classroom. Taking a design-based research approach within the European project FaSMEd, we focus on the ways connected classroom technology may support formative assessment strategies in whole class activities. We will refer to a theoretical framework developed within the FaSMEd project, which relates the development of different formative assessment strategies by different agents (teacher, peers, and the student) to different technology functionalities. In particular, we will focus on the functionalities that allow to submit polls to students, gather the answers from them and show the results (both individual answers and cluster ones) in real time. With reference to the theoretical framework and existing literature, we discuss, how the polls can be used, during classroom activities, to foster the activation of formative assessment strategies.

Keywords: Connected classroom technologies, formative assessment, polls, classroom discussion

INTRODUCTION AND BACKGROUND

Research has highlighted the support given to formative assessment (FA) by the so called Connected classroom technologies (CCT), i.e. networked systems of computers or handheld devices specifically designed to be used in a classroom for interactive teaching and learning (Irving 2006). CCT include: classroom response systems (Roschelle & Pea 2002), networked graphing calculators (Clark-Wilson 2010), and participatory simulations (Ares 2008). Specific features of CCT that make them effective tools for FA are related to the support they may provide in:

1. monitoring students' progress, collecting the content of students' interaction over longer timespans and over multiple sets of classroom participants (Roschelle & Pea 2002) and giving powerful clues to what they are doing, thinking, and understanding (Roschelle et al. 2004);
2. providing students with immediate private feedback, supporting them with appropriate remediation and keeping them oriented on the path to deep conceptual understanding (Irving 2006);
3. fostering positive student's thinking habits, such as arguing for their point of view, creating immersive learning environments that highlight problem-solving processes (Irving 2006);
4. enabling the students taking a more active role in the class discussions and encouraging them to reflect and monitor their own progress (Roschelle & Pea 2002, Ares 2008).

In our research we focused on the way CCT may be exploited for formative assessment during whole class activities. In particular, in this contribution we focus on a specific feature of the CCT we investigated: the possibility of activating *polls*. Polls are a typical characteristic of what research calls Classroom Response System (CRS), which consists of a set of input devices for students, communicating with the software running on the instructor's computer, and enabling the instructor to pose questions to students and take a follow-up poll (Beatty & Gerace 2009). Beatty and Gerace (ibid.) observe that one crucial feature of CRS is that they simultaneously provide anonymity and accountability, support collecting answers from all students in a class, rather than just the few who speak up or are called upon and enable recording the data of students' individual and collective responses for subsequent analysis. They also highlight the flexibility in the use of CRS technology, listing specific instructional purposes connected to its use. Among them: (a) the use of polls for status check, that is to ask students their self-reported degree of confidence in their understanding of a topic; (b) exit poll, that is to poll students to find out which concepts they want to spend more time on; (c) assess prior knowledge, that is to elicit what students know or believe about a topic; (d) provoke thinking, that is to ask a question to get students engaged within a new topic; (e) elicit a misconception; (f) exercise a cognitive skill, that is to engage students in a specific cognitive activity; (g) stimulate discussion with questions having multiple reasonable answers; (h) review, that is to pose questions aimed at remainding students a body of material already covered.

Notwithstanding the potential of these tools, many researchers have stressed that the effectiveness of these technologies depends on the skill of the instructor and on his/her ability to incorporate procedures such as tracking students' progress, keeping students motivated and enhancing reflection with technologies (Irving 2006). Different studies have highlighted that CCT have increased the complexity of the teacher's role with respect to 'orchestrating' the lesson (Clark-Wilson 2010, Roschelle & Pea 2002). Therefore, in order to bring about progress in student participation and achievement, technology must be used in conjunction with particular kinds of teaching strategies.

Beatty and Gerace (2009) developed *technology-enhanced formative assessment (TEFA),* a pedagogical approach for teaching science and mathematics with the aid of a CRS. To help teachers implement FA, the TEFA approach introduces an iterative cycle of question posing, answering, and discussing, which forms a scaffold for structuring whole-class interaction. The essential phases of the cycle are: 1) pose a challenging question to the students; 2) have students wrestle with the question and decide upon a response; 3) use a CRS to collect responses and display a chart of the aggregated responses; 4) elicit from students different reasons and justifications for the chosen responses; 5) develop a student-dominated discussion of the assumptions, perceptions, ideas, and arguments involved; 6) provide a summary, micro-lecture, meta-level comments.

In our research we focus on the use of polls to enhance *effective classroom discussions with FA purposes*. In this contribution we will analyse, in particular, how the processing of students' answers by technology can be exploited to activate different FA strategies. This study is part of a wider design-based research, characterized by cycles of design, enactment, analysis and redesign, where the goal of designing learning environments is intertwined with that of developing new theories (DBRC 2003). The research is carried out in authentic settings (classroom environments), focusing on "interactions that refine our understanding of the learning issues involved" (ibid. p. 5).

FORMATIVE ASSESSMENT WITH TECHNOLOGY: A THEORETICAL FRAMEWORK

FA is conceived as a method of teaching where "evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited" (Black & Wiliam 2009, p. 7).

Taking this perspective, in the FaSMEd project we developed a three-dimensional framework for the design and implementation of technologically-enhanced formative assessment activities (Aldon et al. 2017, Cusi, Morselli and Sabena 2017). The starting point is the work by Wiliam and Thompson (2007), who identified five key strategies for FA: (A) *Clarifying and sharing learning intentions and criteria for success;* (B) *Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding;* (C) *Providing feedback that moves* *learners forward;* (D) *Activating students as instructional resources for one another;* (E) *Activating students as the owners of their own learning.* These FA strategies may be activated by three agents: the teacher, the peers and the student himself. The FaSMEd framework extends this model of FA, taking into account the two dimensions already included (FA strategies and the agents activating such strategies), and adding a further dimension: the functionalities of technology. Technology, indeed, may support the three agents in developing the FA strategies in different ways, which we categorized in three *functionalities*:

(1) *Sending and displaying*, that is the ways in which technology support the communication among the agents of FA processes (e.g. sending and receiving messages and files, displaying and sharing screens or documents to the whole class...).

(2) *Processing and analysing*, that is the ways in which technology supports the processing and the analysis of the data collected during the lessons (e.g. through the sharing of the statistics of students' answers to polls or questionnaires, the feedbacks given directly by the technology to the students when they are performing a test…).

(3) *Providing an interactive environment*, that is when technology enables to create environments in which students can interact to work individually or in group on a task or to explore mathematical/scientific contents (e.g. through the creation of interactive boards to be shared by teacher and students or the use of specific software that provide an environment where it is possible to dynamically explore specific mathematical problems…).

The following chart[1] (fig.1) schematizes the FaSMEd three-dimensional model.

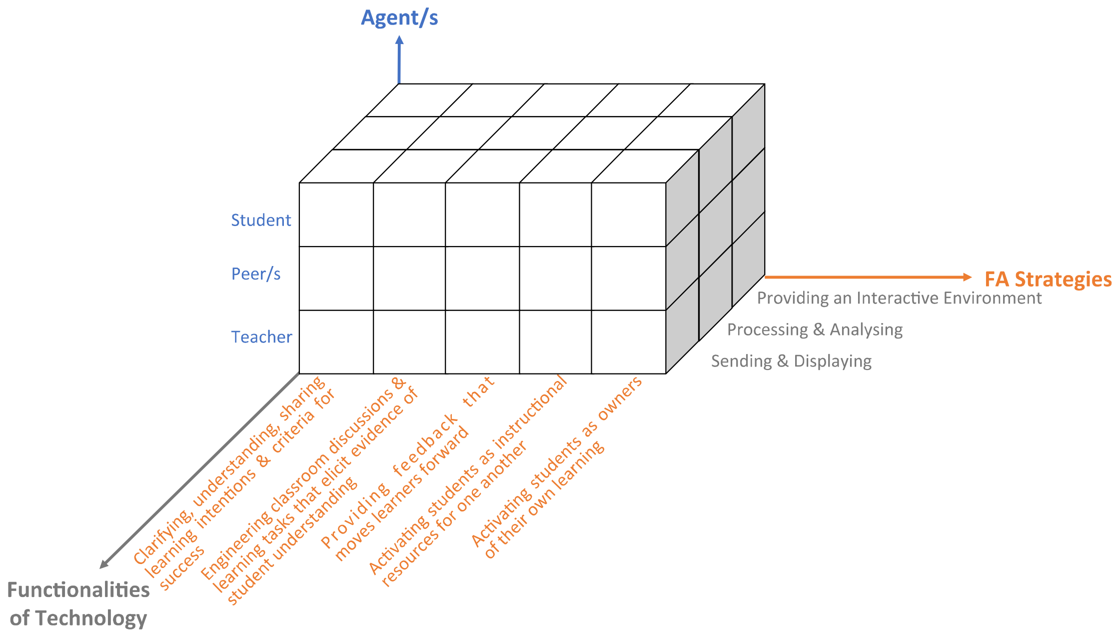

Fig. 1: Chart of the FaSMEd three-dimensional model

[1] We thank D. Wright (Newcastle University) for the digital version of the chart and Hana Ruchniewicz (University Of Duisburg-Essen) for its adaptation.

DESIGNING FA ACTIVITIES WITHIN A CCT ENVIRONMENT

In our design study we adopted a Vygotskyan perspective on the crucial role of the interaction with peers and with an expert in students' learning (Vygotsky 1978). Moreover, we believe that FA has to focus also on metacognitive factors (Schoenfeld 1992). Accordingly, we designed activities aimed at supporting students in (a) making their thinking visible (Collins, Brown & Newmann 1989), through the sharing of their thinking processes with the teacher and the classmates, by means of argumentative processes, (b) developing their ongoing reflections on the learning processes. Effective mathematical discussions (Bartolini Bussi 1998) are considered a key activity, where the teacher plays a key role in planning and promoting fruitful occasions for FA and learning.

Concerning technology, we explored the use of a CCT (provided by a software called IDM-TClass), which connects the students' tablets with the teachers' laptop, allows the students to share their productions and the teacher to easily collect the students' opinions and reflections, during or at the end of an activity, by means of the creation of instant polls.

The use of IDM-TClass was integrated within a set of activities on relations and functions, and their representations (symbolic representations, tables, graphs), adapted from different sources. For each activity, we designed a sequence of worksheets, to be sent to the students' tablets or to be displayed on the IWB (or through the data projector). The worksheets were designed according to *four main categories*: (1) Worksheets introducing a problem and asking one or more questions *(problem worksheets)*; (2) *Helping worksheets*; (3) Worksheets prompting a poll between proposed options (*poll worksheets*); (4) *Worksheets prompting a focused discussion*.

As said before, in this contribution we focus on the creation and use of instant polls, combined with the possibility, offered by the CCT, of showing the results of the polls to all the students. The IDM-TClass software collects all the students' choices and processes them, displaying an analytical record (collection of each answer) as well as a synthetic overview (bar chart). In reference to the analytical framework, we may say that instant polls are used through the support of the "*Processing and Analysing*" functionality of the technology. The possibility of showing the results in real time brings to the fore also the "*Sending and Displaying*" functionality of technology.

In principle, the software enables also to set the time given to students before completing the poll, and offers the opportunity to provide an immediate automatic correction to the student. However, our choice was *not* to provide the immediate automatic correction to student, so that they could be engaged in a subsequent classroom discussion. In tune with Beatty and Gerace's framework (2009), we, in fact, conceived the use of polls as a way of scaffolding whole-class interaction with the aim of fostering the sharing of results and the comparison between students (*FA strategy B*). This is also coherent with our belief on the key role of the teacher and the importance of peer interaction.

During our design experiments, we both implemented *planned polls* that were a priori created to be inserted within each teaching sequence (through *poll worksheets*, which can be used in alternative to *problem worksheets*, where the students are expected to write down a written solution and justification) and *instant polls*, created and implemented on the spot. In the perspective of design-based research, polls created on the spot that revealed fruitful in terms of FA strategies may be inserted in the repertoire of planned polls for the subsequent cycles of experimentation.

Concerning polls, our investigation is guided by the following research question: *What kind of FA strategies can be activated thanks to the use of technology enhanced (planned or instant) polls?*

Due to limits of space, in this paper we focus on planned polls.

DATA ANALYSIS

All the lessons were video-recorded, fields notes were taken, and students' productions (doc files) were collected, building a large amount of data (about 450 hours of class sessions, carried out in collaboration with 20 teachers). Furthermore, teachers were interviewed every two-three lessons and, after each lesson, they were asked to write a report on the effectiveness of the lesson in terms of the activated FA processes and of the support provided by technology. In line with design-based research, the study is carried out through a close collaboration between researchers and teachers, who share the aim of improving practice, taking into account both contextual constraints and research aims.

In the following, we present an excerpt from a class discussion developed starting from the results of a *planned poll.* The example relates to an activity on time-distance graphs adapted from the task sequence "Interpreting time-distance graphs", from the Mathematics Assessment Program (http://map.mathshell.org/materials/lessons.php). From the original source based on paper-and-pencil materials for grade 8, we adapted the activities and created a set of 19 digital worksheets to be used with students from grade 5 to 7. Here we refer to a discussion carried out in grade 7.

The sequence starts with a short text about the walk of a student, Tommaso, from home to the bus stop. This text is accompanied by a time-distance graph, as illustrated in Figure 2:

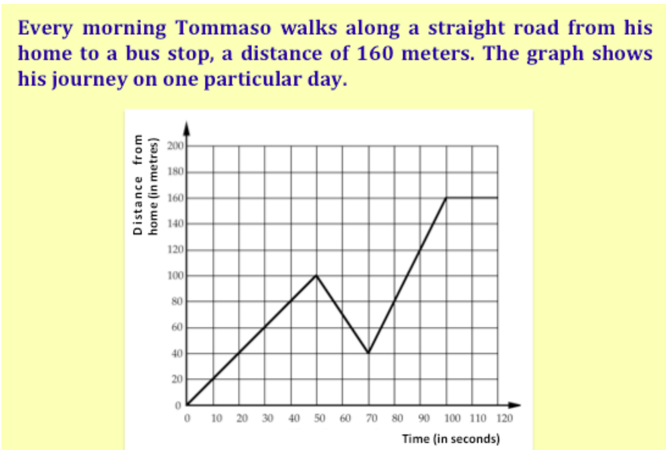

Fig.2: The time-distance graph of Tommaso's walk

Students' interpretation of this graph is guided through questions, posed to them within *problem*, *helping*, and *poll worksheets*. Since the students meet time-distance graphs for the first time through this activity, we designed an introductory activity based on the use of a motion sensor, in which students could explore in a laboratorial way the construction of the graph after a motion experience along a straight line.

Here we focus on an episode concerning the interpretation of the final part of the graph. At first, students were asked via a *problem worksheet* to establish what happens during the last 20 seconds, motivating their answers. During the classroom discussion, a *poll worksheet* was used to focus on the *completeness* of answers (FA strategy A). Specifically, the poll required students to identify which is the most complete among three given answers:

"Some students of another class wrote these answers. Which of them is the most complete?
A) During the last 20s, Tommaso is not walking because we have already said that he has reached the bus stop

B) I think that, during the last 20s, Tommaso is not walking because, from the graph, it is possible to understand that, in the period between 100s and 120s, he is always at the same distance from home, that is 160m

C) I understood that, during the last 20s, Tommaso is not walking because the line of the graph is horizontal."

Students discussed in pairs to answer to the poll. Afterwards, the teacher displayed the distribution of their answers on the IWB: 10% of students chose option A, 50% chose option B and 40% chose option C. Starting from the display of the results, the discussion took place. The teacher exploited the poll worksheet as a way to engineer *effective classroom discussions that elicit evidence of student understanding* (FA strategy B). The following table (table 1) presents selected excerpts from the discussion, analysed according to the FaSMEd framework in the right column.

Excerpts from the class discussion	***Analysis according to the FaSMEd three-dimensional framework***
After a brief analysis of A, justifications B and C are compared. 353) Teacher: let's look at B and C. Let's hear some explanations of those who chose C, why did they chose C, and some motivation of those who chose B. 354) Brown: we chose B because B specifies also that he (*Tommaso*) stayed still from 100 to 120 seconds, while C doesn't say this, saying that they were only 20 seconds they could have been 150, 170, 180 and so on… 355) Silvia: B is the most complete. 356) Teacher: B is the most complete. 357) Mario: for me the B is not right because, we understood that, when we used the motion sensor, let's say, you understand that a person stops when the line is horizontal, and there (justification B) it doesn't say this, then it is not the most complete.	The teacher encourages the students to discuss the reasons behind the choices of the poll. Her aim is to promote a discussion on the completeness of the two options. This is an instance of *FA Strategy* A, since the focus is on the requirements that a complete answer must satisfy. Suggesting that answer B gives more information on the last trait, Brown activates herself as responsible of her learning (*FA strategy E*) and at the same time as instructional resource for her mates (*FA strategy D*). Silvia, echoing Brown, affirms that B is the most complete, thus giving a implicit feedback to Brown (*FA strategy C*). In line 357 the student Mario challenges the former evaluation, activating himself as owner of his own learning (*FA strategy E*) : in his opinion, answer B is not complete because it does not refer to the experience with sensor detectors. This intervention provides a good occasion to discuss again the role and value of the empirical experience with sensors
… 390) Lollo: but if we had not done that activity before… 391) Teacher: the activity with the motion sensor. 392) Lollo: we could not have known that if you are still the line is horizontal	Lollo suggests that one cannot refer to the experience with sensors, since the answer should be intelligible also by a reader who did not do such an experience. Lollo turs himself as instructional resource for his mates (*FA strategy D*). In particular, he gives feedback to Mario (*FA strategy C*). The teacher reformulates Lollo's intervention so as to involve the other students, turning Lollo as a resource for his mates (*FA*

…	*strategy D*). In this way she also activates *FA Strategy C*.
399) Rob: And anyway from the graph you can understand why the distance is always the same but the seconds, let's say, go on… 400) Teacher: ok… then, even if we had not had the experience with the motion sensor, that made you understand in an experimental way that if I stay still the line is horizontal, your classmate [*Rob*] says: "from the graph I can understand it anyway". Why? Rob, could you please repeat it? 401) Rob: because from the graph you can understand that when you don't move, that is to say when there is the horizontal line… 402) Teacher: what does it mean? 403) Rob: the meters remain the same but the seconds go on, let's say.	Rob intervenes, stating that in the horizontal trait the distance from home is always the same. This is a shift from an explanation based on the experience with sensors to a theoretical explanation, based on the meaning of the graph. Rob provides to other students a feedback to move forward (*FA strategy C*), turning himself as an *instructional resource for his classmates* (*FA strategy D*). The teacher reformulates Rob's intervention, giving to all the students a *feedback that moves them forward (FA strategy C)*. Reformulation is also a means to activate Rob as a *resource for his classmates* (*FA strategy D*).
… 413) Teacher: B explains why the line is horizontal, while C just says "the line is horizontal"; B instead explains why the line is horizontal, because the meters remain the same, even if time goes on, isn't it?	As a final intervention, the teacher rephrases the result of the discussion, pointing out what makes answer B more complete. In this way she activates *FA strategy A*.

Table 1: Excerpts from the class discussion and corresponding analysis

The analysis showed a wide range of FA strategies activated by different agents: not only by the teacher, but also by the students themselves. More specifically, since options B and C were both chosen by many students (50% and 40%), the teacher decided to ask students to express the motivation subtended to their choice. In this way, on one side, it was possible to focus on the mistakes subtended to the choice of incorrect answers, making students activate themselves as owners of their own learning (*strategy E*) because they could recognize their own mistakes and reflect on the reasons subtended to them. On the other side, students who chose the correct answer provided their justification, becoming more aware of the reasons why they chose a specific option (again activation of *strategy E*). The students were therefore activated as instructional resources for their mates (*strategy D*) because they gave feedback to each other (*strategy C*) on the reasons why a chosen option is better than the other.

CONCLUSIONS

In this contribution we studied the use of polls for promoting formative assessment in the classroom. The analysis, carried out by means of the FaSMEd analytical framework, showed the emergence of a variety of FA strategies and involved agents, suggesting that planned polls, exploiting the "*Processing and Analysing*" and "*Sending and Displaying*" functionalities of the technology, may turn into a fruitful formative assessment activity.

The outlined pattern may be related to Beatty and Gerace's (2009) TEFA cycle of question posing, answering and discussing. Also in our case, the use of polls may be conceived within a cycle of activities that encompass: solving a problem (and justifying the answer), taking a position in relation to a question in form of poll, commenting the poll results, justifying choices. Our analysis brings even more to the fore the variety of FA strategies that are promoted by the use of the polls, thus giving more insight into each phase of the TEFA cycle.

Although in this paper we confined ourselves to an example of discussion carried out starting from a *planned poll*, we are currently analysing a variety of examples concerning poll use. After three cycles of design, implementation and analysis, we propose a first tentative classification of the polls used during our design experiments, according to their different focus and (consequent) aims: (1) polls that ask to choose the correct answer to a problem, with the aim of promoting a discussion on solving strategies; (2) polls that ask to compare different answers to a problem, with the aim of promoting a meta-discussion on the features of the answers (such as in the example discussed in this paper); (3) polls focused on the difficulties students meet when facing specific kind of tasks or the best strategies to be used to face specific tasks, with the aim of promoting metacognitive reflections; (4) poll focused on students' feelings when facing a specific kind of task or when a particular methodology were adopted during the lessons, with the aim of bringing to the fore also the affective dimension. Referring to the instructional purposes of polls described by Beatty and Gerace's framework (2009), type-1 may be related to "provoke thinking" and "exercise a cognitive skill", whereas type-2 may be linked to "elicit a misconception" and "stimulate discussion with questions having multiple reasonable answers". Types 3 and 4 are of different nature: even if they could be somehow related to "status check", they bring to the fore metacognitive and affective issues that are not so evident in Beatty and Gerace's list. We remark that, in our design, polls are always intended as a starting point for a class discussion and not for individual "revising" or "check status".

Further research will be done on the analysis of the effects of the use of the four types of polls in terms of patterns of FA strategies activated during the class discussion that takes place after each poll. Moreover, we are going to study how the structure of the class discussion is influenced by the results of the processing of data.

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REFERENCES

Aldon, G., Cusi, A., Morselli, F., Panero, M., & Sabena, C. (2017). Formative assessment and technology: reflections developed through the collaboration between teachers and researchers In G. Aldon, F. Hitt, L. Bazzini & U. Gellert, *Mathematics and technology: a CIEAEM source book*. Series 'Advances in Mathematics Education'. Springer International Publishing.

Ares, N. (2008). Cultural practices in networked classroom learning environments. *Computer-Supported Collaborative Learning*, 3, 301–326.

Bartolini Bussi, M. G. (1998). Verbal interaction in mathematics classroom: A Vygotskian analysis. In H. Steinbring, M. G. Bartolini Bussi, & A. Sierpinska (Eds.), *Language and communication in mathematics classroom* (pp. 65–84). Reston, VA: NCTM.

Beatty, I.D, and Gerace, W.J. (2009). Technology-Enhanced Formative Assessment: A Research-Based Pedagogy for Teaching Science with Classroom Response Technology. *Journal of Science Education and Technology*, 18, 146–162.

Black, P., and Wiliam, D. (2009). Developing the theory of formative assessment. *Educational Assessment, Evaluation and Accountability*, 21(1), 5-31.

Clark-Wilson, A. (2010). Emergent pedagogies and the changing role of the teacher in the TI-Nspire Navigator-networked mathematics classroom. *ZDM Mathematics Education*, 42, 747–761.

Collins, A., Brown, J.S., & Newman, S.E. (1989). Cognitive Apprenticeship: Teaching the Crafts of Reading, Writing and Mathematics! In L.B. Resnick (Ed.), *Knowing, Learning, and Instruction: Essays in Honor of Robert Glaser* (pp. 453-494). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cusi, A., Morselli, F. & Sabena, C. (2017). Promoting formative assessment in a connected classroom environment: design and implementation of digital resources. Vol. 49(5), 755–767. *ZDM Mathematics Education*.

DBRC - The Design Based Research Collective (2003). Design-Based Research: An Emerging Paradigm for Educational Inquiry. *Educational Researcher*, 32(1), 5-8.

Irving, K.I. (2006). The Impact of Educational Technology on Student Achievement: Assessment of and for Learning. *Science Educator*, 15(1), pp. 13-20.

Roschelle, J., & Pea, R. (2002). A walk on the WILD side. How wireless handhelds may change computer-supported collaborative learning. *International Journal of Cognition and Technology*, 1(1), 145-168.

Roschelle, J., Penuel, W.R., & Abrahamson, L. (2004). The networked classroom. *Educational Leadership*, 61(5), 50-54.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York: Macmillan.

Vygotsky, L. S. (1978). *Mind in society: The development of higher mental processes*. Cambridge, MA: Harvard University Press.

Wiliam, D., & Thompson, M. (2007). Integrating assessment with instruction: What will it take to make it work? In C. A. Dwyer (Ed.), *The future of assessment: Shaping teaching and learning* (pp. 53–82). Mahwah, NJ: Erlbaum.

25 YEARS OF E-ASSESSMENT AND BEYOND: HOW DID I DO!

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E-assessment is a powerful tool for supporting the learning of mathematics. Early trials began back in 1991 on a local network. Over the past 25 years technical advances have widened and improved its delivery. For the past 10 years MapleTA has been adopted and a huge range of question banks have been developed. Some recent topics include dimensional analysis, fractals, linear programming, Fourier series, oscillatory motion, series solution of ODEs, Laplace transforms and basic solution methods for PDEs. A key element in providing students with feedback on their progress is the "How Did I Do?" option, which allows them to check their answers as they progress through extended problems. The same question is equally relevant when evaluating the effectiveness of e-assessment for many thousands of students over several decades.

Keywords: E-assessment, feedback, modelling, mechanics, calculus

A QUARTER CENTURY OF E-ASSESSMENT ADVANCES

It is over 25 years since e-assessment was first used for mathematics students at the University of Portsmouth on a local network (Figure 1).

Date(s)	Development/Activity/Action	Outcome(s)	Software	Focus
1991	Installation of networked Computer Assisted Assessment (CAA)	First use of summative e-assessment	Question Mark for DOS	Departments
1994	Upgrade of networked CAA to Windows	Delivery of improved functionality, especially graphics/ graphical questions	QM Designer for Windows	Departments
Mid-1996	Beta testing of on-line e-assessment	Feasibility of on-line e-assessment demonstrated	QM Perception Beta + PWS	Dept of Mathematics
Late 1996	Installation of on-line e-assessment Full-time educational technologist appointed	Successful first use of on-line summative e-assessment	QM Perception V1 + PWS	Dept of Mathematics
1997 - 1998	University Project: CAA on the Web (£4500) Purchase of 10 QMP personal licences	Successful pilot use of on-line e-assessment by 10 individual staff covering all faculties	QM Perception V1 + Personal Web Server	University
1999 - 2000	University Project: Framework for the Uptake of CAA (£10,500)	Successful scaling up of on-line assessment with 2 faculty servers Use of WebCT connector proposed	QM Perception V2	University Faculties
April 2001	Proposal for university wide Perception site licence presented to university IT committee	Proposal accepted	QM Perception V3	University
Nov 2001	QM Perception purchased (£25,000) Ongoing annual maintenance (£10,000 per annum)	Full site university licence available	QM Perception V3	University
2002	Full implementation strategy document	Unsuccessful trial set up in Faculty of Technology	QM Perception V3	Faculty of Technology
2003	Purchase of NT server for e-assessment (£5000)	Re-installation of QM Perception	QM Perception V3.4	Faculty of Technology
Mid-2004	Use of Oracle database for storage of questions, assessments and results	Successful trial of full scale Oracle based e-assessment system	QM Perception V3.4 + Oracle	Faculty of Technology
Sept 2004	University On-Line Learning and Assessment Group sets up full-scale pilot	Pilot delivery of diagnostic & summative assessments in mathematics	QM Perception V3.4 + Oracle	Department
2005	Report of successful pilot Oracle based Perception 3.4 Academic PVC identifies Perception as university tool	Use of Web based Perception extended to Science Faculty	QM Perception V3.4 + Oracle	University
2005	Installation of QM Perception V4		QMPV4 + Oracle	University
Oct 2005	Proposal to include e-assessment in academic regulations	Regulations modified		University
2006	Conversion of assessments to QMP V4	Increasing number of assessments delivered	QMPV4 + Oracle	Technology
2007	Use of QM Perception by four Faculties: Technology, Science, CCI and HSS	Successful delivery of formative and summative assessments	QM Perception V3, V4 + Oracle	University

Figure 1. Mathematics e-Assessment Delivery using QuestionMark Software 1991-2007

Although computer algebra systems were not widely available then, it was still possible to author and deliver a variety of standard question types. In 1996 a CAS powered assessment system was developed (McCabe and Watson, 1997) using the Maple kernel within Toolbook authoring software (Figure 2). For the first time ever it was possible to check algebraic question responses with a CAS and develop mathematical questions with a user-friendly interface.

Around the same time the delivery of online assessment was beginning and the main tool used at Portsmouth was QuestionMark Perception (McCabe, 1998), which had no underlying CAS. In 2005 the Department of Mathematics switched to using MapleTA and it has been the primary tool used for e-assessment since then (Figure 3). Initially a local server was used, but in recent years a managed server has proved more convenient and reliable, especially when dealing with product upgrades. Increasing student numbers at Portsmouth (McCabe 2009) made the effort worthwhuile.

The adoption of a commercial product, rather than an open source e-assessment system, such as *STACK* (Sangwin, 2004), *Numbas* (Foster, Perfect and Youd, 2012), *DEWIS* (Gwynllyw and Henderson, 2009) and *Math e.g.*(Greenhow and Kamavi, 2012), has provided stability and the availability of support when it was required.

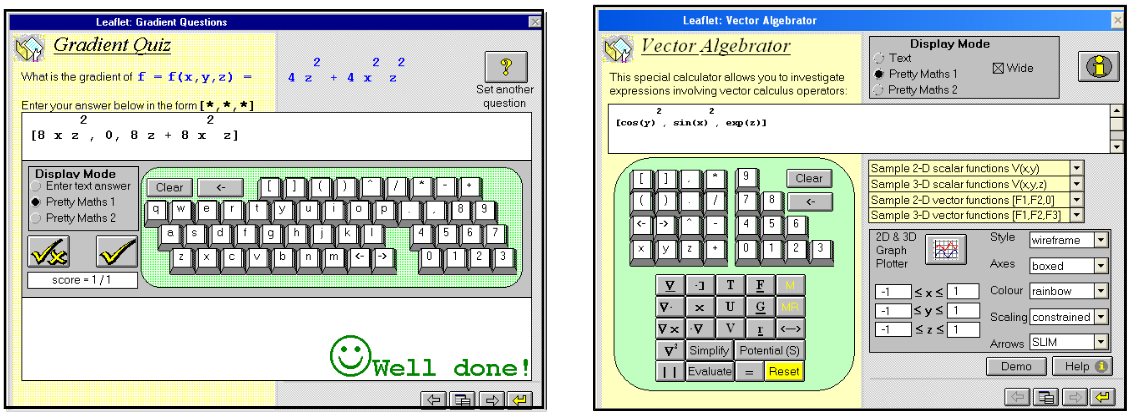

Figure 2. World First Use of CAS for e-Assessment (McCabe and Watson, 1997)

Date(s)	Development/Activity/Action	Focus	Status
Early 2005	MapleTA identified as an appropriate e-assessment tool for maths, science and technical subjects	Department	Investigation
May 2005	MapleTA V2.5 ordered	Department	Investigation
July 2005	Software installed	IS	Testing
Sept 2005	Minor problems rectified	IS	Testing
Oct 2005	Authoring begins	Department	Testing
Nov 2005	First successful delivery of a summative assessment	Department	Pilot
March 2006	Successful delivery of formative and summative assessment on two maths units	Department	Production
June 2007	Successful delivery of formative and summative assessment on four units: 2 maths, 1 science and 1 civil engineering	Faculty	Production
October 2007	MapleTA upgrade to V3	IS	Investigation

Figure 3. Early Mathematics e-Assessment Delivery using MapleTA 2005 - 2007

AN EVOLVING STRATEGY FOR E-ASSESSMENT DELIVERY

The literature on e-assessment has grown considerably over the past 25 years. Timmis et al (2016) provides an up-to-date set of general references for what it calls Technology Enhanced Assessment TEA. Sangwin (2013) is the first textbook specifically on the subject of mathematics e-assessment and many other sources of guidance on e-assessment have been written over the years, e.g. Whitelock (2006), QCA (2007). At Portsmouth it has largely been years of practice and a gradual evolution that has shaped the present strategy for delivering e-assessment.

E-assessment delivery initially focussed on summative tests. Mathematical Models is a typical 1st year mathematics undergraduate course unit, for which MapleTA has been routinely used over the past 10 years. As question banks have increased in size, weekly practice tests with feedback have become the norm. A monthly coursework assessment on each topic, allows students a controlled 24-hour period to complete their work. Although different assessment patterns have been tried out, our experience is that a 40:60 weighting of continuous assessment to a final exam motivates students to work steadily through a course unit and achieve high marks as they progress. The final e-assessment exam lasts 2-hours and is always formally invigilated. Intermediate Calculus, a 2nd year course unit, adopts a similar progressive style of weekly practice e-assessments, monthly 24-hour courseworks, but with a more traditional 2-hour written final exam. The weekly practice assessments often promote flipped learning, with many students using them as the starting point in their study.

EFFICIENT DEVELOPMENT OF NEW QUESTION BANKS

The efficient production of high quality algorithmic questions with feedback has been the key to the successful delivery of e-assessment. The many features of MapleTA have enabled rapid authoring without getting bogged down in technicalities. Three special cases are highlighted here: reverse engineering, randomised components (datasets, functions, equations, graphs, networks, matrices ...) and multipart questions.

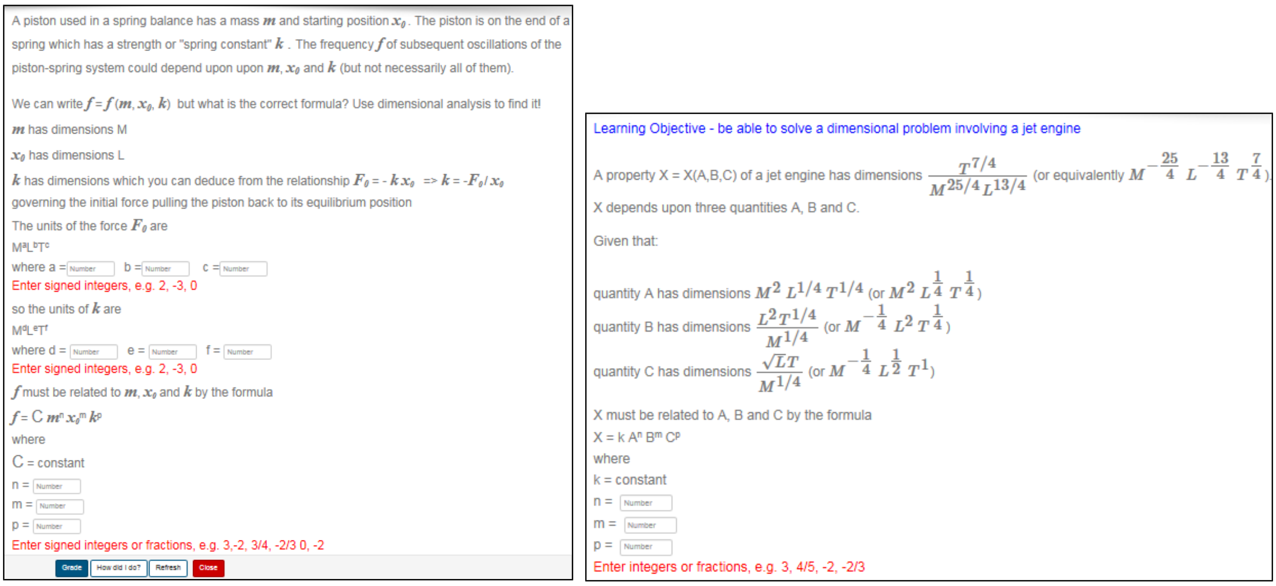

Figure 4. Efficient Question Setting Via Reverse Engineering of Dimensional Analysis

Special Case 1: Dimensional analysis is an extremely useful mathematical technique for solving problems with minimal work, but without a full understanding of the underlying physical processes. It is introduced as part of the Mathematical Models course unit. The left hand screenshot in Figure 4 shows a typical "real-world" question, which leads a student through the solution of a specific problem. The drawback is that finding sufficient realistic dimensional analysis problems to solve and the creation of a question bank is time-consuming. To avoid this, a randomised set of fictitious problems have been developed which allow the technique to be practiced effectively on meaningful questions. To illustrate its implementation, suppose we wish to find an unknown relationship $X = X(A,B,C) = kA^nB^mC^p$. If the dimensions of X, A, B and C are given as $M^{x_1}L^{x_2}T^{x_3}$, $M^{a_1}L^{a_2}T^{a_3}$, $M^{b_1}L^{b_2}T^{b_3}$, $M^{c_1}L^{c_2}T^{c_3}$ respectively, then we deduce that

$$a_1 n + b_1 m + c_1 p = x_1$$
$$a_2 n + b_2 m + c_2 p = x_2$$
$$a_3 n + b_3 m + c_3 p = x_3$$

We could solve for n, m and p using Maple, but cannot easily be assured of user-friendly solutions. Instead the trick is to reverse engineer the question. Rather than setting up a randomised question and solving it, we start with a randomised solution for n, m and p, but then create randomised questions by choosing suitable question parameters. In practice, this simply means randomising a_i, b_i and c_i (i=1..3) and calculating x_1, x_2 and x_3 from the three equations shown above. An important condition, easily implemented in a MapleTA question algorithm, is to ensure that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

is satisfied. A resulting question is shown on the right hand side of Figure 4. The difficulty of the question is readily adjusted by varying the range of randomised parameters and adding more physical quantities, such as temperature. Reverse engineering is an important technique in setting e-assessment questions in mathematics and allows them to be generated far more simply and reliably than the direct approach of solving a randomly generated problem.

Special Case 2: Algorithms with randomised components lie at the heart of most questions. The generation of simple numerical datasets for fractal box counting is a classic example (Figure 5). A simple logarithmic relationship log N(s) = log C + D log s implies that a graph of log N(s) vs. log s will be a straight line with slope D, the fractal dimension. Data can be generated with or without randomised "noise".

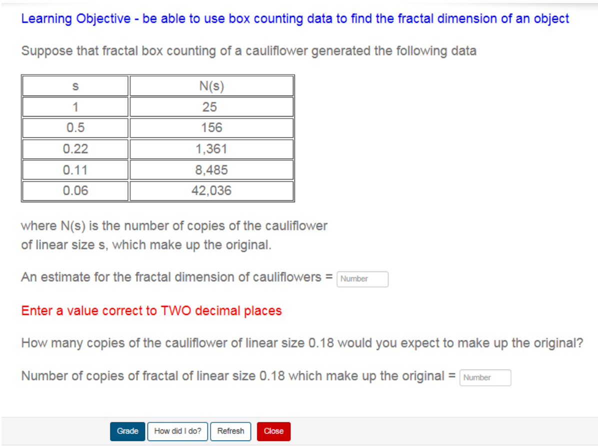

Figure 5. Randomised Datasets for Fractal Box Counting

Large banks of ODE and PDE questions have been developed with randomly generated equations, covering a wide range of types and solution methods.

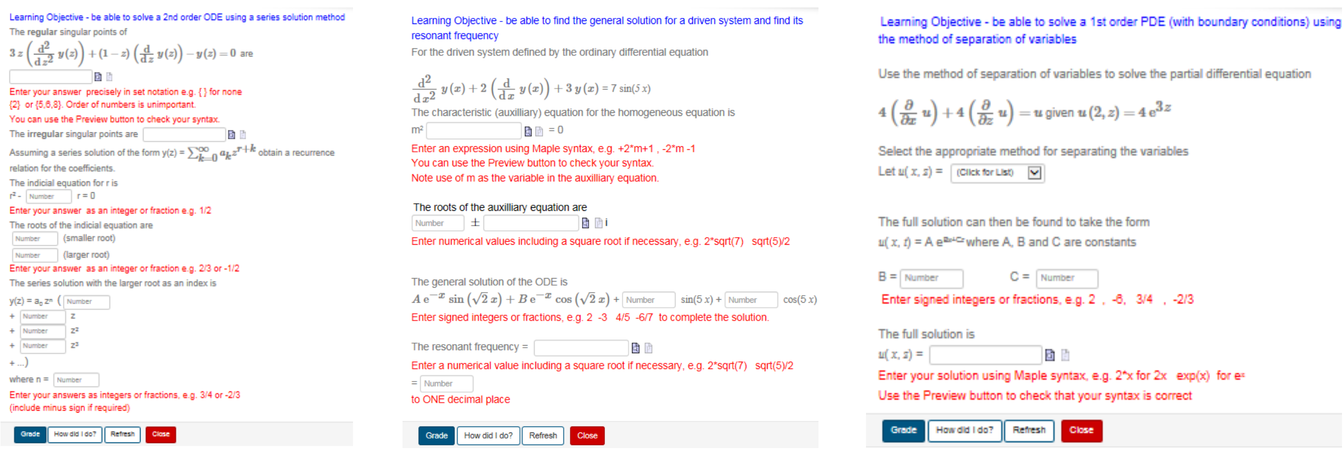

Figure 6. Randomised Equations for ODE and PDE Solution

For these questions, the technique of cloning, i.e. the copying and modification of existing questions, plays an important role in speeding up question production. Often only minor changes are needed to generate a completely different question.

Randomised graphs can be generated in MapleTA very efficiently. The matching question in Figure 7 is an example taken from a linear programming question bank and includes a different, graph for each of the 4 solution possibilities. Any graph that can be generated in Maple can be randomised in MapleTA with minimal effort.

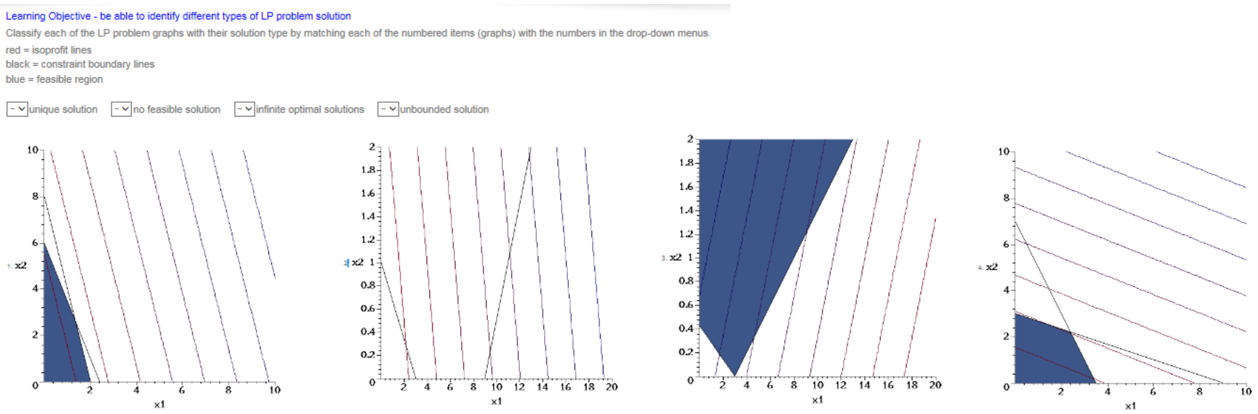

Figure 7. Randomised Graphs for Linear Programming

Other examples of efficient graph plotting, using Maple commands and packages, are shown in Figure 8 below. All the graphs are generated dynamically for each instance of the question with a single command.

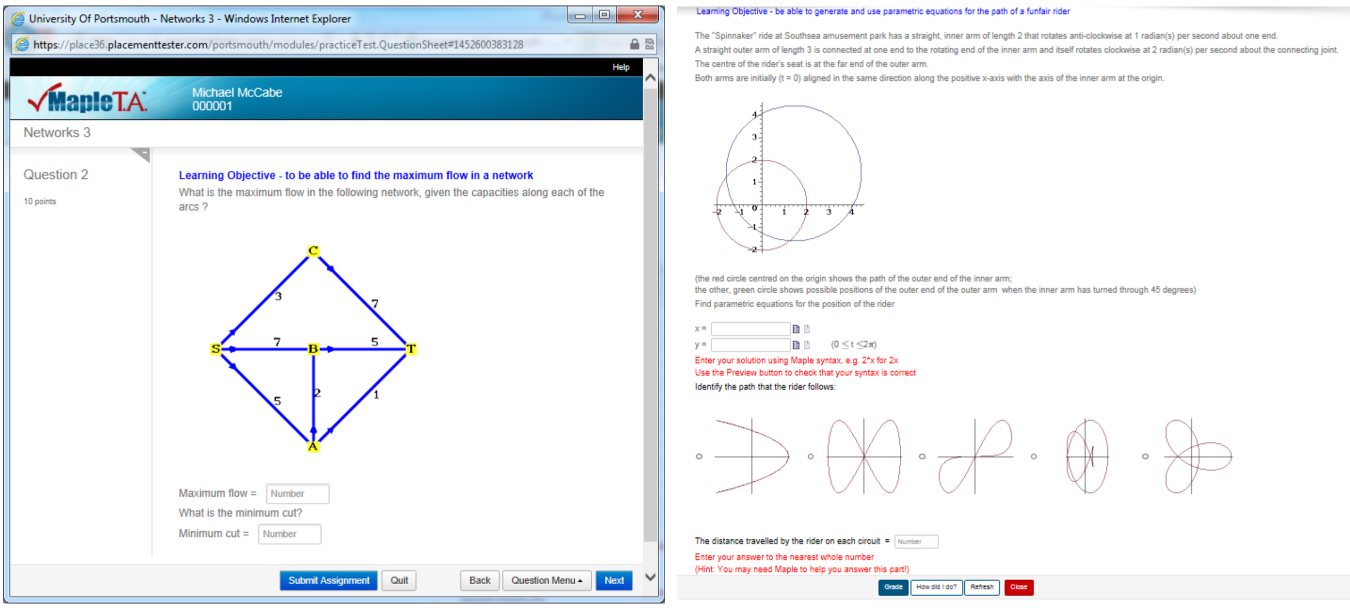

Figure 8. Randomised Graphs for Networks and Parametric Coordinate Problems

Special Case 3: Multi-part questions are used frequently to guide students though common solutions methods. Figure 8 (right) shows the combination of randomised graphs with a multipart question in solving a problem involving parametric coordinates. A further example, shown in Figure 9, is a question which works through statistical hypothesis testing.

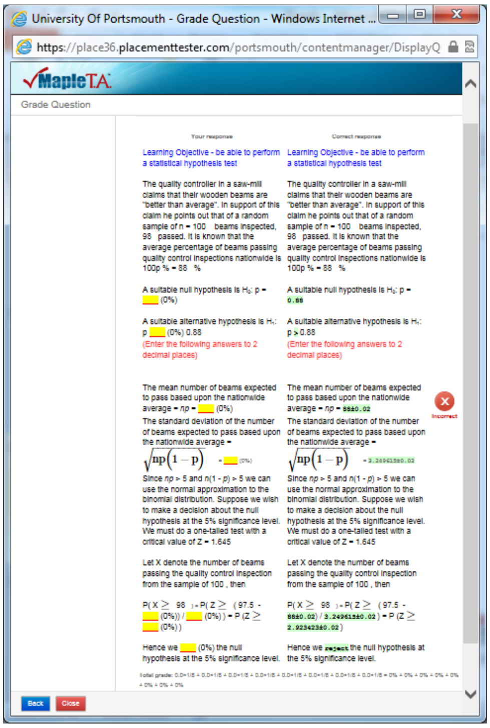

Figure 9. Multi-Part Question for Statistical Hypothesis Testing

Without efficient means of authoring e-assessment questions, their development becomes a slow and unproductive process. The process can become even slower when the time taken to add feedback is taken into account. Experience has shown that reverse engineering, randomised algorithmically generated components and multi-part questions are often the key elements in creating effective, reusable questions.

EFFICIENT AND EFFECTIVE DELIVERY OF FEEDBACK

Basic principles of feedback practice in e-assessment have been well identified over the years, e.g. Nicol and Milligan (2006), but are often extraordinarily difficult to implement. The preparation of detailed, dynamically generated feedback can be extraordinarily time-consuming and may often take longer to produce than the question itself. It is often found to be more efficient to provide traditional written solutions to sample questions. Indeed it can be argued that automated solutions and feedback can eliminate the need to cross-reference different sources of information, such as textbooks, lecture notes and worked examples, making the arena of learning too restricted.

Although students are encouraged to read their lecture notes and worked examples, when they tackle e-assessment questions, some automatically generated feedback is always worthwhile. In MapleTA students can simply ask “How Did I Do?” while they are attempting a multi-part question. For example, when solving an ODE using Laplace transforms (Figure 10), partial solutions can be checked before moving on to later stages of the solution. Thus, mistakes in early parts of the question can be corrected before moving on to the complete solution.

Often a compromise must be reached between time spent on preparing fully automated feedback and more questions for a bank. The pragmatic approach has been to use the "How Did I Do?" option to provide a basic level of feedback without requiring much extra work. When it is routinely available in all questions during weekly practice tests, students usually take full advantage of the help that it gives them.

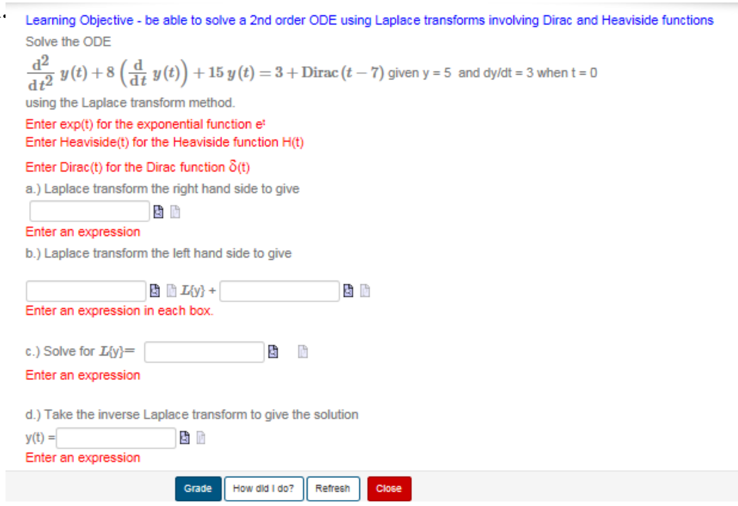

Figure 10. How Did I Do on my Laplace Transform Problem?

THE PRESENT AND FUTURE OF E-ASSESSMENT

Mathematics e-assessment has advanced considerably over a period of more than 25 years, since it was first used at the University of Portsmouth. Software developments have enabled online delivery, CAS checking of responses, randomisation in many different forms, algorithmic question generation, multi-part questions, new question types, targeted feedback and adaptive questions. Changes in the software tools over the first 15 years often made it necessary to abandon existing question banks and write new ones. Maintenance and improvement of question banks is still important, but far less time-consuming than it used to be. For the past 10 years there has been relative stability and the size of question banks has grown (Figure 10) as new topics have been added. With the recent addition of Fourier series, the total number of MapleTA questions now approaches 1000. These are available for MapleTA users of the future, who can also answer the question "How Did I Do?".

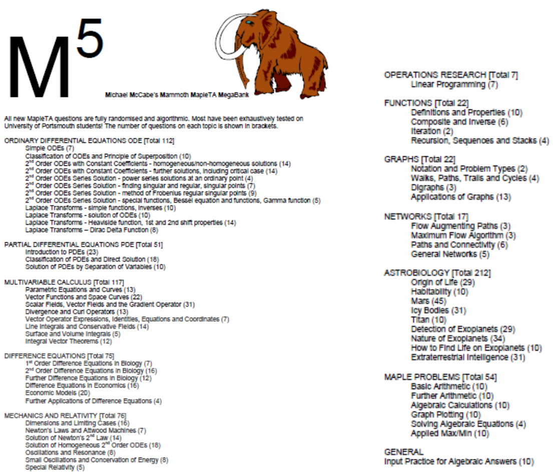

Figure 10. A Megabank of MapleTA Questions

REFERENCES

Greenhow, M. and Kamavi, K. (2012). Maths e.g. - a web assessment application for STEM and beyond, *Proceedings of the HEA STEM Learning and Teaching Conference*.

Gwynllyw, R. and Henderson, K. (2009). DEWIS: a computer aided assessment system for mathematics and statistics, *CETL-MSOR 2008 Conference Proceedings*, pp. 38-44.

McCabe, E.M. and Watson, J. (1997) From MathEdge to Mathwise: The Cutting 'Edge of Interactive Learning and Assessment in Mathematics", Proceedings of the 3rd International Conference on Technology in Maths Teaching, Koblenz

McCabe, E.M. (2009) *The Exponential Growth of Mathematics and Technology at the University of Portsmouth* Teaching Mathematics and Its Applications: An International Journal of the IMA, v28 n4 p222-227 Dec 2009 http://teamat.oxfordjournals.org/content/28/4/222

McCabe,E.M. (1998) CAA Workers Do IT with Perception, Proceedings of the 2nd Annual Computer Assisted Assessment Conference, 105 – 110, Loughborough University, ISBN 0-9533210 1 0

Nicol, D. and Milligan, C. (2006) *Rethinking technology-supported assessment practices in relation to the 7 principles of good feedback practice* D. Nicol & Milligan in "Innovative Assessment in HE" Routledge Falmer http://ewds.strath.ac.uk/REAP/public/Papers/Nicol_Milligan_150905.pdf

QCA, e-Assessment: Guide to Effective Practice (2007) Available online at https://www.e-assessment.com/wp-content/uploads/2014/08/e-assessment_-_guide_to_effective_practice_full_version.pdf

Sangwin, C. (2004). Assessing mathematics automatically using computer algebra and the internet, *Teaching Mathematics and its Applications* **23**(1): 1-14.

Sangwin, C. (2013). *Computer Aided Assessment of Mathematics*, Oxford University Press.

Timmis, S., Broadfoot, P., Sutherland, R. and Oldfield, A. (2016), Rethinking assessment in a digital age: opportunities, challenges and risks. Br Educ Res J, 42: 454–476. doi:10.1002/berj.3215

Whitelock, D. M. & Brasher, A. (2006) *Developing a Roadmap for e-Assessment: Which Way Now? in: D. Myles (Ed) Proceedings of the 10th CAA International Computer Assisted Assessment Conference (Loughborough, UK, Professional Development, Loughborough University), 487–501.*

MAKING GOOD PRACTICE COMMON PRACTICE BY USING COMPUTER AIDED FORMATIVE ASSESSMENT

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Rich technological environments present many opportunities for guided inquiry in the mathematics classroom. In this paper we focus on the role of the teacher supporting the forming and proving of conjectures by the students, during a whole class discussion. We examine the practices of an expert teacher that conducts a classroom discussion based on students' conjectures formed while working in pairs with a dynamic geometry environment (DGE). Specifically, we analyse the way the teacher categorizes the different conjectures, and then addresses them during the whole class discussion. We suggest that this categorization could be offloaded onto a technological platform that would do it automatically, thus making this type of information accessible not only to teachers that could perform this categorization on the spot.

Keywords: Instrumental orchestration, classroom discussions, conjectures, Dynamic Geometry Environment (DGE)

INTRODUCTION

Defining, analysing, and trying to distribute good practice of teachers in the mathematics classroom is an ongoing challenge for the research community (Chazan & Ball, 1999). Guided inquiry tasks are open ended tasks that usually have more than one solution, and often require taking into account various dimensions that were not addressed in previous learning, thus requiring the students to go through a problem-solving process. Promoting and evaluating this process presents challenges for teachers. In the case of computer based guided inquiry, where students are expected to form and reason about conjectures, the primary role of the teacher is to promote and organize discussions (Yerushalmy & Elikan, 2010).

Orchestrating the work of students in a technological environment, referred to by Trouche (2004) as instrumental orchestration, while gathering information about students that could be used for formative assessment, presents challenges for the teachers (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010). In terms of evaluation, formative assessment requires the teacher to draw on information from teaching as feedback to modify accordingly the teaching of the students the information was gathered about (Black & William, 1998). The abundance of data that is created and could be analysed when students engage in rich inquiry tasks on a technological platform presents a challenge for teachers. Some researchers suggest the use of technological platforms for the gathering and display of student answers (Arzarello & Robutti, 2010; Clark-wilson, 2010). Another practice observed by Panero & Aldon (2015) was the combination of automatically collected digital data with tradition pencil and paper work that was used by the teacher in formative assessment in real-time. Another strategy suggested by Olsher, Yerushalmy, and Chazan (2016) would be to offload some of the processing of the data onto a digital platform, automatically categorizing student answers by mathematical characteristics, thus enabling the teacher to have accessible processed data to inform his decision making.

Yet, although data is accessible, and practices are studied, guided inquiry is not a prominent practice in mathematics classroom. One way to address is to explore ways to study and promote *good practice* of teachers in technologically rich environments that present students with guided inquiry.

CONTEXT OF THE STUDY

This study recalls a recorded guided inquiry session with 24 students from grades 9th-10th, working in pairs. The students are using a first generation DGE (Geometric Supposer), which served as a technological platform used to elicit conjectures for about half a year prior to the recorded lesson. The lesson's was planned to summarize major theorems of similarity in triangles. The students are walked through the construction on the board, while the teacher describes the actions, and the students get a printed version of the task as well.

The leading research aim is answering the question whether it is possible to identify good practices about conducting conjecture based discussions in the classroom, and whether the categorization of conjectures in a way that facilitates these practices by providing these categorizations automatically.

METHODOLOGY

In order to address the research question we study a classroom in which the teacher needs to gather, process, and utilize information that is generated by his students while conducting an individual inquiry activity using a DGE. Truoche (2004) uses the term "instrumental orchestration" to describe didactic configurations and the way that they are being exploited in the classroom, and also suggests them as a construct that could "give birth to new instrument systems" (ibid, p.304). In the observed lesson this framework is suitable to describe the way the teacher works with the students answers, and suggest "new instrument systems" whether available within the given environment or supported by different technological platforms.

For the analysis presented in this paper, we have analyzed a recording of a one-hour lesson in a classroom, which serves as our main data source. In addition we draw upon the design principles of the STEP platform for use in classrooms that are equipped with personal digital devices.

In the next part, we describe the task presented to the students. Following that part we analyse the way the teacher orchestrates the discussion surrounding the conjectures raised by the students. Specifically, we analyze the categorization of the conjectures in terms of placement on the blackboard (if at all), and type of treatment they are given by the teacher (i. e. acknowledging the difficulty to prove a certain conjecture, or specifying the underlying constraints). As this orchestration requires a lot of real-time decision making by the teacher, we then examine how the use of automatic analysis tools (e. g. the STEP platform) could offload some of this orchestration, creating new instrument systems thus possibly making this process more accessible for other teachers.

The task

Construct an acute triangle. Draw the altitudes from each one of the triangle points, and mark the feet of the altitudes D, E, F. Label the intersection point G. reflect point G over each side of the triangle. What’s the relationship between triangle DEF which is formed by connecting the feet of each altitude, and the triangle formed by connecting the image points of G, the original triangle, and angles, segments? Investigate anything that you can find. Write out formal conjectures as we have been doing in class.

In figure 1 appears a sketch that resembles the one that was drawn on the blackboard in the recorded lesson.

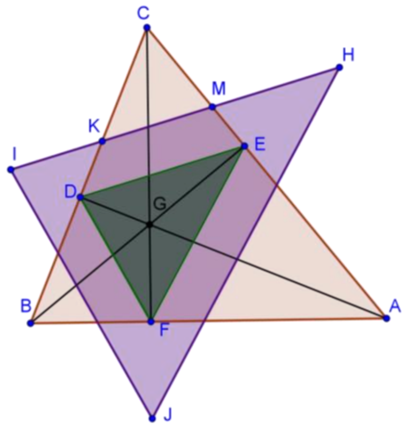

Figure 1. A sketch of the geometric construction discussed in the classroom

DATA ANALYSIS

During the recorder session, we have identified 11 conjectures that were addressed in the classroom (Table 1). Once the seven conjectures were listed, the teacher initiated a discussion aiming to review the conjectures and the argumentation and underlying supposing at the base of each conjecture.

Listed on the left side of the blackboard	Listed on the right side of the blackboard	Conjectures that were raised by students but did not appear on the black board
A1. $\Delta IHJ \sim \Delta DEF$	B1. If ΔABC is isosceles and acute then: $\measuredangle ACB$ is geometric mean of $\measuredangle FED$ $\measuredangle FDE$	C1. $\frac{Sides\ of\ \Delta\ IHJ}{Sides\ of\ \Delta DEF} = 2$
A2. $\frac{Area\ of\ \Delta\ IHJ}{Area\ of\ \Delta DEF} = 4$	B2. Bisector is the same as altitude. $\overline{BE}$, bisects $\measuredangle DEF$ and $\overline{BE}\ extended$ bisects $\measuredangle IHJ$	C2. Corresponding sides are parallel
A3. $\frac{Perimeter\ of\ \Delta\ IHJ}{Perimeter\ of\ \Delta DEF} = 2$	B3 $\measuredangle ACB \cong \measuredangle FED \cong \measuredangle FDE$ (ΔABC is isosceles)	C3. $\Delta JHI \sim \Delta DEF$ bisector of H also bisects E.
A4. $\overline{IH} \parallel \overline{DE}, \overline{EF} \parallel \overline{HJ}, \overline{DF} \parallel \overline{IJ}$		C4. <u>IF ABC is isosceles:</u> Creates two other isosceles triangles.

Table 1. Conjectures raised by students according to their appearance on the black board

When addressing these conjectures, we have identified four strategies used by the teacher. The first strategy can be demonstrated with conjecture A1 (table 1), was to state the conjecture on the left side of the blackboard, and then ask how many of the students agree with the conjecture:

Teacher: You think that triangle IHJ is similar to triangle DEF [writes $\Delta IHJ \sim \Delta DEF$ on the blackboard]. Raise your hand if you believe that's true? [All of the students raise their hand] oh. So everyone does. Great.

The second strategy can be demonstrated using conjectures C1 and A3, was not to write the initial conjecture, but to either refine it by himself (C3 turned into B2) or by involving the students, as can be shown from the following excerpt:

Student 1: Their sides are two to one.

Teacher: The ratio of their sides is two to one.

Students: Perimeter.

Teacher: Perimeter is two to one [writes $\frac{Perimeter\ of\ \Delta\ IHJ}{Perimeter\ of\ \Delta DEF} = 2$ on the blackboard]. The perimeter of triangle IHJ, to the perimeter of triangle DEF is two. Which means the ratio of their sides is also two to one.

The third strategy can be demonstrated using conjecture B1, as to write the conjecture with the additional constraints relevant to it on the right hand side of the board (on the right side of the sketch). In this case the teacher also assigns ownership of this conjecture and the additional constraints to the students that raised it:

Student 2: If triangle ABC is isosceles, then hmm, the measure of the angle ACB equals either of the two base angles in the two smaller triangles. Because those two smaller triangles are also isosceles.

Teacher: You and Jenifer worked a lot with isosceles triangles, didn't you? [Drawing on the blackboard a sketch represented in Figure 2] OK what do you claim?

Student 2: That angle ACB equals, is congruent to angle FED and angle FDE

Teacher: [writes $\measuredangle ACB \cong \measuredangle FED \cong \measuredangle FDE$ (ΔABC is isosceles) on the blackboard].

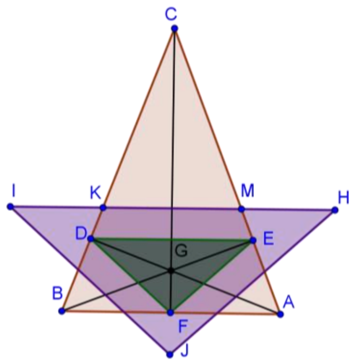

Figure 2. A sketch of the constrained case students addressed in the classroom

The fourth strategy used was in cases that were stated on the right side of the blackboard, but were general. There was one occurrence of a case such as this, conjecture C3 that was raised and then processed to conjecture B2 that was finally written on the board. This conjecture was not one that the lesson plan for this activity prepared the teacher for.

Table 1 suggests rough categories for the conjectures as they were addressed by the teacher. The Conjectures that appeared on the left side of the blackboard (A1-A4) were conjectures that the teacher expected, and went over their justifications in class. The conjectures that did not reach the blackboard (C1-C3) required some additional rephrasing or generalization in order for them to be well defined and represented, and their evolved form eventually appeared on the board. The conjectures that appeared on the right side of the blackboard were either case specific conjectures (B1, B3), or conjectures that were more advanced compared with the learned content.

The teacher then moves on with a reflection, sharing his thoughts and his planning with the students. In this the teacher, a well-established authority figure in the classroom, demonstrated that he was not completely prepared for everything that appeared - on the contrary. He was happy to be surprised by the students' ideas he did not expect. He asks which conjecture the students thought surprised him, and they stated the bisector one (B2), to which the teacher agreed. He then states that he will not address all of the conjectures, but he will do the ones on the left side, stating that these are the ones that everyone found; he then goes over the proofs for all of them. Then he turns to the right hand side of the board, and categorises the conjectures further: conjectures B1 and B3 are referring to the sketch in Figure 2, and are given as homework, but conjecture B2 is referred to as a general conjecture, true for any triangle. The teacher states it might be difficult for them to prove, and gives them additional time and offers hints if they will have difficulties. So the reflection about the "surprising" aspect in students' conjectures serves beyond the issue of authority; this could be the teachers' way to categorize conjectures as being more or less trivial (expected) to be proved.

DISCUSSION

Expert teachers have the skills and knowledge to filter and categorize student answers during the classroom session even in complex situations of inquiry based learning. Yet, this ability is not common practice, especially when gathering information from technologically based platforms (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010). As also appears in the case presented above, the teacher refers to cases that were not expected by him as cases that he might not address in the classroom. Olsher, Yerushalmy, & Chazan (2016) suggest the use of automatic filtering of student responses to make this type of information more accessible for teacher use as means for formative assessment. One example that is suggested by Olsher et al. (2016) is the STEP platform, which enables teachers to predefine mathematical properties of student answers, for the platform to automatically analyze and categorize for increasing the accessibility of the teacher to the student answers.

For the case presented, the categorization of the teacher could be mapped into an automatic filtering scheme. As the topic of this lesson is similarity, many conjectures that address certain characteristics of similarity are expected to be raised: ratio between sides, areas, relationship between corresponding segments (e.g. parallel segments). Even student mistakes that are prominent in the teaching of the similarity could be expected (e.g. mistaking between the ratio of segments and the ratio between areas).

These relations could be predefined and automatically recognized by the platform (e.g. STEP), making relevant data such as: is this relation addressed by the students? If so, by how many of the students? Furthermore, as DGE's are currently even more flexible than the Geometric Supposer in terms of the student's ability to drag pre-constructed figures, categorizing by the mathematical properties stated could be even more important as the students' example spaces potentially grow even wider. One example for filtering student answers is by determining whether they added constraints to the given situation, and by that potentially limited the generality of their answer, such as was demonstrated in conjectures B1 and B3 that were related to isosceles triangles. By defining the expected relations, we are also setting the stage for the unexpected relations to appear. They could easily be addressed by the teacher, and also automatically determine their correctness. By acknowledging that the platform will not identify the entire space of relations that students raised we leave room for student creativity, which is a substantial part of inquiry based activities, but also might keep educators from using automatic assessment platforms. In later sessions teachers could choose to incorporate these relations into the detected relations scheme if they see it fit.

We conclude in suggesting that the automatization of the categorizing and surveying of the student answers, beyond their correctness, could serve as a tool for teachers in their instrumental orchestration of a technologically based guided inquiry learning environment.

REFERENCES

Arzarello, F., & Robutti, O., (2010). Multimodality in multi-representational environments. *ZDM, 42(7),* 715-731.

Black, P., & Wiliam, D., (1998), *Assessment and Classroom Learning,* Assessment in Education, Vol. 5, 7-74.

Chazan, D., & Ball, D.(1999). Beyond being told not to tell. *For the Learning of Mathematics*, 19, 2-10.

Clark-Wilson, A. (2010). Emergent pedagogies and the changing role of the teacher in the TI-Nspire Navigator-networked mathematics classroom. *ZDM,42(7),* 747-761.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: instrumental orchestrations in the technology-rich mathematics classroom, *Educational Studies in Mathematics, 75(2),* 213-234.

Olsher, S., Yerushalmy, M., & Chazan, D. (2016). How might the use of technology in formative assessment support changes in mathematics teaching? *For the Learning of Mathematics, 36(3),* 11-18.

Panero, M., & Aldon, G. (2015). How can technology support effectively formative assessment practices? A preliminary study. In N. Amado & S. Carreira (Eds.) proceedings of the 12th International Conference on Technology in Mathematics Teaching (ICTMT 12). 293-302.

Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for mathematical learning, 9(3),* 281-307.

Yerushalmy, M. & Elikan, S. (2010). Exploring reform ideas of teaching Algebra: Analysis of videotaped episodes and of conversations about them In R. Leikin & R. Zazkis (Eds.), *Learning through Teaching: Developing mathematics teachers' knowledge and expertise in practice*. (pp. 191-207). Springer Netherlands.

CAN I SKETCH A GRAPH BASED ON A GIVEN SITUATION? – DEVELOPING A DIGITAL TOOL FOR FORMATIVE SELF-ASSESSMENT

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This article describes the development of a digital tool for formative self-assessment in a design-based research study. The aim is to create a tool that allows students to self assess their work, rather than having technology evaluate their answers. Thus, learners are provided with a list of typical misconceptions to check their solutions to an open assessment task. This assessment task tests the students' ability to draw a graph based on a given situation. Two case studies in form of task-based interviews with sixteen-year-old students are described. The analysis leads to reconstruction of the learners' formative assessment processes by using a theoretical framework developed in the EU-project FaSMEd. The results show which formative assessment strategies students actively use when working with the digital tool and which functionalities of the technology can be identified.

Keywords: formative self-assessment, role of technology, functions, design-based research

AIM OF THE TOOL

A challenge for the design of a digital tool for student formative self-assessment is that the actual assessment should not be done by the technology. Some digital self-assessment environments generate a set of questions, check the student's answers based on two categories: right or wrong; and then provide the student with feedback in form of the number of correct responses. However, while a student works individually in such environments, he/she does not adopt the role of the assessor. Therefore, the term "self"-assessment refers only to the organisation of the assessment for such tools. In order to move the learning process forward, it is essential for the student to gain information on his/her own understanding of the learning content (Wiliam & Thompson, 2008). Moreover, the active involvement of learners is identified as a common characteristic of effective formative assessment approaches. Investigating their (mis-)conceptions helps students to gain sensitivity for their strengths and weaknesses. In addition, students can discover how to observe and direct their learning processes using metacognitive strategies along with reflection and adopt responsibility for their own learning in the process (Black & Wiliam, 2009; Heritage, 2007). Hence, a key design feature of our tool is a checklist of typical misconceptions related to the mathematical content, which is the change from a situational to a graphical representation of a function, that helps students to become self-assessors. The tool was developed during the design-based research EU-project FaSMEd (Raising Achievement through Formative Assessment in Science and Mathematics Education), which introduced and investigated technology enhanced formative assessment practices (www.fasmed.eu).

THEORETICAL BACKGROUND

Conceptualising formative assessment

Formative assessment (FA) is "the process used by teachers and students to recognize and respond to student learning in order to enhance that learning, during the learning." (Bell & Cowie, 2001, p. 540). It results in the

	Where the learner is going	Where the learner is right now	How to get there
Teacher	1 Clarifying learning intentions and criteria for success	2 Engineering effective class-room discussions and other learning tasks that elicit evidence of student understanding	3 Providing feedback that moves learners forward
Peer	Understanding and sharing learning intentions and criteria for success	4 Activating students as instructional resources for one another	
Learner	Understanding learning intentions and criteria for success	5 Activating students as the owners of their own learning	

Figure 1: Key strategies of FA (Wiliam & Thompson, 2008)

active adaptation of classroom practices to fit students' needs by continuously gathering, interpreting and using evidence about ongoing learning processes (Black & Wiliam, 2009). The required data can be elicited and exploited during the different phases of these processes. Wiliam and Thompson (2008) refer to Ramaprasad (1983) and focus on three central steps in teaching and learning, namely establishing: where the learners are, where the learners are going and how they might get there. The authors state that FA can be conceptualised in five key strategies (Figure 1). These strategies enable teachers, peers and students to close the gap between the students' current understanding and the intended learning goals.

While Wiliam and Thompson (2008) take into account central steps of the learning process and the agents (teacher, peers and learners) who act in the classroom, their framework regards mainly the teacher to be responsible for the process of FA. It is the teacher who creates learning environments to investigate the students' understanding (strategy 2), who gives feedback (strategy 3) and who activates students as resources for one another (strategy 4) and as owners of their own learning (strategy 5). In order to regard all three agents as being able to take responsibility for each of the steps and key strategies, the framework was refined in the FaSMEd project. The FaSMEd framework (Figure 2) allows the characterisation and analysis of technology enhanced FA processes in three dimensions: agent/s, FA strategies and functionalities of technology (www.fasmed.eu; Aldon, Cusi, Morselli, Panero, & Sabena, 2017).

The "agent/s" dimension specifies who is assessing: the student, peer/s, or the teacher. It is important to involve all of the agents in FA as the "assessment activity can help learning if it provides information that teachers and their students can use as feedback in assessing themselves and one another […]" (Black, Harrison, Lee, Marshall & Wiliam, 2004, p.10). Moreover, an active involvement of students by peer and self-assessment is stated as key aspect of FA. It includes opportunities for learners to recognize, reflect upon and react to their own/ their peers' work. This helps them to use metacognitive strategies, interact with multiple approaches to reach a solution and adapt responsibility for their own learning (Black & Wiliam, 2009; Sadler, 1989).

The "FA strategies" dimension of the FaSMEd framework refers to the five key strategies (Wiliam & Thompson, 2008), but understands them in a broader sense by acknowledging that all agents can be responsible for FA. For example, a student can elicit evidence on his/her own understanding (strategy 2) by working and reflecting on assessment tasks, peers can provide effective feedback (strategy 3), or a student can control his/her own learning process using metacognitive activities (strategy 5).

To specify the different functionalities that technology can resume in FA processes, FaSMEd introduced a third dimension to the framework: "functionalities of technology". We distinguish three categories:

(1) *Sending & Displaying*, which

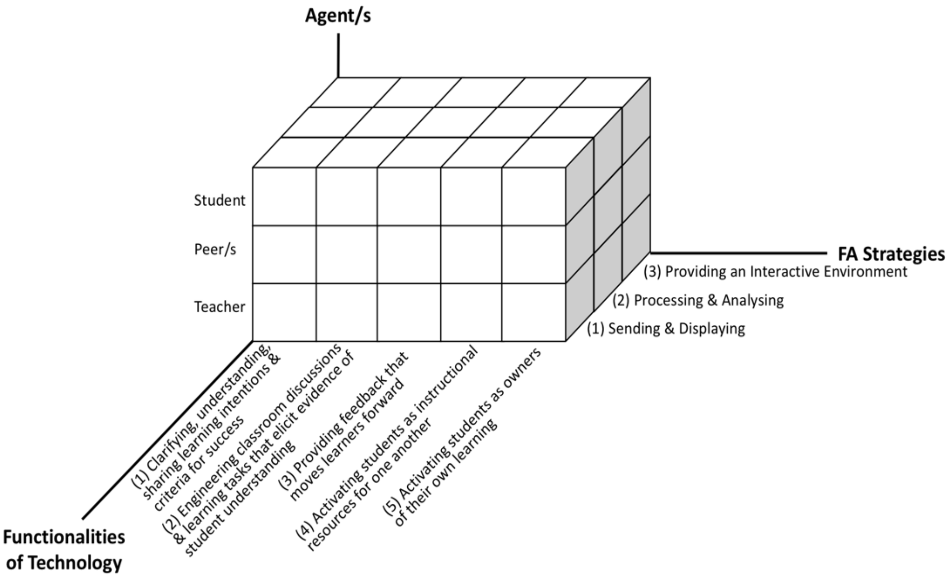

Figure 2: The FaSMEd framework

includes all technologies that support communication by enabling an easy exchange of files and data. For example, the teacher sending questions to individual students' devices or displaying one student's screen to discuss his/her work with the whole class.

(2) *Processing & Analysing* considers technology converting collected data. This includes software that generates feedback and results to an operation or applications which create statistical diagrams of a whole class' solution, for example after a poll.

(3) *Providing an Interactive Environment* refers to technology that enables students to work in a digital environment and lets them explore mathematical or scientific contents interactively. This category includes, for example, shared worksheets, Geogebra files, graph plotting tools, spread sheets or dynamic representations (www.fasmed.eu).

The mathematical content: Functions

During the development of a self-assessment tool, its mathematical content needs careful consideration. Bennett (2011) states that "to realise maximum benefit from formative assessment, new development should focus on conceptualising well-specified approaches […] rooted within specific content domains" (p.5). Therefore, a content analysis needs to evaluate, for example, which competencies or skills students need to master, what a successful performance entails and which conceptual difficulties might occur. This 'a priori' analysis revealed three aspects relating to functions relevant for the tool's development: different mental models that students need to acquire for a comprehensive understanding, translating between mathematical representations and known misconceptions.

The German tradition of subject-matter didactics specifies the idea of mental models in the concept of 'Grundvorstellungen' (GVs). It is used to "characterize mathematical concepts or procedures and their possible interpretations in real-life" (vom Hofe & Blum, 2016, p.230). Thereby, GVs identify different approaches to a content that makes it accessible for students. They describe, which mental models learners have to construct in order to use a mathematical object for describing real-life situations. In this sense, GVs act as mediators between mathematics, reality and the learners' own conceptions (vom Hofe & Blum, 2016). When using the graph of a function to describe a given situation, students have to acquire three GVs for the concept of functions: mapping, covariation and object. In a static view, a function maps one value of an independent quantity to exactly one value of a dependent quantity. The graph of a function can, thus, be seen as a collection of points that originate from uniquely mapping values of one quantity to another. In a more dynamic view, a function describes how two quantities change with each other. Considering a functional relation with this focus allows a graph to embody the simultaneous variation of two quantities. Finally, a function can be seen as a whole new mathematical object. Then, the graph is viewed from a global perspective (Vollrath, 1989).

Besides constructing these three GVs, a comprehensive understanding of the concept requires students to be able to change between different forms of representations of a function (Duval, 1999). Functional relations appear in a range of semiotic representations. Learners encounter them, for instance as situational descriptions, numerical tables or Cartesian graphs. Each of these emphasizes different characteristics of the represented function. Thus, transforming one form into another makes other properties of the same mathematical object explicit (Duval, 1999). What is more, Duval (1999) stresses that mathematical objects are only accessible through their semiotic representations. Therefore, each mathematical activity can be described as a transformation of representations. Duval (1999) differs between treatments, meaning the manipulation within the same semiotic system, and conversions, meaning the change of one representational register to another while preserving the meaning of the initial representation. The author identifies conversions between different registers to be the "threshold of mathematical comprehension for learners […]"

(Duval, 2006, p.128) and concludes that "only students who can perform register change do not confuse a mathematical object with its representation and they can transfer their mathematical knowledge to other contexts different from the one of learning" (Duval, 1999, p.10). Hence, asking students to draw a graph based on a given situation means assessing a key aspect of their understanding of the concept of functions.

As students' mistakes can mirror their conceptual difficulties, typical misconceptions in the field of functions are considered for the development of our digital self-assessment tool. For instance, Clement (1985) states that many students falsely treat the graph of a function as a literal picture of the underlying situation. They use an iconic interpretation of the whole graph or one of its specific features instead of viewing it as an abstract representation of the described functional relation (Clement, 1985). To overcome this mistake, students need opportunities to consider graphs symbolically. Thus, instructions might ask learners to interpret a graph point by point or to describe the change of the dependent quantity for certain intervals. Another example of a typical cognitive issue when graphing functions is the 'swap of axes' labels. This mistake can arise when students name the axes intuitively without regarding mathematical conventions (Busch, 2015). Hadjidemetriou and Williams (2002) even speak of the "pupils' tendency to reverse the x and the y co-ordinates" (p.4). In order to correctly label the axes for a given situation, learners need to understand the functional relation between two quantities from its description and apply the convention to record the independent quantity on the x-axis and the dependent one on the y-axis of a Cartesian coordinate system (Busch, 2015). These are examples of some of the findings on typical misconceptions that were used in the design of our tool that both anticipate certain student difficulties and provide hints to foster the desired competencies.

DESIGN OF THE DIGITAL SELF-ASSESSMENT TOOL

The structure of the tool draws on a set of self-assessment materials originating from the KOSIMA (German acronym for: contexts for meaningful mathematics lessons) project (Barzel, Prediger, Leuders & Hußmann, 2011). Therefore, the tool comprises five parts: *Test*, *Check*, *Info*, *Practice* and *Expand*. These are connected in a hyperlink structure and labelled with different symbols (Figure 3) to support easy learner orientation regarding the tool's use.

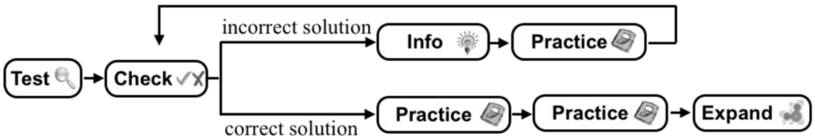

Figure 1: Hyperlink structure of the digital self-assessment tool

The aim is to create a tool that allows students to be self-assessors, that is why the design intends to create a balance between providing enough information as well as autonomy for the learners. The initial step of the self-assessment process is for the student to identify the learning goal. It is specified and made transparent in our tool by the question: "Can I sketch a graph based on a given situation?", which appears on the top of the first screen (Figure 4). The learner is provided with the *Test* task (labelled with a magnifying glass icon). This *Test* presents the story of a boy's bike ride and asks the student to build a graph that shows how the boy's speed changes as a function of the time. Besides labelling the axes by selecting an option from drop-down menus, the learner can build his/her graph out of moveable and adjustable graph segments. These are dragged into the graphing window and placed in any order the student chooses. Furthermore, the slope of the single segments can be altered by the user. After submitting a graph, a sample solution and *Check* are presented to help evaluate the individual answer (Figure 4). The *Check* is labelled with the symbol of a positive and negative check mark. It presents the student with six statements regarding important aspects of the functional relation at hand alongside common mistakes that could arise when solving the *Test* task. For example, one of the *Check*-points addresses the graph's slope: "I realized when the graph is increasing, decreasing or remaining constant.", or another represents the Graph-as-a-picture mistake: "I realized that the graph does not look like the street and the hill." The learner decides for each statement, if it is true for his/her solution, in which case it is marked off. For this diagnostic step, the student's screen not only presents the *Check*-list, but his/her answer as well as a sample solution to make a comparison easy. Thus, the *Check* helps the learner to self-assess his/her solution by presenting criteria for successfully solving the *Test* and by encouraging reflection of one's answer in comparison to the sample solution and *Check*-points. Additionally, the *Check* serves as a directory through the tool's hyperlink structure (Figure 3). This way, the student is encouraged to take further steps to move his/her learning forward.

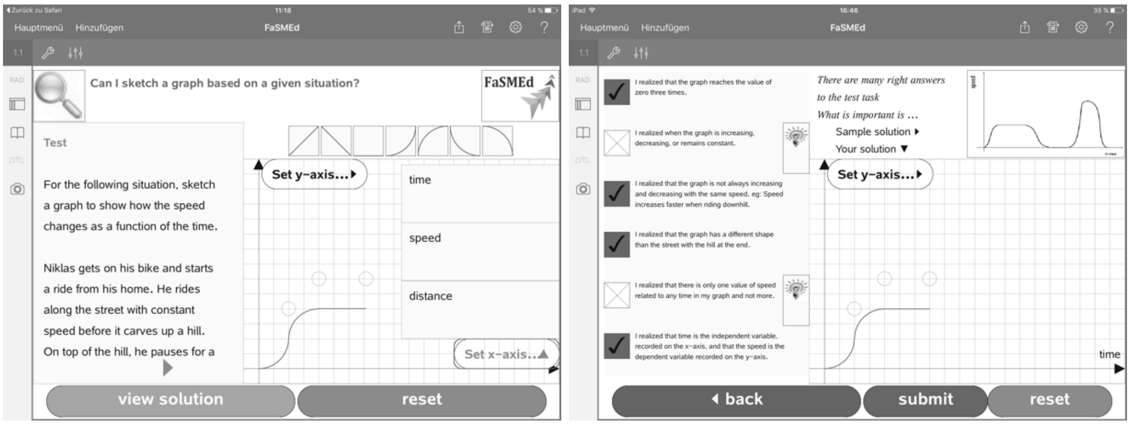

Figure 4: *Test* and *Check* of the digital self-assessment tool

If an error is identified by the learner, he/she can choose to work on the *Info* and *Practice* task corresponding with the *Check*-point's statement. The *Info* is labelled by the symbol of a lightning bulb. It entails a general explanation that is intended to repeat basic classroom contents to overcome the certain mistake. Moreover, the explanation is made accessible by using the time-speed context of the *Test* as an example. In addition, an illustration is included to ensure a visual help and to encourage the learner to change between the two semiotic representations: verbal description as well as Cartesian graph. Then, the *Practice* task lets the student test his/her understanding of the repeated content. It is marked by the picture of an exercise book. Afterwards, the user can go back to the *Check* and work on the next statement. If the sketched graph is stated as correct, two further *Practice* tasks and one *Expand* task with a more complex context are provided. The *Expand* is labelled with a gearwheels icon and, in this case, asks the student to draw two different graphs for the same situation.

Above all, the tool aims to challenge the student to reflect on his/her own solutions and reasoning. This is why, besides offering a *Check*-list, it presents sample solutions for all tasks. It is the learner who decides weather the own answer is correct by comparing it to the sample solution.

METHODOLOGY

The conception and evaluation of the digital self-assessment tool are connected within a design-based research study. This is a "formative approach to research, in which a product or process is envisaged, designed, developed, and refined through cycles of enactment, observation, analysis, and redesign, with systematic feedback from end users" (Swan, 2014, p.148). Here, two different forms of case studies are applied: class trials and student interviews. The purpose of the class trials is to evaluate the effectiveness of the tool's implementation by exploring whether: self-assessment is possible using the tool, the structure is clear, and any technical issues are identified. Hence, class trials are conducted during a lesson where students work on the digital self-assessment tool individually or in pairs. Data is collected in the form of the researcher's notes on the lesson and a classroom discussion about the students' experiences with the tool. In addition, task-based interviews with individual students aim for a more detailed understanding of the learners' FA processes. This is why, students are asked to "think out loud" during their work with the tool and interviewers are instructed to only intervene the students' self-assessment to remind them to verbalise their thoughts or to help with technical issues. At the end, reflecting questions about the students' experience with the tool are asked. The interviews are videoed and transcribed to serve as the main data pool for qualitative analyses. These lead to the reconstruction of FA processes using the FaSMEd framework (Figure 2). Besides generating a well-grounded tool, the aim of the study is to examine the following research questions:

When students work with the digital self-assessment tool:

1) which formative assessment strategies do they use?
2) which functionalities does the technology have within the student's FA processes?

In each cycle of development, the investigation of these questions using the FaSMEd framework (Figure 2) informs the re-design of the tool. On this account, several development cycles took place in the study since 2014. A first pen-and-paper version of the tool was evaluated through interviews with eleven grade eight students from two different secondary schools in Germany.

Following the tool's redevelopment, two digital prototypes were created using different technologies: JACK and TI-Nspire Navigator. JACK is a server-based system for online assessment developed by the Ruhr Institute for Software Technology at the University of Duisburg-Essen. While the software has several useful options, such as being able to generate automatic feedback based on student answers, to create statistical overviews of submitted solutions and to insert tasks with variable contents, the JACK prototype proved to be unfit for implementation of our tool due to three main reasons: First, its hyperlink structure could only be implemented in a restricted way. It was not possible to display the entire *Check*-list at once, but only single *Check*-points. Furthermore, the software has a limited number of task types that are mainly in form of multiple choice or open answer formats. Finally, JACK requires an internet connection, but most schools in Germany do not have access to wireless internet in their classrooms, which would limit its potential use. The second digital prototype was programmed in Lua script using the software TI-Nspire Navigator, which enabled the tool's hyperlink structure to be realized, offline access and a choice of using the tool on a computer or iPad. Moreover, the options for implementing open tasks were greater and dynamic visualisations could be inserted. Hence, the tool's design was implemented only for TI-Nspire Navigator. The subsequent classroom trial of the digital tool run on iPads involving 18 grade ten students led to further redevelopments.

For the following situation, sketch a graph to show how the speed changes as function of the time.

Niklas gets on his bike and starts a ride from his home. He rides along the street with constant speed before it carves up a hill. On top of the hill, he pauses for a few minutes to enjoy the view. After that he drives back down and stops at the bottom of the hill.

The finished digital version was trialled in two grade ten classrooms at two further secondary schools and associated student interviews (one per class) were recorded. Finally, another set of student interviews with two second semester university students were held. The wide range of data in different age groups and schools resulted in a thorough evaluation of the tool's potential and constraints. As it is intended to assess and repeat basic mathematical competencies, its use is not limited to one specific group of learners. First experiences with the tool show that students in all of the tested class levels (grades 8, 10 and university) had similar issues concerning mathematical understanding as well as technical problems. This article focuses on the two single student interviews recorded in grade ten.

RESULTS

Two students' work with the digital tool are presented and their FA processes analysed using the FaSMEd framework (Figure 2). Both learners (S1 & S2) are female and sixteen years old, but visit different secondary schools. Their interviews were chosen for the analysis because they both trialled the digital version of the tool and selected the same *Check*-point regarding switching the x- and y-axis labels to take further steps in their learning. Both students start with the *Test* task (see text box).

S1 built her graph (Figure 5) by dragging moveable graph segments into the graphing window and selecting labels for both axes from drop-down menus. As she solved the assessment task, she evidences her understanding of sketching graphs of given situations (strategy 2) while the tool provides an interactive learning environment (functionality 3). After reading the sample solution out loud, S1 moved to the *Check* and was silent for a while. The interviewer asked what she was thinking about. The student mentioned being unsure about which *Check*-list items to mark off because she "saw in the sample solution that there was another graph and this was missing in [her] own solution." With the "other graph" she means a second hill-shaped part of the graph, which she indicated by gesturing its shape on the screen with her finger. It can be concluded that the *Check* stimulates S1 to assess her answer by comparing her own graph to the sample solution. By reflecting on her answer, S1 uses a metacognitive activity and, thus, adopts some responsibility for her own learning process (strategy 5). The tool displays the information she needs for the diagnostic step in form of the sample solution and *Check*-list (functionality 1). Furthermore, the student decided to evaluate the last statement in the *Check*. It reads *"I realized that the time is the independent variable recorded on the x-axis and that the speed is the dependent variable recorded on the y-axis."* S1 stated that this was not true for her graph, which means that she understands a criterion to successfully solve the *Test* (strategy 1). What is more, she reflects on her solution by comparing it to the *Check*-point statement (strategy 5) and formulates a self-feedback (strategy 3): "The speed and time were wrong because there [she points to x-axis] needs to be the time and there [she points to y-axis] the speed. I did not realize this." Here, the technology is once more functioning as a display of information in the form of the *Check*-point (functionality1).

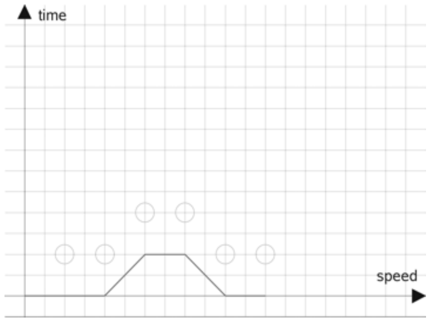

Figure 5: S1's *Test* solution

At that point S1 decided a next step in her learning (strategy 5) when she read the associated *Info*. After the interviewer reminded her of the possibility to do another exercise related to her mistake, S1 worked on the linked *Practice*. This helps her to elicit evidence about her understanding of the independent and dependent quantity of a functional relation (strategy 2). The tool provides the task and sample solution (functionality 1). The task presented the learner with ten different situations describing the functional relation between two quantities. For each one, the learner was asked to assign labels to the axes of a coordinate system (given that he/she imagined drawing a graph based on the situation in the next step). The labels were chosen from a number of given quantities: temperature, distance, speed, time, pressure, concentration, money, and weight. S1 solved six out of ten items correctly. While she seemed to have no difficulties with situations in which time appeared as the independent quantity, she struggled to label the y-axis when time was being dependent on another quantity. For example, in the situation *"In a prepaid contract for cell phones, the time left to make calls depends on the balance (prepaid)."* S1 chose "time" as the label for the x-axis and "money" as the label for the y-axis. However, she explained "if you have a prepaid phone, you can only make calls as long as you have money." Therefore, she grasped the relation in the real-life context but ccouldn't use this knowledge when asked to represent it in form of a graph. Moreover, the student repeated this mistake of 'swapping the axes' even in situations that didn't include time as a quantity. For instance, S1 selected "distance" as the label for the x-axis and "speed" for the y-axis in the situation *"Tim's running speed determines the distance he can travel within half an hour."* Nonetheless, she explained correctly that "the speed specifies how far he can run." A possible explanation for her repeating mistake could be her approach to the task. S1 selected a label for the y-axis first before going on to the x-axis. This could mean that she does not fully understand the conventions of drawing a Cartesian coordinate system. However, her mistake could also originate from a deeper misunderstanding as Hadjidemetriou and Williams (2002) speak of the "pupils' tendency to reverse the x and the y co-ordinates" and their inability to adjust their knowledge in unfamiliar situations" (p.4). This would show a need for further interventions. However, S1 was able to identify two out of her four mistakes by comparing her answers to the sample solution (strategy 5) before she returned to the *Check* and marked off the respective *Check*-point statement.

In summary, S1's work with the digital self-assessment tool can be depicted as shown in Figure 6. She solves a diagnostic task, identifies a mistake by understanding criteria for success, reflecting on her answer and comparing it to a sample solution and displayed statement. She gives herself feedback and decides to take further steps in her learning by revising information on her error and practicing. Though she is not fully able to overcome her mistake, the tool supports S1 to think about her work on a metacognitive level and adopt responsibility for her learning. Thus, S1 uses four FA strategies, while the tool's functionality can be labelled as displaying information or, in case of the *Test* task, providing an interactive environment. Her formative assessment process can be characterised using the FaSMEd framework as shown in Figure 7.

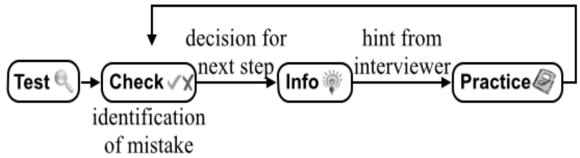

Figure 6: Reconstruction of S1's FA process

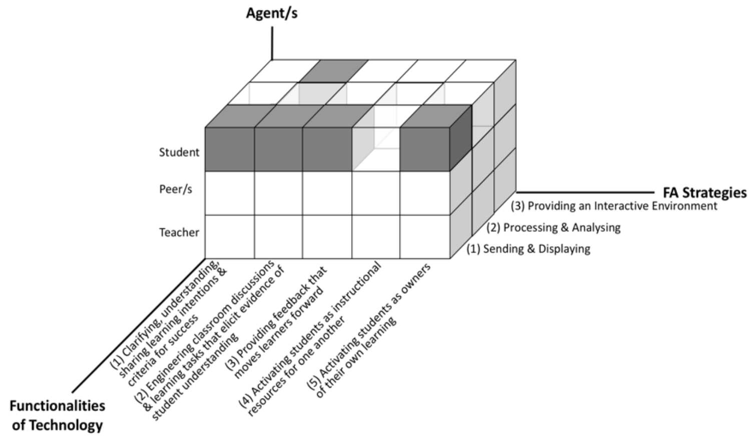

Figure 7: Characterisation of S1's FA process

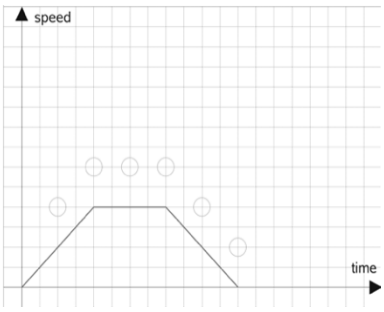

Figure 8: S2's *Test* solution

S2 also sketched a graph (Figure 8) to solve the *Test* and elicit evidence of her understanding (strategy 2) using the tool's interactive graphing window (functionality 3). In the *Check*, she didn't mark off the

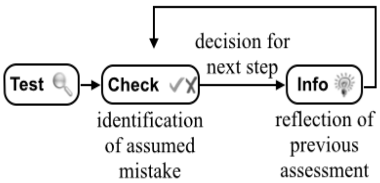

Figure 9: Reconstruction of S2's FA process

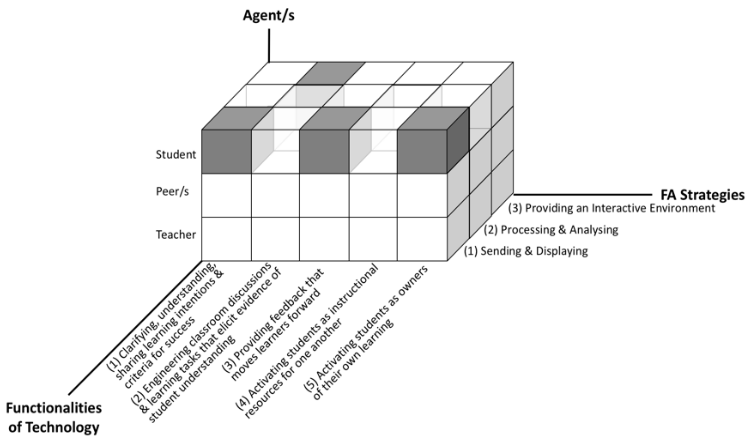

Figure 10: Characterisation of S2's FA process

statement concerning time being the independent and speed being the dependent quantity. Thus, S2 identifies a supposedly error based on the displayed *Check* statement (functionality 1). Even though she labelled the axes correctly, S2 decided to read the *Info* concerning her alleged mistake and is, thus, adopting responsibility for her learning (strategy 5). When reading the *Info*, she realized: “Oh, that is correct as well because I did it in the same way." She not only states a self-feedback (strategy 3), but also compares the displayed information (functionality 1) to her own *Test* answer and reflects on her assessment (strategy 5). Then S2 went back to the *Check* and marked off the statement correcting the error in her previous assessment autonomously. In conclusion she identifies a correct aspect about her work, which means she now understands a criterion for success (strategy 1).

In summary, S2’s work with the digital self-assessment tool can be illustrated as in Figure 9. She works on a diagnostic task, identifies an assumed mistake and decides to gather more information on it. Then S2 identifies an error in her previous self-assessment by comparing her solution of the *Test* to the displayed *Info*. Finally, she corrects her assessment. The analysis shows that within this process, she uses four different formative assessment strategies, while the tool functions mainly as a display of information and for the *Test* provides an interactive environment (Figure 10).

CONCLUSIONS AND FURTHER STEPS

The analysis of the two cases shows that the tool does have the potential to support students’ formative self-assessment concerning their ability to draw a graph based on a given situation. It is the user, who holds the responsibility to identify mistakes and decide on next steps in the learning process. In addition, the tool stimulates students to actively use four different key strategies of formative assessment: the clarification and understanding of criteria for success, eliciting evidence on student understanding, formulating feedback and being activated as the owners of one’s own learning.

However, the case studies highlight some constraints of the digital self-assessment tool, which (in the cyclic process of the study) lead to a redesign that is currently being programmed. In the interviews, it became clear that students are uncertain about assessing themselves as they mentioned that they expect validation from either the teacher or the technology. This is why, the redesign focuses on improving students’ comprehension of the learning goal, namely the change of representation from situation to graph, and simplifying the learners’ self-evaluation. Hence, the static picture of the *Test’s* sample solution will be replaced with a simulation of the described bike ride connected to the sample graph as well as the student’s own solution (Figure 11). Furthermore, all *Practices* will allow simultaneous views of the student’s answer next to a sample solution for easier comparison. The students’ interview statements and S1’s case, in which she was unable to fully overcome her mistake, revealed that it will not be possible for all students working with the tool to (re)learn the change of representation from situation to graph on their own. Further interventions not included in the tool might be necessary. Therefore, the newest version will save the individual student’s work

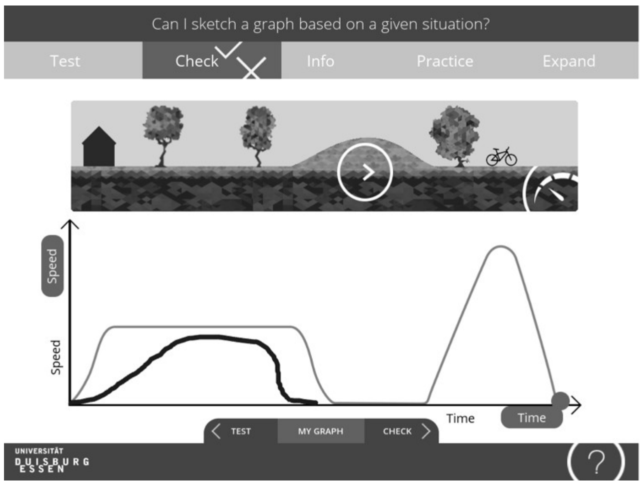

Figure 11: Simulation of bike ride as sample solution of the *Test* task in the tool’s current redesign

and include a teacher functionality to review students' solutions and enable more effective planning of post-assessment classroom interventions by addressing students' needs more directly.

Furthermore, the two cases show that the tool's functionality can mainly be described as displaying information. To increase the interaction between students and tool, the redesign will include dynamic visualisation for most of the *Info* units. These will enable students to click on highlighted segments of a displayed graph to open and read an explanation. In addition, simulations as described for the *Test's* sample solution, that allow to make connections between the real-life situation and the graph of a function, will be used in some of the *Practice* tasks as well.

Finally, the interviews show that more detailed analyses are necessary to gain a deeper understanding of the students' formative self-assessment processes. While working with the digital tool did not help learners to overcome all of their mistakes, it encouraged them to reflect on their own solutions on a metacognitive level. This seems to be the key for students' success in doing self-assessment. Therefore, a category system for a qualitative content analysis of the interviews is currently being developed. It focuses on three main categories regarding the students': metacognitive activities, tool activities and content-related activities. The aim is to observe which metacognitive activities are prompted through which design aspects of the digital self-assessment tool and how this can help the students' conceptual understanding of the content of functions.

REFERENCES

Aldon, G., Cusi, A., Morselli, F., Panero, M., & Sabena, C. (2017). Formative assessment and technology: reflections developed through the collaboration between teachers and researchers. In G. Aldon, F. Hitt, L. Bazzini, U. Gellert (Eds.), *Mathematics and Technology: A C.I.E.A.E.M. Sourcebook* (pp. 551-578). Springer International Publishing.

Barzel, B., Prediger, S. Leuders, T., & Hußmann, S. (2011). Kontexte und Kernprozesse: Aspekte eines theoriegeleiteten und praxiserprobten Schulbuchkonzepts [Contexts and core processes – Aspects of a school book concept guided by theory and practice]. *Beiträge zum Mathematikunterricht*, 71-74.

Bell, B., & Cowie, B. (2001). The characteristics of formative assessment in science education. *Science Education*, 85(5), 536-533.

Bennett, R. E. (2011). Formative assessment: A critical review. *Assessment in Education: Principles, Policy & Practice*, 18(1), 5-25.

Black, P., Harrison, C., Lee, C., Marshall, B., & Wiliam, D. (2004). Working inside the black box: Assessment for learning in the classroom. *Phi Delta Kappan*, 86(1), 8-21.

Black, P., & Wiliam, D. (2009). Developing the theory of formative assessment. *Educational Assessment, Evaluation and Accountability*, *21*(1), 5-31.

Busch, J. (2015). Graphen „laufen" eigene Wege. Was bei Funktionen schiefgehen kann [Graphs "run" own ways. What can fail with functions]. *Mathematik lehren*, 191, 30–32.

Clement, J. (1985). Misconceptions in graphing. In L. Streefland (Ed.), *Proc. 9th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol.1, pp. 369-375). Utrecht: PME.

Duval, R. (1999). Representation, Vision and Visualization: Cognitive Functions in Mathematical Thinking. Basic Issues for Learning. Retrieved from http://files.eric.ed.gov/fulltext/ED466379.pdf.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational studies in mathematics*, *61*(1), 103-131.

Hadjidemetriou, C., & Williams, J. (2002). Children's graphical conceptions. *Research in Mathematics Education*, *4*(1), 69-87.

Heritage, M. (2007). Formative assessment: What do teachers need to know and do? *Phi Delta Kappan*, *89*(2), 140-145.

Ramaprasad, A. (1983). On the definition of feedback. *Systems Research and Behavioral Science*, *28*(1), 4-13.

Sadler, D. R. (1989). Formative assessment and the design of instructional systems. *Instructional science*, *18*(2), 119-144.

Swan, M. (2014). Design research in mathematics education. In *Encyclopedia of mathematics education* (pp. 148-152). Springer Netherlands.

Vollrath, H.-J. (1989). Funktionales Denken [Functional thinking]. *Journal for Didactics of Mathematics*, 10(1), 3-37.

vom Hofe, R., & Blum, W. (2016). "Grundvorstellungen" as a Category of Subject-Matter Didactics. *Journal for Didactics of Mathematics*, *37*(1), 225-254.

Wiliam, D., & Thompson, M. (2008). Integrating assessment with learning: what will it take to make it work? In C. A. Dwyer (Ed.), *The Future of Assessment: Shaping Teaching and Learning* (pp. 53-82). Mahwah, NJ: Erlbaum.

Chapter 4

STUDENTS

STUDENTS' EXPANDING OF THE PYTHEGOREAN THEOREM IN A TECHNOLOGICAL CONTEXT

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Dynamic Mathematical Software (DMS) in general and GeoGebra in particular have attracted the attention of mathematics educators because of their potential to influence student learning. The present research aims to add to the growing research efforts to study the influence of GeoGebra on processes used by dyadic learners to construct knowledge. Specifically, we study the context of three pairs of seventh graders who worked on an inquiry task to construct the relations between areas of squares built on the sides of an obtuse/acute triangle in a specially designed GeoGebra environment. The analysis used the Abstraction in Context (AiC) framework. The findings indicate that the three pairs constructed all the expected knowledge elements, and that one pair constructed unexpected construct. Generally, findings indicate the positive influence of the GeoGebra technological tool on the construction processes.

Keywords: GeoGebra, Construction of knowledge, Pythagorean theorem

INTRODUCTION

Two areas are at the core of the current study: (1) the construction of abstract mathematical knowledge; and (2) the use of dynamic technology in mathematics education. These two domains are examined below.

Construction of abstract mathematical knowledge

Understanding how learners construct abstract mathematical knowledge is a central aim of research in mathematics education. Abstraction in Context (AiC) is a theoretical framework for describing processes of abstraction in different contexts (Dreyfus, Hershkowitz, & Schwarz, 2001). Dreyfus et al. (2001) defined abstraction as a process in which previous mathematical constructs are vertically reorganized into a new structure. The role of context is central to the process of constructing abstract mathematical knowledge. Several contextual factors may influence mathematical abstraction, including the students' prior knowledge, the nature of the task, and interactions with other learners and with technology. Understanding the role of context may lead to a better understanding of abstraction processes. Hence, the present study began with a careful design of the contextual factor – technology applet.

The AiC framework postulates that the genesis of abstraction passes through a three-stage process: the need for a new construct; the emergence of the new construct; and the consolidation of that construct. The emergence of a new construct is described and analyzed by the RBC model: recognizing (R), building-with (B) and constructing (C). Recognizing refers to the learner's realization that a previous knowledge construct is relevant for the situation at hand. Building-with involves combining recognized constructs in order to achieve a localized goal, such as the actualization of a strategy, a justification or a solution to a problem. Constructing consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct.

Some studies have used the AiC framework for investigate the processes of constructing knowledge in technological contexts (Anabousy & Tabach, 2015; Kidron & Dreyfus, 2010; Ofri & Tabach, 2013). These studies demonstrated the positive influence of technological tools. Specifically, Kidron and Dreyfus (2010) studied how instrumentation led to constructing actions and how the roles of the learner and a computer algebra system (CAS) become intertwined during the process of constructing a justification. They showed that certain patterns of epistemic actions were facilitated by the CAS context. Ofri and Tabach (2013) studied knowledge construction among eighth-grade dyads in a GeoGebra environment to explore a problem situation related to functions. They found that the students constructed the targeted knowledge while interacting with a dynamic and multi-representation environment.

The present study aims at tracing processes of constructing abstract mathematical knowledge among three pairs of seventh-grade students engaged in an inquiry task to construct the relations between areas of squares built on the sides of an obtuse/acute triangle in a GeoGebra environment. Also, the present study describes how the technological tool influenced the participants' actions.

Use of dynamic technology in mathematics education: the case of GeoGebra

Numerous studies have shown that dynamic technologies can be used to encourage exploration, conjecture, construction and explanation of geometrical relationships (Jones, 2005). Studies have also shown that the visual characteristics of these technologies may develop the ability to make correct assumptions (Hohenwarter et al., 2008). One such dynamic mathematical software system is GeoGebra, which is specifically designed for learning and teaching mathematics.

The educational potential of GeoGebra has been demonstrated by various studies that examined its effect on learning mathematics (see, for example, Dikovic, 2009). This tool has the potential to encourage student-centred and discovery learning by using interactive explorations to experiment with mathematical ideas (Tran et al., 2014).

As mentioned, the integration of GeoGebra software into teaching and learning mathematics has various benefits (Dikovic 2009; Erkek & Işıksal-Bostan, 2015). On the other hand, this integration also has its disadvantages (Jones, 2005; Erkek & Işıksal-Bostan, 2015). Research on the effectiveness of integrating GeoGebra in teaching and learning mathematics is still limited (Dikovic 2009; Saha et al. 2010). In the present study, GeoGebra was used to build carefully designed applet used in an inquiry task to examine the process of mathematical knowledge construction in this context.

The study was guided by the following research questions:

1. How do seventh-grade students construct the expansion of the Pythagorean Theorem: the case of changing the right-angle triangle to obtuse/acute triangle?
2. How do the purposefully designed GeoGebra-applet influence the construction process?

METHOD

Three pairs of seventh-grade students from the same class participated in the study. According to their teacher, all had high mathematical achievements.

An appropriate GeoGebra applet and an inquiry task concerning the relations between areas of squares built on the sides of an obtuse/acute triangle were designed for the study.

The task presented a mathematical situation (Figure 1). The students were asked to propose a hypothesis regarding the mathematical situation and then to experiment with GeoGebra to verify or refute their hypothesis. Finally, they were asked to explain/justify the constructed mathematical concept/relation. The students worked on the task for about 45-55 minutes, and their work was recorded and transcribed verbatim.

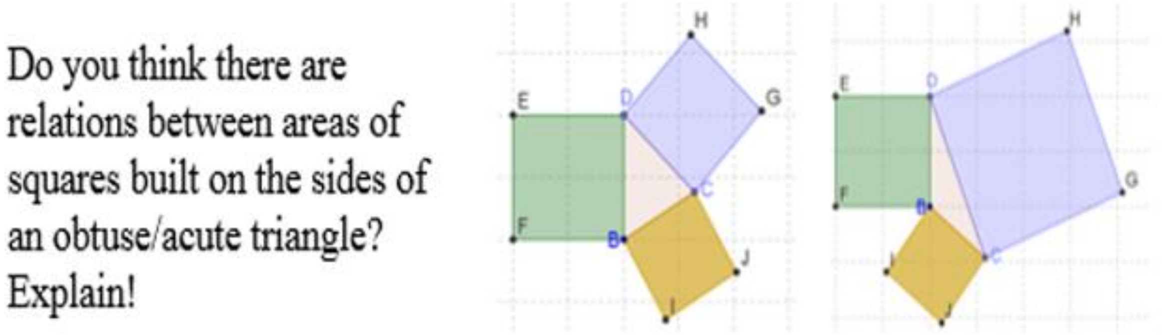

Figure 1: The mathematical situation presented in the task

GeoGebra was selected as the technological environment due to its dynamic nature and ease of use. One GeoGebra applet was built for the study by the first researcher. Figure 2 provides a screenshot of the interface of this applet.

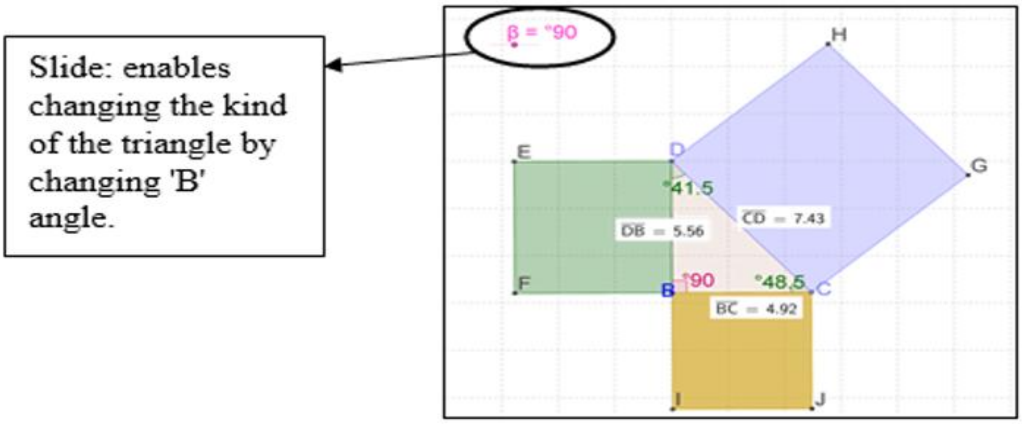

Figure 2: screenshot of the interface of this applet

The expected knowledge elements and sub-elements to be constructed were assumed based upon a-priori analysis. We also considered the Pythagorean Theorem to be a previous knowledge element because of its critical role in the construction processes we assumed would occur. The Pythagorean Theorem was constructed by the three pairs in a task designed by the researchers in a previous study (Anabousy & Tabach, 2015).

Figure 3 shows the a-priori analysis of the connections between the knowledge elements subsequently described. An operational definition was developed for each element to guide the analysis of the students' knowledge constructing activity.

E1: The relations between areas of squares built on the edges of an obtuse triangle.

E2: The relations between areas of squares built on the edges of an acute triangle.

E3: The justification of E1.

E4: The justification of E2.

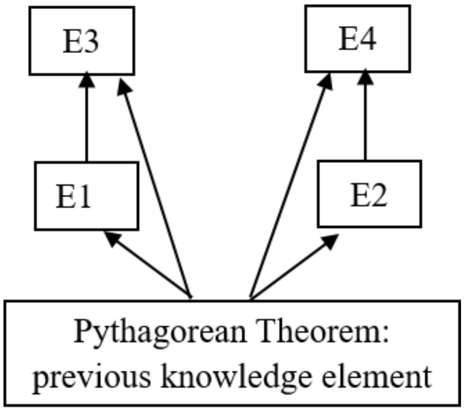

Figure 3: The connections between assumed knowledge elements

Here we consider that expected knowledge elements to be constructed as (Ei), while when the student construct that element we refer to this process as construction (Ci) of the element.

FINDINGS

We describe the process of constructing knowledge graphically and verbally. We also present an episode that shows the constructing of knowledge in the technological context.

To describe the knowledge construction process graphically, we chose the following representations which include all possible cases. When a knowledge element is in the process of being constructed, a solid rectangle appears under it (). When there are no acts of construction, the rectangle is empty (). When the construction is successfully completed, a bold black line () appears at the bottom of the rectangle. When the construction ends partially, a bold red line appears () at the bottom of the rectangle. When an unexpected element is constructed, the rectangle is dashed (). When a knowledge element is constructed incorrectly, the rectangle is red (). Finally, when a knowledge element has not been constructed at all, a solid rectangle framed in red appears ().

We divided the activity into ten segments that are parallel to the activity questions. These segments are arranged chronologically: (1) Recording the areas of squares built on the sides of an obtuse-angled triangle; (2) Identification of the relations between the areas of the squares built on the sides of an obtuse-angled triangle and its generalization; (3) Explanation of the relations between the areas of the squares built on the sides of an obtuse-angled triangle; (4) Problem formulation for exploring the case of the acute-angled triangle; (5) Exploration of one case (of the acute-angled triangle) and generalization; (6) Adjustment of generalization; (7) Checking the students' confidence in their conclusion; (7*) Discovery of another connection (unexpected knowledge element constructed by the first pair); (8) Last formulation of the conclusion; (9) Explanation of the connection between areas of squares built on the sides of an acute-angled triangle. Figure 4 below describes the process of knowledge construction by the three pairs in the activity graphically. In the following figure 4, the vertical axis represents the ten segments, while the horizontal axis represents the knowledge constructs.

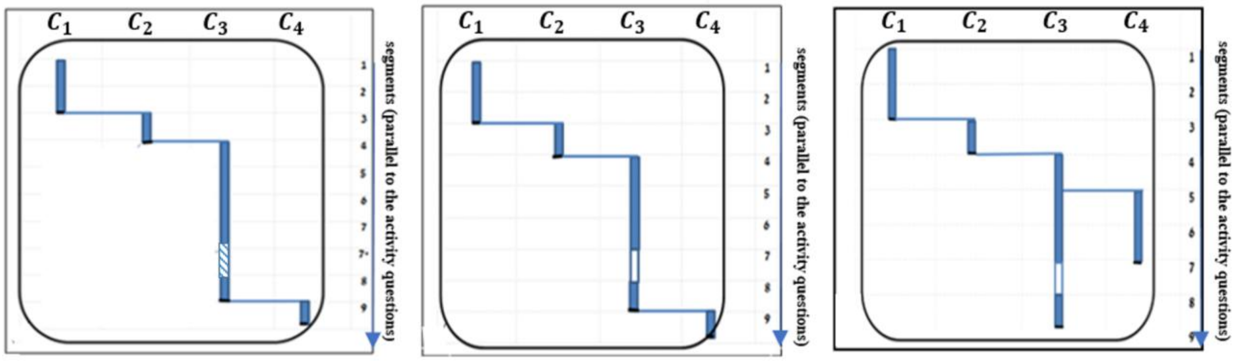

Figure 4a Figure 4b Figure 4c

Figure 4: Knowledge construction in the activity: first, second and third pairs, respectively.

Figure 4 shows the order of construction. As we assumed, the constructs C1, C2, C3, and C4 (usually) were successfully built one after the other by the three pairs. The figure also demonstrates that the first pair produced an unexpected construction: as one of the angles of the triangle approaches 90º, the sum of the areas of the two squares built on both sides of the angle approaches the area of the square built on the opposite side (this process is described in episode 1). Based on Figures 4a-4c, we can also claim that by and large, all the pairs underwent similar construction processes. The process of constructing C4 was similar for the first and the second pairs, and different for the third pair, with the difference manifested in the time it took to build the construct.

Below is episode 1, which shows construction of the unexpected construct by the first pair. Here, R indicates *recognizing* action, B indicates *building-with* action and C indicates *constructing* action.

Episode 1: Construction of the unexpected knowledge element (S1 and S2 are the students)

1.		S1	This is a right angle triangle [presented in fig.5].
2.	B_Pythagorean theorem	S2	No, we cannot depend on the picture. 22+9=27… it's 'more than' relation [she means: the sum of the areas of the two squares built on the edges comprising the acute angle is more than the area of the square built on the edge opposite to the acute angle]. 9+13= 22…ohhh what?!!
3.		S1	Something unusual has happened!
4.		Inter.	What?
5.		S1	S2, You were wrong, this is not 13, it is 14 if we approximate it to an integer [means to approximate 13.8 to integer number]
6.	B	S2	Yes, 14+9=23, it's a 'more than' relation.
7.		Inter.	You do not have to approximate the numbers. Look at this triangle; it is close to being what?

8.	R_right-angle triangle	S2	A right-angle triangle.
9.		Inter.	And what happens to the areas?
10.	B C	S2	Ahhh… the sum of the areas of the two squares built on both sides of the angle, the angle that is approaching 90°, approaches the area of the square built on the opposite side.
11.		S1	That's it, we're finished.

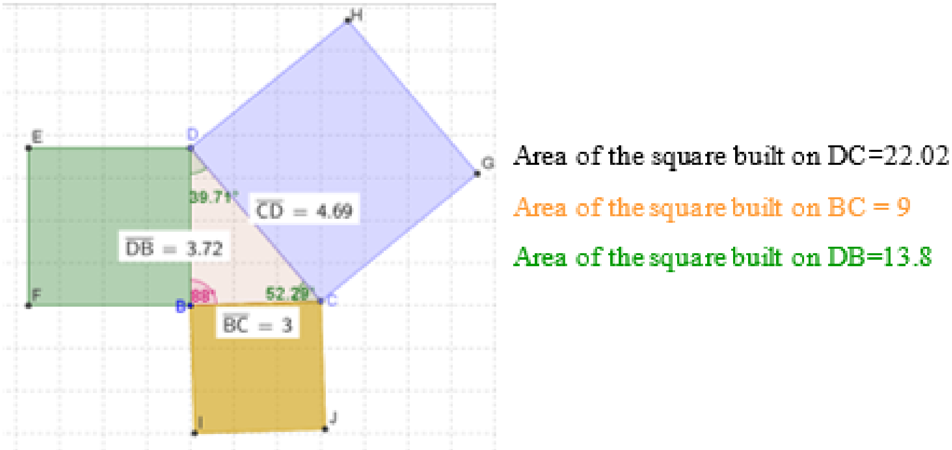

Figure 5: Screen shot from the work of the second pair in episode 1

In this episode, the students investigated an extreme case (see Fig. 5). This investigation enabled the students to move from constructing E2 to constructing the unexpected construct in line 10. The students watched the change in the areas of the squares built on the sides of an acute triangle, and formulated two knowledge constructs: (a) as one of the angles of the triangle approaches 90, the sum of the areas of the two squares built on both sides of the angle approaches the area of the square built on the opposite side [line 10]; and (b) E2 [in discussion after episode 1, not shown].

As mentioned, the knowledge construction processes of these students occurred in a technological context. The technological tool (GeoGebra applet) supported the processes of constructing knowledge, as it enabled students to (1) explore "representative" cases such as triangles with different “types” of side lengths: large/small numbers, fractions and integers and specific extreme cases (e.g., when constructing E1 and E3); (2) transition from one construct to two parallel constructs and constructing an unexpected construct. This transition took place during the construction of E2 by the first pair (see episode 1, lines 1-3); (3) justify/explaining the constructed relations (when constructing E2 and E4). Below we present in details the process of constructing the explanation of E1 and E3 by the first pair.

Constructing C_2 and C_4 (explanation of C_1 and C_3) by the first pair (S1 and S2 are the students): S1 performed the construction immediately at the beginning of the question, with the help of knowledge elements C_1 and C_3. She said, "...in an acute-angled triangle, the right angle decreased [refers to the method they use to obtain an acute-angled triangle]; therefore, the side was reduced, and therefore the area of the square was reduced. And they [squares] were bigger than it, as opposed to the obtuse." S1 tried to explain this to S2, but S2 was not convinced. Eventually, S1 suggested to

S2 to use her hands and began to explain to her what was happening in an acute-angled triangle and in an obtuse-angled triangle, while examining the case of a right-angled triangle. Figure 6 shows an example of S1's explanation of this connection.

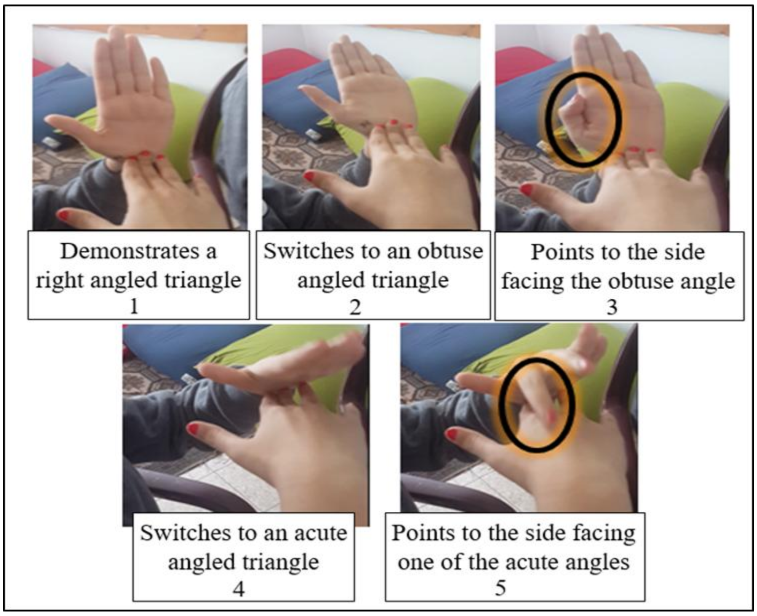

Figure 6: Gestures used by the first pair while explaining the expanded Pythagorean Theorem

DISCUSSION

The present study traced processes of constructing mathematical knowledge by three pairs of seventh-grade students engaging in an inquiry task to construct the relations between areas of squares built on the sides of an obtuse/acute triangle in GeoGebra environment. The findings indicate that the three pairs constructed all the knowledge elements. Successful construction of knowledge in a technological context has also been reported by Ofri and Tabach (2013).

GeoGebra applets supported the knowledge construction sequence: examination of various cases, generalization, proving or explaining. For example, when constructing C_1 (The relations between areas of squares built on the edges of an obtuse triangle), pairs of students discovered the relation by considering many different cases, which was made possible by manipulating the applet. The various cases were not random, as the students selected specific representative cases, such as polygons with sides measured in integers, fractions, large and small numbers. This examination allowed students to make a generalization regarding the relation. Dikovic (2009) also reported on support provided by the technological tool GeoGebra in exploration and generalization activities.

The technological tool also allowed two parallel construct actions to take place simultaneously and it enabled unexpected constructs to be built. For example, when the first pair was constructing C_3 (the relation between the areas of squares built on the sides of an acute-angled triangle), the students observed the change in the areas of the squares built on the sides of an acute-angled, they also examined extreme cases which were provided by the applet and then built unexpectedly two knowledge constructs (for similar findings see Kidron & Dreyfus, 2010)

The construction process was further supported by the technological tool by reducing the need to spend time on calculations, allowing the pairs to focus on searching relations and explanations, for example when constructing C_1 and C_3. The tool provided the areas of the shapes. Many studies have reported similar contributions of the technological tool, for example Becta (2003).

The technological tool also supported the explanations given by the pairs, for example the explanation of C_2. All the pairs argued that when transitioning from a situation in which the triangle is right-angled to one in which the triangle is obtuse-angled, the side opposite the angle will be longer, so that the area of the square built on it will be larger, which will change the relation from equivalence to one of "bigger than." In this case, the structure of the applet and the work using it supported and facilitated the emergence of the explanation provided by the students (Lachmy & Koichu, 2014). Several studies (Ng & Sinclair, 2013) showed students' reliance on gestures and dragging to be multimodal resources for communicating about dynamic aspects of mathematics.

REFERENCES

Anabousy, A., & Tabach, M. (2015). Constructing and Consolidating Mathematical Knowledge in GEOGEBRA environment by a pair of students. In K., Beswick, T., Muir, & J., Wells, (Eds.), *Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 41-48). Hobart, Australia: PME.

Becta. (2003). *What the Research Says about Using ICT in Maths*. UK: Becta ICT Research. NJ: Lawrence Erlbaum Associates.

Dikovic, L. (2009). Applications GeoGebra into teaching some topics of mathematics at the college level. *Computer Science and Information Systems, 6*(2), 191–203.

Dreyfus, T., Hershkowitz, R. & Schwarz, B. (2001). Abstraction in Context II: The case of peer interaction. Cognitive Science Quarterly, 1(3), 307-368.

Erkek, Ö., & Işıksal-Bostan, M. (2015). Is the use of GeoGebra advantageous in the process of argumentation? In K. Konrad, & V. Nada (Eds.), *Proceedings of the ninth Congress of the European Society for Research in Mathematics Education, CERME 9*, (pp. 121-127). Prague, Czech Republic.

Hohenwarter, M., Hohenwarter, J., Kreis, Y., & Lavicza, Z. (2008). Teaching and learning calculus with free dynamic mathematics software GeoGebra. *11th International Congress on Mathematical Education.* Monterrey, Nuevo Leon, Mexico.

Jones, K. (2005). Research on the use of dynamic geometry software: implications for the classroom. In: J. Edwards & D. Wright (Ed), *Integrating ICT into the Mathematics Classroom* (pp. 27-29). Derby: Association of Teachers of Mathematics.

Kidron, I., & Dreyfus, T. (2010). Interacting Parallel Constructions of knowledge in a CAS Context. *International Journal of Computers for Mathematical Learning, 15*, 129-149.

Lachmy, R., & Koichu, B. (2014). The interplay of empirical and deductive reasoning in proving “if” and “only if” statements in a dynamic geometry environment. *The Journal of Mathematical Behavior, 36*, 150-165.

Ng, O.-L., & Sinclair, N. (2013). Gestures and temporality: Children's use of gestures on spatial transformation tasks. *Proceedings of PME 37, 3*, 361-368.

Ofri, O., & Tabach, M. (2013). Knowledge constructing in a computerized environment: Eight grade dyads explore a problem situation. In A. M. Lindmeier, & A. Heinze, (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 401-408). Kiel, Germany: PME.

Saha, R., Ayub, A., & Tarmizi, R. (2010). The effects of GeoGebra on Mathematics achievement: Enlightening coordinate geometry learning. *Procedia Social and Behavioral Science, 8*, 686-693.

Tran, T., Nguyen, N. G., Bui, M.D., & Phan, A. H. (2014). Discovery learning with the help of the Geogebra Dynamic Geometry Software. *International Journal of Learning, Teaching and Educational Research, 7*(1), 44- 57.

LOOKING AT COMPOSITIONS OF REFLECTIONS IN A DGE FROM THINKING MODES AND SEMIOTIC PERSPECTIVES

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The aim of this paper is to investigate whether and how three modes of thinking and semiotic perspectives are compatible for researching the teaching and learning of elementary geometry in a dynamic geometry environment (DGE). It first provides an epistemological analysis of compositions of reflections in a line from geometric, analytic and abstract aspects. Then, it represents a design of a task considering semiotic potential of particular tools in the DGE that was field-tested with a pair of prospective primary school teachers. Further, it discusses how has the double analyses allowed a detailed understanding of the semiotic potential of the designed artefact for the development of all three modes of thinking of the chosen geometric concept for prospective primary school teachers. It finalizes with suggestions for future investigations of development of knowledge of other concepts in geometry through the modes and their support by digital tools.

Keywords: Composition of reflections, DGE, Three modes of thinking, Semiotic perspective, Integrating technology.

INTRODUCTION

One of the themes of the ICTMT 13 refers to mathematics teachers' education and professional development involving the use of technologies. The selection of the most appropriate content for such programmes is not always a trivial task. It not only has to consider local school curricula requirements but also the enhancement of the learning of mathematics itself by bridging different educational levels systematically. Such systematization necessitates deep insights into epistemological and historical evolutions of mathematical concepts, besides the didactical aspects.

This study tries to bring a possible systematization specifically for the concept of congruence transformations into focus of the analysis. We have considered that the theoretical framework for different modes of thinking of mathematical concepts (Sierpinska, 2000) may be suitable to facilitate our aim. In addition, a creation of digital materials considering the semiotic perspective according to Bartolini Bussi and Mariotti (2008) that may support such structured approach brings innovation and opens new questions not only about the efficiency of the suggested teaching materials but also about the effectiveness of linking these chosen theories in analyzing it. Therefore, we consider the following research question. Are the thinking modes and semiotic perspectives compatible for researching the teaching and learning phenomena of concepts in geometry, e.g. composition of reflections in a line, when they take place in a DGE? Along this direction we present results coming from a case study conducted with two prospective teachers.

THEORETICAL FRAMEWORK

In this work, as already announced above, we refer to two theoretical constructs: first, three modes of thinking of concepts in linear algebra (Sierpinska, 2000) which are to be adjusted for the purposes of geometry, and second, theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008). We argue that these two theoretical frameworks may be used for constructing appropriate theoretical foundation for explanations of the teaching and learning elementary geometry; so first we explain each of them.

Epistemological Considerations

As Winter (1976, p. 16) expressed “symmetry and congruence mappings are considered a fundamental idea in the teaching of geometry even in primary school from several aspects: shape, algebraic, esthetical, economic-technical and arithmetical”. Besides, learning complexity of symmetry and rotation notions have also been investigated by researchers (Turgut, Yenilmez, & Anapa, 2014; Xistouri & Pitta-Pantazi, 2006) in different school levels. In order to investigate such phenomena, we hypothesize that three thinking modes that are highly relevant for the research into the teaching and learning of linear algebra could also be useful for studying the development of students’ conceptual understanding in geometry, we have chosen to focus on congruence transformations on a plane, in particular, reflections in a line. An appropriate accommodation of the theoretical constructs about the different modes of description and thinking of concepts in linear algebra into geometry is not straightforward.

Congruence Transformations through the Lenses of Thinking Modes

In this paper we focus on isometries, or congruence transformations of the n-dimensional Euclidean Space, particularly, for n=1,2,3. The types of isometries, e.g. for n=2, $E(2)$ are the identity transformation, translation, rotation about a point, reflection in a line and glide reflection. Every isometry of the Euclidean plane is a bijective distance-preserving map. Two geometric figures are congruent if there exists an isometry, which maps one into the other one, that is: either a rigid motion (translation or rotation), or a composition of a rigid motion and a reflection. Let us propose thinking modes in relation to those expressed above.

Synthetic-Geometric Mode (SGM)

In grades 1 to 4 primary schools, Euclidean plane isometries are generally studied typically with the apparatus of geometry. Starting from observing and discussing in- and out of school contexts, through paper folding and drawing, constructing with straight edge and pair of compasses, pupils gain knowledge about some of the distant-preserving transformations. In this period, usually, due to the level of mathematics, no explicit reference to $E(1)$ or $E(3)$ is made. Mathematical objects such as points, lines, planes or triangles refer to the SGM. In other words, SGM is also considered as a kind of ‘thinking in-action’ (Sierpinska, 2005), i.e., thinking about the objects in coordinate-free geometry, but not about how they are constructed on. Consequently, if a student speaks about geometric objects, for example, points, lines, triangles or basic properties of them, then, those are traces of SGT mode.

Analytic-Arithmetic Mode (AAM)

While geometrical approaches for the introduction to the congruence mappings in school are widely accepted, the analytic-arithmetic mode of thought, though being an inseparable part of the concept, often remains unnoticed. The analytic counterpart that relates to the use of arithmetic language and symbolism is rarely conducted even in lower secondary school mathematics. While drawing, sketching and visualizing refer to the SGM and are typical school activities, thinking of congruence mappings as functions (from the plane in itself) in an analytic-arithmetic mode, persists out of the scope in school. Within the context of elementary geometry, representing objects as a system or using formulas to describe the action can be considered as a kind of AAT mode.

Analytic-Structural mode (ASM)

The set of isometries of the Euclidean plane $E(2)$ with the operation composition of functions forms a group (closure, associative, identity and invertibility properties). A glance on the historical evolution shows that the Euclidean groups E(n) of n—dimensional Euclidean space are among the oldest and most studied, at least implicitly for n=2,3, long before the concept of group was introduced. This historical geometrical conduction, prior the algebraic and the abstract, seems to be reflected in mathematics school curricula and textbooks designs even today. In primary schools, the abstractness is largely decreased. Yet, in our opinion, this knowledge is also relevant for teacher education and teacher professional development programs. Ignorance of any of the modes may prevent pupils from further earlier cognitive development. For an illustration, incomplete pupil's acquisition of reflection in a line in grade 3 may occur as a result of a teacher's insufficient personal resources about reflection regarding components of teacher's knowledge as reported by Donevska-Todorova (2016). "Interestingly, students mostly do not use symmetry to explain a particular conjecture" (De Villiers, 2004, p. 713) about a geometric figure (e.g., isosceles trapezoid) by dragging even in cases when it has been constructed by means of line reflection. Both prospective and practicing mathematics teachers usually require substantial assistance with the formal defining (e.g. of an isosceles trapezoid) but they do indeed develop abilities of descriptive and constructive defining (De Villiers, 2004, p. 722).

With respect to traces of AST mode can be considered as emergence of thinking about mathematical objects and conjecturing about the *action*, and/or making generalizations about the mathematical properties. For example, within the scope of this paper, thinking about congruence and group of functions such as identity function, i.e., inverse, associative and other properties of the function can be considered as traces of AST mode.

Theory of Semiotic Mediation (TSM)

The TSM proposed by (Bartolini Bussi & Mariotti, 2008), not only aims to construct mathematical meanings in a social communicative environment (where the teacher has a role of a mediator), but also to analyse teaching-learning process with a semiotic lens. In the TSM, in mediation process, the teacher focuses on specific artefacts and intentionally but carefully uses them to guide students' personal meanings to desired, culturally accepted mathematical meanings. At the same time, the teacher analyses possible evolution of signs that foster students' learning. Consequently, TSM bases on two key notions: (i) *semiotic potential* of an artefact and (ii) design of *didactic cycles*. The first refers to epistemological and didactical analysis of the artefact's evocative power to stimulate emergence of meaningful mathematics (Mariotti, 2013), while the second refers to (carefully) design the teaching-learning environment, specifically in the light of the epistemological learning route elaborated in the first phase.

A complex semiosis could be observed when the students interact with the artefact. In order to classify the signs that emerge, Bartolini Bussi and Mariotti (2008) have identified three type of signs: *artefact signs* (AS)*, mathematical signs* (MS) *and pivot signs* (PS). AS immediately emerge when the student uses the artefact, and they are generally in relation to practical observations, specifically about the artefact. MS refer to mathematical meanings that are accepted by the community by generalizing and/or expressing a conjecture, a definition or a proposition. PS underline interpretative link between personal meanings and MS sometimes including hybrid expressions.

METHODOLOGY

The participants of this case study are two (sophomore level) prospective teachers (A, B, both nineteen years old females) from a department of primary education. Regarding the mathematical content, the students had experience mainly in algebra, e.g. relations, functions, and (*2D* and *3D*) geometry, e.g. geometric transformations and their representations and notations as functions (independent-dependent variables, etc.). However, they did not have any experience with compositions of reflections. Regarding didactical considerations, the participants had experience with a dynamic geometry system (DGS), e.g., GeoGebra. They were familiar with fundamental tools, their roles and distinctive property of any DGS: initial drawings (independent objects) can be dragged but constructed (dependent) objects cannot.

Task-based interviews were video-recorded, where screen-recorder software worked synchronously. The interview lasted half an hour and the collected data coming from two videos and students' productions were analysed through a double lens, first referring to the thinking modes, and second, from a semiotic lens.

Semiotic Potential of Specific Functions and Tools of a DGS and the Task

The mathematical context embedded in aforementioned three thinking modes in elementary geometry provided us to consider *compositions of reflections* on the Euclidean plane $\mathbf{R}^2$ in a specific DGS GeoGebra. We have considered scalene triangles and compositions of two reflections in a line. First, a triangle ABC was reflected (σ_1) according to line l (the black line in Figure 1), by this way obtaining the triangle $A'B'C'$. Next, the triangle $A'B'C'$ was reflected (σ_2) according to a line g (the purple line in Figure 1) resulting with a triangle $A''B''C''$. Consequently, with respect to the position of the lines, one can refer to three separate cases: (1) when the axes of reflections coincide (Figure 1a), (2) when the axes are parallel (Figure 1b) and (3) when the axes intersect in a point (Figure 1c, 1d).

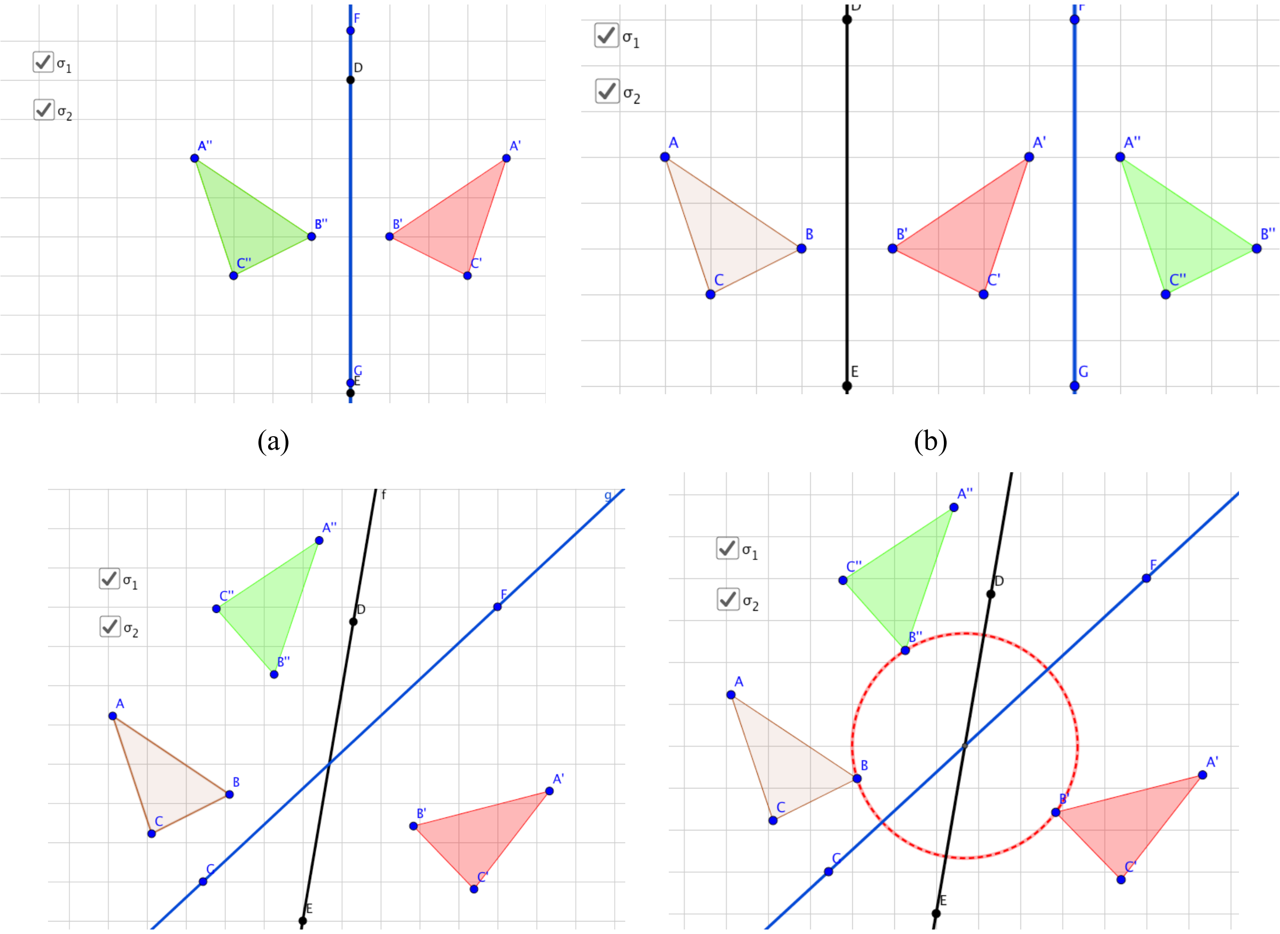


Figure 1: (a), (b), (c), (d) Three cases for compositions of reflections

With the terminology of the ASM, one could implicitly refer to properties of a group, in particular: *closure* (cases on Figure 1b: translation for a vector and Figure 1d: rotation where the intersection point of the reflection lines is the center of the rotation, both being isometries) and *neutral element,* i.e. identity transformation (case in the Figure 1a). We hypothesize that the following tools and functions of the DGS (in this case GeoGebra) have semiotic potential for creating meaning for such cases, i.e., mathematical notions expressed above:

– *Dragging* function enables the students to move and manipulate free (independent) geometric figures, by this way creates an environment for exploring different situations.

However, dragging function of any DGS does not work for constructed (dependent) objects, which also provides the notion of co-variance of the objects.

– *Grid* function of any DGS constructs parallel lines on the geometric plane, which can contribute students' observation of distances between initial and reflected objects.

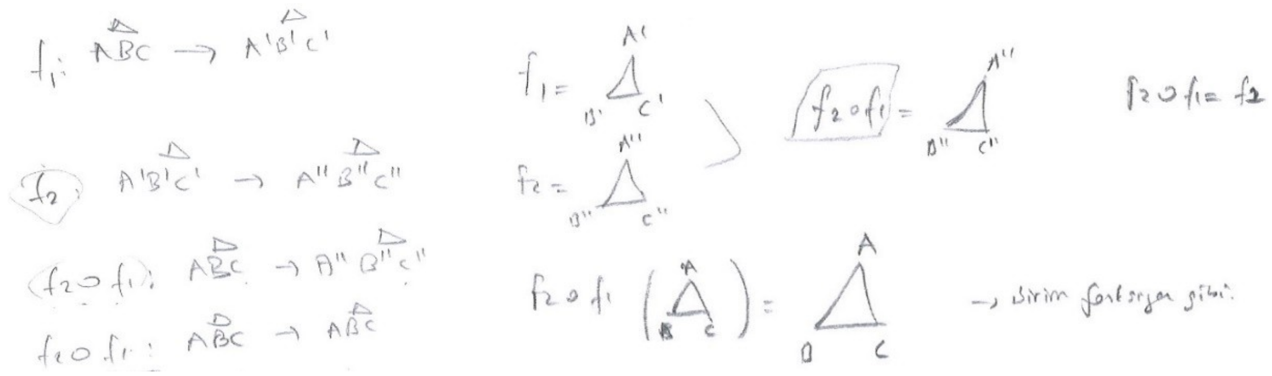
Following this, the task delivered to the students was: Step I: Click on σ_1, drag the points or the line *l*. Explain your observations mathematically. Step II: Click on σ_2 and follow the first step. Step III: Explain the relationships between triangles and generalize your findings.

ANALYSIS WITH A DOUBLE LENS

The discussion started by asking the students to follow steps of the task, what is a composition of two reflections in a line. In the first step, they focused on the reflection in the line *l* and realized that points *A, B* and *C* can be dragged. The following excerpt in Table 1 (Unit I) was drawn from this discussion (I: Interviewer), where we also provide first step of a double analysis.

Unit I	Thinking Modes Analysis	Semiotic Analysis
[14] B: … A, B and C can be dragged. Then this triangle [*points ABC triangle*] can be dragged. [15] I: Did you check the other points? [16] A: But it is depended… [17] B: Yes, because this [*points A'B'C' triangle*] is depended on initial one… [18] A: Exactly. I mean there is a transformation here something like that [*writes* $f(\Delta)=f'$]… [19] B: Yes. Something like that. When *x* varies, then *y* varies you know. Nevertheless, here we have triangles as variables … we can write [*writes* f(x)=y, *x* independent]. Because *y* is dependent on *x*, here this triangle [*means A'B'C' triangle*] is depended on the initial triangle [*explains pointing on the window*]. Therefore, we cannot drag this. … [22] A: a reflection transformation… [23] B: Actually, points are transforming. Then triangle is transforming, and then we have a new triangle…	– In [14-17], students speak in-action, i.e., about movements of geometric objects can be dragged on the screen. Actually, they speak about what they observed when they drag the moveable objects. Those are *traces* of being in SGM, although they mention the notion of dependent-independent variables. – In [18-19], the students move forward from SGM to AAM by beginning to use symbolic language of the action. They express the σ_1 reflection through function *f* and express independent and dependent triangles on the screen by explaining which can be dragged. – In [22-23], the students characterized transformation and related situation with their pre-knowledge. B's explanation reflects her thinking about transformation as a mathematical object, which can be referred as being in ASM.	- The students' immediate observations are due to the use of the artefact. For instance, not only verbal expressions such that "can be dragged", "this triangle", "is depended on", "depended on", "cannot drag", but also their gestures for pointing triangles can be considered as AS. - There appears a specific PS here: the notion of function, which contributes students to emerge their personal meanings with their observations coming from the artefact. In other words, the PS function mediates the emergence of a mathematical characterization: "… points are transforming. Then the triangle transforming…" that can be considered as a mathematical expression and also a manifestation of MS.

Table 1: Double analysis of the Unit I



(a) (b)

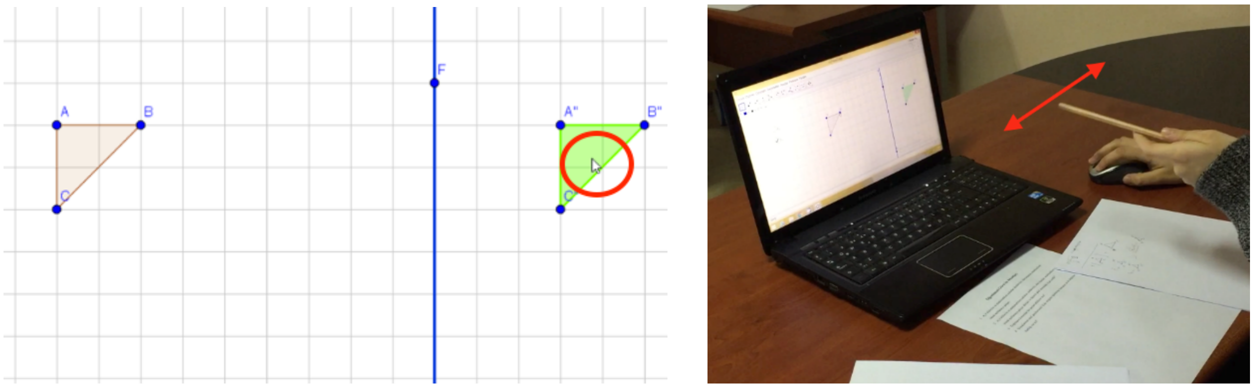
Figure 2: (a) A's mathematical expressions, (b) B's mathematical expressions

Then, the students were asked to follow the second step of the task and express what they observed by clicking σ_1 and σ_2 reflections. They immediately observed the second reflection with respect to the line *g*. The students used an interesting terminology to represent composition of reflections. The following excerpt (Unit II in Table 2) shows the discussion and the second step of the double analysis.

Unit II	Thinking Modes Analysis	Semiotic Analysis
[35] A: Could you drag line *l*? … The second transformation is also depended on the movement of line *l*. [*B drags the lines*] Actually, this is… [36] B: This [*means the final triangle* $A''B''C''$] is our new dependent variable. [37] A: [*Pointing the* $A''B''C''$ *triangle*] Dependent variable is changed… … [40] I: How can we express this situation mathematically? [41] A: For example, let me show the second transformation with f_2 [*she writes the second row of* Figure 2a] … But, finally we have [*she writes the third row of* Figure 2a]; because of the two composite reflections. [42] B: Yes. [*She writes synchronously to* A, see the first row of Figure 2b].	– In [35-37], the students, again, speak about the movements and draggable points and lines. They characterize "new" dependent – independent variables. However, they are aware that the dependent variable "is changed". – After the teacher's intervention [40], the students immediately relate the compositions of reflections with composite functions. They use mathematical expressions of the composite of reflections [41-42]. These all over imply that the students are in the AAM.	- Students discuss about artefact's feedbacks about dragging (e.g., "drag line *l*", "movement of line *l*"), which were AS. However, their observations trigger to emergence of a new PS: *independent–dependent* variables in the compositions of reflections (e.g., "new dependent variable"). - Finally, they use mathematical representations to express their mathematical meanings (*use of a triangle as a dependent or independent variable* in Figure 2) about compositions of reflections. This can be considered as an example of how AS transform into MS.

Table 2: A double analysis of the Unit II

Further, the discussion continued about the three different positions of the lines, which affect the positions of the initial and final triangles. Therefore, the students were asked to unclick the first reflection and discuss the mathematical situation on the screen (see Excerpt III in Table 3).

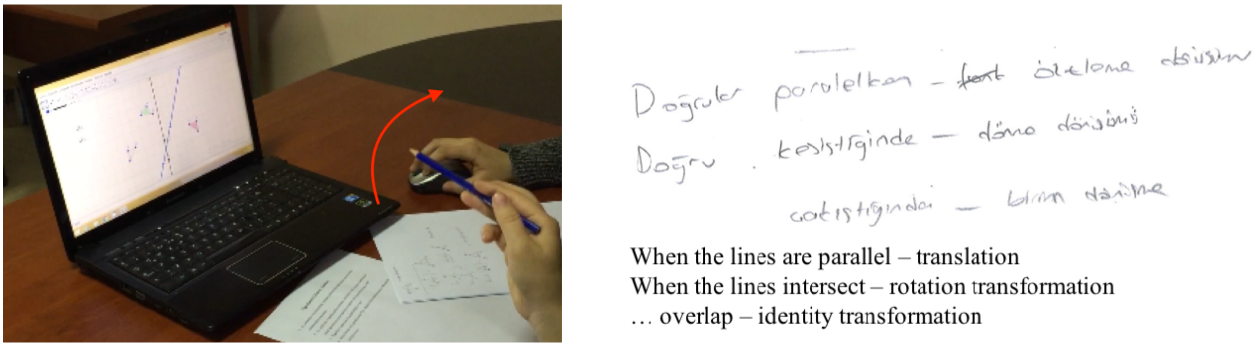


(a) (b)

Figure 3: (a) B’s cursor mimics, (b) B’s translation gesture with pencil

Unit III	Thinking Modes Analysis	Semiotic Analysis
[45] I: Ok, right. Please unclick σ_1, … what do you observe when you drag *ABC* triangle? [46] B: … This [*mimics with cursor*, see Figure 3a] is not a reflection… The distance …For example, we have *ABC* triangle, but it seems like translated into [*she gestures with pencil*, see Figure 3b] *A"B"C"* triangle… … [48] I: Ok. Please click on σ_1 … Check the position of such lines! [49] B: They are now parallel… [50] A: They can intersect, either can be parallel and they can overlap. [51] B: Let’s move this [*she drags the line l onto line g*]. [52] A: The initial and final triangles overlapped! [53] B: Like functions… [54] I: How can we express this situation mathematically? [55] B: One-to-one and onto …? [*She writes the last line of* Figure 2b]… Is this identity function? … [56] A: … [*She writes the last line of* Figure 2a]. … [67] B: Let’s intersect the lines… [*she drags continuously and tries to understand the situation*] [68] A. Here … [*gestures with* pencil, see Figure 4a]. Like a… [*B drags the points and lines*] The final triangle is rotated around initial triangle. Yes this is now a rotation… [*They together write their conclusions*, see Figure 4b]…	– In [46], the student use her spatial perception and therefore express their observations in the case of the axes are parallel, even she finds the translation of the triangles, but not mathematically. Because she does not mention any *translation vector*. Also, in [49-52], the students speak about their observations on the screen, not about mathematical necessities. All those are traces of being in SGM. – However, in [53], B relates the situation with functions. She also realizes that such kind of reflection might be similar to identity function and have one-to-one and onto properties. A also uses a similar notion. Since, suffice it to say that, in this point, they are in AAM, since they does not generalize the situation [55-56], and does not mention how this could be possible. –They finally explore the case when the reflection axes intersect. They analyze the three cases and make a generalization [67-78], which seem a kind of having ASM.	- In [46], mimicking with cursor, “the distance”, “*ABC* triangle”, “translated into” and also B’s gesture with pencil and *A"B"C"* triangle are AS. - In [49-52], there appear several AS. For example, “parallel”, “intersect”, “overlap”, “move this”, and “the initial and final triangles overlapped” are also AS. - In [53] pivot signs “function” and “identity function” and also specific expressions in Figure 2 appear, which show interpretative link with classification of composition of reflections with respect to axes. - They finally categorize the cases, and characterize composition of reflections with respect to positions of axes. They express their conjectures with validation through dragging, e.g., “yes this is now a rotation”, which can be accepted as traces MS.

Table 3: A double analysis of the Unit III



(a) (b)

Figure 4: (a) A's rotation gesture, (b) Students' conclusions (translated into English)

FINDINGS AND DISCUSSION

Based on the double analysis in Table 1, we have found out that the PS of the artifact have mediated an appearance of MS related to congruence of triangles in a SGM of thinking. Then, in contrast to our expectations that the most frequent mode would be the SGM, it was the AAM, which dominated indeed. A reason for it may be the students' pre-knowledge about transformations. Yet, the analysis in Table 2 shows that it may also be a result of the semiotic potential of the created DGE to stimulate an emergence of MS due to interactions within the artifact. Further, our double analysis has shown that the occurrence of the ASM, manifested through an axiomatic property of a group, e.g. the identity transformation on the plane (Figure 2a, on the bottom), could be influenced by the potentials of the design (Table 3). An observation of the students' written materials, leads to a conclusion that though the symbolic language is not fully developed, the AAM was influenced by the SGM of thinking (triangles occur as variables on Figure 2) showing that the design has contributed to changes from one into another mode of thinking. This analysis has leaded us to propose a diagram of possible links between the three modes of thinking and the potentials of the DGS for the emergence of the three signs (Figure 5).

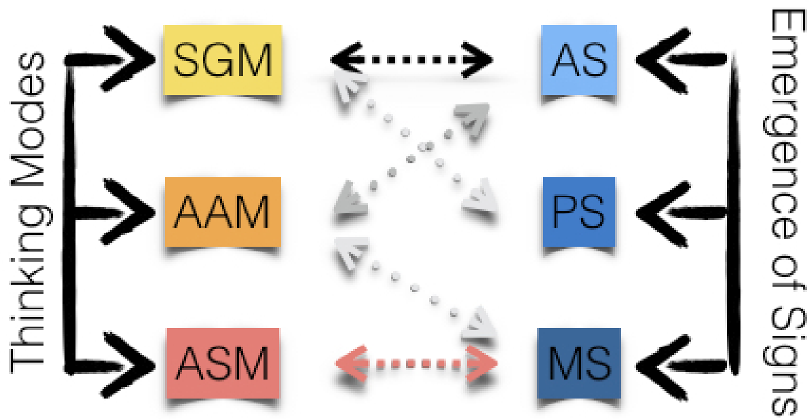


Figure 5: Relationships between the modes of thinking and emergence of signs

CONCLUSIONS

In this paper, we have considered the research question, 'Are the thinking modes and semiotic perspectives compatible for researching the teaching and learning of composition of reflections in a line taking place in a DGE?' Firstly, we have shown that the "borrowed" terminology related to the three modes of thinking of concepts in linear algebra (Sierpinska, 2000) may be meaningful for studying the teaching and learning processes of certain concepts in elementary geometry, in this case, congruence of reflections in a line. Additional exemplary geometric concepts are required in order to investigate whether (or to which extent) a "nested diagram" of three modes of thinking in linear algebra (Donevska-Todorova, 2017) is also suitable or adjustable for studying concepts in geometry on a local level.

Secondly, the analysis from a semiotic perspective has shown how students' personal meanings transformed into mathematical meanings, and how gestures contributed to emergence of students' thinking. One interesting point in the semiotic analysis was how gestures contributed students' thinking and emergence of mathematical meanings. Another point was about affirmative result of the semiotic potential of the specific functions and tools of a DGS, which confirmed that those could be considered as a tool of semiotic mediation that is consisted within the recent literature (Turgut, 2015, 2017).

Finally, our double analysis has provided affirmative insights into existing relationships between the three modes of thinking and the potentials of the tools of the designed DGE (Figure 5). As seen from Figure 5, SGM, sometimes were in relation to both AS and PS, while AAM mode also implied both AS and MS. But interestingly, ASM separated from AS and PS and was directly in relation to MS. Complexity of analysis tools of the thinking modes and semiotic perspectives also appeared in a recent study with respect to learning the notion of parameter in linear algebra (Turgut & Drijvers, 2016). We express our awareness of the affordance and limitations of this diagram for interpreting explicit relationships between the modes and the semiotic potentials by pointing out that such confirmations require further research.

REFERENCES

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of International Research in Mathematics Education* (2. ed., pp. 746–783). Mahwah, NJ: Erlbaum.

De Villiers, M. (2004). Using dynamic geometry to expand mathematics teachers' understanding of proof. *International Journal of Mathematical Education in Science and Technology*, *35*(5), 703–724.

Donevska-Todorova, A. (2016). *What (Should) Teachers Assess in Elementary Geometry?* Paper presented at the Third ERME Topic Conference, Berlin, 2016.

Donevska-Todorova, A. (2017, to appear). Fostering Students' Competences in Linear Algebra with Digital Resources. In Stewart, S., Andrews-Larson, C., Berman, A. & Zandieh, M. (Eds.). *Challenges in Teaching Linear Algebra*. Springer.

Mariotti, M. A. (2013). Introducing students to geometric theorems: how the teacher can exploit the semiotic potential of a DGS. *ZDM-Mathematics Education, 45*(3), 441–452.

Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 209–246). The Netherlands: Kluwer Academic Publishers.

Sierpinska, A. (2005). On practical and theoretical thinking and other false dichotomies in mathematics education. In M. G. Hoffmann, J. Lenhard, & F. Seeger (Eds.), *Activity and Sign* (pp. 117–135). US: Springer.

Winter, H. (1976). Was soll Geometrie in der Grundschule? *Zentralblatt der Didaktik der Mathematik*, *8*(1/2), 14–18.

GAMIFYING MATH TRAILS WITH THE MATHCITYMAP APP: IMPACT OF POINTS AND LEADERBOARD ON INTRINSIC MOTIVATION

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Gamification in education describes the application of game elements in the design of learning processes. The MathCityMap project, which consists of a web portal and a gamified application for smartphones, combines the idea of math trails with the possibilities of mobile devices. To evaluate the impact of points and leaderboard on intrinsic motivation a pilot study has been conducted. The results suggest that there is no significant difference between these two game elements. However, gender seems to play an important role on the impact of gamification on intrinsic motivation.

Keywords: math trails, app, gamification, motivation, gender

INTRODUCTION

The British Department of Education and Skills recommends doing more lessons outside the classroom. Learning outside can "nurture creativity, develop skills, improve attitude to learning, stimulate and improve motivation" just to name a few (DfES, 2006). One suitable way to implement learning outdoors in the math classroom is the math trail concept. The math trail idea was born in Australia in the early 80s (Blane & Clark, 1984). A math trail consists of a set of mathematical outdoor tasks or problems in walking distance. Tasks like "What is the height of the building?" or "How much water is in the pond?" are bound to real objects in the environment and therefore often authentic and motivating. To answer this kind of questions it is necessary for the student to measure (enactive action), to translate the problem into a mathematical model (abstraction) and to calculate the answer (cognitive action). The connection of these three cognitive levels is valuable, because one is more likely to remember the learned later (Rösler, 2011). The trail guide (Shoaf, Pollak & Schneider, 2004) is a booklet, which contains a map that shows mathematically interesting places and the description of the tasks.

Although mobile devices and computers are widely used in every aspect of our daily lives (especially among pupils), they play just a little role in education (Chen & Kinshuk, 2005). Going on a math trail could greatly benefit from using mobile devices, because they allow learning to occur in an authentic context and extend to real environments. At the Goethe-University of Frankfurt / Main we started the MathCityMap Project (MCM), which combines traditional math trails with the opportunities of new technologies. In 2013 first ideas have been made concrete (Ludwig, Jesberg, Weiss, 2013), but it took until 2016 to finally launch a web portal and a mobile application. These are mainly for teachers and their students to use in class, but everyone is free to use it.

In the summer term 2016, we had the opportunity to observe some school classes going on a math trail with the MCM app. Besides many positive observations, we also made two negative observations: (1) answers were often guessed, (2) there is a motivational obstacle to begin working on the tasks (for example expressed in walking slowly to the first task).

THEORETICAL BACKGROUND

Motivation

The most basic distinction in Self-Determination Theory (SDT) is between intrinsic motivation, which refers to doing something because it is inherently interesting or enjoyable, and extrinsic motivation, which refers to doing something because it leads to a separable outcome (Ryan & Deci, 2000). Most activities in school are not inherently interesting and therefore must initially be externally prompted. A person that faces an activity due to external regulations might experience the activity's intrinsically interesting properties, resulting in an orientation shift (Ryan & Deci, 2000). The Intrinsic Motivation Inventories (IMI, 1994) define interest and enjoyment as a central measure of intrinsic motivation. For Ryan & Deci (2000) the source of intrinsically motivated behaviour lies in satisfying psychological needs namely competence, autonomy and relatedness. A higher intrinsic motivation manifests in personal, cognitive, emotional and behavioural engagement (Fredricks, Blumenfeld & Paris, 2004), which are desirable attitudes towards learning.

Gamification

Gamification describes the application of game elements in a non-game context to manipulate the behaviour of users towards a certain goal (Fuchs et al. 2014). The term gamification started to occur more frequently from 2010 mainly in marketing, where gamification is used to increase the customers brand loyalty. Huotari & Hamari (2012) divide gamification into three parts: (1) implementation of game elements in non-game activities, (2) resulting psychological changes and (3) visible changes in the user's behaviour. One main goal of gamification is to modify a serious activity, which is bound to a particular purpose (in our case that could be working on a math trail task), so that it appears more game-like and therefore is more inherently interesting to the user resulting in a higher intrinsic motivation and engagement (Hamari et al. 2014).
Game elements are often different types of feedback on the user's action like points, levels, leaderboards, badges and quests (Zichermann & Cunningham, 2011). Although the gamification concept seems suitable to improve psychological aspects in non-game activities, many projects may fail or will not meet the expectations due to poor understanding of how to design gamification (Morschheuser et al. 2017). This is also the case for gamification in education. Dicheva & Dichev (2015) analysed the outcomes of gamification projects in education in the period July 2014 – June 2015 and conclude:

> […] papers that report positive results are only 24%, while those reporting negative results – 7% and the inconclusive – 49%. Thus from 41 papers only 10 can be considered as evidence of positive effects for gamification in education […] (Dicheva & Dichev, 2015, p. 7).

Prior to implementing game elements, Morschheuser et al. (2017) recommend to analyse the projects target group, the conditions and the inherent activities. The result of the analysis is the definition of goals that gamification should achieve. The next step is to design and implement game elements based on the defined goals. Finally, evaluation and monitoring is useful to make further improvements.

Gamification in math education

"Gamification in education refers to the introduction of game elements and gameful experiences in the design of learning processes" (Dicheva & Dichev, 2015). The number of papers about gamification in education grows: 34 papers in the period January 2010 – June 2014 and 41 papers in the period July 2014 – June 2015.

One example of gamification in math education is Attali & Arieli-Attali (2014) “Gamification in assessment: Do points affect test performance?”. The assessment is based on a mathematical online test with 100 questions from grade six to eight (e.g. fraction addition). Participants were randomly assigned to three groups: (1) control group (no gamification), (2) experimental group 1 and (3) experimental group 2. The experimental group 1 could earn up to 10 points per question depending on the time needed to answer the question, whereas the experimental group 2 could earn up to 10 + 5 points (10 for a correct answer and up to 5 additional points depending on the speed). Results show that “the point manipulation had no effect on the main performance outcome, response accuracy” (Attali & Arieli-Attali, 2014). Whereas the response time decreased significantly, but the effect sizes were small. In addition, no differences between female and male participants were found.

GPS-based applications in math education

Two examples of applications in math education, that already successfully use mobile GPS-data, are Wijers, Jonker & Drijvers (2010), who developed a game which allows students to walk along the shape of geometric objects outside the school, and Sollervall and de la Iglesia, who have developed a GPS-based mobile application for embodiment of geometry (Sollervall & de la Iglesia, 2015).

The MathCityMap project

The intention of the MathCityMap (MCM) project is to automate many steps in the creation of the math trail booklet/guide and to provide a collection of tasks and trails that can be freely used or just viewed to get inspiration for own tasks. Furthermore, it gives users (e.g. groups of pupils) the possibility to go on a math trail more independent by using mobile devices’ GPS functions to find the tasks location, by giving feedback on the users answer and by providing hints in the case that one got stuck at a particular task. The core of the MCM project can be divided into two parts, the MCM web portal and the MCM app.

MCM web portal - www.mathcitymap.eu

The web portal is a math trail management system. After a short registration, the user can view public trails and tasks or create his own tasks and trails by typing in the necessary data (e.g. position, the task itself, the answer, an image of the object etc.) into a form. For every math trail, the math trail booklet can be downloaded as PDF or accessed via the MCM App (see Figure 1). It contains all tasks information, a map overview and a title page.

MCM app for mobile devices

The MCM app allows the user to access math trails created with the web portal. The trail data, such as images and map tiles, can be downloaded to the mobile device. After this procedure, it is possible to use a trail without internet connection (see Figure 2). This design decision minimizes technical issues when using the app without mobile internet or in an area with low connectivity. Furthermore, the app offers an open street map overview for orientation purposes, feedback on the entered answers and a stepped hint system. The hint system has the purpose to enable pupils to solve the tasks independently and additionally has a positive impact on learning performance, learning experience and communication (Franke-Braun, Schmidt-Weigand, Stäudel, & Wodzinski, 2008).

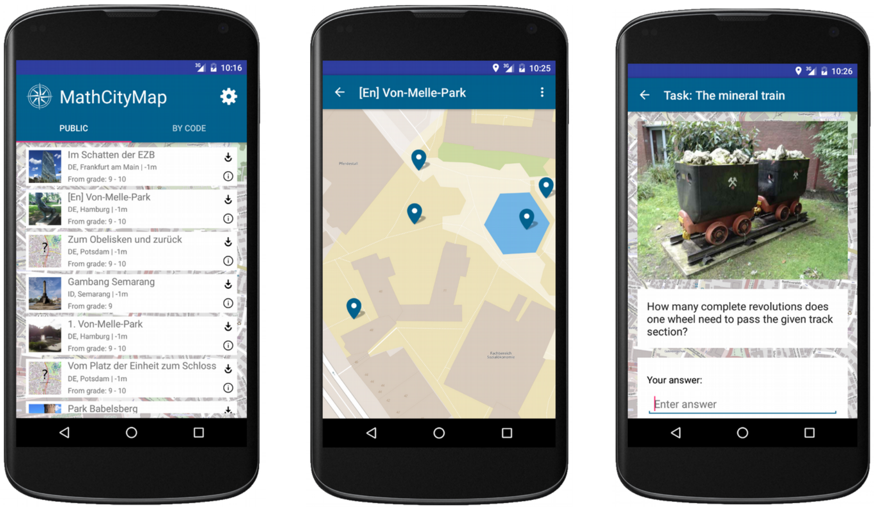


Figure 1: Screenshots of the MCM App

To describe the pedagogical functionality of MCM, we use the model by Drijvers, Boon and Van Reeuwijk (2010). It divides digital technologies into three groups of didactical functionalities: (a) do mathematics, (b) practice skills, (c) develop concepts. MCM offers mathematical tasks at real life objects where the user mainly can practice his skills.

GAMIFYING THE MATHCITYMAP APP

Following Morschheuser et al. (2017), we have analysed the MCM project prior to implementing gamification.

Analysis of the MCM supported math trail activity

Secondary school students, who are familiar with using smartphones and apps, are the target group of our project. A math trail in school is usually used irregularly (e.g. day's hike, project days). In our approach, students collaborate in groups of three (one is using the MCM app, one is responsible for measuring and the last one is responsible for taking notes) and walk the math trail independently during math classes.

The math trail activity is divided into sub activities that are titled "working on a task". Each sub activity consists of the following sequence: (1) finding the task's location; (2) reading the task description; (3) collecting data; (4) transform task into mathematic model; (5) calculating the answer; (6) entering answer into the app and getting feedback; (7) optionally, taking hints and retry. During step (1), (2), (6) and (7) students use the MCM app.

Gamification goals

The gamification goals are based on the negative observations that were mentioned in the introduction of this article.

(1). Prevent students from guessing answers

(2). Increase intrinsic motivation for working on math trail tasks (decrease time that passes when walking from one task to the next).

Implementation

To prevent guessing we have decided to implement (1) points that the user is rewarded when answering a task correctly. When the user guesses too often, the maximum amount of possible points decreases. The second gamification is the (2) local leaderboard, which is based on the points gamification. The difference between a global and local leaderboard is that the first displays all users so that it is possible to see one's absolute ranking. The latter displays only the user's rank in comparison to the user in front and the user behind him. Additionally, we have added a computer player who is always the last.

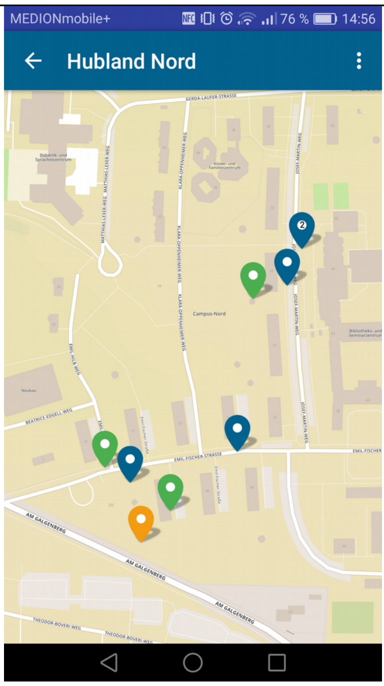
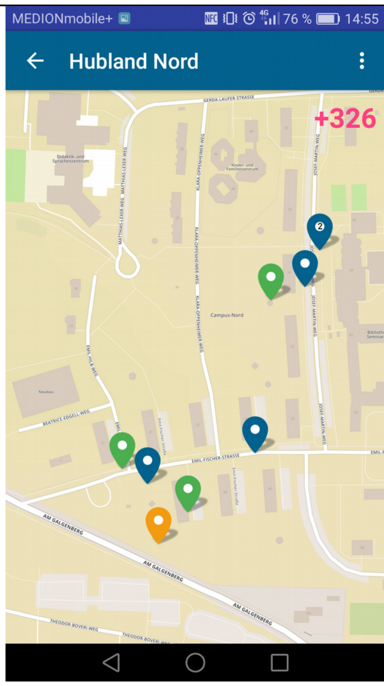
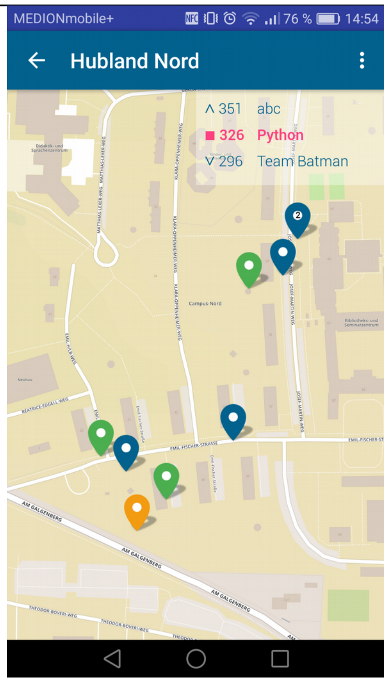
0: No gamification	1: Points	2: Local leaderboard
		

Table 1: Types of gamification in MathCityMap.

Research Question

Is there a difference in student's intrinsic motivation while walking a math trail using the MCM app with points or with leaderboard gamification?

METHODOLOGY

In December 2016, we conducted a pilot study with two ninth grade school classes (n = 47) comparing the intrinsic motivation between points (g1) and leaderboard (g2) gamification.

Study design

In the first 15 minutes, the participants learned how to walk a math trail with MCM. The functionalities of the app and the rules were explained. Subsequently, they had 90 minutes to work on the tasks independently in groups of three. The tasks were mainly about cylinders. Finally, they were asked to fill in a translated version of the Intrinsic Motivation Inventory (IMI, 1994) questionnaire. In this case, both groups were experimental groups. The first group walked the math trail with points gamification (g1), whereas the second group used the leaderboard gamification (g2).

Questionnaire

The used IMI questionnaire consisted of twenty-two 7-point Likert scale items that can be assigned to four sub scales. The sub scales represent positive or negative indicators for intrinsic motivation (see table 2). The students had to indicate how true the statements were for them (not at all true – very true).

Sub scale	Example item
Interest / Enjoyment (positive)	This activity was fun to do.
Perceived Competence (positive)	I am satisfied with my performance at this task.
Perceived Choice (positive)	I believe I had some choice about doing this activity.
Pressure/Tension (negative)	I felt pressured while doing these.

Table 2: Sub scales and example items (IMI, 1994).

RESULTS

An independent-samples t-test was conducted to compare intrinsic motivation for walking a math trail using the MCM app with points gamification (g1) and leaderboard gamification (g2). There was no significant difference in the scores of any sub scale:

Interest / Enjoyment: g1 (M=3.6, SD=1.4) and g2 (M=3.8, SD=1.1); t (45) = -.542, p = .59.
Perceived Competence: g1 (M=3.7, SD=1.5) and g2 (M=3.5, SD=1.5); t (45) = .352, p = .72.
Perceived Choice: g1 (M=3.8, SD=1.6) and g2 (M=4.2, SD=1.2); t (45) = -.815, p = .42.
Pressure / Tension: g1 (M=2.8, SD=1.3) and g2 (M=3.2, SD=1.4); t (45) = -.789, p = .43.

At the first glance, these results suggest that the two types of gamification do not differ in how they impact intrinsic motivation. However, when taking the sex of the participants into account the results of the sub scale Interest / Enjoyment do change.

	Gamification	Sex	M	SD	N
Interest /Enjoyment	Points	male	3,4290	1,34171	10
		female	3,7031	1,49078	13
	Leaderboard	male	4,1869	,96772	13
		female	3,2991	,99534	11

Table 3: Statistics of gamification and sex as independent variables

A two-way analysis of variance was conducted on the influence of two independent variables (gamification, sex) on the Interest / Enjoyment sub scale. The interaction effect was not significant, F(1,43) = 2.63, p = .112.

Finally, two independent-samples t-tests were conducted to compare the Interest / Enjoyment sub scale with the combination of gamification type and sex. The first test compared female participants with points gamification (M=3.7, SD=1.5) and male participants with points gamification (M=3.4, SD=1.3). No significant difference in score of the sub scale could be found, t(21) = -.456, p = .653. The second test compared female participants with leaderboard gamification (M=3.3, SD=1.0) and male participants with leaderboard gamification (M=4.2, SD=.97). There was a significant difference in the interest sub scale score, t(22) = 2.2, p = .04. These results suggest that the impact of points gamification (g1) on intrinsic motivation does not differ for ninth grade female and male students. Whereas the leaderboard gamification (g2) impacts the intrinsic motivation different depending on the sex of the participant.

DISCUSSION

In the conducted pilot study, no significant difference in intrinsic motivation between points and leaderboard gamification was found. However, the results indicate that points gamification influences the interest / enjoyment sub scale of male and female students equally (cf. Attali & Arieli-Attali, 2014) since no significant difference in their scores could be found. Whereas leaderboard gamification leads to a significant higher interest / enjoyment sub scale score for male students (M=4.2) compared to female students (M=3.3).

Mathematic classroom rates (at least in Germany) as a male domain (Budde, 2009). The results suggest that different gamification types might influence this issue in a positive or in a negative way. Prior to implementing gamification in math classroom, it should be considered carefully that it might favour one group and discriminate the other.

Prospects

The main study with 25 ninth-grade classes will be conducted in May / June 2017. The classes will participate in a pre-test and be divided into three groups: (g0) control group; (g1) points gamification; (g2) leaderboard gamification. Additionally, to the impact on intrinsic motivation, the evaluation of the gamification goals will be examined.

REFERENCES

Attali, Y. & Arieli-Attali, M. (2015). Gamification in assessment: Do points affect test performance? In Computers & Education (83), p. 57-63.

Blane, D.C. & Clark, D. (1984). *A Mathematics Trail Around the City of Melbourne*, Monash Mathematics Education Centre, Monash University.

Budde, J. (2009). Mathematikunterricht und Geschlecht - Empirische Ergebnisse und pädagogische Ansätze. Bonn, Berlin: Bundesministerium für Bildung und Forschung (BMBF).

Chen, J. & Kinshuk (2005). Mobile technology in educational services. J. Educ. Multimed. Hypermed., 14, 91–109.

Department for Education and Skills (DfES). (2006). *Learning Outside the Classroom MANIFESTO*, DfES Publications, Nottingham, United Kingdom.

Dicheva, D. & Dichev, C (2015). Gamification in Education: Where Are We in 2015? In proceedings of the World Conference on E-Learning (E-LEARN 2015), Kona, Hawaii, October 19-22, 2015.

Drijvers, P., Boon, P., & Van Reeuwijk (2010). Algebra and technology. In P. Drijvers (Ed.), *Secondary algebra education. Revisiting topics and themes and exploring the unknown* (pp. 179-202). Rotterdam, The Netherlands: Sense.

Franke-Braun, G., Schmidt-Weigand, F., Stäudel, L., & Wodzinski, R. (2008). *Aufgaben mit gestuften Lern-hilfen - ein besonderes Aufgabenformat zur kognitiven Aktivierung der Schülerinnen und Schüler und zur Intensivierung der sachbezogenen Kommunikation.* In R.

Messer, & W. Blum, Lernumgebungen auf dem Prüfstand. Zwischenergebnisse aus den Forschungsprojekten. (pp. 27-42). Kassel: Kassel Univ. Press

Fredricks, J., Blumenfeld, P., & Paris, A. (2004). School Engagement: Potential of the Concept, State of the Evidence.

Fuchs, M., Fizek, S., Ruffino, P. & Schrape, N. (2014). RETHINKING GAMIFICATION, meson press, p.10 f

Hamari, J., Koivisto, J. & Sarsa, H. (2014). Does gamification work? – A literature re-view of empirical studies on gamification. In proceedings of the 47th Annual Hawaii International Conference on System Sciences, Hawaii, USA, January 6-9, 2014.

Huotari, K., & Hamari, J. (2012). Defining Gamification - A Service Marketing Perspective. In Proceedings of the 16th International Academic Mindtrek Conference (pp.17–22). Tampere, Finland, October 3–5, 2012. ACM, New York, NY, USA.

Intrinsic Motivation Inventory. (1994). Intrinsic Motivation Inventory (IMI). *The Intrinsic Motivation Inventory, Scale description*. URL: http://selfdeterminationtheory.org/intrinsic-motivation-inventory/ (23.11.2016)

Ludwig, M., Jesberg, J., David Weiß (2013). *MathCityMap – eine faszinierende Belebung der Idee mathematischer Wanderpfade.* In: Praxis der Mathematik (53), p. 14-19

Morschheuser, B., Werder, K., Hamari, J., & Abe, J. (2017). How to gamify? A method for designing gamification. In Proceedings of the 50th Annual Hawaii International Conference on System Sciences (HICSS), Hawaii, USA, January 4-7, 2017.

Rösler, F. (2011). Psychophysologie der Kognition, Spektrum Akademischer Verlag, Heidelberg, p.109 ff

Ryan, R. & Deci, E. (2000). Intrinsic and Extrinsic Motivations: Classic Definitions and New Directions. Contemporary Educational Psychology, 25, 54-67

Shoaf, M., Pollak, H. & Schneider, J. (2004). *Math Trails*. Lexington: COMAP

Sollervall, H. & de La Iglesia, D.G. (2015) *Designing a didactical situation with mobile and webtechnologies.* Konrad Krainer; Nada Vondrova. CERME9 - Ninth Congress of the European Society for Research in Mathematics Education, Feb 2015, Prague, Czech Republic. pp.2410-2417, Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education.

Wijers, M., Jonker, V., & Drijvers, P. (2010). *Exploring mathematics outside the classroom*. ZDM Mathematics Education, pp. 789-799.

Zichermann, G., & Cunningham, C. (2011). Gamification by Design: Implementing Game Mechanics in Web and Mobile Apps, first ed. Sebastopol, CA: O'Reilly Media.

"POWER OF SPEED" OR "DISCOVERY OF SLOWNESS": TECHNOLOGY-ASSISTED GUIDED DISCOVERY TO INVESTIGATE THE ROLE OF PARAMETERS IN QUADRATIC FUNCTIONS

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This paper reports an intervention-study where students investigated the role of parameters in quadratic functions through technology-assisted guided discovery. The intervention had three experimental groups of students, each of which used a different type of visualisation *(sliders, "drag mode", or a function plotter) and one control group. The study provides insight into whether technology-assisted discovery learning supports the conceptualization and understanding of the role of parameters in quadratic functions. Qualitative analysis of students' work investigated the potential and constraints of each of the four different approaches for visualisation. Initial findings showed that technology does support the students in their learning, with the dynamic visualisation groups (drag mode and sliders) showing greater understanding than the function plotter group.*

Keywords: quadratic functions, dynamic visualisations, parameter, discovery learning, technology

BACKGROUND

During the study, the students take part in a self-paced guided discovery learning about the concept of parameters in the field of quadratic functions. Mosston and Ashworth (2008) describe guided discovery as a "convergent process that leads the learner to discover a predetermined target" (p. 214) whereas Gerver and Sgroi (2003) stress the importance that guided discovery lessons "have a story line that inherently engages the participant" (p. 6). The lessons also need an "Aha! Component" (Gerver & Sgroi, 2003), so at some point in the lessons the students realize that they discovered mathematical ideas through their exploration. Even though minimal guidance is widely promoted, Kirschner, Sweller & Clark (2006) state that there is no research supporting the use of instruction using minimal guidance. However, Alfieri, Brooks, Aldrich, and Tenenbaum (2011) in a meta study of 164 studies of discovery learning found that enhanced discovery learning, for example through use of guided tasks, was indeed beneficial for learning.

In order for students to develop conceptual understanding of parameters it is important to have an understanding of both variables and functions, as well as being able to change between different representations.

Variables can be used differently in different contexts. For example, a variable can be used as a placeholder or as an unknown, depending on the context of a problem. Küchemann (1981) described six ways children use letters in mathematical tasks: *letter evaluated, not used, used as an object, used as a specific unknown, used as a generalised number* and *used as a variable*. Usiskin (1988) however, describes the variable as a "symbol for an element of a replacement set" (p. 9) but also states that there are different views possible. He distinguishes variables as *pattern generalizers, unknowns, parameters* and *arbitrary marks* on paper (Usiskin, 1988). Küchemann and Usiskin use different terms for the same concept, for example Küchemanns' *letter evaluated* and Usiskin's *unknown* both describe the same use of a variable. In contrast, Malle (1993) only describes three roles of variables namely, variables as *unknowns, generalized number* and *changing variable*. Some authors, for example Usiskin (1988), view parameters as a specific role of variables, whereas others, for example Drijvers (2003), describe parameters as meta-variables with several meanings themselves. Drijvers distinguishes between parameter as a *placeholder*, as a *generalizer*, as a *changing quantity* and as an *unknown*. The view of a parameter as a *placeholder* can result in consideration of specific values, one by one, for the parameter; in a graphic model, each time the parameter is changed it is visualized as one graph being replaced by another. However, graphs can also be viewed dynamically, and therefore the parameter as a *changing quantity* can be observed via dynamic software (e.g. graphics calculators, or computer software) while the parameter is continuously changed to take a number of values. Lastly, a parameter as a *generalizer* can be visualized by a family of functions (Drijvers, 2003). Drijvers (2003) suggests a hierarchy for the understanding of parameters, with parameter as a placeholder associated with a lower level of understanding than both parameter as a changing quantity and an unknown, while understanding of a parameter as a generalizer shows a higher level of understanding. Students need to be able to distinguish between parameters and variables, but this presents difficulties as the distinction is context related and parameters cannot be explained without second order structures (Bloedy-Vinner, 2001). Bardini, Radford, and Sabena (2005) describe this as "the paradoxical epistemic nature" (p. 130) of parameters, which makes it difficult for students to understand. Therefore, parameters need to be addressed in a variety of ways (Bardini et al., 2005).

In addition to understanding the concepts of variables, it is necessary for students to understand the Grundvorstellungen associated with functions. Grundvorstellungen and the development of Grundvorstellungen are terms used in the German literature to describe the connection between the mathematical concepts, real contexts and the students' mental models (Blum, 2004). Grundvorstellungen includes normative, descriptive and constructive aspects, where the normative aspects of Grundvorstellungen can be used to determine, what a full understanding of a particular mathematical concept should include. These normative aspects are derived through a subject matter analysis (vom Hofe & Blum, 2016). Descriptive aspects of Grundvorstellungen are used to describe the student's mental representation and these can include misconceptions or partial understandings (vom Hofe & Blum, 2016). Grundvorstellungen can be constructed through teaching, hence the constructive aspects of Grundvorstellungen. Blum (2004) identified the Grundvorstellungen of a function as mapping, covariation and object. Mapping is aligned with a static view, where one quantity is matched with another. Covariation, however is more aligned with a dynamic view, where a change in one quantity is observed when the other quantity changes and this dynamic view can be supported through the use of technology. Object as a whole is a global view of the function as one object (vom Hofe & Blum, 2016), which is different to the other two ideas of mapping and covariation, as the global features of the function are considered. vom Hofe, Kleine, Wartha, Blum, and Pekrun (2005) describe the linking of different Grundvorstellungen as an essential requirement for developing mathematical understanding. For the study presented here, it is therefore crucial that the three Grundvorstellungen of functions are developed, when students are learning about the differences between an original graph and the resultant graph as one of the parameters of a function equation changes.

As mathematical objects cannot be accessed without the use of representations, understanding the concept of a function is closely intertwined with being able to change from one representation to another (Duval, 2006). Duval (2006) describes being able to change between representations as a "critical threshold for progress in learning" (p. 107). He distinguishes two kinds of transformations of representations: treatments and conversions. Treatments occur within one register, for example solving an equation, whereas conversions transform one register into another, for example graphing a function from its equation (Duval, 2006). Using multiple representations have been found to be important in the teaching and learning of all mathematical concepts, with Kaput (1992) describing the change between different representations as a part of "true mathematical activity" (p. 524). Penglase and Arnold (1996) in their review of research on graphics calculators pointed out that this change between different representations can be supported through use of graphics calculators. Kaput (1992) described that the automatic linking possible through use of technology (so that a change in one representation immediately occurs in the other) can help students to visualize the connections between representations in a different way than a static change of representation. Some authors describe this linking of different representations as essential for developing a full understanding of concepts (e.g. Thomas, 2008; Duncan, 2010). This linking of representations can be supported through the use of digital technologies (Ferrara, Pratt & Robutti, 2005).

Using general-purpose technology like dynamic geometry software, spreadsheets or function plotters to explore mathematical connections is included in the core curricula in North Rhine Westfalia in Germany (MSW, 2007). It is not specified when or how it should be used, but starting from 2017 all students in North Rhine Westfalia need to use graphics calculator in their final upper secondary school exams ("Abitur"). Many studies have shown that dynamic technology can be beneficial for learning, however Zbiek, Heid, Blume, and Dick (2007) pose the question, whether the use of sliders obscures, rather than enhances, the understanding of a connection between the value of the parameter and the changing graph. Drijvers (2003) outlines that when students work with sliders, the students are often not successful in explaining the effects of the sliders as they only examine the effects superficially. However, Drijvers (2004) found that concerning the parameter as a changing quantity, students achieved a higher understanding through the use of sliders regarding his proposed hierarchy of parameter roles.

RESEARCH QUESTIONS

The main research question of this project is:

How can technology-assisted guided discovery support the conceptualization of parameters in the field of quadratic functions?

This paper reports on three sub-questions:

- Is it possible for a sample of 379 grade 9 students to support the learning of the concept of parameters in the field of quadratic functions through technology-assisted guided discovery?
- What insight into students' exploration and testing of hypotheses can be generated?
- What limitations of the study are visible?

METHODS, METHODOLOGY AND DESIGN OF THE STUDY

Participants

This study is a control group design intervention study with three experimental groups and one control group. 14 classes of grade 9 with a total of 379 students participated in the study. The 14 classes were from 8 different upper secondary schools from 5 different cities in two states in Germany (13 classes in North-Rhine Westphalia and 1 class in Thuringia). Ten teachers taught a single class and two teachers taught two classes each. In each school, where there was more than one class participating in the study, each class was assigned to a different group. A pen-and-paper technology-free pretest was conducted in all classes before the intervention to collect baseline data on pre-requisite algebra knowledge and skills for the intervention. This showed that the control group and experimental groups had similar skills in the field of linear functions and equations. Further information from the pretest is not reported in this paper.

Intervention: structure of the groups

All students took part in an intervention which involved the use of a given technology (using either a scientific calculator or TI-Nspire CX CAS) to investigate the role of parameters in the vertex form, $f(x)=a\cdot(x-b)^2+c$, of a quadratic function. Table 1 provides an overview of the groups in the study, the type of visualisations and screenshots of pre-prepared files.

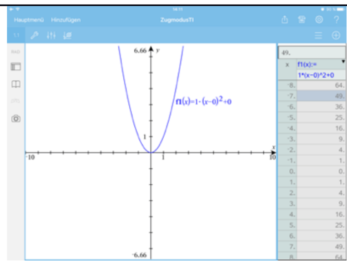
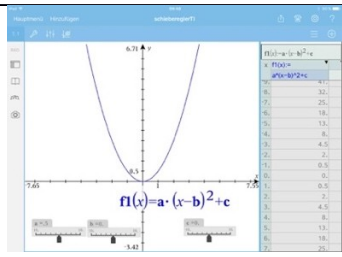
Group	Type of Visualisation: change of Parameters	Pre-prepared file provided to students
Control group "without visualisation" (n=71)	Function tables on scientific calculators, sketching graphs by hand	No pre-prepared file
Experimental group 1: "function plotter" (n=67)	Plotting functions and displaying function tables.	No pre-prepared file
Experimental group 2: "drag mode"(n=85)	Pre-prepared file, manipulation of the graphs by dragging the graph, dynamically linked function equation and tables	 **Figure 1: Screenshot of the "drag mode" file**
Experimental group 3: "sliders" (n=130)	Pre-prepared file, manipulation of the graphs through sliders for each parameter, dynamically linked function tables	 **Figure 2: Screenshot of the "sliders"-file**

Table 1: overview of the groups in the study

The **control group "without visualisation"** was only allowed to use a scientific calculator without graphing functions during the intervention, while the three experimental groups had access to the TI-Nspire CX CAS as handhelds or as an app on iPads and were able to use different features of the technology; namely, function plotters, drag mode or sliders.

- **"function plotter"-group**: students were allowed to use the "function plotter" freely and could choose how many and which functions to plot and produce function tables for.
- **"drag mode" group**: students were able to "drag" the graph and manipulate the form and position of the parabola. For each graph produced, the function equation and function table were displayed on screen and dynamically linked to the graph.
- **"sliders"-group**: students could change the parameters in the vertex form with sliders to manipulate the form and position of the parabola. For each graph produced, a general vertex form equation was displayed, without the values showing. The values for the parameters could be read from the sliders. The function table was shown and dynamically linked to the graph.

Intervention lessons:

The intervention consisted of three lessons of 45 minutes designed by the first and second named researchers. The first lesson involved an introduction to the use of technology features needed for lessons 2 & 3. This lesson was run by the teacher, or the first-named researcher where requested. This lesson is not reported here.

Lessons 2 & 3: Self-paced technology assisted guided discovery

Lessons 2 and 3 were conducted as a double lesson and were designed as a self-paced learning environment with guided discovery and the design process was empirically grounded. All students worked on the same worksheet which did not contain any instructions about technology use, but the students in the experimental groups were given a help-sheet for the use of TI-Nspire CX CAS System, where syntax for the most needed features was provided. Additionally, a pre-prepared file (refer to Table 1) was provided to each of the students in the "drag mode" and "sliders" groups. The main worksheet was structured in four parts, where students were asked to:

Part 1: Describe one example of a transformed parabola and explain the differences compared to a standard parabola.

Part 2: Explore the influence of the different parameters a, b, c in the vertex form $f(x) = a \cdot (x-b)^2 + c$ one by one.

Part 3: Find explanations for the change in the graphs.

Part 4: Write a summary sheet in a form of a cheat sheet of everything learned during the lessons.

For parts 2 and 3 of the worksheet students were provided with a number of guiding questions on the worksheet. For example, in Part 2, the students were asked to first look at $f(x) = x^2 + c$ and change the value c.

Data

In 13 of the classes, one pair of students was videographed during the intervention and asked to think aloud and discuss their work with each other; these students (n=26) were told that it was not necessary to produce a summary sheet, however 22 students out of the 26 produced summary sheets. The videos were transcribed and qualitatively analysed. The first named researcher was present in these classes and wrote lesson observation notes. Summary sheets (for an example refer to Erreur : source de la référence non trouvée) were also collected from all other students (n=331), who produced the sheets in pairs or groups of three. In total, 178 summary sheets from 353 students were collected for analysis. The lesson observation notes, videos, video transcripts and the summary sheets provided data for this study.

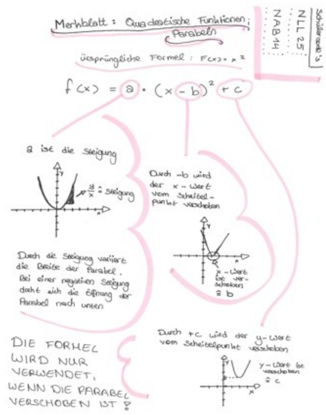


Figure 3: summary sheet example

A coding manual for the summary sheets was developed based on the qualitative content analysis of the summary sheets from two classes. Four main categories (language use, representation, structure, content) were set by the first and second named researchers and 84 sub-categories developed from the data. All summary sheets were coded by three master's thesis candidates after intensive instruction by the first named researcher. The codes for each summary sheet were assigned to each student in the group who produced the sheet. After the initial coding a first analysis was conducted and the results led to a refinement of the coding manual. During this refinement, some sub-categories were combined and some were coded in a slightly different way. After the revision of the coding manual the first named researcher then recoded all summary sheets. The results concerning the summary-sheets presented in this paper are based on the second round of coding.

RESULTS

Three different aspects of the results will be reported in this paper corresponding to the three sub-questions:

- technology-assisted guided discovery can support the learning of the concept of parameters,
- technology-assisted guided discovery can support students' exploration,
- technology-assisted guided discovery can support the testing of their hypotheses.

Additionally, a few constraints of the study will be represented.

Technology-assisted guided discovery can support learning of the concept of parameters

The following results are based on an analysis of the summary sheets. The analysis suggests that the dynamic visualisation using the "drag mode" and "sliders" supported students' investigation of the role of parameters more than static visualisation using the "function plotter". There are nearly no differences in the summary sheets results between the "function plotter" group and the "without visualisation" group. In nearly all of the sub-categories developed in the coding process of the summary sheets, the "drag mode" or "sliders" group gave the most *appropriate* answers, the "function plotter" and "without visualisation" group gave nearly always about the same amount of *appropriate* or *inappropriate* answers. In order to be classified as *appropriate* the responses on a summary sheet had to include ideas which were either already correct or which showed a preliminary or developing understanding.

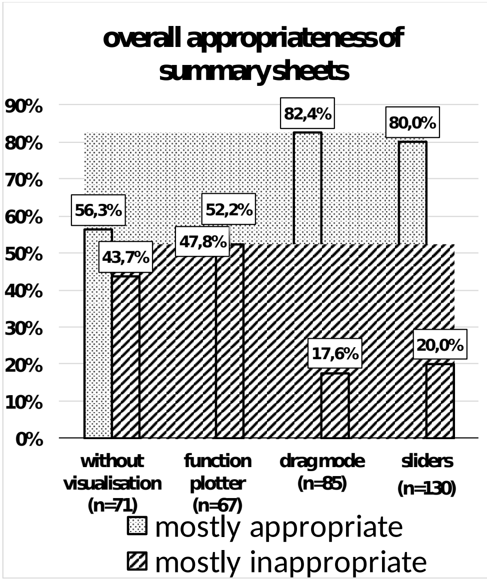


Figure 4: overall appropriateness of the summary sheets

The sub-category "Overall appropriateness" was used to classify the sheets according to the grade of correctness. Each sheet was considered as a whole and classified as *mostly appropriate* or *mostly inappropriate.* In order to be classified as *mostly appropriate* the summary sheets needed to contain more than 50% appropriate answers. The sheets of the "drag mode"- and the "sliders"-group gave *mostly appropriate* answers, whereas the "without visualisation"- and "function plotter" groups had approximately the same number of summary sheets in the two categories (refer to Erreur : source de la référence non trouvée). The dynamic visualisation through the use of "drag mode" and "sliders" seem to support development of students' knowledge of parameters more than by using static visualisation like function plotters or sketching the graphs by hand. This is evident through the ability of 80% or more of the students in the "drag mode"- and "sliders"-groups (refer to Erreur : source de la référence non trouvée) to provide statements classified *mostly appropriate*.

Another interesting point supporting the result that "drag mode"- and "sliders"-were more beneficial than a "function plotter" or no use of visualisation is that the only students who noted on their summary sheets that there is a special case when a=0 and the graph is a horizontal straight line, were in those two groups (i.e. "drag mode" or "sliders"). This special case (i.e. a=0) was recorded on the summary sheets of 37 students across the two groups, and of these 35 students noted this special case in an *appropriate* manner.

From the video and lesson observation notes, there was evidence that some students appeared to stumble across this special case by accident when using the technology to give negative values for the parameters; in moving from positive to negative values for the parameter 'a' students had to pass a=0, prompting them to further explore this case.

Technology-assisted guided discovery can support student exploration

The videos and lesson observations showed that in all three experimental groups and in the control group, students used a variety of different approaches to the task. Even though the task was pre-structured, a number of pairs in the "drag mode"- and "sliders"-groups explored their own relevant examples, rather than those provided, which still enabled them to explore the influence of the parameters using the iPads or handhelds.

For example, one pair of students (in the "sliders" group) initially followed the pre-structured task for describing the example of a transformed parabola, but three minutes after beginning their work they started exploring on their own, investigating examples which weren't suggested on the worksheet. This exploration was prompted when students were changing the values of the sliders to replicate the values in the example and they noted that changing parameters caused transformations. After doing this for a while (about 1 minute), the following statements were made (translated by the first-named author, text in italics describe the movements of the students).

TNA24: so a describes *(points with pen to slider for a)* this open and close, how wide it is open and how far it is closed *(IAR20 moves finger to slider b, then onto the equation)*

TNA24: b describes…ahm right or left *(moves pen to right and left, while IAR20 moves slider for b)*

IAR20: mh exactly…exactly

TNA24: c describes up or down, so move up. Yes but how can you, four is a *(writes something)*, isn't it?...but wait…

The transcript above is one example of how the students used the technology to explore and discover the influence of the different parameters in a very short time by manipulating the sliders to reproduce the given example. Concurrently, the students observed the change in the graph and hypothesized a generalisation for the effect of the parameter.

Technology-assisted guided discovery can support students to test their hypotheses

The same students used the technology to test their hypotheses. The students were working on Part 2 of the worksheet and as suggested they were investigating $f(x)=x^2+c$ for different values for c. To do this they had put all sliders to zero first and then changed the one for c to 4.9. After zooming out so the graph was displayed the following statements were made (translated by the first-named author, italics describe movements of students).

TNA24: Oh didn't think that, ha but wait ahm, four point 9 (*points with pen on slider of c)* ah minus yes

IAR20: yes so when you, through c

TNA24: through c

IAR20: it is determined if it is a parabola, is the function

TNA24: a parabola, yes ok

IAR20 then changed the slider for c to 3.3 (refer to Erreur : source de la référence non trouvée) with stops in between around zero, while TNA24 states that it just moves, IAR20 replies:

IAR20: no wait, wait, no, c doesn't determine that, that must be something else, because otherwise it would have changed

TNA24: correct

IAR20: Then it has to be one of the other, so a or b

TNA24: so c only determines *(TNA24 points to the graph)*

IAR20: the y-intercept

They discuss for a minute about the vertex points of the graph and then change the slider for c back to zero and go on to the next part:

IAR20: now comes a *(moves slider for a)*

TNA24: a…that determines it, so a determines if it is a graph or not

IAR20: if it is a parabola or not

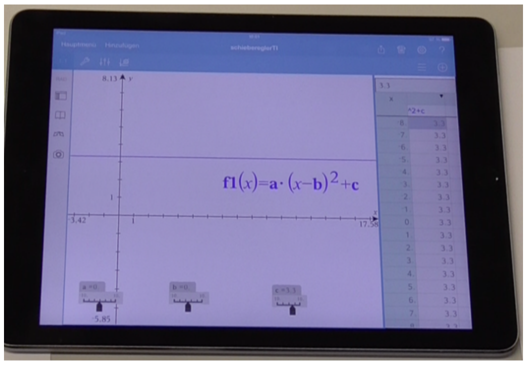


Figure 5: screenshot of the file the students are working on

They changed the slider for 'a' to zero and back again and then decided that 'a' determines if the graph is a parabola or not. These transcript passages show that the students used the technology to test and falsify their statement that c determines whether the graph is a parabola.

Limitations of the study

The videos, the summary sheets and the lesson observations show that a lot of the students were able to find out the effect on the graph, when one of the parameters is changed, but throughout all experimental groups and the control group reasons (correct or incorrect) for the transformations were not provided by most students. Some students tried arguing using knowledge of multiplication of fractions to explain the changes in the function tables when 'a' was changed, but they were largely unsuccessful in providing appropriate reasoning.

In order to support the students in finding explanations for the effect of changing 'c', there were two statements given in part 3 on the pre-structured worksheet concerning the shape of the graph, one of which was false. In addition, part 3 provided support for consideration of parameter 'b' by providing two correct statements asking students to reconstruct them. Unfortunately, this aspect of the pre-structured worksheet caused confusion for some students. When students realised that one of the statements for c was wrong they tried to find out which one of the two statements for parameter b was also wrong, rather than attempt to reconstruct the two correct statements as requested.

There were some technical difficulties which impacted students' ability to focus on the underlying mathematical ideas. A number of the students in the "drag mode" group had difficulties with the accuracies of the given file. Due to the programming, the numbers in the function equation were displayed with two decimal places and the table with up to three decimal places. It was nearly impossible to get whole numbers as values for the parameters while dragging the graph. Hints from the teachers that the students should only try to achieve approximate values did not help much. The students lost considerable time trying to achieve exact values, which distracted them from the task.

The difficulties achieving exact values made comparing values in different function tables very difficult.

A similar problem occurred in the "sliders"-group, where sliders could only be manipulated in 0.1 steps and the function tables value were displayed with two decimal places, resulting in students' confusion again caused by accuracy of the displays. In addition, sliders were too small for some students to work with accurately and it took considerable time for some students to move sliders to obtain desired values. Despite this, it was observed that during the intervention lessons students tended to get more proficient with the sliders while working on the tasks, so the size issue could just be related to familiarity with the use of sliders.

DISCUSSION

The technology-assisted learning of the concepts of parameters in quadratic functions is possible in multiple ways, with all three different approaches having some potentials and constraints. Overall the "drag mode" seemed the most suitable to support the learning of the concept of parameter. Use of sliders was also found to support students' learning. The two experimental groups using "drag mode" and "sliders" were, on the whole, able to produce much more appropriate summary sheets than the control group, so it could be argued that the dynamic visualisation approaches have greater benefits for students' abilities to provide explanations for the effects of parameters on graphs than the static visualisation in the "function plotter" and control group. So in the frame of this study Zbiek et al.'s (2007) conjecture that the slider obscures rather than enhances students' understanding was not observed. On the contrary, the dynamic visualisation in the two groups "drag mode" and "sliders" proved to be beneficial for the investigation of the role of parameters. Even though this study involved a short intervention and did not explore the persistence of the learning gains, the results seemed promising. Overall, the study suggests that technology-assisted guided discovery is beneficial for the conceptualization of parameters.

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REFERENCES

Alfieri, L., Brooks, P. J., Aldrich, N. J., & Tenenbaum, H. R. (2011). Does discovery-based instruction enhance learning? *Journal of Educational Psychology, 103*(1), 1-18. doi:10.1037/a0021017

Bardini, C., Radford, L., & Sabena, C. (2005). Struggling with variables, parameters, and indeterminate objects or how to go insane in mathematics. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 129-136. Melbourne: PME.

Bloedy-Vinner, H. (2001). Beyond unknowns and variables - parameters and dummy variables in high school algebra. In R. Sutherland, T. Rojano, A. Bell & R. Lins (Eds.), *Perspectives on school algebra* (pp. 177-189). Dordrecht, Netherlands: Kluwer Academic Publishers.

Blum, W. (2004). On the role of „Grundvorstellungen" for Reality-Related-Proofs – Examples and Reflections. In M. A. Mariotti (Ed.), *CERME-3 – Proceedings of the Third Conference of the European Society for Research in Mathematics Education*. Universitá di Pisa.

Drijvers, P. (2003). *Learning algebra in a computer algebra environment: design research on the understanding of the concept of parameter*. Utrecht: CD- Press, Center for Science and Mathematics Education.

Drijvers, P. (2004). Learning Algebra in a Computer Algebra Environment. *International Journal for Technology in Mahtematics Education, 11*(3), 77-89.

Duncan, A. G. (2010). Teachers' views on dynamically linked multiple representations, pedagogical practices and students' understanding of mathematics using TI-Nspire in Scottish secondary schools. *ZDM, 42*(7), 763-774. doi:10.1007/s11858-010-0273-6

Duval, R. (2006). A Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics. *Educational Studies in Mathematics, 61*(1-2), 103-131. doi:10.1007/s10649-006-0400-z

Ferrara, F., Pratt, D. & Robutti, O. (2006). The role and uses of technologies for the teaching of algebra and calculus. In A. Gutiérrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education. Past, Present and Future* (pp. 237-273). Rotterdam: Sense.

Gerver, R. K. & Sgroi, R. J. (2003). Creating and Using Guided-Discovery Lessons. *The Mathematics Teacher, 96*(1), 6-13.

Kaput, J. (1992). Technology and mathematics education. In D. Grouws (Ed.), *A Handbook of Research on Mathematics Teaching and Learning* (pp. 515–556). New York: MacMillan

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why Minimal Guidance During Instruction Does Not Work: An Analysis of the Failure of Constructivist, Discovery, Problem-Based, Experiential, and Inquiry-Based Teaching. *Educational Psychologist, 41*(2), 75-86. doi:10.1207/s15326985 ep4102_1

Küchemann, D.E. (1981). Algebra. In: K.M. Hart, (Ed.), *Children's understanding of mathematics: 11 - 16* (pp. 102 – 119). London: J. Murray.

Malle, G. (1993). *Didaktische Probleme der elementaren Algebra.* Braunschweig, Wiesbaden: Vieweg.

Ministerium für Schule und Weiterbildung des Landes Nordrhein-Westfalen [MSW] (2007) (Ed.). Kernlehrplan für das Gymnasium – Sekundarstufe I (G8) in Nordrhein-Westfalen Mathematik. Frechen: Ritterbach.

Mosston, M. & Ashworth, S. (2008). *Teaching physical education: First online edition*. Retrieved from http://spectrumofteachingstyles.org/NEW2/wp-content/themes/sots/img/Teaching_Physical_Edu_1st_Online.pdf

Penglase, M. & Arnold, S. (1996). The graphics calculator in mathematics education: a critical review of recent research. *Mathematics Education Research Journal, 8*(1), 58–90. doi:10.1007/bf03355481

Thomas, M.O.J. (2008). Conceptual representations and versatile mathematical thinking. In M. Niss (Ed.), *ICMI-10 proceedings and regular lectures*. Copenhagen, Denmark.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A.F. Coxford (Ed.), *The ideas of algebra, K-12 (1988 Yearbook of the NCTM)* (pp. 8-19). Reston, VA: NCTM.

Vom Hofe, R. & Blum, W. (2016). "Grundvorstellungen" as a Category of Subject-Matter Didactics. *Journal für Mathematikdidaktik 37(S1),* 225-254.

Hofe, R. vom, Kleine, M., Wartha, S., Blum, W., & Pekrun, R. (2005). On the role of „Grundvorstellungen" for the development of mathematical literacy – First results of the longitudinal Study PALMA. *Mediterranean Journal for the research in Mathematics Education, 4*(2), 67-84.

Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on Technology in Mathematics Education – A Perspective of Constructs. In F. Lester (Ed.), *Second Handbook of research on mathematics teaching and learning.* (pp. 1169-1207). Charlotte, NC: Information Age.

DYNAMIC GEOMETRY SOFTWARE IN MATHEMATICAL MODELLING: ABOUT THE ROLE OF PROGRAMME-RELATED SELF-EFFICACY AND ATTITUDES TOWARDS LEARNING WITH THE SOFTWARE

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Mathematical modelling is a complex process consisting of several steps, which can also be carried out with the use of digital tools. This paper takes a closer look on how students perceive the DGS GeoGebra when learning mathematical modelling, how their confidence in their tool competencies changes when using the software to do modelling, and if the learning outcome concerning modelling competencies is influenced by programme-related self-efficacy or attitudes towards learning with the digital tool. Results from both qualitative and quantitative evaluations of a study with approx. 300 grade 9 students are reported.

Keywords: DGS, Modelling, Self-efficacy, Attitudes

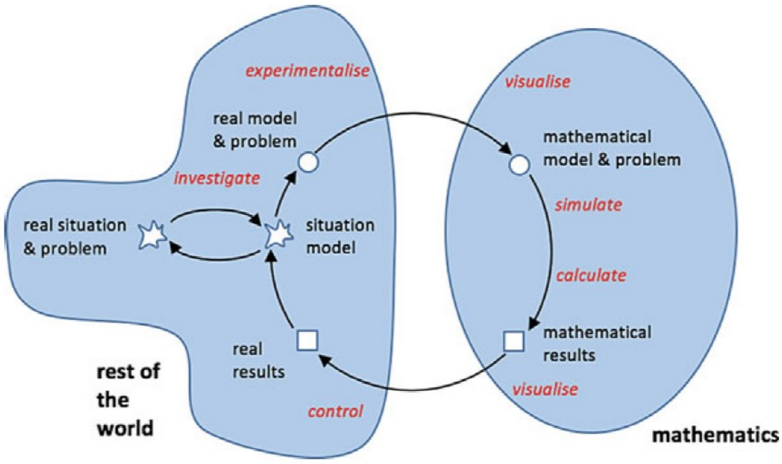
THEORETICAL BACKGROUND

Modelling with Dynamic Geometry Software

Mathematical modelling is a complex process in which a problem in a real-world situation must be understood and simplified and then translated into the world of mathematics to be solved by mathematical means. The found mathematical results then must be related back to the real-world problem and finally be reflected and checked for plausibility. If this check indicates that the found results do not yet represent a satisfying solution, the steps of the modelling process can or should be repeated until a satisfying solution is found (Blum 2015). That is why this process often is displayed as a cycle (see Figure 1), even though the real process of solving a modelling problem does not necessarily have to strictly follow this cycle (Borromeo-Ferri 2006). Modelling consists thus of different steps, most prominently among them are simplifying, mathematising, interpreting and validating. Being competent in modelling therefore means, in a comprehensive sense, being able to construct and to use or apply mathematical models by carrying out appropriate steps as well as to analyse or to compare given models (Blum et al. 2007). The abilities to carry out a certain step of the modelling process respectively are also called sub-competencies of modelling (Blum 2015).

It is also possible to make use of digital tools while modelling. Depending on the kind of modelling problems, spreadsheets, computer-algebra-systems or dynamic geometry software (DGS) may not only support or take on the mathematical work but also visualise models, simulate real-world processes or be used to control mathematical results (Siller & Greefrath 2010). When modelling with a DGS, the software can for example be used to draw or construct geometric models or to measure specific quantities needed to solve a problem. The dynamics of the software are especially useful for a flexible adaptation of already constructed models either with the aim of simulating possible solutions or of improving the used model. Thus, not only the step of mathematising may benefit from the use of a digital tool. Steps like validating or reflecting, which require the adaptation or adjustment of mathematical models, may profit from the use of a digital tool as well (Greefrath, Siller & Weitendorf 2011).

Fig. 1. Possible use of digital tools for modelling (Greefrath et al. 2011)



When using a DGS, the different actions that can be carried out, can be classified into a scheme of operations (Mackrell 2011, Sedig & Sumner 2006). The most obvious actions done with a DGS are probably those in which a new object is constructed using both already existing objects and the available DGS-tools. These construction-operations are supplemented by object-operations in which no new object is created, but existing objects are changed. For example with the help of a drag-mode, it is possible to rearrange objects, to simulate dynamic processes or to vary different parameters. Both construction- and object-operations can enhance the modelling process: in the step of mathematising or validating for example, geometric models can be build using construction-operations and be reflected using object-operations. Additionally, view-operations, in which a constructed object is displayed in an alternative way, e.g. from a different angle, can support reflections regarding the model fit.

But even though a DGS might be supportive for different modelling steps, mathematical modelling remains a cognitively demanding activity, which requires not only mathematical knowledge but also concept ideas, appropriate beliefs, attitudes and extra-mathematical knowledge (Blum 2015). Modelling with digital tools additionally requires skills in certain software tools (Siller & Greefrath 2010). When students are learning modelling with a digital tool, it is possible that two learning processes take place at the same time. On the one hand students have to learn how to deal with complex tasks like modelling problems and on the other hand they have to cope with software to which they might not yet be fully accustomed. Little is known on how students perceive the used instrument when learning modelling with a digital tool, what difficulties they encounter and what strategies they pursue to take full advantage of the instrument's power.

Programme-Related Self-Efficacy and Attitudes Towards DGS

It is known from the Social Cognitive Theory that self-efficacy (SEF), which is the belief that one has the ability to perform a particular action, has a strong influence on behaviour (Bandura 2012). When confronted with difficulties, those individuals with a low confidence in their abilities are easily discouraged whereas more confident students will intensify their efforts (Igbaria & Iivari 1995). This concept was extended to the context of computer software. Studies in the nineties already showed the important role of computer-SEF in performances using information technologies (e.g. Compeau & Higgins 1995, Gist, Schwoerer & Rosen 1989). Individuals with a high computer SEF use the computer more, derive more of their use of computers and are able to exploit management support better (Igbaria & Iivari 1995). While computer SEF means general confidence in one's own abilities to work with a computer, independent from specific programmes or software, specific computer self-efficacy or programme-related self-efficacy describes one's own beliefs about being able to operate a specific software like for example a certain DGS (Agarwal, Sambamurthy & Stair 2000). It is yet unknown if persons with a higher programme-related SEF also benefit more of learning mathematical modelling with the use of a DGS than persons with a lower SEF concerning the development of modelling competencies. This could be the case because the former perhaps take more advantage of the tools a DGS offers while searching for mathematical models or adapting them. Additionally, user attitudes towards the computer or specific software can moderate the outcome of training programmes (Torkzadeh, Plfughoeft, & Hall 1999) or the individual's SEF (Torkzadeh & Dyke 2002). It is yet unknown, if the attitudes towards software also moderate the outcome on programmes where not the computer usage itself is trained, but the computer just serves as a medium to learn, e.g. mathematical content. Therefore, we take a closer look on the following questions in this paper:

RESEARCH QUESTIONS

1.) How do students perceive the DGS GeoGebra when learning mathematical modelling and especially, what difficulties do students encounter?

2.) Does the students' programme-related SEF or their attitudes towards the software change when learning mathematical modelling with a DGS?

3.) Is there a relationship between students' programme related SEF or their attitudes towards the used software and their growth of modelling competency when learning modelling with a DGS?

METHODOLOGICAL BACKGROUND

Design of the Study

To answer these research questions, we conducted an intervention study with a pre- and post-test as well as a four-lesson intervention in which students worked on geometric modelling tasks with the help of the DGS GeoGebra. A total of 328 grade 9 students in 15 different classes took part in this study, which was carried out in their regular mathematics lessons. During four consecutive math lessons, which were held in computer labs at their schools, the students worked in pairs on a geometric modelling task that was implemented in GeoGebra. Even though the participating teachers had to make sure that their students had already worked with the software GeoGebra before the beginning of the project, the teaching unit began with a short revision of useful symbols and constructions in GeoGebra to ensure basic knowledge about possible commands in the software. After this programme related revision, students worked independently on different modelling problems with the software. At the end of the respective lessons different solutions were projected and discussed. Before and after the teaching unit all students filled out a modelling test which measured their modelling competencies and a questionnaire concerning their confidence in their abilities to operate GeoGebra and their attitudes towards this software.

During the teaching unit, the students worked on four different modelling tasks. While the first of the used tasks was structured by different instructions and served as an introduction into the various steps of the modelling process, the remaining three tasks were rather open problems with several correct answers. For example, one of the tasks dealt with finding market areas of supermarkets in Berlin to determine where a new branch could be opened. With the help of GeoGebra, it is possible to try out different models. For example students may neglected the network of roads and find areas by constructing midperpendiculars between different branches of the supermarket. Alternatively, the existing roads can be taken into account and assumptions on the number of residents in different streets can be made. In this case it is sensible to choose polygons as market areas which can be adapted to their assumed number of residents.

All lessons started with a short presentation of the problem by the teacher to the class. Following, students worked in pairs on one computer. They had both a working sheet which presented the task in a written format and a DGS-file that contained the necessary graphics, e.g. a map of Berlin in the task described above. Students were asked to use the DGS GeoGebra and to write down their solution processes as comprehensible as possible. After a working time of approximately 25 minutes, several students presented their results or suggestions with the help of a projector. The whole class discussed and compared different solutions with a special focus on the different steps in the modelling process.

During the intervention, the teachers gave as little help as possible but gave students freedom to work independently and to make their own decisions. If the teachers intervened, they mostly gave strategic help or helped with issues with the software, e.g. helping to save the files in the right places. To prepare teachers for the intervention they were instructed with detailed materials including lesson plans in several meetings. During these meetings they were also prepared to typical questions and sensitized for students' modelling processes.

Questionnaires and Test Instrument

To assess students' programme-related SEF, which is their confidence in their own tool competency, an adaption of the CUSE-D questionnaire (Spannnagel & Bescherer 2009) was used. Since this questionnaire, originally developed by Cassidy and Eachus (2002), aims to measure general computer-related SEF, but not the confidence in one's own ability to operate a specific software, the used items were adapted to be more specifically related to GeoGebra. Students had to express their agreement to ten statements like "I think working with GeoGebra is easy" or "I think of myself as a skilled user of GeoGebra" on a rating scale with six categories. The lowest score of 10 points implied no confidence in the own tool competencies, the highest score possible of 60 points implied a very high confidence. The internal consistency of this scale was Cronbach's $\alpha=0.82$ for the pre-test and $\alpha=0.92$ for the post-test. The attitudes towards learning with GeoGebra were measured analogously. Exemplary items are "GeoGebra is a good help for learning" or "Using GeoGebra makes learning more interesting". Even though only five items were used, the reliability for this scale was as good as for the first with Cronbach's $\alpha = 0.87$ for the pre-test and $\alpha = 0.90$ for the post-test. The minimal score of 5 indicates strong disapproval of the software whereas the highest score of 30 indicates strong approval and a very positive attitude towards the software.

The modelling competencies were measured by a newly constructed test instrument that consisted of multiple choice or short-answer questions and focused on different steps of the modelling process. To avoid that students had to answer to the same modelling items twice but nevertheless to be able to use the same items in the pre- and in the post-test, a multi-matrix design was used, which had to be evaluated within the frame of Item Response Theory. With the help of this theory, Weighted Likelihood Estimators for the modelling sub-competencies at the different points of measurement could be estimated. As explained above, the steps of the modelling process differ regarding the use of a DGS. Therefore, we analysed two different aspects of modelling competency separately: On the one hand the sub-competency Mathematising/Validating (MV) in which the DGS was used to build, try out and compare different mathematical models and on the other hand a sub-competency Simplifying/Interpreting (SI) where the DGS did not play an equally active role. The estimators for these two dimensions could be determined with a reliability of $\alpha = 0.70$ for SI both in the pre-and post-test, and $\alpha = 0.71$ for MV in the pre-test and $\alpha = 0.73$ in the post-test. 40 % of the tests were rated by two independent coders. The interrater-reliability lay within a range of $.81 \leq \kappa \leq .95$ (Cohen's Kappa). Both the modelling test and the questionnaire were answered within 45 minutes in the math lessons directly before and after the teaching unit. For these lessons, no computers were needed since both the test and the questionnaire were purely in a paper-pencil format.

Interviews

During the teaching unit, the desktops of student pairs from six different classes (n = 12) were filmed and their conversations were recorded. These recordings and films were analysed to identify scenes where the students encountered difficulties during the modelling process. After the intervention, six pairs of students were confronted with their respective scenes and questioned in a semi-structured interview. The focus of this interview was on difficulties the students perceived while working on modelling tasks with GeoGebra, on their strategies to overcome these difficulties as well as on their attitudes towards GeoGebra and digital tools in general.

Methods of Evaluation

To evaluate the interviews, they were transcribed and coded in accordance with the summarizing qualitative content analysis. To do so, all transcribed interviews were line-serially analysed to inductively build different categories (Deeken 2016). In a final analysis, all interviews were coded.

Since the intervention took place in the regular classes, the quantitative data is structured in clusters. Students who are in the same class are likely to be more similar to their classmates than to students from different classes. Ignoring this structure when statistically evaluating the intervention would lead to distorted standard errors and thus to incorrect tests of significance. That is why we decided to correct these errors by using the programme Mplus and the *type = complex*-Option. With Mplus, we calculated Wald's t-tests to analyse possible change in programme-related SEF and attitudes towards the programme from pre- to post-test. To analyse a possible relationship between the growth of modelling competency and SEF and attitudes towards the software respectively, we calculated multiple regressions using the post-test values as dependent variable and pre-test values, gender, SEF and attitudes toward GeoGebra in the post-test as covariates.

FINDINGS

Results of the Interviews

The analysis of students' remarks in the interviews concerning their perception of GeoGebra revealed several different categories: required working time, insecurity, calculating device, operation, precision and usefulness. Students saw a connection between the use of the DGS and the time they spent working on the task. While some students found working with the DGS to be "quick and easy", others said looking for complex models was "time-consuming and complicated". The latter was especially the case when students still felt insecure with the software. In those cases, students were for example "slightly annoyed by those appearing numbers you had to hide by clicking on the buttons on the right". Often, these difficulties were due to missing knowledge about the geometric rules of constructions that lay behind the software's commands. All students agreed that working with the software became easier once they felt more familiar with it. They also perceived the software supportive of calculations. Some students remarked that they "had no clue how to calculate" the surface of a non-regular polygon without the software. They found the DGS to be practical, some saw an even bigger potential for the software in more complex tasks than those used in the study. Other students stressed the software's precision. One student compared her work in GeoGebra with constructions on paper: "A pencil goes blunt and you have to sharpen it while working. And if you want to erase it, it does not go away completely. That's much easier with GeoGebra". The aspects of changing models, adapting models or restarting a modelling process are also remarked by several students. "You can delete things quickly without having to restart completely" a student says. He goes on: "We went through the different options in the programme to find something to model the figures in the best way possible". Like him, several students mentioned the opportunity to be inspired by the commands implemented in the software while searching for suitable mathematical models. They "just tried out different things without having to ask someone" and "were able to find solutions on [their] own". Thus, the mathematics lessons became "a welcome change to regular math classes" and GeoGebra "a sound assistance".

Change in programme-related Self-Efficacy and Attitudes

A total of 289 students answered to all items measuring the programme-related SEF in both the pre- and the post-test. The confidence in tool competencies increases from a mean of 38.63 (SD = 8.12) in the pre-test to a mean of 44.41 (SD = 9.71) in the post-test. The Wald's t-test reveals that this difference of 5.46 is significant ($t(1) = 74.64, p < .001$). The effect size Cohen's $d = 0.61$ indicates a medium effect. The attitudes towards the programme remain relatively stable with a mean in the pre-test of 20.26 (SD = 5.74) and 20.42 (SD = 6.42) in the post-test. The Wald-test shows that this difference is not significant ($t(1) = 0.182, p = .670$), Cohen's $d = 0.03$ also indicates no effect. It thus can be stated that the four-lesson intervention where students worked on modelling tasks with a DGS lead to a significant improvement in their programme-related SEF while their attitudes towards the software remained the same.

Table 1. Descriptive Statistics for Pre- and Post-Test

	pre-test			post-test		
	N	M	SD	N	M	SD
SEF	277	38.86	8.06	308	44.45	9.49
Att	282	20.62	5.65	311	20.66	6.28
MV	320	0.00	0.72	320	0.11	0.80
SI	320	-0.04	0.75	320	0.01	0.84

N =Number of participants; M = Mean; SD = Standard Deviation; MV=Mathematising/Validating; SI=Simplifying/Interpreting; SEF=programme-related self-efficacy (measured in post-test); Att = attitude towards the software (higher value=more positive attitude, measured in post-test);

Relationship between Modelling and programme-related Self-Efficacy or Attitudes

As it can be seen in Table 2, there is a significant correlation between the programme-related SEF and the competency Mathematising/Validating (MV), both measured after the intervention, but not between the programme-related SEF and the competency Simplifying/Interpreting (SI). The more confident a person in their abilities to operate the software is, the better their result in MV in the post-test is. But this is also valid for their results in the pre-test, as the correlation between MV in the pre-test and SEF is significant as well. Persons who feel more confident in their tool competency after the unit also achieved a higher score in the pre-test. Concerning SI the correlation with SEF is significant for the pre-test only. Apparently, students who have a higher competence in SI at the beginning of the teaching unit also have a higher programme-related SEF. This seems to change during the teaching unit so that at the end no relationship between SEF and SI-competencies can be seen.

The attitudes towards the software and the programme-related SEF are strongly correlated. The more confident a person in their own competencies in using the software is, the more positively they see the software. It can also be seen that neither MV nor SI are correlated with gender but SEF and attitudes are. Boys tend to be more confident in their own abilities to operate the software and see the software more positive.

Table 2. Correlations of the used variables in the regression models

	MV_post	MV_pre	SEF	Att		SI_post	SI_pre	SEF	Att
MV_pre	0.41***				SI pre	0.46***			
SEF	0.20***	0.15**			SEF	0.10	0.15**		
Att	0.07	0.03	0.62***		Att	0.04	0.06	0.62***	
gender	0.12	0.05	0.26***	0.16**	gender	0.09	0.06	0.26***	0.16**

**p<0.01; ***p<0.001 (MV=Mathematising/Validating; SI=Simplifying/Interpreting; SEF=programme-related self-efficacy (measured in post-test); Att = attitude towards the software (higher value=more positive attitude, measured in post-test); gender: 1= boys, 0 = girls)

To analyse the influence of programme-related SEF and the attitudes towards the software on the modelling competencies independently from differences of competencies that already existed before the teaching unit began, two multiple regression-models were calculated. With help of the first model we analysed if programme-related SEF or attitudes towards the software were significant predictors of the achievement in the post-test in the dimension MV when controlled for both achievement in the pre-test and gender. The second model examined analogously their influence on the independent variable SI.

As can be seen in Table 3, only the score in the pre-test and the programme-related SEF are significant predictors of the achievement in the post-test concerning MV. Persons who feel more confident in using GeoGebra also improved their competencies in MV more, regardless of their gender. The attitude towards the software was no significant predictor of the post-test achievement in MV. The development of this modelling competency seems to be independent from the students' perception of the programme. Even when they did not recognize the software as a useful instrument for learning, they were able to build up the modelling competency MV by modelling with it. The standardized regression weight indicates a small effect size (β=.15). And on the other hand, a positive view on the software did not automatically lead to a stronger improvement in the modelling competency. This is also valid for the competency SI.

For the achievement in SI in the post-test, only the score in the pre-test is a significant predictor. Persons who are more confident in their tool competencies thus do not achieve higher scores in the SI part of the modelling post-test, when adjusted for the pre-test scores. Equally the attitude towards the software had no influence on their achievement either. With these regression models 19.4 % and 22 % respectively of the total variance can be explained.

Table 3. Multiple Regressions on MV_post and SI_post

model	coefficient	b	SE	β	p	R^2
1 (criterium: MV_post)	Intercept	-0.37	0.23	-.46	.12	19.4 %
	MV_pre	0.43	0.08	.39	< .001	
	SEF	0.01	0.00	.15	<.001	
	Att	-0.01	0.01	-.05	.54	
	gender (1=boys)	0.12	0.10	.15	.23	
2 (criterium: SI_post)	Intercept	-0.10	0.17	-.12	.57	22.0 %
	SI_pre	0.50	0.05	.46	<.001	
	SEF	0.00	0.01	.03	.64	
	Att	-0.00	0.01	-.02	.72	
	gender (1=boys)	0.11	0.08	.13	.18	

MV = Mathematising/Validating; SI = Simplifying/Interpreting; SEF = programme-related self-efficacy (measured in post-test); Att = attitude towards the software (higher value = more positive attitude, measured in post-test); gender: 1= boys, 0 = girls

Summary, Discussion and Outlook

To sum up the findings it can be stated that students did recognize possible benefits of working with a DGS as they stressed the software's precision und usefulness. But they also experienced difficulties, mainly due to missing either mathematical or software-related knowledge. Students stated that after having worked with the software during the four intervention lessons they felt more secure and more confident in their tool abilities. This impression can be confirmed by the quantitative data. We have seen that even the short period of four lessons in which students worked with a DGS led to a significant improvement in their programme-related SEF that was sustained even three month after the teaching unit. The attitudes towards learning with the software though remain stable throughout all points of intervention. Apparently students did not see the software more positively even though they felt more secure with it.

Comparing the two dimensions of modelling competency, only the development of the competency MV is influenced by the programme-related SEF when modelling is learned with the help of a DGS. This is in accord with the theoretical considerations that there are different phases of the modelling process where the software can play different roles. While trying out different models, adapting them or searching for alternative useful mathematics the DGS plays an important role. It serves not only as tool to visualise models, but it also gives inspirations on what kind of mathematics could be worth trying out to find a solution to the given problem. A possible explanation for the relationship between programme-related SEF and the development of MV is that persons who feel confident in the software can concentrate more on the step of mathematising or validating. Perhaps those persons can profit more of the benefits that the software offers which then leads to a greater improvement in the modelling competence. This assumption is supported by the results of the qualitative study where students recognised possible benefits of the software but also saw the need of basic knowledge in operating the DGS. Nevertheless, students' difficulties were not solely to be attributed to the software. The combination of the qualitative with the quantitative data showed that especially those students who improved their competencies in mathematising and validating often also named difficulties that resulted from the task itself and not from problems in operating the software. But of course the students who were interviewed is just a small sub-sample and certainly not representative. The intervention study, even though conducted with a large number of students was limited to students of higher-achieving schools in grade nine. This was mainly due to practical reasons as this group of students was most likely to have already worked with a DGS. A focus on complete novices or experts in the software when modelling with a DGS might be useful to reflect the results found in this study.

But nevertheless, this study gives a first insight into the complex interplay between modelling and factors like programme-related SEF and attitudes when modelling with a DGS. Yet there is still a need for research on how modelling with digital tools can be successfully learned, which premises should be fulfilled so that digital tools can be used in a profitable way when learning modelling and finally on the effects of digital tools on the development of modelling competencies.

An important point also lays in the design of tasks when digital tools are available. In our research, the used tasks could still be solved without the use of a digital tool. This reflects the usual practice in classrooms of just expanding students' tools in solving tasks and is the best basis to understand what changes in a working process are caused by the software. Equally interesting is the question how modelling processes change, when the problem cannot be tackled without a digital tool. Perhaps in those tasks the role of computer competencies has to be taken in account even more.

In our up-coming studies, a special focus will be given to the role of difficulties when modelling with a DGS. As the interviews in this study have shown, students remark problems that could be traced back either to a lack of software-knowledge or to barriers in the modelling process independently from the tool. As the confidence in tool-competencies seemed to have a bigger impact on the development of the competencies mathematising and validating than on other competencies, the hypothesis arises that in those phases of the modelling process more software-related difficulties might occur. Detailed observations of students modelling processes with special focus on their difficulties will give more insight into the different role of the software during different phases of modelling. The observation of students' modelling processes in this study already revealed interesting scenes where difficulties in using the software (e.g. how to construct a circle) led to a deeper mathematical understanding of the problem (e.g. is it generally possible to construct a circle in the given situation). We hope by this means to reach a better understanding of how a digital tool and the process of learning how to use a tool can be used in a promising way to foster modelling competencies.

REFERENCES

Agarwal, R., Sambamurthy, V., & Stair, R. M. (2000). Research Report: The Evolving Relationship Between General and Specific Computer Self-Efficacy—An Empirical Assessment. *Information Systems Research, 11*(4).418-430.

Bandura, A. (2012). *Self-efficacy: The exercise of control*. New York, NY: Freeman.

Blum W. (2011) Can Modelling Be Taught and Learnt? Some Answers from Empirical Research. In: Kaiser G., Blum W., Borromeo Ferri R., Stillman G. (eds), *Trends in Teaching and Learning of Mathematical Modelling. International Perspectives on the Teaching and Learning of Mathematical Modelling*, Vol 1. (p.15-30). Dordrecht: Springer.

Blum, W. (2015). Quality Teaching of Mathematical Modelling: What Do We Know, What Can We Do? In: Cho, S.J. (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education - Intellectual and Attitudinal Challenges* (p. 73-96). New York: Springer.

Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modelling process. In: *Zentralblatt für Didaktik der Mathematik*, 38 (2), 86-95.

Cassidy, S & Eachus, P. (2002). Developing the computer user self-efficacy (CUSE) scale: Investigating the relationship between computer self-efficacy, gender and experience with computers. *Journal of Educational Computing Research*, 26(2), 169-189.

Compeau, D. R., Higgins, C. A. (1995). Computer self-efficacy: Development of a measure and initial test. *MIS Quart*. 19. 189–211.

Deeken, J. (2016). Modellierungsprozesse mit GeoGebra aus Sicht von Schülerinnen und Schülern in Klasse 9 – eine qualitative Interviewstudie. Unpublished Masterthesis. Westfälische Wilhelms-Universität: Münster.

Gist, M., Schwoerer, C. & Rosen, B. (1989). Effects of alternative training methods on self-efficacy and performance in computer software training. *Journal of Applied Psychology*, 74(6). 884–891.

Greefrath G, Siller H, Weitendorf J. (2011). Modelling considering the Influence of Technology. In Kaiser G, Blum W, Borromeo Ferri R, Stillman G (Eds.), *Trends in Teaching and Learning of Mathematical Modelling* (p. 315-329). Dordrecht: Springer.

Igbaria, M., & Iivari, J. (1995). The Effects of Self-Efficacy on Computer Usage. *Omega*, 23 (6), 587-605.

Siller, H.-St. & Greefrath, G. (2010): Mathematical modelling in class regarding to technology. In Durand-Guerrier, V.; Soury-Lavergne, S. & Arzarello, F. (eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (p. 2136-2145). Lyon: IRNP.

Spannagel, C. & Bescherer, C. (2009). Computerbezogene Selbstwirksamkeitserwartung in Lehrveranstaltungen mit Computernutzung. *Notes on Educational Informatics - Section A: Concepts and Techniques, 5*(1), 23-43.

Torkzadeh, G., & Van Dyke, T. (2002). Effects of training on Internet self-efficacy and computer user attitudes. In: *Computers in Human Behavior*, 18(5), 479–494.

Torkzadeh, G., Plfughoeft, K., & Hall, L. (1999). Computer self-efficacy, training effectiveness and user attitudes: an empirical study. *Behaviour & Information Technology*, 18, 4 299–309.

Sedig, K. & Sumner, M. (2006). Characterizing Interaction with Visual Mathematical Representations. International Journal of Computers for Mathematical Learning, 11, 1–55.

Mackrell, K. (2011). Design decisions in interactive geometry software. ZDM Mathematics Education, 43, 373–387.

FEEDBACK IN A COMPUTER-BASED LEARNING ENVIRONMENT ABOUT QUADRATIC FUNCTIONS:

RESEARCH DESIGN AND PILOT STUDY

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Even though feedback is an essential part of computer-based learning environments (CBLEs), it is still not clear what effect different types of feedback have on students math achievement and on their abilities to assess their own achievement. To approach this problem we are currently planning a quasi-experimental study with grade eight or nine students using a CBLE called "Discover Quadratic Functions". This paper gives insight both into existing research concerning the design of CBLE as well as concerning different types of feedback and into the theoretical principals that guided the CBLE design. Additionally, results of a qualitative pilot study as well as the design of the up-coming main study are presented.

Keywords: Computer-Based Learning Environment, Quadratic Functions, Feedback (*Response*), *Mathematics Achievement*

THEORETICAL BACKGROUND

The theoretical basis for our research consists of three main topics. Firstly theoretical and empirical findings about CBLEs are reported with a focus on the definition of CBLEs and basic design principles. Secondly, the current state of research regarding feedback is discussed with a focus on the relationship between feedback and achievement and finally the chosen mathematical subject of quadratic functions is explained from a didactical point of view highlighting both mental mathematical representations (*Grundvorstellungen*) and learning difficulties as well as possibilities to avoid or overcome them.

Computer-Based Learning Environments

In search for a definition of CBLEs, it gets obvious that this phrase commonly is used as generic term for computer- and web-based proposals (cf. Baker, D'Mello, Rodrigo, & Graesser 2010; Balacheff & Kaput 1996; Isaacs & Senge 1992). Roth (2015) gives a more precise definition of CBLEs based on mediawiki software. According to him, CBLEs provide structured pathways with a well-matched sequence of tasks, which invites learners to work self-regulated and self-reliantly (Roth 2015). Aside from interactive materials like (GeoGebra-)applets, which are a central content of CBLEs, the integration of retrievable help and the presentation of results are promising ways to support students' working processes in a CBLE (Roth 2015). Wiesner & Wiesner-Steiner (2015) explored central functions of those CBLEs in a qualitative study where they interviewed, among others, experts about central functions regarding the technical and didactical level. Their findings imply that experts gave great account to the (technical) integration of dynamic tools and availability of direct feedback. Concerning the didactic level especially included metacognitive activities and reflexion tasks were pointed out. Suchlike CBLEs are for example available on the German OER-website *ZUM-Wiki* (https://wiki.zum.de/wiki/Hauptseite), which is called on to yield a well-kept surrounding (Vollrath & Roth 2012). The learning environments are subjected to the creative commons licence CC-by-sa 3.0, so that everybody is invited to copy and even change them on condition that the authors name(s) are mentioned and the licence stays the same. Considerably, all *ZUM-Wiki* CBLE versions are saved online and can be re-activated.

Feedback

There are different models and understandings of feedback. Boud and Molloy (2013) for example distinguish a unilateral from a multilateral view on feedback. In the first, feedback is understood as a "one way transmission" (Boud & Molloy 2013, p. 701) whereby the teacher acts as "driver of feedback" (ibid., p. 698). In contrast, the multilateral view attributes a key role to the learners. This view is shared by several other authors (e.g. Nicol & Macfarlane-Dick 2006, Sadler 1989). For example, Sadler (1989) names feedback only "dangling data" (p. 121), if one does not investigate and monitor if and how feedback effect on students behaviour. In addition, Nicol and Macfarlane-Dick (2006) see a connection between self-regulated work and feedback. Thus feedback "can help students take control of their own learning, i.e. become self-regulated learners" (Nicol & Macfarlane-Dick, p. 199). According to this, they worked out seven principles of good feedback practice, such as "[it] facilitates the development of self-assessment (reflection) in learning" (Nicol & Macfarlane-Dick, p. 205). Beside the feedback *practice*, the possible *types* of feedback are of interest for our research. The complexity of feedback is called a specific aspect of effective feedback, whereby studies about this particular field have yielded inconsistent results (Mory 2004, cf. Nelson & Schunn 2008, Shute 2008). The impact of feedback for the process of learning is investigated frequently (cf. Black & Wiliam 1998; Shute 2008; Hattie 2009). Hattie (2009) for example identified feedback as one of the top influences on achievement in school. The intensity of this influence differs depending on the kind of provided feedback. While *knowledge of the correct response feedback* (KCR) has shown similar effects to *no feedback*, variants of feedback that have a *multiple-try* function (MTF) have shown a positive gain on learning (Attali 2015; cf. Shute 2008; Niegemann 2008). MTF offers the possibility to re-think results and thus to remove mistakes. Provided additional help may structure and foster this processes (Attali 2015). Furthermore, Attali (2015) names the effects of providing "explanations for the correct answers" (p. 266), which means a combination of KCR and explanations, as "an interesting area for future research" (p. 266). Dempsey, Driscoll and Swidell (1993) have defined this kind of feedback earlier and allocated it as *elaborated feedback* (EF). Kulhavy and Stock (1989) categorised basically three elaboration types: "(a) task specific, (b) instruction based and (c) extra-instructional" (p. 286). Shute (2008) offers some more types of EF such as giving hints that guide the learners. She says that in general EF "provides information about particular responses or behaviours beyond their accuracy" (Shute 2008, p. 157). In addition to the portrayed variants of feedback, reference should also be made to other influencing factors such as prior knowledge and the point of time when feedback is given (Shute 2008, cf. Mory 2004). The research outcomes concerning the proper time for feedback are divergent, although different meta-analysis tend to foster immediate feedback (cf. Bangert-Drowns et al. 1991; Shute 2008; Niegemann 2008).

Quadratic Functions

Quadratic functions play a central role in German secondary math education (NRW Ministry for Schools and Further Education 2007) and there is wide range of didactical considerations of this topic. One important current revolves around the idea of mental mathematical representations, in German *Grundvorstellungen*. Doorman, Drijvers, Gravemeijer, Boon, and Reed (2012) mention three of them concerning the function concept: "functions as an input-output assignment", "functions as a dynamic process of co-variation" and "functions as a mathematical object" (p. 1246). Besides, many others have specified these concepts as well (e.g. Vollrath 1989;

Malle 2000; Greefrath, Oldenburg, Siller, Ulm, Weigand 2016). The input-output concept refers to the attribution of a domain element to a single element of the target set, co-variation records how changes of one quantity involve modifications of another one and object perceptions means seeing functions as a single object, which describes a relationship as a whole. Functions as mathematical objects are primarily discussed in German upper school (Greefrath et al. 2016). Hence, lower secondary education especially addresses the other two concepts. Zaslavsky (1997) indicates five "cognitive obstacles" (p. 20) concerning quadratic functions (Zaslavsky 1997, p. 30–33):

> "Obstacle 1. The interpretation of graphical information (pictorial entailments) […]
> Obstacle 2. The relation between a quadratic function and a quadratic equation […]
> Obstacle 3. The analogy between a quadratic function and a linear function" […]
> Obstacle 4. The seeming change in form of a quadratic function whose parameter is zero […]
> Obstacle 5. The over-emphasis on only one coordinate of special points".

Furthermore, Nitsch (2015) reports about learning difficulties in the field of representational changes upon functional relationships. One of the difficulties she reveals within her study refers to the understanding of parameter impacts on the graphical representation of quadratic functions (Nitsch 2015). Systematic variation of parameters may be a connecting factor to foster understanding (cf. Vollrath & Roth 2012). It can for instance be offered by the possibility to use a slider in dynamic geometry applets.

RESEARCH QUESTIONS

Based on the portrayed research findings regarding variants of feedback and in view of the possible learning difficulties in dealing with quadratic functions, it is interesting to investigate the impact on self-rating as well as achievement.

1. Is students' self-rating better if they work with a CBLE including feedback towards the correct solution in combination with explanations, than it is when students receive feedback without explanations (research based on quadratic functions)?

2. Does a computer-based learning environment, including feedback towards the correct solution in combination with explanations, have greater benefit on students' math achievement in comparison to students' achievement when they receive feedback without explanations (research based on quadratic functions)?

In preparation for the main study, in which the research questions above will be examined, a pilot study pays attention to the evaluation and enhancement of the designed CBLE about quadratic functions.

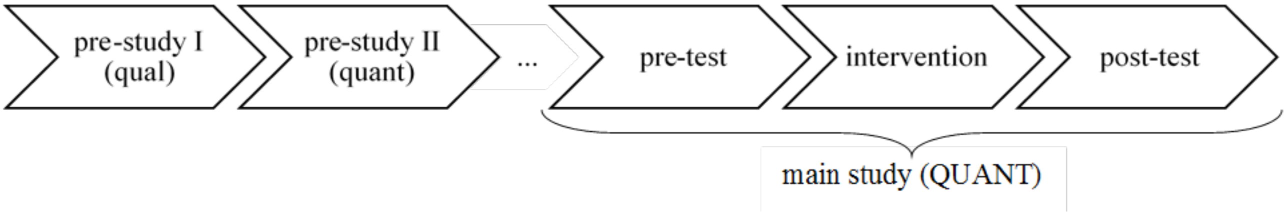
0.1 How do students perceive their self-reliant work with the CBLE? What do they think about the given steering measurements?

0.2 Which metacognitive contents of the CBLE are estimated supportive by students for the process of learning?

DESIGN

In the following sections, two different types of design are described. On the one hand the research design and on the other hand, the underlying thoughts regarding the designed CBLE about quadratic functions are illustrated. The research design, especially the main study design, depicts our plans and might be adapted due to supplementary requirements. Currently we are conducting some preliminary studies with various focuses. Concerning the CBLE design, some general reflections, exemplary variants of integrated feedback and two inserted exercise formats are described in the following paragraph. In our research the definition by Roth (2015), concerning CBLEs based on mediawiki software is used.

Research Design



The research questions already implicate that the main study will focus on different types of feedback, which can be integrated in CBLEs, and especially on their influence on students' math achievement. As a further interest, we want to examine if there is a link between received feedback and self-assessment in the sense of self-rating the own achievement. A quasi-experimental study with a control group design (1x1) is planned. Both the control and the experimental group receive the same CBLE and self-assessment scales. Since the scales have to be in accordance with the CBLE contents, no pre-existing questionnaire can be used and the scales are going to be developed too. The experimental group receives immediate feedback about the correct answer of a task included in the CBLE. This feedback combines KCR with explanations and a kind of MTF. It has to be discussed if the included feedback can be ranked as EF. The control group gets another variant of feedback in CBLEs, comparable to KCR, which mainly means that there are no explanations about the procedure of solution and no prompts to try a task again. Our aim is to examine if the type of feedback in the experimental group is as effective as the theory leads us to think. As mentioned above, KCR has shown similar effects to giving no feedback, whereby the combination with explanations and a multiple-try function may show positive effects on learning (p. 2). But since the risk remains that students use the feedback in a nonreflective way and just copy the prompted results, it is also possible that the experimental group's gain in math achievement stays behind the control group's. To measure the math achievement before and after students' work in the CBLE, we are developing two tests that are connected via anchor items. While in the pre-test the focus lays on functional thinking and linear functions with only a few items on quadratic functions integrated to measure previous knowledge, the post-test contains items measuring functional thinking as well as knowledge about quadratic functions. Those items are designed to be comparable to the pre-test items concerning linear functions. A duration of six school lessons for the intervention is intended between pre- and post-test. Learners are supposed to work on their own respectively in groups of two students with the CBLE. Teachers' role is to be attendant as advisor especially for technical questions, but stay out of students' work in total. Before the conduction of our main study, we are conducting several preliminary studies. On the one hand, we conduct qualitative pre-studies to test and enhance the designed CBLE (pre-study I). One of these studies with focus on students' working experience in the CBLE is described below. Others will be expert interviews as well as monitoring students' while working with the entire CBLE. On the other hand, a quantitative pilot run is planned to check the quality of the used achievement test as well as to get a first impression of what kind of learning progress can be expected in the main study (pre-study II).

Computer-Based Learning Environment *Discover Quadratic Functions*

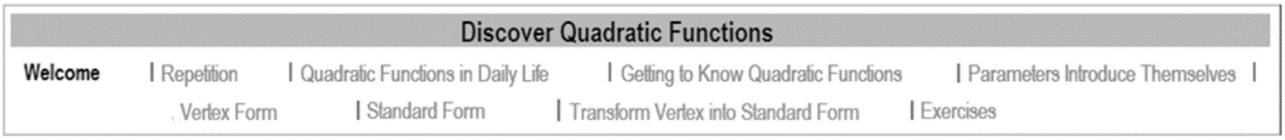


Figure 1. Index of the created CBLE *Discover Quadratic Functions*. (Translated)

At the moment, the CBLE consists of eight chapters shown in figure 1. After a short introduction (*Welcome*), which combines technical instructions with a declaration of required- and goal-competencies, the learners are encouraged to work with the CBLE. If they are not sure about whether they already possess the needed competencies, they have the opportunity to work on tasks that repeat basic knowledge (*Repetition*). Otherwise, they may enter the next chapters *Quadratic Functions in Daily Life* and *Getting to Know Quadratic Functions*. The formers thematic priority is motivation, whereby the latter introduces simple quadratic functions ($f(x) = x^2$). The next step is to work with parameters (*Parameters Introduce Themselves*) and thus to discover the *Vertex Form.* Afterwards the *Standard Form* is thematised as well as the proceeding to *Transform Vertex into Standard Form.* The CBLE ends with further *Exercises* about all included subjects. Within one chapter, the learners can decide about the order of the exercises and the time they spend on each. Regarding the interactive tasks, a repetition is possible too (cf. figure 2). Furthermore, within some application tasks the learners can adapt the level of difficulty (cf. figure 3). Nevertheless, since this CBLE introduces a new subject area for learners, the flexibility is limited in contrast to CBLEs that are built for e.g. repetition. Besides to working at the computer, students are needed to write into a notebook while working with the CBLE. Some tasks explicitly ask for paper-pencil work ensuring that learners do not forget how to draw a graph by hand for example. In addition, students may gather mnemonic sentences or complete self-assessment scales in there.

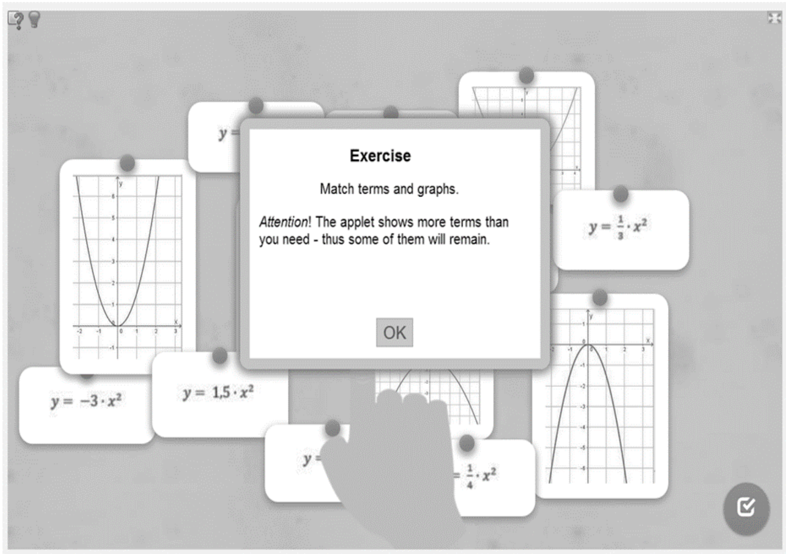


Figure 2. Exemplary inner-mathematical task about matching terms and graphs. (Translated)

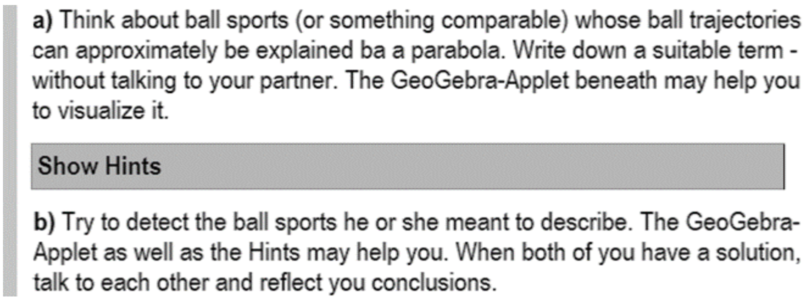


Figure 3. Exemplary application task about quadratic functions. During the exercise, students need to work in pairs. (Translated)

There are different exercise formats integrated in the CBLE. Besides being interactive or not, the exercises can be distinguished into inner-mathematical or application tasks. The inner-mathematical tasks serve as an introduction to get to know the subtopics of quadratic functions and to consolidate new skills. The application exercises are included to deepen students' knowledge. In figure 2, an inner-mathematical task of the CBLE is shown. It exposes an interactive applet in which the learners should match quadratic terms and parabolas. After they have finished, they can check their results by clicking the button downright. Correct answers are marked green, wrong ones red. If no matched pairs remain, their solution is omitted. This exercise is also exemplary for allowing and encouraging multiple tries. Application tasks need more skills than draw on taught issues. Figure 3 for example requires knowledge about how to modify quadratic functions and creativity as well. The shown exercise is about finding a suitable term for a freely chosen ball sports. In the following tasks the learners exchange terms with a partner who tries to detect the underlying sports. Last step is to explain ones decisions and to reflect it together in pairs.

Since the CBLE is built to investigate the impact of feedback during the main study, figure 4 shows different variants of feedback which are applied. The rationale behind choosing exactly these feedback variants are mainly due to the possibilities delivered by *ZUM-Wiki*. On the one hand, interactive applets are bound. As mentioned above, this kind of tasks has a control button whose activation shows if the entered solution is right or should be reconsidered (figure 4a). On the other hand, hints and solutions are integrated. They are hidden until the learners activate them by mouse click (figure 4b). For further information about the concrete kinds of feedback behind it and the rationale for using these variants, see paragraph "Research Design" (p. 4).

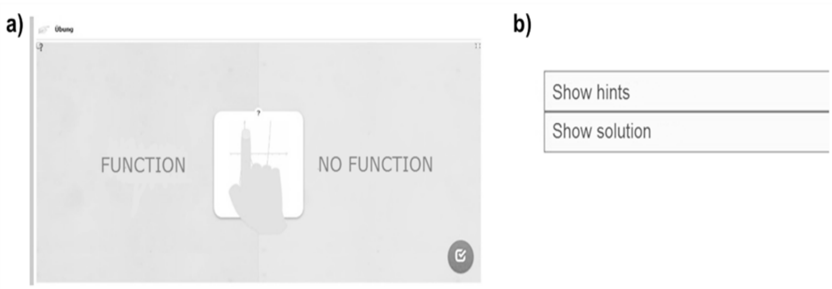


Figure 4. Exemplary variants of feedback that are included in the designed CBLE: a) Bound interactive applet with a control button. b) Hidden help and solution. (Translated)

PILOT STUDY

In preparation for the main study, the designed CBLE is tested. The portrayed pre-study is qualitative with the aim to evaluate and enhance the designed learning environment (pre-study I). It has been performed in cooperation with Sur (2017) in the course of his master thesis at the University of Münster. Six ninth grade students of a high school (Gymnasium) in North Rhine-Westphalia have participated within this study. They worked for approximately 45 minutes with the CBLE chapter *Vertex Form* [1] and were afterwards questioned in guided interviews with a duration of about 15 minutes each. The interviews were transcribed and coded with MAXQDA. The coding was based on summarizing content analysis according to Mayring (2010), whereby the used procedure can be declined to have a focus on inductive coding with some deductive approaches.

The guided interviews contained questions related to research questions no. 0.1 and 0.2 (p. 3). The questions were verbalised in an open way that animated the students to speak freely. For example, they were asked: "You have now worked on your own with the CBLE for 45 minutes. Which parts of the CBLE supported your work? a) How did you use it? b) In which way did it support your working process?” (Sur 2017, p. XV, translated).

FINDINGS AND DISCUSSION

Students’ comments were summarized to the intended categories *steering measurements and participant activity* (combined to the generic term *self-reliance*) and *metacognition*. These two categories concern research questions no. 0.1 and 0.2 and findings relying on them are herein depicted in brief. Since the results of the pilot study are organised in categories, their depiction will follow the same structure. First some remarkable citations are shown (translated; names modified). Afterwards positive aspects the students’ named as well as some of their supplementary remarks are summarized. With reference to the rationale for research questions 0.1 and 0.2, each paragraph has integrated some exemplary résumé on how the CBLE will be revised due to the presented results. As final remark it should be emphasized that the shown findings just express the individual meaning of a small group of students (n=6). There is no aspiration to generalize them, but they serve as a starting point for a complete evaluation of the developed CBLE. Besides they are going to be supplemented by expert interviews.

Self-Reliance

Isabell: I liked to work self-reliantly and yes it is something different from only being present in classroom and absorb thinks like a sponge.

Felix: This partner work. That was good; it was not working all by myself, but to have the possibility to compare how others work.

According to their own statements, students liked to work actively with the contents of the CBLE. They also welcomed the variability of some tasks, for example, when it was their turn to choose three of five pictures to work with (according CBLE chapter [1], exercise 1). Occasionally, students wished to be more assisted by the teacher, especially at the beginning. Tasks, which included the need to work in pairs accommodated the students and have been highlighted (cf. Felix’ citation). In addition, the immediate feedback was mentioned as being helpful. One of the students remarked that she felt pressed for time.

With reference to the last point, we are going to provide a weekly schedule in future. It is to be hoped that this overview will facilitate time-management. Assistance by teacher is difficult to manage during the quantitative study, because of non-evaluable influences. A more detailed briefing in combination with extended CBLE-usage, as well as the according weekly schedule may perhaps promote the students’ self-reliant working processes. Since the students are going to work self-reliantly for a long time during the main study, it may be thought of organizing them in pairs for the whole time.

Metacognition

Mia: I liked the possibility to self-control my results.

Marcus: I see slight risks because of the integrated feedback. Perhaps one looks immediately for the solution.

Metacognitive components of the CBLE are the integrated transparent goals, self-assessment scales, hints and suggested solutions. Students mainly underlined the transparent goals. They named them helpful for understanding why to work on the following tasks. The self-assessment scales have been presented as open questions, which is why some of the students have had some phrasing difficulties as they stated. Nonetheless, this activity was highlighted assistant for the process of learning and for teachers' acknowledgement as well. Students in this study made not much use of hints. A probable reason is that the learners already knew quadratic functions and used the CBLE chapter for repetition. However, they remarked that the hints might be an enrichment for students who are getting to know quadratic functions. Finally, students had different opinions on the suggested solutions. Most students liked the possibility to self-control their results (cf. Mia's citation), but saw associated risks as well (cf. Marcus' citation). According to the students' statements, detecting mistakes was facilitated because of the retrievable feedback.

Regarding these findings, we are going to transfer the self-assessment scales into a closer format, e.g. a checklist. On the one hand, we thereby hope to foster the students' self-rating activity. On the other hand, it will also be easier to analyse the filled scales by statistical means. The mentioned risk, namely using the feedback unreflectively, is exactly what we want to investigate in our main study.

SUMMARY AND PROSPECT

Within this paper, we reported on preparations we made for an upcoming study about the impact of different kinds of feedback integrated in a CBLE about quadratic functions on students' math achievement and self-assessment. Framed by the theoretical background, we drew up our research questions and informed about the planned main study as well as about several pre-studies. Furthermore, design choices concerning the content of the CBLE and the included feedback variants have been presented. Thus far, we conducted a pilot study with the aim to evaluate and enhance the CBLE (pre-study I). Based on the findings of the reported pilot study and to gain more information about how our adaptions and assumptions work in total, further qualitative pre-studies will accompany the quantitative pilot-run (pre-study II) which will take place in November 2017.

NOTES

1. See ZUM-Wiki link: https://wiki.zum.de/wiki/Quadratische_Funktionen_erkunden/Die_Scheitelpunktform (in the version of 2016-11-29), only available in German language.

REFERENCES

Attali, Y. (2015). Effects of multiple-try feedback and question type during mathematics problem solving on performance in similar problems. *Computers & Education, 86*, 260–267.

Baker, R.S.J.d., D'Mello, S.K., Rodrigo, M.M.T., Graesser, A.C. (2010). Better to Be Frustrated Than Bored: The Incidence, Persistence, and Impact of Learners' Cognitive-Affective States During Interactions with Three Different Computer-Based Learning Environments. *International Journal of Human-Computer Studies, 68*(4), 223–241.

Balacheff, N., Kaput, J.J. (1996). Computer-Based Learning Environments in Mathematics. In A.J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, C. Laborde (Eds.), *International Handbook of Mathematics Education: Part 1* (pp. 469–501), doi:10.1007/978-94-009-1465-0_14.

Bangert-Drowns, R.L., Kulik, C.-L.C., Kulik, J.A., Morgan, M.T. (1991). The Instructional Effect of Feedback in Test-Like Events. *Review of Educational Research, 61*(2), 213–238, doi:10.3102/00346543061002213.

Black, P., Wiliam, D. (1998). Assessment and Classroom Learning. *Assessment in Education: Principles, Policy & Practice*, *5*(1), 7–74, doi:10.1080/0969595980050102.

Boud, D., Molloy, E. (2013). Rethinking models of feedback for learning: the challenge of design. *Assessment & Evaluation in Higher Education, 38*(*6*), 698–712.

Dempsey, J. V., Driscoll, M. P., Swindell, L. K. (1993). Text-based feedback. In J. V. Dempsey and G. C. Sales (Eds.), *Interactive instruction and feedback* (pp. 21–54). Englewood Cliffs, NJ: Educational Technology.

Publications.Doorman, M., Drijvers, P., Gravemeijer, K., Boon, P., Reed, H. (2012). Tool Use and the Development of the Function Concept: From Repeated Calculations to Functional Thinking. *International Journal of Science and Mathematics Education, 10*(6)*,* 1243–1267.

Greefrath, G., Oldenburg, R., Siller, H.-S., Ulm, V., Weigand, H.-G. (2016). *Didaktik der Analysis. Aspekte und Grundvorstellungen zentraler Begriffe.* Berlin and Heidelberg: Springer Spektrum.

Hattie, J.A.C. (2009). *Visible learning: A synthesis of 800+ meta-analyses on achievement.* Abington: Routledge.

Isaacs, W., Senge, P. (1992). Overcoming Limits to Learning in Computer-Based Learning Environments. *European Journal of Operational Research, 59*(1), 183–196, doi:10.1016/0377-2217(92)90014-Z.

Kulhavy, R. W., Stock, W. A. (1989). Feedback in Written Instruction: The Place of Response Certitude. *Educational Psychology Review, 1*(*4*), 279–308.

Kulhavy, R. W., Wagner, W. (1993). Feedback in programmed instruction: Historical context and implications for practice. In J. V. Dempsey & G. C. Sales (Eds.), *Interactive instruction and feedback* (pp. 3–20). Englewood Cliffs, NJ: Educational Technology Publications.

Malle, G. (2000). Zwei Aspekte von Funktionen: Zuordnung und Kovariation. *mathematik lehren, 118*, 57–62.

Mayring, P. (2010). *Qualitative Inhaltsanalyse. Grundlagen und Techniken* (11., revised edition). Weinheim and Basel: Beltz.

Mory, E. H. (2004). Feedback research revisited. In D.H. Jonassen (Eds.), *Handbook of Research on Educational Communications and Technology: Second Edition* (pp. 745-783). Taylor&Francis.

Nicol, D. J., Macfarlane-Dick, D. (2006). Formative assessment and self-regulated learning: a model and seven principles of good feedback practice. *Studies in Higher Education, 31*(*2*), 199–218.

Niegemann, H. M.; Domagk, S.; Hessel, S.; Hein, A.; Hupfer, M., & Zobel, A. (2008). *Kompendium multimediales Lernen*. Heidelberg: Springer.

Nitsch, R. (2015). *Diagnose von Lernschwierigkeiten im Bereich funktionaler Zusammenhänge. Eine Studie zu typischen Fehlermustern bei Darstellungswechseln.* Wiesbaden: Springer Spektrum.

NRW Ministry for Schools and Further Education (2007). *Kernlehrplan für das Gymnasium – Sekundarstufe I (G8) in Nordrhein-Westfalen. Mathematik.* Retrieved from http://www.schulentwicklung.nrw.de/lehrplaene/upload/lehrplaene_download/gymnasium_g8/gym8_mathematik.pdf.

Roth, J. (2015). Lernpfade: Definition, Gestaltungskriterien und Unterrichtseinsatz. In J. Roth, E. Süss-Stepancik & H. Wiesner (Eds.), *Medienvielfalt im Mathematikunterricht. Lernpfade als Weg zum Ziel* (pp. 3–25). Wiesbaden: Springer Spektrum.

Sadler, D. R. (1989). Formative assessment and the design of instructional systems. *Instructional Science, 18,* 119–144.

Shute, V. J. (2008). Focus on Formative Feedback. *Review of Educational Research, 78*(1), 153–189, doi:10.3102/0034654307313795.

Sur, C. (2017). *Selbstgesteuertes Lernen mit Lernpfaden im Mathematikunterricht aus Sicht der Lernenden – Eine qualitative Untersuchung auf Basis von Interviews.* Unpublished master thesis, University of Münster.

Taras, M. (2003). To Feedback or Not to Feedback in Student Self-Assessment. *Assessment & Evaluation in Higher Education, 28*(5), 549–565, doi:10.1080/0260293032000120415.

Vollrath, H.-J., Roth, J. (2012). *Grundlagen des Mathematikunterrichts in der Sekundarstufe* (2. edition). Heidelberg: Springer Spektrum.

Vollrath, H.-J. (1989). Funktionales Denken. *Journal für Mathematikdidaktik, 10*(1)*,* 3–37.

Wiesner, H., Wiesner-Steiner, A. (2015). Einschätzungen zu Lernpfaden – Eine empirische Exploration. In J. Roth, E. Süss-Stepancik & H. Wiesner (Eds.), *Medienvielfalt im Mathematikunterricht. Lernpfade als Weg zum Ziel* (pp. 27–45). Wiesbaden: Springer Spektrum.

Zaslavsky, O. (1997). Conceptual Obstacles in the Learning of Quadratic Funtions. *Focus on Learning Problems in Mathematics Winter Edition, 19*(1), 20–44.

EXPLOITING POTENTIALS OF DYNAMIC REPRESENTATIONS OF FUNCTIONS WITH PARALLEL AXES

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The concept of function has a central role both at school and in everyday situations. Several studies revealed that it is hard for students to think of functions and graphs in terms of covariation and this could contribute to their struggles in Calculus. The emergence of available technologies has fostered new teaching and learning approaches to overcome students' difficulties and some of them concerns the use of dynamic algebra and geometry software programs to experience the dependence relation and to explore functions as covariation. In this paper we describe a particular representation of functions with parallel axes and the analysis of a protocol in which four students work together on a problem that involves the exploration of a function represented in a dynamic interactive file. The analysis has been carried out to explore the potential of the proposed dynamic representation of functions that incorporates the semantic domain of space, time and movement.

Keywords: dynamic algebra and geometry software, dragging, function.

INTRODUCTION

The concept of function is very important both in secondary school and university mathematics but it also has a central role in everyday situations. For a long time, this notion has been at the core of several studies in mathematics education, and a rich literature has revealed students' difficulties in understanding the concept in all its aspects (Vinner & Dreyfus, 1989; Tall, 1991; Dubinsky & Harel, 1992). Difficulties in interpreting the dependence relation as a dynamic relation between covarying quantities are widely reported (Goldenberg et al., 1992; Carlson et al., 2002) and also difficulties in manipulating graphs and recognizing functions' properties from graphs (Carlson & Oehrtman, 2005).

Indeed, the tendency to think of functions and graphs as static objects, rather than as dynamic processes, may contribute to students' struggles in the learning of Calculus (Ng, 2016). At the same time the emergence of a variety of new available software has fostered new teaching and learning approaches. Therefore, we can find several studies about the use of technology to manipulate multiple representations of functions (Healy & Sinclair, 2007; Sinclair et al., 2009).

Falcade et al. (2007) suggest that the use of a dynamic algebra and geometry software, such as GeoGebra, allows students to experience functions as covariation, that is a crucial aspect of the idea of function (Confrey & Smith, 1995; Tall, 1996). According to these assumptions we are interested in studying students' cognitive processes involved in working with functions represented in a dynamic environment.

In this paper we describe a particular dynamic representation of functions and some results from a pilot study conducted last year. This study is part of a larger research project whose focus is investigating how certain aspects of the mathematical concept of function could be supported by such dynamic representation. Moreover, we are interested in exploring the semiotic potential of the representation of functions with parallel axes (Bartolini Bussi & Mariotti, 2008), to gain insight into how to exploit it didactically.

DYNAGRAPHS

Dynagraphs, as they have been referred to by Goldenberg et al. (1992), are particular representations of functions obtained by using a dynamic software, which consist in representing both the x- and y-axes horizontally, in one dimension, unlike the Cartesian graphs which represent functions in two dimensions. The underlying assumption is that this kind of representation can support a dynamic conception of functions because it draws attention to variables' variations and movements and to the relation between these variations.

We now propose a description of our development of this idea, implemented within the algebra and geometry software GeoGebra. In particular, we designed a sequence of activities aimed at making the representation of functions in the Cartesian plane rich in meanings. We start with a kind of dynagraph and evolve its design, through a sequence of activities, in order to reach the Cartesian graph.

As we can see in Figure 1 the first dynagraph has one horizontal line, with 0 and 1 marked, and two little ticks that can move on it in this way: one of them represents the independent variable and can always be dragged, the other one represents the dependent variable, it cannot be directly dragged, but it moves depending on the movements of the independent tick. We note that the variables are represented by ticks and not by points, because a point is usually seen as a pair of coordinates, while a tick better expresses the idea of "value". Moreover, there are two points marked on the line that determine the unit segment, to highlight that it is the real number line.

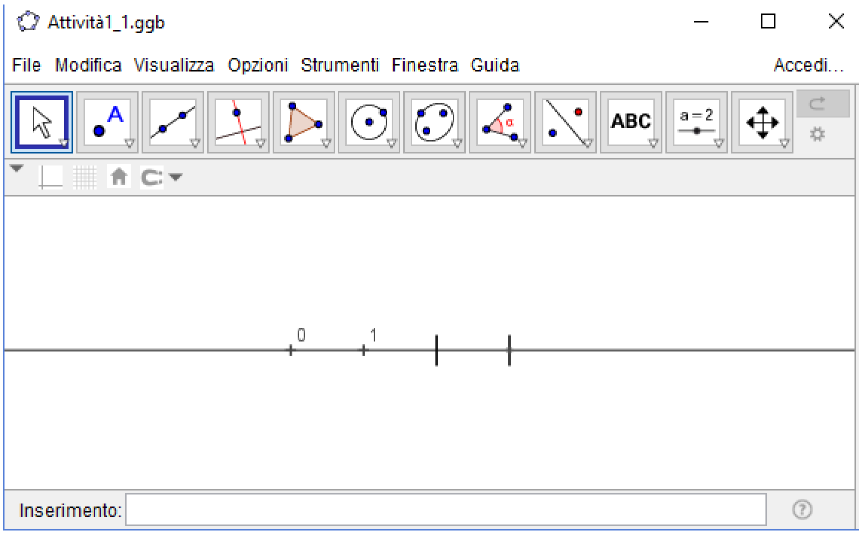

Figure 1. Dynagraph

The design of our representations also allows to separate out the two variables, that is, to create a copy of the line in order to have one notch on each line. So what can be seen on the screen changes because there is a fixed horizontal line, representing the x-axis, and its double, representing the y-axis, that can be dragged up and down maintaining the parallelism, and the alignment of the origin.

Thanks to the dragging and the design of the dynamic files there is the opportunity to rotate the y-axis, joining the zeros and making it orthogonal to the x-axis; obtaining a representation that includes the Cartesian axes on which two ticks can move. As described above, the tick on the x-axis can be directly dragged while the other one moves depending on it. The following step for the construction of the Cartesian graph of a function consists of the construction of the point (x, f(x)) and, finally, by activating the trace tool on this point and dragging the independent variable we obtain the graph, as showed in Figure 2. In the rest of this paper we will only discuss activities with parallel axes, investigating their semiotic and didactic potential.

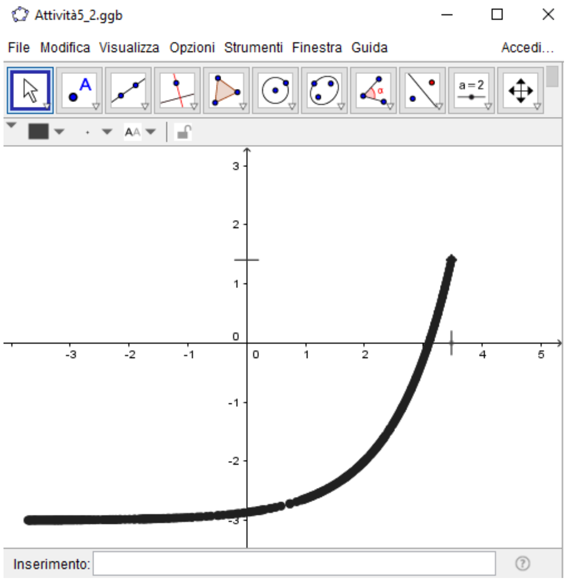

Figure 2. Cartesian graph

PILOT STUDY

As we mentioned above, a first experimentation was conducted last year in a 10th grade of an Italian High School for Math and Science, where we introduced students to the function concept through dynagraphs. Students worked in pairs on pre-designed dynamic interactive files that they were asked to explore. The tasks proposed in these files were open, in order to support students' explorations, and working in pairs was to foster their speaking aloud and explaining their reasoning to each other. Lessons were video-recorded through two cameras.

The foundational goal of the pilot study was to build the mathematical meaning of functional dependence, as a relation between two covarying quantities: one depending on the other one. We expected to start from the relation between the movement of the two ticks bounded to the lines.

Starting from the representation of function on one horizontal line, we designed a sequence of activities that led to the Cartesian graph of functions, following a trajectory like the one described in the previous section. We proposed several examples, including not everywhere defined functions and discontinuous functions, in order to support the production of situated signs related to the mathematical concepts of domain, limit, continuity and asymptote. We also expected that this kind of one-dimensional representation would foster the description of relative movements of the ticks and comparisons between possible walks followed by the ticks on the lines. For example, students could recognize symmetry or concordant movements that we would identify as situated signs for monotonicity's properties of functions. Speaking about advanced mathematical concepts, we also expected that a description of change in speed could be read mathematically as an attention to the slope of the function, that is its derivative.

Our choice to let the user decide to see two distinct lines or to have them overlap is led by the observation that we think there could be some cases for which it is convenient to have two separated axes (for example to explore functions like $f(x) = \sqrt{x}f(x) = \sqrt{x}$ or $g(x) = |x|g(x) = |x|$) and some other cases in which it is easier to work with one line with two variables moving along it (for example to determine $f(x) = xf(x) = x$).

Analysis of an activity

The Theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008) describes the semiotic potential of an artifact as follows:

> On the one hand, personal meanings are related to the use of the artifact, in particular in relation to the aim of accomplishing the task; on the other hand, mathematical meanings may be related to the artifact and its use. This double semiotic relationship is named the *semiotic potential of an artifact*.

According to this definition we analyze the semiotic potential of the representation of functions with parallel axes focusing on the embedded knowledge and the utilizations schemes that students employ when exploring the dynamic files.

In the first lesson of the pilot study, after the exploration of a linear function, students are asked to explore the dynagraph of the function $f(x) = \frac{1}{x-4}f(x) = \frac{1}{x-4}$ and to write down their observations. We chose to give them this not everywhere defined function in order to introduce them to the mathematical concept of domain, while exploring the dependence relation between the two variables. We expected that the particular behaviour of the dependent variable in a neighborhood of the vertical asymptote $x = 4x = 4$ could have supported the employment of a language referring to the movement and to the relation between the movements of the two ticks. Moreover we expected that students would have noticed the existence of the horizontal asymptote $y = 0y = 0$ in terms of changing in speed of the tick representing the dependent variable.

In the next sections we analyze some excerpts from this lesson with the goal of recognizing instances in which the semiotic potential of dynagraphs seems to be exploited.

Excerpt 1

The following transcript is a dialogue between four students who are interacting with a GeoGebra file that represents the dynagraph of the function $\frac{1\ \ 1}{x-4x-4}$. We chose this excerpt from the first lesson because in it students frequently use words that refer to variables' movements and to the relation between these movements. Moreover, as we expected, the function's behaviour in a neighborhood of the vertical asymptote $x = 4x = 4$ causes students' astonishment and some interesting observations.

1	Gian:	Oh no, it is going crazy!
2	Fra:	Look there, it dashes backwards
3	Gian:	It makes certain leaps!
4	Fra:	Ah, but are they three points here?
5	Dar:	What? Here there is back to the future!
6	Fra:	Eh eh, there are three points guys
7	Rob:	No
8	Fra:	Or not?
9	Gian:	This one doesn't move, and the meeting point is the same, it doesn't change.

10	Fra:	No no they are two, indeed I tried to make some changes but they are equal, actually they are the same

As we can read from the dialogue, the discontinuity of the function is something very interesting for these students, because when they drag the independent variable they see the dependent one disappear from one side of the screen and then re-appear from the other side of the screen. They try to interpret this phenomenon by using a "continuous" interpretation. Fra supposes that there could be three points, possibly because he does not accept that one point can run off on one side and come back from the other side. But another interesting fact is that he modifies the tick representing the dependent variable in order to convince himself that the points are two and not three: dragging the independent tick he always sees the same output. Therefore the feedback is directly given by the software, and by useful manipulations made on the file.

Let us now look at a sentence that we consider as a first sign, situated in the context of the dynamic file, of the mathematical concept of domain of the function. It is important to observe that the representation of the function with parallel axes requires the following interpretation of the domain: this needs to be read on the y-axis because the independent variable can always be dragged, bounded to its line. So we could say that a point on the x-axis belongs to the domain of the function if it has a corresponding output on the y-axis.

As we can read from Rob's words there are different aspects of the semiotic potential of the representation of the function that come to light. Indeed, he refers to time (after a moment), to space (upper, below) and to movement (a range of movement).

24	Rob:	After a moment, the upper point moves only in a certain range of movement of the point below.

The next excerpt concerns a description of the asymptotic behavior of the function when x tends to infinity.

Excerpt 2

71	Fra:	But do you see how it dashes away? Look!
72	Dar:	Try to move a bit further backwards, look, it still moves very little.
73	Fra:	It continues to move
74	Dar:	Do you see? it moves a little bit
75	Fra:	Yes, it is moving a little bit
76	Dar:	Look, it moves here
77	Rob:	Nothing is moving, where do you see that it moves?
78	Fra:	It moves you're right, yes
79	Rob:	No, here it does not move
80	Fra:	Yes Rob it moves, look!
81	Rob:	Zoom in zoom in, so we can see it. And then it makes certain leaps…
82	Fra:	It leaps it leaps!
83	Rob:	Look, it has leapt to one side
84	Fra:	And then it stops
85	Rob:	That's it, from here on it is fixed, look

Recalling the verb (to dash away) used previously as well, Fra underlines the unexpected acceleration of the dependent variable (71). Then the other students observe that $f(x)f(x)$ makes some leaps (81) when dragging the x in a neighborhood of the point where the function is not defined. The semiotic potential of this dynagraph comes into play in the mathematical concept of derivative: by dragging the x-tick in a neighborhood of $x = 4$ the $f(x)f(x)$-tick leaps, which corresponds to a function having a very high slope.

Then students discuss about the function's behavior for x tending to negative infinity. Dar suggests that $f(x)f(x)$ still moves when x is dragged backwards (72), that is x tending towards negative infinity; and Fra agrees (82). But Rob prefers to zoom in because it seems to him the x-tick to be fixed and he would convince himself of the contrary. Again the semiotic potential of the dynagraph comes into play, supporting with respect to the mathematical concept of limit; aspects of such potential can be observed in students' words and actions. In particular, zooming in students can observe the function's behavior for ever smaller variations of the independent variable.

In the last sentence (85) Rob refers to x values bigger than zero and far from it, and we are sure of it because we see from the video that he is dragging the x-tick to the right on its line.

Before the discussion we would just notice that the analysis of the two excerpts reveals some interesting considerations consistent with the *a priori* analysis of the designed activity.

DISCUSSION

The studies on the interaction between humans, technology and mathematics have to take into account a variety of aspects: the relation between the teacher and the technology in the mediation of mathematical knowledge, or how this knowledge is influenced by constraints and actions allowed in the technological environment, and several other components that are involved. In this paper we have presented a study to better understand the explorations of functional dependence in a dynamic algebra and geometry environment. In particular, we have analysed aspects of the semiotic potential of the representation of functions with parallel axes, presenting some excerpts from a pilot study conducted last year.

We noticed that students' descriptions of dynagraphs are rich in references to movement, time and space. We think that it could be fostered by the dynamic environment, by the task that requires for the exploration, and by the possibility of dragging. However such richness could also be affected by the fact that the students never met the concept of function (in high school) before, so they have not yet developed a formal mathematical vocabulary about functions, so the use of these terms becomes necessary for them.

From the analyses we can infer that introducing students to functions through dynagraphs seems to promote a covariational view of functions, seen as relations between the movements of quantities that are varying in an interval of real numbers. In the same way also some mathematical properties of functions are conceived dynamically, for example Rob identifies the domain of the function as a certain range of movement of the independent variable. Consistent with our expectation about this kind of one-dimensional representation, that it would have fostered the description of relative movements of the ticks and comparisons between possible walks followed by the ticks on the lines, we also notice students frequent use of verbs strictly related to movement and speed (2, 3, 9, Excerpt 2).

Finally, we highlight students' creativity, revealed by their use of the tools offered by the software, for example Fra changes the $f(x)f(x)$-tick's visualization and Rob zooms in to convince himself that $f(x)f(x)$ moves on.

In a future study it could be interesting to investigate whether students' conceptions of functions evolve, and if so how. In particular we are interested in analyzing students' use of references to movement and time when they are taught the mathematical definitions: do they disappear or do they last? How do students deal with these dynamic terms together with the static definition of function?

Along the lines of the design of these activities, it could be interesting to design some new activities concerning other properties of functions in order to gain a deeper insight into possible exploits of the semiotic potential of functions' representation with parallel axes.

REFERENCES

Bartolini Bussi M. G., & Mariotti M. A., (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English et al. (Eds.), *Handbook of International Research in Mathematics Education,* second edition, (pp. 746-783). New York and London: Routledge.

Carlson M., Jacobs S., Coe E., Larsen S., & Hsu E., (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378.

Carlson M., & Oehrtman M., (2005). Key aspects of knowing and learning the concept of function. Research Sampler 9. MAA Notes.

Confrey J., & Smith E., (1995). Splitting, covariation and their role in the development of exponential function. *Journal for Research in Mathematics Education, 26,* 66–86.

Dubinsky E., & Harel G., (1992). The nature of the process conception of function. In G. Harel and E. Dubinsky (Eds.), *The concept of function: aspects of epistemology and pedagogy*, (pp. 85–106). MAA Notes.

Falcade R., Laborde C., & Mariotti M.A., (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics*, *66*, 317–333.

Goldenberg E. P., Lewis P., & O'Keefe J., (1992). Dynamic representation and the development of an understanding of functions. In G. Harel and E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy, 25*. MAA Notes.

Healy L., & Sinclair N., (2007). If this is your mathematics, what are your stories? *International Journal of Computers for Mathematics Learning*.

Ng O., (2016). Comparing calculus communication across static and dynamic environments using a multimodal approach. *Digital Experiences in Mathematics Education*.

Sinclair N., Healy, L., & Reis Sales C., (2009). Time for telling stories: Narrative thinking with Dynamic Geometry. *ZDM*, *41*, 441–452.

Tall D., (1991) The psychology of advanced mathematical thinking, In: D. Tall (Eds.) *Advanced Mathematical thinking,* 3–21. Dordrecht: Kluwer.

Tall D., (1996). Function and Calculus, In: A.J. Bishop et al. (Eds.) *International Handbook of Mathematics Education,* (pp. 289–325). The Netherlands: Kluwer Academic Publishers.

Vinner S., & Dreyfus T., (1989). Images and Definitions for the Concept of Function. *Journal for Research in Mathematics Education*, *20*(4), 356–366. National Council of Teachers of Mathematics.

REASONING STRATEGIES FOR CONJECTURE ELABORATION IN DGE

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The present study analyzes students' reasoning strategies for elaboration of conjectures when working in a Dynamic Geometry Environment (DGE). We observed 18 pairs of ten-graders in a private school in Lebanon, while working on open geometrical proof problems using Dynamic Geometry Software (DGS), namely GeoGebra. The analysis revealed three reasoning strategies employed by students. In the first two, the students worked on satisfying the presumed premise of the conjecture in the figure and identifying / validating the conclusion, either by observing the figure at hand (strategy 1) or by dragging the figure to validate the conclusion across different instances (strategy 2). Conversely, in the third strategy, the students worked on satisfying the conclusion of the conjecture in the figure and observing it to identify the premise that corresponds. Each strategy entails the use of different construction tools and types of constructions which affect the correctness of the resulting conjecture.

Keywords: secondary – conjecture – dynamic geometry – reasoning

SETTING THE CONTEXT

The proving process involves two sub-processes: conjecture elaboration and proof development. These two processes become particularly more explicit in Dynamic Geometry Environments (DGEs) since the nature of the first process in DGE is radically different from pencil-and-paper environments, which consequently affects the way the second process evolves. Dynamic draggable constructions strongly affect the proving process by mediating the type of conjectures developed (Sinclair & Robutti, 2012).

Extensive research (Hölzl, 2001; Laborde & Sträßer, 2010; Laborde & Laborde, 2011; Laborde, 2005) has been conducted on the potentialities of the dragging tool, which resulted in understanding this tool as a pedagogical tool conducive to mathematical reasoning (Jones, 1998), particularly in the process of conjecture formation in geometry. The epistemic potential of the dragging tool lies in its relationship with the discernment of invariants (Leung, Baccaglini-Frank, & Mariotti, 2013). According to Mariotti (2014), dragging acts as a mediator between geometrical invariants and logical statements. In fact, dragging to elaborate a conjecture is a complex process as it requires the interpretation of perceptual data by analyzing the image in order to identify a geometrically significant relationship between its elements and properties. For example, when dragging to search for consequences, students need to interpret the geometrical dependence between direct invariants (i.e. invariant properties observed between independent elements) and indirect invariants (i.e. invariant properties observed between dependent elements) as the logical dependence between the premise and the conclusion of a conditional statement.

Given that in the literature the primary focus in the process of conjecture elaboration has been on the role of dragging, this study aims at a more comprehensive analysis of the process of conjecture elaboration within DGE by shifting the focus from the dragging tool to include the different reasoning strategies, construction tools and types of constructions employed by students. The study consisted of a series of observations of 18 pairs of ten-graders (15-17 years old) who worked on open geometrical proof problems within a DGE, namely Geogebra. Data were collected using video-recording and collection of materials, including any paper trace (sketches, scribblings, proof formulations) generated by students, together with their GeoGebra files. This paper presents students' work on two problems previously used by Olivero (2002) and Arzarello et. al. (2002) respectively:

Problem 1 (P1) (Olivero, 2002)

1) Let ABCD be a quadrilateral. Consider the bisectors of its internal angles and the intersection points H, K, L, and M of pairs of consecutive bisectors.
2) Drag ABCD, considering different configurations, and explore how HKLM changes in relation to ABCD.
3) Write down conjectures and prove them.

Problem 2 (P2) (Arzarello et. al., 2002)

Given a triangle ABC, consider P the midpoint of [AB] and the two triangles APC and PCB. 1) Explore the properties of the triangle ABC which are necessary so that both APC and PCB are isosceles (in this case, the triangle ABC is called "separable").

2) Write down conjectures and prove them.

STRATEGIES FOR CONJECTURE ELABORATION

Through observing and analyzing the students' work, it was possible to identify three reasoning strategies that were used for conjecture generation and that describe the way students moved from exploring the changes of the dynamic geometrical figure to identifying relational properties of the figure. A sentence is considered to be a conjecture when stated in the form "If… then…." the first part (if…) is referred to as premise and the second part (then…) as conclusion. Upon analyzing students' work, the following strategies for conjecturing were identified.

Strategy 1 – Forward Static Observation (FSO)

This strategy consisted in building a figure and satisfying the presumed premise, then observing a static instance of the figure to conclude the conjecture. Students construct the figure while incorporating the properties provided in the presumed premise, then observe, without any manipulation, a single static instance of the figure obtained, to identify invariants and to formulate a conjecture. The constructed figure may be robust or visually adjusted.

The inferences made while using the FSO strategy are of two types:

- *Type 1 inference – visual verification*: When the conclusion was a priori known by the students, i.e. provided by the statement of the problem (as in problem 2 where students knew they had to find a separable triangle ABC), students incorporated the presumed premise and observed whether the single instance of the geometric figure obtained met the conclusion. In such a case, a conjecture was developed; if not, then the premise was rejected based on the counterexample.
- *Type 2 inference – visual speculation*: When the conclusion was not provided within the problem (as in problem 1 where students did not know which shape of HKLM they will obtain), then, after incorporating the presumed premise, students identified the conclusion based on a single instance of the geometric figure and formulated a conjecture.

The following examples illustrate different cases of use of the FSO strategy for conjecturing. The code between parentheses refers to the problem being solved (P1 or P2).

Example 1 (P1). Many students who were working on problem 1 started by investigating the case - premise: "If ABCD is a square". They constructed a robust square using the *Regular Polygon* tool (Figure 1). Not knowing what the conclusion should be (type 2 inference), they observed the figure and developed the conjecture: "If ABCD is a square then H, K, L and M coincide".

Even though the conjecture was based on the mere observation of a single figure, the fact that it was a robust construction provided greater validity and reliability for the conjecture.

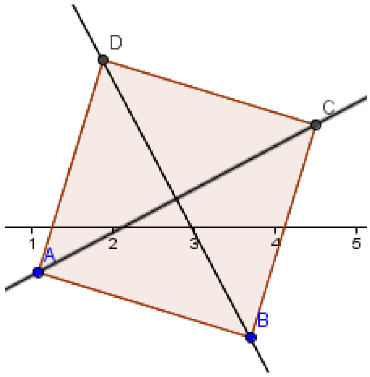

Figure 1. Constructing a robust square and observing a static drawing to conclude

Example 2 (P1). Another pair of students constructed a parallelogram using the Parallel Line tool, then the angle bisectors of the angles and dragged A, B and D to form a square by visual adjustment (Figure 2). Similarly to example 1, they did not know what the conclusion should be (type 2 inference), so they observed the figure and developed the following conjecture: “If ABCD is a square then HKLM is also a square”.

However, in contrast to example 1, their conjecture was based on a single instance of a soft figure obtained by visual adjustment, in which case the drawing was inaccurate, yielded a rectangle instead of a square, and led to an incorrect conjecture.

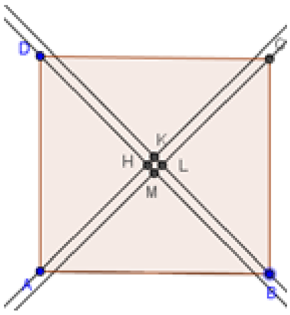

Figure 2. Constructing a square and observing a static drawing to conclude

Example 3 (P1). One of the observed pairs of students dragged the vertices of the scalene polygon ABCD to make it a trapezoid, based only on visual adjustment (Figure 3). As they did not know what conclusion to expect (type 2 inference), they observed the figure and formulated the conjecture: “If ABCD is a trapezoid then H, K, L and M coincide”.

The students observed in a single figure that H, K, L and M coincided and generalized the result to the entire class of trapezoids ABCD without validating the conjecture in additional instances of the figure. Thus they were not able to observe the different types of quadrilaterals HKLM obtained for different types of trapezoid ABCD.

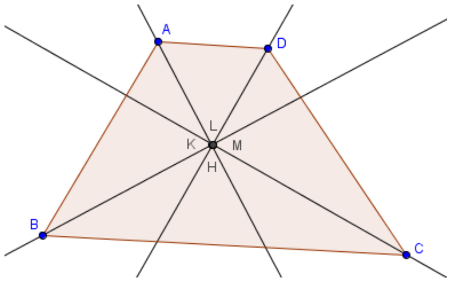

Figure 3. Constructing a trapezoid and observing a static drawing to conclude

Example 4 (P2). A pair of students investigated the premise, “ABC equilateral triangle”, by drawing a scalene triangle ABC, constructing the perpendicular bisector of [AC], dragging B onto it to make the triangle isosceles, displaying the measure of $\widehat{ABC}$, and dragging B along the perpendicular bisector until having $\widehat{ABC}$ = 60º (Figure 4). Since the students were given the conclusion they are supposed to reach, i.e. ABC separable (type 1 inference) they rejected the premise since in that figure ABC was not separable.

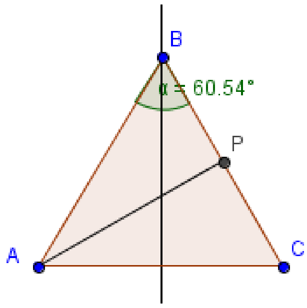

Figure 4. Constructing an equilateral triangle and observing a static drawing to conclude

Strategy 2 – Forward Dynamic Observation (FDO)

This second strategy consisted in constructing a robust figure satisfying the presumed premise, then dragging to search for the invariant properties and conclude the conjecture. If the invariants were observed across dragging, the conclusion is identified and a conjecture is developed (examples 5 and 6); if not then the premise is rejected based on a multitude of counterexamples (examples 7 and 8). To use this strategy, the construction is required to be robust in order to hold under dragging.

Example 5 (P2). A pair of students wanted to explore if the triangle ABC is separable when it is right at C. They built a robust figure that satisfied their premise (i.e. ABC right) using perpendicular lines (CA) and (CB) (Figure 5). They dragged the independent points to identify invariant properties across dragging (i.e. two equal sides for each of the triangles CPA and CPB) which was their conclusion.

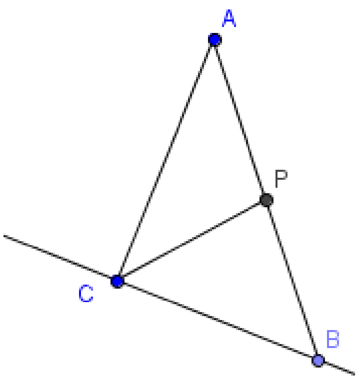

Figure 5. Constructing a robust right triangle and dragging to identify invariants

Example 6 (P1). One pair of students chose to investigate the premise "If ABCD is a parallelogram" to determine its conclusion. They constructed a robust parallelogram using parallel lines (Figure 6). They dragged A, B and D and identified the invariant properties of HKLM and deduced that it is a rectangle. They formulated the conjecture: "If ABCD is a parallelogram then HKLM is a rectangle".

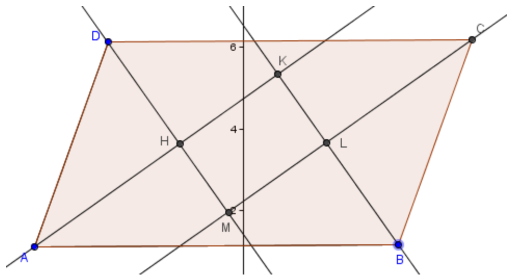

Figure 6. Constructing a robust parallelogram and dragging to identify invariants

Example 7 (P1). One of the observed pairs of students, attempted to investigate the premise: "If ABCD is a trapezoid". They constructed a robust trapezoid using the *Parallel Line* tool. They dragged the vertices of ABCD (Figure 7) but were not able to identify any invariant property for HKLM through dragging. Thus the case of the trapezoid was rejected.

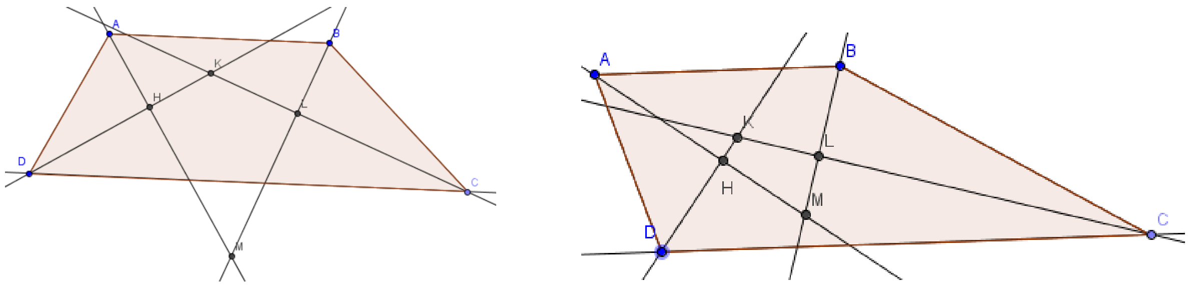

Figure 7. Constructing a robust trapezoid and dragging to identify invariants

Example 8 (P2). A pair of students wanted to explore if the triangle ABC is separable when it is isosceles. So they built a robust figure that satisfied their premise (i.e. ABC isosceles) by placing a point F on the perpendicular bisector of a segment [DE] and formed a robust isosceles triangle (Figure 8). They dragged the independent point D to identify invariants of the conclusion i.e. two equal sides for each of the triangles DGF and GFE. Given that the conclusion could not be met, the premise was rejected.

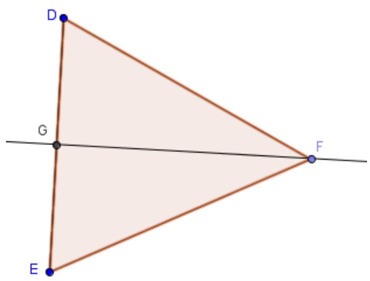

Figure 8. Constructing a robust isosceles triangle and dragging to identify invariants

Strategy 3 – Backward Static Observation (BSO)

This third strategy consisted of incorporating the conclusion of the desired conjecture in the figure and observing that single static instance of the figure to discover the corresponding premise. Students' assumption is that, if they are able to incorporate the properties of the conclusion into the construction, the drawing should reveal the premise for which they are looking. If it is not possible to incorporate the properties of the conclusion in the construction, the premise should be rejected.

In problem 2, although the conclusion is given by the problem, it can be further developed into sub-conclusions; CPA and CPB can be considered simultaneously isosceles at different vertices. Students have actually attempted testing different sub-conclusions.

Example 9 (P2). A pair of students satisfied the conclusion of the conjecture in their figure by making ABC separable (APC and PCB isosceles at P and C respectively). To do so, they sketched a scalene triangle ABC, constructed the perpendicular bisectors of [AC] and [BP], and then dragged the vertices of ABC to bring P and C simultaneously onto the respective perpendicular bisectors (Figure 9). They were able to observe in the figure that the conclusion (i.e. APC and PCB isosceles at P and C respectively) is satisfied when ABC is a right triangle, thus completing their conjecture. We note that, in this case, PCB is equilateral, but students were not aware of this fact.

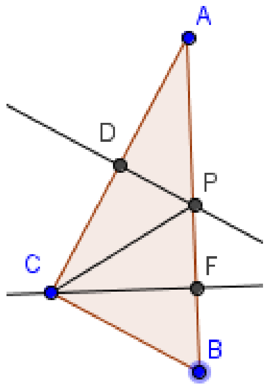

Figure 9. Satisfying the conclusion "ABC separable" and identifying the premise "ABC right triangle"

Example 10 (P1). After developing the conjectures "If ABCD is a square then H, K, L, and M coincide" and "If ABCD is a rhombus then H, K, L, and M coincide", one pair of students attempted to investigate under which condition (in general) do the points H, K, L and M coincide. They dragged the vertices of ABCD to form a new quadrilateral ABCD where H, K, L and M coincided; that is they satisfied the conclusion "H, K, L and M coincide" in the figure. They formed a kite shape (Figure 10). However, they were not able to identify its nature and thus were not able to develop a general conjecture on the nature of ABCD for which H, K, L and M coincide.

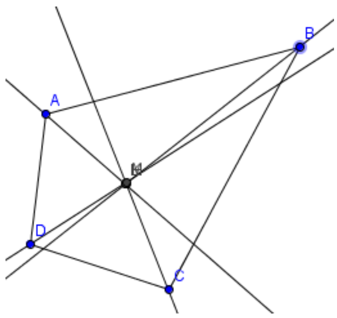

Figure 10. Satisfying the conclusion "H, K, L and M coincide"

Example 11 (P2). One pair of students thought about investigating whether there is a specific premise, i.e. nature of ABC, which satisfies the conclusion "ABC separable at C", that is CA = CP and CP = CB. They constructed AC = 5; CB = 5 and connected A and B. Then they constructed CP = 5 and tried to drag P onto [AB] but were not able to do so (Figure 11). Thus they concluded that it is an impossible case.

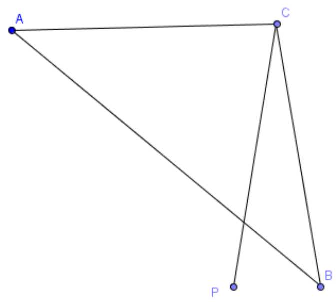

Figure 11. Satisfying the conclusion “ABC separable at C” to identify the premise

CONCLUSION

The analysis of students’ work provided in the study allowed us to identify three strategies (Figure 12) for the elaboration of conjectures based on the way students move from exploring the different instances of the figure to identifying relational properties of the figure and thus developing conjectures. In the first two strategies the students work on satisfying the premise in the figure and identifying the conclusion by observing the static figure at hand (strategy 1 - FSO) or by dragging a robust figure to validate the conclusion across different instances (strategy 2 - FDO). Conversely, students can also work on satisfying the conclusion in a soft figure and observing it, without any manipulations, to deduce the premise that corresponds (strategy 3 - BSO).

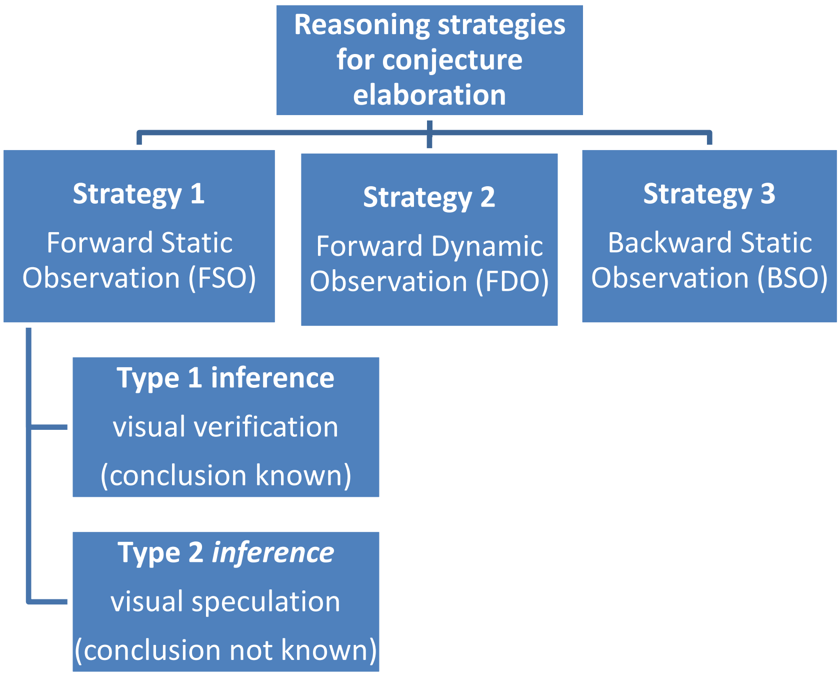

Figure 12. Reasoning strategies for conjecture elaboration

The weakness of FSO is caused by the elaboration of the conjecture based on a single instance of each case; students are only observing the figure they constructed and formulating the conjecture based on that single instance of the figure which, in many cases, happened to have additional properties leading to a misguided conclusion.

In FDO, the use of robust constructions and dragging tool lead to the creation of a powerful instrument for conjecture generation since the use of robust constructions results in valid drawings, which eliminates ambiguous results. Also, when dragging robust constructions, the premise can be verified in a multitude of figures. Thus the elaboration of a conjecture or the rejection of the premise is based on a multitude of instances of the same case.

In contrast to the first and second strategy, in BSO the students work their way backwards from conclusion to premise, which is not always an easy task. Most students preferred strategies 1 and 2, that is testing different premises to find the one that satisfied the conclusion instead of incorporating the conclusion into the construction and letting the figure reveal the premise i.e. strategy 3. However, based on a single instance of the figure, students may consider a certain observed property to be the premise, when in fact it is not. The premise has to be induced from invariants across dragging.

The identification of these three strategies induces thinking about a fourth possible strategy that was not observed in the participating students' work but that we can see as a possible one. When working from premise to conclusion, students used both soft (FSO) and robust (strategy 2) constructions. However, when working from conclusion to premise, only soft constructions were used (strategy 3). Therefore, we develop this potential fourth strategy in order to hypothetically describe the work from conclusion to premise using a robust construction. The strategy would be named "Backward Dynamic Observation (BDO)" and would consist in satisfying the conclusion by construction and dragging to deduce the premise that led to the desired conclusion. The use of robust constructions is required to ensure the validity of the figure and allow the elaboration of the conjecture based on a multitude of instances of the same case through the use of dragging. More research with a larger sample of students is needed to validate the fourth potential strategy.

REFERENCES

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practices in Cabri environments. *ZDM, 34*, 66-72.

Hölzl, R. (2001). Using dynamic geometry software to add contrast to geometric situations–a case study. *International Journal of Computers for Mathematical Learning, 6*, 63-86.

Jones, K. (1998). Deductive and intuitive approaches to solving geometrical problems. In C. Mammana, & V. Villani, *Perspectives on the Teaching of Geometry for the 21st century* (pp. 78-83). Dordrecht: Kluwer Academid Publishers.

Laborde, C. (2005). Robust and soft constructions: two sides of the use of dynamic geometry environments. In H.-C. L.-C. Sung-Chi Chu, *Proceedings of the Tenth Asian Technology Conference in Mathematics* (pp. 22-35). South Korea: Advanced Technology Council in Mathematics, Incorporated.

Laborde, C., & Laborde, J. M. (2011). *Interactivity in dynamic mathematics environments: What Does that Mean?* Retrieved from ATCM:

http://atcm.mathandtech.org/ep2011/invited_papers/3272011_19113.pdf

Laborde, C., & Sträßer, R. (2010). Place and use of new technology in the teaching of mathematics: ICMI activities in the past 25 years. *ZDM, 42*, 121-133.

Leung, A., Baccaglini-Frank, A., & Mariotti, M. (2013). Discernment of invariants in dynamic geometry environments. *Educational Studies in Mathematics, 84*, 439-460.

Mariotti, M. A. (2014). Transforming images in a DGS: The semiotic potential of the dragging tool for introducing the notion of conditional statement. In S. Rezat, M. Hattermann, & A. Peter-Koop, *Transformation - A Fundamental Idea of Mathematics Education* (S. 155-172). NY: Springer.

Olivero, F. (2002) The proving process within a dynamic geometry environment. Doctoral dissertation. UK: University of Bristol.

Sinclair, N., & Robutti, O. (2012). Technology and the Role of Proof: The Case of Dynamic Geometry. In M. K.-K. Clements, *Third international handbook of mathematics education* (S. 571-596). New York: Springer.

ALGEBRA STRUCTURE SENSE IN A WEB ENVIRONMENT: DESIGN AND TESTING OF THE *EXPRESSION MACHINE*

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The article reports on outcomes from a study that aims to investigate the role of affordances, level-up and feedback in the web environment Expression Machine in developing the algebra structure sense of tertiary education students. Algebraic substitution is the main procedure involved in the way the software works. Its design and testing methodology are based on the Human - Computer Interaction aspect of Activity Theory. From this approach, the study redefines the notion of algebra structure sense formulated in previous works. Results from the experimental sessions show that some features of the environment favor the development of sutudents structure sense, specifically when they deal with substitution and factorization tasks. At the same time, it was possible to identify aspects to be improved, for instance, adding categories of tasks with increasing structural complexity and less visually salient, which may require a greater cognitive demand from students.

Keywords: Algebra structure sense, web environment, human-computer interaction, activity theory.

INTRODUCTION

Once they have overcome the difficulties of learning the rules of syntax, as well as those for understanding the semantics of symbols and the conventions of algebraic notation, students in tertiary education face the challenge of recognizing the basic structures of algebraic expressions in complex transformational algebra tasks. Hoch & Dreyfus call this recognition ability 'algebra structure sense' and they define it as a set of abilities that involve: 1) recognizing a familiar structure in its simplest form; 2) dealing with a compound term as an entity and, by performing the appropriate substitutions, recognizing a familiar structure within a more complex form; and 3) choosing appropriate manipulations for a better use of structure (Hoch & Dreyfus, 2007, pg. 436).

As of the above definition, these authors designed activities to be used during teaching interviews with 11th grade students. A pre-post test scheme and an analysis of the interview protocols revealed that the students made progress regarding the development of structure sense in specific cases, such as applying the rule $a^2 - b^2 = (a + b)(a - b)$ in compound expressions such as $(x + 8)^2 - (x - 7)^2$ or $x^2 - (x + 1)^4$. However, the results also showed that students found factoring variants like $(x + 3)^4 - (x - 3)^4$ extremely challenging, despite their having the support of the researcher. This shows that the research area is emerging, especially when investigating how to teach structure sense. Formulation of the project 'Developing structure sense with digital applications' was largely inspired by the work of Hock and Dreyfus. This project intends to deepen the research of the learning and teaching of the structural aspects of symbolic algebra within technology environments. The study reported here was undertaken in the framework of this project and its main goal is to investigate the role of *affordances, level-up* and *feedback* [1] in the virtual environment *Expression Machine* in developing the algebra structure sense of tertiary education students. This article briefly describes the design features of the *Expression Machine* and reports its testing results, specifically with tasks that involve algebraic substitution and factorization [2].

BACKGROUND

Structure Sense

Prior to the work of Hock & Dreyfus, the topic of structure sense in algebra was studied by A. Arcavi, who tried to characterize symbol sense by extrapolating part of the information from number sense. The latter is conceived as 'a non-algorithmic sense of numbers', based primarily on an understanding of its nature and the nature of its operations, as well as the need to examine the good sense of its results and related effects. The author establishes parallels with this conception by referring to symbol sense as the complementary relationship between algebraic manipulation and 'seeing through' algebraic expressions (i.e *seeing the unseen*, Arcavi, 1994).

On the one hand, in 2004 D. Kirschner found that students spontaneously respond to the visual patterns of algebraic expressions (visual salience) independently of the declarative rules, which suggests that typical errors like a + x / b + x = a/b reflect the predominance of visual aspects over the declarative knowledge of algebraic rules. According to this author, the receptive disposition of students to the visual structure of rules, independently of their intellectual commitment to the declarative content, is at odds with the habitual cognitive presumption that human intellectual abilities rely on the acquisition or development of algorithms and well structured rules (Kirshner, 2004, pg. 4). From this, Kirshner concludes that absent an understanding of the structural fundamentals, what students register is something about the visual shape of correct and incorrect applications, and that eventually, with persistence, the visual pattern recognition processes become sufficiently refined that they may restrict incorrect applications (2004, pg.42).

On the other hand, Sfard and Linchevski (1994) have documented the persistence of students to remain within the procedural aspects of algebra. That is, students tend to interpret algebraic expressions as calculation processes, and after repeatedly applying a procedure (or algorithm), they see them as objects, something upon which to reflect. They call this phenomenon reification. From this perspective, according to these authors, algebraic expressions have a dual process/object nature.

In very different ways, the studies of Hock & Dreyfus, Arcavi, Sfard & Linchevski, and Kirshner state that independently of what is meant by the nature of structure sense, implementing it in symbol manipulation tasks is enormously complex. Furthermore, the conclusions reached by these authors suggest that teaching structure sense is extremely challenging. The research shown in this paper intends to face this challenge by using the potential of technological resources for the learning of mathematics.

Algebra Learning with Technology

There is currently a broad repertoire of technological tools that can be used to teach algebra at various school levels, most notably; Computer Algebra Systems (CAS), Spreadsheets, Aplusix and the widely used Geogebra, which combines several mathematical representations (graphical, algebraic and geometric). In most cases, learning activities are focused on topics of functions, equations and graphs, as well as the use of CAS to verify the results of solving equations or performing algebraic expression transformations by hand (for instance, simplifying or developing expressions). The literature reporting results of research undertaken using these tools for teaching and learning algebra is significant and provides evidence of their great didactic potential. However, the literature on the use of software designed for teaching specific topics is less abundant. Some examples of these types of environments are: *eXpresser,* especially designed to foster generalization processes in algebra (https://migenproject.wordpress.com/using-migen/); *Virtual Balance*, used to teach the solving of linear equations (Rojano & Martínez, 2009); and the program *DragonBox* (www.dragonbox.com) which has the features of serious games [3], developed around entertainment in solving algebraic tasks and integrating these activities in the game.

STRUCTURE SENSE AND THE *EXPRESSION MACHINE*

To undertake the study reported here, the web environment *Expression Machine* (EM) was designed. EM is an ad-hoc tool for developing structure sense among tertiary education students, inspired by *serious games* and *touch* applications, with virtually no instrumentation time (training time at the use of the artifact level). The guiding resarch questions are:

1. Is it possible to guide students, through *affordances* and *feedback,* towards actions that allow them to perform tasks fostering development of an algebra structure sense?
2. Specifically, what features of *affordances* and *feedback* in a virtual environment foster students developing a structure sense for algebraic substitution and factoring expressions?

Theoretical Elements

Regarding the notion of structure sense underlying the design of EM, the principle of algebraic substitution is the main consideration, which allows for equivalent symbols to be used interchangeably, so one may be used instead of the other in an algebraic expression; a variable may be replaced by an expression and vice versa (Freudenthal, 1983, pg.483). In terms of the way that a structure sense may be acquired or developed, we resort to the idea, on the one hand, that meanings arise during usage and activity in practices that are shared socially within a community (the second Wittgenstein, 1988); and on the other hand, the idea that meanings are associated with training, following rules and *seeing how* (Huemer 2006).

EM software design and testing methodology are based on the Human - Computer Interaction (HCI) aspect of Activity Theory, with special emphasis on the notion of *affordances* or preconditions for action.

In the Activity Theory (AT), activity in general -not just human activity- but rather the activity of any subject, is understood as an intentional interaction of the subject with the world –a process in which mutual transformations take place between the "subject-object" poles. In this theory, the subject-object relationship is a starting point and it is interpreted as a non-direct relationship, that is to say, that it is mediated by language and artifacts, and as a non-symmetrical relationship because in it the subject holds the initiative and command.

In the field of HCI and of designing digital artifacts, the foregoing is translated into having the relationship between two components of a large scale system be asymmetrical, given that the interaction is begun and undertaken by the subject so as to cover its needs (Kaptelinin & Nardi, 2006, pg. 30). As such, an activity consists of a person or several persons doing something toward attainment of some end. In the field of learning, according to Knutti (1996), an activity is a way of doing that is oriented towards an objective, and learning is strongly linked to the doing and the social system in which the doing takes place. From this perspective, technologies are not a means by which knowledge is transmitted to a user, rather a tool that provides structure and mediates learning through activity (DeVane & Squire, 2012, pg. 242).

The AT envisages learning technologies not as 'teaching machines', but rather as 'a support system for learning by doing'. Learning is not only accomplished through observation, but also by 'doing', and learning technologies serve to support and structure those tasks (Knutti, 1996, pg. 26).

Taking this approach, the study redefines algebra structure sense in terms of actions, as follows: A student demonstrates having algebra structure sense if, in order to solve an algebraic manipulation task efficiently, the student performs a combination of the following actions: a) Recognition of structures (for instance, recognizes notable products); b) See-how, that is, switching between various forms of an expression, to take advantage of the structures (learn to see sub-expressions as an object or entity); c) Substitution (whether internal or explicit); and d) Timely application of known algebraic identities.

Characteristics of *Expression Machine*

EM is a web application that was developed for users to learn, through experimentation and practice, the rules that the machine uses to generate tasks and, in time, acquire an algebra structure sense, in terms of actions a) to d). It incorporates school algebra rules such as algebraic substitution and equivalence of expressions. The interactive sequence was designed as of a scheme where the elements of the machine are the *input* (two expressions IE1 and IE2), *process* (a generating expression GE) and *output* (a resulting expression OE after substituting IE1 and IE2 in GE) (see Figure 1a).

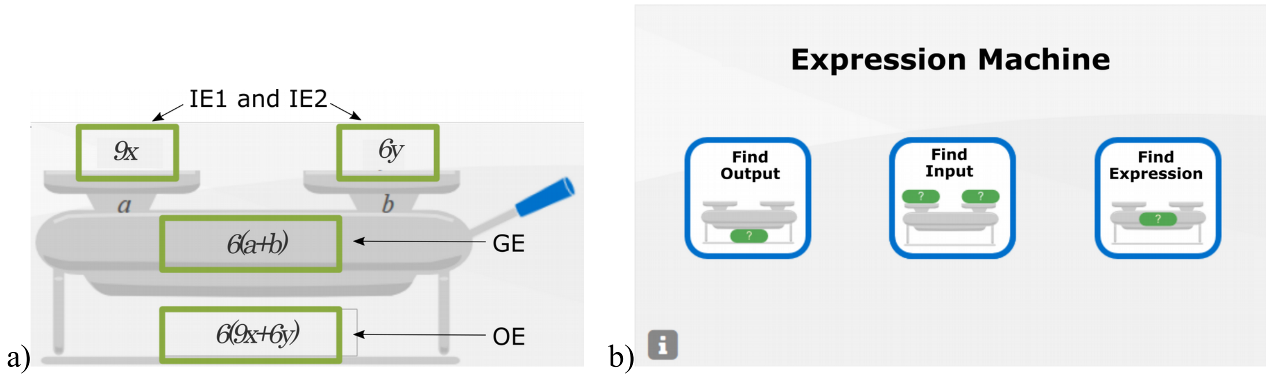

Figure 1. a) Expression Machine b) Main screen

In all cases, the expression machine generates an output expression upon substituting the input expressions in the generating expression. For instance, if the input expressions are $9x$ and $6y$, and if the generating expression is 6(a+b), then the machine will output $6(9x+6y)$ as it substitutes a with $9x$ and b with $6y$. The activities proposed with this machine are of three types, which can be accessed through the main screen (see Figure 1b). These activities are:

1. *Conjecture input expressions*. Given an output and a generating expression, students must describe two input expressions that would produce such output expression.
2. *Predict the output expression*. Given the input expressions and a generating expression, students must describe the output expression the machine would produce. Students are asked to write down the expression they believe the machine will produce. They may then process the expressions and get the machine to produce an output expression. With this activity, in addition to practicing algebraic substitution, students may strengthen their knowledge on equivalent expressions, as the machine may give an equivalent expression that is syntactically different

from their prediction. Feedback from the machine will clarify that the expression they input is correct and equivalent to that given by the machine.

3. *Conjecture generating expressions.* Given input and output expressions, students must conjecture a generating expression that will make the machine produce the given output expression. They may key in the generating expression into the machine to prove that it really works.

In most of the cases, interaction with EM requires intensive algebraic manipulations to be performed by hand (Muñoz & Rojano, 2014).

EXPERIMENTAL WORK WITH EM

The EM was tested with a group of 35 tertiary level students in a Mexico City public school. A pre-post test scheme (based on the questionnaires of Hoch and Dreyfus) was applied to assess participants' mastery of symbolic manipulation and their structure sense level. Participants had a period of interaction with the EM between tests. 16/35 of the students correctly solved 15 or more of the 32 items in the pre-test, and seven of the students took part in the experimental in-person sessions (with the participation of the researcher to briefly explain operation of the EM). In that 1.5-hour session, the students worked intensively on the three types of EM activities ('find the output', 'find the input', and 'find the generating expression', see Figure 1b) and they worked through different levels of complexity in terms of the algebraic expressions involved. 'The doing' of the students included paper and pencil algebraic manipulations to solve the three types of EM activities. Their actions were recorded during this experimental period, and the method described in Bødker (1995) was used to analyze recordings that detected *focus shifts* and *breakdowns* [4].

After the in-person experimental session, they were given the URL of the EM web application for them to use it freely at home over the course of one week. The post-test was applied at the end of that week of home use. In summary, two types of data were collected, as follows: 1) data collected with application of the pre and post-tests, where the written algebraic productions of students were analyzed; and 2) the material entailed in the video-tape of the interactive experimental (in person) session with the EM, together with the respective paper and pencil productions of the students, which show the algebra manipulations that they undertook in order to solve the EM exercises (see Figure 2).

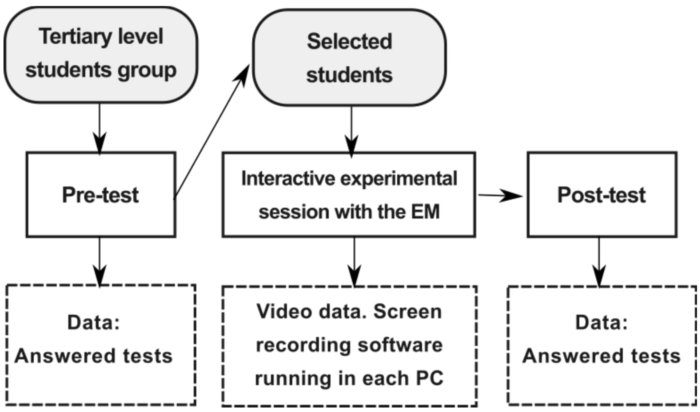

Figure 2. Types of collected data

Pre – Post Test Results

As a group, students improved significantly between the pre and the post test in both items of algebraic manipulation, and in items related to structure sense (according to the re-definition of structure sense formulated in this study in terms of actions a)-d)). One performance of particular merit was noted (Bedani), as it showed algebraic skills that surpassed those of the rest of the group. In order to illustrate this progress, two extracts taken from the productions of Edwin in the pre and post tests are presented.

In problem 9 (Figure 3) Edwin shows a good level of algebraic manipulation in the pre-test but fails to recognize *5-x* as an entity. However, he correctly factorizes *7-y* in the post-test in order to obtain a product of two binomials, and quickly solves the problem. In the latter case, actions a) and b) become evident. Similarly, in problem 12 of the pre-test, Edwin applies several rules and even tries to assign values to the variables but fails to solve the problem. In contrast, during the post-test, he identifies the product *xy* as a single entity and makes an explicit substitution in order to find the solution using the general formula (see Figure 4). Here Edwin performs actions such as those described in b) and c).

It is noteworthy that the substitution technique was not explicitly taught to the students in any of the activities of their experimental session with the EM: Nevertheless, this technique is included in EM (that is, substitution is the process used by the machine).

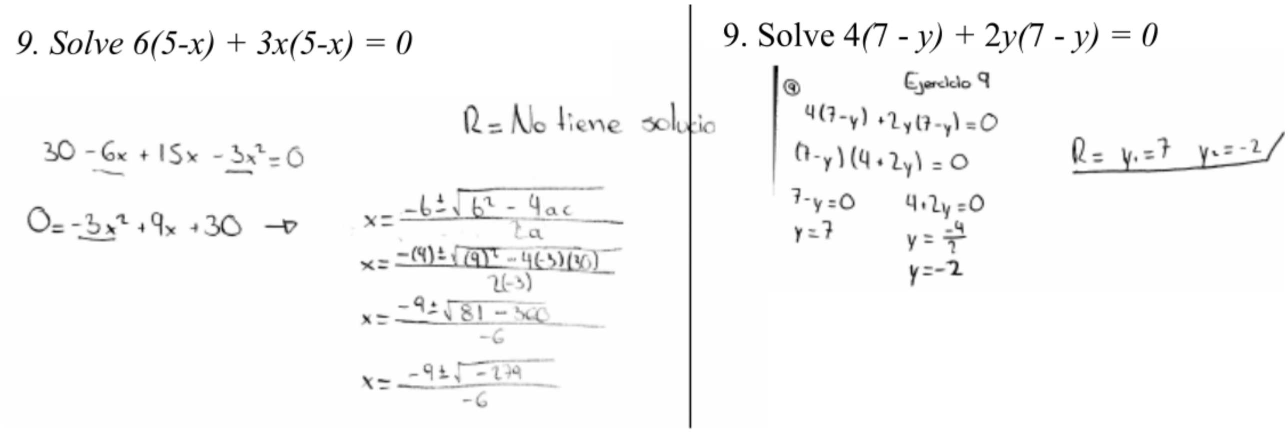

Figure 3. Problem 9, (Edwin *pre-test*/Edwin *post-test*)

12. Solve for xy: $9x^2y^2 + 6xy + 1 = 0$ | 12. Solve for ab: $4a^2b^2 - 4ab + 1 = 0$

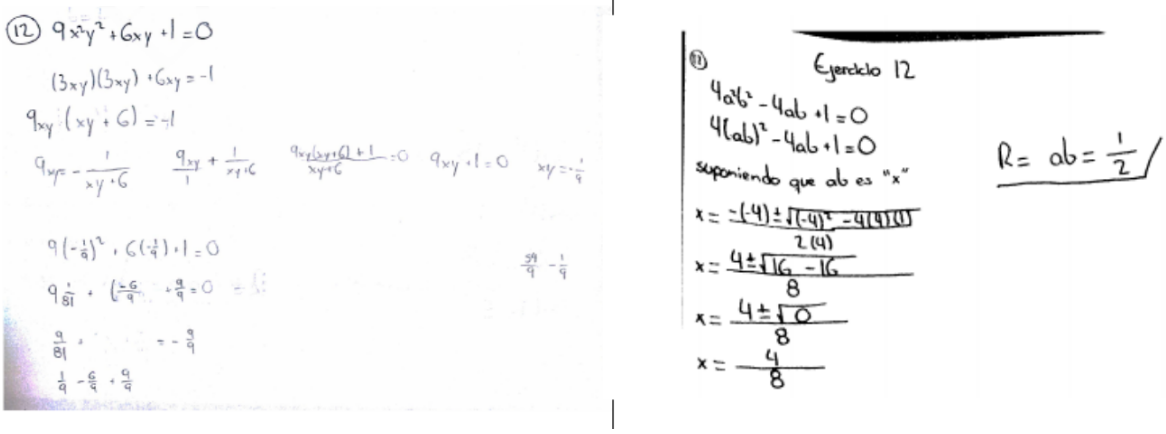

Figure 4 Problem 12 Edwin. (Edwin *pre-test*/Edwin *post-test*)

Results from the Interactive Experimental Session with the EM

The experimental session with the EM lasted for approximately 90 minutes and 4 groups of data corresponding to 3 activities were gathered: 2 answer sheets on the functioning of the machine prior to using it (activity 1), and 10 minutes using the machine (activity 2); video recording of on-screen interactions (activities 2 and 3); and audio clip of the group solving exercises with the EM.

Activity 1: Consisted of projecting the EM on a screen and asking: What do you think the machine is doing? and, how does it work? Three students mention substitution in their answers, although in some cases implicitly. David, Jenifer and Bedani use the verbs elaborate, substitute and assign respectively. Only Bedani answered the question about how the machine works and she did so correctly. *Activity 2:* The EM is designed to not require a manual in order to learn how to use it. Instrumentation is achieved through *affordances, feedback* and progressively encountering levels. Therefore, the second activity consisted of a 10-minute-long free exploration. Here it was observed that, in less than 10 minutes, not only did the students learn to use the machine, but they also obtained a fairly accurate idea of its functioning. On average, it took 1 minute and 44 seconds to correctly solve the first exercise. *Activity 3*: 40 minutes of free exploration of the EM. The analysis was centered around the *breakdowns* of student interactions with the machine. Three types of *breakdowns* were identified: one was associated with the process/object duality and the other two were associated with feedback.

Exercise 3/22 of the *Find Input (FI)* scene is analyzed below. Although this exercise is simple, it's challenge lies in the generating expression being a sum and the output expression being a product, that is, they don't have the same structure. This implies that a solution requires for the product xy (output expression) to be considered as a single entity. While six of the seven students attempted a solution, only three of them managed to identify the product xy as a single entity on the first try, two did so after several attempts, and one was trapped in an operational or procedural conception.

Fabiola's case illustrates a quick transition to identifying the expression as an object. She failed to solve exercise 3/22 on the first try, probably because she didn't see xy as a single entity (Figure 5a) and then she abandons the problem. At 17:53 however, she returns; writing down something at minute 19 (Figure 5b). Upon isolating the variables, she clearly parses xy as a process and not as an object. On this try, she fails to recognize *xy* as a single entity. However, she offers a correct solution at the end of minute 21 (Figure 5c).

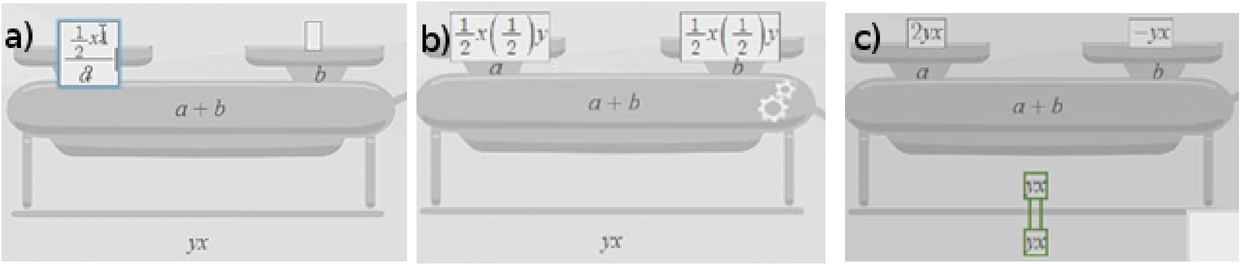

Figure 5 Extracts of Fabiola's Interaction Exercise 3/22 of the FI

This is Jennifer's interaction with exercise 20/22 of *Find Input (*Figure 6). It can be observed that she is unable to solve the problem, possibly due to her trying to guess the answer following the visual salience of the *output.* Jennifer tries to combine sub-expressions of 8x(x+1) to solve it but fails to provide a correct answer.

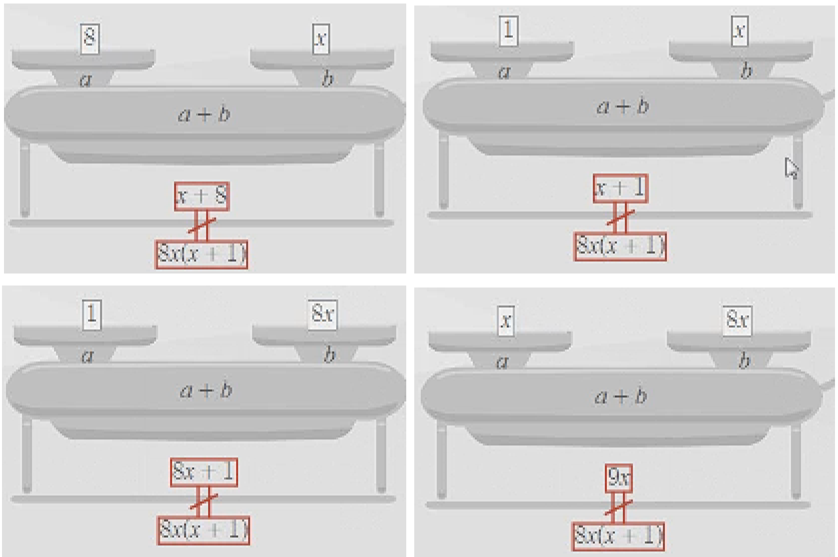

Figure 6. Extracts of Jennifer's Interaction Exercise 20/22 of the FI

The expected solution was to transform the product $8x(x+1)$ into the sum $8x^2+8x$ and put each term on a plate; or rather to identify $x(x+1)$ as a single entity and put a multiple of the expression on each plate so that they can be added to get $8x(x+1)$ (It is noteworthy that other exercises of the *output* scenario already show this as a sum, which simplifies the task).

CONCLUSIONS AND FINAL COMMENTS

The results of the analysis of students interaction with EM suggest a positive answer to research question 1. First, as the EM is essentially an algebraic substitution machine, it favors users adopting that technique. However, the activities proposed also require other actions, such as structure recognition, switching between various forms of an expression or application of known algebraic identities. This is noticeably seen in Edwin's case, who spontaneously applies the change of variable technique to solve two post-test problems. Or Fabiola who, after seveal failed attempts, is able to see product *xy* as an entity and correctly solves the problem. In this sense, and given that the students showed improvement in their structure sense as a group, we can say that the EM facilitates and fosters development of algebra structure sense, where the latter is conceived in terms of actions a)-d).

Regarding research question 2, the HCI aspect of Activity Theory suggests that minimalist design, *affordances, feedback* and *level-up* (task design by levels of complexity) had a positive usability effect on students. This is evidenced and confirmed by the nearly null instrumentation period required. In addition, students continued to explore and solve the exercises without intervention from the teacher or researcher.

Level-up is one of the distinctive features of EM, and the results of this research suggest that tasks involving a greater level of difficulty and complexity should be included in the future, in order to trigger stress and *breakdowns* (Bodker) and in turn expand students' learning experience.

The experimental work showed that visual salience often causes students to solve the tasks quickly, without carrying out a structural analysis of the algebraic expression (an analysis based on declarative rules, according to Kirshner). This motivates the inclusion (in a future EM version) of less visually salient levels that require greater cognitive demand from students.

Notes

1. The term *affordances* is used in the sense of Norman (2002), that is to say as suggestions or invitations (of the artifact) for usage possibilities. The broad and general meaning of *feedback* is adopted as 'information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding' (Hattie & Timperley, 2007, pg. 81). In the particular case of technology learning environments, the agent is the computer program, which provides feedback to the learner's performance based on its ability to interact with the latter.
2. The design and development of EM was funded by Conacyt – Mexico (V. Munoz-Porras doctoral dissertation, 2015). We want to thank students and authorities of CCH Vallejo school for the facilities to carry out the experimental work of the study.
3. A *serious* game is a game designed with a purpose other than pure entertainment. Its design explicitly emphasizes the added pedagogical value of fun and competition (Wikipedia, consulted on February 22, 2017, https://en.wikipedia.org/wiki/Serious_game).
4. Bødker (1996, pg. 6) uses the term *breakdown* when the learning activity is interrupted because something did not happen as it was expected to (for example, if a button is pressed, but nothing happens). That same author uses the term *focus shift* when interruption of the activity is more deliberate and does not necessarily happen due to a system failure, for instance when the teacher wants to explain something in particular about the operation of an artifact. A *breakdown* is the perception of a discrepancy between our expectations and what actually happens in the world. A breakdown causes a focus shift from the object of the activity mediated by the artifact to the artifact itself.

REFERENCES

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the Learning of Mathematics* 14 (3), pg. 42-47.

Bødker, S. (1995). Context and Consciousness. En B. A. Nardi (ed.), (pgs. 147-174). Massachusetts Institute of Technology. (ISBN: 0-262-14058-6.)

DeVane, B., & Squire, K. D. (2012). 10 Activity Theory in the Learning Technologies. *Theoretical foundations of learning environments*.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: Kluwer - Reidel.

Hoch, M. & Dreyfus, T. (2007). Recognising an algebraic structure. in D. P. Pantazi & G. Philippou (ed.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pgs. 436-445).

Hattie, J. & Timperly, H. (2007). The power of feedback. Rev. Educ. Res., 77, 81-112.

Huemer, W. (2006). The Transition from Causes to Norms: Wittgenstein on Training. In M. Kober (ed.), (pgs. 205-225). Rodopi.

Kaptelinin, V. & Nardi, B. A. (2006). *Acting with technology: activity theory and interaction design.* The MIT Press.

Kirshner, D. (2004). Visual salience of algebraic transformations. Journal for Research in Mathematics Education, 35 (4), 224–257.

Kuutti, K. (1996). Activity theory as a potential framework for human-computer interaction research. *Context and consciousness: Activity theory and human-computer interaction*, 17--44.

Muñoz, V. (2015). *Herramienta ad hoc para el sentido de la estructura en álgebra.* Doctoral dissertation, Centro de Investigación y de Estudios Avanzados (Cinvestav) IPN, Mexico.

Muñoz, V. & Rojano, T. (2014). Algebraic Expression Machine. A web ad hoc learning environment for developing structure sense. In G. Futchek & C. Kynigos (eds.) *Proceedings of the 3rd International Constructionism Conference 2014* (pp 540-541). Ósterreichishe Computer Gesellschaft (OCG) (ISBN: 9783854033011).

Norman, D. (2002). The Design of Everyday Things. Basic Books.

Rojano, T. & Martinez, M. (2009). From concrete modeling to algebraic syntax: learning to solve linear equations with a virtual balance. En S. L. Swars, D. W. Stinson, y S. Lemons-Smith (Eds.), Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Georgia State University: Atlanta, GA.

Sfard, A. & Linchevski, L. (1994). The gains and the pitfalls of reification - The case of algebra. *Educational Studies in Mathematics* 26 (2), 191-228.

Wittgenstein, L. (1988). Investigaciones Filosóficas. Mexico: Instituto de Investigaciones Filosóficas UNAM.

CENTRAL AND PARALLEL PROJECTIONS OF REGULAR SURFACES: GEOMETRIC CONSTRUCTIONS USING 3D MODELING SOFTWARE

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The contribution addresses the constructions of central and parallel projections of regular surfaces which can be regarded as a subarea of descriptive geometry. My aim is to increase the interest of students in classical and descriptive geometry primarily through 3D computer modeling. I have been seeking to establish a stronger connection between descriptive geometry and its practical applications and to extend descriptive geometry with knowledge of computer graphics and computer geometry. In order to provide insight into more complex geometric problems and to increase the interest in geometry, I have integrated 3D computer modeling in my lectures and seminars. Geometry is a necessary component of many engineering processes such as the development of innovative graphics software or the design of complex industrial and architectural structures. My aim is to show that the principles and knowledge of classical and descriptive geometry are the stepping stones for solving tasks in practice.

Keywords: central and parallel projections; regular surfaces; descriptive geometry; 3D computer modeling; computer geometry

MOTIVATION

I have been teaching classical geometry, descriptive geometry and computational geometry at the Faculty of Mathematics and Physics (Charles University) in the Czech Republic for several years. I also supervise bachelor and master theses on various geometric topics. The motivation for studying geometry can be found in many branches (building practice, engineering and construction practice, architectural and industrial design, production industries …) and geometry represents one of the highly demanding fields of mathematical science which require logical thinking and which also strongly stimulates spatial imagination. The study of geometry represents an ongoing challenge in terms of research and practice.

In this contribution I will focus on modern teaching methods of descriptive geometry and I will put these methods in contrast to traditional hand sketching and drawing activities. Computer-aided education of geometry will be demonstrated on examples of central and parallel projections of regular surfaces.

Descriptive geometry (Paré et al., 1996; Pottmann et al., 2007; Robertson, 1966) represents a subarea of classical geometry and deals with the representation of three-dimensional objects in two dimensions. The typical task in descriptive geometry is to represent three-dimensional objects on a two-dimensional planar surface or to reconstruct three-dimensional objects from the two-dimensional image i.e. result of the projection. Descriptive geometry deals with those representations which are one-to-one correspondent. From a historical point of view, the development of descriptive geometry reached its greatest height in the last century. Nevertheless, even despite today's innovative approaches and continuous development of modern computer technology and equipment, descriptive geometry has not lost its importance. The role of descriptive geometry in practice is irreplaceable in such branches in which correct visualization is crucial. To be able to project some three-dimensional object and get the two-dimensional result the construction methods require good knowledge of the fundamental geometry, the properties of geometrical objects in the plane and in the space, and their relations. This means that the study of descriptive geometry represents the significant stepping stones for solving geometric tasks in real practice.

In my research, I investigate innovative methods of explaining complex concepts in teaching of geometry (specifically descriptive, classical, computational geometry) at Czech colleges and their impacts on students' successes. The innovation in explanation and didactic methods include 3D computer modeling and interactive software visualization. My aim is to stimulate the interest of students in geometry, to increase their motivation, to improve their understanding of geometry, to improve the methods of teaching geometry currently in use, to help students achieve better results in examinations and to promote the practical use of geometry. I seek to make traditional topics from classical and descriptive geometry more attractive to students by updating the current methods of teaching geometry. The new teaching methods are aimed at strengthening the connection between classical geometry and the practical applications thereof on the one hand and extending classical and descriptive geometry into computer graphics and computational geometry on the other. The connection between classical geometry and 3D computer modeling is intuitively understood by students.

In this contribution I will describe possible activities, examples of outputs from 3D modeling software and the combination of descriptive and computational geometry. I will explain what computer-aided education of geometry means in my lessons. I have been using described activities in university classroom practice for several years. The main research question within this article is to discuss either use 3D computer modeling software for creation of 3D models and animations or hand sketching and drawing activities within specific task in education of descriptive geometry or how to combine them. The possibilities of 3D computer modeling in education of geometry will be shown and discussed.

I also find very important to clarify what is my aim for the future. I plan to deal with the following explorations and gather the results regarding computer-aided education in university classroom:

- the evaluation of success rate of university students attending geometric courses in the last few years (before and after when the innovative methods were implemented),
- questionnaire survey which was conducted among university students attending the courses and lectures on geometric topics where computer aided education was realized,
- my survey revealed that the modern type of computer-aided education was adopted very positively among students and according to higher students' interest in geometric topics within the research projects and qualification theses they seem to be more motivated,
- using computers in education of geometry is an efficient aid because geometrical and mathematical software (*GeoGebra*, *Rhinoceros*, *Mathematica*, *Maple*,…) allow us to deal with more complex task even in classroom practice,
- proper functions and tools in geometrical software develop creativity and imagination of students; on the other hand, the ability to use geometrical or mathematical software is not equal to the knowledge of geometry.

The evaluation of these surveys is a very complex and long-lasting task but first results show that our efforts to improve geometry education are successful. Some results of my didactic survey have been already published, (Surynková, 2013, 2015).

In this article we will show the examples of using modeling and graphics software in teaching geometry. Computer-aided education of descriptive geometry is demonstrated on typical tasks from descriptive geometry - the constructions of the two-dimensional results of central and parallel projections.

The rest of the paper is organized as follows. In Section 3D modeling software versus traditional teaching methods of descriptive geometry we introduce novel methods of using modern software in a classroom practice and remind the importance of traditional approaches including hand sketching and drawing. In Section Parallel and central projections of regular surfaces we show examples of geometric tasks regarding these topics including students' work. Discussion, summary, and future work are given lastly.

3D MODELING SOFTWARE VERSUS TRADITIONAL TEACHING METHODS OF DESCRIPTIVE GEOMETRY

Geometry in general ranks among the demanding subjects in secondary schools and colleges. One possibility how to stimulate the interest of students in geometry is to show them that the challenges of this Information Age can be addressed by means of geometry. The practical applications of geometry include computer-aided architectural and industrial design. Geometry is essential to the manufacturing, engineering and construction industries; the digitization of real objects using 3D scanning; digital surface reconstruction from point clouds; the replication of the shapes of real-world objects using 3D printing; computer graphics and many more, (Eilam, 2005; Foley et al., 1995; Hoschek and Lasser, 1993; Lipson et al., 2013; Sarkar, 2015). All these applications can be characterized by combinations of geometric principles. The extension of descriptive geometry, and geometry in general, into 3D computer modeling is a very promising approach how to increase student's motivation and to improve the methods of teaching geometry currently in use.

There exist a wide range of professional graphics software and environments which provide the required user input tools, and speed up production and are commonly used in the process of designing, design documentation and construction for modeling and drawing (Farin et al., 2002). We can use similar software in teaching of descriptive geometry.

I have integrated 3D computer modeling in my descriptive geometry lessons and seminars and I work mainly with the *Rhinoceros* (*NURBS Modeling for Windows*) software which is a commercial NURBS-based 3D modeling tool, (McNeel, 1999), commonly used in the process of designing, design documentation and construction. It is not necessary to work only with *Rhinoceros* or with expensive CAD applications. As there exist the number of inexpensive or free software applications for geometry and mathematics, students and teachers can use them. One of the most widespread free geometrical tools is mathematics and geometry dynamic software *GeoGebra*. The great advantage of *GeoGebra* is the possibility to change dynamically the parameters of the designed geometrical objects.

I have been teaching descriptive geometry and related subjects for more than ten years and my personal experiences show that teaching and studying geometry must be accompanied by traditional methods i.e. hand sketching and drawing on the blackboard and on the sheets of paper. Drawing and sketching helps us to develop our precision skills and patience and we rely on these tools when developing of our initial ideas and finding solutions to geometrical problems. In my lectures and seminars I combine the both approaches to the teaching of descriptive geometry - the traditional geometry teaching methods and procedures (sketching and drawing activities) and modern computer-aided education using digital modeling tools.

PARALLEL AND CENTRAL PROJECTIONS OF REGULAR SURFACES

Let us show the examples of concrete topics in descriptive geometry where 3D computer modeling can be used. I use 3D computer modeling to create 3D models of geometric objects and situations in the three-dimensional space, I demonstrate geometrical constructions of regular surfaces and I also use digital tools for creation of central and parallel projection of surfaces. I use these outputs during my lessons as illustrations of geometrical properties of studied objects which can help my students understand geometrical problems in intuitive and natural way. Students can discover geometrical principles and properties of objects more easily.

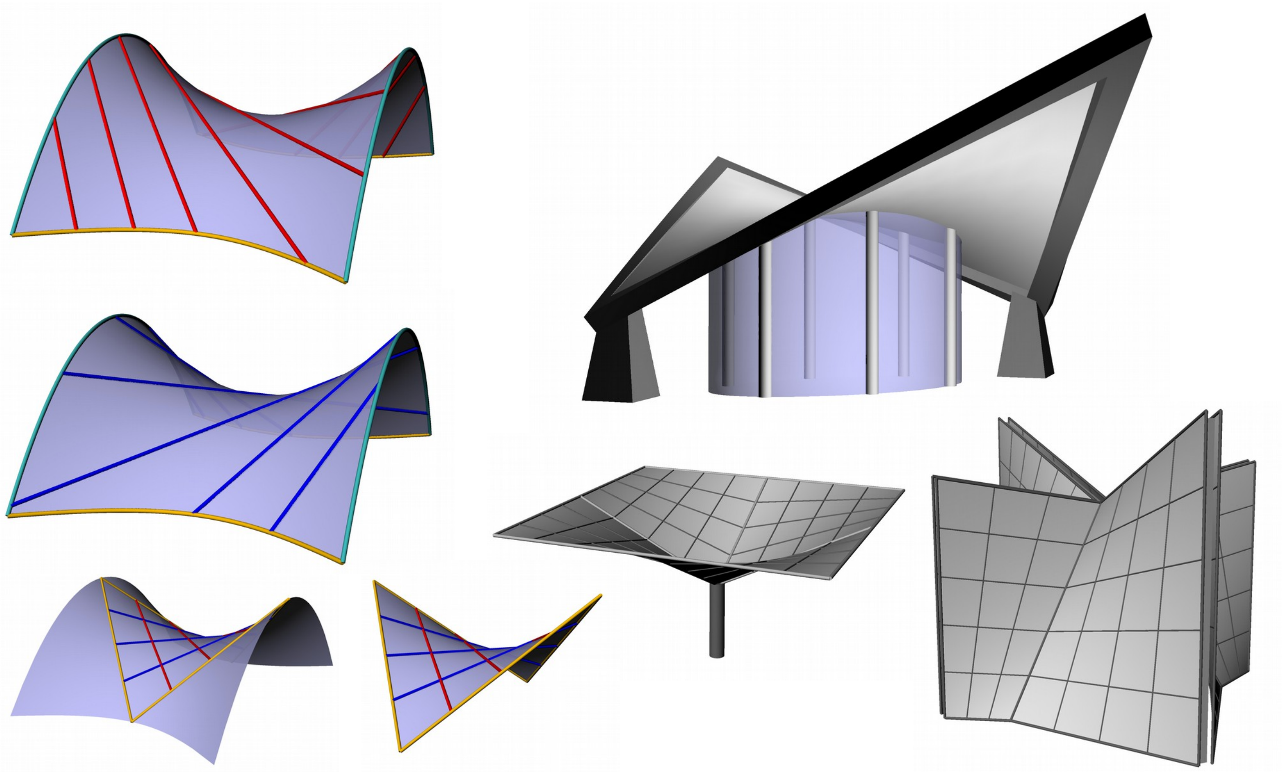
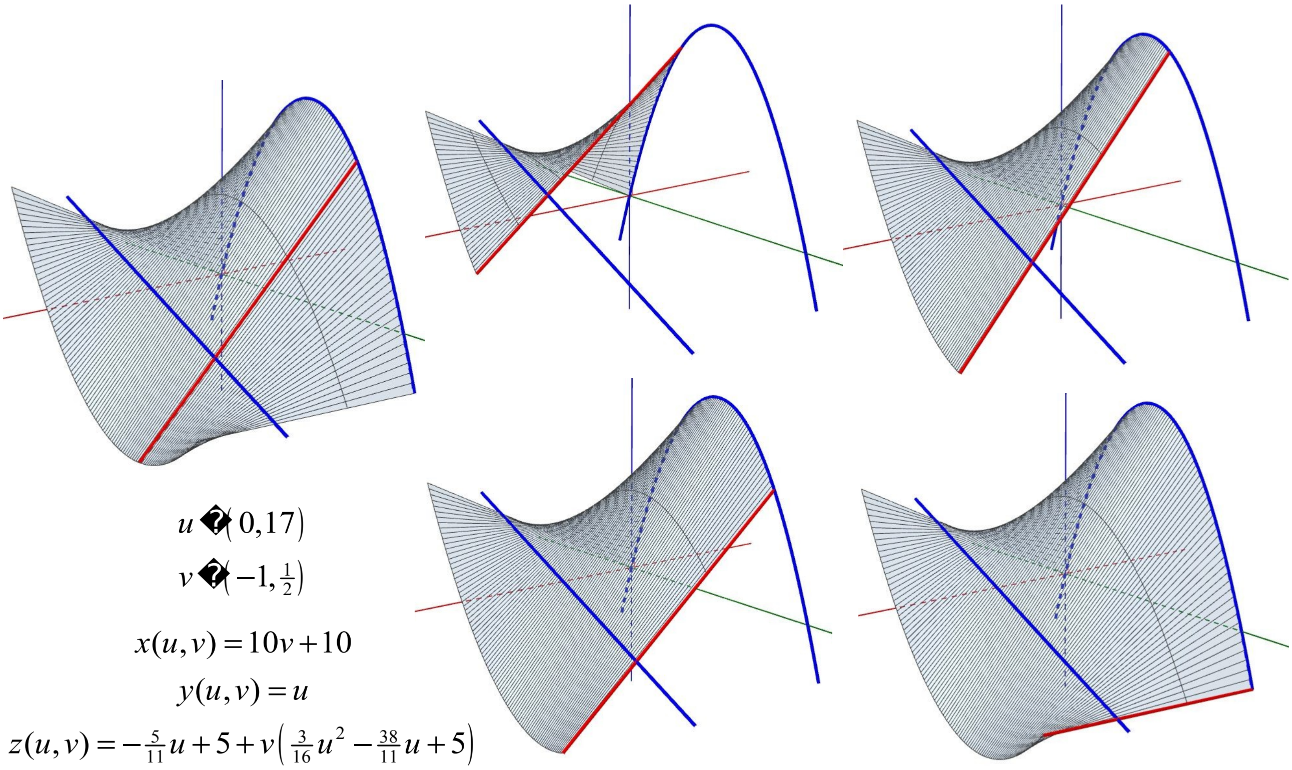


Figure 1: Ruled surfaces - determination and practical applications.

When I am teaching the topic of regular surfaces, firstly, a theoretical explication regarding the determination of surfaces in the three-dimensional space is provided and here the illustrations from the 3D computer modeling software can be used with potentially great success. The virtual model of the spatial situation and 3D virtual models of surfaces make a significant contribution to the development of spatial imagination. Besides the determination and the properties of regular surfaces I also present the practical usage of studied surfaces to students. Figure 1 shows an example of ruled surfaces. Firstly, the illustration of the determination of a surface is provided; secondly, the practical usage of this type of surfaces is shown. All these pictures are created as three-dimensional objects in the *Rhinoceros* software. So when I am teaching, I can show these 3D models to my students directly in 3D modeling software, I can change the view of a designed object and we can observe the spatial objects from different positions.

As has been already pointed out, my aim is to integrate the knowledge of computer graphics and computer geometry into my descriptive geometry lessons. Figure 2 shows an example of the determination of a ruled surface using *GeoGebra*. In *GeoGebra* we can use the animation for the demonstration of sequential construction of a surface. In this case it is necessary to know the mathematical description of a studied surface. So creating all these illustrations does not mean only "drawing nice pictures"; we have to know how to describe a surface in software environment. We can for example use a parameterization or synthetic construction of studied surfaces.



$$u \in (0,17)$$
$$v \in \left(-1,\tfrac{1}{2}\right)$$
$$x(u,v) = 10v + 10$$
$$y(u,v) = u$$
$$z(u,v) = -\tfrac{5}{11}u + 5 + v\left(\tfrac{3}{16}u^2 - \tfrac{38}{11}u + 5\right)$$

Figure 2: Example of the determination of a ruled surface using *GeoGebra*.

The typical task in descriptive geometry is to construct central or parallel projection (a two-dimensional image) of a surface or of its parts. An example of projections of a ruled surface is given in Figure 3. Firstly, the situation in the three-dimensional space is provided (i.e. 3D model), secondly two types of projections are shown - central and parallel projection of the same ruled surface (i.e. planar images). If we want to project complicated 3D object in parallel or central projections from general view point (i.e. we do not consider now simple top or front view of an object), we have to find a proper position in the three-dimensional space from which we can clearly observe this object. I use 3D modeling software for the determination this position and also for the automatic construction of the two-dimensional result of the projection. This is very useful tool because it is very difficult to estimate the proper view point of a surface manually. We can also use the determination of projection for testing purposes, i.e. the task for students is to construct the two-dimensional image of a surface in given projection from given inputs. Students can solve the tasks using 3D modeling software or they can draw the solutions by hand. It means that in this stage they work just in the plane. When using software, it is necessary to construct the silhouette of the surface; if drawn by hand, the aim is to depict some of the important curves on the surface. In both cases, the result is a planar image. An example of a geometric task is given in Figure 4, parallel projection and the determination of a ruled surface are given and the task is to construct an image of a surface (the result of the projection) and an intersection curve of a surface and a given cutting plane. 3D model of the constructed surface is also presented.

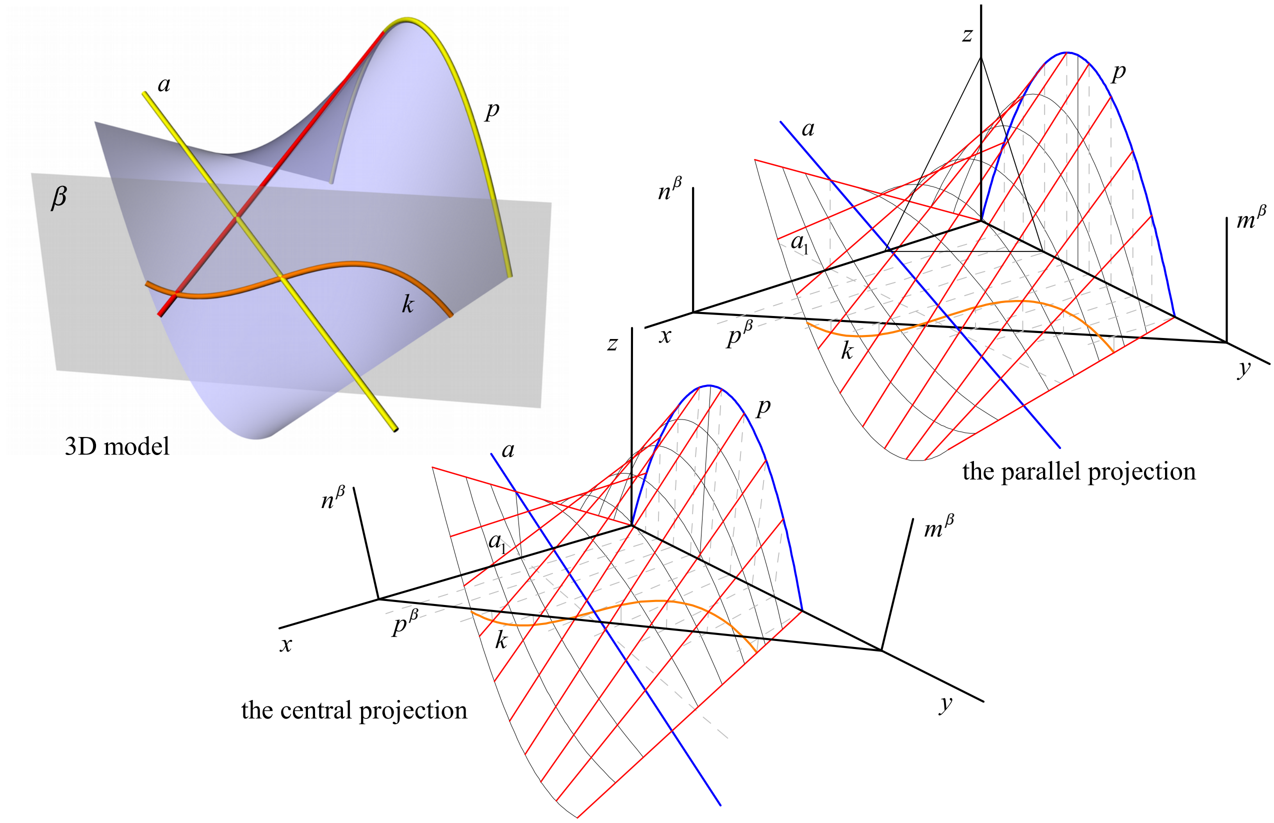


Figure 3: Example of projections of a ruled surface. The situation in the three-dimensional space, central and parallel projection of the same ruled surface.

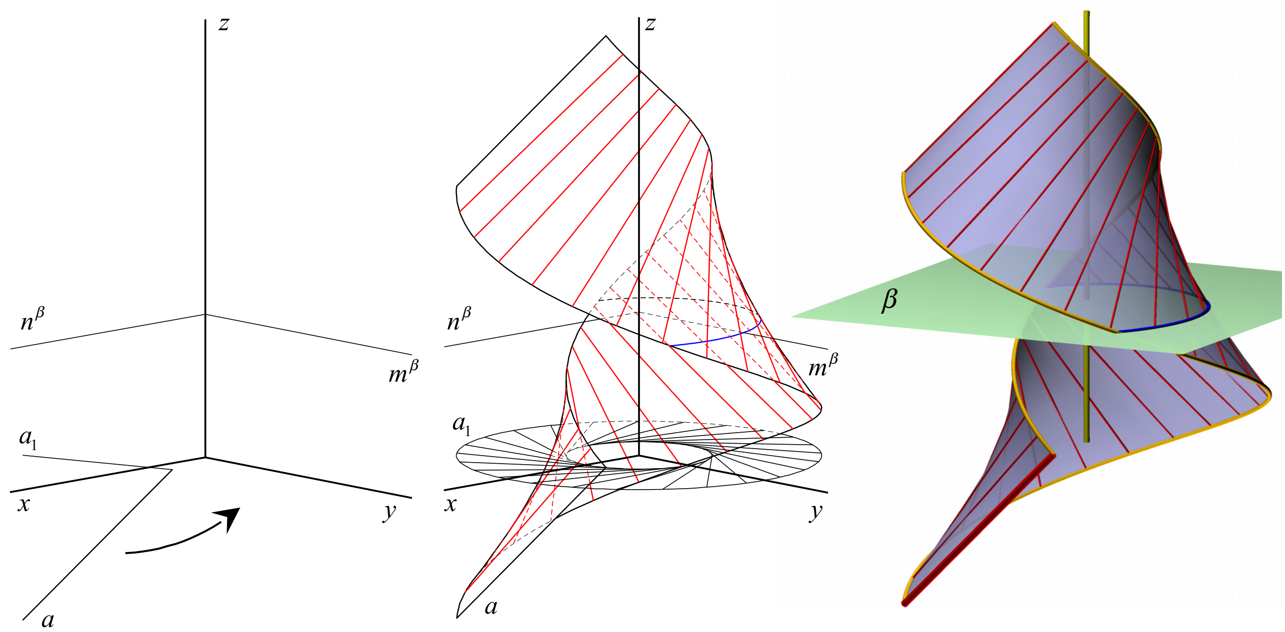


Figure 4: The determination of a parallel projection and a ruled surface. The result of the projection, an intersection curve of a surface and a given cutting plane, and 3D model of the spatial situation.

Students' work

My students meet 3D computer modeling within compulsory lessons of descriptive geometry and also can attend the seminars of applied descriptive geometry where practical applications of descriptive geometry and 3D modeling are mentioned and discussed. Students also use practically 3D modeling software during these lessons and seminars and can create themselves the outputs - 3D computer models and planar constructions. I am also supervising bachelor and master theses on various geometric topics where my students can use 3D modeling software. Figure 5 shows several examples of students' work.

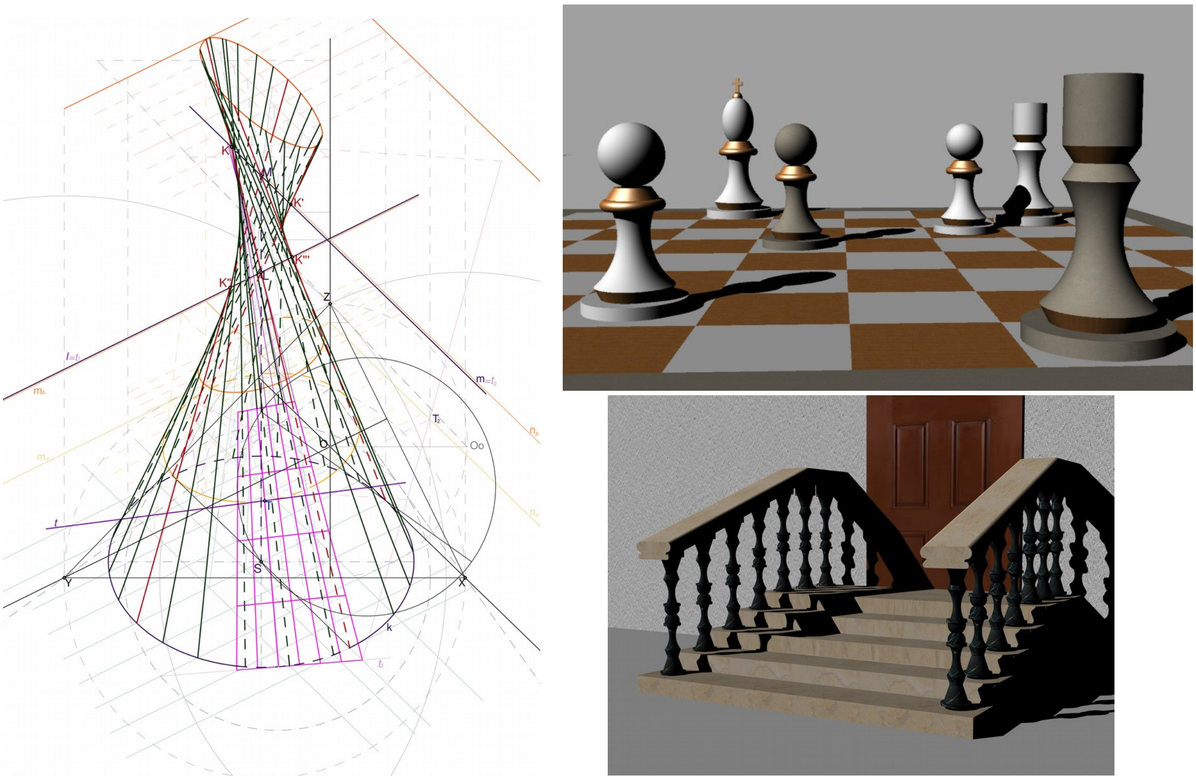


Figure 5: Students' work.

DISCUSSION, CONCLUSION, AND FUTURE WORK

I presented the possible methods of innovation in teaching descriptive geometry including 3D computer modeling and demonstrated this approach on examples of parallel and central projections of regular surfaces.

I have been teaching using described modern methods, activities and outputs for several years. When I started using it my aim was to minimize or even abandon the hand sketching and drawing. I had to change my mind very soon. According to my personal experiences computer aided classroom practice must be always accompanied by traditional explanation of geometry - i.e. hand sketching and drawing on the blackboard (lecturer and students) and on the sheets of paper (students) otherwise the students have the problems with taking notes during the lecture and understanding the individual steps of geometric constructions.

For instance, the projections which were shown in Figures 3 and 4, cannot be demonstrated to students only in this way. It is always necessary to show the construction of all lines, curves and points one by one. It even means that it is not enough when these objects just „appear" on the screen one by one, students have to see the process of their drawings. In these situations is usually better to draw the constructions by hand on the blackboard because students can easily follow the instructor. The computer outputs can be shown at the same time.

3D computer modeling is an efficient aid for creating the animations as has been shown in Figure 2. These animations can be still accompanied with physical models on which we can demonstrate a sequential construction of geometric objects. According to my experiences there will be always students in a classroom who prefer more physical models which can touch by hand.

Regarding the future work I plan to focus on preparing a new textbook on descriptive geometry and to continue on the creation of study materials, 3D computer models, and another outputs for descriptive geometry. Besides these plans I would like to deal with the didactic survey and explorations which were introduced in this article, moreover to discuss following questions:

- advantages and disadvantages of computer-aided education of geometry (in general of mathematics),
- computer-aided education support and develop students' skills for future occupation in technical fields and branches,
- students' hand sketching and drawing (the both - in the classroom and during home preparation) is still necessary in currently digital era.

REFERENCES

Eilam, E. (2005). Reversing: Secrets of Reverse Engineering. Wiley.

Farin, G., Hoschek, J., Kim, M.-S. (2002). Handbook of Computer Aided Geometric Design. The Netherlands: Elsevier Science.

Foley, J. D., Dam, A., Feiner, S. K., Hughes, J. F. (1995). Computer Graphics: principles and practice. Addison-Wesley Publishing Company

Hoschek, J., Lasser, D. (1993). Fundamentals of Computer Aided Geometric Design. Second edition, USA: A K Peters.

Lipson, H., Kurman, M. Fabricated (2013). The New World of 3D Printing. First edition, USA: John Wiley & Sons.

McNeel, R. (1999). Rhino NURBS 3D Modelling. New Riders.

Paré, E.G., Loving, R. O., Hill, I. L., Paré, R. C. (1996). Descriptive Geometry. Peachpit Press.

Pottmann, H., Asperl, A., Hofer, M. & Kilian A. (2007). Architectural Geometry. Exton, USA: Bentley Institute Press.

Robertson, R. G. (1966). Descriptive Geometry. Pitman Press.

Sarkar, J. (2015). Computer Aided Design: A conceptual Approach. First edition, USA: CRC Press.

Surynková, P. (2015). Reflections of Developments in Educational Techniques in the Design of a New Textbook on Descriptive Geometry. Proceedings of the 12th International Conference on Technology in Mathematics Teaching (ICTMT12), pp. 320-327, University of Algarve.

Surynková, P. (2013). Recent Advances in the Application of 3D Geometric Modeling Software with Focus on Linear Perspective. Proceedings of the eleventh International Conference on Technology in Mathematics Teaching (ICTMT11), pp. 336-337, University of Bari Aldo Moro.

SPATIAL–SEMIOTIC ANALYSIS OF AN EIGHTH GRADE STUDENT'S USE OF 3D MODELLING SOFTWARE

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The aim of this paper is to analyse the emergence of spatial–semiotic resources attached to an eighth-grade student's use of 3D modelling software while solving certain spatial tasks. The data comes from a task-based interview and it is analysed within a spatial-semiotic lens, including different kinds of resources not only based on the discourse, but also based on extra-linguistic expressions such as that sketches and gestures. The results of the study show that generally the student's reasoning steps explored a viewpoint for adding or removing cubes by use of the 'orbit' and 'select' tools, using ready-made mental pictures derived from completed steps, linking 2D and 3D representations through spatial visualisation and spatial orientation, emergence of spatial vocabulary including his strategies and generalizations.

Keywords: Spatial thinking, Spatial–Semiotic lens, 3D modelling software, Multimodal paradigm.

INTRODUCTION

The acts of thinking, constructing and expressing meaning through digital technologies are generally beyond words, but they can also be interlaced with our gestures, mimics and sometimes with specific sketches. Consequently, involvement of our sensory-motor functions' productions in our communication can be considered to be a *multimodal* process (Arzarello & Robutti, 2008). Following a multimodal paradigm, to interpret specific *signs* that emerge in communicating and/or expressing meaning, *semiotic perspectives* have received robust attention from mathematics educators (Arzarello, 2008; Godino, Batanero, & Font, 2007; Ng & Sinclair, 2013; Presmeg, Radford, Roth, & Kadunz, 2016).

Spatial thinking is a core concept in the teaching and learning of mathematics, which can be defined as an amalgam of different sub-skills in relation to geometric reasoning. Because of its importance, a number of epistemological analyses were conducted to elaborate and explain how individuals think spatially when they commence a mathematical task, and specific *spatial images* (Presmeg, 1986) and specific *processes* for 3D geometry and visualization (Bishop, 1983; Gutiérrez, 1996; Yakimanskaya, 1991) have been defined by researchers.

In this work, we acknowledge a combination of two paradigms, namely the synergy between the semiotic perspective-multimodal paradigm and spatial thinking, and consider the following research question: what kind of spatial–semiotic resources emerges when an eight-grade student solves spatial tasks with a 3D modelling software?

CONCEPTUAL FRAMEWORK

In order to analyse classroom activities with a *spatial–semiotic* lens (S-SL), Turgut (2017) proposes a conceptual framework based on the hypothesis that thinking spatially in a 3D modelling software environment is also multimodal. S-SL combines three theoretical constructs (i) *mental images* (Presmeg, 1986), (ii) *interpret figural information* (IFI) and *visual processing* (VP) (Bishop, 1983), and (iii) *Action*, *Production* and *Communication* space and the notion of *semiotic bundle* (Arzarello, 2008). These are used to look at the emergence of signs linked to spatial thinking. Following the multimodal paradigm, S-SL frames classroom productions that include specific signs, such as words, gestures, sketches and acts and so on, which are attachments to students', as well as the teacher's, spatial thinking processes. To do so, S-SL distinguishes spatial thinking as two major processes; IFI and VP. IFI includes the emergence of spatial vocabulary and the interpretation of visual images, while VP includes the emergence of Concrete Images (CI), Kinaesthetic Images (KI) and Dynamic Images (DI). CI can be considered as pictures in the visual memory, whereas KI refers to physical movements, and DI covers conceiving and manipulating dynamic mental images (Presmeg, 1986; Turgut, 2017). Figure 1 summarizes the S-SL and its components.

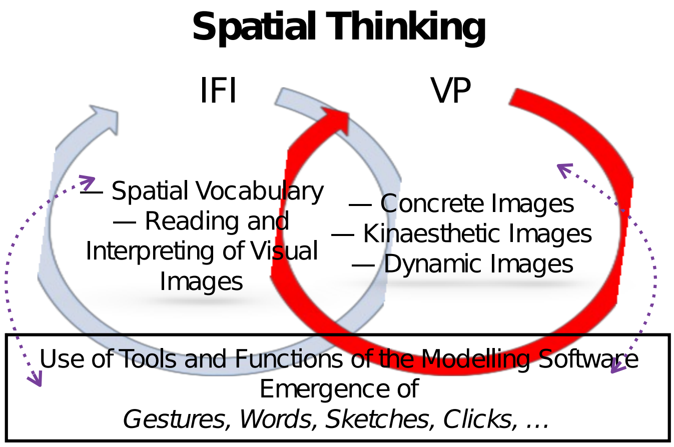


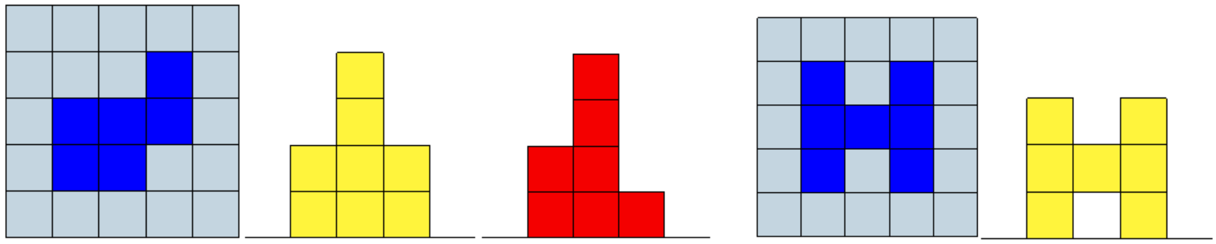
Fig. 1: S-SL with its components (modified from Turgut, 2017, p. 183)

Within the context of the present paper, we identify two strategies under IFI; a *spatial–analytic* strategy, meaning focusing on parts of the object, and a *spatial–holistic* strategy, which refers to comprehending and reasoning on the object as a whole. S-SL offers analysis on the emergence of signs through the notion of semiotic bundle (Arzarello, 2008), which constitutes two different, but complementary analysis tools; a *synchronic analysis* and a *diachronic analysis*. Synchronic analysis refers to 'the relationships among different semiotic resources simultaneously activated by the subjects at a certain moment', while a diachronic analysis means the 'evolution of signs activated by the subject in successive moments' (Arzarello, Paola, Robutti, & Sabena, 2009, p. 100).

METHODOLOGY

A task-based interview was conducted with an eighth grader, Atakan (pseudonym), who has a moderate level performance in mathematics. He has a desktop computer in his home and, as a result, he is competent in the use of basic computer tools. In order to research Atakan's spatial reasoning process, we considered 3D modelling software SketchUp® (SU) as an artefact, which is originally designed for engineering and model building. It should be noted that Atakan has experience in the use of SU since, as a part of a larger study he carried out 3D geometry tasks with the same software when he was in 7th grade.

In the context of acquisitions described in the Turkish middle school mathematics curriculum, we prepared two interrelated but different tasks. During the interview, we first proposed three (top, front and right) views of a building (Figure 2a, 2b, 2c) made up of unit cubes and asked Atakan to construct the building. This initial task included two main steps; (i) *constructing the building* using concrete unit cubes provided and (ii) using virtual cubes within SU that provides a zero-gravity environment with the aim of *making alternative 3D buildings*. In the second task, we asked Atakan to complete 3D buildings within only the SU environment according to top and front views given on the paper (Figure 2d, 2e).



(a) (b) (c) (d) (e)

Fig. 2. (a) Top, (b) front, (c) right views in 1st task; (d) top, (e) front views in 2nd task

The video-recorded interview lasted about an hour. In order to capture signs, we used two cameras in different positions as well as screen recorder software. A thematic analysis (Braun & Clarke, 2006) was employed covering all the collected data to elaborate Atakan's reasoning steps.

SPATIAL-SEMIOTIC ANALYSIS OF THE DATA

For the sake of presenting an evolution of the student's reasoning, we first briefly present a macro analysis of the initial step of Task 1. As the first step, Atakan built the first floor of the building in a way to provide the top view (building blocks parallel to the ground) to form the structure with concrete cubes in accordance with the views given in the worksheet. In the second step, he built the cube block in a vertical position relative to the ground) to form the front view without changing the top view. In the third step, he compared the right view of the structure (with the right view given in the worksheet) changing the viewpoint by bending. Finally, in the fourth step, without changing the top and front views, he put a cube in an appropriate place to complete the right view. By the end of the process, Atakan had built a structure using twelve cubes.

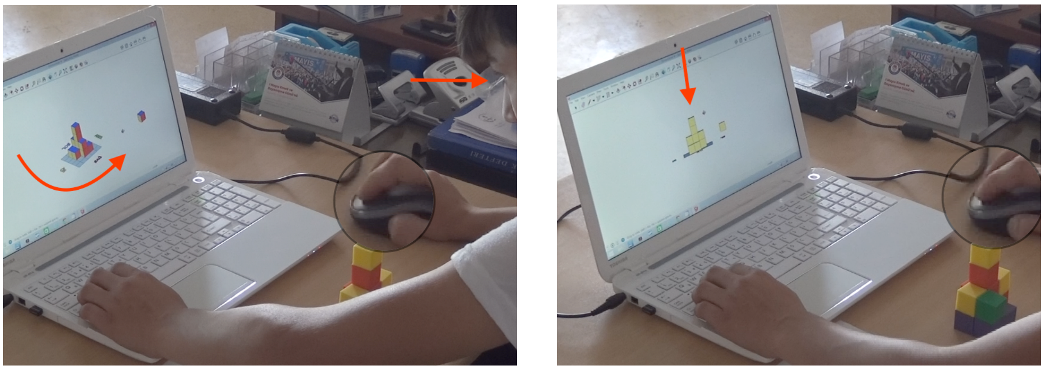
Synchronic and Diachronic Analyses of the Second Step of Task 1

Several spatial–semiotic resources appeared synchronously, when Atakan solved the second step of the Task 1 through SU. Table 1 briefly provides a summary of the most frequent spatial-semiotic resources categorized under the IFI and VP processes (SV: Spatial Vocabulary).

IFI		VP		
Spatial–Analytic	Spatial–Holistic	CI	KI	DI
–Exploring an appropriate viewpoint to add or remove a cube –Adding block cubes which are parallel or vertical to the base to obtain a top view –Focusing single views of the object	–Evaluating the object from different viewpoints –Reasoning which cubes that can be removed without changing the views –SV: expressing why front views isolated his strategies –SV: expressing strategies in relation to top and right views	–Using a mental picture derived from the paper and concrete object –Basing an obtained mental image in the completed (reasoning) step(s)	–Using the *Orbit* tool to complete different steps –Adding, moving or removing the cubes using the *Select* tool –Using the cursor for pointing out cubes or the object while explaining the situation	–Linking 2D and 3D representations mentally –Mental rotation with respect to given directions –Spatial orientation with respect to different viewpoints

Table 1. An overview of spatial–semiotic resources attached to the reasoning steps of Task 1

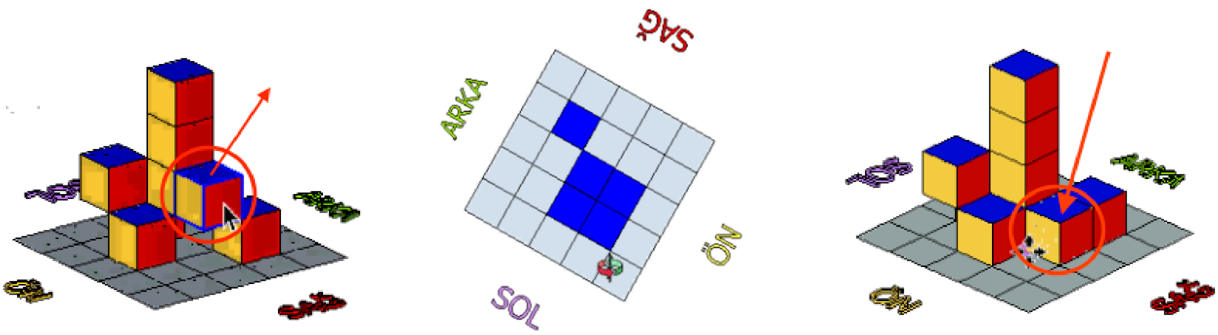
In order to present the emergence of specific resources expressed in Table 1, in the following statements, we summarize Atakan's reasoning steps for Task 1. At first, he repeated the steps in the initial part of Task 1 to create a representation of the structure (formed with twelve concrete unit cubes) in SU. In this process, by making use of the tool 'orbit', he made reasoning (using the tool slowly) about the procedures to be applied (KI, DI). He searched for a viewpoint appropriate to cube addition (using the tool fast) (KI) (Figure 3a), and he evaluated the top, front and right views of the structure he had formed (using the tool fast) (KI, CI) (Figure 3b). In the second part of Task 1, it was seen that without changing the top, front and right views, Atakan deleted a cube from the first floor in the process of transition from a 12-cube structure to an 11-cube structure (DI, KI), deleted a cube from the second floor in the process of transition to a 10-cube structure (DI, KI), deleted a cube from the first floor in the process of transition to a 9-cube structure (DI, KI), deleted a cube from the second floor in the process of transition to an 8-cube structure (DI, KI) and evaluated the views of the new structure at the end of each step (CI, KI).



(a) (b)

Fig. 3: (a) Atakan's exploration for a viewpoint (b) Evaluating the object from different viewpoints

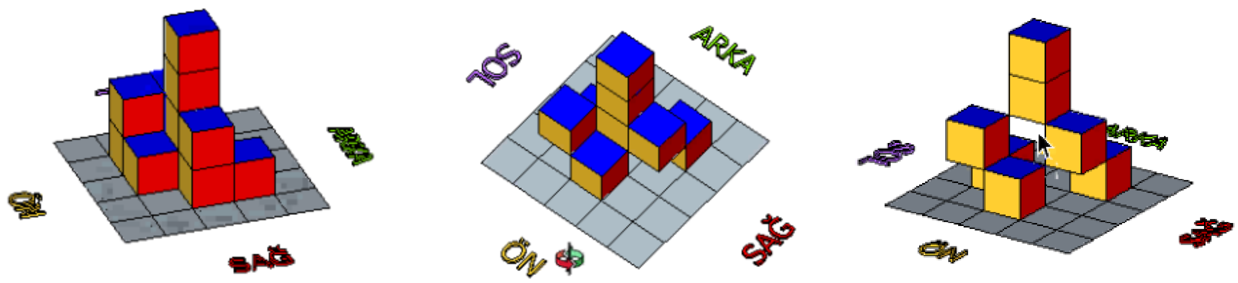
In the third part of Task 1, it was seen that without making any changes in the top and right views, Atakan changed the top view by deleting a wrong cube from the second floor in the process of transition from an 8-cube structure to a 7-cube structure (KI) (Figure 4a), recognized the wrong strategy in the second step (CI) (Figure 4b) and placed the cube (he had deleted) unintentionally in the first floor rather than in the second floor (Figure 4c) while trying to cancel this deletion (KI).



(a) (b) (c)

Fig. 4: (a) Deleting wrong cube, (b) Evaluating the top view, (c) Replacing the deleted cube

In the third step, Atakan examined the structure he had formed previously with concrete cubes when he failed to develop a strategy for transition from the 8-cube structure to the 7-cube structure, and he returned back to the 11-cube structure by adding cubes (KI) (Figure 5a). In the following steps, the participant used the cube-deletion strategy, respectively, to form 8-cube structure (Figure 5b), and finally to form the 7-cube structure (Figure 5c) that provided the top and right views and reached the correct result (DI, KI, CI). In the 2nd and 3rd parts of Task 1, Atakan, with the help of 'orbit'; (i) did reasoning in relation to solution strategies (using the tool slowly) (KI, DI), (ii) evaluated the views of the new structures formed (using the tool fast) (KI, CI) and (iii) searched for a viewpoint appropriate to cube-deletion and cube-addition (using the tool quickly) (KI). When the researchers asked Atakan why he had returned back to the 11-cube structure from the 8-cube structure, he replied, "*Well, it didn't work. I had formed according to the front view… the previous shape*" (SV).



(a) (b) (c)

Fig. 5: (a) 11-cube building, (b) 8-cube building, (c) 7-cube building

In order to summarize a combination of synchronic and diachronic analyses of Task 1, i.e., to articulate specific signs with respect to evolution of reasoning, we borrow the notion of *semiotic chain* in (Bartolini Bussi & Mariotti, 2008) and express Figure 6 to overview an evolution of Atakan's reasoning process.

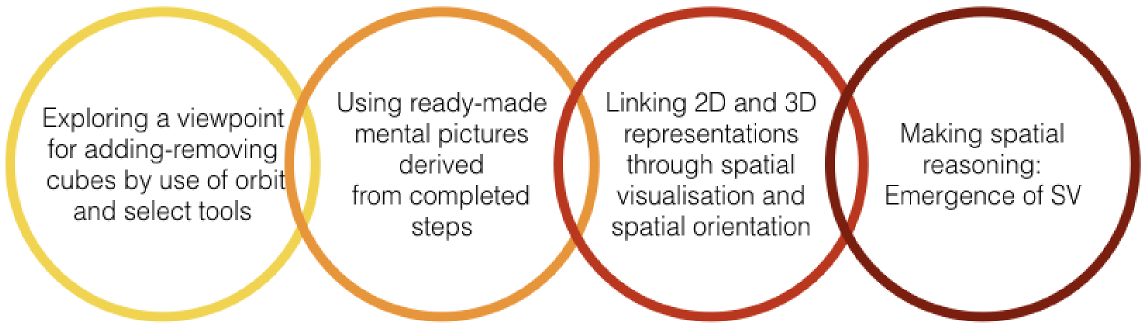
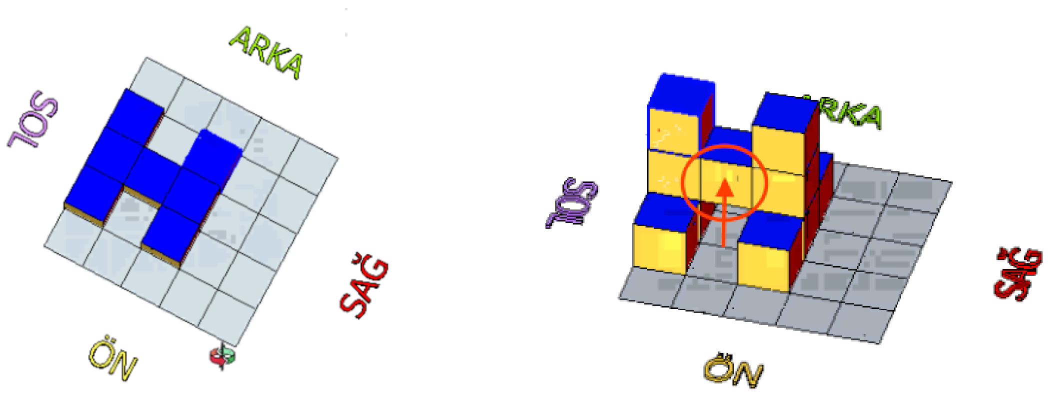


Fig. 6: A semiotic chain shows an evolution of Atakan's reasoning for Task 1

Synchronic and Diachronic Analyses of Task 2

Because the aims of the second step of the second task and the third task are close, the emergence of spatial-semiotic resources was similar to Table 1. However, in the second task, Atakan's strategies differed, where in this case two views of the building were provided on paper. Therefore, he exploited his experience coming from the first task and, in this way, he developed new insight for exploring the situation and all of this changed the IFI and VP columns in Table 1. Another fact is that, in the present case, the SV is more apparent compared to Task 2. Table 2 summarizes the emergence of specific signs.

Atakan first focused on building the first floor of the structure to form the top view in the first step and formed the cube-block in a position parallel to the ground. In this process, Atakan changed the viewpoint on the screen with the help of 'orbit' (fast use) to add the cubes where needed (KI). In addition, the participant considered the direction codes on the screen and rotated the image for the top view in his mind as appropriate to the direction codes while building the first floor of the structure (DI) (Figure 7a).



(a) (b)

Fig. 7: (a) Initial structure, (b) Moving the cube from the first to second floor

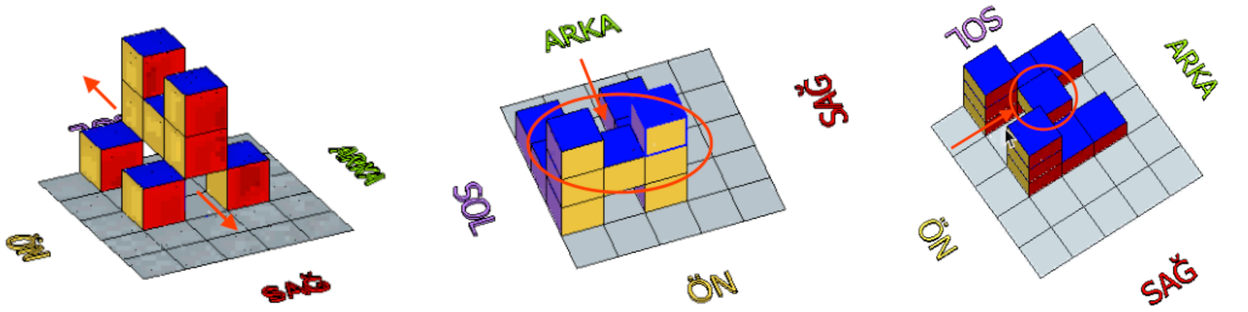
In the second step, Atakan focused on the second and third floors of the structure as appropriate to the front view given in the worksheet and realized that the first floor he had formed in the first step provided the top view but not the front view (CI). In addition, he carried one of the cubes from the first floor to the second floor (DI, KI) (Figure 7b).

IFI		VP		
Spatial-Analytic	Spatial-Holistic	CI	KI	DI
–Exploring an appropriate viewpoint to add or remove a cube –Adding block cubes which are parallel or vertical to the base to obtain a top view –Focusing single views of the object –SV: emphasis on a partial solution strategy –Work (temporarily) on the cubes that satisfy a front view while not satisfying a top view –SV: evaluating the different views part by part –SV: explaining why he could not develop a strategy for removing cubes with respect to the floors of the cubes –Focusing only on removal of the cubes, and as a result of this, failing to visualise of the object with 7 cubes	–Evaluating the object from different viewpoints –Reasoning on the cubes that can be removed but which do not change the views –SV: explaining and pointing out the cubes that can be moved but also satisfying the views –Determining symmetric cubes satisfying two different views when they are deleted –SV: reasoning on the relationship between the front and rear views – Building the object from the beginning for developing new strategies –SV: a new strategy for moving cubes on the third floor –SV: generalizing strategy of removing cubes to satisfy top view	–Using a mental picture derived from the paper –Basing single views (top, front and so on) of the object – Basing obtained mental images in the completed (reasoning) step(s)	–Using the ‘*Orbit*’ tool to complete different steps –Adding, moving or removing the (symmetric) cubes and/or *blocks* using the ‘*Select*’ tool –Using the cursor for pointing out cubes or the object while explaining the situation –Using the zoom in-zoom out tool –Deleting the whole object	–Linking 2D and 3D representations mentally –Mental rotation with respect to given directions –Spatial orientation with respect to different viewpoints –Visualising new views of the object when some cubes are moved –Visualising different views synchronously in the case of removing and/or moving the cubes

Table 2. A summary of spatial–semiotic resources attached to reasoning steps of Task 2

Following this process, Atakan built a cube block in a vertical position to the ground (KI) and formed a structure that provided the front view. In addition, it was seen that Atakan searched for a viewpoint appropriate to cube addition with the help of ‘orbit’ (fast use) (KI) and evaluated views of the structure (CI). In the second part of Task 2, Atakan focused on building the structures that provided top and front views using fewer cubes. In this process, Atakan focused on symmetrical cube pairs on the right and left sides that did not change the top and front views when deleted (DI) and he deleted two symmetrical cubes from the first floor (KI) (Figure 8a). Following this, while the participant evaluated the top and front views of the new structure with the help of ‘orbit’ (fast use) (CI, KI), the researchers asked him whether there was an alternative solution, which included nine cubes. Within the scope of this question, it was seen that Atakan initially replaced again the two symmetrical cubes he had deleted (KI) and then simultaneously examined the 11-cube structure and the views given in the worksheet to produce new strategies (DI). In such a way, it was also seen that Atakan examined the structure from different viewpoints with the help of ‘orbit’ (slow use) (KI), searched for the cubes that would not change the views when deleted, and failed to produce solution strategies.

Therefore, the researchers asked Atakan whether he would be able to form an alternative 11-cube structure with the same top and front views. Within the scope of this question, to begin with, Atakan simultaneously examined the 11-cube structure and the views given in the worksheet (DI) and then said the block which formed the second and third floors could be moved one unit backward or one unit forward (DI, SV). In the following step, he moved this cube block one unit forward (KI) (Figure 8b). Following this, Atakan, with the help of ‘orbit’ (fast use), evaluated the top and front views of the structure (CI, KI) and saw that the top view had changed. As a result, he moved one cube on the second floor to provide the top view (KI) (Figure 8c). Following this step, Atakan evaluated the views with the help of ‘orbit’ (CI, KI), realized that the top view was again wrong and deleted one cube in the second floor, which changed the top view (KI). Following this strategy, in which the participant did not change the top and front views, he evaluated the views with the help of “orbit” again (fast use) (KI, CI) and said that the alternative 11-cube structure was complete (SV).



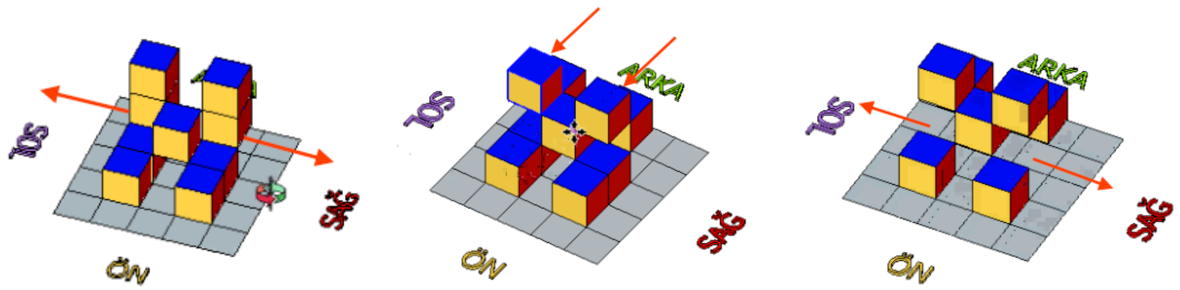
(a) (b) (c)

Fig. 8: (a) Deleting symmetrical cubes, (b) Moving cube block forward, (c) Moving the cube backward

In the next part, the researchers asked Atakan whether he could work on the structure and form an alternative 9-cube structure. Within the scope of this question, the participant, with the help of ‘orbit’ (slow use), searched for symmetrical cubes, which would not change the top and front views when deleted (KI, CI, DI) and said that he failed to find a strategy to form such a structure (SV). In the following process, Atakan continued his search with ‘orbit’ (slow use) (KI) and realized that there would be no change in the top and front views when two symmetrical cubes in the first floor were deleted. The participant deleted the symmetrical cubes he had determined (KI), and he then evaluated the top and front views of the new structure with the help of ‘orbit’ (fast use) (CI, KI).

In the final part, the researchers asked Atakan whether he could form a 7-cube structure without changing the top and front views. Within the scope of this question the participant, with the help of ‘orbit’ (slow use), searched for cubes that would not change the top and front views when deleted (KI, CI, DI). As a result, Atakan reasoned in relation to the 9-cube structure and the views in the worksheet (DI), but failed to develop a strategy to form the 7-cube structure at the end of the process. Eventually, he deleted all the cubes on the screen to re-form the 11-cube structure (KI, SV). Atakan, who started building the structure again, this time formed the cube block in a vertical position to the ground to complete the front view (CI, KI). This block was built in such a way as to form the rear of the structure differently from his previous structures.

In the next part, the participant examined the structure from the top with the help of 'orbit' (fast use) (KI) and saw that one of the cubes he had added to the second floor changed the top view (CI). Therefore, he moved this cube one unit forward (DI, KI). Following this, Atakan worked on the cube block in a position parallel to the ground and built the first floor (KI) in such a way as to complete the top view without changing the front view (CI, DI). As a result, he completed the 11-cube structure. The participant deleted two symmetrical cubes from the first floor during transition to the 9-cube structure (KI) (Figure 9a). Next, he used the 'zoom' tool to examine the structure in more detail (KI). Lastly, with the help of 'orbit' (slow use), he searched for cubes he could delete to make transition to the 7-cube structure (KI). In this process, Atakan did reasoning in relation to the 9-cube structure and regarding views given in the worksheet (CI, DI). He said that he did not make transition to the structure with the cube-deletion strategy as demanded in the question (DI, SV). In this respect, when the researchers asked Atakan whether he had developed his thinking strategy based on a cube-deletion strategy, he responded positively to this question and said he would think about the structure a bit more and move two symmetrical cubes in the third floor one unit forward. He then added that these cubes would hang in the air at the end of the process without changing the views (DI, SV). Following this, the participant moved the symmetrical cubes in the third floor one unit forward (KI) (Figure 9b). Next, he added the cubes to places where he wanted and examined the structure with the tool of 'zoom' (KI). After this, he deleted the symmetrical cubes in the first floor, which were under the symmetrical cubes he had moved forward (DI, KI) (Figure 9c). In the last step, Atakan evaluated the top and front views with the help of 'orbit' (fast use) (CI, KI).



(a) (b) (c)

Fig. 9: a, b, c Process of transition from 9-cubes building to 7-cubes building

At the end of the solution process, the researchers asked Atakan to explain his reasoning processes, and he said that the cubes placed under one cube in the upper floors were not visible from the top view and that deleting the cubes below would not change the top view (SV). In this respect, Atakan reported, "*from the top view, we see the upper cubes, and the ones below are not visible. If we take the ones below, those at the top look the same*". In addition, Atakan stated that he evaluated how simultaneously the deletion process, which did not change the top view, did not change the front view (SV). In this respect, Atakan said, "*When we did not move to the front and if I take these* (showing the symmetrical cubes he had deleted from the first floor in the last step)*, then these* (coming to the top view rapidly with the help of 'orbit') *would have looked as if they had been removed* (pointing to the procedure that changed the top view). *In addition, if I had taken these* (showing the symmetrical cubes in the second floor at the back) ... *they would have remained at the back* (showing the symmetrical cubes in the third floor he had moved one unit forward) … *Then they would have looked* ... (taking the front view rapidly with the help of 'orbit' and showing the spaces that would appear in the front view at the end of the process)."

Figure 10 refers to a combination of synchronic and diachronic analyses of Atakan's reasoning processes associated with Task 2.

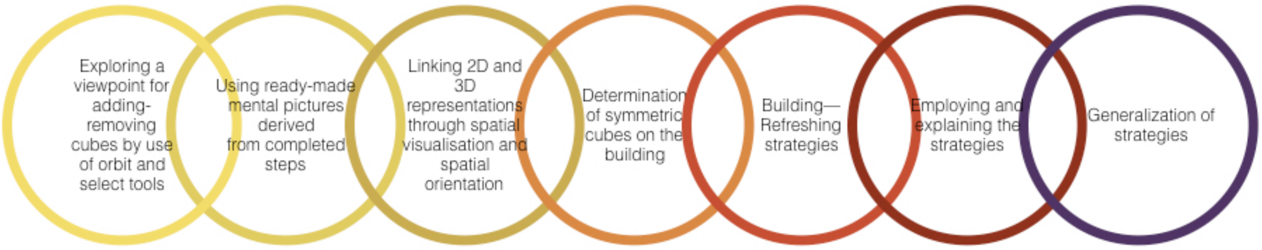


Fig. 10: A semiotic chain shows an evolution of Atakan's reasoning for Task 2

CONCLUSIONS

In this paper, we consider the following research question: 'What kind of spatial-semiotic resources emerge when an eighth-grade student solves spatial tasks using 3D modelling software?' Spatial-semiotic analyses of the data obtained provided us with a detailed understanding of student's spatial reasoning processes in SU. In the first task, the student easily built the structure with concrete unit cubes whose different views provided on the paper. In the second task, the student's reasoning steps appeared with an emphasis on a spatial-analytic strategy based on exploring a viewpoint for adding or removing cubes, using ready-made mental pictures, linking 2D and 3D representations through spatial visualisation and spatial orientation, and an emergence of spatial vocabulary, including his strategies. However, in the third task, certain specific reasoning steps appeared as spatial-holistic strategies more than in the previous task, such that focusing on an environment with zero-gravity, symmetric cubes, and constructing and explaining strategies, interlaced into completed steps in the second task. Within the context of our study, gestures were limited to in the use of specific tools ('orbit', 'select', mimicking with cursor, ctrl+v, delete and 'zoom'). There did not appear to be any gestures independent of the artefact (mouse and keyboard), such as hand movements, tracing with a finger and so on.

In terms of the obtained results, we finally summarize the synergies among the KI, CI, DI and VP and IFI processes through Figure 11, which are a theoretical contribution and an attachment to Figure 1.

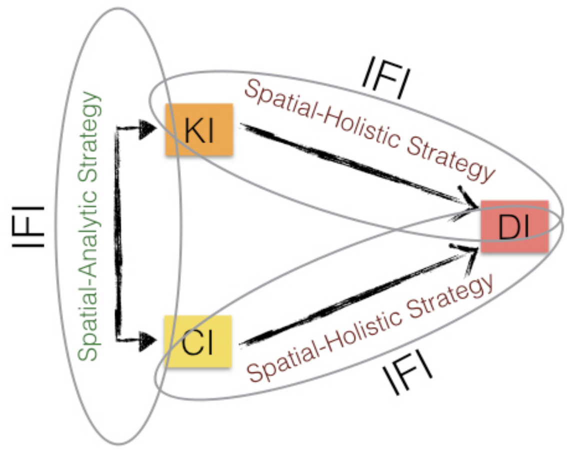


Fig. 11: Synergies between spatial thinking processes

Figure 11 implies that spatial–analytic and spatial–holistic strategies that we consider in this paper commonly intertwined with IFI process and emergence of KI, CI and DI. IFI process always emerged when the student solved spatial tasks and this appears to be that IFI is *the core element* in spatial thinking and creation of DI. The emergence of signs confirmed that the student's initial strategy was spatial–analytic, and specific images were KI and CI. The next step was emergence of DI in terms of spatial–holistic strategy and IFI process. However, these results come from only an eighth grader's result, it will be meaningful to explore a group of students' results to discuss articulation of Figure 1 and Figure 11 in a further research.

REFERENCES

Arzarello, F. (2008). Mathematical landscapes and their inhabitants: perceptions, languages, theories. In E. Emborg & M. Niss (Eds.), *Proceedings of the 10th International Congress of Mathematical Education* (pp. 158–181). Copenhagen: ICMI.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97–109.

Arzarello, F., & Robutti, O. (2008). Framing the embodied mind approach within a multimodal paradigm. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of International Research in Mathematics Education* (2 ed., pp. 720–749). Mahwah, NJ: Erlbaum.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of International Research in Mathematics Education* (2. ed., pp. 746–783). Mahwah, NJ: Erlbaum.

Bishop, A. (1983). Spatial abilities and mathematics thinking. In M. Zweng, T. Green, J. Kilpatrick, H. Pollak, & M. Suydam (Eds.), *Proceedings of the Fourth International Congress on Mathematical Education* (pp. 176–178). Boston: Birkhäuser.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*, 77–101.

Godino, J. D., Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM-Mathematics Education, 39*(1), 127–135.

Gutiérrez, A. (1996). Visualization in 3-dimensional geometry: In search of a framework. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (pp. 3–19). Valencia: Universidad de Valencia.

Ng, O.-L., & Sinclair, N. (2013). Gestures and temporality: Children's use of gestures on spatial transformation tasks. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 361–368). Kiel, Germany: PME.

Presmeg, N. C. (1986). Visualization in high school mathematics. *For the Learning of Mathematics, 6*(3), 42–46.

Presmeg, N. C., Radford, L., Roth, W.-M., & Kadunz, G. (2016). *Semiotics in Mathematics Education*. Switzerland: Springer International Publishing.

Turgut, M. (2017). A Spatial–Semiotic Framework in the Context of Information and Communication Technologies (ICTs). In M. S. Khine (Ed.), *Visual–spatial Ability in STEM Education: Transforming Research into Practice* (pp. 173–194). Switzerland: Springer International Publishing.

Yakimanskaya, I. S. (1991). *The development of spatial thinking in schoolchildren* (Vol. 3). Reston, USA: NCTM.

Workshops

MATHEMATICS IN PRE-SERVICE TEACHER EDUCATION AND THE QUALITY OF LEARNING: AN EXPERIENCE WITH PAPER PLANES, SMARTPHONES AND GEOGEBRA

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This poster presents an instance of mathematics modelling by students on a geometry activity in a pre-service teacher training course analysed through a model designed to access the mathematical thought and the quality of student leaning outcomes. This analytical model, supported by the SOLO taxonomy, uses Activity Theory as a contextual framework that integrates the different relations, namely advanced mathematical thinking concepts like procept and proceptual divide. Results allowed us to see three different pathways taken by the students to solve the same problem.

Keywords: Advanced mathematical thinking, geometry, proceptual divide, quality of learning, SOLO taxonomy.

INTRODUCTION

Portfolio assessment brings an open evaluation method into the mathematical classroom and allows the mathematical abilities of the students to grow. In this study, students use paper planes, smartphones and dynamic geometry software (geogebra) to find the equation that describes the flight path of the paper planes and were given two classes to solve them and to explain in detail their solution process. This solution process involves brainstorming sessions centred on the best solution, and the detailed explanation necessarily involved in self-regulated learning processes. This teaching method aims to extend the mathematical knowledge of future teachers, involving them in activities more open and less structured than the traditional ones.

The data presented here were chosen because it highlights three different path and took different approaches to the same problem. Data was studied using the analytical model that highlights these differences and integrates SOLO taxonomy (Biggs & Collis, 1982) with the advanced mathematical thinking theories and concepts of Tall (2002) alongside the conceptualization of the proceptual divide (Gray & Tall, 1994), and activity theory (Engeström, 2001) as a contextual structure. The SOLO taxonomy allows us to identify five progressive levels of understanding from the *prestructural* (lowest level), through the *unistructural*, the *multistructural*, the *relational* to the *extended abstract* (highest level).

The task statement asks to find the equation that describes the flight path of a paper plane, using smartphones and geogebra, writing a step-by-step report of the activity. In the first class students started by folding the paper planes (exploring somehow the art of origami), all the instructions were given using geometry concepts like area, midpoint, segment bisector and so on. With the planes built, the group moved outside the classroom and, using the cameras of their smartphones, photographed the various moments of the flight of the planes since its launch, the flight itself and the landing. Photographs sequences were drawn according to contemplate various angles, forces and aircraft launch positions and capture the movement of the same in flight and subsequent landings.

After the images were collected, they were recorded on the computers and using geogebra, the flight sequence of the various planes was simulated using a Cartesian axis and the plane's nozzle as a point of reference for marking points. With the complete sequence of the flight scanned in points in the software the described curve of the airplane was worked in order to find one, or more equations of the function described by the airplane. This trajectory (usually a parabola - represented by a second degree equation) can be shown and students can control the parameters of the same, appropriating algebraic concepts. In this case, factors like the idea of air resistance and elevation were not contemplated, which would allow other types of explorations (which were not the objective of this task).

FINAL REMARKS

This form of mathematical modelling becomes attractive and allows students to visualize mathematics to take shape as a real activity, evading the notion of common sense that mathematical concepts do not represent reality. Geogebra was introduced in the task, and students learned their handling by trial-and-error. To finalize the exploration task, hypothetical situations were created with the software that served as an exercise to study the characteristics of a quadratic function and the parameters of its variables. Experience had good adhesion by the students and were in some expectant form on the final results, some of the comments involved the strangeness of a math class have to go out of the classroom (which, in most cases it was an absolute novelty).

The results of the analysis identify three levels of response: a first level, where the construction of the function fails to recognize the graphical movement of the various parameters of the quadratic function, identify the vertex, place the parabola in the correct place, but show difficulties in adjusting the aperture of the parabola (roots of the quadratic equation) to the curve of the airplane were classified as of the prestructural level; a second level where, when debating with the same problems of the previous level, they use the calculation (with paper-and-pencil or with graphing calculator - even having the software of dynamic geometry available) to adjust the parameters of the function and were classified as multistructural level, possibly relational and; a third level where using the functions of the selectors adjusted the parabola to the plane curve, arriving with some ease to the quadratic equation classified as relational level, possibly extended abstract.

Given the constraints, such as the lack of knowledge of the software and the lack of habituation to tasks of this kind, allowed the students to have a different view of what can be an attractive math class for the students, which is evidenced with some of the conclusions of the students themselves:

> Student A: We can also verify that math can be fun and does not have to be just worked indoors and with the use of new technologies we can catch attention and perform a certain type of exercises easier (student report).

REFERENCES

Biggs, J., & Collis, K. (1982). *Evaluating the quality of learning*. London: Academic Press.

Engeström, Y. (2001). Expansive learning at work: toward an activity theoretical reconceptualization. *Journal of education and work*, *1*(14), 133-156.

Gray, E., & Tall, D. (1994). Duality, ambiguity and flexibility: a proceptual view of simple arithmetics. *The journal for research in mathematics education*, *2*(26), 115-141.

Tall, D. (Ed.). (2002). *Advanced mathematical thinking*. New York: Kluwer Academic Press.

THERE IS MORE THAN ONE FLIPPED CLASSROOM

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The Flipped Classroom pedagogy has been developed for being responsive, student-centered and promoting self-directed learning. Three years ago, we started an international research project aimed at understanding how the FC can be implemented by secondary school math teachers through the use of a MOOC (Massive Open Online Course) developed at the Polytechnic of Milan. In particular, we focus on the teachers' use of MOOC videos. A variety of scenarios emerged from our direct classroom observations and work-in-team with the teachers. In this paper we propose a sketch of such a variety of FC implementations.

Keywords: MOOC; teaching with technology; student-centered learning; teachers' beliefs.

INTRODUCTION AND THEORETICAL FRAMEWORK

Flipped Classroom (FC) is mostly associated with university setting and it is commonly known as a method that arranges the lecturing part of the teaching as homework through videos. This is considered the out-of-class part of the FC and as such can be seen as a case of technology used for teaching and learning. When students come to class, FC features the students' learning in a student-centered manner, using various problem-solving activities in small groups (Bergmann & Sams, 2012). This is considered as the in-class part of FC. Both parts are vital for the FC learning model to work.

The out-of-class video learning "primes" the students for the in-class active phase. This happens with some difficulties, as Fredriksen, Hadjerrouit, Monaghan and Rensaa (in press) have singled out: (a) the students expect to be "taught" by the teacher; (b) the students express preference towards solving the exercises in solitude; (c) preparation through video lessons requires discipline; (d) the need to express mathematical problems verbally requires fluency in discourse; (e) group work requires social skills to be developed. We would say that MOOC videos are a kind of technology that promotes self-directed learning: the quality of student group collaboration and understanding, and overall the quality of the lesson, depends on how the students grasp the mathematics in the videos and on how they work out-of-class (Fredriksen *et al.*, in press). The main idea of FC is that, by saving time in introducing material, a teacher obtains an opportunity to challenge the students at both a collaborative and conceptual level through well-designed mathematical activities (Wan, 2015).

To sum up, we draw on evidence in literature that FC is a student-centered pedagogy that prompts students to go beyond rote-learning and may even promote conceptual understanding. At the same time, we acknowledge that FC is all but easy to implement in the classroom since it necessitates a significant change in the classroom's rules and practices. Moreover, we recall that recent research reports on an existence of a considerable gap between the learning potential of technology and actual teaching practices, a gap that is both qualitative and quantitative (for an elaborated review see Bretscher, 2014). The quantitative gap is understood in terms of the limited impact that new technologies have on classroom practices compared to the huge amount of money and time spent on technology and teachers' training to use it. The qualitative gap refers to the majority of teachers who use technology in a transmissive or teacher-centered way compared to the ones who exploit it for learner-directed activities. Despite curriculum changes, professional development and substantial financial investment, mathematics classroom practices are often still surprisingly similar to those practiced decades ago (McCloskey, 2014). Windschitl and Sahl (2002) have identified two factors that appear to be crucial to the ways in which teachers adopt or resist changes: (i) their beliefs about learners, about what counts as good teaching in their institutional culture, and about the role of technology in learning; (ii) the resources available at school. Thus, it becomes crucial to investigate teachers' goals, resources and orientations towards FC in general and towards MOOC videos in particular is.

In order to analyse teachers' attitudes towards technology (MOOC videos, in our case) and student-centered lessons, their goals, their knowledge and the resources they have at disposal, we refer to Schoenfeld's (2011) theoretical lens, which focuses on teachers' beliefs, goals and resources during in-the-moment classroom decision making. The basic assumption of Schoenfeld's framework is that beliefs and orientations are an essential factor shaping teachers' decision-making, and thus shaping their behavior and professional development. In Schoenfeld's view, teachers' behaviors also depend on their goals and goals recruit resources (including: knowledge, materials, personal and interpersonal skills and connections):

> Every sequence of actions can be seen as consistent with a series of goal prioritizations that are grounded in the teacher's beliefs and orientations, and the selection, once a goal has been given highest priority, of resources intended to help achieve that goal (Schoenfeld, 2011, p. 460).

A goal, whether explicit or tacit and unarticulated, is something that a teacher wants to accomplish. Resources include all kinds of 'goods' that are available for a teacher. For example, the tools in the classroom; students' knowledge; teachers' knowledge, interpersonal skills and relations with students.

RESEARCH QUESTIONS AND METHODOLOGY

The research questions we aim at addressing are: (1) how do secondary school math teachers plan to and actually integrate MOOC videos in their classrooms? (2) How do teachers promote self-directed and student-centered learning when using MOOC videos? In a pilot study with two teachers (Andrà, Brunetto & Kontorovich, under review), we compare and contrast their goals, resources and orientations. At the present stage of the research, we have at disposal more data coming from a larger set of teachers who participated in the research project. The data for this paper is concerned with the final weeks of the first semester, when the teachers need to arrange suitable activities to recap mathematics studied during the school year and to help students to end the semester with good marks. Thus, we proposed teachers to integrate MOOC videos, accompanying exercises and their solutions into their classrooms. Introducing new technology for recalling not new mathematical content was aimed at preventing students from facing a double difficulty: the difficulty of adjusting to a new way of teaching and learning and the difficulty of engaging with unfamiliar mathematics.

As a part of our project, twelve secondary teachers chose 1-3 MOOC videos to work with from a MOOC course made of 84 videos, covering different topics (arithmetic, sets, logics, algebra, analytic geometry, exponential and logarithms, trigonometry, probability and statistics). For each topic, 3-6 videos recap mathematical theory (definitions, properties and theorems), and procedures (algorithms and computations). Among the teachers who participated in the study, Nadia (N), Francesca (F) and Elisabetta (E) teach in three grade-12 classes, where it was necessary to recall exponential and logarithms in the first two ones, and the use of Excel spreadsheet for descriptive statistical analysis in the third one. Following Schoenfeld, we explored goals, resources and orientations through teachers' lesson images and conducted lessons, paying specific attention to unplanned decisions that were made. N teaches in a school where math lessons are delivered 6 hours/week, while F and E 3 hours/week.

DATA ANALYSIS AND RESULTS

Table 1 contains excerpts of the three teachers' goals, orientations and resources, in particular their description of the classes in relation with how they expect they will work with the MOOC and their orientations towards MOOC.

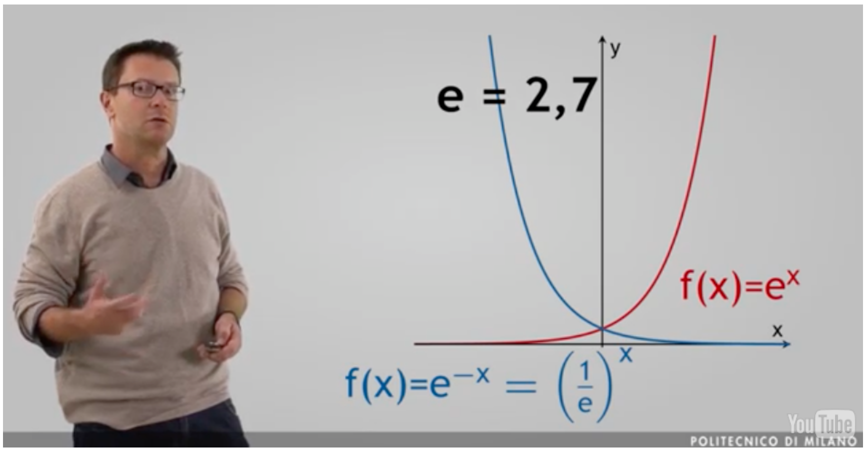
	Nadia (N)	Francesca (F)	Elisabetta (E)
Topic	Exponential and logarithms		Descriptive statistics
Math lessons	6 hours/week	3 hours/week	
Goals	I want my students to do not panic if some steps in a procedure are not made explicit, if one cannot grasp something at first or if different parameters are used.	[My goal is] To recap exponentials and logarithms. I also have non-math goals: to favour autonomy, to stimulate curiosity and to provoke critical thinking towards multimedia resources.	I want my students to become able to use Excel spreadsheet to compute the relevant statistics for data. They should become able to understand a video even if it uses different symbols and different words.
Resources: Excerpts from teachers' descriptions of their students	They are used to the FC and I expect that their major difficulties will be with logging in and with the organization of the courseware.	It's a class of only girls and they are really cooperative and collaborative with me. Some of them are good in math.	It's not easy to engage this classroom in math activities: one girl is the leader of the class and she wants to be the best at everything. If someone shows her ability, she punishes her mate.
Orientations: Teachers' feelings towards MOOC	I feel good with technology. I am interested in MOOC since the graphics are really good and the quality of videos is excellent. The advantages of FC are to save time that can be invested in group activities and the students can hear the voices of more than one teacher, so that they access different ways of dealing with the same math concept.	I feel good with technology. The advantages of using MOOC are: saving time, better understanding since the students can stop the videos, favouring the students' self-confidence with technology. Video-lessons are attended at home, where students are comfortable, but at the same time there's a risk they won't work, compromising the efficacy of FC. A drawback is the impossibility to make questions and to receive answers from the teacher in the video. This flaw can be dealt the day after, at school, with their teacher.	I feel good with technology. I believe that the topics that can be better introduced with MOOC videos are those that are procedural. In this way, the MOOC interferes less with the way the teacher wants to introduce the mathematical theory. For example, I would like to introduce the logarithmic function in class and leave the students work at home with translations of the function.

Table 1: teachers' goals, resources and orientations.

N and E's major *goal* is to enhance students' ability to operate and learn from MOOC videos. They mention a potential difficulty that can arise from MOOC lecturer using terminology and symbols that are different from the ones used in a classroom. Moreover, N says that her students are used to watch math videos at home, while for E's students it is going to be the first time. At the same time, N and E share a similar goal, namely that their students become fluent with mathematics discussed in the videos. F's goals can be classified into long-term goals within Schoenfled's view, since she also wants to develop critical thinking. F's reflections on her students make an impression that she sees her students as more collaborative. E, on the other hand, describes her classroom as "difficult", given that a girl plays the role of a leader. With respect to considering MOOC as a *resource*, the teachers are aware of the possibility for using it to save time in the class. They are also aware that videos are not interactive: there exist a chance that the students will not engage with videos at home. Notably, E's *orientations* propose that she wants to be the one who (re)introduces the mathematical concepts, consequently, she prefers to use the MOOC videos for recalling procedures.

Lesson images

Both N and F assigned the same video to be watched at home. In this 3-minutes long video, the



graph and some properties of an exponential function f(x)=a^x are explained. The particular cases of e^x and $1/e^x$ are also discussed (Figure 1 shows a snapshot from the video). The logarithmic function is recalled as well.

Figure 1: a snapshot from the video assigned by N and F.

Nadia. "[...]I will assign the videos on exponential and logarithmic functions to be watched at home and I will assign some questions to be answered as well. I will assign the theoretical video which recalls definitions and properties, and two practical videos which show the solution of exercises. The questions I will assign to my students will enable them to reflect on how a video can be watched, which questions can one pose to oneself, how exercises can be solved. I want to see my students' answers in advance, hence I will collect their work through emails. In class, I will start from the part in the video where the graphs of exp(x) and of exp(-x) are shown simultaneously. I will ask my students to draw the graph of f(x) and f(-x) for the following functions: a parabola of the form ax^2+bx+c, sin(x) and cos(x). This will prompt the students to notice symmetries in some cases and I will introduce the definition of an even function focusing on the features on the examples drawn by the students. The students will work in groups". In N's lesson image, her *goal* about her students' ability to watch the video and be able to understand emerges in her intention to invite them to reflect on how to access the video content: in fact, she says that "the questions I will assign to my students will enable them to reflect on how a video can be watched, which questions can one pose to oneself, how exercises can be solved". Nadia also mentions the good quality of graphs and in fact she wants to exploit one of the graphs in the video to introduce the definition of even function: this speaks to N's *resources*.

Francesca. "In a previous lesson, I will show my students how to access the MOOC and I will assign them the exercises in the MOOC, both ones that have a solution provided in videos and those which required to be solved in solitude. In class, I will discuss with students' solutions, which I will collect via email in advance, and we will do more exercises". In F's lesson image, we notice that she plans to spend a lesson commenting the videos ("I will discuss with students' solutions"), watched at home and to do more exercises. This is in line with her *orientation* about the MOOC, namely that the students don't have the possibility to ask questions to the teacher in the video, but this can be done in class the day after.

The video assigned by E regards an exercise about the grades taken during an exam by 32 students and it shows how to compute the mean, the median, the standard deviation of the given data.

Elisabetta. "I want my students to become confident with Excel spreadsheet. In a previous lesson, I will introduce the main statistical measures: the mean, the median, the variance, the standard deviation, the absolute deviation. Then, I will introduce the software and main commands for working with data and computing these descriptive statistics. Then, I will assign to watch related MOOC videos as a homework and I will ask my students to do an exercise that is similar to the one presented in the videos. There is some difference between my lesson and the videos: we use different terms and the video does not address the absolute deviation. I want to see if in their solutions students will follow what we did in class, or what was done in the video". Similarly to other teachers, E wants to use the MOOC videos as a *resource* for recapping some concepts and reinforcing students' knowledge. Differently from the other teachers, however, E explicitly says this in her lesson image. E's choice of a procedural video is in line with her *orientation* about the use of MOOC, namely that it is more suited for exercises while the teacher should be left free to introduce the concepts in class. Finally, we comment on the differences between the MOOC videos and E's lesson and on her way to detect whether the students will follow the former or the latter: we see this comment from E in line with her *goal* that the students should become able to understand the video even if it differs from what they have seen in class.

Implemented lessons

Nadia. Even though N did not plan to show the videos in the classroom, she noted that many students did not sent her their homework in advance. She also suspected that the majority of the students did not watch the video. Hence, she started the lesson with the video (saying: "it will last just for a few minutes, to show the video won't compromise the lesson"), and then the students worked in groups. They sketched the graphs of exp(x) and exp(-x), of sin(x) and sin(-x), of cos(x) and cos(-x), while Nadia navigated the class and engaged in conversations with the groups. She invited them to find out general features of the drawn functions and she introduced the definition of an even function.

Francesca. Like N, F also notes that some of her students did not send her their homework and in class she asked them why. She also asked how the students coped with the assignments. The students replied that the videos were clear but they experienced difficulties with assigned exercises and requested teacher's assistance. Hence, the teacher engages the classroom in a rich discussion about "how to do". The students actively engaged in the discussion, which aligns with F's opinion about her students' cooperative mood. We also notice that her way of conducting the lesson stimulates the students' critical thinking, since many times during the lesson they were not satisfied with the procedure recapped by F and wanted also to recap "why to do so".

Elisabetta. All her students submitted their homework before the lesson and E says this with satisfaction at the beginning of the lesson. After reviewing the submissions, however, she noticed that some students have followed her lesson while others have followed the video. She, thus, engages in a frontal lesson in which she poses questions to the students in order to better know how much they grasped the out-of-class activity. The first half of the lesson can be summarised as the teacher posing questions and the students avoiding to answer, while the teacher solved exercises from the homework showing all the steps and elaborating on the concepts involved. At some point, one student asked for clarifications on the formula to compute standard deviation. The students spent the rest of the lesson in posing questions to the teacher, making-sense of her explanations and replying to the questions the teacher in turn made them.

Links between the lesson image and the implemented lesson (...and more data)

We have commented that N and E are concerned that a terminology used in the video might hinder their students' engagement with the content. In N's lesson image, she was planning to deal with the concern through inviting students to reflect on the experiences. N also mentions the good technical quality of graphs and in fact she wants to exploit one of the graphs in the video to introduce the definition of an even function (see Figure 1). In N's implemented lesson, she starts watching the video, since she noticed that few students have sent her their homework. Her orientation is that "this won't *compromise* the lesson" and we see something deep in this comment from N: we see an acknowledgement that it would be a 'deviation from the plan' and as such it is a decision made to deal with students' lack of homework, but at the same time we also see that N's orientation is that to show the video in class is in line with her *goals*. N's orientation is completely different from E's one, who chose not to show the video and to make questions about it in class, so that the students who have done their homework will both have the opportunity to shine and to help their peers even if those did not watch the video had not seen it. E explicitly says that she does not want her students to think that even if they do not make their home-work there would be an in-class opportunity to cope with this lack of work. We have noticed that N and E share similar goals (i.e., to allow their students to be able to grasp the mathematics in the videos), but the decisions that they make to cope with their students' lack of home-work are rather opposite.

N's decision-making led to a student-centered lesson, while E delivered a frontal lesson. The students' lack of homework is, thus, addressed in two different ways which can be explained with teachers' orientations towards teaching and learning: N assigns a paper for each students and arranges the class so that each one can work individually but at the same time s/he can ask for her/his mates' help or N's help. This overcomes one of the possible tensions highlighted by Fredriksen *et al.*, namely that students prefer to work in solitude. E arranges a frontal lesson, which is not as teacher-centered as it appears at a first glance: her lesson, in fact, is responsive of the students' feedback provided in the homework sent to E in advance. She highlights the terminology that is not clear for the students, she commented on some students' mistakes and she designed the lesson accordingly. We can say that this particular use of MOOC videos prompts even the teachers who prefer teacher-centered lessons to arrange *responsive frontal lessons*, since they are allowed to know in advance their students' difficulties. This can be seen as an interesting feature of FC.

F, being not concerned about her students' ability to access the video content, engages the students in a classroom discussion that is rich and at the same time challenging for the students. One of her students, in fact, at a certain point says that she finds it difficult to follow what her peers say and propose, and asks the teacher to make a summary to clarify the ideas that have emerged. This request speaks both to the genuity of the classroom discussion and to this student's will to understand. Given that the class has a weak mathematical curriculum, this can not to be taken for granted. F is reaching her goal, namely to stimulate curiosity and critical thinking. Like E, also F does not show the video in class. Like E, also F engages in a frontal lesson to correct the exercises the students have found difficult to solve in solitude. Differently from E, however, F's lesson results in a classroom discussion rather than a frontal delivery of mathematical content.

We would like to summarise the three teachers' lessons as follows: her way of using MOOC videos allows N to design a *groupwork activity* aimed discovering the features of even function. Her (different) way of using MOOC videos allows E to implement a *frontal lesson that is responsive* to students' difficulties as they emerged from their homework. Her (different) way of using MOOC videos allows F to carry out a *classroom discussion*, which is not planned: the students' need to correct the homework emerges as the lesson starts. For E, it was possible to plan a responsive lesson (while for F it was not), because E's students sent their homework in advance, while F coped with her students' lack of homework: F arranges an *unplanned* responsive lesson.

DISCUSSION

As a response to our research questions, we can propose that secondary school teachers can use MOOC videos in their classrooms in different ways (question 1): they assign videos as homework and design group activities for introducing new concepts, or they design a classroom activity that allow to pinpoint the differences in terminology and symbols between the teacher's lesson and the video, or they plan to do more exercises on the basis of the out-of-class activity. Self-directed and student-centered learning (question 2) can be seen as an intrinsic feature of FC: frontal lessons are responsive, classroom discussion in informed by out-of-class activity and group work is primed by conceptual work done at home.

To our understanding, FC is a combination of a learning setting and activities. The students were asked to watch videos at home, but if a teacher solves or even answers questions in the class, is the classroom flipped? This remains an open question, but the three teachers in our study used MOOC videos to create new forms of classroom. Definitely it was not a classical version of FC, but this was to some extent expected, because teachers and students *learn to flip*. We would say that E implements FC: she assigns out-of-class work and she arranges an in-class activity that draws on the students' homework. By not showing the video in class, she provokes the students' self-directed learning, since they are not able to follow if they had not watched the video. Also F implements FC: she does not show the videos in class and she implements a lesson that is responsive to her students' needs, namely instead of making challenging exercises as per her lesson image, she spends the majority of time showing the procedure to solve the exercises assigned as homework and replying to the students' questions. E's students learn that they need to do the out-of-class activity in order to be able to follow the in-class lesson. F's students learn that if they are not able to understand the out-of-class activity, her teacher is there to help them in class. N shows the video in class and she comments the video, connecting the students' work with the content of the video. As such, we would say that N's lesson is a case of co-teaching instead of FC. She co-teaches with the teacher in the video and she implements a student-centered lesson.

REFERENCES

Andrà, C., Brunetto, D., and Kontorovich, I. (under review). How to combine technology in secondary math classroom? A case of MOOC videos.

Bergmann, J., & Sams, A. (2012). *Flip your classroom, reach every student in every class every day*. Washington, DC: International Society for Technology in Education.

Bretscher, N. (2014). Exploring the quantitative and qualitative gaps between expectation and implementation: a survey of English mathematics teachers' uses of ICT. In: A. Clark-Wilson, N. Sinclair & O. Robutti (Eds.), *The mathematics teacher in the digital era*. Springer.

Fredriksen, H., Hadjerrouit, S., Monaghan, J., Rensaa, R.J. (in press). Exploring tensions in a mathematical course for engineers utilizing a flipped classroom approach. In: *Proceedings of CERME10*, Dublin, February 1-5, 2017.

McCloskey, A. (2014). The promise of ritual: a lens for understanding persistent practices in mathematics classrooms. *Educational Studies in Mathematics, 86*(1), 19–38.

Schoenfeld, A. H. (2011). Toward professional development for teachers grounded in a theory of decision making. *ZDM, 43*(4), 457-469.

Wan, N. (2015). Technology Integration and the Flipped Classroom. In: N. Wan (Ed.), *New Digital Technology in Education* (pp. 149-170). Switzerland: Springer International Publishing.

Chapter 5

TEACHER

MOOC FOR MATHEMATICS TEACHER TRAINING: DESIGN PRINCIPLES AND ASSESSMENT

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This paper reports on an ongoing international research about MOOCs for in-service mathematics teacher training. We describe and analyse two different experiences of this kind: the Italian MOOC Geometria and the French MOOC eFAN Maths. Both MOOCs aimed at supporting the teachers' professional development through a suitable mediation of technology and at triggering as much as possible the teachers' engagement so that they could develop from a non-community towards one or more communities of practice. As authors of this paper, we are members of the trainers' team of the respective MOOCs and we also participated in the design. Starting from our methodological choices, we want to propose some reflections about design principles of MOOCs for mathematics teacher training in order to foster participation and collaboration among trainees and to efficiently assess this kind of engagement.

Keywords: MOOC, design principles, project-based assessment, completion rate

INTRODUCTION

The Massive Online Open Courses (MOOCs) are becoming widespread as a training tool in universities (Pomerol et al., 2015) and management institutions (Porter, 2015) and also for in-service teacher training, but not so much for mathematics teacher training.

Our paper describes and analyses a double experience gained in France and in Italy with MOOCs of this kind, delivered to mathematics teachers, mainly from secondary schools, with the aim of increasing their professional competencies and improving their classroom practices.

In fact, there are at least three main problems in MOOCs: (i) How to trigger an active participation? (ii) How to assess in a trustable way the efficiency of the MOOC? and (iii) What type of technology is better to use in order to get as positive results as possible in the previous two points? This last issue assumes a specific connotation in the case of mathematics teachers. Our paper shows that there are different ways to catch the interest of trainees and we explore, compare and discuss them considering two different design approaches.

For the assessment issue, a first crude evaluation estimate consists in considering the drop-out rate. The literature suggests a mean of 95% rate (Bayne & Ross, 2013). In our cases the figures are completely different: in France it was about 88% (second experience), and in Italy 64% (first experience). In other joint papers (Taranto et al., submitted chapter; Panero et al., 2017; Taranto et al., 2017) we have described how a subtler analysis can help to develop a more sophisticated assessment of the level and nature of participation in a MOOC for in-service mathematics teacher training.

For the technology, we sketch how some 2.0 open source devices and software or professional social network, suitably organized and exploited, can trigger and support an active participation of the trainees in the MOOCs activities.

Based on the analysis of the two experiences our paper faces the following two research questions:

(i) *What design principles are useful to mediate teachers' professional development courses with technology?*

(ii) *How to assess the impact of such courses on mathematics teachers' engagement?*

AIMS AND DESIGN PRINCIPLES OF MOOC

In this section we explain the specific aims of each MOOC. They have been achieved taking into consideration precise methodologies and design principles: (i) promote educational innovation; (ii) stimulate reflection on the use of technology in the classroom and with the students; (iii) create communities of practices (Wenger, 1998) and support the sharing.

Italian MOOC: The MOOC Geometria

MOOC Geometria is a MOOC on Geometry, for training in-service mathematics teachers of secondary school (both lower and higher). 424 participants enrolled in it, all teachers in secondary school, from all over Italy. It was delivered on a Moodle platform called DIFIMA[1] during 8 weeks: from October 2015 to January 2016. MOOC Geometria was designed by experienced teachers of secondary school in collaboration with some researchers in Mathematics Education from the Department of Mathematics of Turin University; the same team took care of delivering the course. These experienced teachers were trained in Mathematics Education and in innovation basing on the didactical material of the m@t.abel project (https://goo.gl/Q30Dn0), a plurennial National Program that pushed innovation in mathematics teaching basing on concrete activities proposed to teachers and discussed with them in suitable training e-courses. The following needs had been identified: awareness of the role of training in teaching activities; willingness of developing best practices of innovation using software; reconsidering in terms of learning the sharing practices of social media most used by the students. Hence, it was decided to offer the opportunity of an authentic development experience designed for a larger group of teachers that could have become a community of practices (Wenger, 1998): that is the idea of the Italian MOOC Geometria.

In particular, five specific modules on geometric contents were created. The activities had a weekly basis and the duration of each section varied from 1 to 2 weeks (depending on the treated topics). All the activities are based on mathematics laboratory and MERLO[2] assessment tools. As pointed out above, they are inspired by m@t.abel project and are transposed in a digital format following the E-tivity framework (Salmon, 2013). The E-tivity are designed before opening the MOOC to participants. They provide learners with an effective scaffolding to support them in achieving the learning outcomes: in fact, they promote a learner-centred task and problem-based approach to online learning (moving away from content-centric design) and find easily purposeful ways of using freely available, topical and/or game-based resources within the learning design.

To motivate participants to contribute and consolidate ideas in a focused way, and, at the same time, to collaborate and communicate, specific technological tools were selected. There are only open

[1] DIFIMA: Didactics of Physics and Mathematics (http://difima.i-learn.unito.it/), hosted by the Department of Mathematics of Turin University

[2] MERLO: Meaning Equivalence Reusable Learning Object (Arzarello et al., 2015)

source tools in the MOOC (e.g. Geogebra, Dynamic Geometry System), thus respecting the Open in the MOOC acronym and, above all, enabling teachers to easily fit in with them in their teaching practices. In the design we took into account also the TPACK model (Mishra & Koehler, 2006) with the intention of enlarging the mathematical technological knowledge of the trained teachers. In particular, with respect to the 7Cs (Conole, 2014), a great attention was given to "Communicate" and to "Collaborate", focusing on the choice of the best tool to be used both for a catchy and easy online access presentation of a selected content and for supporting the communication and collaboration among the participants in the course. In fact, specific communication message boards from web 1.0 to web 2.0 were selected (Forum, Padlet - https://it.padlet.com/, Tricider - https://www.tricider.com/). Trainers reduce their interventions in this space as much as possible for fostering the development of an interactive only-trainees community. However, trainers were "behind the scenes": they sent weekly emails to inform all participants about the progress of their experience training; they also intervened when technical problems came up (sometimes even with an email to a single person). Real moments of contact with the trainees were the three webinars. They are online meetings in which an expert shares with the participants some issues about the research in mathematics education and focuses on some questions that could be raised during the previous weeks in the MOOC. During the webinars the participants had the opportunity of taking part in a chat in synchronous way. All of the three webinars had a high participation (from 90 participants in the first one to 50 in the last one) and consensus by the trainees, who posed many questions and doubts.

French MOOC: MOOC eFAN Maths

MOOC eFAN Maths (*Enseigner et Former avec le Numérique en Mathématiques – Teach and train with digital technology in mathematics*) is a MOOC about teaching mathematics with technology, for training in-service mathematics teachers and teacher educators, particularly from secondary school. The second season of the MOOC eFAN Maths was delivered on the French MOOCs national platform, called FUN (*France Université Numérique*) from the 8th of March to mid-April 2016. The MOOC eFAN Maths was organised in five weeks, each proposing three video-lessons on key concepts of technology in mathematics education, one multiple-choice test per lesson, an activity related to the theme of the week and a few articles for in-depth study. The examples discussed in the video-lessons were selected and adapted from different European research projects (e.g., FaSMEd[3], MC Squared[4]) with a focus on the use of technology supporting formative assessment and enhancing creative mathematical thinking.

The trainers were also members of the designers' team of the MOOC, composed of experienced secondary school teachers and researchers in Mathematics Education working at the Ifé (French Institute of Education) of the Ecole Normale Supérieure de Lyon. The designers were motivated by a double institutional aim: to support teachers and teacher educators in understanding and implementing the new French curriculum (applicable since September 2016 in all French primary and secondary schools) and to promote collaboration within the French-speaking mathematics education community.

The designers' methodological choices can be explained according to some of the "pillars of an accompanied auto-training" introduced by Carré (2003). Inspired by these pillars, the designers tried to manage and equilibrate the interplay of the *individual project* with which each trainee enrols in a

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[3] Formative Assessment in Science and Mathematics Education (fp7/2007-2013 grant agreement n.612337).

[4] Mathematical Creativity Squared (ICT-2013.8.1 "A Computational Environment to Stimulate and Enhance Creative Designs for Mathematical Creativity", Project 610467).

training, the *pedagogical contract* between trainers and trainees, the trainees' *pre-training* to use some particular tools (such as the possibility to access tutorials), the role of the *trainer as a facilitator*, and the presence of an *open environment*. The designers provided an open environment to encourage trainees' participation in the training. Only free open-source tools were presented, so that teachers could easily find and appropriate them. Moreover, to foster collaboration between trainees, they were invited to join a professional social network for teachers, called Viaéduc (www.viaeduc.fr), where trainees could gather together around a shared project constituting public groups (so that any trainee could read the work of any other group and follow any discussion). Some trainers worked as community managers: they helped trainees to solve technical problems, such as creating an account on Viaéduc; they made tutorials for using FUN and Viaéduc platforms; they created and regularly updated a list with all the trainees' ongoing projects to help teachers to find a project to join; they recalled the tasks to be done week by week. Furthermore, every week began with a quick video titled "From one week to the other" in order to bridge two consecutive weeks of the MOOC. Finally, to cultivate and induce the generation of trainees' groups as communities of practice, one trainer per group followed the development of the group project from the inside, intervening to encourage and trigger the collaborative work (Panero et al., 2017). This represents also a special condition of the pedagogical contract between trainees and trainers.

The total number of participants enrolled in the MOOC on the FUN platform was 2572, mostly French-speaking mathematics teachers and teacher educators interested in the use of technology. However, only 737 trainees decided to join Viaéduc and work on collaborative projects.

PROJECT-BASED ASSESSMENT

In this section we illustrate the activities expected by the participants in each MOOC. In both MOOCs we chose a project-based methodology for assessing the trainees' participation, but articulating it in different ways, and both turned out to be efficient.

Italian MOOC: Project Work with Learning Designer

Every week the trainees had an individual work and interfaced themselves with methodologies at different levels, in order to collect their weekly badges: watching a video where an expert introduced a conceptual knot of the week; watching a "cartoon video" with some guidelines to carry out the units; reading the geometry activities based on mathematics laboratory (and possibly experimenting them in their classroom). Moreover, they had to use the suitable communication message boards (Forum, Padlet, Tricider) to express opinions about the content of the course, make a comparison between colleagues, and benefit from experiences or ways of thinking of others. There were a collaborative climate and, surprisingly, some of them started to voluntarily share material created by themselves and that they were using in their lessons. This is certainly an aspect that does not occur in a traditional training course.

The MOOC design included as a final module two production activities: the design of a teaching activity (or Project Work, hereafter PW) and the review (or Peer Review, hereafter PR) of a project designed by a colleague. The time available to perform these last activities was two weeks. For all those who took part in all MOOC stages (that is, accomplishing all tasks for collecting all weekly badges and accomplishing the PW and PR), a participation certificate was issued by the Math Department of the University of Turin.

It was not necessary that the PW was experienced in class to carry out a PR: it was an activity to be done remotely, demonstrating teaching competencies and experience. In fact, PW and PR have been designed to give the participants the opportunity to get involved in the MOOC activities in terms of methodology, creativity, and with the aim of sharing and discussing them in the community. Each trainee could choose individually a geometrical theme taking a cue from those of the MOOC or even choosing a new one. A lot of freedom was given in the design of the PW both because as teachers the trainees had surely already had experience with these activities (and we as trainers did not want to influence them) and to give space to their creativity. Moreover, the PW had to be done through a web-based tool, the Learning Designer (hereafter LD) designed by D. Laurillard (2012). LD is a software that guides and encourages the planning of the lesson: it is characterized by a standard format that allows the integration of technologies (the teacher can include links to material that s/he has produced or that is on the net); it allows you to have an overview of the teaching/learning dynamics centered on the student and allows sharing of what you have produced online. In order to familiarize the trainees with LD we created both a video tutorial (suggested via link) and a paper-based tutorial: they were made available two weeks before the opening of the last module. For this tutorial supports we detected 411 readings: these figures show its utility.

It was considered important to ask the trainees to make a PR of a colleague's PW in order to have an analysis from an educational point of view, generated by the eyes of a teacher who had no other aim than the analysis of the asset itself. The instructions for the PR were given in a more specific way compared with the PW. We specified the review criteria to follow, because we wanted to focus attention on the main aspects of each educational intervention and, for the purposes of the MOOC itself, on a conscious use of instruments and of digital software.

The deadline had been announced as "sharp" because we wanted to allow everyone to be able to continue with the peer review. However in the forum dedicated to technical problems, some trainees expressed the need of having more time available to accomplish their PW. As trainers-designers, we are taking this feedback into account for the next season of the MOOC.

In the last module of the MOOC, the participants were asked to complete a final questionnaire: we could so receive a feedback on their experience of distance learning, as well as their impressions about the latest activities (for more information see: Taranto et al., 2016).

French MOOC: collaborative project on Viaéduc

Week by week, the proposed activities aimed to support trainees in the design of a mathematical task integrating the use of a digital tool. The phases of the project design consisted of: a) a description of the mathematical task; b) a toolkit made of digital or non-digital artefacts and resources with the related usage schemes within the designed task; c) an analysis of the students' expected mathematical activity and interactions with the artefacts; d) an analysis of the teacher's role in orchestrating the situation in the classroom.

The activities of the weeks devoted to students' and teacher's role relied on two grids designed, uploaded and commented by the trainers. They helped to analyse the designed mathematical task and the role of technology from the point of view both of the students and of the teacher. They consisted in guiding questions grounded on the instrumental approach (Artigue, 2002, Rabardel, 1995) and on the instrumental orchestration (Trouche, 2004), which were both introduced in the lessons delivered in the corresponding weeks. To encourage collaboration, trainees were invited to work on the proposed activities in a collaborative way, by forming groups around common interests for a mathematical theme on Viaéduc a platform that essentially allows members to post comments, to create groups, to create and publish documents and to comment/recommend/share them. Group members can work collaboratively either asynchronously, being authors of the same online document, or synchronously, writing on the same online collaborative board (*padlet*).

The project, collaboratively written, went through two phases of evaluation: a peer evaluation with the possibility of improving the work basing on the received feedback, and a trainers' evaluation (the evaluator was the trainer who followed the group from the inside). An evaluation grid was constructed by the trainers to encompass all the phases of the project design, developed in the MOOC week after week. This grid was structured around the following four criteria: 1) Accuracy of the definition and description of the project; 2) Relevance of the mobilised digital tools and resources with respect to the educational goals of the designed mathematical task; 3) Relevance of the analysis of the students' expected mathematical activity; 4) Relevance of the analysis of the teacher's role.

For each criterion, some guiding questions were proposed with a double objective: to foster the production of justified feedback and to deepen the reflection carried out in the previous weeks of the MOOC. The grid finally asked for a brief global feedback on the project and some suggestions to improve the work. Each trainee was invited to use the grid individually to evaluate the project of another group, by answering each guiding question with an evaluation: very good, satisfactory, fragile or insufficient, accompanied by a justification. The community managers gradually collected feedback and comments in a table and shared it in a specific space on Viaéduc, called "Project evaluation", so that all the trainees could access them.

After the MOOC, a questionnaire was sent to all the enrolled participants to get feedback on such an experience of distance training, with a particular focus on the collaborative project and collaborative tools of Viaéduc. As trainer-designers, we are taking this feedback into account for the third season of the MOOC.

DISCUSSION AND CONCLUSIONS

In this paper we have introduced and compared two different MOOCs for training in-service secondary schools mathematics teachers, one in France and one in Italy. The courses were designed according to a different structure in the two countries, because of the different institutional school backgrounds and traditions, but they had two common goals: (i) to foster the professional development of teachers through a suitable mediation of technology; (ii) to trigger as much as possible the engagement of participants in order that they could develop from a non-community towards one or more communities of practice (and possibly of enquire). These two goals are related to the two research questions listed above and put forward some challenging methodological issues for the research teams: the design principles and the assessment of teachers' engagement.

For the design both teams had to hypothesize a "common" zone of proximal development (Vygotsky, 1978) of participating teachers with respect to their pedagogical and mathematical knowledge mediated by technology – what Mishra and Koehler (2006) call TPACK – so that the proposed activities could be interesting for the majority of trainees and introduced them to situations they were able to approach and elaborate. The MOOCs were also a training opportunity for sharing the results and the reflections about research projects with the community of teachers. But the major related problem was to transpose such an information into the MOOC environment, namely we had to transpose the usual methodology of training courses into images, words, videos, and nothing else: we had so to choose friendly open source tools that could be easily available and that could be easily used in the trainees' activities in their own classrooms. We had to support the developing community not imposing the team's presence but being vigilant and ready to intervene promptly in case some help is required. Implementing some webinars in the MOOC, where the "expert" could communicate through a video-chat with the trainees, as well as proposing a trainer per group as a personal tutor, had the purpose (and effect) of making trainees feel accompanied and become faithful followers. As said above, the positive effects of such a complex design were tangible: in fact, while the literature says that the percentage of people who complete a MOOC is about 5% (Bayne & Ross, 2013), in the French MOOC it was 12%; in the Italian MOOC Geometria it was 36%; and in the successive Italian MOOC Numeri it was 43%.

To concretely check the possible development of communities of practice, some more creativity from the designers' teams was required: it is not an easy task to extract data from the MOOC environment, where the researchers must base only on the stored traces and messages that the participants leave on the MOOC devices, on the tasks they upload and on their answers to the questionnaires. Of course this second issue is strictly linked to the previous one: having data easy to access strongly depend on the type of activities required to the participants and to their willingness to do them. Hence a first filter consists in checking if the trainee has accomplished all the required tasks. For this, a good strategy could be using a gamification context within the training: e.g., in Italian case, each week a badge was automatically released to those who did everything: the sequence of the got badges certificated the level of participation to the course. A second important evaluation tool is the elaboration of a final project, where the trainee could show how she was able to apply what had been presented in the course. This second evaluation was based in both MOOCs on a peer review, complemented in France with a trainers' evaluation of the project necessary for delivering the university certificate. We took care of this aspect, and we recommend to do it as MOOC designers, because obtaining such a certificate of completion by universities can be an important stimulus for teachers to engage in distance training. The (relatively) good percentage of people who ended the MOOCs shows that this goal was positively achieved in both cases.

Of course not everything was rosy in our experience. In both MOOCs we realized that a project-based methodology can create a gap between the timeline of the MOOC (videos, quizzes, activities) and the timeline of the project, which can destabilize the trainee in some cases. For these reasons, in the following seasons of the MOOCs, the time factor has been taken into greater account, leading also to modify some aspects of the design.

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REFERENCES

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, 7, 245–274.

Arzarello, F., Robutti, O., & Carante, P. (2015). MERLO: a new tool and a new challenge in mathematics teaching and learning. In Beswick, K., Muir, T., & Wells, J. (Eds.). Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education, Vol. 2, pp. 57-64. Hobart, Australia: PME.

Carré, P. (2003). L'autoformation accompagnée en APP ou les sept piliers revisités. *Les ateliers de pédagogie personnalisée ou l'autoformation en actes*, P. Carré et M. Tétart (dir.), Paris, L'Harmattan.

Conole, G. (2014). The 7Cs of learning design a new approach to rethinking design practice. In *Proceedings of the 9th International Conference on Networked Learning* (pp. 502-509).

Laurillard, D. (2012). Teaching as a design science. London: Routledge.

Mishra, P., & Koehler, M. J., (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers college record*, 108(6), 1017-1054.

Panero, M., Aldon, G., Trgalová, J. & Trouche, L. (2017). Analysing MOOCs in terms of their potential for teacher collaboration: the French experience. Presented to TWG15 of the 10th Conference of European Research on Mathematics Education (CERME). Dublin, Ireland.

Pomerol, J. C., Epelboin, Y., & Thoury, C. (2015). *MOOCs: Design, use and business models*. John Wiley & Sons.

Porter, S. (2015). *To MOOC or not to MOOC: How can online learning help to build the future of higher education?*. Chandos Publishing.

Rabardel, P. (1995). *Les hommes et les technologies ; approche cognitive des instruments contemporains*. Armand Colin.

Salmon, G. (2013). *E-tivities: The key to active online learning.* Abingdon, UK: Kogan Page

Taranto, E., Alberti, V., Labasin, S., Arzarello, F., Gaido, S. (2016). Project Work e Peer Review: attività conclusive di un MOOC di matematica per la formazione docente. Atti di EMEMITALIA2016

Taranto, E., Arzarello, F., Robutti, O., Alberti, V., & Labasin, S. (submitted). A MOOC on Geometry for teachers designed by teachers. In V. Hoyos, J. Silverman, R. Barcelos-Amaral, R. Vogel (Eds.), *Distance Learning, E-Learning and Blended Learning of Mathematics: Advances in the Research of Distance Mathematics Education Mediated by Technology* (pp. XX-XX). Springer

Taranto, E., Arzarello, F., Robutti, O., Alberti, V., Labasin, S., & Gaido, S. (2017). Analysing MOOCs in terms of their potential for teacher collaboration: the Italian experience. Presented to TWG15 of the 10th Conference of European Research on Mathematics Education (CERME). Dublin, Ireland.

Trouche, L. (2004). Managing complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning*, 9, 281–307.

Vygotsky, L. (1978). Mind in society. Cambridge, MA: Harvard University Press.

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity.* Cambridge university press.

THE EFFECT OF COLLABORATIVE COMPUTERIZED LEARNING USING GEOGEBRA ON THE DEVELOPMENT OF CONCEPT IMAGES OF THE ANGLE AMONG SEVENTH GRADERS

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This research intends to investigate the effect of collaborative computerized learning using GeoGebra on the development of the concept images of angle among seventh graders who were engaged with computerized collaborative activities that encouraged the development of five types of the angle concept images: verbal, authentic-life, graphical, numeric and dynamic. The research sample consisted of eight seventh grade students who worked collaboratively in groups of two. Two tests (a pre-test and a post-test) were administered to examine the development of students' concept images of the angle. In addition, interviews were held with the participants to study this development. The Constant Comparison Method was used to analyze the data. The results showed positive effects of the visual and dynamic use of GeoGebra, as well as of the collaborative learning on the development of participants' concept images of the angle, especially the dynamic one.

Keywords: Angle, Concept Image, Dynamic Image, Collaborative Learning, Computerized Learning

INTRODUCTION

If students do not adequately understand concepts related to angles, their cognitive understanding of subsequent topics in geometry in particular and daily life activities connected with geometric concepts in general will be affected negatively (Alkan & Altun, 1998). Moreover, the concept of angle will never be fully grasped until students can identify and distinguish it easily, and consequently; use it better in their life. This can only be achieved through displaying the angle dynamically by the use of technological tools such as GeoGebra, Applets, etc., which promote and facilitate learning geometry. Once technology is used properly, the learning environment becomes suitable for visual and dynamic investigations, through which students will grasp geometrical concepts better (NCTM, 1989). In the present research, we try to find how GeoGebra helps seventh grade students develop the concept of angle.

In addition to the use of technological tools that promote students' investigations of mathematical concepts, collaborative learning encourages students to participate, argue with each other, and raise questions for discussions, resulting in internalizing mathematical concepts. Furthermore, during collaborative investigation, wrong concepts are raised for discussion in order to be corrected. This is better than getting the right answers directly in order to conceive the mathematical concept definitions.

Concept image

A concept image refers to the total cognitive structures that are associated with the concept and includes all the concept images of the individual, as well as the associated properties and processes which have been built up over the years through experiences of all kinds; they are likely to change depending on the different stimuli the individual receives (Tall & Vinner, 1981). It is possible for the concept image to be totally different from the formal concept which is scientifically acceptable. This difference may lead to the emergence of knowledge-based conflict (Lambertus, 2007). Vinner and Dreyfus (1989) stated that students divide concepts into definition and concept images, without relating properly the formal definition to their concept images.

The concept image develops based on the type of ideas being learnt by students at different stages. More specifically, they change according to students' experiences, examples they had, and their awareness of mathematical concepts (Tall, 1992; Vinner & Dreyfus, 1989). Concept images can be incarnated into verbal forms; however, these verbal forms are not the first thing that stimulates students' memories (Vinner, 1991).

The concept image of angle

Based on what had previously been mentioned, when students are exposed to technology, they are introduced to new experiences and various examples, due to the fact that concepts are dynamically presented with different forms and different representations. Consequently, technological tools such as GeoGebra are conceptual tools that support students in developing multiple concept images associated with mathematical concepts (Battista, 2002; Choi-Koh, 1999; Dixon, 1997). Among these concept images is the dynamic representation of the mathematical concepts.

Mathematical curricula usually lack dynamic representation of the angle concept, which is normally seen as a fixed entity by students. For instance, students do not realize what happens to the angle after rotating or extending one of its rays. Moreover, most curricula concentrate on four concept images of the angle: verbal, authentic-life, graphical and numeric. Therefore, the teaching unit that we built to develop the various concept images of the angle took in consideration also the dynamic concept image of the angle, hoping to improve the perception of this concept.

There is no doubt that the concept of angle is multifaceted, therefore, researchers had given different definitions for the concept of angle (e.g. Lo, Gaddis & Henderson, 1996). These definitions include the rotation of a ray around its endpoint, the geometrical shape formed when two rays meet at their endpoints, the area enclosed between two rays that meet in one endpoint, etc. For this reason, students may have a set of different images for the concept of angle.

Some researchers (e.g., Clements & Battista, 1992; Mitchelmore & White, 2000), pointed out that students do not realize angle as a rotation of a ray, therefore, their perception of the concept is partial and superficial rather than complete and deep. Mitchelmore and White (2000) conducted a study to identify the way students from grades (2 - 8) define the angle concept. They found that the students looked at an angle as a point and two arms. Many students in the eighth grade still do not relate rotation to the concept of angle. Therefore, these students have wrong perception of the angle concept, as they believe that the length of the angle arms affects its value.

Several studies showed that technology helps represent the angle dynamically, which enables students to notice the rotation of the ray and how this affects the value of the angle (Battista, 2002; Choi-Koh, 1999; Dixon, 1997; Kakihana & Shimizu, 1994).

The authors of the current study reviewed the mathematics curriculum of the seventh grade in Israel, and they found that this curriculum presents four different types of representations of the angle concept: verbal, authentic-life, graphical and numeric. In more detail, according to this curriculum, the seventh grade students should possess different meanings related to angles including: defining and representing angles verbally, representing angles in daily life experiences, drawing angles and measuring angles. This study focused on the development of the angle ideas and representations among seventh grade students after carrying out a number of computerized collaborative activities.

Research question

What is the effect of computerized collaborative learning based on GeoGebra on the development of concept images of the angle concept among seventh graders?

RESEARCH METHODOLOGY

The research procedure, context and participants

The participants consisted of eight female seventh graders (12-13 years old) who were grouped into four pairs. The students were of various levels including weak, good, and excellent students. A pre-test and a post-test were built to measure the development of the different types of representations of angle concept including: the verbal, the authentic-life, the graphical, the numeric and the dynamic. After teaching the unit, the students were interviewed individually in order to identify their development of the angle concept images.

The research tests

A pre-test and a post-test were administered to identify the concept images of angle among seventh graders. The researchers used the curriculum of the elementary school in Israel to build the pre-test and the post-test based on the angle ideas in the curriculum. According to this curriculum, the sixth grade students should possess different meanings related to angles including: Defining and representing angles verbally, representing angles in daily life experiences, drawing angles and measuring and calculating angles. The unit developed includes also the dynamic image of angles. The students took the pre-test before carrying out the collaborative computerized activities included in the teaching unit. Later, they took another test (a post-test), where the results of the two tests were compared to arrive at the development of students' concept images in the five representations.

Data analysis method

The Constant Comparison Method was used to obtain various themes related to the manifestation of the five representations of the angle. Then, the participating students' concept images of the angle manifested in the post-test were compared with those in the pre-test.

The teaching unit

We chose GeoGebra for its visual and dynamic features that can assist the students in investigating and discovering independently and collaboratively various mathematical representations of the angle concept and its components. The students were also given the opportunity to construct various angle representations through performing teacher-guided activities. More specifically, we developed a unit that utilizes computerized collaborative learning using GeoGebra and based on Guided Discovery. The unit included seven activities that aim to develop different representations of the angle concept. The activities were based on investigation to help the students discover the different representations of the angle. It also included construction activities to help them construct mathematical objects related to the angle concept in order to understand the angle manifestations. The students were also asked to observe the effect of the dynamic changes on the angle representations. The collaborative aspect was also stressed through the instructions in the activities before and during performing the activities. For example, some of the teacher's instructions were:

A) When solving a problem, think aloud and describe to your mate the solution steps asking her for help.
B) When your mate performs a move in GeoGebra, ask her to explain this move and discuss it with her.
C) Discuss together and write a proper definition of the angle.

D) Explain this phenomenon to your mate and discuss it with her.
E) Discuss, with your mate, the other angles you see in the pyramid, and explain why these are considered angles.
F) Construct an angle using an icon in GeoGebra and give your mate a chance to construct other angles herself.
G) Raise a conjecture and ask your mate to verify its correctness.
H) Formulate a definition of the angle based on the way you constructed it in GeoGebra; then ask your mate to check how exact the definition is, by constructing an angle in GeoGebra using this definition, or constructing an example of a shape which is not an angle but satisfies the conditions of the definition.

RESEARCH RESULTS

The main objective of this research was to identify the effect of computerized collaborative learning using GeoGebra on the development of the images of the angle concept among seventh graders. In this section we shall summarize the development of the images of the angle concept due to collaboratively being engaged with the angle concept with the help of GeoGebra.

Verbal image

The research results showed significant development of verbal perception of the ray concept and, consequently, the angle concept. Before the experiment, the students could not define the ray correctly saying that it is the distance between two fixed points. While after the experiment, all the students defined both the ray and the angle correctly based on their graphical manipulation of these concepts in GeoGebra. It was clear that the students' development of the graphical perception of the ray and the angle led to the correct verbal perception of these concepts. For instance, the students draw the ray in GeoGebra starting with an endpoint, so they defined it as a straight line which has an endpoint from one side only.

Authentic-life image

During the experiment, the students developed various meanings of the authentic-life representation of the angle. For instance before the experiment, the students expressed the angle in daily life only verbally as the corner of a door, a window, or cover of an open book. They also talked only about right angles in daily life, and did not mention any dynamic ones. However, the students, after the experiment, had developed deeper understanding of the angle concept to the extent that they started to talk about different types of angles in daily life both verbally and graphically. For instance, they began to talk about and draw angles formed by an open window, the angle made by the back and the base of a chair, the angle formed between the pen and the sheet of paper, the angle of an open fridge door, the angle of an open laptop, the angle formed by the hands of a clock, or the subtended angle formed by the observer's eye when looking at an object (for example: a building).

Furthermore, the students developed wide perception of the angle concept as a dynamic one that can be moved and controlled based on their daily needs. For instance, the angles formed when they open a laptop, when they move a pen on a sheet of paper, when they move their heads, also angles formed by the movement of the hands of a clock, or subtended angles formed by the observer's eye when looking at an object. They could notice the change in the angle's value according to the movement of its sides or its vertex as in the subtended angle case. The study also showed that prior to the experiment, the students mentioned only two important aspects of angles in daily life: beauty and organization. Yet after the experiment, the students listed four further important aspects in performing daily tasks: household chores, practicing sports and writing and reading. All in all, the development the student had with respect to the angle concept resulted in developing students' perception of its vital roles in their daily lives.

Graphical image

There was a noticeable development in students' abilities to perceive graphically the angle concept. For example, the participants mentioned only three types of angles before the experiment (acute, obtuse and right). Furthermore, the students were not able to draw all three types of angles in one geometrical shape. However, after using GeoGebra, and through collaborative leaning, the students stated five different types of angles (acute, obtuse, right, straight and reflex) and managed to draw all of them using one geometrical shape.

Numeric image

Before the experiment, the students stated that shrinking or extending the side of an angle affects its value considerably; yet after the experiment, the students denied this claim and stated that the change in the length of the angle sides does not affect its value, because angles are formed by rotating a ray, and the lengthening or shortening of that ray does not affect the size of the angle formed by its rotation.

Dynamic image

The students of the seventh grade were not capable of dealing with the problems related to the dynamic perception of angles on the pre-test. For example, they could not imagine the effect of moving one of the angle's components such as the vertex or a side on the angle's value. After the experiment, the students managed to overcome problems of this type on the post-test to the extent that they provided valid verbal and graphical justifications for most of them based mainly on the idea that an angle is a rotation of a ray.

After carrying out the activities, the students stressed, during the interviews, that before using GeoGebra it was very difficult for them to imagine what happens to the angle when they stretch or shrink one of its sides, or move its side or its vertex. On the other hand, when they started to use it, they were able to imagine every possible change that might happen when moving any part of the angle. Therefore, they began to realize the importance of angles in their daily lives and how dynamic they are. For example, they started noticing the change in the value of the subtended angle of a particular object to the observer's eye when approaching or moving away from it, or the change in the value and the shape of the angle when a laptop screen is moved in order to use it in a comfortable position and to see the objects on the screen properly.

DISCUSSION

The study results showed that using computerized collaborative environment helped students develop their images of the angle concept, mainly the dynamic and daily-life images. Before the experiment, these students managed to express the angle concept mainly verbally, considered only right angles and did not view angles as dynamic objects. This indicates that the participants had partial perception of the angle, which was reflected also in the examples they gave in the pre-test. After working visually, dynamically and collaboratively with angles, we notice that the students' perception of angles got more profound through expressing various kinds of angles from their daily life verbally, graphically and dynamically. This advancement achieves one of the objectives of the NCTM (2000), which emphasizes the importance of linking mathematics with students' daily-life as well as their personal matters. Moreover, the development of students' perception of the angle as dynamic object could be due to the use of the dynamic features of GeoGebra, such as dragging that the students utilized when carrying out the activities (Anabousi, Daher, & Baya'a, 2012).

Furthermore, students' interactions with dynamic diagrams, in our case angles in GeoGebra, enabled them to see how dragging changes the angle graphically as well as its measures, which allowed the participating students to find characteristics of the angle that might remain hidden in static diagrams of the angle (Gonzalez & Herbst, 2009). Moreover, these interactions with dynamic angles constituted strategic use of a content-specific mathematics technology that supported the students in exploring and identifying the geometric concept with which they worked (NCTM, 2015); i.e. the angle.

With respect to students' graphical images of angles, it is possible to attribute students' development in recognizing various types of angles and drawing them in the same shape to a number of factors including: the construction process using GeoGebra and the dragging process which helped students notice how angles change. Besides, the activities instructions (e.g. moving one side of the angle and identifying the type of the resulted angle) helped the students develop various concept images of the angle. In addition, collaboration and discussion helped in increasing students' awareness of various aspects of the angle concept. All these interpretations are consistent with the claim that the concept images of a particular concept are the result of students' experiences, including examples, of this mathematical concept (Tall, 1992; Vinner & Dreyfus, 1989). Furthermore, the teacher's actions and interactions with the students, including the activity instructions, presented her as an orchestrator and coach of the strategic use of GeoGebra for the learning of the angle concept (NCTM, 2015).

In terms of the numeric image of angles, the students stated, before the experiment, that the extension or shrinking of the ray of the angle affects the angle's size. After the students' engagement with angles collaboratively utilizing GeoGebra, they noticed that neither the extension nor the shrinking affects the size of the angle, because an angle appears as a consequence of rotating a ray and has nothing to do with the decrease or increase of the two sides of the angle. This development of students' numeric concept of the angle was supported by the software potentiality of measurement that enabled the students to verify their claims and substantiate them (Gonzalez, 2009). Furthermore, These findings emphasize the claim of Biber, Tuna and Korkmaz (2013) who concluded that when students have meaningful, real experiences of measuring and comparing angles via different methods, they become more likely to perceive the standard definition of the angle, and understand that the angle value is affected by the amount of the rotation of its ray rather than the length of its sides. For this matter, our study concentrated on the use of technology and the implementation of collaborative learning strategies for treating students' misconceptions of angles.

As far as the dynamic image of the angle is concerned, the students' had developed deeper understanding of this image due to the use of GeoGebra in solving the problems in the activities as the students emphasized during the interviews. They stressed that GeoGebra helped them imagine and perceive the angle concept deeper, especially when they realized it as an object resulting from rotating a ray. This development of their dynamic concept of angles depended on working with GeoGebra which helps display ideas and concepts visually and dynamically through various representations that are mostly related to geometry and algebra (Anabousi, Daher, & Baya'a, 2012). Generally speaking, using technology encourages looking at angles as an act of rotating a ray around its endpoint, and thus helps overcome misconceptions related to the angle concept, as well as widens the meanings associated with the mathematical concept (Battista, 2002; Chazan, 1988; Choi-Koh, 1999; Dixon, 1997; Kakihana & Shimizu, 1994).

In addition to the arguments above, the research results show the advantages of students' collaboration in groups or pairs. The participating students in the present research were urged to collaborate with and criticize each other through discussions and comments. This collaboration supported the students in carrying out the activities successfully (Gellert, 2014). The students managed to define the concepts correctly and gradually based on arguments and constructions

(Herbst, Gonzalez & Macke, 2005). These results show that it is necessary to allow students to construct, explore and investigate mathematical concepts by themselves, as well as to encourage them to develop their knowledge of mathematics through collaboration with each other.

CONCLUSION AND RECOMMENDATIONS

The multiple visual and dynamic features of GeoGebra proved to be very helpful in collaborative settings in deepening students' understanding of the angle concept as an object which can be moved and controlled. This led to widening their images of the angle, especially as they realized that different types of angles could be created when a ray is rotated. This what the students lacked prior to the experiment, for they were not able to express the angle created dynamically from rotation. The deficiency of not being able to see the angle as a rotation of a ray was stressed in previous studies, concluding that it led to partial understanding of angles and misconceptions included in the images of students of the angle concept. Here technology enabled seeing the angle as a rotation, which helped the students develop new image of the angle, that of rotation.

Specifically, collaboration among group members led to lively mathematical discussions about the different images related to the angle concept. These discussions helped reach a concise and accurate definition of what an angle is, and consequently, led to understanding deeply the various images of the angle concept. The results of the present research agree with the consideration that small-group discussions that involve students and their teacher and that focus on mathematical meanings through problem solving offer great potential for debate and argument (Gellert, 2014).

We recommend that teachers should share different examples of angles in the various angle images, especially the dynamic one. This could happen when encouraging students to construct and manipulate visually and dynamically angles in GeoGebra. Teachers' participation with students in discussing ideas about angles, after and during students' constructions of angles with GeoGebra, helps the students arrive at an acceptable accurate definition of the angle close as possible to the standard one.

REFERENCES

Alkan, H., & Altun, M. (1998). *Mathematics teaching*. Eskiúehir: Open Education Faculty Publications.

Anabousi, A., Daher, W., & Baya'a, N. (2012). GeoGebra in the mathematics classroom. *Jamea'a*, *16*, 3-54.

Battista, M. T. (2002). Learning geometry in a dynamic computer environment. *Teaching Children Mathematics*, *8,* 333-339.

Biber, C., Tuna, A. & Korkmaz, S. (2013). The Mistakes and the misconceptions of the eighth grade students on the subject of angles. *European Journal of Science and Mathematics Education 1*(2), 50-59.

Chazan, D. (1988). *Similarity: Exploring the understanding of a geometric concept*. (Technical Report No. 88-15). Educational Technology Center, Cambridge, MA.

Choi-Koh, S. S. (1999). A Student's learning of geometry using computer. *Journal of Educational Research, 92*, 301-311.

Clements, D. H. & Battista, M. T. (1992). Geometry and spatial reasoning. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 420-466). Reston, VA: National Council of Teachers of Mathematics.

Dixon, J. K. (1997). Computer use and visualization in students' construction of reflection and rotation concepts. *School Science and Mathematics, 97*, 352-358.

Gonzalez, G. & Herbst, P. G. (2009). Students' conceptions of congruency through the use of dynamic geometry software. *International Journal of Computers for Mathematics Learning*, *14*, 153–182.

Gellert A. (2014). Students discussing mathematics in small-group interactions: opportunities for discursive negotiation processes focused on contentious mathematical issues. *ZDM, 46*(6), 855-869.

Herbst, P., Gonzalez, G. & Macke, M. (2005). How can geometry students understand what it means to define in mathematics? *The Mathematics Educator*, *15*(2), 17-24.

Kakihana, K. & Shimizu, K. (1994). The roles of measurement in proof problems analysis of students' activities in geometric computer environment. In D. Kirshner (Ed.), *Proceeding of the 18th conference of the international group for the psychology of mathematics education (Vol. 3*, pp. 81-88).

Lambertus, A. J. (2007). *Students understanding of function concept: concept image and concept definitions* (Unpublished Master Theses). North Carolina State University.

Lo, J-J., Gaddis, K. & Henderson, D. (1996). Building upon student experience in a college geometry course. *For the learning of mathematics, 16*(1), 34–40.

Mitchelmore, M. & White, P. (2000). Development of angle concepts by progressive abstraction and generalization. *Educational Studies in Mathematics*, *41*, 209–238.

National Council of Teachers of Mathematics (NCTM) (1989). *Curriculum and evaluation standards for school mathematics*. Reston, Va.: NCTM.

NCTM (2000). *Principles and Standards for School Mathematics*. Reston, Va.: NCTM.

NCTM (2015). *Strategic use of technology in teaching and learning mathematics: A position of the National Council of Teachers of Mathematics*. Reston, Va.: Author.

Tall, D. (1992). The transition to advanced mathematical thinking: functions, limits, infinity. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 495-514). New York: Macmillan Publishing Company.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*. *12*, 151-169.

Vinner, S. (1991). The role of definitions in the teaching and learning of Mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65-81).

Vinner, S. & Dreyfus, T. (1989).Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356-366.

TEACHING WITH GEOGEBRA: RESOURCE SYSTEMS OF MATHEMATICS TEACHERS

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This paper reports from a study of use of the dynamic mathematics software, GeoGebra. This study used Ruthven's (2009) Structuring Features of Classroom Practice model to analyse the classroom practices of three teachers in English secondary schools. Here the particular focus is on how teachers manage their resource system with the use of GeoGebra. The main conclusion was that the stage teachers were at in terms of learning to teach with this software indicated differences in regard to establishing a functioning resource system especially in teaching the operation of the software, preparing the dynamic files and the choice of tasks. It became evident that the most experienced teacher was more fluent in managing his system of resources.

Keywords: classroom practice, mathematics teaching, resource system, dynamic mathematics software

INTRODUCTION

Although the influence of new digital technologies has been increasing over recent decades, their incorporation into mathematics education has been slow. It has become apparent that teachers have a central role in the integration of technology in mathematics classrooms, and that this issue needs more attention by research. Understanding the process that teachers go through to appropriate technology effectively into their instruction is of crucial importance to help facilitate successful integration. In this report the concept of 'resource system' will be used to describe a central aspect of the instructional environment of classrooms bearing on the use of technology. Ruthven (2009) has drawn attention to this concept as one of five key structuring features of classroom practice (alongside working environment, resource system, curriculum script and time economy), which shape the use of technology in lessons and the kinds of professional knowledge required. Resource system relates to collection of didactical tools and materials in use, and coordination of use towards subject activity and curricular goals (Ruthven, 2014).

With the influx of new technologies in schools, on the one hand classrooms are filled with resources, which can be considered as providing more opportunities for learning. On the other hand, the usability and adaptability of new technologies can present challenges for some teachers and students since "resources are not self explanatory objects with mathematics shining clearly through them" (Adler, 2000, p. 207). Researchers have shown that the challenge of adaptability of dynamic geometry software (DGS) lead the majority of teachers to use them in the more conventional types as "a reduced and static use of the possibilities of the software", and "the absence of autonomous experimentation by students" (Laborde, 2001, p. 299). Additionally, Ruthven et al. (2008) argues that there is an 'interpretative flexibility' in the incorporation of dynamic geometry software into educational practices. The researchers highlight the difference between how mathematics educators interpret DGS to be used in classrooms (e.g. more open student exploration) and how more typical teachers use it in their teaching.

> Where dynamic geometry has entered mainstream classrooms, it appears to be used to support more established forms of pedagogical practice, notably student activity directed towards empirical confirmation of standard curricular results, often through guided discovery, as already prevalent in the teaching of geometry in many educational systems. (Ruthven et al., 2008, p. 314)

In terms of the design of tasks, Laborde's (2001) study indicated that it was challenging for teachers to go beyond textbook tasks when using DGS. Development of task design scaled from almost traditional geometry tasks, to tasks that could only be approached in a dynamic geometry environment. The former indicated tasks typically applied in a paper-and-pencil environment often supported by a textbook and by applying available tools such as compasses and ruler. Using DGS on such tasks only facilitated drawing the shapes more accurately and quickly. In this regard, Laborde emphasised the fact that integration of DGS into mathematics teaching is a lengthy process. On the other hand, Monaghan's (2004) study indicated that most of the teachers managed to leave aside their textbooks in their technology-based lessons. However, the tasks in the worksheets "emphasised students' management of the computer software per se" as mentioned before. Similarly, findings from Erfjord's (2011) study investigating teachers' initial use of DGS in Norway indicated that two teachers used DGS with prepared instructional material focusing on the technical aspects of using DGS and had little explicit focus on the mathematics by mostly demoing for students what they needed to do in technical terms. This offered students a good background to further develop their ability to use compasses. Dragging mode was utilised for checking the accurate use of DGS rather than for provoking mathematical interpretation with the aim of improving students' abilities of using compasses. Furthermore, Assude (2005) also reported that in terms of use of compasses and Cabri there was not "any major changes with regard to the broad types of tasks: construction, description and property identification." (p. 192). Similarly, Ruthven, Hennessy, and Deaney (2005) argue that the typical use of DGS proposed by teachers in England is to let students work with geometrical properties utilising the dynamic dragging-function; further that many teachers in their teaching tried to control and constrain student work in order to avoid students spending too much time on the exploratory affordances in the DGS.

In this respect, this paper examines how three mathematics teachers establish their resource system with the use of GeoGebra –a dynamic mathematics software- in ordinary classrooms. The specific research question is: "What aspects of teachers professional knowledge emerge in relation to working with their resource system?"

RESEARCH CONTEXT

This paper reports on the 'resource system' aspect of the classroom practice of three English secondary-school mathematics teachers, associated with their use of GeoGebra for mathematics teaching. GeoGebra is an open-source educational software package, which provides dynamic mathematical representations (Hohenwarter & Preiner, 2007). We chose to study GeoGebra lessons in particular because of the interest that this software has attracted amongst teachers in England. After teachers agreed to participate in the research, the first author visited their schools to discuss their timetable and to find out when they were planning to make some significant use of this software. Observations and interviews then took place at a time agreed in advance with each teacher at his or her convenience. In this paper we focus on a type of mathematical topic which all three teachers chose to teach with GeoGebra, concerned with geometrical transformations.

The three teachers participating in this research have rather different profiles:

− an experienced teacher (pseudonym Chris) and Advanced Skills Teacher (a recognised grade of classroom teacher within the English school system, also taking special responsibility for leading professional development), who utilises new technologies in a progressive way in mathematics instruction. He taught the topic over a series of four lessons to a high attaining Year 9 class.

− an experienced teacher (pseudonym Susan) in terms of mathematics teaching but a novice technology user who is in her early stages in using GeoGebra. It was the first time she had integrated GeoGebra into her teaching of this topic to the extent of actively involving students in dynamic exploration. She taught the topic over two lessons to a lower set Year 7 class.

− a less experienced teacher (pseudonym Tom) in terms of mathematics teaching (4 years teaching experience) but familiar with the use of GeoGebra in mathematics teaching. He started using GeoGebra regularly in his teaching from the beginning of his career. He taught this topic over a period of three lessons to a Year 10 class (set 2 out of 4).

Classroom Observations: A semi-structured, non-participant observation approach was adopted for which the SFCP framework as an interpretative lens provided guidelines.

Teacher Interviews: Semi-structured post-lesson interviews were conducted in order to clarify the observed lessons and the professional thinking behind them according to the key themes of the SFCP framework.

RESULTS AND DISCUSSION

The resource system for all the teachers consisted of prepared dynamic files involving use of GeoGebra.

Susan employed GeoGebra with a pre-developed (paper) worksheet including four tasks, accompanied by prepared dynamic versions of each task for students to experiment with in GeoGebra. She borrowed these tasks from a Geogebra website. The purpose of these lessons was for students to learn a specific type of transformation, reflection. Susan's use of GeoGebra for this topic aimed at easing students' learning by permiting them to observe and think about the processes involved. In addition, Susan reported that GeoGebra speeded the learning process up by allowing the students to engage with more examples that were unusual to see in the traditional paper-and-pencil environment. She used the school's Virtual Learning Environment (VLE) for students to download/upload the tasks. Use of the VLE also aimed at sharing students' work with the whole class. All the reflection files required, first, students' predictions of where they thought the shapes would end up after reflection, second, students checking their predictions in GeoGebra to see if they were correct. In addition, for some tasks, students were told to move some of the points at the corner of the shapes and observe the changes.

Tom used teacher-created files involving use of GeoGebra for these lessons. Tom considered the dynamic and instant nature of GeoGebra as essential characteristics. With the aid of GeoGebra, students could be more curious and see what happens in a diagram, which Tom believed was distinctive to Dynamic Mathematics Software use in comparison with traditional methods. One of the prepared files was the Reflection file. Students were asked to draw the reflections of a number of shapes by using the polygon tool, and then check their answers by using the reflection tool to see if they were right or wrong.

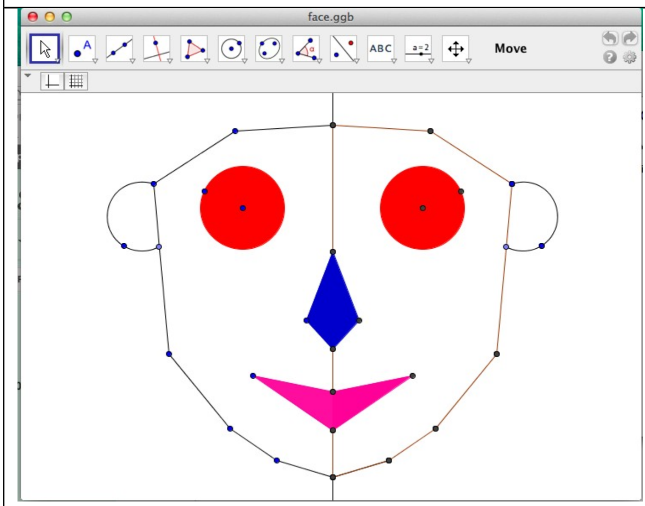
Chris also used teacher-created dynamic files involving use of GeoGebra. As an example, in the second lesson the teacher set up a file, where there was an object and its image locked so that the students were not able to move anything. The assignment was to locate the line of symmetry. Working with these prepared dynamic files, Chris encouraged his students to explore different paths to reach the same mathematical solution and placed less emphasis on finding the exact answer. In this case, he considered the use of GeoGebra for this topic as a tool to help students' thinking.

Four subthemes emerged, reflecting different aspects of teachers' professional knowledge in relation to working with this resource system.

Operation of the software

Susan's class was novice in GeoGebra use; therefore she allocated some lesson time to showing students the general set up of GeoGebra and the VLE, prior to the Transformations lessons, for students to get familiar with the technology. At the beginning of the first lesson, Susan did a technical demo on the IWB in order to show students how to go onto the VLE and find the GeoGebra files for students to download and work on and then upload back the finished activity to the VLE. In addition, she explained the tasks on the paper worksheet and the dynamic version of those tasks found through the VLE. Then, Susan provided students with a task (that was related only to technical aspects of the software) for them to accustomed to the software (see Table 1).

Table 1. The face activity and instructions for the activity

	• Reduce the size of his eyes • Make him look sad • Give him two earrings (chose a shape: polygon or circle). • Make one on the left and then reflect it in the line using the reflect tool • Give him eyebrows (use line segment and reflect it in the line)

Tom demonstrated and explained a number of GeoGebra techniques to students when and as necessary during different segments of his lessons. For instance, at the beginning of the first lesson, Tom demonstrated some new skills that are as following: how to copy and paste, which was different in the new GeoGebra version, how to use a vector from a point in GeoGebra, how to set a slider. His class had used GeoGebra before and had the basic knowledge of the software, such as how to find and open GeoGebra and get access to the tool bar.

Chris began the lesson by demonstrating how to operate the software: how to open the software, where to find the prepared files for transformations, where the related icons were located, as well as how to use the dragging tool. Then, he provided students with a dynamic file to explore the software by themselves the students were using GeoGebra for the first time for this particular topic (see Figure 1). With this dynamic file, he required students not only to learn the software but also "to look for interesting things" in the diagram and "explore the mathematics".

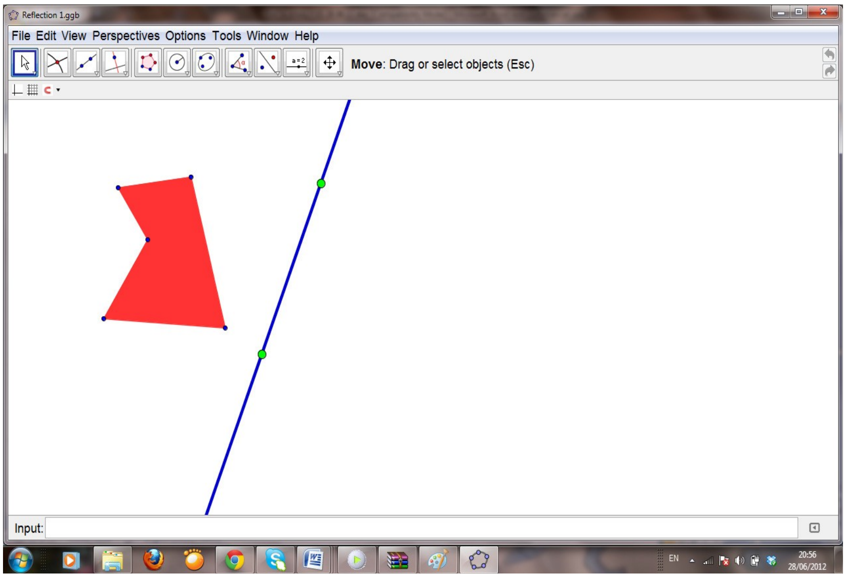


Figure 1. Reflection file that students worked on during Lesson 1

Prepared files involving use of GeoGebra

All teachers provided students with pre-designed dynamic files so that students could work on transformation concepts.

Level of professional knowledge underpinning pre-designed dynamic files was varried:

- Susan borrowed a pre-developed (paper) worksheet and accompanied by dynamic files for students to experiment with in GeoGebra. According to Susan, the use of such worksheet "helps the teacher" by acting as a "crutch" especially for those new to learning to teach with GeoGebra.

- Tom designed his own tasks involving GeoGebra use. On the basis of his experience, Tom found starting from a blank page "too much" for students as "there is quite a lot going on at the background of the program". He prepared dynamic files for students "to get them just see something in action, understand what the idea of translation or rotation is and to be able to then visualise their imagination."

- Chris had well-established teaching repertoires with the use of GeoGebra and planned investigative lessons. He graded students' use of technology by providing prepared files for the first five lessons and then for the last two lessons asking them to start from blank pages to create their own files with the guide of instructions. This implied that Chris saw value in eventually having the students create their own files. As believed by Chris, working with prepared files to start with gave the students more time to focus on the mathematics and their thinking, and thus increased their learning time.

These case studies illustrated a relationship between the technology experience level of teachers and the use of ready-made resources. Susan as a novice technology-using teacher found it more appropriate to start with files and worksheets already prepared by someone else: this gave her more confidence and also alleviated the time pressures she felt. On the other hand, Tom, who has made more use of GeoGebra in teaching than Susan, created the dynamic files himself taking into account of his previous experience. Chris, who is the most experienced of the three in teaching with GeoGebra, also managed students use of GeoGebra gradually by allowing them to gain confidence in working from scratch with GeoGebra. This development of activities appeared very similar to the progression reported by Laborde (2001). Following four teachers for three years to understand how teachers novice to technology integrated it into their teaching, Laborde found that they gradually granted control of technology and the learning environment to students.

Choice of tasks

Tom and Chris designed the tasks for students themselves whereas Susan borrowed a pre-made worksheet from a GeoGebra website. In this sense, Susan was at the stage where she needed pre-made worksheets/tasks to incorporate GeoGebra into her lessons so that she could follow those worksheets to set a lesson agenda. The tasks designed by Chris were more open ended in the sense that they allowed the students to discover different ways of solving a problem. Furthermore, Chris encouraged the students to take various directions and focused on students' thinking rather than the result itself, which in turn enabled him to create whole class discussion of different ideas and approaches. However, the tasks prepared by Tom were more structured in that there was not room for students to consider from multiple perspectives. In addition, he mostly relied on questions from a textbook which he considered the best he had ever used because it was very close to the current examination specifications. This implied that he wanted his students to practice with questions that could also directly prepare them for examinations.

In this sense, Tom and Susan both tended to use closed-ended tasks with the focus on students getting the questions correct. However. Chris used open ended tasks that allowed students to discover different ways of solving a problem. He had much richer repertoire of interactional moves related to open ended GeoGebra based tasks.

Again, Laborde (2001) emphasised that teachers experienced the design of tasks to be employed with computer software as a challenging process (exemplified in her study by the Dynamic Geometry Software, Cabri). Both Laborde (2001) and Monaghan (2004) found that most teachers used and designed worksheets characterised as having elements of control or guidance, which were usually in accordance with teachers' use of tasks from textbooks in non-technology lessons. Thus, Susan's and Tom's choices to use files/questions intended to closely guide and support their teaching of the lessons and the individual student work in GeoGebra are not surprising. However, Chris, as an experienced teacher and technology user, moved beyond that and came up with his own resources, which were not inspired by paper-and-pencil activities. This suggested that he had come to the point that he no longer found tasks that replicated those from textbooks to be learning productive.

Promoting independent learning through the use of software

The tasks to be utilised with GeoGebra for Transformations lessons attempted to stimulate students' use of a trial and refinement process. All the teachers pointed to the ways in which use of the software could facilitate the process of self-testing and self-correcting. Hence, the main new aspect that these case studies provided evidence for was that activity formats depended more on processes through which the students made a prediction and the technology provided feedback on it rather than (as the teachers pointed out) the teacher or other students validating – or invalidating – it. That shifted the role of the teacher towards becoming an organizer/observer of this process. In addition, technology appeared to be conducive, even essential, to this activity format providing for the necessary levels of interactivity and immediate feedback.

In this connection, Predict-and-test was part of the logic of all the teachers' lesson agendas for handling this topic. In Tom's and Susan's cases, choice of the tasks making use of GeoGebra aimed to enable pupils to check the results of Transformations already done 'by hand'. In this regard, they had limited professional knowledge regarding interactional moves around tasks. However, as has already been discussed, Chris provided students with more open-ended tasks that allowed students to have different ideas and encouraged them to test their ideas with the dynamic software. The aim was to promote students' skills "in terms of their development as a mathematician". He was already used to discursive open ended approach.

In summary, the three teachers clearly demonstrated how establishing a functioning resource system was an important issue for them and they illustrated different stages in achieving that (see Table 2). Susan was at the stage where consideration of the resource system was the focus as she and the students needed to learn how to use the particular software. On the other hand, Tom was more confident in operating the software and constructed his own files for students to work on. In Chris's case, he was at the stage where a coherent set of resources was already developed as a functioning resource system and knew how to prepare his students in order for them to be able to use those. In this sense, it is possible to see the transition to actually creating geometric constructions and other dynamic figures from scratch as teachers gain more confidence in operating the software. In terms of design of technology-based tasks, these case studies illustrated that the experience level also suggests that there is a change from regular forms of extension activity to things that need high-level thinking. Considering the most experienced teacher in the study, it is clear that for teaching the topic Chris placed far more emphasis on the development of high-level mathematical reasoning/thinking in comparison with Susan and Tom. Furthermore, between Tom and Susan, the evidence highlights that Tom, drawing on his experience, put more thought into designing his task by focusing on students' thinking.

Table 2. Characterisation of Novice/Expert teachers that relates to resource system

	Susan (limited)	Tom (developing)	Chris (sophisticated)
Resource System	Just embarked on establishing a functioning resource system with GeoGebra for her classes. Prioritized mainly developing instrumental knowledge to make it part of the resource system, first for herself personally and then also for her students. Started with files and worksheets already prepared by someone else: this gave her more confidence and also alleviated the time pressures she felt.	Had confidence in operating the software since he previously made use of GeoGebra in teaching. Created dynamic files himself taking into account of his previous experience so that students could focus on mathematical content rather than dealing with complexity of software.	Had already established a set of resources as a functioning resource system. Managed students use of GeoGebra gradually by allowing them to gain confidence in working from scratch with GeoGebra. Designed more open-ended tasks that allowed students to discover different ways of solving a problem.

REFERENCES

Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education*, 3(3), 205–224.

Assude, T. (2005). Time management in the work economy of a class. A case study: Integration of Cabri in primary school mathematics teaching. Educational Studies in Mathematics, 59(2), 183-203.

Drijvers, P. (2012). Teachers transforming resources into orchestrations. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), From Text to 'Lived' Resources (pp. 265–281). New York: Springer.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: instrumental orchestrations in the technology-rich mathematics classroom. Educational Studies in Mathematics, 75(2), 213–234.

Erfjord, I. (2011). Teachers' Initial Orchestration of Students' Dynamic Geometry Software Use: Consequences for Students' Opportunities to Learn Mathematics. *Technology, Knowledge and Learning*, 16(1), 35–54.

Laborde, C. (2001). Integration of technology in the design of geometry tasks with Cabri-geometry. *International Journal of Computers for Mathematical Learning*, 6(3), 283–317.

Monaghan, J. (2004). Teachers' activities in technology-based mathematics lessons. International Journal of Computers for Mathematical Learning, 9(3), 327–357.

Ruthven, K. (2009). Towards a naturalistic conceptualisation of technology integration in classroom practice: The example of school mathematics. Education & Didactique, 3(1), 131–159.

Ruthven, K. (2014). Frameworks for analysing the expertise that underpins successful integration of digital technologies into everyday teaching practice. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The Mathematics Teacher in the Digital Era* (pp. 373–393). Berlin: Springer.

Ruthven, K., Hennessy, S., & Deaney, R. (2005). Incorporating dynamic geometry systems into secondary mathematics education: Didactical perspectives and practices of teachers. In D. Wright (Ed.) *Moving on with Dynamic Geometry* (pp. 138-158). Derby: Association of Teachers of Mathematics.

Ruthven, K., Hennessy, S., & Deaney, R. (2008). Constructions of dynamic geometry: A study of the interpretative flexibility of educational software in classroom practice. *Computers & Education*, 51(1), 297–317.

Ruthven, K., Deaney, R., & Hennessy, S. (2009). Using graphing software to teach about algebraic forms: A study of technology-supported practice in secondary-school mathematics. Educational Studies in Mathematics, 71(3), 279-297.

PLANNING TO TEACH LOWER SECONDARY MATHEMATICS WITH DYNAMIC MATHEMATICAL TECHNOLOGY: QUALITY FEATURES OF LESSON PLANS

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Set within the context of the longitudinal Cornerstone Maths project in England. we adapt Thomas and Hong's theoretical framework (mathematical) 'pedagogic technology knowledge' (MPTK, Thomas & Hong, 2013) to explore teachers' espoused knowledge to teach with dynamic mathematical technology in lower secondary mathematics. We conclude a set of eight 'quality features' of such plans, and highlight how each of these features can provide a dynamic insight into teachers' MPTK development over time.

Keywords: Dynamic mathematical technology; linear functions; algebraic variable, geometric similarity; mathematical pedagogic technology knowledge (MPTK)

INTRODUCTION

The Cornerstone Maths project, which began in England in 2011 has researched the design, implementation and impacts (on both students and teachers) of a set of three digitally enhanced curriculum units for lower secondary mathematics: algebraic variable; linear functions; and geometric similarity. These are three topics that are considered hard to teach and for which a body of evidence exists to suggest that dynamic mathematical technologies can enhance students' understandings. This earlier work is widely reported (Clark-Wilson & Hoyles, 2017; Clark-Wilson, Hoyles, & Noss, 2015; Clark-Wilson, Hoyles, Noss, Vahey, & Roschelle, 2015; Hoyles, Noss, Vahey, & Roschelle, 2013). In this paper, we focus on a strand of work that is motivated by the research question: What mathematical pedagogic technology knowledge is desirable for teachers to integrate dynamic visual technologies in their teaching of these concepts? This required an articulation of such knowledge and a methodological design that would have legitimacy within the context of a 15-month long professional development project.

THEORETICAL FRAMEWORK

Conceptualising teacher knowledge

A major shortcoming of widely adopted frameworks that conceptualise teacher knowledge such as Ball et al's 'Mathematical Knowledge for Teaching' (MKT, Ball, Hill, & Bass, 2005) and Rowland et al's 'knowledge quartet' (Rowland, Huckstep, & Thwaites, 2005) is that they have not evolved from researching teaching scenarios in which dynamic mathematical technologies were present. We define 'dynamic mathematical technologies' as those that offer different mathematical representations (geometric shapes, graphs, tables, algebraic expressions) that teachers and pupils can manipulate and by doing so, engage with the underlying mathematical concepts and relationships. Consequently, such frameworks pay no attention to the particular aspects of a teachers' knowledge for planning and teaching lessons with such technologies.

Hence, we looked to a broader framework that includes MKT, but also conceives knowledge as a dynamic construct that considers both cognitive and affective aspects of knowledge and that had emanated from research into teachers' developing use of DMT in classrooms. We adopted a framework developed by Thomas and colleagues, '(Mathematical)[1] Pedagogical Technology Knowledge', (Thomas & Hong, 2013; Thomas & Palmer, 2014), henceforth we call MPTK, as shown in Erreur : source de la référence non trouvée.

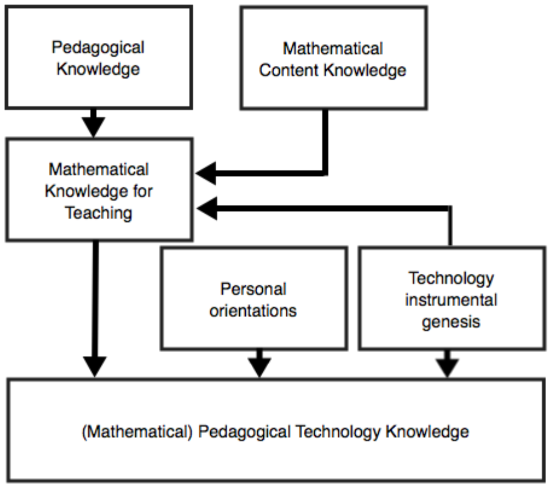

Figure 1 Components of (Mathematical) Pedagogical Technology Knowledge (Thomas & Hong, 2013)

This is a theoretical construct with the following components:

- **Pedagogical knowledge**: First suggested by Shulman (1987), this is a teacher's knowledge of the 'broad principles and strategies of classroom management and organization that appear to transcend subject matter' (p. 8).
- **Mathematical content knowledge**: A teacher's own knowledge of mathematics.
- **Mathematical Knowledge for Teaching (MKT)**: This combination of a teacher's pedagogical knowledge alongside their mathematical content knowledge was first defined as MKT by Ball, Hill and Bass (2005).
- **Personal orientations:** The teachers' affective variables, that is their goals, attitudes, dispositions, beliefs, values, tastes and preferences, as described by Schoenfeld (2011, p.29), also incorporating their perceptions of the nature of mathematical knowledge and how it should be learned (with and without technology).
- **Technology instrumental genesis**: Rooted in activity theory, this is the process through which the teacher makes actions and decisions through which the technological tool is adapted to accomplish a particular mathematical task (Drijvers & Trouche, 2008; Guin & Trouche, 2002). Furthermore, for teachers, this genesis incorporates the development of the teachers' understanding of pupils' processes of instrumental genesis, whereby pupils become familiar with the affordances of the technology and can begin to use it in mathematically productive ways (Haspekian, 2005).

Landmark activities

An important construct that underpinned both our theoretical framing and informed our research methodology is that of 'landmark activities'. We define landmark activities as those which provoke

1 Thomas & Hong do not include Mathematical in their description of PTK, as, we conjecture their 'overarching world' is Mathematics. For clarity, we add Mathematical to the PTK, so henceforward call it MPTK.

a rethinking of the mathematics or an extension of previously held ideas – the ‘aha’ moments that show surprise - and provide evidence of pupils’ developing appreciation of the underlying concept. Although each curriculum unit includes several potential landmark activities, the research team selected one activity from each unit, which became the focus for teachers’ planning, teaching and subsequent reflection within our adapted ‘lesson study’ approach. In this way, the landmark activities act as boundary objects for the study (A more detailed account of landmark activities can be found in Clark-Wilson, Hoyles, & Noss, 2015).

METHODOLOGY

The project involved 209 teachers from 48 London secondary schools in the period Jan 2014 - July 2016. Teachers were either self-selecting or nominated by their school and they or their schools chose for them to be involved in up to three cycles of professional development that enabled them to plan, teach and evaluate a landmark activity from each of the three Cornerstone Maths curriculum units. Some schools opted to send the same teacher(s) to two or three of these cycles. Briefly, the teachers participated in a one-day face to face session that: introduced them to the curriculum topic; supported them to become instrumented with the CM software; provided an opportunity for a lesson planning activity; and inducted them to the project’s online community. The teachers were also invited to give their ethical consent for their data to be used within the study (n=111, 53%).

During the lesson planning exercise, which the teachers carried out in pairs, a common lesson planning proforma was adopted that captured the following information:

- Contextual information about the class: (age, ability level):
- Teacher’s preparation notes:
- Pupils' prior experience/skills with the software:
- Key intended learning outcomes for the pupils:
- Description of the planned phases of the lesson that included the teachers intended actions and the anticipated pupils’ responses to these.

Furthermore, a critical aspect of the methodology was that all teachers shared their lesson plans within the project online community, what was visible to all participants. The teachers were actively encouraged to review each other’s plans and to adopt or adapt the text, as they thought useful. The teachers were encouraged to produce the best plan possible, although it was understood that, as they taught the CM curriculum tasks that preceded the identified landmark activity, they would most probably want to review and revise their plan in the light of this classroom experience.

The lesson plans were evaluated according to the following set of ‘desirable’ features:

Feature 1: Describes teachers’ actions and questions (not involving the DMT).

Feature 2: Describes pupils' actions on DMT.

Feature 3: Supports pupils in their instrumental genesis of the DMT, as appropriate to the activities.

Feature 4: Refers to the mathematical concept at stake (i.e. variables, functions, geometric objects).

Feature 5: Describes acting on and connecting mathematical representations.

Feature 6: Uses mathematical vocabulary.

Feature 7: Uses technological/contextual vocabulary.

Feature 8: Includes planned teacher use of the DMT.

These eight features had been developed a priori by the researchers as a means to arrive at a 'quality score' of between zero and eight for each plan, depending on whether the plan included particular feature. Hence it was possible to arrive at quantitative indications of quality in addition to the more obvious qualitative analysis that could be deduced from the plans.

FINDINGS

82% of the teachers surveyed (n=111) reported that they had never or only occasionally used a mathematical technology in their key stage 3 teaching, with only 33% reporting that they felt confident or very confident to do so. From this data, we conclude that, for many of the teachers, the lesson plans were their first ever plan for this type of lesson.

The analysis of the teachers' lesson plans provided an insight their MPKT as they prepared to teach the lessons. We begin by presenting our findings with respect to the first of the three CM curriculum topics (algebraic variable) to highlight the nature of the resulting data and then describe the cross-topic analysis that led to a more general set of outcomes.

Algebraic variable

Twenty-eight lesson plans that had been produced in pairs and trios by 74 teachers were analysed and the frequencies of each feature is shown in Table 2.

Feature of lesson plan	Frequency	% (n=28 plans)
1. Explicit descriptions of teachers' actions/questions	16	57%
2. Explicit descriptions of pupils' actions on DMT during the lesson	9	32%
3. Appreciation of pupils' instrumental knowledge (i.e. prior skills with software, progression of skills in lesson)	12	43%
4. Explicit reference to variables (i.e. creating, naming, acting on)	11	39%
5. Explicit reference to acting on reps (i.e. dragging/moving sliders)	17	61%
6. Explicit use of mathematical vocabulary	18	64%
7. Explicit use of technological/contextual vocabulary	18	64%
8. Includes planned plenary phases that involved teacher use of software	5	18%

Table 2 Algebraic variable: Summary of lesson plan analysis (28 Lesson plans)

An exemplification of high quality planning for the algebraic variable research lesson in relation to each of the desirable features (taken from the complete set of lesson plans) is provided in Table 3.

Feature of lesson plan	Exemplification from teachers' plans
1. Explicit descriptions of teachers' actions/questions	"Encourage pupils to play the pattern again and ask does it correspond to your pattern if you change the number of blocks?"

2. Explicit descriptions of pupils' actions on DMT during the lesson	"Encourage students to use slider - ask them how you can make both sliders move at the same time. What will they need to do the variables?"
3. Appreciation of pupils' instrumental knowledge (i.e. prior skills with software, progression of skills in lesson)	"Remind students how to use the software – recap Investigation 1. i.e. Blocking and patterning. (Lock student screens)"
4. Explicit reference to variables (i.e. creating, naming, acting on)	"Ensure all pupils start to introduce a variable for their blocks ('unlock' the no of blocks column)".
5. Explicit reference to acting on reps (i.e. dragging/moving sliders)	"[Ask] What is the purpose of the slider? What impact is it having when you slide along the bar?"
6. Explicit use of mathematical vocabulary	"[Ask] How can we check if our orange and green blocks increase in the same way?"
7. Explicit use of technological/contextual vocabulary	"Ask students to create a table snapshot, starting from 1 block. What do students notice about the total number of lights?"
8. Includes planned plenary phases that involved teacher use of software	"demonstrate how the Blocks and Pattern should have been made. What does the slider do?"

Table 3 Algebraic variable: Exemplification of the features of high quality lesson plans

The plans were of a highly variable quality and it was notable that only five plans included six or more of the desirable features, which suggests that the teachers had very little prior experience of a lesson planning approach that emphasised their own actions and words, rather than solely a plan of what their pupils would be expected to do. Within the plans, approximately two thirds of the plans included references to actions on the dynamic slider and approximately one fifth of the lessons plans included a planned plenary phase that involved the teacher's use of the software.

Development of lesson plans over time

The above process was replicated for the two subsequent curriculum topics (linear functions and geometric similarity) and distributions of the quality scores produced as shown in Figures 2, 3 and 4.

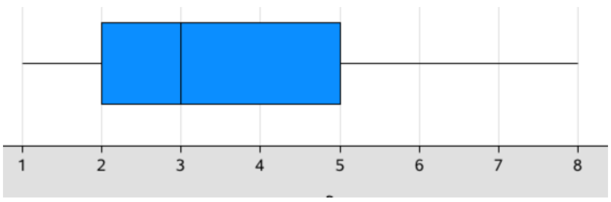

Figure 2 Algebraic variable: Distribution of quality scores for lesson plans (n=27, = 3.9 SD = 1.8)

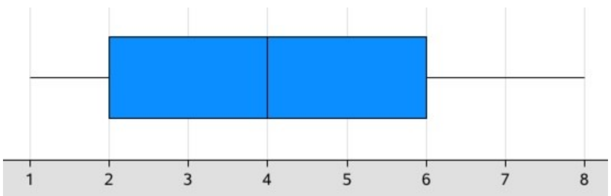

Figure 2 Linear functions: Distribution of quality scores for lesson plans (n=42, $\overline{x}$ = 4.2

SD = 2.1)

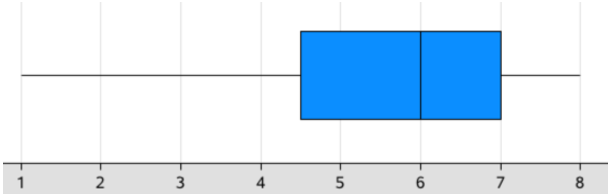

Figure 3 Geometric similarity: Distribution of quality scores for lesson plans (n=21, $\bar{x} = 5.5$, SD = 1.8).

These mean average and median quality scores show very clearly the development in the quality of the teachers’ lesson plans over time as they participated in the repeated PD cycles as both scores increased. This is substantiated by the qualitative analysis of the lesson plan text, examples of which will be shared in the conference presentation. A summary table of the frequencies of quality features across the three topic areas is also informative (Table 4).

	estionsTeachers’ actions and	Pupils' actions on DMT	l genesisSupports for pupils'	nceptFocus on mathematical	l Actions on representations	Uses maths vocabulary	bularyUses technical and/or	DMT
Algebraic variable (n=28)	57%	43%	32%	39%	61%	64%	64%	18%
Linear functions (n=42)	69%	38%	45%	57%	26%	62%	31%	24%
Geometric similarity (n=21)	95%	33%	86%	71%	62%	86%	71%	43%

Table 4 Comparison of lesson plan quality features by topic

The nature of the individual landmark activities did provoke a need for teachers to plan in ways that might privilege particular features, i.e. the geometric similarity landmark activity required increasingly more detailed definitions, which would by necessity privilege the use of mathematical language. However, given that the three PD cycles took place over the period of a year, the following conclusions can be made:

- Teachers became much more aware of the need to plan what they were going to do and say during the lessons and particularly during the whole class plenaries around the important mathematical ideas.
- Teachers became more mindful of the need to provide support for pupils to make sense of the DMT such that they could use it in mathematically productive ways beyond only the initial lessons (i.e. support the pupils’ instrumental genesis more explicitly).
- Teachers paid increasing attention to the mathematical concept at stake.

- Teachers were more explicit in their plans to convene whole class plenaries to focus on the mathematics at stake (with more teachers considering how they would use the DMT to support this work).

DISCUSSION

The teachers' lesson plans provide an insight into their espoused MPTK. Furthermore, the features of the lesson plans can be mapped to the components of the teachers' knowledge as shown in Figure 2.

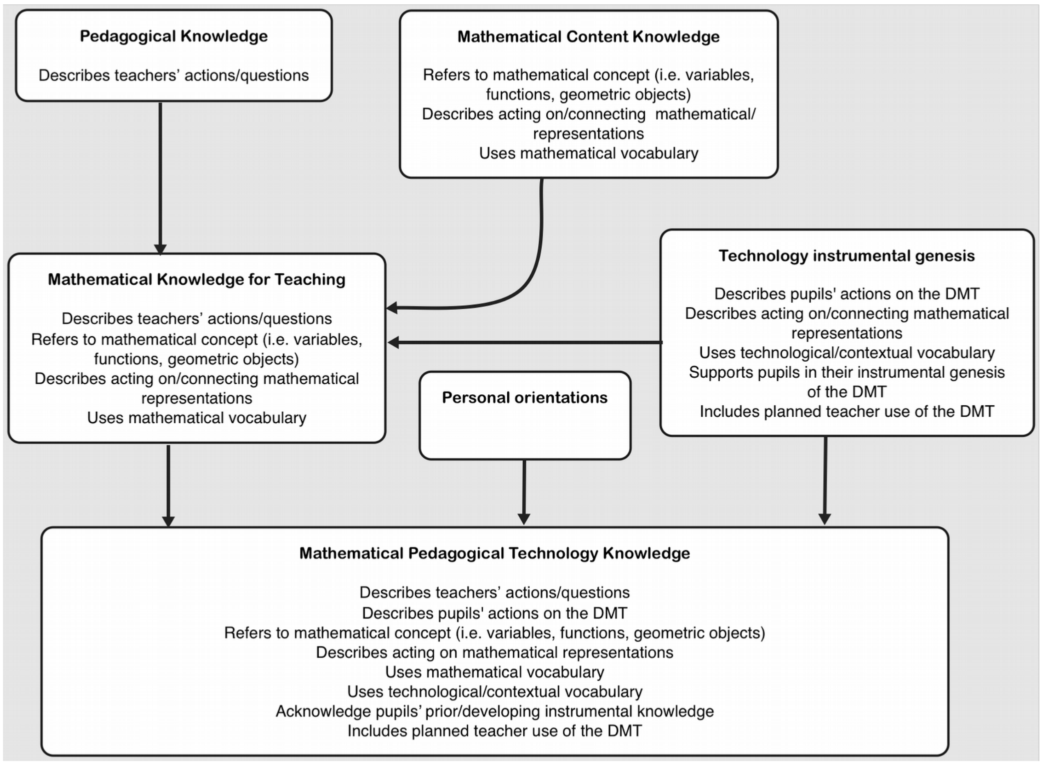

Figure 2 Features of CM landmark activity lesson plans and their relationship to a teacher's MPTK.

This provides an indication of the key elements of planning lessons with technology that concern the development of pupils' instrumental geneses – a significant element of teachers' knowledge that should be developed within teacher education and professional development programmes.

ACKNOWLEDGEMENTS

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REFERENCES

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing Mathematics for Teaching: Who Knows Mathematics Well Enough To Teach Third Grade, and How Can We Decide? *American Educator*(Fall 2005), 14-46.

Clark-Wilson, A., & Hoyles, C. (2017). *Developing teachers' mathematical knowledge for teaching through engagement with key mathematical concepts using dynamic digital technology [Nuffield Foundation funded project 2014-2017]*. Retrieved from London:

Clark-Wilson, A., Hoyles, C., & Noss, R. (2015). Scaling mathematics teachers' professional development in relation to technology: Probing the fidelity of implementation through landmark activities. In K. Krainer & N. Vondrovà (Eds.), *Proceedings of the Ninth Conference of the European Society for Research in Mathematics Education* (pp. 2333-2339). Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME.

Clark-Wilson, A., Hoyles, C., Noss, R., Vahey, P., & Roschelle, J. (2015). Scaling a technology-based innovation: Windows on the evolution of mathematics teachers' practices. *ZDM Mathematics Education, 47*(1), 79-92. doi:10.1007/s11858-014-0635-6

Drijvers, P., & Trouche, L. (2008). From artifacts to instruments: A theoretical framework behind the orchestra metaphor. In G. Blume & K. Heid (Eds.), *Research on technology in the learning and teaching of mathematics: Syntheses, cases and perspectives. Vol 2 Cases and perspectives* (pp. 363-392). Charlotte: National Council of Teachers of Mathematics/Information Age Publishing.

Guin, D., & Trouche, L. (2002). Mastering by the teacher of the instrumental genesis in CAS environments: necessity of instrumental orchestrations. *ZDM Mathematics Education, 34*(5), pp. 204 - 211.

Haspekian, M. (2005). An "Instrumental Approach" to study the integration of a computer tool into mathematics teaching: The case of spreadsheets. *International Journal of Computers for Mathematical Learning, 10*(2), 109-141.

Hoyles, C., Noss, R., Vahey, P., & Roschelle, J. (2013). Cornerstone Mathematics: Designing digital technology for teacher adaptation and scaling. *ZDM Mathematics Education, 45*(7), 1057-1070. doi:10.1007/s11858-013-0540-4

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education, 8*, pp. 255-281.

Schoenfeld, A. (2011). *How we think. A theory of goal-oriented decision making and its educational applications*. New York: Routledge.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 56*(1), 1-22.

Thomas, M. O. J., & Hong, Y. Y. (2013). Teacher integration of technology into mathematics learning. *International Journal for Technology in Mathematics Education, 20*(2), 69-84.

Thomas, M. O. J., & Palmer, J. (2014). Teaching with digital technology: Obstacles and opportunities. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The Mathematics Teacher in the Digital Era: An International Perspective on Technology Focused Professional Development* (pp. 71-89). Dordrecht: Springer.

PRE-SERVICE TEACHERS' PREPARATION AS A CATALYST FOR THE ACCEPTANCE OF DIGITAL TOOLS FOR TEACHING MATHEMATICS AND SCIENCE

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The present research examines whether the pre-service teachers' preparation in using digital tools in their teaching develops their acceptance of these tools as teaching tools. Here, acceptance is measured in terms of the constructs of the technology acceptance model (TAM) introduced by Davis. It also examines the mediation of self-efficacy, anxiety of using digital tools for teaching mathematics and science and enjoyment of this use between the constructs of acceptance of digital tools for teaching. We used questionnaires that are part of TAM. Forty eight mathematics and science pre-service teachers participated in the study. We analyzed the collected data using SPSS 21. The research results indicate that the pre-service teachers' preparation resulted in significant differences in their scores of affective and usage constructs associated with their acceptance of digital tools for mathematics and science teaching, except in the scores of anxiety.

Keywords: Pre-service teachers, preparation model, digital tools, mathematics, science

INTRODUCTION

A main factor in the use of technological tools in the mathematics classroom is the teacher (Thomas & Palmer, 2014), which necessitates educating pre-service teachers in using these tools in their teaching in the training schools. This education would encourage them to use these tools in their future teaching of the subject matter. In the present paper, we describe a model for preparing pre-service teachers in the use of digital tools in teaching and the effect of this preparation on some affective and behavioral aspects of the usage of these tools in the classroom. Two of the factors that affect teachers' use of technological tools in their teaching are their orientations towards this use and the value of this use (Thomas & Palmer, 2014). In the present research, we are interested in the previous two constructs, among other constructs, as constructs that could affect teachers' use of digital tools in the classroom. We utilize the Technology Acceptance Model (TAM) of Davis (1989) as a framework for such analysis. We are aware that other frameworks could be used to analyze the studied issue (e.g., Getenet, Beswick & Callingham, 2015), but we chose the TAM framework because it suits the analysis of the acceptance of digital tools for teaching the subject matter, which is one aspect of pedagogical technological knowledge (PTK) (Hong & Thomas, 2006). This serves studying teachers' professional development in integration technology in the classroom, using different theoretical frameworks, which serves understanding teachers' learning from different perspectives.

Technology Acceptance Model (TAM)

One of the most widely used models for technology adoption and usage is the 'Technology Acceptance Model' (TAM) developed by Davis (1989). Similar to other technology acceptance models, TAM assumes that users choose to employ a specific technology based on individual cost-benefit considerations (Compeau, Higgins, & Huff, 1999). Specifically, TAM assumes that perceived ease-of-use (PEOU) and perceived usefulness (PU) determine the user's acceptance of a technology. Davis , Bagozzi, & Warshaw (1989) describes PEOU as the degree to which the user expects the technology to be free of effort, while PU is the individual's subjective perception of the technology as increasing performance within an organizational context. As shown in Figure 1, TAM suggests that the user's actual usage of a particular system develops over four stages, where external variables as individual abilities and situational constraints influence technology usage through their impact on the PEOU and PU. Both factors affect a user's attitude towards the technology, which in turn influences the user's intention to use the technology. Furthermore, there is a direct impact of perceived usefulness on the user's intention to use the technology, which could mean that even if the individual has a negative attitude towards a technology, this could be overcome by a positive belief about the technology's usefulness, which finally leads to a positive usage intention (Röcker, 2009).

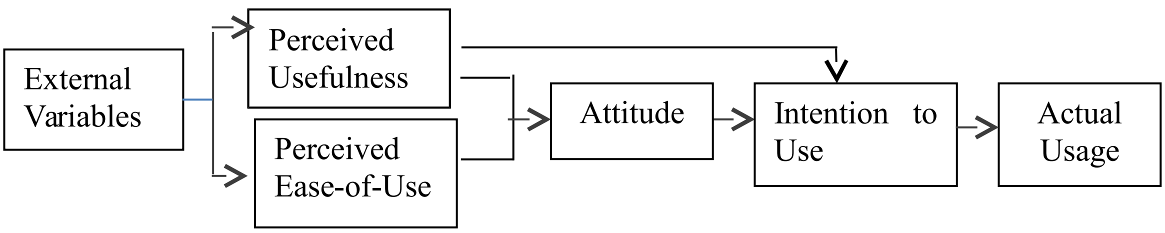

Figure 1: Original Technology Acceptance Model as in Davis et al. (1989)

Alenezi, Karim and Veloo (2010) added 'enjoyment', 'computer anxiety', 'computer self-efficacy' (CSE) and 'internet experience' to PEOU and PU to explain users' attitudes towards using E-learning. They found that computer anxiety, CSE and Enjoyment significantly influenced students' intention to use E-learning, while the Internet experience insignificantly influenced them. Alenezi et al. (2010), in contrast to Shih and Huang (2009), found that attitude was confirmed to mediate the relationship between PU, PEOU and the users' behavioral intention. Results from Yi and Hwang (2003) highlighted the important roles of self-efficacy, enjoyment, and learning goals orientation (as components of intrinsic motivation) in determining the actual use of web-based information systems. In the present research, we examined whether self-efficacy, enjoyment and anxiety mediate between the constructs of accepting digital tools for teaching mathematics and science.
In addition to the said above, though different theoretical frameworks are suggested today to study teachers' acceptance of digital tools in mathematics and science classes, TAM is still worldwide used to study mathematics and science teachers' acceptance of digital tools as tools in their classrooms (e.g., Pittalis & Christou, 2011; Frikkie & Ogunniyi, 2016). Theoretical frameworks like TAM are especially needed when studying the acceptance of technology for teaching by teachers of different disciplines (e.g., Padmavathi, 2016), as the case in the present research.

Research rationale and goals

Different studies examined students' acceptance of technological tools, but little research examined pre-service teachers' acceptance of digital tools for mathematics and science teaching, as a result of their preparation, especially when affective constructs are considered as mediators for this acceptance. The present study attempts to do so, considering self-efficacy, enjoyment and anxiety when using digital tools in teaching mathematics or science, as personal constructs that could mediate between the different constructs of pre-service teachers' acceptance of using digital tools in their teaching. Doing so, one of the present study goals is to examine whether pre-service teachers' preparation in using digital tools in their teaching in the training schools, develops their acceptance of these tools as teaching tools. Here, acceptance is measured in terms of the TAM components, i.e. PEOU, PU, CSE, attitude and future use. As mentioned above, another goal is to examine the mediation of self-efficacy, anxiety of using digital tools for teaching mathematics and science and enjoyment of this use between the constructs of the pre-service teachers' acceptance of digital tools for their teaching.

Research questions

1. Does the preparation of pre-service teachers in the use of digital tools increase their acceptance of these tools for teaching mathematics and science?
2. Does the preparation result in significant differences in the scores of the different constructs of pre-service teachers' acceptance of digital tools (for teaching mathematics and science) according to the pre-service teachers' specialization, computer-ability and computer-use?
3. Do CSE, anxiety of using digital tools for teaching mathematics and science and enjoyment of this use mediate between PEOU and PU on one side and attitude towards this use on the other side?
4. Do CSE, anxiety of using digital tools for teaching mathematics and science and enjoyment of this use mediate between attitude towards this use on one side and intention to use on the other side?

METHODOLOGY

Research context and participants

The current research accompanies the preparation of third year pre-service teachers to use digital tools effectively in teaching mathematics and science. The preparation of the pre-service teachers in the college lasts four years, where the third year is the year in which the pre-service teachers are prepared to use digital tools in their teaching in the training schools. In their first year, the pre-service teachers participate in a course that focuses on technological skills as skills to use editors and spreadsheets. In their second year, the pre-service teachers participate in a course that focuses on integrating technology in teaching the discipline, as mathematics or science. In the third year, the pre-service teachers are expected to practice the integration of digital tools in their teaching in the training schools.

We administered questionnaires to measure the advancement of the pre-service teachers' acceptance of digital tools in teaching mathematics and science. Forty eight pre-service teachers majoring in mathematics and science teaching (twenty four in each discipline) completed the questionnaires at the beginning and end of the preparation.

Research instruments

The questionnaire is based on Davis' Technology Acceptance Model (Davis, 1989), with 7-point Likert items, was composed to conduct the research. Since TAM variations in different previous studies were all reliable, factor analysis was not carried out in this study. Instead, face validity and reliability (Cronbach's Alpha) calculations were carried out. The six following scales were used: digital tools usefulness (9 items), digital tools self-efficacy (5 items), digital tools ease-of-use (3 items), attitude towards digital tools (3 items), intention-to-use and use of digital tools (3 items).

The added constructs (self-efficacy, enjoyment and anxiety were taken from Alenezi, Karim and Veloo (2010).

Statistical exams

The questionnaire had two parts. The first part collected personal information as specialization, computer-ability, and computer-use, while the second part was composed of the six scales. The questionnaire was translated for the first time to Arabic language before administering them to the pre-service teachers. The questionnaire underwent validity and reliability exam.

Face validity: The Arabic translation of the questionnaire was given to a group of pre-service teachers who were required to examine if the scales' statements were understandable to the reader. Some items of the scales were restated to clarify their ambiguity.

Content validity: The questionnaire was given to a group of experts (five college pre-service teachers) who were required to examine whether the questionnaires' items cover the full domain of the different educational constructs and whether they cover constructs other than the appropriate ones. The experts gave no remarks that necessitated the modification of the TAM questionnaire.

The scales' reliability: The scores of the pre-service teachers in the TAM questionnaires were used to compute Cronbach alpha of the various TAM constructs. The computations resulted in values that ranged between .82 and .91, which are considered acceptable reliability scores.

Data processing: Research question 1 was answered using paired-samples t-test. Research question 2 was answered using ANOVA. Research questions 3 and 4 were answered using the four steps of Baron and Kenny (1986).

RESULTS

The preparation model:

The pre-service teachers' preparation was based on two theoretical frameworks: the community of inquiry framework (Jaworski, 2005) and practice-based or practice-oriented professional development framework (Ball & Bass, 2003; Ponte, 2012). The utilization of the two previous frameworks targeted developing mathematics and science pre-service teachers' practice in working with digital tools for teaching. More specifically, it targeted developing pre-service teachers' practices with digital tools, starting from selection of a digital tool for a specific topic, continuing to preparing lesson plans that utilize the tool, teaching a specific topic with the selected digital tool, reflecting on this teaching and improving the lesson plan as a result of the reflection.

The preparation model depended on the interaction between the pre-service teachers in an electronic forum designed for discussing the selection of digital tools for specific topics, utilizing them in lesson plans and implementing them in the mathematics and science classrooms. The four authors of the paper functioned as educators of the pre-service teachers in the forums, discussing with them issues that the pre-service teachers or the educators raised. The discussion also happened at office hours between the pre-service teachers and their educators.

In more detail, the pre-service teachers' preparation concentrated on two aspects. First, knowing the tool technically (technological knowledge) and pedagogically (pedagogical knowledge) and being able to suggest it for teaching a mathematical or scientific content (one aspect of PTK). Second, being able to start from a specific content, and select and integrate appropriate digital tools for its teaching (another aspect of PTK). In more detail, each pre-service teacher had to learn at least two digital tools technically by himself/herself and prepare user guides (PDF file or digital book) for other teachers that include description of the most significant operations in these digital tools. Furthermore, the pre-service teacher had to record video clips of screen shots while performing operations in these digital tools in order to explain for the users how to perform these operations.

Moreover, each pre-service teacher was required to prepare pedagogical materials of how to use the digital tools in teaching mathematics or science, and then present the materials in the training workshop and afterwards in the electronic forum. Following that, all the materials were uploaded to internet sites that were constructed by the pre-service teachers. An internet site was constructed by the pre-service teachers' educators that included all the materials prepared by the pre-service teachers, where these materials constituted a data bank for digital tools. In addition, each pre-service teacher was requested to prepare at least two lessons for teaching mathematics or science using three digital tools from the data bank. These lessons had to involve also collaborative learning and investigations that encourage the use of higher order thinking skills. In addition, each pre-service teacher chose a subject in a digital textbook for teaching mathematics or science, and added layers on it that connect to pedagogical activities based on using digital tools from the data bank site. All these issues were discussed by all the pre-service teachers in the electronic forum, as well as in the office hours of the pre-service teachers' educators.

All this happened in the first semester. In the second semester, the pre-service teachers were asked to experiment with the prepared materials and lessons in their training schools, reflect on their experimenting and then improve the lesson plans they built before. All the previous steps were practice-based and intended to develop the actual practice of the pre-service teachers of using digital tools in the science and mathematics classroom. Figure 2 describes the teachers' preparation model that we followed to encourage the pre-service teachers to adopt digital tools for teaching mathematics and science.

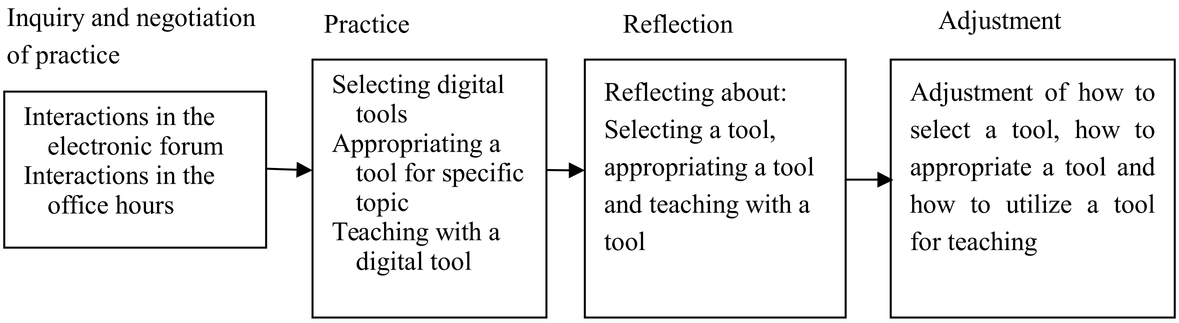

Figure 2: Teachers' preparation model for adopting digital tools for teaching

The effect of the preparation model on the participating pre-service teachers' acceptance of digital tools for mathematics and science teaching

The present research examined the influence of using a specific model (described in the methods section) to prepare pre-service teachers for using digital tools through one academic year. Doing so, we computed means and standard deviations of the scores of the different constructs of acceptance digital tools for mathematics and science teaching. We also ran paired t-test to examine whether the differences between the scores before and after the intervention are significant. Table 1 shows the results of the previous computations.

Table 1: Means, standard deviations, T value of the scores of the different constructs of acceptance of digital tools for mathematics teaching (N=48)

Category	Pre-Mean (SD)	Post- Mean (SD)	T value
Ease of use	5.23 (1.29)	6.05 (1.13)	5.31***
Usefulness	5.69 (1.11)	6.15 (1.11)	2.80**
Attitude	5.72 (1.19)	6.18 (1.13)	2.82**
Intention to use	5.83 (1.11)	6.18 (1.20)	2.39*
Use	5.14 (1.48)	6.18 (1.18)	4.21***
Self-efficacy	5.80 (1.13)	6.16 (1.15)	2.23*
Anxiety	3.00 (1.83)	3.33 (2.28)	0.79
Enjoyment	5.71 (1.31)	6.15 (1.20)	2.58*

*p<.05, **p<.01, ***p<0.001

Table 1 shows that using a specific model (described in the methods section) to prepare pre-service teachers for using digital tools through one academic year resulted in significant differences in the scores of the different components of the technology acceptance model used in the present research, except in the scores of anxiety.

We measured the effect size related to the differences that resulted from the intervention, using Cohen's (1988), where 0.8 is considered a large effect size, 0.5 is considered a medium effect size and 0.2 a weak one. Doing so, we found that the effect size of the intervention for "ease of use" (d = 0.78) was found to be a large effect, while the effect size of the intervention for "usefulness" (0.405), "use" (0.406) and "intention to use" (0.607) were found to be medium effects. In addition, the effect size of the intervention for "attitude" (0.33), "self-efficacy" (0.33) and "enjoyment" (0.37) were found to be weak effects.

Another goal of the present research was to examine whether the independent variables (specialization, computer-ability, and computer-use) influenced the results of preparing the pre-service teachers utilizing a model which is bases on the community of inquiry framework, together with a practice-based framework. Doing so, we ran mixed way ANOVA which showed no significant interaction at the level of 0.05 or lower.

A third goal of the present study was to examine the mediation of enjoyment, anxiety and self-efficacy between the predictors ease-of-use and usefulness, and the outcomes attitude and use. At the beginning, Pearson correlations were computed, which showed non-significant correlations with anxiety. So, anxiety was not considered as mediator construct. Moreover, self-efficacy and enjoyment were examined as mediators between ease-of-use and usefulness and between attitudes (First mediation). Afterwards, self-efficacy and enjoyment were examined as mediators between attitude and the intention to use digital tools in teaching (second mediation).

In Step 1 of examining the first mediation, the regression of attitude as outcome on the predictors, ease of use and usefulness scores, ignoring the mediator, was significant, b =.94, t(47) = 1.71, p =.000 for ease of use and b=0.96, t(47)=24.28, p=0.000 for usefulness. Step 2 showed that the regression of the mediators' scores, self-efficacy and enjoyment, on the predictors, was also significant, b = 0.91, t(47) = 15.22, p =.000 for self-efficacy on ease-of-use; b = .97, t(47) = 26.35, p =.000 for self-efficacy on usefulness; b = .92, t(47) = 16.13, p =.000 for enjoyment on ease-of-use; b = .97, t(47) = 28.39, p =.000 for enjoyment on usefulness. Step 3 of the mediation process showed that the regression of attitude scores on the mediators was also significant, b=.96, t(47)=24.19, p=.000 on self-efficacy and b=.95, t(47)=20.39, p=.000 on enjoyment. Step 4 of the mediation process showed that the regression of the attitude on the ease-of-use controlling for enjoyment as a mediator was also significant, b =.46, t(47) = 4.57, p =.000. This shows partial mediation of enjoyment as the effect of ease-of-use has dropped from .94 to .46. Partial mediation was obtained too for self-efficacy as mediator between attitude and ease-of-use, b =.39, t(47) = 4.81, p =.000, where the effect of ease-of-use has dropped from .94 to .39. The same computations in step 4 were carried out for the mediators between attitude, as outcome, and usefulness, as predictor. These computations showed partial mediation regarding self-efficacy and no mediation regarding enjoyment.

As to the second mediation, computing for mediation effects for intention to use as outcome and attitude towards use as predictor, the first three steps showed significant results for self-efficacy and enjoyment as mediators. Step 4 showed partial mediation of self-efficacy, where the effect of attitude on the intention to use has dropped from .95 to .31. It also showed partial mediation of enjoyment, where the effect of attitude on the intention to use has dropped from .95 to .36. The previous results are illustrated in Figure 3.

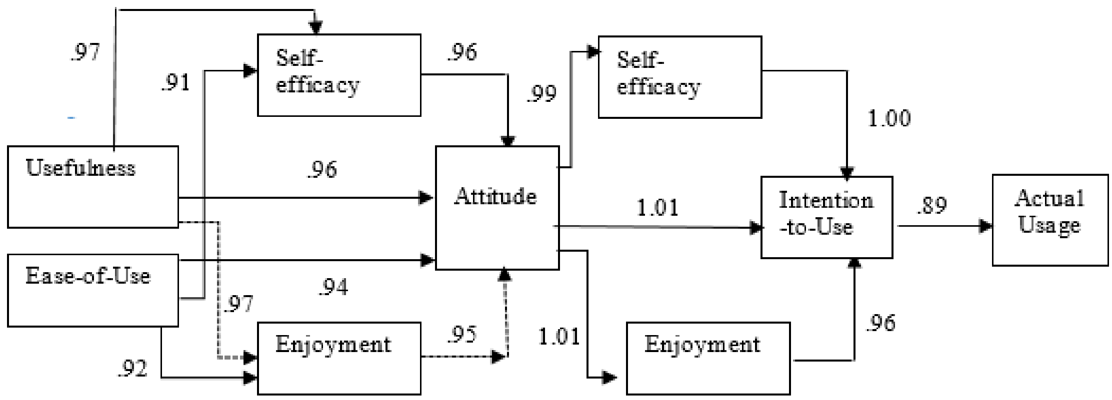

Figure 3: Regression analysis of the TAM constructs for pre-service teachers acceptance of digital tools for teaching

DISCUSSION AND CONCLUSIONS:

The present research examined the influence of using a specific model in the preparation of pre-service teachers for using digital tools through community of inquiry practices and practice-based professional development. The research results indicate that the preparation resulted in significant differences in the scores of the different constructs, except anxiety, of the acceptance of digital tools for teaching mathematics and science. These results show the effectiveness of the community of inquiry practices (Jaworski, 2005) regarding the acceptance of technology for teaching, where these practices included synchronous as well as asynchronous means of communication. It is our conclusion that both means are essential in pre-service teachers' education. The results also show the effectiveness of practice-based professional development (Ball & Bass, 2003) which included the two effective means (Spector et al., 2008): pre-service teachers' access to the digital tools, and the opportunities they had during the preparation year to utilize these tools in their teaching. Regarding the insignificant differences in the scores of anxiety, these scores were already low before the preparation as a result of the pre-service teachers' preparation in their first two years of study, so it is natural that they stayed low after the preparation.

Running mixed way ANOVA showed that the independent variables (specialization, computer-ability, computer-use) did not interact significantly with the intervention. This insignificant interaction indicates that the intervention influenced positively all the pre-service teachers, and not only part of them. We expected that the intervention would benefit the science pre-service teachers more because the mathematics pre-service teachers who are specialized in computers too. It seems that this did not happen because the intervention was involved with technological pedagogical content knowledge which was developed in the two groups of pre-serves teachers as a result of preparing them to integrate digital tools in teaching during one year. This has not much to do with the technological knowledge that the mathematics pre-service teachers are engaged with during their computer specialization.

The anxiety scores, being low, resulted in insignificant correlations with the other variables, what excluded anxiety from being a mediator between the variables of technology acceptance. Self-efficacy proved to be a partial mediator between ease-of-use and attitude, as well as between usefulness and attitude. Moreover, self-efficacy proved to be a partial mediator between attitude and intention-to-use. At the same time, enjoyment proved to be a partial mediator between ease-of-use and attitude, but not between usefulness and attitude. At the same time, enjoyment proved to be a partial mediator between attitude and intention-to-use. These results show the importance of paying attention to affective and psychological constructs as mediators or moderators between other constructs of pre-service and in-service teachers' acceptance of technological tools for teaching. Thomas and Palmer (2014) studied the teachers' confidence-to-use of technologies and their beliefs in the value of technology as constructs that affect their use of technological tools in teaching mathematics. Here self-efficacy is related to the confidence-to-use construct. Nevertheless, more attention to the various affective and psychological constructs are needed in order to study the issue of technology use in teaching mathematics and science, as well as teachers' education regarding this use.

REFERENCES

Alenezi, A. R., Karim, A. MA., & Veloo, A. (2010). An empirical investigation into the role of enjoyment, computer anxiety, computer self-efficacy and internet experience in influencing the

students' intention to use e-learning: a case study from Saudi Arabian governmental universities. *The Turkish Online Journal of Educational Technology*, *9*(4), 22-34.
Baron, R. M., & Kenny, D. A. (1986). The moderator-mediator variable distinction in social psychological research: Conceptual, strategic, and statistical considerations. *Journal of Personality and Social Psychology, 51* (6), 1173-1182.
Compeau, D., Higgins, C., & Huff, S. (1999). Social cognitive theory and individual reactions to computing technology: A longitudinal study. *MIS Quarterly, 23*(2), 145-158.
Davis, F.D. (1989). Perceived usefulness, perceived ease of use and user acceptance of information technology. *MIS Quarterly, 13*(3), 319–339
Davis, F. D., Bagozzi, R. P., & Warshaw, P. R. (1989). User acceptance of computer technology: a comparison of two theoretical models. *Management Science, 35*(8), 982-1003.
Frikkie, G., & Ogunniyi, M. (2016). Teachers' perceptions on the use of ICT in a CAL environment to enhance the conception of science concepts. *Universal Journal of Educational Research, 4*(1), 151-156
Getenet, S. T., Beswick, K., & Callingham, R. (2015). Conceptualising technology integrated mathematics teaching: The STAMP knowledge framework. In K. Beswick, T. Muir, & J. Wells (Eds.), *Proceedings of PME 39*, Vol (2), pp. 321–328. Hobart, Australia: PME.
Hong, Y. Y., & Thomas, M. O. J. (2006). Factors influencing teacher integration of graphic calculators in teaching. *Proceedings of the 11th Asian technology conference in mathematics* (pp. 234–243). Hong Kong.
Jaworski, B. (2005). Learning communities in mathematics: Creating an inquiry community between teachers and didacticians. In R. Barwell and A. Noyes (Eds.), *Research in Mathematics Education: Papers of the British Society for Research into Learning Mathematics*, Vol. 7 (pp. 101-119). London: BSRLM.
Padmavathi, M. (2016). A study of student-teachers' readiness to use computers in teaching: an empirical study. *I-manager's Journal on School Educational Technology, 11*(3), 29-39.
Pittalis, M., & Christou, C. (2011). Extending the technology acceptance model to assess secondary school teachers' intention to use Cabri in geometry teaching, in: M. Pytlak, R. Rolland, & E. Swoboda (eds.), *Proceedings of CERME 7* (2318–2327), Rzeszów, Poland: University of Rzeszów.
Ponte, J.P. (2012). A practice-oriented professional development programme to support the introduction of a new mathematics curriculum in Portugal. *Journal of Mathematics Teacher Education, 15*, 317–327
Röcker, C. (2009). Perceived Usefulness and Perceived Ease-of-Use of Ambient Intelligence Applications in Office Environments. In: M. Kuroso (Ed.), *Proceedings of the 13th International Conference on Human-Computer Interaction* (pp. 1052 - 1061). Heidelberg, Germany: Springer Verlag.
Shih, Y. Y., & Huang, S. S. (2009). The actual usage of ERP systems: An extended technology acceptance perspective. *Journal of Research and Practice in Information Technology, 41*(3), 263–276.
Spector, J. M., E. Jakubowski, et al. (2008). *Investigating pre-service mathematics teachers' initial use of the next-Generation TI-Nspire graphing calculators: A case study*. Tallahassee, FL: Florida Center for Research on STEM education (FCR-STEM), Learning Systems Institute, Florida State University.
Thomas, M. O. J., & Palmer, J. (2014). Teaching with digital technology: Obstacles and opportunities. In A. Clark-Wilson, O. Robutti & N. Sinclair (Eds.), *The Mathematics Teacher in the Digital Era: An International Perspective on Technology Focused Professional Development* (pp. 71-89). Dordrecht: Springer.
Yi M. Y., Hwang Y. (2003). Predicting the use of web-based information systems: self-efficacy, enjoyment, learning goal orientation, and the technology acceptance model. *International Journal of Human-Computer Studies, 59(*4), 431-449.

EFFECT OF GEOGEBRA COLLABORATIVE AND ITEARTIVE PROFESSIONAL DEVELOPMENT ON IN-SERVICE SECONDARY MATHEMATICS TEACHERS' TPACK

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This study aims at understanding the effect of collaborative and iterative GeoGebra intervention on in-service mathematics teachers GeoGebra adoption in their teaching and the factors that mediate that adoption. This article is one out of four parts of the study. The type of the study is a multiple case studying in depth the effect of a GeoGebra (a free mathematics software) intervention on the Technological Pedagogical Content Knowledge (TPACK) of in-service mathematics teachers in secondary schools who follow the Lebanese curriculum. The methodology used is Design-Based Research that focuses on working closely with practitioners in collaborative and iterative manner in the real context to add principles to theory and practice. Results showed an increase in the level of TPACK domains of teachers especially in their student-centered teaching approach.

Keywords: In-service secondary teachers PD, GeoGebra, TPACK.

INTRODUCTION

To effectively integrate technology in their classrooms mathematics teachers need to have good mathematical content knowledge, technological knowledge, pedagogical knowledge, and more importantly a mix of all of them as TPACK. In addition, they need to know what barriers they might face when they integrate technology in their classes and how to overcome them.

Literature review

The previous research on using technology in teaching of mathematics is so rich. For example between 2012 and 2017 there were more than 15000 articles about GeoGebra, more than 5000 on mathematics teachers' TPACK. But there is around hundred combining both (Google scholar advanced search, August 2017). Very few of these hundred studies address in-service secondary mathematics teachers and their practices. After reading and analyzing all relevant literature the following was found. First, research on technology integration lately is focusing on the design, implementation, and impact of tasks that are intended for prospective and practicing teachers for their professional learning concerning technology use in classrooms. (European society for Research in Mathematics Education ERME book, in print). However, "research on the application of these theories [theories on technology integration and teachers' knowledge] within the design of tasks intended for teachers' professional learning initiatives is still in its infancy." (ibid)
Second, in a recent meta-analysis concerning all the articles written about the issue of using technology in upper secondary mathematics education reported four striking issues (Hegedus,et.al., 2017). One of them was the gap between teachers' needs and the teacher education contents which is an under-represented issue in the field of mathematics education research. Third, there is evidence of GeoGebra being used extensively around the globe; it has been translated into fifty four languages and has been used by approximately more than millions of teachers worldwide for more than 15 years now (Hohenwarter, GeoGebra Global Gathering 2017). However, systematic enquiries into the effectiveness of GeoGebra in teaching practices are limited. (Lu, 2008). One of the studies concerning GeoGebra and TPACK worked with 44 prospective secondary mathematics teachers enrolled in two methods courses. One of the results was that creating dynamic activities is essential to the development of teachers' TPACK. One of their recommendations was that we can deepen prospective teachers' knowledge of teaching and learning mathematics with technology by creating a rich and collaborative learning environment and challenging them with new problems, new pedagogies, and new solutions associated with the use of technology. Fourth, it terms of TPACK it was found that the *Technology, Pedagogy and Content Knowledge* (TPACK) framework

(Mishra & Koehler, 2006) is a dominant frame used to address teachers' professional knowledge and skills (ERME book). Recently, despite the large number of researches done on TPACK, one research recommended "further research through participant observation studies…to ascertain the nature, magnitude, and direction of the interaction among teachers' TPACK elements and the reality of school contexts" (Handal , Campbell, Cavanagh, Kelly, & Petocz, 2013, p. 36). The changes in teachers' knowledge can lead to the changes in their classroom practices and that these changes can be reliably measured by the TPACK survey (Shin, Koehler, Mishra, Schmidt, Baran, & Thompson, 2009). Finally, though there has been a flowering of research on TPACK and its measurement, the review indicates that there is still much to be done particularly in the area of measuring how TPACK works in different disciplinary contexts. The quality of research has also been patchy, and there is a clear need for better-designed studies and instruments. (Koehler, Mishra, Kereluik, Shin, & Graham, 2014)

Summing up, this research aims to study how a collaborative and iterative work with in-service mathematics teachers affects their GeoGebra integration level in their teaching. Accordingly, this study aims to answer the following research questions:

1. How does a GeoGebra intervention done cooperatively and iteratively affect in-service secondary mathematics teachers' TPACK regarding integrating GeoGebra in their teaching?
2. How do participants' Valsiner's three zones mediate the impact of the intervention on teachers' TPACK regarding GeoGebra integration in their teaching?

Theoretical framework

In this study we have selected three theories namely: The zone theory, the diffusion of innovation theory, and TPACK (Technological Pedagogical Content Knowledge).

The Zone Theory states that the factors affecting technology usage by teachers are categorized into three zones namely zone of proximal development (ZPD, includes skill, experience, and general pedagogical beliefs), zone of free movement (ZFM, includes access to hardware..., support, curriculum and assessment requirements, students...) and zone of promoted action (ZPA, includes pre-service education, practicum and professional development) (Goos et al., 2010). TPACK claim that whenever new technology is to be effectively employed, teachers need to develop dynamic equilibrium among three elements, namely, technology, pedagogy, and content (Mishra and Kohler , 2006). The diffusion of innovation model describes the stages a person goes through when making the decision to adopt or reject a new technology (or innovation). It includes the following stages: (a) knowledge, (b) persuasion, (c) decision, (d) implementation, and (e) confirmation. Niess et.al. (2009) combined TPACK and the diffusion of innovation theories to obtain a new model called the "TPACK development model". In this study we have used that model with Valsiners' zones.

Methodology

Design Based Research (DBR) methodology in three iterations was used in this study over two stages (Figure 1). The first stage is the pre-intervention stage. This stage was dedicated to understanding the situation of integrating GeoGebra in the Lebanese curriculum, piloting the GeoGebra activities and testing the instruments. Six workshops were conducted over two years and a pilot study with two teachers. At the end of this stage four teachers (other than the ones in the pilot study) were selected as cases for the study. After selecting the participants 3 hour-workshop was conducted by the researcher with the four participants to make sure all participants acquired the basic features of the software (GeoGebra). In addition, as a group we collaborated in discussing the topics in the secondary mathematics Lebanese curriculum that could be better taught with the use of GeoGebra. We found that GeoGebra can be used in 37 different lessons of the secondary Lebanese curriculum. The second stage was the intervention stage which was made up of two iterations. In this stage collaboration was one-to-one between the researcher and each of the participants. In the first iteration, the participating teachers decided which lesson they wanted to teach with GeoGebra in accordance with their school mathematics scope and sequence. They were provided with a ready-made GeoGebra activities (made by the researcher) to be implemented in their classes. In the second iteration, teachers adapted already made GeoGebra activities and/or made their own GeoGebra activities. Three visits were conducted with each participant at his/her own school and according to his/her free time. The first visit was to prepare for the first lesson. The second visit was to evaluate the first lesson and prepare for the second lesson. Analysis of data collected from the instruments was done before starting the second iteration as required by Design Based research. The last visit was to evaluate the second lesson and give a general overview of the whole experience.

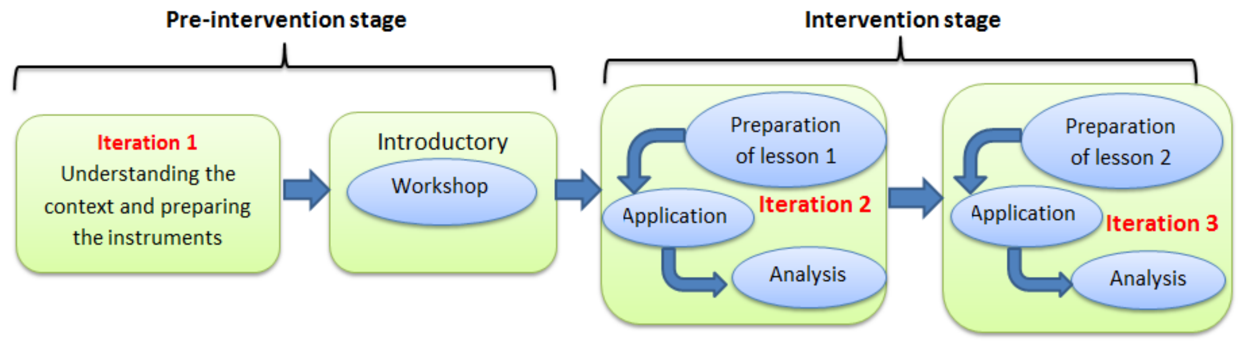

Figure 1. **The stages of the study**

Instruments

For the pre-intervention phase, three questionnaires were administered by the participating teachers: (1) Demographics questionnaire, (2) The Technological Pedagogical Content Knowledge Development Level Questionnaire TPACKDLQ (Form 1), (3) Barriers (grouped in zones) in Using Technology Questionnaire BUTQ (Form 1). The purpose of these questionnaires was to measure teachers' current TPACK integration level of the GeoGebra software in their teaching and the barriers that affect their technology integration. The questionnaires were adapted from TPACK development model (Niess, et.al, 2009). After conducting the first lesson, semi-structured interview in parallel form of the previous pre-intervention instruments TPACKDLI (Form 2) and BUTSI (Form 2) but combined were used to measure the impact of the intervention on teachers' TPACK and to find out to what extent the zones could mediate that effect. In addition, another instrument was used to assess the GeoGebra activity itself .The instrument is Lesson Assessment Criteria semi-structured Interview (LACI) which is based on instrument by Harris, Grandgenett & Hofer (2010).

Niess et.al (2009) combined the four categories of TPACK: (a) curriculum and assessment, (b) learning, (c) teaching, and (d) access, with the five levels of the diffusion of innovation theory: (a) recognizing (knowledge), (b) accepting (persuasion), (c) adapting (decision), (d) exploring (implementation), and (e) advancing (confirmation). The results of this combination are eleven domains that constitute the TPACK development model. Each domain is made up of a five-scale that measures teachers' level of integrating a particular technology in teaching and learning mathematics. In this study we adapted this instrument to ask specifically about GeoGebra.

For the impact of the intervention we were interested in the change of the TPACK integration level of GeoGebra at the end of each implementation, whereas for the dynamicity we were interested in the pattern in which this change happened in between the implementation stages: 'before implementation', 'after implementation 1', and 'after implementation 2'. The dynamicity could be: (1) static: there was no change in the level in between the implementation stages or (2) dynamic: there was a change in the level in between the implementation stages. In this sense after the two lessons a general pattern could be static (across the 2 iterations), dynamic, (across the 2 iterations) static (no change) then dynamic (change) or dynamic (change) then static (no change).

Participants

In the last (sixth) workshop conducted by the researcher for the study attendees were given the pre-intervention questionnaires mentioned above and another questionnaire that measures the extent of using the GeoGebra in their practices. Based on the answers, for the practice instrument, the values were 0 (never use GeoGebra), 1(sometimes use GeoGebra), and 2(most of the time use GeoGebra). The average of all the questions was calculated. An average within the range [0, 0.7[is considered low integration level, an average between [0.7, 1.3] is moderate integration level, and between] 1.3, 2] a high integration level. Similarly the average for each zone was calculated in the zone questionnaire that consists of 27 questions. Based on these results, four cases were selected (Pseudonyms: Tima, Sara, Amani, and Hazem) in a way that they differ among themselves in practice level and/ or in at least one barrier level. Table 1 represents the characteristics of each participant.

Name	Age	Highest degree	Teaching experience	Practice level	ZFM	ZPA	ZPD
Amani	50-55	BS	25 years	Low	Moderate	Moderate	Low
Tima	23-26	Masters +TD	2 years	Moderate	Low	Moderate	Not*
Sara	33-40	BS	7 years	Moderate	Moderate	Low	Not
Hazem	41-50	Masters	31 years	High	Moderate	Not	Not

Table 1. Participants domographics, practice and zones level
****Not: the zone is not considered as a barrier to GeoGbera integration***

GeoGebra modules

The criteria used for lesson selection are based on the criteria identified by Angeli & Valanides (2009) called ICT-TPCK. The GeoGebra activities were prepared by the researcher and tested on both students and teachers. The activities were designed based on the following criteria: Each activity: 1) should be student centered, 2) can be conducted by students in a computer lab or elsewhere (classroom or at home), 3) allows student to discover the concept or theorem under study, 4) includes immediate application of the concept under study, 5) does not require prior knowledge of the software.

Each teacher selected an activity according to his/her scope and sequence, so each teacher applied a different GeoGebra activity. Table 2 shows which activities applied by each teacher. An example of one activity is provided at the end of the article.

	Activity 1	Activity 2
Amani	Sign of quadratic polynomials	Derivative
Tima	vectors	3D
Hazem	Equation of a straight line	Thales Theorem
Sara	Translation of functions	Vectors

Table 2. **The intervention activities conducted by each of the participating teachers**

RESULTS

The median of all eleven domains for each participant before, after first implementation, and after second implementation were calculated and the results are shown as box plots in Figure 2.
It was found that before the intervention the median result for all cases is the *adapting* level, after

the first implementation the median −¿ for most cases −¿ improved to the *exploring* level but many subcategory levels stayed in the *adapting*, whereas after the second implementation the median −¿ for most cases −¿ stayed at the *exploring* level but with less subcategory levels on *adapting* and more on the *advancing* level.

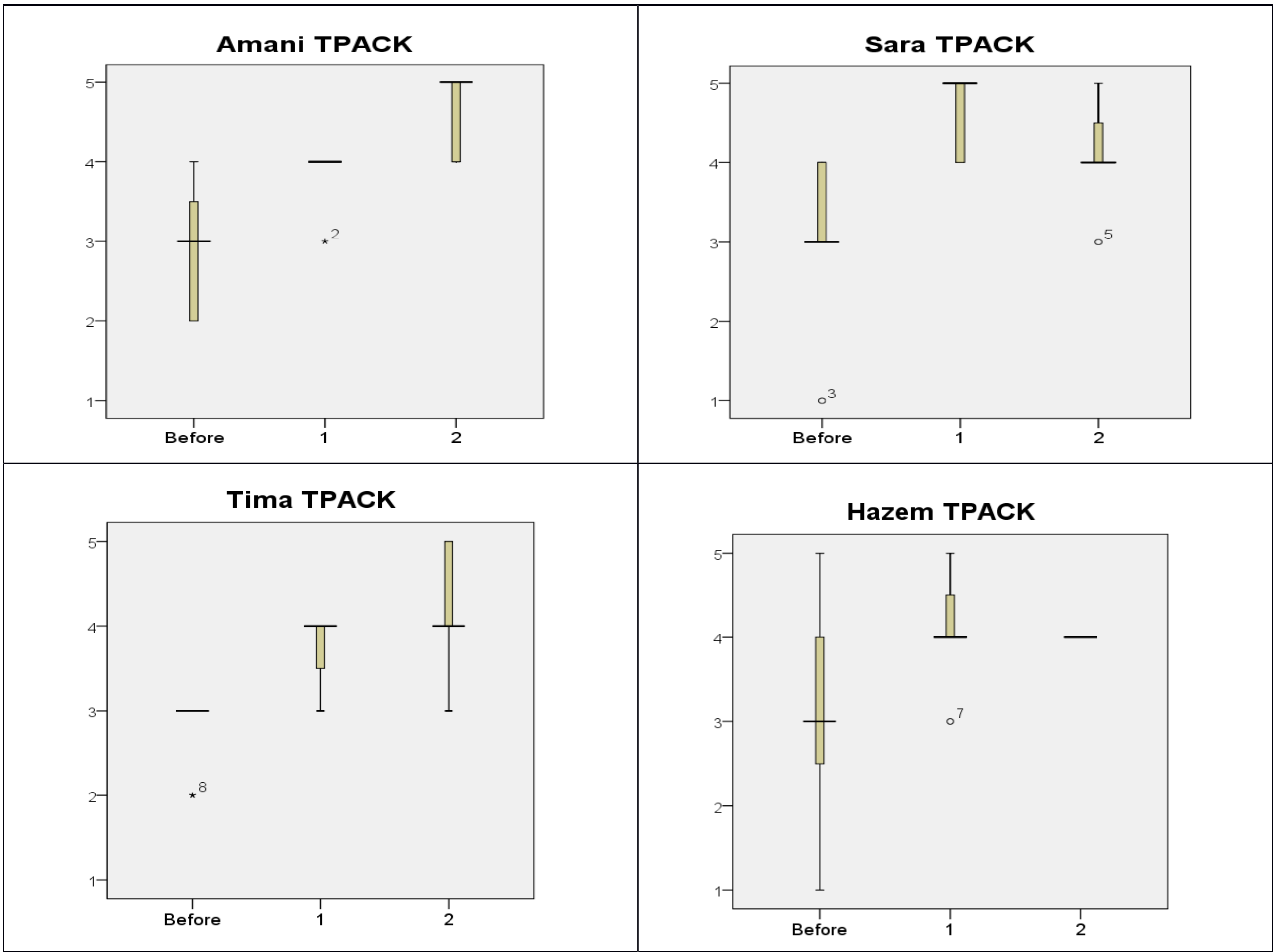

Figure 2. The effect of the intervention on participants' TPACK level

Iterations: Before: before the intervention; 1: after first implementation; 2: after second implementation
Levels: 1 *recognizing*, 2 *accepting*, 3 *adapting*, 4 *exploring*, and 5 *advancing*

Based on the details of the results and to answer the research questions we can say the intervention done iteratively and cooperatively improved all TPACK domains for all the participants: (1) teachers learned and experienced how to effectively integrate GeoGebra in their curriculum, (2) they started using some kind of GeoGebra assessment questions, (3) they started using more student-centered activities that require more critical thinking skills, and (4) they sensed they need more professional development in GeoGebra and in technology integration in general.
Most of the mediating zone factors were overcome except some related to the zone of free movement (ZFM) that came in the way of reaching higher levels. Those ZFM factors were: (1) *curriculum requirements* because not all lessons are appropriate to be taught with technology, (2) *lack of hardware (or not enough), and* (3) *students' motivation.*
For the dynamicity of change it was found that eight of the eleven domains of TPACK development model were dynamic then static, two domains were dynamic specifically the professional

development and overcoming barriers, and lastly one static (to a certain extent) domain which is mathematical teaching.

For most cases before the intervention, most participants' subcategories were in *adapting* level and after two iterations they reached either the *exploring* or the *advancing* level (highest TPACK level). The change was mostly dynamic with change happening immediately after the first implementation. There were some assisting factors to higher TPACK levels mainly *collaboration* (ZPA), *increase in knowledge and skills* (ZPD), and some ZFM factors like *availability of hardware, curriculum requirements and students' motivation*. The limiting factors were mainly ZFM factors such as: *not enough available or accessible hardware, lack of time to prepare and conduct GeoGebra activities, students' motivation, and curriculum requirements…* and one ZPD factor *lack or not enough skill.*

DISCUSSION

It was clear from the dynamicity of change the direct effect of the intervention on teachers' TPACK in all its eleven domains. Not only it raised the adoption level of GeoGebra in their teaching but also kept that adoption high. In addition, as a direct impact on collaboration and applying GeoGebra activities in their teaching teacher felt the need of learning more about this software and how to effectively integrate it in their teaching. In fact, many after the intervention attended other workshops conducted by the researcher and kept on using GeoGebra in their teaching. Sometimes teachers' perception of the barriers is not related to reality and that we have seen when they changed their list of barriers before the intervention and after. One important factor acted sometimes as encourager and sometimes as barrier to higher integration level was students' motivation.

RECOMMENDATIONS

In terms of the research questions, can we say that the effect of the design as collaborative and iterative manner is more powerful than one-day-workshops? Does the effect of such intervention affect teachers' knowledge more than teachers' practices? The study is an ongoing one and much to be discussed in its four parts to get a clearer and better picture of the integration of technology problem.

REFERENCES

Angeli, C. & Valanides, N. (2009). Epistemological and methodological issues for the conceptualization, development, and assessment of ICT–TPCK: Advances in technological pedagogical content knowledge (TPCK). *Computers & Education* 52, 154 −¿ 168. Retrieved from http://teaching.cycu.edu.tw/pdf/2009_TPCK.pdf

Goos, M., Soury-Lavergne, S., Assude, T., Brown, J., Ming Kong, C., Glover, D., … Sinclair, M. (2010). Teachers and Teaching: Theoretical perspectives and issues concerning classroom implementation. In C., Hoyles, & J. -b. Lagrange (Eds.), *Mathematics education and technology-rethinking the terrain* (pp. 311 −¿ 328). New York: Springer.

Handal, B., Campbell, C., Cavanagh, M., Petocz, P. & Kelly, N. (2013). Technological pedagogical content knowledge of secondary mathematics teachers. *Contemporary Issues in Technology and Teacher Education*, *13*(1), 22-40.

Hegedus, S., Laborde, C., Brady, C., Dalton, S., Siller, H. S., Tabach, M., ... & Moreno-Armella, L. (2017). Uses of Technology in Upper Secondary Mathematics Education. In *Uses of Technology in Upper Secondary Mathematics Education* (pp. 1-36). Springer International Publishing.

Koehler, M. J., Mishra, P., Kereluik, K., Shin, T. S., & Graham, C. R. (2014). The technological pedagogical content knowledge framework. In *Handbook of research on educational communications and technology* (pp. 101-111). Springer New York.

Lu, Y. W. A. (2008). Linking Geometry and Algebra: A multiple-case study of Upper-Secondary mathematics teacher's conceptions and practices of GeoGebra in England and Taiwan. *Unpublished Master's thesis, Cambridge: University of Cambridge, UK*.

Mishra, P., & Koehler, M. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record, 108(6),* 1017-1054.

Niess, M. L., Ronau, R. N., Shafer, K. G., Driskell, S. O., Harper S. R., Johnston, C., Browning, C., Özgün-Koca, S. A., & Kersaint, G. (2009). Mathematics teacher TPACK standards and development model. Contemporary Issues in Technology and Teacher Education, 9(1), 4-24

Shin, T., Koehler, M., Mishra, P., Schmidt, D., Baran, E., & Thompson, A. (2009, March). Changing technological pedagogical content knowledge (TPACK) through course experiences. In *Society for Information Technology & Teacher Education International Conference* (pp. 4152-4159). Association for the Advancement of Computing in Education (AACE).

Example of a part of an activity

Objective: Understand the definition of derivative.

Given the function *f* defined by: $f(x) = x^2 + 3x - 4$. Let (P) be its representative curve in an orthonormal system.

A) In the input bar type $f(x) = x^2 + 3x - 4$. Let A(-2,-6) and B (*x*, *f(x)*) be two points of (P).
B) Join A and B by a straight line and specify its slope.
C) Change the position of B and note what is happening to the slope of (AB).
D) Can *x* be -2 in the slope of (AB)? _________.
E) Can it be near -2? _____________.
F) As point B approaches point A, the slope of (AB) approaches_____.
G) Prove that the slope of (AB) expressed in terms of *x* is: Slope of (AB) = $\dfrac{x^2 + 3x + 2}{x + 2}$.
H) Calculate $\lim_{x \to -2} \dfrac{x^2 + 3x + 2}{x + 2} =$ What can you deduce?_______________.
I) Can we call the line (AB), with respect to (P), in this case a tangent? _______________.

Conclusion: The slope of the tangent to (P) at point A of (P) with abscissa *x* =*-2* is equal to $\lim_{x \to -2} \dfrac{f(x) - f(-2)}{x - (-2)}$. It is also called the **derivative** of *f* at *x* = *-2*.

GEOGEBRA AND NUMERICAL REPRESENTATIONS: A PROPOSAL INVOLVING FUNDAMENTAL THEOREM OF ARITHMETIC

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This paper reports a qualitative research whose subjects were Elementary School Teachers who took part in a workshop about primality of positive integers and the Fundamental Theorem of Arithmetic (FTA). These topics was dealt with from different technological perspectives and analysed under a theoretical proposal connected to the concepts of transparency and opacity of numerical representations and to the "humans-with-media" approach. The interactions occurred in a Post Graduate Program in Mathematics Education and they consisted of two activities created to ask subjects which numbers in a random list would be prime. The analysis showed that participants had difficulties with FTA, which led them to adopt strategies with a high cognitive cost and make mistakes. Likewise, data showed that hindrances were overcome based on the educational proposal planned from a configuration of humans-with-GeoGebra.

Keywords: fundamental theorem of arithmetic; numerical representations; digital technologies in education; human-with-media; GeoGebra.

INTRODUCTION

The task of recognizing whether a positive integer is prime may seem simple and not very interesting in terms of teaching or researching in the field of Mathematics Education. Considering that a natural number is prime if it can be divided uniquely by itself and by one, we might think that there are no associated complexities. Ultimately, it just seems to be a question of finding a divisor between 2 and the square root of the number tested for primality or of applying the so-called ‘divisibility rules’. Such a search, however, can have quite a high operational cost. For example, if the number is 30847, we must consider that its first divisor is 109 and that there is only another divisor apart from the number itself and 1, which is 283.

In order to deal with such problem, this work describes a research whose participants were a group of basic education teachers from public schools, involved in the projects “Mathematics Teaching and Learning Processes in Technological Environments”, constituted through a partnership between the Pontifical Catholic Universities (PUC) of Sao Paulo (Brazil) and of Lima (Peru), and “Technologies and Mathematical Education: researches on fluency in devices, tools, artefacts and interfaces”, also linked to the PUC Sao Paulo [1]. In the proposed activities, teachers had to identify whether several numbers were prime, in situations where divisibility rules, for example, were an inefficient strategy and where FTA knowledge would be relevant. Through the chosen research instruments – a question about the subject ‘primality of natural numbers’ and an application built with GeoGebra – we sought to highlight the strategies used by the subjects to solve the problems, their ideas about the representation of natural numbers and the influence of the type of technology on the mobilization of the mathematical knowledge at issue. In this respect, arguments related to the number representation arise, as well as discourses related to the use of technology in Mathematics Education, which leads to the following theoretical treatment.

NUMERICAL REPRESENTATIONS

One of the core concepts addressed in this research refers to transparency and opacity of numerical representations. In this respect, the study of Zazkis and Liljedahl (2004) mentions the role of representations in the field of natural numbers. In their paper, the authors discussed data obtained from a research conducted with basic education teachers in training, which focused on subjects understanding of prime numbers, so as to detect the factors that interfere with such understanding. The argument used in the analysis of collected data is that the lack of transparency of prime numbers representation is a hindrance to understand them. This idea was inspired from the paper of Lesh, Behr and Post (1987). When referring to different representations of rational numbers, the authors show that they "incorporate" mathematical structures, meaning that they represent them materially. Thus, representational systems can be regarded as opaque or transparent: a transparent representation has no more or less meaning than the ideas or structures it represents, while an opaque representation emphasizes some aspects of the ideas or structures it represents while hiding others. When having different representational possibilities, a didactic strategy should, for example, capitalise on the strengths of a specific representational system and minimize its weaknesses.

Expanding Lesh, Behr and Post's proposal (1987), Zazkis and Gadowsky (2001) introduce the notion of relative transparency and opaqueness, focusing on numerical representations. The authors suggest, on their paper, "that all representations of numbers are opaque in the sense that they always hide some of the features of a number, although they might reveal other, with respect to which they would be 'transparent'" (p.45). As an example, the authors provide a list with the following items: (a) 2162, (b) 363, (c) 3 x 15552, (d) 5 x 7 x 31 x 43 + 1, (e) 12 x 3000 + 12 x 888. The authors mention that such expressions do not seem to represent the same number, 46656, pointing that each representation shifts our attention to different properties of number.

Based on the above-mentioned concepts, the authors claim that all numeric representations are opaque, but they have transparent features. In relation to the work we describe here, activities involving the features of numerical representations were presented to the subjects by using different means and technological interfaces. For this reason, a discussion about the use of technologies in Mathematics Education becomes necessary in this paper.

ABOUT THE USE OF TECHNOLOGIES IN MATHEMATICS EDUCATION

Technologies have been part of educational processes from time immemorial, if we consider, like Lévy (1993) that transmutation of temporalities in Human History brought about different instruments; some of them prevailed over the others depending on the evolutionary character of the society in a given time. Thus, orality, writing and information technology are enrolled as technologies of intelligence. The ascendancy of information technology did not suppress previous technological proposals, but constituted, in relation to them, a feature of redefining functions and, in the last resort, of convergence. Thus, it is unavoidable to associate the use of some technology in processes of teaching or learning: a construction of knowledge and its forms of access, therefore, were always linked to more or less material tools of technological nature.

From this perspective, we can conjecture that the process of knowledge construction involves people and technologies associated in some way. However, technologies are not part of this association as substitutes for human capacities, not even as a supplement to them, but as reorganizers of human thought (Tikhomirov, 1981). In this sense, computer applications, for example, allow for unusual forms of mediation, delegating to the computer the role of tool for human mental activity, with functions similar to those performed by language in vygotskinian logics. Similar reasonings apply to contemporary devices such as tablets or cell phones.

Such considerations support the claim that, in Mathematics, learning is a process involving technologies that are somehow integrated with people what allows intentions, strategies, plans and conjectures come into play. According to Borba and Villarreal (2005), such integration must be of such an order that it excludes any attempt to see these items, people and technologies, as separate groups. Therefore, for these authors, mathematical knowledge is formed from a collective of humans-with-media, considering that media reorganize people's thinking and that the presence of different technologies conditions the production of different forms of knowledge. Thus, in the research described here, the two activities explained below attempted to investigate how teachers in continuing education comprise numerical representations related to prime numbers and FTA, based on the use of different technologies at different times: during the first activity, teachers-with-pencil-and-paper; in the second activity, teachers-with-computers-and-GeoGebra.

METHODOLOGICAL APPROACH

The participants of this qualitative research are eight basic school teachers in São Paulo and Pará states, all of them voluntary in workshops carried out in the framework of the projects mentioned at the start of this text. The research we describe here was conducted in one of the computing labs of the abovementioned institution, in a single session which lasted around four hours. Among the teachers so described, five work in primary and secondary education while three of them work only in primary education. Moreover, all of them have completed their bachelor's degree in Mathematics, three of them are attending an Academic Master in Mathematics Education and two have completed their Specialization in Mathematics Education.

Participants in this study were invited to working with two kinds of activities involving knowledge on primality in the framework of the Theory of Numbers. The first activity consisted of an issue to be solved individually (here, the response had to be presented in writing): "Consider F = 151 x 157. Is F a prime number? Circle YES/NO, and explain your decision" (Zazkis & Liljedahl, 2004, p.169). To answer to the abovementioned question, students would have to write down the option 'No', since the representation in question, with transparent features, shows that F is composite, and it even relates the component prime factors. Teachers should resort to the Fundamental Theorem of Arithmetic (FTA) to conclude that the said decomposition would be unique, except by the order.

If they considered another strategy, subjects could perform the multiplication contained in the question, obtaining 23707, an opaque representation in relation to the detection of the composite character of the number. From this other representation, teachers could test with several possible divisors, although this was not necessary to solve the problem. It might even happen that some teacher attempted to divide 23707 by the prime numbers from 3 onwards, giving up after some attempts, since natural numbers below 151 are not divisors of 23707 – in this case, this teacher could even erroneously state that the number would be prime.

The next activity was carried out immediately after the first one. Individually, the eight teachers had in front of them a GeoGebra screen containing only one button with the word "Números" ("Numbers") on it. They were all informed that the application would draw nine numbers when they pressed the said button. Then, researcher instructed teachers with regard to the result they would see at clicking the button: nine numbers would be shown on the Algebra View of the software. They had to state, by writing on a blank paper that had been handed out and in a lapse of 20 minutes, whether each drawn number was prime or not. In the meantime, they should not click on the button with label "Coisa" ("Do something") [2] on it (Figure 1) shown after the draw.

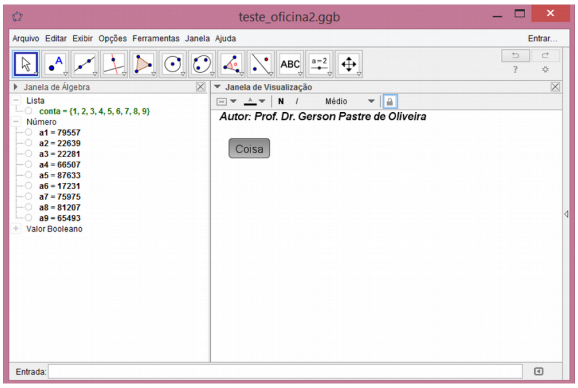

Figure 1. Screen with drawn numbers (developed by the author)

Regarding the numbers, it is worth noting that the construction of the javascript code that carried out the draw took into consideration the random selection of numbers in the range of 1001 to 99999 (Figure 1). The purpose was to restrict the direct application of rules and division algorithms. Despite this, these resources could be successfully used in some cases (such as 47301, which is divisible by 3) and fail in others (such as 39203, which is not prime, but whose prime factors are 197 and 199). In theory, the representations provided by the software were opaque with respect to the primality of the numbers, at least up to this point of the experiment.

Once the time allowed was over, regardless of the amount of resolutions among the nine proposals, subjects were invited to click on button "Coisa". After this action, the Algebra View of GeoGebra showed the decomposition of each of the nine numbers into prime factors, in the form of lists (Figure 2) – in the case of primes just the number itself was presented.

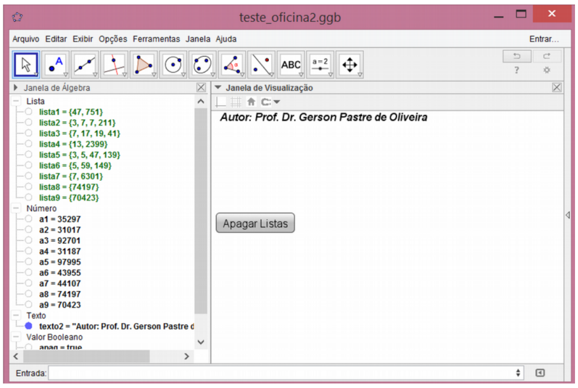

Figure 2. Numbers and their compositions in prime factors (developed by the author)

Immediately after this action and still individually, participants were invited to revisit their previous answers in the light of the new data obtained with the factorization carried out in GeoGebra. At that occasion, subjects were expected to relate the lists obtained with the numbers drawn previously and use the FTA so as to decide on the primality or not of each one. Ten minutes were allowed for this part of the activity, after which there would be a debate involving the subjects and the researcher. The button "Apagar Listas" ("Erase all") (Figure 2) could be used to go back to the initial screen, which would allow repeating the experience as many times as considered necessary by participants.

ANALYSIS

At the very beginning of first activity, Teach2 after trying some division operations with pencil and paper stated that 23707 would 'probably' be prime. Other four participants also stated, wrongly, that the number in question would be prime, using similar strategies:

- Teach3 did countless division operations and ended up stating that "23707 is prime, because it can be divided by itself and by one". In this way, Teach3 showed he is not aware that this criterion does not distinguish between prime and composite numbers;
- Teach4, after several attempts using division operations, concluded that 23707 would be a prime number because it "ends in 7 and 7 is prime";
- For Teach5, as the divisibility tests by 2, 3, 5, 7, 11 and 13 'failed' (divisions had a remainder other than zero), the number in question would be prime – in this case the teacher indicates he believes that "decomposition into prime factors means decomposition into small prime factors" (Zazkis & Campbell, 1996, p. 215);
- In the case of Teach8, number 23707 "is prime, because it is an odd number and it cannot be divided by its square root or any other prime number". Like Teach5, Teach8 limited the universe of prime numbers to the interval between 2 and 13 and it evidenced several confusions involving the concepts of perfect square numbers and odd numbers.

The representation of F provided in the question formulation had transparent features in relation to primality, because it presented the number by means of its unique decomposition in prime factors. However, the abovementioned teachers did not use this idea as expressed in the FTA, which points out the fact that a numeric representation with features that make it transparent can be kept opaque when mathematical knowledge about it is not mobilized by the individual. Similar results were observed in the work of Zazkis and Liljedahl (2004), which leads us to consider the need to use didactic strategies capable of reinforcing the transparent characteristics of this representational system, as advocated by Zazkis and Gadowvsky (2001). This was done in the second activity.

Furthermore, some participants correctly stated that F would not be prime. According to Teach1, "F is divisible by 151 and 157, which makes it non-prime". Participants Teach6 and Teach7 said, in a similar way, that F had other divisors besides itself and 1, which would make it non-prime. Nevertheless, none of the three participants that answered correctly did show any sign of using the FTA in their conjectures: when questioned about the possibility of F having other divisors apart from the ones mentioned, all of them said it was possible but that they would have to test numbers up to a certain limit (according to Teach1 up to the number square root; according to Teach6 and Teach7, up to half the number). Another aspect worth highlighting is that none of the teachers showed awareness of the fact that factors 151 and 157 represented prime numbers.

From a different perspective, the technological component of the collective humans-with-pen-and-paper, even if intensively used, does not seem to support cognitive movements related to a change of strategies in this activity – in the case of participants who exhausted attempts with divisibility algorithms – or to the use of formal notions in Mathematics, such as the FTA – in the case of participants who stated that 23707 might have other divisors.

Regarding the second activity, carried out in GeoGebra, participants accessed the application that was available on the computers in the institution's lab and drew the nine numbers, as shown on figure 1, without further questions. From this moment, teachers had 20 minutes to decide which numbers would be primes. All participants alleged, initially, that time was too tight and that numbers would be too big (odd numbers between 1001 and 99999) to be able to provide an answer. The researcher said that they should state as many results as possible in this lapse of time, which could not be extended.

Until that moment, the technological aspect in the collective teachers-with-GeoGebra did not have great influence on the issue of transparency of the numeric representation regarding primality, because the program interface in question only provided odd random numbers within the said limits. Thus, as drawn numbers were potentially different, the amount of correct or wrong results showed significant differences among the subjects. Those whose drawn numbers allowed direct application of divisibility rules or tests with 'small' prime divisors (3 to 13) obtained more hits than those whose drawn numbers were 11741 (59 x 199) or 31753 (113 x 281), for example. Generally, such numbers were wrongly stated to be prime. Even when prime numbers were identified, as 17231, a doubt used to remain, as in the case of Teach5, who wrote, next to the said number, "I think it is prime. I tested up to 13". As we have already seen, this strategy, in the case of numbers whose prime factors are all higher than 13, is not efficient.

Among the teachers' talks during the 20 minutes allowed for the activity, several references were made to the numbers 'difficult form', a clear attempt to refer to their representation, which is clearly opaque in terms of the feature 'primality'. Teach6 even got to question the goal of GeoGebra in that context, as the application did not seem, according to him, "to make things easier for those who tried to solve the problem".

Faced with perplexity caused by the proposal, the researcher, after the time allowed, proceeded to coordinating a debate with the participants, whose main motivation was raising the conjectures and strategies that subjects had proposed in order to verify the primality of their nine numbers. None of the participants said to have tried or even thought about obtaining the factorization of the numbers to be tested for primality in order to using knowledge on FTA. Once the debate was ended, the researcher said that subjects could click on button 'Coisa', which would show, for each raffled number, the respective list of component prime factors (figure 2). After this, in 10 minutes, teachers had to review their answers. Immediately after clicking on the button, teachers had to construe the data showed on the screen.

When they realized the correspondence between the lists provided and drawn numbers (list1 referred to number a1, list2 referred to number a2, and so on), most teachers started searching for relations among the said components:

Teach6: Professor, I would like to review my answers.

Researcher: Yes, why so?

Teach6: Because I realized something that I had not seen before… the lists… they are the factors of each number...

Teach1: By multiplying the numbers in the lists we obtain the drawn numbers…

Teach4: It´s true, but there are cases where only one number appears… these numbers are primes, as we can only multiply by one!

Teach3: [does not seeming convinced] Professor, I´ll draw the numbers again… [after repeating the draw and factorization] It's true! Prime numbers do not have factors, only composite numbers do.

Teach4: A prime number's factors are itself and one...

Teach7: I was thinking... In my case, one of the numbers is 88739... Factorization is shown as 7, 7 and 1811. I could as well write 49 and 1811, right?

Teach8: [after some discussion with pairs] I think that 49 can appear, but 49 is not prime, and lists show prime factors of numbers. The idea is that only prime numbers appear. We can see that all the factors, in all the numbers, are prime.

Teach4: You are right. Any number can be written as a product of prime factors! This is it! Wow! First question was obvious! There is only one decomposition into prime numbers for each number.

Teach8: It is the Fundamental Theorem of Arithmetic!

After these observations, subjects could tell which numbers were prime and which were not, by repeating raffling and the whole procedure several times. As noted by Borba and Villarreal (2005), visualization and experimentation were important factors in the new strategy adopted by subjects from configuration humans-with-GeoGebra. According to these authors, such items allow, among other actions, for example, to invest in generating conjectures about the problems at issue (and testing them through countless examples), bring to light some results which were not known before the experiments and testing different ways of collecting results. The access to visual components, in the consolidation of the results of actions carried out by people-with-GeoGebra, became a way to transform the understanding they had about the problems at issue.

Another aspect that must be taken into account in the configuration teachers-with-GeoGebra is the dynamism of digital technologies that has been seen here as a possibility to manipulate the parameters, attributes or values which served the constitution and/or definition of a mathematical construct in a computerized context. Faced the possibilities open by this resource, a fundamental investigative movement to mathematics finds consistent subsidies in the development, testing and validation (or refutation) of conjectures. This could be largely seen in the experiment we describe here, when teachers invested, through experimenting and visualizing, in the procedure repetition, using the regularities observed in factorizations – and dynamically obtained - as a means to support the reorganization of ideas on the primality of the presented numbers. All these factors collaborated to mobilise knowledge on FTA for the solving of the problem.

FINAL CONSIDERATIONS

The discussion following the last activity was quite fruitful: participants stated that, in activity 1 they had not realized that the 'way' in which the number was written (its representation) allowed the question to be answered directly by mobilizing knowledge related to the FTA.

Regarding activity 2, participants said that numbers were not expressed in a 'convenient way' (transparent representation), i.e., according to them, they were 'big numbers' that were not decomposed in prime factors. Teachers mentioned the fact that they had spent the 20 minutes to try and say which of the numbers were prime, but if they had the appropriate representation in factors and had remembered the FTA, they would have done this much faster. This last feature was perceived when they clicked on the second button, causing the decomposition of the numbers in prime factors to appear. The participants concluded that, when there were other factors besides the number itself, the number in question would not be prime. Moreover, teachers highlighted the importance of FTA knowledge and of the use of GeoGebra in the procedures, saying that this would be a good way to approach the topic in classroom.

Moreover, in this context, the Geogebra´s application would work as a 'calculator to decompose in prime factors', turning a numeric representation, opaque in relation to primality, into a transparent representation, provided that the knowledge about the FTA has been mobilized. In this case, the configuration teachers-with-GeoGebra contributed in a more efficient way to direct the problem-solving effort towards a strategy that brings more chances of success. Dialogues show, although only to some extent, the renegotiation of meanings, the conjectural reformulations and the reorganization of thinking which allowed the right answer to come out. Another feature that must be highlighted is that representations of prime numbers can give opportunity to transparent features emerge, as soon as the appreciation of underlying meanings and concepts is taken into account. When this does not occur, the tendency is to call for large, expensive solutions in cognitive terms. To investigate the nature of these processes and develop proposals in order to avoid such difficulties to remain among basic school teachers seems to be an important challenge, open to new researches.

NOTES

1. Research projects supported by FAPESP and CNPq, respectively (Brazilian scientific development agencies).

2. The button title could then be 'Factorize', but the idea was to promote a research where subjects were the authors of the hypothesis formulated to solve the problem: the use of this title might imply that teachers needed, compulsorily, to do the factorization of the numbers, which would compromise their autonomy.

REFERENCES

Borba, M. C, & Villarreal, M. (2005). *Humans-with-media and the reorganization of mathematical thinking: Information and communication technologies, modelling, visualization and experimentation*. New York: Springer.

Lesh, R., Behr, M., & Post, T. (1987). Rational Number Relations and Proportions. In C. Janiver (Ed.). *Problems of Representations in the Teaching and Learning of Mathematics* (pp. 41-58). Hillsdale: Lawrence Erlbaum.

Lévy, P. (1993). *As tecnologias da inteligência: O futuro do pensamento na era da informática*. Rio de Janeiro: Editora 34.

Tikhomirov, O. K. (1981). The psychological consequences of computerization. In Wertsch, J. V. (Ed.). *The concept of activity in soviet psychology* (pp. 256–278). New York: M. E. Sharpe.

Zazkis, R., & Campbell, S. R. (2006). Number Theory in mathematics education research. In Zazkis, R.; Campbell, S. (Eds.). *Number Theory in mathematics education: perspectives and prospects* (pp. 1-18). New Jersey: Lawrence Erlbaum Press.

Zazkis, R., & Campbell. S. R. (1996). Prime decomposition: Understanding uniqueness. *Journal of Mathematical Behavior*, 15, 207-218.

Zazkis, R., & Gadowsky, K. (2001). Attending to transparent features of opaque representations of natural numbers. In Cuoco, A. (Ed.). *NCTM 2001 Yearbook: The roles of representation in school mathematics* (pp. 41-52). Reston: NCTM.

Zazkis, R., & Liljedahl, P. (2004). Understanding primes: The role of representation. *Journal for Research in Mathematics Education*, 35, 164-186.

Zazkis, R. (2002). Language of number theory: metaphor and rigor. In Campbell, S. R., Zazkis, R. (Eds.). *Learning and teaching number theory: Research in cognition and instruction* (pp. 83-96). Westport: Ablex Publishing.

ANALYZING THE TEACHER'S KNOWLEDGE FOR TEACHING MATHEMATICS WITH TECHNOLOGY

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The teacher's knowledge has long been viewed as a strong influence on the students' learning. Several authors have sought to develop procedures to assess this knowledge, but this has proved to be a complex task. In this paper I present an outline of a conceptualization to analyze the teacher's knowledge, based on the model of the Knowledge for Teaching Mathematics with Technology (KTMT) and a set of tasks. These tasks are chosen by the teacher taking into account the potential of the tasks to take advantage of the technology's potential. The analysis of the teacher's KTMT is based on the characteristics of the tasks chosen by the teacher; the balance established between the representations provided by the technology that the tasks advocate; the way how the tasks pay attention to the new issue of seeking for a suitable viewing window; and also the way how the tasks take into account the expectable difficulties of the students in the process of looking for the window.

Keywords: Professional knowledge; KTMT; technology.

INTRODUCTION

The teacher's knowledge is considered as an important requirement for high-quality teaching (Fauskanger, 2015). And many are the authors who have been dedicated to developing characterizations of this knowledge, identifying important aspects of the knowledge required to teach and developing models that articulate specific knowledge in a global and comprehensive structure. And closely related to this intention to characterize the teacher's knowledge, is the desire to assess the knowledge effectively held by the teacher. And this is an issue that has proved to be complex. Several authors (such as Fauskanger (2015) and Schmidt et al. (2009)) point out weaknesses and even question the reliability of the results achieved through the application of some of the instruments developed. There are also several authors who criticize the options taken to assess the teacher's knowledge (such as Rocha (2010) and Schmidt et al. (2009)). According to them, these options are too demanding, in terms of the time required to implement and in terms of the resources required to achieve them.

This article intends to present a theoretical conceptualization to analyze the knowledge of the teacher in a context of technology use. It is a work still in progress, based on the model of Knowledge for Teaching Mathematics with Technology (KTMT) and that is based on the analysis of the tasks proposed by the teacher to the students. At this stage of the work it is only considered the teaching of Functions with the graphing calculator.

The structure of the paper includes a section devoted to a summary presentation of KTMT, followed by a brief critical analysis of the main options taken by two authors who developed tools or strategies to assess the teacher's knowledge. It is then presented the conceptualization that is the object of this article and justified the options assumed. Finally, to clarify the ideas presented, an hypothetical example of application of this conceptualization is discussed. The data presented in this last part are real and have been collected in the course of another study. This means that the tasks were actually implemented by one teacher. However, this is not a real example of application of this conceptualization, once the tasks were selected by the researcher and not by the teacher.

KNOWLEDGE FOR TEACHING MATHEMATICS WITH TECHNOLOGY – KTMT

A look at the knowledge models developed so far, such as Mathematical Knowledge for Teaching (MKT) by Hill et al. (2007) or the TPACK from Mishra and Koehler (2006), suggests the knowledge of Mathematics, Teaching-Learning, Technology, and Curriculum as important domains. These are the basic domains of KTMT, being the Curriculum viewed in a transversal way, influential over all the others domains.

Besides those, KTMT particularly values two sets of inter-domain knowledge developed at the confluence of more than one domain: Mathematics and Technology Knowledge (MTK) and Teaching-Learning and Technology Knowledge (TLTK). This is new knowledge that goes beyond the intersection between knowledge of the base domains. The MTK focuses on the knowledge of how technology influences Mathematics, enhancing or constraining certain aspects. The TLTK focuses on how technology interferes with the teaching-learning process, enhancing or constraining certain approaches.

One of the main intentions behind the design of KTMT, which distinguishes it from other existing models, is to integrate into a single model the research on professional knowledge and on the integration of technology into professional practice. That is why MTK necessarily includes:

- Knowledge of technology's mathematics fidelity, i.e., knowledge of the level of agreement between the results of the Mathematics and the results of the mathematics as presented by the technology;
- Knowledge of the new emphasis that technology puts on the mathematical content (e.g., more intuitive approaches encouraging or requiring a different domain of the influence of the values represented in the coordinate axes on the shape of the displayed graph);
- Knowledge of new sequences of content;
- Representational fluency, involving knowledge of different representations, of how to relate them and how to alternate between different representations and between different forms of the same representation.

And TLTK necessarily includes:

- Knowledge of new issues that technology requires students to deal, including the difficulties they face when using technology and that arise from such use;
- Knowledge of mathematical concordance of the proposed tasks, i.e., the alignment between the mathematics the teacher intended the students to work on and the mathematics the students actually worked;
- Knowledge of the potential of technology for the teaching and learning of mathematics, including knowledge of different types of work and teacher roles that technology becomes possible, knowledge of ways of articulating them and knowledge of the contribution they can bring to mathematics learning.

Finally, KTMT includes Integrated Knowledge (IK). A knowledge held by the teacher that simultaneously articulates knowledge in the base domains and in the two sets of inter-domain knowledge. This is a knowledge developed from the interaction between all the domains and that is characterized by its comprehensive and global nature and at the same time by its particularity, in the sense that it is this knowledge that allows the teacher to maximize the specific potentialities of the technology to provide a better mathematical learning to the students. It is this knowledge that is the true essence of KTMT.

Figure 1 presents a schematic representation of the KTMT model, trying to highlight its pyramidal structure. The colors intend to illustrate the process of development of new knowledge at a higher level. The intention is to represent, for example, the MKT as a new knowledge developing from the knowledge on Mathematics and on Technology. MKT is a new knowledge and not just the intersection of the base knowledge on Mathematics and Technology, in the same sense that orange is a new color developing from red and yellow.

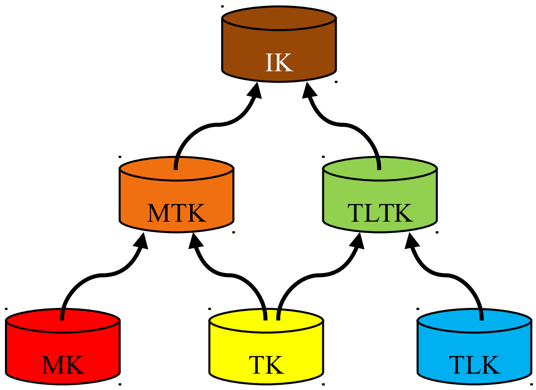


Figure 1. Schematic representation of KTMT model.

ANALYSIS OF THE PROFESSIONAL KNOWLEDGE OF THE TEACHER

The analysis of the professional knowledge held by the teacher has been a concern for several authors, but it has proved to be a complex task.

Angeli and Valanides (2009) assess the TPACK held by the teacher focusing on the task developed by him and a set of five criteria: (1) identification of topics to be taught with technology where the additional value brought by it is patent; (2) identification of representations to transform the content to be taught so that it becomes understandable to the students and in cases where it would be difficult to do so on the basis of traditional methods; (3) identification of teaching strategies that would be difficult or impossible to implement based on traditional means; (4) selection of appropriate technology; (5) identification of appropriate strategies for the introduction of technology in the classroom. The assessment process they use involves expert assessment, peer assessment, and self-assessment, which turns its implementation complex in terms of the structure it requires.

Niess et al. (2009) propose a model of professional development based on a characterization of the use and concerns of the teacher in relation to technology. The authors present four themes (curriculum and assessment, learning, teaching and access), five stages (recognizing, accepting, adapting, exploring and advancing) and a set of descriptors and examples. They do not, however, clarify the reasons behind the choice of these themes to conceptualize the development of knowledge based on a model that is organized in very different domains (knowledge of content, pedagogy, technology and intersections between them). Moreover, as they themselves recognize, the model does not allow a global analysis of professional knowledge, being possible to find teachers at different levels depending on the theme considered.

ANALYZING KTMT BASED ON THE TASKS PROPOSED BY THE TEACHER

The implications of technology's use on mathematics teaching and learning are well recognized (Graham et al., 2003). And Dunham (2000) points to the differences on the tasks proposed and on the students' work resulting from those proposals as the main impact that the graphing calculator can have on teaching. Therefore, it seems pertinent to consider the tasks proposed by the teacher as the basis for the analysis of his KTMT.

According to Goos and Geiger (2000), teaching approaches that emphasize problem solving and exploration, find in this technology a natural and mathematically powerful partner. Increasing the work around open questions and the exploration of concepts by the students is one of the possible consequences of technology integration (Cavanagh, 2006; Graham et al., 2003). The graphing calculator also allows the students to work on data collected by others as well as on data collected by the students themselves (White, 2009). In this sense, the tasks have the potential to elucidate about how the teacher takes advantage of the potential of technology for the teaching and learning of Mathematics.

Ponte (2005) classifies the tasks based on the level of demand that the task places on the students and on the level of structure, taking into account the context of the task (strictly mathematical or from reality). The author then distinguishes the tasks in exercises, problems, explorations or investigations, where the explorations correspond to investigations with a lower degree of difficulty and the modeling tasks are regarded as problems or investigations, depending on the level of structure.

Laborde (2001) considers the tasks in a context of technology use and classifies them into: (1) tasks that are facilitated by the technology, but are not modified by it; (2) tasks where technology facilitates exploration and analysis; (3) tasks that can be done with paper and pencil, but where technology comes to allow new approaches; (4) tasks that cannot be performed without the technology. And the author organizes these types into two groups, depending on how the tasks are facilitated by the technology, but could continue to be implemented without using it; or are modified by it, as in the tasks in which real phenomena are modeled or deductions are made from a set of observations.

Therefore it seems pertinent to start from an analysis of the tasks proposed to access the knowledge on the potential of technology for the teaching and learning of Mathematics held by the teacher and, consequently, his TLTK.

One of the features of the graphing calculator is to allow access to multiple representations (Kaput, 1992), which makes it possible to establish or reinforce connections in a way that would not be possible without the support of technology (Cavanagh & Mitchelmore, 2003), articulating numerical or tabular, symbolic or algebraic and graphical representations (Goos & Benninson, 2008) and fostering the development of a better understanding of Functions (Burril, 2008). As Kaput (1989) points out, the connection between different representations creates a global vision, which is more than the joining of the knowledge relative to each of the representations. And the author emphasizes how the technology allows a full exploration of the numerical and graphic approaches in a way that until then was not possible, thus favoring an integrated approach of the different representations and consequently the development of a deeper understanding. Thus the use of multiple representations has the potential to turn learning in a meaningful and effective experience.

Despite the importance of working with different representations and the teachers' concern about articulating and balancing their use, Molenje and Doerr (2006) found that the use of algebraic and graphic representations are dominant with respect to numerical representation. Moreover, when teachers actually use all the three representations, there tends to be a pattern in the way they do it. This pattern in the alternation between representations tends to be copied by the students, and this makes it difficult for students to develop the desired fluency between the different representations.

In this sense, an analysis of how the tasks proposed require the use of different representations and, when they do, an analysis of the characteristics that can be identified in their use, is an indicator of the representational fluency of the teacher and, consequently, of his MTK.

The integration of the graphing calculator in the teaching and learning of Mathematics forces the students to take mathematical decisions that until then they never had to face (Cavanagh & Mitchelmore, 2003). As the authors point out, the graphing calculator requires the students to make an adequate choice of the scale and of the values of the viewing window, to know how to deal with situations in which no graph is observed, or in which only a partial view appears. This is mathematical content that the students were not used to deal with and, in this sense, the way how the teacher includes it in the tasks proposed to the students may be another indicator of his MTK.

According to Cavanagh and Mitchelmore (2003), the main difficulty faced by the students when the technology becomes available in the classroom concerns to the process of finding a suitable viewing window. And the authors attribute the origin of this difficulty to the previous mathematical experience of the students, where all the graphs were drawn with paper and pencil using a referential where the values represented were almost always the same. This highlights the attention that the teacher needs to give to these issues, reflecting on the new emphases that the use of technology tends to place on mathematical contents. It is therefore crucial that the teacher manages appropriately the form and, above all, the moment when the students face these difficulties. Actually, according to Cavanagh (2006), it is fundamental that the teacher promotes a gradual contact with potentially problematic situations, ensuring that the students do not face them too soon. And an analysis of the tasks proposed will undoubtedly allow us to understand how the teacher considers this question. It will thus be another way of accessing elements of his TLTK.

We have thus identified a set of aspects that allow us to access elements of teacher's KTMT and, consequently, to assess the teacher's professional knowledge from an analysis of the tasks proposed by the teacher (see figure 2 for a synthesis).

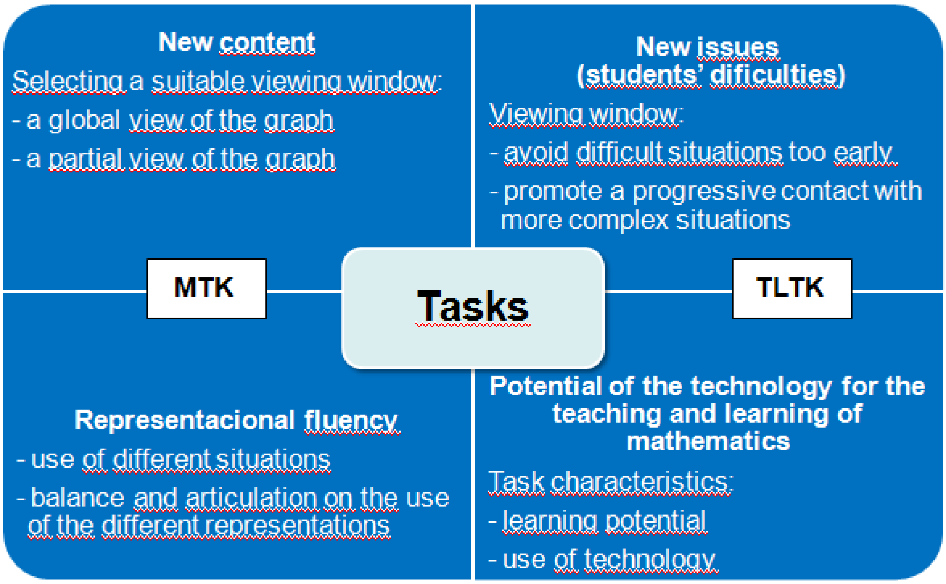


Figure 2. Synthesis of aspects to analyze in the tasks to access KTMT.

EXAMPLE

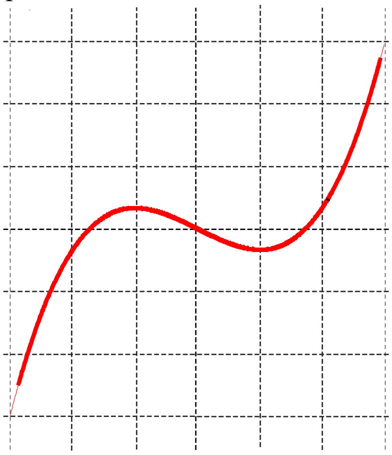
Let us suppose that a teacher selects four tasks to illustrate the work done by his students of a 10^{th} grade class when they study functions using the graphing calculator.

The first of the tasks selected was implemented in the third lesson (L3) and asked the students to study some families of quadratic functions: ax^2, ax^2+c, $a(x-h)^2$, $a(x-h)^2+k$ ($a\neq0$). Students should observe graphs of their choice of functions in each of the families under study and conjecture about the effect on the graph of changing the parameters. Then they are asked to report their conclusions.

In the fourth lesson (L4), the task consists on a situation with real context, where the number of bacteria, in thousands, of a certain colony, is given by $N(h)=-h^2 + 4h + 9$, where h represents the elapsed time. Questions relating to the number of bacteria at certain times and to the period of time when the number of bacteria is higher or lower than certain values are then placed. The students are expected to analyze the function and calculate function values, find the object of a certain image and calculate the zeros of the function.

In the task proposed on the tenth lesson (L10), the students were given a graph on paper (see figure 3) and informed that the function represented is a 3^{rd} degree polynomial function with a root for $x=\sqrt{3}$. Then they are asked to draw the referential so that the function is odd. After finding that the expression of the function is $f(x)= 1/6\ x\ (x^2-3)$ (using analytical methods), the students were asked to find the relative maximum and minimum of the function.

Figure 3. Graphical representation of the function on the task of 10^{th} lesson.



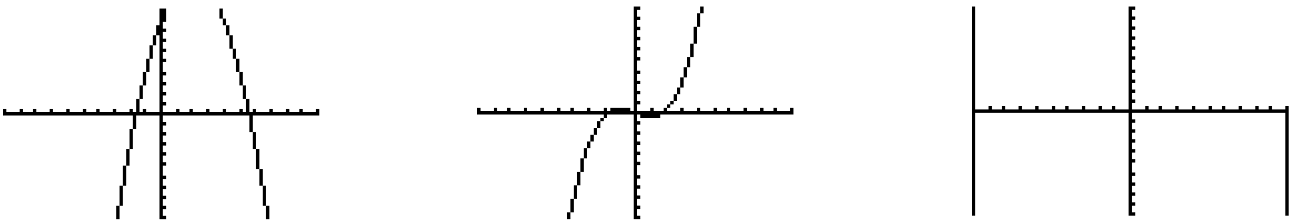
In the task proposed on the twelfth lesson (L12), the students were asked to graphically solve the inequality $x^3 - 100x \leq 10x^2 + 100x$.

The analysis of the tasks supposedly chosen by this teacher shows a reduced diversity in what concerns to the type of tasks. Adopting the classification developed by Ponte (2005), the task proposed at the 3^{rd} lesson (L3) can be classified as an exploration. The other three tasks are exercises where the students rely on already known strategies. In all the cases the students used some procedure to get the intended answer from the calculator (find a zero or a maximum, calculate a value of the function or the object of a certain image, find the intersection points of two functions).

In what concerns the relevance of the technology to solve the tasks, in most of these tasks the technology does not assume a central role. Dunham (2000) points to the potential of technology to allow the exploitation of different kinds of tasks, as the main impact of technology integration. But in this set of tasks, sometimes the use of technology is not even required to solve the task. That is the case of tasks on the 4^{th} and 12^{th} lessons (L4 and L12). However, solving the tasks with and without the technology is slightly different. According to the classification of Laborde (2001) for tasks in a context of technology integration, those are tasks that can be done with paper and pencil, but where the technology comes to allow new approaches. The task proposed on the 3rd lesson (L3) is a task that cannot be done without the technology. Actually, it is the technology that allows the students to explore the different graphs, trying to find the impact of changing the parameters on the graph of the function. The task solved on 10th lesson (L10) requires the use of technology to calculate the maximum and minimum values of the function because the students have not learned the analytical way of doing it.

Therefore it seems that the teacher's choice of tasks does not take into account the different types of tasks available, neither the potential of the technology to change the characteristics of the students work and to promote students' mathematical understanding.

Concerning the way in which the tasks require the students to deal with the viewing window, it is possible to identify mainly two situations. The task proposed on the 3rd lesson (L3) leaves to the students the decision about the functions to represent graphically. However, if they do not choose big values for the parameters, it will be possible to represent the functions using the standard window of the calculator. On the 4th lesson (L4), the task includes a function whose graphic is not completely visible on the standard window (see figure 4a), nevertheless it is possible to understand the behavior of the function from that view of the graph and to solve the task without changing the viewing window. On the task from lesson 10 (L10), the graph of the function in the standard window is somehow compressed around the x-axis (see figure 4b), but once again it is possible to intuit the behavior of the function and to solve the task without any change on the window. So this three tasks can be solved using the standard window and do not require knowledge about a suitable choice of the viewing window. The task proposed on the 12th lesson (L12) is different. In this case the standard window shows two lines that do not allow an understanding of the two functions represented (see figure 4c). Although the image immediately suggests that this is not a good representation of the graph, finding a suitable window is not trivial.



(a) (b) (c)

Figure 4. Graphic representation of the functions from 3rd, 4th and 10th lessons on the standard window.

The approach to the viewing window, suggested by these tasks, indicates that the teacher is aware that finding a suitable viewing window is a complex task for the students. As a consequence the teacher seems to avoid complex situations too early (as recommended by Cavanagh (2006)). Nevertheless, there is no evidence about the teacher's knowledge on the different types of situations related to finding a suitable viewing window. In fact, the tasks do not include situations of hidden behavior, incomplete view or partial view, suggesting that the teacher could have moved from situations where the standard window allows the students to solve the task (such as the tasks in L3, L4 and L10), to situations of simultaneous incomplete and partial view (the task in L12). There is also no evidence about the teacher's knowledge in what concerns the different mathematical knowledge needed to deal with each of the different types of situations.

In what concerns the work around the representations, it is again possible to identify two situations related to how the different representations are articulated. On the task proposed on the 3rd lesson (L3), the students start from an algebraic representation and move to a graphic representation, starting a cycle of alternation between these two representations, trying to make sense of the impact of the parameter on the graph. On the other three tasks proposed the students start from an algebraic representation, move to a graph representation and then to a numerical representation (on the graph). This option suggests a preference for a pattern in the alternation between representations. It is also evident an absence of the use of tabular representation.

Figure 5 presents a schematic synthesis of the knowledge of this teacher.

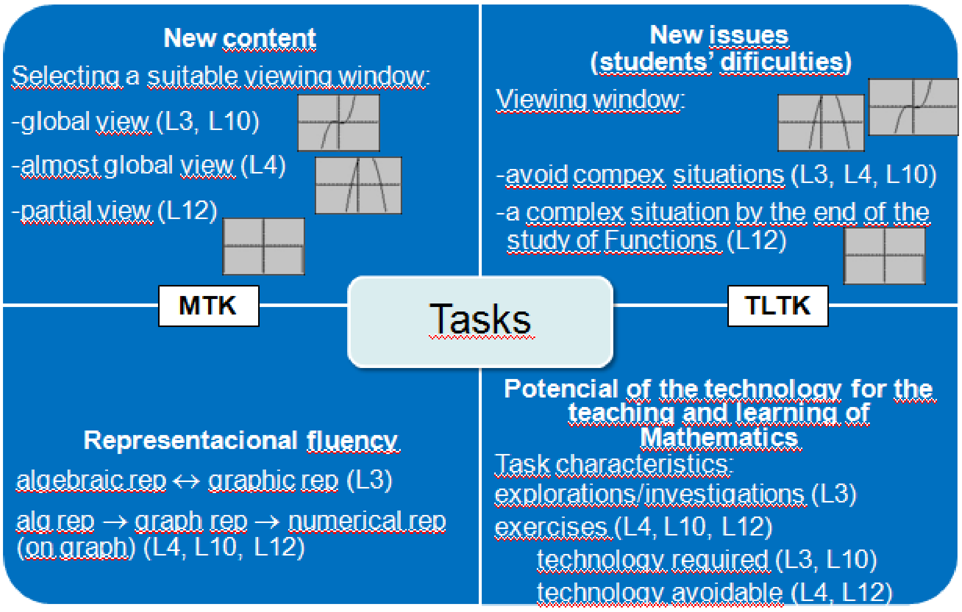


Figure 5. Synthesis of the teacher's KTMT analysis.

This teacher evidences having knowledge about important aspects, however it is possible to identify domains where his KTMT could be developed. That is, domains that should be considered when planning a professional development program for this teacher. That is the case of developing a deeper knowledge on the type of tasks taking advantage of the potentialities of the technology for the learning of mathematics; of the different types of situations requiring a change on the viewing window and of the mathematical knowledge needed on each one; and of the different ways of articulate and balance the representations available on the calculator.

CONCLUSION

In this article we present a proposal for the analysis of the professional knowledge of the teacher based on the KTMT's model and a set of tasks chosen by the teacher as representative of what was done with the students using technology. A reflection based on the fictitious example presented suggests that this proposal has the potential to access aspects of professional knowledge, nevertheless some criticism of the options assumed is inevitable. And the main one is related to the way how this proposal for analyzing the teacher's knowledge ignores all the aspects related to the implementation of the tasks.

In fact, although important, the tasks proposed alone do not characterize the teacher's practice. As Boaler (2003) emphasizes, one must take into account the complexity of practice and keep in mind that the same task can support different practices. In other words, we cannot forget that even the most meritorious task does not necessarily constitute the inevitable source of a productive learning environment, since what might at first be a rich question, that appeals to the students' exploration can, depending on the teacher's implementation, lose its potential and become a trivial exercise.

And if it is undeniable that the absence of elements regarding the implementation of the tasks by the teacher is a reality, it is not so clear that it is effectively a significant limitation. In the worst case (where the implementation of the task in the classroom reduces its potential), we will always have a majorization of the teacher's professional knowledge. And this because if there are references to a reduction of the cognitive demand of the task during its implementation, the opposite does not seem to be usual. And this majorization of the teacher's knowledge will continue to be useful and relevant information.

This proposal intends to assume a simple structure, still allowing access to relevant information. It aims to avoid complex analyzes carried out by multiple actors, as in the case of Angeli and Valanides (2009); or multiple interpretations of the teacher's knowledge according to the domain considered, as in the case of Niess et al. (2009).

Moreover, this is not the only proposal which does not include aspects relating to the teacher's practice. The work of Hill et al. (2007), which is based on a closed-ended questionnaire to which teachers respond, is well known. The number of authors who developed their research based on the ideas of Hill et al. (2007) suggests that it is possible to develop useful strategies to analyze the teacher's knowledge that do not include elements from the classroom practice.

This is, however, a proposal that is still in an early stage of its development. It is now important to refine it and to analyze in more depth the contributions it can bring, in particular by considering its integration within existing ones.

REFERENCES

Angeli, C., & Valanides, N. (2009). Epistemological and methodological issues for the conceptualization, development, and assessment of ICT-TPCK: advances in technological pedagogical content knowledge (TPCK). *Computers & Education*, 52, 154-168.

Boaler, J. (2003). Studying and capturing the compexity of practice – the case of the "dance of agency". In N. Pateman, B. Dougherty & J. Zilliox (Eds.), *Proceedings of the 27th PME International Conference* (vol.1, pp. 3-16). Hawai: PME.

Burril, G. (2008). The role of handheld technology in teaching and learning secondary school mathematics. In *Proceedings of ICME 11*. Monterrey, México: ICME.

Cavanagh, M. (2006). Enhancing teachers' knowledge of students' thinking: the case of graphics calculator graphs. In P. Jeffery (Ed.), *Creative dissent, constructive solutions*. Parramatta, NSW: AARE.

Cavanagh, M., & Mitchelmore, M. (2003). Graphics calculators in the learning of mathematics: teacher understandings and classroom practices. *Mathematics Teacher Education and Development*, 5, 3-18.

Dunham, P. (2000). Hand-held calculators in mathematics education: a research perspective. In E. Laughbaum (Ed.), *Hand-held technology in mathematics and science education: a collection of papers* (pp. 39-47). Columbus, OH: The Ohio State University.

Fauskanger, J. (2015). Challenges in measuring teacher's knowledge. *Educational Studies in Mathematics*, 90, 57-73.

Goos, M., & Bennison, A. (2008). Surveying the technology landscape: teachers' use of technology in secondary mathematics classrooms. *Mathematics Education Research Journal*, 20(3), 102-130.

Goos, M., & Geiger, V. (2000). Towards new models of teaching and learning in technology enriched mathematics classrooms. In W-C. Yang, S-C. Chu & J-C. Chuan (Eds.), *Proceedings of the Fifth Asian Technology Conference in Mathematics*. Chiang Mai, Thailand: ATCM Inc.

Graham, T., Headlam, C., Honey, S., Sharp, J., & Smith, A. (2003). The use of graphics calculators by students in an examination: what do they really do?. *International Journal of Mathematical Education in Science and Technology*, 34(3), 319-334.

Hill, H., Sleep, L., Lewis, J., & Ball, D. (2007). Assessing teachers' mathematical knowledge: what knowledge matters and what evidence counts?. In F. Lester, Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 111-156). Charlotte, NC: NCTM, IAP.

Kaput, J. (1989). Linking representations in the symbol systems of algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 167-194). Reston, Va: NCTM.

Kaput, J. (1992). Technology and mathematics education. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515-556). New York: Macmillan.

Laborde, C. (2001). Integration of technology in design of geometry tasks with Cabri-geometry. *International Journal of Computers for Mathematical Learning*, 6, 283-317.

Mishra, P., & Koehler, M. (2006). Technological pedagogical content knowledge: a framework for teacher knowledge. *Teachers College Record*, 108(6), 1017-1054.

Molenje, L., & Doerr, H. (2006). High school mathematics teachers' use of multiple representations when teaching functions in graphing calculator environments. In S. Alatorre, J. Cortina, M. Sáiz & A. Méndez (Eds.), *Proceedings of the 28th annual meeting of NA-PME*. Mérida, México: Universidad Pedagógica Nacional.

Niess, M. et al. (2009). Mathematics teacher TPACK standards and development model. *Contemporary Issues in Technology and Teacher Education*, 9(1), 4-24.

Ponte, J. (2005). Gestão curricular em Matemática. In GTI (Eds.), *O professor e o desenvolvimento curricular* (pp. 11-34). Lisboa: APM.

Rocha, H. (2010). Ensinar Matemática com a tecnologia: uma nova conceptualização do conhecimento e das etapas subjacentes ao seu desenvolvimento. In *Actas I Enconto Internacional TIC e Educação* (pp. 121-126). Lisboa: IE-UL.

Schmidt, D. et al. (2009). TPACK: The development and validation of an assessment instrument for preservice teacher. *Journal of Research on Technology in Education*, 42(2), 123-149.

White, A. (2009). Graphics calculator in the secondary mathematics classroom, pedagogical tool or just a gadget?. In *Proceedings of the Conference Redesigning pedagogy: research, policy, practice*. Singapore: National Institute of Education.

A CASE STUDY OF A SECONDARY MATHEMATICS TEACHER'S CLASSROOM PRACTICE WITH WEB-BASED DYNAMIC MATHEMATICAL SOFTWARE

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The broad purpose of this research is to gain a holistic understanding of teachers' integration of digital technology into classroom practices. This research examined a case of a secondary mathematics teacher's classroom teaching in England in which a web-based dynamic mathematical tool, the Cornerstone Maths software, was used in the teaching of geometric similarity. One of the contemporary theoretical frameworks, the Structuring Features of Classroom Practice (Ruthven, 2014), guided my research with its five different components related to the integration of new technologies into classroom practice. Classroom observation and semi-structured post-lesson teacher interview was employed for data collection. The analysis shows how the teacher used the dynamic mathematical software to teach geometric similarity in terms of five underlined features of classroom practice within the framework.

Keywords: secondary mathematics teachers, classroom teaching practice, dynamic mathematical technology, the SFCP framework

INTRODUCTION

Over the last two decades, the mathematics education community has paid increasing attention to the integration of digital technology into classrooms, aiming to exploit the full potential of technological developments into mathematics pedagogy. To date, there has been a considerable amount of research focusing on digital technology integration in the mathematics classroom. While some studies have concentrated on the impact of particular technologies on students' understanding of mathematical ideas (Saha, Ayub, & Tarmizi, 2010), others have focused on the interactions between teachers and students, and technological tools (Mason, 2014). Due to the fact that technology integration is a complex process that encompasses various relations that are not easily describable and analysable, researchers have started to focus on teachers and their integration of digital technologies into classroom practice (Clark-Wilson et al., 2014). More recently, there appears a need for further research investigating holistically the process of incorporating digital mathematical tools into classroom practice so as to both develop a more comprehensive understanding of the process of integration and improve the emergent theoretical frameworks (Artigue, 2010).

In this sense, through a contemporary theoretical framework (the Structuring Features of Classroom Practice) (Ruthven, 2014), my research [1] seeks to develop a better comprehensive understanding of dynamic geometry software (DGS) integration into classroom teaching, in respect of how a new web-based dynamic tool, the Cornerstone Maths (CM) software, interacts with various features of classroom practice in the hands of a secondary mathematics teacher teaching geometric similarity. Therefore, this present study aims to address the following research question:

- How does a secondary mathematics teacher's integration of dynamic mathematical technology into classroom teaching to promote students' understanding of geometric similarity interact with the five factors of classroom practice identified in the SFCP framework?

THEORETICAL FRAMEWORK

The SFCP framework was chosen as the most appropriate theoretical model for this study as it has potential to help comprehensively analyse the teacher's classroom practice using digital technology through its productive lens that focuses directly on five structuring components of classroom practice, namely: working environment, resource system, activity structure, curriculum script, and time economy.

Since this framework promises "a system of constructs closer to the lived world of teacher experience and classroom practice" (Ruthven, 2012, p.100) that other frameworks mostly neglect, it is particularly helpful in elucidating the gap between the potential of digital technology use outlined by research and the reality of teachers' use of digital technology in classroom practice and also in understanding how and why mathematics teachers use digital technology in classroom teaching to support students' understanding of mathematics (Bozkurt & Ruthven, 2016). However, I should note a limitation with regard to the SFCP framework, which is that the ideas which lead to the development of the components of the framework partly emanated from research studies which are non-specific to mathematics education. Despite this fact, I employed this framework to guide my research.

Working Environment

The incorporation of digital technology into teaching practice makes a number of demands on teachers in working environments, which involve changes of location and physical layout, modification of classroom organisation and routines. Teachers are required to make sure that the general technical infrastructure available in the working environment functions properly, that students, tools and materials are wisely arranged and that students are seated appropriately for individual and group work.

Resource System

The incorporation of digital technology into teaching practice might lead to difficulties for teachers in terms of building a coherent resource system of compatible didactical elements. Teaching with digital technology thus entails establishing suitable techniques and norms and handling double instrumentation, where technologies already in use can efficiently perform alongside new digital technologies. The employment of new digital technologies also calls for an environment in which students are allowed to familiarise themselves with core techniques and to explore a wider range of technical possibilities.

Activity Structure

The incorporation of digital technology into teaching practice presents some changes in the format of activities and creates new classroom routines. Teachers are therefore obliged to develop new structures to encourage interaction between students, teacher, and new technological tools and to identify appropriate (re)specifications of roles, in order to exploit new technological resources.

Curriculum Script

The incorporation of digital technology into teaching practice requires teachers to develop a curriculum script through a loosely ordered model of aims, resources and activities for teaching a particular topic, the treatment of potential emergent issues, and alternative paths of action.

Time Economy

The incorporation of new digital technology into teaching practice has implications in terms of the use of time in the classroom since there is a time cost related to the innovation itself (Ruthven, 2014). That is why, in order to reduce the "time cost" in classroom practice, teachers are required to recalibrate their timing for digital technology-enriched classroom practice.

RESEARCH CONTEXT

The qualitative case study approach was employed in this research to gain a comprehensive understanding of how the teacher uses a particular dynamic technology in classroom teaching. This method enabled me to understand the case by highlighting why things happen as they do, and allow me to interpret findings through "an in-depth investigation of the interdependencies of parts and of patterns that emerge" (Sturman, 1994, p.61).

The Cornerstone Maths project

The present study concerns with one of the Cornerstone Maths (CM) project teachers' use of the CM software in the actual classroom teaching. The project was devised at the University College London Institute of Education in England. This multi-year project aims to raise mathematics teachers' awareness of the potential benefits of digital technologies to ensure that these technological tools are routinely employed by them in the classroom for effective teaching and learning of mathematics. It covers three web-based dynamic units pertaining to "hard-to teach and learn" topics within Key Stage 3 of England's new National Curriculum (geometric similarity, linear functions, and algebraic patterns and expressions). The project provides secondary mathematics teachers with professional development alongside resources such as web-based dynamic mathematical software, student workbook and teacher guidebook (see Clark-Wilson & Hoyles, 2014 for more detailed information about the CM project).

The term CM software is used in this paper to refer to a new DGS tool. This software was designed as web-based DGS (Clark-Wilson, Hoyles, & Noss, 2015), drawing on the design affordances of DGS. Apart from common functionality with DGS (e.g. dragging, measuring), the CM software also offers other functionalities such as the ratio checker by which students can compare the ratios of the lengths of the corresponding sides of two geometric figures to determine if the figures are mathematically similar.

Participant

A secondary mathematics teacher from the active community of teachers formed by the CM project appeared to represent a suitable case for this study because those have started to integrate the CM software and its materials into their classroom practice after involving in the professional development course provided. A secondary mathematics teacher who has experience in teaching using digital technology was identified for my case study research through the project coordinator, Alison Clark-Wilson. Having agreed to participate in my research, the teacher scheduled the teaching of geometric similarity with the CM dynamic tool within his mathematics classroom setting during the school term (2015-2016). I will call the teacher as Joseph (pseudonym name) here forth.

Joseph was a reasonably experienced teacher in terms of teaching mathematics with digital technologies. He had 4-5 years of mathematics teaching experience in a secondary school in London. He was also responsible for developing schemes of work and lesson plans in line with the curriculum objectives of the school. He was following a Masters' degree in mathematics education especially to improve his competencies in exploiting the mathematical fidelity of available digital technologies. This indicates that Joseph is rather enthusiastic about benefiting from digital technologies to enhance students' understanding of mathematics.

Data Collection

Classroom observations took place in times scheduled in advance with Joseph in his classroom (two separate lessons in different days). An observation protocol was designed based on five elements of the SFCP theoretical framework and employed in the research. My primary focus of observation was on Joseph's integration of the CM software into his classroom teaching rather than on his students. He was informed of the focus of the observation. The audio recorder was placed somewhere in the classroom to be used as support in my writing up of some aspects of the observations.

Following the observations, semi-structured post-lesson interview was conducted with Joseph. An interview protocol with more and less structured questions was developed according to each of the components of the SFCP framework to ensure the questions covered all of the components. The interview lasted about 40 minutes and was digitally audio-recorded. A laptop computer was also used during the interview to enable Joseph to articulate his thoughts on the CM software activities in the dynamic environment.

RESULTS

The findings are presented according to five structuring features of classroom practice, namely: working environment, resource system, activity structure, curriculum script, and time economy, respectively.

Working Environment

In the two sessions, which I observed, Joseph was covering for his colleague who was on maternity leave at the time. The lessons took place in a classroom regularly used by his colleague. In the class, there was an interactive smart board along with an ordinary whiteboard and teacher's main computer. There were rows of tables and chairs at which students sat. The students used laptops and iPads at their tables, with each device being shared by two students.

In the interview, the teacher, Joseph, stated that the U-shaped classroom layout is the best layout for him in terms of the provision of computers. He was, however, not able to teach in the computer room where computers are placed in a U-shape, since the computer room had been booked by his colleague. Joseph claimed that the U-shape classroom arrangement allows teachers to better monitor students' screens and facilitate group work with the students in the middle of the classroom.

> I would like to have computers all-round the sides. It means that I can see more easily what is on their screens from where I am standing. I think that is the most flexible arrangement because groups, they can sit in the middle. We can do group work and then we go away and use the computers and then come back to the middle.

This result echoes Bozkurt and Ruthven's (2015) findings in relation to the advantages of a U-shaped computer room.

In the first lesson, the students were provided with laptops; however, in the second lesson, Joseph preferred to use iPads instead of laptops due to the layout of the classroom. In the interview, he reported that since he was not able to monitor the screens of the students' laptops during the whole-class lesson instruction in the first lesson, he had to either circulate around the classroom or ask the students to lower the screens of their laptops. He emphasised that, in the first lesson, he could not ask the students to push the screens of their laptops down, as they would need to log in again, so he decided to use iPads in the second lesson.

> With the iPads, what I like is that you can just put the face down because what is difficult with the students is getting them not to look at the screen and to focus on you when they

> have these devices. So putting it face down as a way of moving that...which you cannot do so much with laptops because if you lower the laptop screens, you have to log in again. It takes a lot of time so iPads are better from that perspective.

This finding indicated that teachers need to develop fall-back strategies to deal with possible contingencies in technology-integrated classrooms. Joseph turned to the use of iPads rather than laptops for the second lesson, as iPads can be used in a way that made them less distracting for the students especially during plenary discussion. Given that there is a strong connection between confidence in using technology in the classroom and the teachers' experience which supports the use of the pedagogical use of the digital technology (Thomas & Palmer, 2014), it can be claimed that Joseph's experience provided him with the flexibility of choosing the most suitable device at the time.

Resource System

Joseph appreciated both the CM dynamic mathematical software with the structured twelve dynamic investigations with regard to geometric similarity and its two ready-to-use booklets: one was for the students and included tasks and instructions on how to engage with software-based tasks focusing on geometric similarity, and the other was for teachers and contained a wide range of information involving some implementation suggestions related to the tasks and possible student answers. Since he found CM resources sufficient, he did not adapt any other resources for his lessons.

According to Thomas and Palmer (2014), there is still considerable need for classroom resources delivering good ideas for teachers to incorporate into classroom practice. Joseph's case suggests that providing more reliable access to technological resources might be important in terms of promoting their use by teachers in the classroom.

The focus of Joseph's lessons was on determining whether an image is the enlargement of an original shape through numbers and using a scale factor. He took advantage of the dynamic software and the booklets throughout the lessons. For example, in light of the instructions in the booklets, the students were asked to watch an animation, which aimed to enable them to explore what copies are always mathematically similar to the original object by pausing the animation to rotate and translate the shapes to investigate. They were also encouraged to use dynamic measurements and comparisons and the ratio checker. The students used the CM software to discover some ideas, such as "the relationship between mathematically similar shapes", "the scale factor" and "the relationship between corresponding sides", formulating some conjectures and then checked and verified them in the dynamic environment. During his reflections, Joseph mentioned those outcomes as benefits of using CM resources as follows:

> It is the kind of discovering for themselves if the scale factor of two is being doubled or the scale factor of three is being tripled and that actual half [inaudible]. You know, they have constructed the understanding, which I think research shows that this is much more powerful than we are just telling them. Conjecturing both what the impact of scale factor was and then being able to check it and verify it for themselves was, for me, the key aspects of the lessons, which was purely driving by their interaction with the mathematical world.

The extent to which teachers make students use digital technologies depends on how teachers view them in terms of advancing students' understanding (Ruthven, Hennessy, & Deaney, 2008). Joseph's experience of teaching with digital technologies and his awareness of the potential of them to enhance students' understanding led him to conduct the lessons in this manner.

Joseph also showed an awareness of the difficulties that the students might encounter when interacting with the CM software investigation activities. At the beginning of the two lessons, he demonstrated some technical features of the software and how the tasks on the booklet should be used along with the dynamic software; i.e. how the lengths of the objectives could be measured and coloured or how the measurements of corresponding sides could be dragged. In the interview, he stated that since he had experienced a problem using the measurement and colouring buttons, he underscored a few technical features for students before starting to teach the main content.

Overall, it can be said that the case indicates that the teacher has not yet integrated CM resources into his resource system. The CM materials are still stand-alone resources for the teacher, which is not surprising because he has only used them in his teaching practices a few times.

Activity Structure

Joseph broke down his lessons roughly into three sections. In the first section, before introducing the tasks, the students were asked to explain what they had learnt in the previous lessons. His aim was to remind the students of the prerequisite knowledge needed for that day's lessons. In the second section, he adapted a sequence of the activities, which allowed the students to discover particular objectives pertaining to the topic in the CM software environment in pairs, through "predict-test-explain" sequences (Ruthven, 2014, p.387). Throughout most of this section, the students used the web-based dynamic tool by engaging with the investigation activities in the booklet. However, to create a whole class discussion, Joseph sometimes asked the students to stop using the software.

In his reflection on the lesson, Joseph stated that when he gives lessons in the computer room, he often selects a couple of students' work to share with the whole class on their screens. He feels that this method is quite powerful to draw the students' attention to mathematical facts**.**

> In my other [computer] room, what is possible to do is to pick one student's screen and put it up on front of everybody's screen which is really cool because students are more interested in another student's work. They really care more. So, that is quite a powerful thing to do.

He also emphasised the fact that allowing students to work in pairs in the CM dynamic environment helped them learn from one another.

> Having them in pairs like this, I think, helps them [overlap] by making them explain and then one pair can be slower and they can learn from the neighbour. I guess the software provides hints for them when they have a disagreement, they can use the software to check...always check and see the results.

In the third section, the key ideas raised by the lessons were highlighted by Joseph by asking several probing questions to the students. In the students' responses, he paid attention to the appropriateness of the mathematical language they used, to support the development of their mathematical language.

Curriculum Script

Joseph did not prepare a specific lesson plan demonstrating his curriculum script (CS) for these lessons. The classroom observations and post-lesson interview helped me sketch out Joseph's CS on the topic of geometric similarity.

He told that he developed his lesson plan in line with the guidance provided by the CM teacher booklet. The booklet, for example, provided him with the key ideas that students should develop an understanding of in the lessons such as "the heights and the widths of mathematically similar rectangles are related to a common multiplier" and "scale factor is the multiplier by which the lengths in the original shape result in the enlargement". Additionally, the booklet indicated where students might get confused about geometrical facts, for instance, "some pupils may confuse height and width, which does not necessarily affect the relationship between the shapes, but this approach might be confusing in discussion with the whole class. If this happens, you can briefly mention the role of conventions in mathematics".

In his reflection on CS, he drew attention to why such a booklet is useful for helping teachers develop a lesson plan.

> Adapting and using these things [the CM software and the booklets] is quite difficult. And, I mean, teaching in London schools is a lot of pressure to get the results [and to] get through the curriculum. So what is good about the Cornerstone [Maths booklets] is that they help me a lot like think how to use these table things [a part of an activity in the booklet] effectively and I guess the bridging activities are in the big club, you know the focus of attention is on the right things. The hard thinking has already been done for teachers.

This quotation implies that incorporating new dynamic digital technologies into their curriculum script (CS) might be a challenge for teachers for two reasons. The first reason is that teachers are required to teach all the topics mentioned in the national curriculum over the school term, which means that they do not have enough time to devote to designing their CS around digital technology tools. Second, the schemes of learning (SOL) developed by schools do not generally take the use of technologies into consideration, which leads them to have difficulty in integrating digital tools and related resources into lessons. Thus, curriculums should be designed with the use of digital technologies and resources in mind, thus providing teachers with "explicitly designat[ed] topics, problems or investigations to be delivered using specific educational software" (Sinclair et al., 2010, p.67).

Overall, according to Joseph, the students were able to gain an understanding of hard-to-grasp ideas of geometric similarity through the dynamic environment, which the CM software offers, and with the guidance of the teacher booklet. This finding is in agreement with those obtained by Ruthven (2009) regarding the accuracy, speed and manipulative facilities of dynamic software in support of students' explorations of various cases.

Time Economy

One issue arising from Joseph's lessons was that organising the students' access to the web-based CM activities took up a considerable amount of the available teaching time. Joseph asked the students to open the CM web-based activities by typing the CM website address (http://cornerstonemaths.co.uk/) on the board when the software needed to be used at the beginning of the lessons. The students then spent some time accessing the dynamic activities. In the interview, Joseph referred to this issue, observing that as well as getting the technological devices set up, having to choose between student A or B and teacher A or B [2] before accessing the CM activities was time-consuming and confusing for the students.

> I think, one thing [which] is annoying with the CM software is that you have to choose student A, and teacher A. That seems to be confusing to them [the students], because that does not seem to be anything.

His well-established lesson routines also meant that his lessons started, proceeded, and ended in a timely and purposeful manner. He devoted a reasonable amount of time to familiarising the students with the core technical software features that they may need. This enabled him to optimise the didactic return on time investment during the students' interaction with the software.

Furthermore, according to Joseph, the feedback provided by the CM software facilitated the students' learning and reduced the time he had to spend for revising the students' work.

CONCLUDING REMARKS

Through the Structuring Features of Classroom Practice (SFCP) framework, this study investigated a case of classroom practice of a secondary school teacher having experience in using digital technology in the teaching of mathematics.

The key finding related to the working environment was that the teacher valued the U-shape classroom arrangement, as it allowed him to monitor students' screens better; thus facilitating classroom management, and offering the space to conduct group work without using technology. In terms of resource system, the study revealed that teachers may need ready-to-use resources "with good ideas" to assist them to effectively plan and conduct their lessons since they do not have time to prepare such resources themselves for use in technology-embedded lessons. The teacher in this study used a structured activity format and spent time on the students' own discovery of facts in relation to geometric similarity through the use of dynamic activities and on maintaining a balance between individual and group work. Additionally, the teacher mentioned various benefits of using the CM software: the software enabled the students to discover geometric facts through positing and testing conjectures and provided feedback. Thus, it increased the speed of learning by focusing the students' attention on the geometric relationships involved and enabling teachers to devote more time to making students more active.

In terms of the conceptual framework used in this study, it assisted me in analysing the data by making particular aspects apparent and workable. However, I sometimes struggled to decide under which component a piece of data should be examined. Even though this struggle could be caused by the intertwined nature of teachers' classroom practice, I believe that the components of this framework need to be clarified further to avoid the danger of providing researchers with "a coarse-grained tool" (Ruthven, 2014, p.380).

In terms of suggestions for further research, it would be beneficial to carry out longitudinal systematic studies, which explore the use of digital technologies by mathematics teachers, who have varying levels of experience in using technology to teach, because teachers' confidence and competence can influence how teachers use digital technologies in classroom practice (Bretscher, 2014). It would also be helpful for researchers to employ the combination of several frameworks which have different theoretical lenses in parallel, to carry out a much deeper and fruitful investigation of teachers' classroom practice using digital technologies.

Notes

1. This paper was developed based on a part of the author's Master's thesis at the University College London (UCL) Institute of Education.

2. At the beginning, in order to access the CM software activities, students need to select the group of their teacher and then their names.

REFERENCES

Artigue, M. (2010). The future of teaching and learning mathematics with digital technologies. In C. Hoyles, & J.-B. Lagrange (Eds.), *Mathematics Education and Technology-Rethinking the Terrain: The 17th ICMI Study* (pp. 463-475). New York: Springer.

Bozkurt, G., & Ruthven, K. (2015). Expert and novice teachers' classroom practices in a technological environment. In K. Krainer, & N. Vondrová (Ed.), *CERME9 Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 2319-2325). Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME.

Bozkurt, G., & Ruthven, K. (2016). Classroom-based professional expertise: A mathematics teacher's practice with technology. *Educational Studies in Mathematics*, 94(3), 309–328.

Bretscher, N. (2014). Exploring the quantitative and qualitative gap between expectation and implementation: A survey of English mathematics teachers' uses of ICT. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The Mathematics Teacher in the Digital Era: An International Perspective on Technology Focused Professional Development* (pp. 43-70). Dordrecht: Springer.

Clark-Wilson, A., & Hoyles, C. (2014). Blended learning and e-learning support within the Cornerstone Maths Project. In K. Maaß, B. Barzel, G. W. Törner, E. Schäfer, & K. Reitz-Koncebovski (Ed.), *Proceedings of the Conference hosted jointly by the project mascil (mathematics and science for life) and the German Centre for Mathematics Education (DZLM)* (pp. 158-166). Essen: WTM.

Clark-Wilson, A., Aldon, G., Cusi, A., Goss, M., Haspekian, M., Robutti, O., & Thomas, M. (2014).The challenges of teaching mathematics with digital technologies – the evolving role of the teacher. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.) *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 1 pp. 87-116). Vancouver, Canada: PME.

Clark-Wilson, A., Hoyles, C., & Noss, R. (2015). Scaling mathematics teachers' professional development in relation to technology -probing the fidelity of implementation through landmark activities. In K. Krainer, & N. Vondrová (Ed.), *CERME9 Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 2333-2339). Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME.

Mason, J. (2014). Interactions between teacher, student, software and mathematics: Getting a purchase on learning with technology. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The Mathematics Teacher in the Digital Era: An International Perspective on Technology Focused Professional Development* (pp. 11-40). Dordrecht: Springer.

Ruthven, K. (2009). An investigate lesson with dynamic geometry: A case study of key structuring features of technology integration in classroom practice. *Proceedings of the Sixth European Conference on Research on Mathematics Education*, (pp. 369-378).

Ruthven, K. (2012). Constituting digital tools and materials as classroom resources: The example of dynamic geometry. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From Text to "Lived" Resources: Mathematics Curriculum Materials and Teacher Development* (pp. 83–103). New York: Springer.

Ruthven, K. (2014). Frameworks for analysing the expertise that underpins successful integration of digital technologies into everyday teaching practice. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The Mathematics Teacher in the Digital Era: An International Perspective on Technology Focused Professional Development* (pp. 373-393). Dordrecht: Springer.

Ruthven, K., Hennessy, S., & Deaney, R. (2008). Constructions of dynamic geometry: A study of the interpretative flexibility of educational software in classroom practice. *Computers & Education, 51*(1), 297-317.

Saha, R. A., Ayub, A. F., & Tarmizi, R. A. (2010). The effects of GeoGebra on mathematics achievement: Enlightening coordinate geometry learning. *Procedia Social and Behavioural Sciences, 8*, 686-692.

Sinclair, N., Arzarello, F., Gaisman, M. T., Lozano, M. D., Dagiene, V., Behrooz, E., & Jackiw, N. (2010). Implementing digital technologies at a national scale. In C. Hoyles, & J. B. Lagrange (Eds.), *Mathematics Education and Technology-Rethinking the Terrain: The 17th ICMI* (pp. 61-78). New York: Springer.

Sturman, A. (1994). Case study methods. In J. P. Keeves (Eds.), *Educational Research, Methodology, and Measurement: An International Handbook (* 2nd ed.) (pp. 61-66). Oxford: Permagon.

Thomas, M. O., & Palmer, J. M. (2014). Teaching with digital technology: Obstacles and opportunities. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The Mathematics Teacher in the Digital Era: An International Perspective on Technology Focused Professional Development* (pp. 71-90). Dordrecht: Springer.

A CLASSIFICATION OF RESOURCES USED BY MATHEMATICS TEACHERS IN AN ENGLISH HIGH SCHOOL

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This paper provides a classification of resources used by mathematics teachers in an English high school. It is based on data analysis from ongoing PhD research exploring mathematics teachers' appropriation of digital resources and the impact on classroom practices in selected schools. This paper reports a way of making sense of the myriad curricular and digital resources that are increasingly available to the teachers in planning and enacting their teaching and assessing their students' understandings in the context of their every day practices. The classification has potential to aid understandings of teachers' appropriation of resources for teaching mathematics.

Keywords: Coding data, Mathematics, Resources, Teachers

INTRODUCTION

This paper examines the resources used by four teachers in a Mathematics department in a state school in England. It presents and discusses a way to classify resources used by teachers and explores similarities and differences in teachers' use of resources using this classification. The paper is structured as follows. We begin with a review of literature on mathematics teachers and resources and the theoretical framework which guides the PhD research. We then present the context of our research, the school and the four teachers. The methodology of the research (the means of collecting and analysing data) is then outlined which is taken a step further in the next section which present a 'logical classification' of codings obtained in data analysis. We then present the results, the resources used by the four teachers in preparing to teach and in teaching; we present these using the 'logical classification'. The paper ends with a discussion of the classification and of the resources the four teachers used.

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

The literature on the importance and relevance of the use of curricular and digital resources in mathematics teaching and learning has matured over the new millennium. Our understanding of resources aligns with Adler's (2000, p. 7) reconceptualization of resources "It is possible to think about resource as the verb re-source, to source again or differently". Within educational settings, 'curricular resources' (Stylianides, 2016) or 'curriculum material' (Remillard, 2005), include all the materials (digital or physical) that teachers appropriate in and for their teaching, with the textbooks been the most dominant resource internationally. In the context of mathematics teaching and learning Pepin, Gueudet, & Trouche define "mathematics teaching resources as all the resources which are developed and used by teachers (and pupils) in their interaction with mathematics in/for teaching and learning, inside and outside the classroom" (2013, p.929). A more recent publication, Monaghan, Trouche and Borwein (2016) documents the major milestones in the studies on teachers' integration of digital technology and the ongoing research efforts focusing on the appropriation of curricular and digital resources in the context of practice. We appropriate the above views in our research and consider mathematics teaching resources as including:

- Text resources, such as curriculum materials: mathematic textbooks, teacher curricular guides, teachers' worksheets, spreadsheet, posters and syllabi.

- ICT resources, such hardware and software: laptops, iPads, applets, e-textbooks, games, Geogebra, blogs and learning platforms.
- Other material resources, such as students' handheld white boards, manipulatives and calculators.

The above suggests that the construct *resource* is understood in the context of mathematics teaching and learning as everything that supports and facilitates teachers' practices but the practice takes place in a context and in a community that need to be considered to account for actual use (and variation in use) of resources by teachers. In the preparation for teaching, teachers select, use, combine and modify, bookmark and save a variety of resources over time into a structured set of teacher's resources, this is referred to as teacher's *resource system*. Bozkurt and Ruthven (2015) show how digital resources structure teachers' planning and classroom practices. They identify five key features in the structuring process: *working environment, resource system, activity format, curriculum script and time economy*. Our research takes into consideration this milieu in which teacher's practice with mathematics teaching resources takes place. The milieu in preparing to teach often includes people and we consider people (face-to-face and online) as resources when they support teachers' practices.

In the analysis of the data in this study we combine an activity theoretic approach (Engeström, 1987) with the more recent 'documentational approach' (Gueudet and Trouche, 2009) from the French didactics as theoretical tools for developing an understanding of the teachers' appropriation of resources in the context of planning and enacting their lessons and assessment. These provide coherent multiple interpretative perspectives to be simultaneously considered in the processes of data collection and analysis.

CONTEXT

Data for analyses presented in this paper were gathered in the context of the aforementioned PhD research exploring mathematics teachers' appropriation of digital resources in selected High schools in the UK. The unit of analysis is the teachers' nested activity contexts. The mathematics departments in the selected schools are the broad setting since teachers usually undertake their practices within that collective context. This environment consists of overlapping layers of interactions: the whole school environment, the Mathematics Department, classrooms, curricular and digital resources available for mathematics teachers for planning, for teaching and for assessment. The four teachers considered in this paper volunteered to participate in the project and each has more than 5 years of teaching experience. The overall structure of these teachers' lessons was a three-part-lesson: *Starter phase*- to engage students and bridge learning from previous lessons into the current; *main part of the lesson*- for the development and consolidation of new learning and; the *plenary*- for extension and assessment (Jones & Edwards, 2005; Beere, 2012). The school, for students aged 11-18 years, hosts one of the *maths hubs* (a collaborative national networks of schools' initiative) where the use of digital resources are encouraged and supported. The schools hosted many visitors over the data collection year including mathematics teachers from China (hence the reference to 'Chinese teacher' in Table 1 below). The students were in mixed ability classes. The classification we provide in this paper is based on these four teachers from this school though we believe it could be used more widely.

METHODOLOGY

A qualitative case study approach (Creswell, 2013) was adopted. Purposive sampling was used in selecting seven teachers from three schools based on the use of curricular and digital resources, access, proximity and the opportunity to observe teachers' practice with mathematics teaching resources in natural settings. Data collection was undertaken during the 2015-2016 school year through periodic whole day school visits. Data were collected from a range of sources: audio-recorded semi-structured interviews; classroom observations using an adapted systematic classroom analysis notation for mathematics lessons (SCAN) (Beeby, Burkhardt & Fraser, 1979); recordings of teachers accessing digital resources, enabled by screen capture software (SnagIt, https://www.techsmith.com/snagit.html); researchers' field notes; and the collation of documents to which the teachers made reference. In this paper, the interviews are used as primary data sources, complemented by the screen capture data.

The four teachers from the school described above are the case units of analysis: Katie, James, Emily and Joe (pseudonyms). They were selected from teachers willing to be involved in the study based on their commitment to the of ethos the Mathematics department in the school. They taught in a context where a wide range of curricular and digital resources are easily available for use. Transcripts of the interviews and screen capture data were coded and thematically mapped, constantly grouped and regrouped into categories (we use the term 'category' for a set of codes).

Our classification of resources (for these four teachers)

The coding process described above produced a lot of codes/categories. There were four stages in the development from initial coding/categories to the classification in Table 1 below, these were: (i) initial coding by the first author; (ii) discussion and refinement of the initial coding by both authors; (iii) an informal inter-coder reliability session between the first author and a teacher from a non-study school which also produced a slight refinement of the coding; (iv) a second meeting of the two authors in which a 'logical classification', which we now discuss, was superimposed on the post stage iii categories.

Open coding is an interesting, and often useful, activity but it is very subjective (even when more than one coder is involved). In discussing this in stage iv we saw that the codes could be divided logically; 'human' – 'non-human' provides a partition of all resources a teacher may use. Taking this logical division further we can: partition human resources into those where there is 'physical contact' and those where there is 'not physical contact'; similarly, non-human resources can be partitioned into those which are 'electronic' and those which are 'non-electronic'. Our final division is to partition: electronic resources into 'hardware' and 'not hardware' (notice that 'hardware' and 'software' is not a logical partition); and non-electronic resources into those created by the individual teacher under consideration ('individual', e.g. Katie) and those which were not created by the individual teacher under consideration ('not individual'). Note that a worksheet created by Katie and used by Katie and James would be coded 'individual' for Katie but 'not individual' for James.

Note that further divisions are possible. For example, 'human, physical contact' could be partitioned into 'formal' (e.g. within a scheduled meeting that has an agenda) or 'informal' but we found the classification provided in Table 1 was sufficient to accommodate all of the codings developed in stage iii; it was also manageable, relatively easy for the two authors to code in an identical manner.

RESULTS

Table 1 provides a summary of resources used by the four teachers.

Human		Non-Human			
Physical contact	Not Physical Contact	Electronic		Non-Electronic	
		Hardware	Not Hardware	Individual	Not Individual
KATIE Personnel CPD TeachMeet Chinese teacher	Podcast Social-networking Twitter	iPads IWB HWB	Resource banks_ Mangahigh Task-spec websites applets Gcsepod.com Resourceaholic TES.com Music	Paper-based resources Workbooks Worksheets Posters	Resource banks Paper-based resources textbook
JAMES Personnel TeachMeet CPD	Podcast Facebook Twitter Blogs	iPads Laptops IWB	Resource banks_ Mangahigh applets Resourceaholic TES.com GeoGebra Mathspad.co.uk Desmos Coberttmaths.com	Paper -based resources Workbooks Worksheets Posters	Resource banks Paper-based resources
EMILY Personnel TA TeachMeet CPD	Twitter	Calculators iPads Laptops IWB HWB	Wordwall Ttrockstars Mathsbox.com Code buster King of Maths QR code Stopwatch	Paper -based resources Workbooks Worksheets Number line	Resource banks Paper-based resources
JOE Personnel TeachMeet CPD	Facebook Twitter	Calculators iPads Laptops IWB HWB	TES.com Socrative Mathswatch.co.uk Plickers QR codes 10ticks Virtual manipulatives	Paper -based resources Workbooks Worksheets	Resource banks Paper-based resources

Table 1 Resources used by the four teachers (Katie, James, Emily and Joe)

Table 1 is structured with the columns representing the partitions we described in the previous section and the rows representing the four teachers. Many of the terms (e.g. twitter, iPad), we feel, need no explanation. Abbreviations used are: CPD – continued professional development; IWB – interactive whiteboard; HWB – handheld manual whiteboard; and TA – teaching assistant. 'TeachMeet' are informal but organised opportunities for teaching to meet to share 'good practice'. We now explain software used by teachers under five terms in common use in English schools. Endnotes provide links to websites for specific resources.

Resource Banks

There are three types of resources banks. (i) The individual mathematics teachers' resources on an iPad and flash drive (ii) The Shared resource bank of the mathematics department.)iii) Online resource banks, some of which are commercial and some are free. Online resources banks include: Gcsepod.com[i], Resourceaholic[ii], TES.com[iii], Mathspad.co.uk, Mathswatch.co.uk, 10ticks[iv] and Mathsbox.com. Here teaching resources of various sort can be accessed and used by teachers, parents and students.

Applets

These are small applications that performs specific task. They run within the scope of a dedicated widget engine and are designed to be placed on a web page as a plug-in auxiliary application. Applets used by these four teachers include Plickers, Socrative (mathematics specific) and King of Maths.

Dynamic Mathematics Software

These are open-source software that afford teachers and students dynamically linked multiple mathematical representations tools to help create models of real situations and links algebra and geometry representational systems simultaneously. In this category *Geogebra* and *Desmos*[v] were used by one of the teachers.

General Purpose software

This is often a suite of software in the form of an integrated package like MS Works incorporating spreadsheet and presentation software like power point. Wordwall belong to this class.

Data-capture Software

This are simple but powerful tools that allow the teacher capture, collate and analyse data in real-time for a whole class formative assessment. Plickers and Socrative were used for this purpose.

DISCUSSION

We first comment on our classification and then consider similarities and differences over the four teachers.

As mentioned above, open coding is often useful but it subjective. The partitioning in our classification is not subjective, it employs the law of the excluded middle: in classical mathematical logic, for any well-defined statement *A*, '*A* or *not A*' is true. Our classification of resources was not designed or used to replace the open codes generated in stage iii but as a means to present the resources contained in the open codes. We think this is a case of *having your cake and eating it*. With research on teachers' use of resources on the rise we offer this logical classification to fellow researchers as a means to present the results of our research in similar formats. We offer this as a 'malleable template'. The partitions we used in Table 1 suited, in our opinion, the data we collected and analysed; a different set of partitions may be more suitable for a different research study. We now move on to similarities and differences that can be observed in Table 1.

In exploring the similarities and differences the classroom setting is worth considering first. The school of study has a new ultra-modern building, all the classrooms are spacious and equipped with an IWB, an adjacent chalk board, a laptop, a projector and every student is given a handheld writing board (HWB). The mathematics teachers have a growing shared bank of mathematics teaching resources where peer-reviewed resources are stored and are accessible to every member of the Mathematics department as a 'collective resources system'; this a 'go-to area' in lesson planning. There are class sets of iPads with an accessible iPad storage and charging trolley within the department for student use. This is the structuring context for understanding the similarity and differences amongst the teachers.

Similarities exist across the four teachers in their access to the CPD, teach-meet (a periodic whole school teachers' meeting to share experiences and expertise). All four teachers use social media and Twitter (see https://twitter.com/hashtag/mathschat?lang=en) in particular. The entries for this in Table 1 are for the specific purpose of planning lessons. The use of a mathematics teacher dedicated social networking media as a tool for communication, queries and sharing of resources is a regular feature among these teachers. There are many similarities with regard to non-human resources. Under *hardware*, all used iPads and IWBs. Under *not hardware* Resource banks, Mangahigh, TES.com and Resourceaholic are all used by two or more of the four teachers. With regard to *non-human & non-electronic* resources, the cells in the *individual* and *non-individual* columns of Table 1 are almost identical; this, we posit, is related to what we say above of the Mathematics department as a collective resources system.

With regard to differences between the four teachers and the resources they use we first note that the similarities far outweigh the differences. But we comment on differences with regard to 'doing mathematics' and to 'uniqueness'. With regard to 'doing mathematics', four resources that stand out to us as different to the rest are: calculators, Desmos, Geogebra and, to a lesser extent, virtual manipulatives. These are resources which students or teachers can use flexibly to explore mathematical relationships[vi] as opposed to being shown how to do mathematics by someone else. With regard to the four teachers and the two electronic columns contain these four resources we can see that: Katie uses neither; James uses non-hardware (software actually); Emily uses hardware; and Joe uses both. Are these individual differences? This leads us to 'uniqueness'.

Each of teachers has something unique in their use of other resources. For instance: only Katie has used a Chinese teacher as a source of ideas and uses GCSEpod.com; only James is recorded to have used blogs, GeoGebra, and Desmos in his lessons; only Emily makes use of Ttrockstars, code buster and King of Maths; only Joe uses Plickers, 10ticks, Socrative and virtual manipulatives. We also note with interest that the differences only exist with regard to digital electronic resources and we wonder whether this is the case in other European countries (as our perception is that 'The Mathematics Department' in English/UK high schools is a more homogeneous community of practice than it is in many European countries). Whatever the answer to our speculations, the uniqueness of each teacher seems to be related to the teacher's confidence in the use of digital resources and the use of those resources that help attend to the needs of the students in their engagement with mathematics.

NOTES

REFERENCES

Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education*, 3, 205–224.

Beeby, T., Burkhardt, H., & Fraser, R. (1979). *Systematic Classroom-analysis-notation for mathematics lessons*: SCAN. Shell Centre for Mathematical Education, University of Nottingham

Beere, J. (2012). *The Perfect Ofsted Lesson-revised and updated*. Crown House Publishing.

Bozkurt, G., & Ruthven, K. (2015, February). Expert and novice teachers' classroom practices in a technological environment. *In CERME 9-Ninth Congress of the European Society for Research in Mathematics Education* (pp. 2319-2325).

Creswell, J. W. (2013). *Qualitative inquiry and research design: Choosing among five approaches* (3rd Ed.). Thousand Oaks, CA: Sage

Engeström, Y. (1987). *Learning by Expanding: An Activity-theoretical Approach to Developmental Research*. Orienta-Konsultit, Helsinki.

Gueudet, G., & Trouche, L. (2009) towards new documentation systems for teachers? *Educational Studies in Mathematics*, *71*(3), 199-218.

Jones, K., & Edwards, R. (2005). *Planning for mathematics learning. Learning to Teach Mathematics in the Secondary School*: *A Companion to School Experience*, 93.

Monaghan, J., Trouche, L., & Borwein, J. M. (2016). *Tools and Mathematics*. Springer International Publishing.

Pepin, B., Gueudet, G., & Trouche, L. (2013). Re-sourcing teachers' work and interactions: a collective perspective on resources, their use and transformation. *ZDM*, 45(7), 929-943.

Remillard, J. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, 75(2), 211–246.

Stylianides, G. (2016). Curricular resources and classroom use: The case of mathematics. Oxford University Press.

i https://www.gcsepod.com/
ii http://www.resourceaholic.com/
iii https://www.tes.com/teaching-resources
iv http://www.10ticks.co.uk/
v https://www.desmos.com/
vi Some apps allow this too.

DOCUMENTATION EXPERTISE AND ITS DEVELOPMENT WITH DOCUMENTATION EXPERIENCE IN COLLECTIVES: A FRENCH CASE OF COLLECTIVE LESSON PEPRATION ON ALGORITHMIC

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With a background of great changes of curriculum reform in France, there comes the new challenge for both (1) the new teaching contents such as the new content of algorithmic and (2) their working way in collectives, such as inter-discipline teaching practice and three-year-cycles. In this study, we are interest to see how the mathematics teachers react to the new challenges from a resource view. The two notions of documentation experience and documentation expertise are used in a case study of Anna and Cindy in our research.

Keywords: resource, mathematics teachers, documentation work, documentation experience, documentation expertise

INTRODUCTION

From September 2016, the middle schools in France started to carry out the new curriculum program with a series of great changes: the curriculum is organized on three-year-cycles without details of what has to be done in each level of the cycle, besides, the last two grades in primary school (Grade 4 and Grade 5) and the first grade in lower secondary school (Grade 6) are put in cycle 3, which demands more cooperation and communication between teachers within the same school and the primary schools. The components of computer sciences, algorithmic and programing, firstly appeared in cycle 4, with the contents separated into mathematics and technology. To contribute to the idea of interdisciplinary teaching practice, each school has to choose at least three interdisciplinary teaching activities, thus for example, algorithmic and programing becomes a choice for mathematics teachers and technology teachers. Great changes also come with great challenges: the French middle school mathematics teachers never teach or be trained how to teach this topic before, lots of teaching resources needs to be developed; and cooperation between teachers from different disciplines needs to be tempered because although it is stated that “teaching is collaborating” in French Dictionary of Pedagogy (Buisson, 1911), most of teachers in France prefer working individually than collectively.

With an aim to explore how mathematics teachers react to new challenges, we situated our study in a time of French curriculum change, with a French middle school math teacher, Anna, and her collective work with her colleague, Cindy in a lesson preparation on algorithmic and programing as a case study. Without any teaching and training experience, to teach algorithmic and programing successfully, teachers have to integrate their available and potential resources. We assumed that there remains some expertise in teachers’ resources integration, which is closely linked to the resources accumulated from their experience in both learning and working. As cooperation among teachers is considered important for teacher professional development (Hargreaves, 1995), we pay a specific attention on the collective aspect. We are interested to see: How does teachers’ experience in collectives benefit their resource work when preparing a new lesson? How does teachers’ collective work contribute to their expertise in resource work?

In the following sections, we will start from our theoretical frameworks, introducing the concepts of *documentation experience* and *documentation expertise*, and then with a more precisely stated research questions, we proposed our research methodology and tools. In the fourth part, we will analysis the French case from documentation experience and documentation expertise. As a conclusion part, a summarization of the relationship between documentation experience and documentation expertise will be presented.

THEORETICAL FRAMEWORKS

Our study based on the *Documentational Approach to Didactics* (DAD) (Gueudet & Trouche, 2008). In this section, we will present our theoretical choices for analyzing collective dimension in teachers’ work with resources, with two new concepts, documentational experience and documentation expertise, for analyzing teachers work with resources.

With an interest on analyzing and understanding teacher professional development, we chose a view of resource: according to *DAD*, the first theory framework of our study. Professional development is defined as the results of an interrelated process of incorporating new resources to the teacher's work, the development of her knowledge for teaching and her relationships with other participants of this process. The interactions between teachers and resources, including restricting, selecting, modifying, adapting, saving and sharing off, are defined as *documentation work* in DAD, and *resource* could be anything with the potential to *re-source* teachers' activity (Adler, 2000), and the structured set of resources a teacher is working on/with is named her resource system. Along with teachers' integrating continuously new resources, there is a process of generating the schemes of developing a resource, incorporating the resource produced and its usage, thus defined as a *document* in DAD. Scheme is defined by Vergnaud (1998) as an invariant organization of activity for a given class of situations, comprising goals, rules of action, knowledge and possible inferences. The framework of DAD provides a resource-view on teachers' work, both individually and collectively, with the elements to be analyzed: resource, goal(s), rules of action, and knowledge.

While taking DAD as a framework to see the resource perspective on teachers' work, there also comes the question of the nature of teacher work: their work is neither isolated nor individual, but part of society, their documentation work is connected to others, and culturally and socially situated (Gueudet *et al.,* 2013). Thus to explore the factors of collective, we adapt two different frameworks to explore how do documentation experience in collectives nourish documentation expertise:

- The *thought collective* (Fleck, 1934), for analyzing teachers collective work over time, the broad definition of *thought collective* proposed exists when "two or more people are exchanging thoughts" (p. 44) and generates a *thought style* "characterized by standard features in the problems of interest to a thought collective, by the judgment which the thought collective considers evident, and by the methods which it applies as a means of cognition" (p. 99);
- *Activity theory* (Engeström, 1987), for understanding and analyzing teachers' collective activity with a structure of situating three components (subject, instruments, object) into the background collective (the role of the subject in the activity, the rule of the collective, and the labor division in the activity), which links the teachers' activity with the social and cultural elements of the collective where the activity occurs. It also echo DAD from its principles as (1) collective, artifact-mediated and object oriented; (2) multi-voicedness which calls on listening to the other members besides the subject in the activity; (3) historicity, in both the history of the collective and the history of teachers' activities in this collective; (4) the central role of contradictions as sources of change and development.

The open definition of collective of Fleck (1936) allow us to analyze collective work of different nature: formal or informal, regulated or not, stable or transit, required or voluntary, collaborative or cooperative, etc. We use the term cooperation and collaboration as Roschelle and Teasley (1995, p. 70): cooperation "is accomplished by the division of labor among participants" and collaboration "as the mutual engagement of participants in a coordinated effort to solve the problem together".

In teacher's day-to-day activity documentation work happens: when she decides to join a new collective, or meets new resources, or has to face a curriculum change, or has to take into account students with special needs. And in these activities teachers accumulate their *experience* working with resources. And we kept the definition of experience of Pastré (2005), which could be understood as the *accumulation of the past* as *the accumulation* and *appropriation of the past* by the subject. As we focus on the resource-aspect of working experience, we call this experience related to teachers' documentation work as *documentational experience*. For analyzing teacher's documentation experience accumulated in collective work we first, collective *thought style* and for that we proposed three points inspired in Fleck (1936) definition*:* collective's interest *standard features* (CSF) (targeted audience, objectives, missions, types of resources designed, etc.), collective's *judgment* (CJ) about mathematics and teaching mathematics (pedagogical assumptions, point of view about mathematics digital resources, institutional purposes for teaching mathematics, etc.)

and collective's *methods* (CM) for creating resources (functioning mode, member status, type of interaction, etc.). Still following the reflective investigation perspective, we use self-collective's description (charts, status...). After, we identify in Anna discus what she did in these collectives in relation to new curricular changes.

In a similar way, we proposed another new notion of *documentation expertise (DE)* as teacher's expertise in their documentation work. In Berliner's study (1988), expertise "is *specific to a domain*, and to particular contexts in domains, which is developed over hundreds and thousands of hours". Key elements of expertise are linked with teaching problems solving efficiently and creatively with a wide range of knowledge and experience (Sternberg & Horvath, 1995), or more precisely, teachers with "adaptive expertise" were proposed as "specialists in retrieving, organizing, utilizing, and reconsidering their professional knowledge and beliefs" (Avalos, 2011). Drawing from the definitions of expertise and documentation work, DE is defined preliminarily as the abilities and related knowledge to deal with the whole process of interacting (retrieving, selecting, organizing, modifying, adapting, creating and sharing off) with resources. It is considered as a developing resource-aspect state of teacher professional development.

In this way, we re-formulate our research questions as: for preparing a new lesson on algorithmic,

1. How does teachers' documentation experience accumulated in collectives benefit their resource work?
2. How does their collective work contribute their documentation expertise?

We hold a hypothesis that in front of new challenges, teachers will integrate the resources from their corresponding documentation experience into their current work, documentation expertise could be seen and developed through collective work by teachers' mutual interactions on documentation experience, such as conflicts, complementation, questions and answers.

METHODOLOGY

Based on DAD, our study mainly adapted the reflective investigation, which involves the teachers as part of the study throughout the whole data collection, with the four principles of "*long-term follow-up"*, "*in- and out-of-class follow-up"*, "*reflective follow-up*" and "*broad collection of the material resources"* (Gueudet *et al.,* 2013, p. 27). We will present our methodological design in three parts. First, we present our tools based in the principles of reflective investigation. After, we will present the collective putting in evidence the choice the concept of documentation-working mate.

Reflective investigation tools

Different from the traditional investigation, reflective investigation involves the teachers as part of the study throughout the whole data collection, which means the teachers are not only the data provider, but also the data creator. In this way we have two types of tools for following the teachers:

(1) To know how the teachers organize and represent their available resources, the tool of "*schematic representation of the resource system (SRRS)*" (ibid, P. 28) was kept, where a teacher is asked to draw a schematic representation of the structure of the resources she uses. To be noticed that this SRRS is not a final one, but improved, complemented, and reorganized continuously with the development of teachers' cognition on their resource system.

(2) Online *"Reflective Investigation Box (RI Box)"* was built and shared between us and the targeted teachers, in which the teachers could share their resources used during the activities (such as lesson plans, screenshots of blackboard writing etc.), and respond the questions (either about resources in RI Box, or any other questions) from the researcher regularly. Considering the using habits of the targeted teachers, we use Dropbox to support the RI box.

Documentation-working mate

To understand how teachers develop DE during their daily work, it is needed to situate the targeted teacher into a collective. Following the principle of "multi-voicedness" of AT, a new notion of *documentation-working mate* is proposed here as someone who works closely with the targeted teacher, with mutual influences on their documentation work and DE development. *Mate* in Oxford Dictionary infers "a fellow member of joint occupant of a specific thing, like table-mate", with an "origin related to meat (the underlying concept being that of eating together)" (see in https://en.oxforddictionaries.com/definition/mate) . The reason to choose "mate" but not "peer" as in "peer education" (Turner & Shepherd, 1999) is that "mate" breaks the boundary of age and education/professional background. For a given teacher, her documentation working could be a colleague with similar working experience in her school, or someone from a totally different working context as an university or research institute etc. In each case of this study, a *documentation-working mate* will be selected according to the targeted teacher: they form a smallest but closet collective, and the documentation-working mate will be followed in the same way as the targeted teacher.

With a broad meaning of "resource", a variety of information from the teachers is considered in our study: emails, CVs, published papers and articles, blogs etc. Observations and interviews are intertwined: while observing school activities (such as classroom teaching or teachers meetings), field notes were taken by the researcher.

A two step case study analysis

Designed as a case study, two mathematics teachers were selected from a same middle school in France, Anna as the main teacher, and Cindy as her documentation-working mate. Anna is selected because of her rich working experience in collectives, and she proposed Cindy as her documentation-working mate because: they are both experienced teachers (started to work together since 1990) and they both participate in several collectives; they have cooperated with each other in the same school over ten years (since 2006); they both work as part-time in education research collectives and they are willing to attend our research.

A two-step case study analysis includes a preliminary analysis from individual level, and a deep analysis from collective level:

-The preliminary study mainly focuses on the main teacher, Anna's resource work experience in collectives, as well the resources and resource habits gained in these collectives. We will identify the *thought style* of the collectives that Anna participated and how they support her preparation for new curricular changes;

-As for the deep analysis, we adapted video analysis on a collective lesson preparation between Anna and Cindy, from two dimensions: the documentation expertise shown in the collective work, and how it is developed by their interactions. Details will be presented in the following section.

CASE STUDY AND ANALYSIS

Graduated in 1989, Anna passed her CAPES (Certificate of Secondary Education Professional Qualification) exam in 1990, after one year's pre-service teacher education, she got her first position in a middle school of urban Paris, a "famous" school for the tricky problem students, till 1995. From 1995 to 2005, she worked in a middle school in Lyon. Then since 2005, she starts to work in the current middle school with three classes. And, we will present our analysis in two parties: first, we will analyze Anna documentational experience working in different collectives, and second we will analyze Anna documentation-working mate with Cindy.

Anna documentational experience nourishing itself of a plurality of collective

In this section, we will address the research question: *how does teachers' documentation experience accumulated in collectives benefit their resource work?*

We present our analyses about Anna collective works for preparing for new curricular changes in two parts: first, we will present her work in three regulated collectives, in which we could infer some elements about their thought style in their site; second, we will present her work in one informal collective, in which we follow Anna works inside it, them we infer they thought style.

We start for older regulated collective that she is member the SÉSAMES, in which teachers and researchers discussing Algebra. Anna met SÉSAMES coordinator, Sylvie Coppé, in her school and she invites Anna to join the collective. Anna hesitated, because Algebra was not her favorite notion to teach. However, she enjoyed this collective in 2006. For observing SÉSAMES site (http://pegame.ens-lyon.fr/) we could infer their though style as:

- Teachers and researchers thinking about resources for teaching Algebra and promoting teacher's training (CSF); mathematics for solving problems and teaching mathematics basing in activity of research by students; collaboratively and voluntary work between teachers and researchers, principles for creating resources (CM).

SÉSAMES is an important collective for Anna documentational experience, during more then 10 years of work she created a hug resource system to teach Algebra. And for the new curricular program, the members started to review all resources and think about some programs of calculus to be used in algorithmic. For example, one important resource that they will use for teaching is the *Mise en train* (MET). Resources MET are created for exploring one notion in a progressivity way, during many sections, that was applied in the fifteen first minutes of class lesson. This resource was used in others collectives: IREM, APMEP, ANNA-CINDY collective, among others.

In the national French institute of research, IREM, teachers and researchers research together how to improve teaching mathematics. And Anna uses their resources since she started to teach. And in SÉSAMES she met some members of this collective then she decided to join them too in 2010. For observing IREM site (http://math.univ-lyon1.fr/irem/) we could infer their thought style as:

- Articulated work between research and practice looking for diffusing research results and promoting teacher's training (CSF); mathematics in live and teaching mathematics malleable (CJ); collaboratively work for designing resources, not have specific principles for creating resources, but created resources with didactical advices (CM).

IREM have discussed teaching by competences since 2006 when the new common core was implemented in France. This common core is the base of new curricular change, then Anna discussed there how teaching by competences. Also she designs new activities to teach in the interdisciplinary way. And in this collective, Anna had access to many resources about teaching algorithmic exchanging with members.

In the teacher's professional association, APMEP, in which teachers working together for thinking about teaching mathematics. Anna also uses their resources (booklets with activities, articles in their site, etc.) since that she started to teach. And in 2012 she goes from simple consumer of resources to the member of the association. For observing APMEP site (http://www.apmep.fr/) we could infer they though style as:

- Gathering teachers teaching mathematics from pre-primary schools to University and promoting teacher's training (CSF); being a force of proposal for improving mathematics teaching and providing math teachers with rich resources (from a didactical and epistemological point of view) (CJ); working voluntarily and collaboratively without hierarchy, not having specific principles for creating resources, but created resources with didactical advices (CM).

Anna re-interprets and takes position about curricular changes in this collective. They have meeting in which they review all notions proposed in curricular program, in particular, they discuss about algorithmic and programming. In this collective, she shared in their site some resources that she found online for supporting teaching in the new reform.

The last collective that we will explore here is ANNA-CINDY collective, in which they always prepared their lesson together. We recorded their lesson preparation for teaching algorithmic and programming, and we use that for infer some elements about their tough style:

- Anna-Cindy collective though style: prepare their lesson together in a moment the exchange their experience about teaching (CSF); about algorithmic and programming they look for activities to teach algorithmic as thinking about mathematics, they did not want to teach as used one software (CJ); gathered all possible resources that may be interesting to use (many textbooks, sites, etc.); discuss them and take all decisions together collaboratively exchanging their experience (CM).

In this collective Anna prepare her activities for teaching this new content that will applied in class and shared her experience about lesson implementation. In the next section, we present more elements about resources designed in this lesson preparation.

Finally, in this section, we evidenced that Anna use collective work as one network for preparing her teaching in the new curricular program. Specially, about her documentation experience for teaching algorithmic and programming, she: interprets curriculum proposes in APMEP, discuss some activities for teaching Algebra in SÉSAMES, exchange activities in IREM and design her planning activities with Cindy.

A deep analysis on a collective lesson preparation of Anna and Cindy

In May 2016, a video of collective lesson preparation between Anna and Cindy on algorithmic was filmed, when it was the time to prepare their teaching plans for the following academic year, and decide the textbooks to be used. This collective lesson preparation lasted for one hour, which was proposed by Anna, with the reason that they were used to prepare lessons together. An email with three questions was sent before their collective work: What are the difficulties for teaching this topic? What resources do you have and lack of? Why do you prefer to work together?

The first transcription was shared with Anna through Google document, in which we marked our confusions and questions in the video, particularly the name of the resources that are unclear for us. Then with the second transcription, we discuss with Anna face-to-face, mainly on the source of the resource appeared in their collective lesson preparation. This section will analyze it from two dimensions: the documentation expertise (schemes) shown, and how it is developed in the collective work:

(1) Some schemes of resource work could be found in this collective activity:

The schemes of retrieving resources, which are also based on the schemes of resources management or storage. It could be an ability to make the use of the available resources. In the eyes of Cindy, Anna herself is already a kind of resource: “When I need some resources, Anna is always the first choice”. Her documentation expertise could be traced back to her online working habit, in both the organization and preparation: with various high qualified website resources, she has Google drive to shared documents and agendas with her colleagues in SESAMES team, Dropbox with her colleagues in her school. Meanwhile she stored her personal resources in Dropbox and Evernote, in which the documents are classified by the name of different collectives and projects. She has also some online platform like Pixies and Viaéduc to collect and store her favorite resources so that when she needs some resources she could find them easily.

The schemes of selecting resources, which rely on the understanding on activity goals, related concepts, and their teaching practices. For Anna, she is clear that the first lesson preparation of algorithmic should be an introduction with some activities. She has her own understanding of algorithmic, which is different from the explanations in the official program, and this is the basic for her critical thinking on the official resources and the suggestions from the inspectors. The critical thinking in resources selecting also relies on the confidence and proficient knowledge about their teaching practice, for example, when Anna and Cindy were reading the goals of algorithmic in a textbook, they doubted that the goals written (“encourage the students to understand the variables…”) impossible, because “it is a notion in information”, so “it is better to change the name”.

The schemes of modifying and adapting resources need the teacher’s understanding of the situation requirements, and technology skills. Such trends appear more obvious on Anna, she has no personal office space, so she has to take her laptop all the day. According to an interview, as a mathematics teacher in middle school, Anna does not need to learn very complex software, and her first big challenge was the whiteboard when her school equipped it in each classroom, and she had to learn how to use it, which cost her almost one year. She explained happily that her students learned much quicker and often assisted her. This is also an open mind or a kind of curious towards new things, and new changes.

The schemes of resources sharing, which is not a spirit of contributing others, but an efficient way of mutual benefiting. Taking Anna and Cindy as an example, Cindy used to say directly that when she had some problems in searching information and resources, she will turn to Anna and she always got her answer. And also from the observation of their school meetings or co-training in service teachers, Cindy seems to be strong to propose her ideas, comments, and suggestions in a clear and reasonable way. The sharing off of resources is not only an action of throwing the resources into the common area, but a carefully maintained, regular refreshed and re-organized, just like the common folder named "le cours" shared among Anna, Cindy and other mathematic colleagues.

Besides, for both Anna and Cindy, there is a kind of flexibility in integrating resources into different roles. This could be evidenced by their presentation on the difficulties and available resources at the beginning. They both traced back from teaching experience to training experience, and then to their learning experience in university and students' activities they ever organized. This kind of scheme is not only integrating resources from different sources (or collectives) to their current task, but also their current work could be the future resources for other tasks. For example, when teaching the "line, segment and half-line" chapter, she arranged several drawing exercises, which come from IREM website, as classroom exercises for students. Then she collected and took pictures of their work, and uploaded to the school webpage for inter-discipline students' masterpieces; she also adapts these as examples to her teacher-training work, shared and discussed with other teachers.

(2) Seeing from the dialogues between Anna and Cindy, their interactions were classified in three types: disagree and conflicts, agreements and complements, questions and answers. The interactions are considered as a mutual way in developing documentation expertise:

The conflicts could be seen when they hold different ideas, but between Anna and Cindy, there are not obvious or strong conflicts seeing from their dialogues. Taking their first ideas about the Scratch as an example: At the beginning, Cindy seems to agree with teaching Scratch according to the suggestions in the program, when she heard the word "but…" from Anna, she tried to remind her that the inspectors also suggested to use Scratch. When Anna explained that algorithmic should be a kind of thinking rather than using a software usage, Cindy seems to change her ideas, she reacted with "Hum" and "Yes". Later she commented that almost the whole activities suggested in the program are centered with Scratch, then Anna re-stated her idea that she do not want to teach Scratch. However, in the end, they decided to arrange a computer lesson for the students to let them explore Scratch. This could be seen as a process in exchanging their ideas, and influences on each other.

There are more agreements and complementation in this collective work, and they appeared more tacit agreement when Anna and Cindy were discussing the textbooks, they read textbooks on their own, they had their division for these 13 textbooks, and they shared the valuable parts, and exchanged he doubtful points.

In this collective work, the questions and answers happened when one teacher did not know something, and the other explained it. Anna plays more roles on solving the questions proposed by Cindy. For example, when reading the Sésamath textbook, Cindy asked Anna "How do you understand 'some languages are not used in a declared way'?", Anna proposed an example of Python, the equal ("=") is not the equal that we know normally, "it is specific, but it has a different meaning". Also, when Cindy proposed the "idea of dance" in the document of creative computing, she also explained the source and author of the document, it is the first time heard by Anna, but she learned this after it is explained.

DISCUSSION

After a two-step analysis on Anna's documentation experience in collectives and her documentation expertise shown and developed in collective work with Cindy, we find some answers to our research questions: documentation experience in collectives for the teachers not only bring resources to them, but also some thought styles that guide resource design; she takes resources from one collective to another, for example, the resource MET; while in new collective documentation work, ANNA-CINDY collective develop their documentation expertise through trying to adapt those resources and thought styles into their new work. When exploring an individual teacher's professional develop, it is needed to situate her/his work in not only her personal documentation work, but also her documentation experience in collectives, because interactions within collectives could be a crucial way of learning, and developing her documentation expertise.

REFERENCES

Adler, J. (2000). Conceptualizing resources as a theme for teacher education. *Journal of Mathematics Teacher Education, 3*, 205–224.

Avalos, B. (2011). Teacher professional development in teaching and teacher education over ten years. *Teaching and Teacher Education, 27*(1), 10-20.

Buisson, F. (Ed.) (1911). *Nouveau Dictionnaire de Pédagogie et d'instruction primaire*. Paris: Hachette. Retreived in September 2015 at: http://www.inrp.fr/edition-electronique/lodel/dictionnaire-ferdinand-buisson/

Berliner, D.C. (1988). Describing the behaviors and documenting the accomplishments of expert teachers. *Bulletin of Science Technology and Society*, *24*(3), 200-212.

Engeström, Y. (1987). *Learning by expanding. An activity-theoretical approach to developmental research*. Helsinki: Orienta-Konsultit.

Fleck, L. (1981). *Genesis and Development of a Scientific* Fact. Chicago: University of Chicago Press (original edition, 1934).

Gueudet, G., & Trouche, L. (2008). Du travail documentaire des enseignants: genèses, collectifs, communautés. Le cas des mathématiques. *Education et didactique*, 2(3), 7-33.

Gueudet, G., Pepin, B., & Trouche, L. (2013). Collective work with resources: an essential dimension for teacher documentation. *ZDM - The International Journal on Mathematics Education, 45*(7), 1003-1016.

Hargreaves, A. (1995). Beyond Collaboration: Critical Teacher Development in the Postmodern Age. In J. Smyth (Ed.), *Critical Discourse on Teacher Development* (pp. 149-173). London: Cassell,

Roschelle, J., & Teasley S. D. (1995). The construction of shared knowledge in collaborative problem solving, In O'Malley C. (dir.), *CSCL* (pp. 69-97). Springer-Verlag.

Pastré, P. (2005). Genèse et identité. In P. Rabardel, & P. Pastré (Eds), *Modèles du sujet pour la conception: dialectique activités développement* (pp. 231-260). Toulouse: Octarès.

Sternberg, R. J. & Horvath, J. A. (1995). A prototype view of expert teaching. *Educational Researcher, 24*(6), 9-17.

Turner, G. & Shepherd, J. (1999). A method in search of a theory: peer education and health promotion[J]. *Health Education Researc*h, 14 (2), 235-247.

Vergnaud, G. (1998). Toward a cognitive theory of practice. In A. Sierpinska, & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 227–241). Dordrecht: Kluwer.

IN SEARCH FOR STANDARDS: TEACHING MATHEMATICS IN TECHNOLOGICAL ENVIRONMENT

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In the literature review on teacher education aimed at integrating technology to mathematics lessons we presented in the topic study group 43 at the 13th ICME conference, we brought to the fore a lack of standards for teaching mathematics with technology, which is, in our view, an issue for the actors of teacher education. In this paper, we tackle this issue by presenting existing ICT standards at the international and national levels and analyse them through the lenses of TPACK model and double instrumental genesis. We argue that the existing standards are too general, as they are neither school level, nor subject matter specific. We call the mathematics education community to take this issue in consideration.

Keywords: Mathematics teachers' knowledge for ICT; Standards for teaching with ICT; TPACK model, Double instrumental genesis

INTRODUCTION

Teacher education was one of the four central themes discussed at ICME 13 congress in the Topic Study Group 43, *Uses of technology in upper secondary mathematics education (age 14 to 19)*. In our contribution to this theme (Tabach & Trgalová, 2016), we noticed, in a number of research papers, a disappointment with the outcomes of teacher education programs. The gap between teachers' needs and the teacher education contents is deemed as the main reason. This brings to the fore a necessity for teacher educators to understand better what teachers need to know in order to use efficiently ICT, which raises the issue of ICT competency standards. We thus searched for an institutional framework regarding teachers' knowledge for teaching mathematics with technology. Surprisingly, we could hardly find such standards for mathematics teachers or even for teachers in general. Therefore, we recommended that "*Elaboration of ICT standards for mathematics teacher education might become one of the goals of the mathematics education international community*" (Hegedus at el., 2016, p. 30).

In this paper, we aim to expand on the issue of standards that we consider crucial for teacher education: standards aiming at teachers, and specifically mathematics teachers working with ICT. We relate to both international and national levels.

THEORETICAL PERSPECTIVE

Several researchers suggested theoretical frameworks for examining and analyzing teacher knowledge in general. In theorizing about the unique knowledge needed for teaching with digital technology, Mishra and Koehler (2006) introduced the concept of Technological Pedagogical Content Knowledge (TPCK or TPACK): the knowledge and skills teachers need to meaningfully integrate technology into instruction in specific content areas.

The notion of TPCK emerged from Shulman's (1986) construct of pedagogical content knowledge (PCK). Shulman rejected the view of Content knowledge (CK) and Pedagogical knowledge (PK) as two distinct bodies of knowledge, and suggested a partial overlap between them. This overlap implies a unique type of knowledge, specific for teachers, PCK. Along similar line of thoughts, Mishra and Koehler (2006) suggested an additional body of knowledge, Technological, which partially overlaps CK and PK. The resulted image of teachers' knowledge is captured in Fig. 1, and includes seven bodies of knowledge.

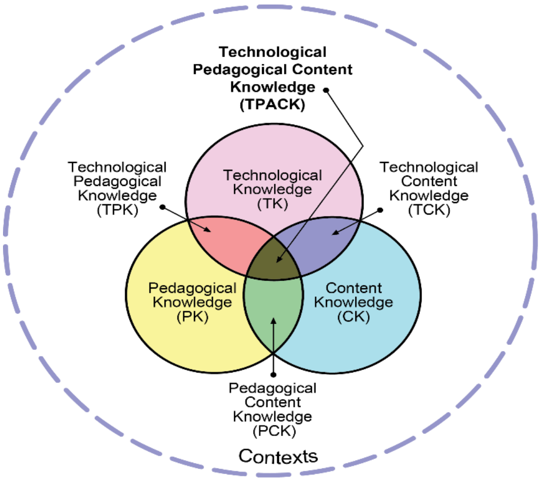

Figure 1: TPACK (with permission from TPACK.org)

The TPACK framework was used by many researchers and several different interpretations are currently accepted (Voogt et al., 2012): T(PCK) as extended PCK; TPCK as a unique and distinct body of knowledge; and TP(A)CK as the interplay between three domains of knowledge and their intersections. In the current paper we adopt the latter view.

The theoretical construct of *double instrumental genesis* (Haspekian, 2011) brings forth the implication of both personal and professional instrumental geneses in teachers using ICT. While the first is related to the development, from a given artefact, of a teacher's personal instrument for mathematical activity, the second results in a professional instrument for her didactical activity. These two processes mobilize knowledge of the artefact (TK), ability to solve mathematical problems using it (TCK), to orchestrate ICT-supported learning situations (TPK) and to teach mathematics with ICT (TPACK).

METHODS

In this paper, we review institutional documents in an attempt to answer the following questions: What knowledge standards are set for teachers working in technological environments? What are the specificities for mathematics teachers that are unique to this sub-group of teachers?

Two types of data sources were available for us. At the international level, we searched the web for organizations that published documents on the subject. We found the UNESCO ICT Competency Framework for Teachers (2011) and the International Society for Technology in Education (ISTE[1]) Standards-T (2008), both sources relate to teachers in general. At the national level, we consider the NCTM (2001) from the US, specific for teaching mathematics, yet focused mainly on standards for learning, as well as available documents from France and Israel to have a wider national perspective.

While reading each of the data sources, we tried to relate them to one of the four knowledge areas that pertain to technology, as reflected by the TPACK framework.

[1] International Society for Technology in Education, http://iste.org

FINDINGS

ICT standards around the world

UNESCO ICT Competency Framework for Teachers (ICT-CFT) (2011) sets out "*the competencies required to teach effectively with ICT*" (p. 3). The framework stresses that

> "it is not enough for teachers to have ICT competencies and be able to teach them to their students. Teachers need to be able to help the students become collaborative, problem solving, creative learners through using ICT so they will be effective citizens and members of the workforce" (ibid.).

The Framework is therefore organized in three different approaches to teaching corresponding to three stages of ICT integration. The first is Technology Literacy "*enabling students to use ICT in order to learn more efficiently*", the second is Knowledge Deepening "*enabling students to acquire in-depth knowledge of their school subjects and apply it to complex, real-world problems*" and the third is Knowledge Creation "*enabling students, citizens and the workforce they become, to create the new knowledge required for more harmonious, fulfilling and prosperous societies*" (p. 3). It is interesting to note that these stages are formulated in terms of students' abilities to exploit the ICT potential as a result of the ways teachers use ICT. All aspects of teachers' work, namely understanding ICT in education, curriculum and assessment, pedagogy, ICT, organization and administration, and teacher professional learning, are addressed at the three stages (Fig. 2).

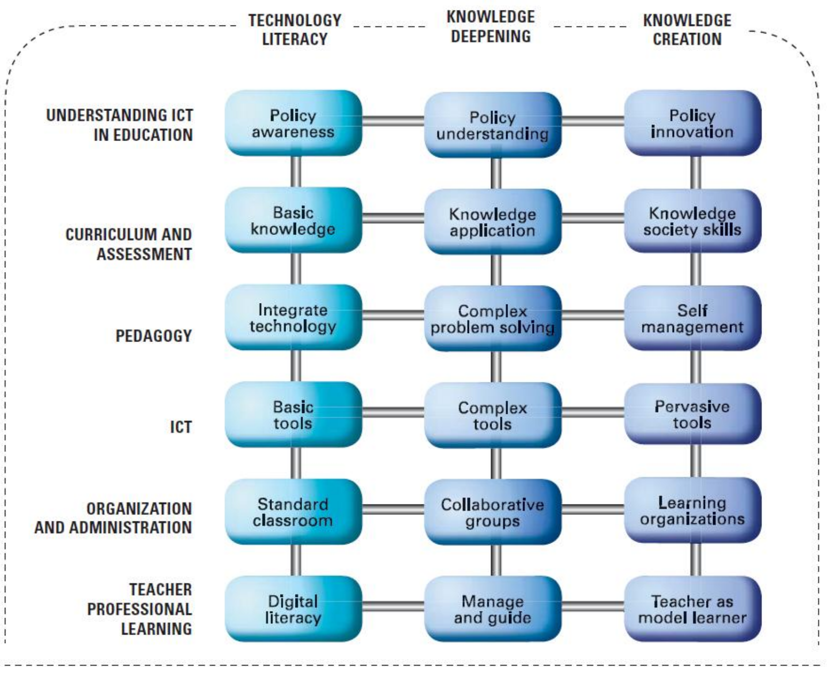

Figure 2. The UNESCO ICT Competency Framework for Teachers (UNESCO, 2011, p. 13)

The authors of the UNESCO framework claim that

> "[t]he successful integration of ICT into the classroom will depend on the ability of teachers to structure the learning environment in new ways, to merge new technology with a new pedagogy, to develop socially active classrooms, encouraging co-operative interaction, collaborative learning and group work. This requires a different set of classroom management skills. The teaching skills of the future will include the ability to develop innovative ways of using technology to enhance the learning environment, and to encourage technology literacy, knowledge deepening and knowledge creation" (ibid., p. 8).

The Framework specifies competencies teachers need in all aspects of their work. At the level of Technology Literacy,

> “teacher competences […] include basic digital literacy skills and digital citizenship, along with the ability to select and use appropriate off–the-shelf educational tutorials, games, drill-and-practice software, and web content in computer laboratories or with limited classroom facilities to complement standard curriculum objectives, assessment approaches, unit plans, and didactic teaching methods. Teachers must also be able to use ICT to manage classroom data and support their own professional learning.” (ibid., p. 10).

Referring to the TPACK model, we may consider “basic digital literacy” as part of TK and the ability to select appropriate resources to “complement […] standard didactic teaching methods” as part of TPACK. TPK and TCK are mentioned together with the TPACK at the further level, Knowledge Deepening:

> “teacher competences […] include the ability to manage information, structure problem tasks, and integrate open-ended software tools and subject-specific applications [TCK] with student-centred teaching methods and collaborative projects in support of students’ in-depth understanding of key concepts [TPACK] and their application to complex, real-world problems. To support collaborative projects, teachers should use networked and web-based resources to help students collaborate, access information [TPK], and communicate with external experts to analyze and solve their selected problems. Teachers should also be able to use ICT to create and monitor individual and group student project plans, as well as to access information and experts and collaborate with other teachers to support their own professional learning’ (ibid., p. 11).

Finally, at the level of Knowledge Creation, teachers

> “will be able to design ICT-based learning resources and environments; use ICT to support the development of knowledge creation and the critical thinking skills of students [TPACK]; support students’ continuous, reflective learning [TPK]; and create knowledge communities for students and colleagues” (ibid., p. 14).

The UNESCO document provides examples of syllabi for teacher education that demonstrate ways how to operationalize the ICT competency framework. In the Table 1, a few examples of tasks suggested in the syllabi at the three levels, technology literacy (TL), knowledge deepening (KD) and knowledge creation (KC), of teachers’ competencies are given, organized according to the TPACK model and the double instrumental genesis concept.

These examples of teachers’ competencies show that the UNESCO ICT framework takes into account both teacher’s personal and professional ICT knowledge and skills, although the first are only present at the TL and KD levels, the teachers at the KC level being certainly considered as having a sufficient personal mastery of technology. All technology-related categories of the TPACK model are present, although the TPACK itself is not specific to whatever subject matter.

	Personal instrumental genesis	Professional instrumental genesis
Teachers should be able to…	TL - Describe the purpose and basic function of graphics software and use a graphics software package to create a simple graphic display (TK)	TL - Identify the appropriate and inappropriate social arrangements for using various technologies (TPK)
	TL - Use common communication and collaboration technologies, such as text messaging, video conferencing, and web-based collaboration and social environments (TK)	TL - Match specific curriculum standards to particular software packages and computer applications and describe how these standards are supported by these applications (TCK)
	TL - Use ICT resources to support their own acquisition of subject matter and pedagogical knowledge (TCK, TPK)	TL - Incorporate appropriate ICT activities into lesson plans so as to support students' acquisition of school subject matter knowledge (TPACK)
	KD - Identify or design complex, real-world problems and structure them in a way that incorporates key subject matter concepts and serves as the basis for student projects (TCK)	KD - Structure unit plans and classroom activities so that open-ended tools and subject-specific applications will support students in their reasoning with, talking about, and use of key subject matter concepts and processes while they collaborate to solve complex problems (TPACK)
	KD - Operate various open-ended software packages appropriate to their subject matter area, such as visualization, data analysis, role-play simulations, and online references (TCK)	KC - Help students reflect on their own learning (TPK)

Table 1. Examples of teachers' competencies mentioned in the UNESCO ICT framework.

The ISTE Standards-T (2008) define five skills teachers "*need to teach, work and learn in the digital age*":

> (1) "Teachers use their knowledge of subject matter, teaching and learning, and technology to facilitate experiences that advance student learning, creativity, and innovation", (2) "Teachers design, develop, and evaluate authentic learning experiences and assessments incorporating contemporary tools and resources", (3) "Teachers exhibit knowledge, skills, and work processes representative of an innovative professional", (4) Teachers [...] exhibit legal and ethical behavior in their professional practices", and (5) "Teachers continuously improve their professional practice […], exhibit leadership in their school and professional community by promoting and demonstrating the effective use of digital tools and resources".

These skills are rather general and relate to various aspects of teacher profession. These skills do not relate to TK per-se. It seems that in these standards, teachers' TK is taken as a starting point. Also, as the standards are not subject specific, they do not relate to TCK. In fact, this set of skills is about TPK. Note that the standards encompass various aspects of teacher's profession – designing, teaching, evaluating, leading their peers in school and in their professional community, as well as legal behavior. A hint that some adaptation to the content taught is needed can be found in the beginning – "*Teachers use their knowledge of subject matter…*". Yet, it is not directly conveying that adaptation of these skills to different content areas within K-12, namely focusing on TPACK, may yield different results for different subject matters.

To summarize, at the international level the standards mostly aim at teachers in general, with no specific adaptation to any school subject. As a result, the documents refer to teachers' TPK, rather than TCK or TPACK, which are only referred to by evoking “didactic teaching methods” or “support of students’ in-depth understanding of key concepts”.

ICT standards at the national level

The National Council of Teachers of Mathematics (NCTM) organization published document statement and positions aimed to the US, yet these documents are influential beyond the national level. In many countries, they serve as model for the national documentation. The NCTM relates specifically to mathematics teachers, as can be viewed from an explicit relation to mathematics, as well as to digital tools specific to mathematics.

NCTM (2011) claims that

> “Programs in teacher education and professional development must continually update practitioners’ knowledge of technology and its application to support learning. This work with practitioners should include the development of mathematics lessons that take advantage of technology-rich environments and the integration of digital tools in daily instruction, instilling an appreciation for the power of technology and its potential impact on students’ understanding and use of mathematics.”

The NCTM position toward technology in mathematics education emphasizes three conditions for an efficient integration of technology, which should guide the development of teacher education programs: teachers’ awareness of the technology added value in terms of students’ understanding of mathematics, which is about TPACK; teachers’ continuous upgrading of their knowledge of technology and its use in teaching, which relates both to teachers TK and TPK; and designing teaching resources taking advantage of affordances of digital tools, which is about TPACK.

In a position statement from 2015 the NCTM further stated that

> "Effective teachers optimize the potential of technology to develop students' understanding, stimulate their interest, and increase their proficiency in mathematics. When teachers use technology strategically, they can provide greater access to mathematics for all students."

The document further relates to particular technologies to be used, from mathematical and non-mathematical domains:

> "Content-specific mathematics technologies include computer algebra systems; dynamic geometry environments; interactive applets; handheld computation, data collection, and analysis devices; and computer-based applications. Content-neutral technologies include communication and collaboration tools, adaptive technologies, and Web-based digital media."

Teachers are viewed as orchestrators and coaches of strategic use, and their leading consideration should stem from the mathematics they are teaching. Technology is used at the service of mathematics. Although not specifically stated, it seems that for the NCTM, TCK, TPK and TK all play central role in the knowledge teachers need to have in order to teach with ICT. This impression is enhanced by the fact that in most of the publication, ICT appears at the background rather than up front.

The situation in Israel is quite different in terms of teachers' standards for teaching in ICT environment in general, and for mathematics teachers in particular. At the national level of preservice teacher education, the only reference is made to the 21th century skills in general. In other words, they refer to TK which is expected from all citizens and are not particular to teachers. At the mathematics education level, again there are no particular standards as to what do mathematics teachers need to know. This is not typical, as the Israeli ministry of education is very centralistic in its approach.

France was, until 2014, one of the European countries in which a certificate of digital skills, called "certificate of computer science and Internet", was required to become a primary or a secondary teacher. Since 2014, this certification is integrated in the preservice teacher education. This certification was created in 2010 to vouch for professional skills in the pedagogical use of digital technologies, common and necessary to all teachers and trainers for the exercise of their profession. National standards of competencies related to the certification comprise two main parts: (A) general skills related to the exercise of the profession, and (B) skills needed for ICT integration into the teaching practice. The general skills (part A) are organized in three domains: "A1 - mastery of professional digital environment" (e.g., select and use the most appropriate tools to communicate with the actors and users of the education system), "A2 - development of skills for lifelong learning" (e.g., use online resources or distance learning devices for self-training), and "A3 - professional responsibility in the education system" (e.g., take into account the laws and requirements for professional use of ICT). The skills for ICT integration are classified in four domains: "B1 - Networking with the use of collaborative tools" (e.g., search, produce, index, and share documents, information, resources in a digital environment), "B2 - design and preparation of teaching content and learning situations" (e.g., design learning and assessment situations using software that is general or specific to the subject matter, field and school level), "B3 – pedagogical enactment" (e.g., manage diverse learning situations by taking advantage of the potential of ICT (group, individual, small groups work), and "B4 - implementation of assessment techniques" (e.g., use assessment and pedagogical monitoring tools). While the skills from the part A refer mostly to TK, those from the part B refer to TCK, TPK and TPACK. Numerous intersections can be found between the French national and UNESCO international standards, mainly in considering various aspects of teachers' profession, not only reducing it to their classroom activity, leading to taking into account both personal and professional mastery of ICT. Like the other standards presented above, the French ones are common to all teachers, whatever their school level and the subject matter taught.

CONCLUSION

In the current paper, we asked two connected questions: What knowledge standards are set for teachers working in technological environments? What are the specificities for mathematics teachers that are unique to this sub-group of teachers? To answer the two questions, we searched for institutional documents, both at the world-wide level and at the national level. In the findings section we detailed our analysis of the few documents we found, through the lenses of the TPACK framework for teachers' knowledge and the double instrumental genesis concept. We found a document composed by the UNESCO with elaborated ICT standards for teachers in general – regardless of the subject matter or grade level. The second document was composed by the International Society for Technology in Education organization, again at the general level. We were surprised to find only these two documents. We would like to point out that the two documents did not address any specific grade level – as if the standards for teaching in an ICT environment at any grade level were the same. Also, the documents did not address any specific subject matter, or did not suggest that particular adaptations are needed for teaching various school subjects.

At the national level we searched for documentation from three countries –US, France and Israel. There are profound differences between the three countries in terms of national level standards for teaching with ICT as well as some striking similarities: like at the international level, both in Israel and France the reference is made only to teaching in general, with no relation to specific age level or subject domain. Yet, while in Israel there is some reference to 21th ICT skills needed for any citizen, centering on TK, in France we saw awareness to both the personal TK as well as professional knowledge needed for teaching, in line with the double instrumental approach. The findings from the US are different in the sense that the standards specifically aim at teaching mathematics. Indeed, the analysis shows that these standards relate to all types of TPACK knowledge. However, they lack specifications.

We are currently at a time of change in terms of teachers' technological knowledge – the newcomers to this profession are expected to be more skillful at the personal level than the veterans. Yet, we think that teachers' mastery of ICT, both in terms of TK and TCK should not be taken for granted. Rather, this personal level in the double instrumental genesis should be addressed by standards. Moreover, we call for much more elaborated sets of standards for teaching in ICT for different age groups and school subjects, to allow for the professional level of instrumental genesis to be promoted.

REFERENCES

Haspekian, M. (2011). The co-construction of a mathematical and a didactical instrument. In M. Pytlak, E. Swoboda, & T. Rowland (Eds.), *Proceedings of the seventh Congress of the European Society for Research in Mathematics Education. CERME 7*, Rzeszow.

Hegedus, S., Laborde, C., Brady, C., Dalton, S., Siller, H. S., Tabach, M., Trgalova, J., & Moreno-Armella, L. (2016). *Uses of Technology in Upper Secondary Mathematics Education*. Springer.

ISTE (2008). *ISTE Standards for Teachers*. Online http://www.iste.org/standards/standards/standards-for-teachers [Retrieved February 20, 2017].

NCTM (2011). Technology in Teaching and Learning Mathematics. A Position of the National Council of *Teachers of Mathematics*. Online http://www.nctm.org/Standards-and-Positions/Position-Statements/Technology-in-Teaching-and-Learning-Mathematics/ [Retrieved September 17, 2015]

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for integrating technology in teacher knowledge. *Teacher College Records, 108*(6), 1017-1054.

Shulman, L. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Tabach, M., & Trgalová, J. (2016). Teacher education: what, how and why? *Invited paper presented in the TSG43 at the ICME13 congress*, July 2016, Hamburg.

UNESCO (2011). *UNESCO ICT competency Framework for Teachers*. Online http://unesdoc.unesco.org/images/0021/002134/213475E.pdf [Retrieved February 20, 2017].

Voogt, J., Fisser, P., Pareja Roblin, N., Tondeur, J. & van Braak, J. (2012). *Technological pedagogical content knowledge – a review of the literature*. *Journal of Computer Assisted Learning*, 29(2), 109-206.

Workshops

MATHMAGIC: THE ENCOUNTER BETWEEN COMPUTATIONAL AND MATHEMATICAL THINKING

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Computer Science is becoming more present in the curricula of the schools. Some countries already have changed their programs to offer subjects like algorithms, logic and programming languages since the beginning of early cycles of fundamental education. However, the contents are strongly related with mathematics, and for this reason, mathematics teachers need to update themselves to be able to teach the fundamentals of computer science. For this reason, we offer on this workshop an introduction to the development of computational thinking from a mathematical trick with cards. At the end of the workshop, the participants will be able to write their own code, solving a mathematical problem.

Keywords: computer science; mathematics; algorithms; computational thinking; mathematical thinking;

THE PROBLEM

The problem to be solved during the Workshop will be to model a magic trick from the Figure below:

3	7	23	19
17	9	11	31
5	27	21	13
1	25	29	15

6	5	20	14
28	23	15	13
31	29	21	22
4	30	7	12

14	10	15	31
11	27	29	13
26	28	9	12
8	30	25	24

14	26	10	27
3	30	15	18
31	6	19	23
2	7	11	22

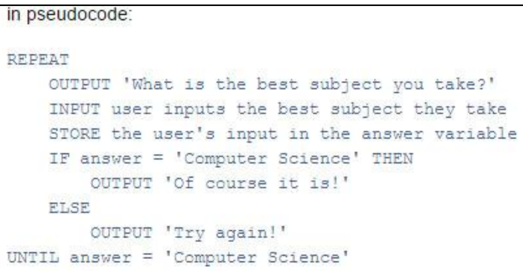
17	23	19	25
27	30	29	18
22	21	26	24
16	28	31	20

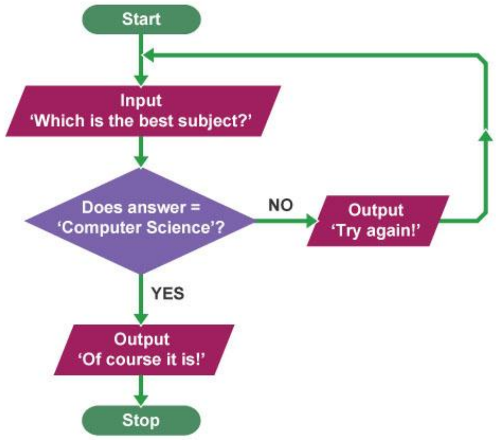
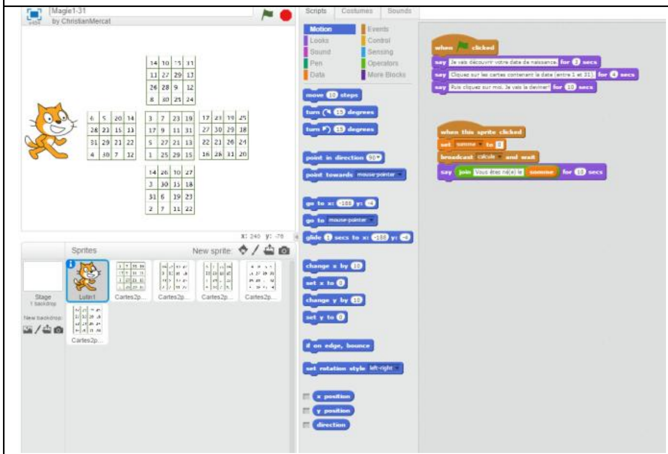
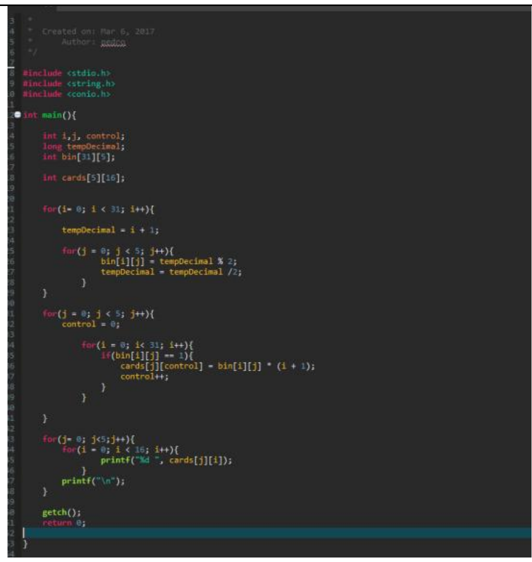
Figure 1: Cards

The magical trick is to guess a number between 1 and 31 indicating in which card the number is present. For instance, if one choses the cards: 1, 2, 3 and 5 in the previous order, it's possible to say that the chosen number was 29. Of course, I won't reveal the trick here. This is one of the exercises of the Workshop. Also, the participants will learn:

- Writing an algorithm in pseudocode: Describing the thoughts and steps to solve a problem in natural language, explaining the reasoning to achieve the solution;
- Creating an algorithm in visual language (Scratch): Understand the scope of a computer program, create variables, methods and operations and printing the result. At the end of this stage the participant will have the game in a digital version.
- Creating a program in programming language (Python): Going a little bit deeper the participant will be able to create the algorithm to generate the cards used in the magical trick writing down a Python program.

Then, is expected that participants achieve the following levels:

<table>
<tr>
<td>
in pseudocode:

REPEAT
OUTPUT 'What is the best subject you take?'
INPUT user inputs the best subject they take
STORE the user's input in the answer variable
IF answer = 'Computer Science' THEN
OUTPUT 'Of course it is!'
ELSE
OUTPUT 'Try again!'
UNTIL answer = 'Computer Science'</td>
<td>
</td>
</tr>
<tr>
<td>Pseudocode – Level 1</td>
<td>Flowcharts – Level 2</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Visual Programming – Model – Level 3</td>
<td>Coding – Abstraction and Generalization – Level 4</td>
</tr>
</table>

The participants shall bring their own computers and the time expected is about 4 hours.

REFERENCES

images:

http://www.bbc.co.uk/education/guides/z3bq7ty/revision/2

https://scratch.mit.edu/projects/111975655/

MOVING, COMPARING, TRANSFORMING GRAPHS: A BODILY APPROACH TO FUNCTIONS

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The workshop aims at presenting and discussing activities in which graphing motion technology named WiiGraph is used. The activities offer possible lines of (inter)action within the classroom to introduce discourses about the concept of function, graph sense, transformations, all through modelling motion. These lines might be followed at different school levels subjected to suitable task design (for example, we are going to carry out similar activities with grade 4, grade 7 and grade 10 students). The software allows for working with graphs of many different types. It leverages two remote controllers of the Nintendo Wii to detect and graphically display the location of two users as they move along lifesize number lines. Embodied interactions with the software are the ground for gaining insights into temporo-spatial mathematical relationships and covariational reasoning. We will discuss these aspects in relation to task design.

Keywords: Functional relationships; Graphs; Movement; Time; Task design

A BODILY APPROACH TO FUNCTIONS

Graphing motion activities have been largely investigated in mathematics education research since the 90s, through the use of motion sensor and other technology (e.g. Nemirovsky et al. 1998; Yerushalmy & Shternberg, 2005; Radford, 2009). Researchers have been studying the ways in which the interaction with this kind of tools may stimulate mathematical thinking while taking advantage of perceptuomotor activity. Even though different researchers have had different conceptions of function, these studies generally share the vision of covariation as a foundation for function in mathematics (see Thompson & Carlson, 2017). Our focus here is on highlighting features of graphing motion activities with a specific technology: a new software application, named WiiGraph [1]. This technology allows for the creation of different types of graphs while two users are moving each a controller of the Nintendo Wii (Wii Remote or Wiimote). Initially, we used it with the idea of introducing secondary school students to variational and covariational reasoning. Drawing on Nemirovsky and colleagues (2013), WiiGraph is a mathematical instrument, that is, "a material and semiotic device together with a set of embodied practices that enable the user to produce, transform, or elaborate on expressive forms (e.g., graphs, equations, diagrams, or mathematical talk) that are acknowledged within the culture of mathematics." (p. 376). Implicating movements of the controllers by two people in an interaction space, activities with WiiGraph also implicate bodily proprioceptive and kinaesthetic experiences both with the devices in use and with the graphical lines and symbolic operations provided by the technology. Nemirovsky et al. (2013) unfold the powerful idea of mathematical instrument to speak about fluent use and mathematical expertise as inseparable from perceptual and motor aspects implied in the activity with the tool. While these researchers are interested in studying fluency with the instrument in the informal context of a scientific exhibition, we focus on the more formal context of the mathematics classroom. In the design of tasks and intervention that we propose in the workshop, the vision of Nemirovsky and colleagues helped us, as researchers and educators, to draw attention to the kind of engagement and practices that activity with the technology might favour within the classroom (e.g., strategic thinking, competitive and collaborative dynamics, use of material resources, etc.; see e.g. Ferrara & Ferrari, 2015a; Ferrara & Ferrari, 2016). We centre on these aspects as a way of discussing challenging lines of flight on covariation, function and families of functions and the issue of designing activities for students from the early years to secondary school. Attending closely to the perceptual and motoric aspects of the experiences, we are interested in offering insights into the ways in which creating and thinking of graphs and functions might change through these experiences and into the new meanings that might emerge from the activities.

WIIGRAPH

To the aim stated above, we focus here on two main options of WiiGraph: *Line* and *Versus*. *Line* furnishes in real time two position-time graphs that capture the distance of the Wiimotes over time from a sensor (origin of the reference system). Time and spatial ranges can be set and modified for the Cartesian axes. This in turn implies specific time interval for the motion to be performed (e.g. 30 seconds) and space constraints for the two users' movements (e.g. at most 10 feet far from the sensor) within the interaction space. Labelled a and b the two distances, the software displays the lines $a(t)$ and $b(t)$ differently coloured on the screen (Figure 1 left). Additionally, selecting the *Make your own Maze!* modality allows for the creation of a target maze to be traversed through the movements. The maze can be built choosing a number of inflection points, a certain value for its thickness and tension, thus a particular graphical arrangement for the maze, which appears on the screen as a tick light blue line. At the end of the session, each user gets a score based on the traversal rate of the created graph with respect to the maze.

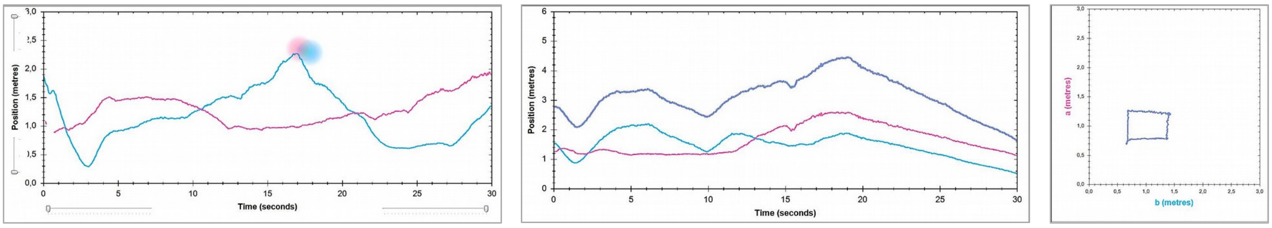

Figure 1. *Line* graphs; *Line* graphs and $a+b$ operation; a rectangle in *Versus*

Within the *Operation* modality, a third coloured graph is shown on the screen: in particular, the addition $a+b$ implies a third graph of position over time that depicts in real time $a(t)+b(t)$ (Figure 1 middle); in a word, the graph shows instant by instant the sum of the two distances. It is also possible to choose among the other simple mathematical operations (subtraction/multiplication/division), with an analogous result (a third graph with the chosen characteristics).

The second option we draw attention to is *Versus*, which allows for the creation of a single graph on a Cartesian plane with isometric axes, depending on both users' movement. *Versus* plots at each time t an ordered pair of the positions of the two controllers, showing the line $b(a)$ and leaving time implicit, therefore giving a motion trajectory. Practically, (Figure 1 right) vertical displacement in the graph corresponds to one user's movement, horizontal displacement to the other user's movement.

ACTIVITIES OF GRAPHING MOTION(S): MOVING, COMPARING, TRANSFORMING

In this section, we discuss insights coming from activities with WiiGraph that we carried out through some teaching experiments in Italian classrooms during the last three years. While we recognize that the use of WiiGraph engenders mathematical discourse similar to work with other motion detectors—which have been explored in the literature, we are interested in the ways in which we can exploit the potential of WiiGraph through the design of tasks. We believe that this technology might permit novel reasoning about variation and covariation in the context of graphing motion, therefore new ways of exploring mathematical relationships. In fact, the software requires that two people move in the same interaction space, in the same time interval. In the meanwhile, there are at least two graphs on the screen, which "move" together while originating in real time on the same Cartesian plane. When two students move with the devices, relationships between movements are captured through the relationships between the graphs that are created. Therefore, we can think of activities as (mainly) unfolding along two dimensions. One dimension is concerned with types of bodily engagement of the users with the technology, the other dimension regards how the concepts of graph and function can be grounded on aspects of covariation, coordination and plane transformation. In particular, the ideas shown in the following arise from five different teaching experiment that involved classes of grade 4, 9, 7 and 10 students.

Line* and *Versus

In a first exploration phase, using *Line* option, students might be simply asked to move and make conjectures about the meaning of the lines created on the screen. They might be challenged to move in order to get a couple of graphs with specific shape (like two straight lines or two curved lines). But, profiting of the potential of having two graphs on a single Cartesian plane, they might also be asked to think of ways of moving to obtain two graphs with related shapes (like two parallel slanted straight lines, two meeting straight lines or two translated gibbous lines). Thinking of two parallel straight lines in terms of vertical translation of one to obtain the other, for example, opens room for discussing relationships among the two graphs from a qualitative point of view. This also offers occasions for exploring bodily ways of moving that express given constraints (like parallelism and straightness): for instance, pairs of students have actually been asked to find ways of coordinating together to get the parallel straight lines. While this is rather trivial in the case of horizontal straight lines, it is not in the case of slanted straight lines, in which the two students have to try to keep the same pace. In our experiences, we observed some students drawing attention to each other, in order to maintain their relative positions while moving; some others instead held their hands to keep stretched arms (and fixed distance between them; Figure 2). Different coordination strategies embodied the need of preserving distance among the users to achieve the desired configuration on screen. Students involved in such explorations can give kinaesthetic definitions to vertical translation of graphs, whose mathematical counterpart is the idea of a constant vector that describes a rigid motion.

Figure 2. Bodily movements to capture graphical translation

Using the *Make Your Own Maze!* modality, a maze is added to the graphical space offering a visible shape as the goal of graphing motion for the students. In this case, learners might be challenged to move in pairs to traverse the maze as precise as possible, eventually engaging them in competitive interactions. Being in the challenge means to focus on both quantitative and qualitative aspects of the graphical notation, which are relevant to pursue the highest score. In addition, students might be asked to think of difficult graphs for their mates to match, and to describe features in terms of changes in direction, speed and position. In previous case studies (e.g. Ferrara & Ferrari, 2015b), we have seen how the challenging situations offered by the *Make Your Own Maze!* actually involve degrees of covert/overt coordination among the students. The ways that bodies partake in the creation of the graphs do have a role in the perception and thinking of speeds and shapes.

Selecting the *Operation* modality, for example working with the sum, the students have to shift attention to the relationships among the three graphs on the screen. Beyond this, it is challenging for the students to think of different bodily ways of producing a given sum—imagined or made present through the *Make your own Maze!* modality—like an horizontal straight line. Of course, an horizontal straight line can easily be obtained summing up two suitable horizontal straight lines, however one can discover and discuss further possibilities by summing up pairs of suitable slanted straight lines, etc. Another intriguing experience is that of moving keeping fixed distance (like in the case of two parallel slanted straight lines) and discovering that the sum does not preserve slope. Again, learners enter the realm of collaborative interactions, looking for suitable coordination between their movements.

New possibilities in terms of collaboration and coordinated movement are given by the *Versus* option that displays a single graph. In fact, one of the most interesting challenges for a *Versus* graph involves the creation of plane shapes, like rectangles, rhombuses and circles, or closed lines (see again Figure 1 right). Different ways of bodily coordination are assembled in different graphical lines with specific qualities: for example, moving at the same pace in the same direction produces a piece of line segment slanted by 45 degrees. All of this makes room for focusing on the crucial role of time as independent variable, as well as connecting the spatial relationships in *Versus*, which essentially are motion trajectories, with the space-time relationships in *Line*, which are functional relationships. The kinaesthetic ways of interacting with this option implicate an extended perception of movement, which goes beyond the perception of each student's movement to incorporate the composition of the coordinated movements of the students (de Freitas *et al*., 2017).

Notes

1. WiiGraph has been developed by R. Nemirovsky (Manchester Metropolitan University) and some colleagues (C. Bryant, M. Meloney, B. Rhodehamel) from the Center of Research in Mathematics and Science Education of San Diego State University.

REFERENCES

de Freitas, E., Ferrara, F. & Ferrari, G. (2017). The coordinated movement of a learning assemblage: Secondary school students exploring Wii graphing technology. In E. Faggiano, F. Ferrara & A. Montone (Eds.), *Innovation and Technology Enhancing Mathematics Education*. Springer Verlag. (forthcoming)

Ferrara, F. & Ferrari, G. (2015a). Becoming mathematical subjects by playing mathematical instruments: Gibbous lines with WiiGraph. In S. Carreira & N. Amado (Eds.), *Proceedings of the 12th International Conference on Technology in Mathematics Teaching* (pp. 378-386). Faro, Portugal: Faculty of Sciences and Technology of the University of Algarve.

Ferrara, F. & Ferrari, G. (2015b). Being collaborative, being rivals: Playing WiiGraph in the mathematics classroom. In B. Di Paola & C. Sabena (Eds.), Proceedings of the 67th Conference of the Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques, *Quaderni di Ricerca in Didattica (Scienze Matematiche), 25*(2), 461-472. Palermo, Italy: G.R.I.M., Dipartimento di Matematica e Applicazioni. ISSN: 1592-4424. Available at: http://math.unipa.it/~grim/CIEAEM%2067_Proceedings_QRDM_Issue%2025,%20Suppl.2_WG3B.pdf

Ferrara, F. & Ferrari, G. (2016). Traversing Mathematical Places. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, 251-258. Szeged, Hungary: PME.

Nemirovsky, R., Tierney, C. & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction, 16*(2), 119-172.

Nemirovsky, R., Kelton, M.L. & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education, 44*(2), 372–415. doi: 10.5951/jresematheduc.44.2.0372

Radford, L. (2009) Signifying relative motion: Time, space and the semiotics of Cartesian graphs. In W.M. Roth (Ed.), *Mathematical Representations at the Interface of the Body and Culture* (pp. 45-69). Charlotte, NC: Information Age Publishers.

Yerushalmy, M. & Shternberg, B. (2005). Chapter 3: Epistemological and cognitive aspects of time: A tool perspective. *Journal for Research in Mathematics Education, Monograph, 13*. Retrieved from http://www.jstor.org/stable/30037731

Dynamic Technology for Simulating a Scientific Inquiry for Learning - Teaching Pre-Calculus concept

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In the proposed workshop, we are interested to share with mathematics educators a possible use for dynamic technology to simulate a scientific inquiry for teaching-learning pre-calculus topics. Inspired by the logic of inquiry approach and the variation theory, we designed a set of activities and methodology that aim to bridge the gap between the formal world of mathematics and the real-life situations. Participants of the workshop will experience the activities and will engage in the methodology for conducting classroom practices. Results from our ongoing research would be shared with the participants and insights could be discussed that would emerge from the experience.

Rational

Calculus is considered to be one of the most important topics in mathematics, and it is included in several curricula worldwide. Calculus - the study of how things change - is inherent in many other topics and in many real-life situations. It is difficult to imagine that there could be any other dynamic phenomena which is unable to be modelled by means of calculus. Nurturing of calculus thinking can result in the productive integration of citizens into modern society. Exposing students to the logical structure of calculus concepts may help develop the logical thinking of the students. This type of thinking is essential in dealing with the challenges that citizens face in the 21st century. Furthermore, encouraging students to model mathematically real-life situations may help them integrate within society as citizens who are able to make intellectual decisions. Our research is aimed at understanding the ways of teaching-learning of pre-calculus concepts in a technologically rich environment that simulate a scientific inquiry of real-life situations.

Often, as seen in Italy as well as in Israel, calculus is taught in the upper secondary schools in a formal way as a set of rules and strategies for investigating functions and computing areas bound between functions. This kind of teaching-learning essentially concentrates on the formal world of mathematics which poses a barrier for the sense of mathematical statements (Arzarello, 2016a, b). Several scholars have criticized this kind of teaching and claimed that it is not a guarantee for boosting the understanding of calculus concepts, and even found it to be a barrier for understanding calculus when it is learned at a university level (e.g. Thompson et al; Swidan & Yerushalmy, 2016; Broussard, 2011). Also the Italian official curriculum (www.indire.it/lucabas/lkmw_file/licei2010/indicazioni_nuovo_impaginato/_decreto_indicazioni_nazionali.pdf) points out the necessity of presenting examples where mathematical models of different phenomena are emphasized.

To overcome the barrier that existed between the formal world and the real-life situation, Arzarello (2016a) has suggested a 'virtuous cycle' model. The model consists of four formal and informal intertwined aspects: (1) Aspects of the real situation represented in the formal system. (2) Treatment within a formal system / Conversions between systems. (3) Interpretation of the results of the formal system in the real situation. And (4) Interpretation/theorization of the real situation through the theoretical lens. Since the formal and informal aspects are deeply intertwined in the

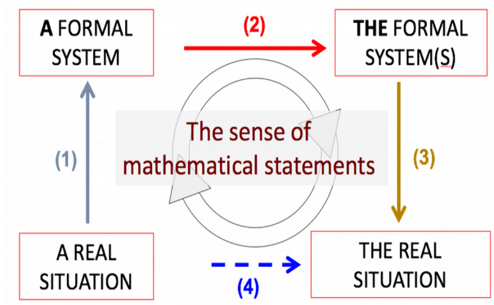

Figure 1. Virtuous Cycle (Adapted from Arzarello (2016a))

mathematical reasoning, Arzarello (*ibid.)* argued that a major teaching goal should be to operationalize this virtuous cycle in classroom practices.

However, the main question remains as to how to apply the virtuous cycle in classroom practices, that ensures a deep understanding of the mathematical concepts. This question should guide the discussion of the workshop. Inspired by the logic of inquiry approach (Hintikka, 1998), which generally viewed scientific inquiry and knowledge construction as a question-answer process, and the variation theory (Marton and Tsui, 2004) which defines learning as a change in the way something is seen, experienced or understood. We have designed a set of activities and suggest a methodology (Method of Variation Inquiry) which may facilitate the conceptual learning of pre-calculus concepts and to engage students in a scientific inquiry.

Hooke's Law Activity

Task 1

Your task is to watch the Hooke's Law clip (Fig. 2) and to answer the following questions:
What attracted your attention when you watched the clip? Write as many observations as you can?
After watching to the Hooke's Law clip, write as many hypothesis as possible about the content of the clip, which you may wish to discuss later.

Task 2

Your task is to explore how the change of the mass affects the extension of the spring.

A. Can you make a conjecture about how the change of the mass should affect the extension of the spring?
B. Open the Hooke's Law **1 applet** (Fig. 3). Change the mass to verify or refute the conjectures you raised in (A). Did your conjectures change? If yes, why? If not, justify and prove your conjectures.
C. How do the differences between the y values of the points on the graph change when varying the the mass. Why? Is your conjecture always true? Can you prove it?
D. Can you find an equation that represents the sketch of the spring? Why or why not? Justify your answer.

Task 3

Your task is to explore how the change of the spring elasticity affects its extension.

A. Hypothesize how the elastic of the spring affects its extension. Why?
B. Open the Hooke's Law **2 applet** (Fig. 4), vary the elasticity of the spring, and change the mass. Refute or verify the hypotheses you raised in (A). Justify and prove your hypotheses.
C. Why does the function graphs change as the elasticity of the spring varies? Discuss with your classmates which aspects changed and which ones remained invariant.
D. Follow your interaction with the applet. Write new hypotheses based on your experience with the applet. Which hypotheses can and can't you justify? Why?

Task 4

Challenge your classmates by asking them questions. You win the game if you ask a question about the Hooke's Law, which your classmate cannot answer.
Use either applet to ask your questions.

Figure 1. Hook's Law activity that was given to the students

Figure 2. Video clip that the students watch as the starting point for the inquiry

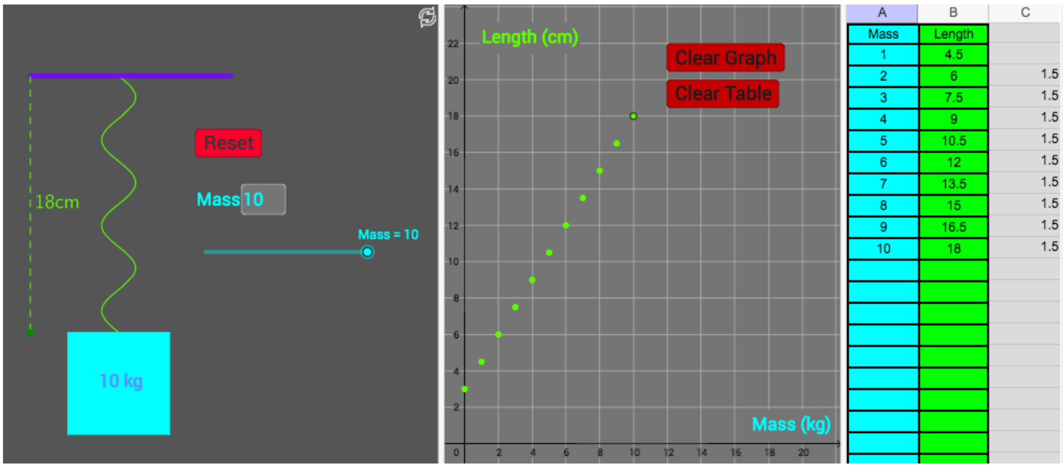

Figure 3. Interface of the Hooke's Law 1 applet

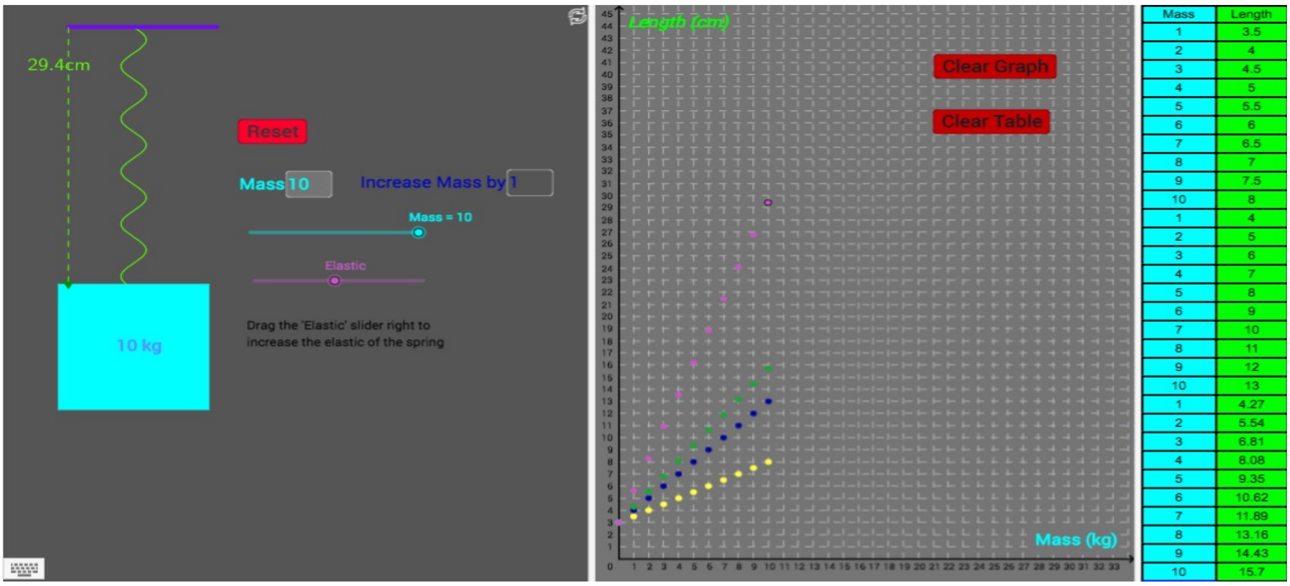

Figure 4. Interface of the Hooke's Law 2 applet

The Method of Variation Inquiry (MVI) is based on the idea of creating inquiry modes for students by changing and challenging the observations they make during the learning process. The MVI model consists of six levels:

- Level 0. Choosing a starting point.
- Level 1. Listing the observations.
- Level 2. Collective discussion (asking questions, reacting to observations). The conclusion should be If (Oi) then (Dj). The explanation why the sentence is or is not correct is shared.
- Level 3. What if the statement is not correct? (~Oi)
- Level 4. Collective discussion: the conclusion should be If (~Oi) then (Dj)*. The explanation why the sentence is or is not correct is shared.
- Level 5. Meta-reflection:
  - If (Oi) then (Dj)? ▭ (Rk)
  - If (~ Oi) then (Dj)* ▭(Rk)*

Questions and Goals

The workshop will address the question of how the designed activities and the suggested methodology could be applied in classrooms across different cultures. In addition, epistemological and cognitive opportunities as well as obstacles that are raised during the application of the proposed activities and methodology in classrooms could be discussed. We will use examples from our ongoing research to provoke discussions of these questions. These examples are derived from our analysis of a beta experiment conducted in a scientific oriented school in Italy, in which 10th grade students are working in small groups as well as a teacher who conducts a collective discussion using the suggested methodology. In our beta experiment, we have noticed learning opportunities that encouraged a deep understanding of pre-calculus ideas. We hope participants in the workshop will notice additional opportunities (obstacles) which would allow them to engage in discussions of the implications of such results for improving the teaching-learning pre-calculus concepts in classrooms.

Planned Activities

a.	Introducing the 'Method of Variation Inquiry': theory and practice.	15 min
b.	Experiencing the methodology in small groups through solving the designed activities.	20 min
c.	Discussing the opportunities and obstacles of the methodology and the activities.	15 min
d.	Introducing the finer logic of inquiry model: theory and practice	20 min
e.	Analyzing transcripts using the finer model of logic of inquiry	20 min
f.	Collective discussing of how the methodology and the activities could be applied in classrooms	20 min

REFERENCES

Arzarello, F. (2016a). How to promote mathematical thinking in the classroom. Lecture delivered at the 3rd Italian Summer School in Math Education, UMI-CIIM, Bardonecchia, 2016. Downloadable from: http://www.umi-ciim.it/attivita-della-ciim/scuole-estive/3a-scuola-estiva-2016/materiali/

Arzarello, F. (2016b). Tra sensate esperienze e necessarie dimostrazioni: quale senso matematico alle pratiche in classe? In: a cura di D'Amore, B. e Sbaragli, S., *La matematica e la sua didattica. Convegno del trentennale Atti del Convegno "Incontri con la Matematica" n.30 - Castel S. Pietro Terme*, Relazione Generale.

Bressoud, D. M. (2011). Historical reflections on teaching the fundamental theorem of integral calculus. American Mathematical Monthly, 118(2), 99-115. doi: 10.4169/amer.math.monthly.118.02.099

Hintikka, J. (1998). *The principles of mathematics revisited*. Cambridge University Press.

Marton, F., & Tsui, A. B. M. (2004). *Classroom discourse and the space of learning*. In P. P. M. Chik, P. Y. Ko, & M. L. Lo (Eds). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Swidan, O., & Yerushalmy, M. (2016). Conceptual structure of the accumulation function in an interactive and multiple-linked representational environment. *International Journal of Research in Undergraduate Mathematics Education*, *2*(1), 30-58.

Thompson, P. W., Byerley, C., & Hatfield, N. (2013). A Conceptual approach to calculus made possible by technology. *Computers in the Schools, 30*(1–2), 124–147.

Chapter 6

INNOVATION

THE TRANSPOSITION OF COUNTING SITUATIONS IN A VIRTUAL ENVIRONMENT

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The mathematics education research is increasingly focused on different didactical hypotheses for constructing teaching and learning situations involving the decimal principle of the numeration system. One of these situations is, for example, counting a big collection of objects through the tangible manipulation. In this paper we introduce the simulating device, "Simbûchettes", for analysing its potential concerning this situation with respect to the tangible material. In particular, we will show that "Simbûchettes" preserves all of the techniques we identified in the tangible world and it allows to mobilise other techniques strongly grounding on the decimal principle of the numeration system that we rarely observed with the tangible material.

Keywords: TAD, simulating device design, tangible manipulation, decimal number system

INTRODUCTION

Learning how decimal number system works has an important role in understanding several areas of mathematics: the calculus, the conversion of units of measurement, the decimal numbers, etc. The decimal number system is the product of the articulation between two different principles: the decimal and the positional one (Serfati, 2005). The "positional principle" allows associating a rank within a string of digits to each numbers unit (ones, tens, hundreds, thousands…). In other words, the position of a digit in the number determines its value. The "decimal principle" explains the relations between different numbers units in a number: each unit is equal to ten units of the next lower rank (for example, 1 ten=10 ones, 1 hundred=10 tens= 100 ones; 1 thousand=10 hundreds= 100 tens=1000 ones …).

THEORETICAL PERSPECTIVES

According to Tempier (2010), the decimal principle is considered as a source of learning difficulties. However, this aspect is necessary for understanding the numeration system. This principle can be taken into account, for example, through activities involving grouping by tens and exchanges. These rules state that 10 elements of a numbers unit can be grouped and exchanged with one element of the numbers unit of the next bigger rank. Moreover, one element of a numbers unit can be ungrouped and exchanged with 10 elements of the numbers unit of the next lower rank.

Difficulties related to teaching the numeration system at primary school

Research carried out by Bednarz and Janvier (1984) has shown students' difficulties related to understanding the decimal principle of the numeration system:

- *"difficulty in seeing groups of tens and their role in the canonical form of written-numbers, despite the important place that this canonical form takes in teaching";*

1. *"difficulty in seeing the relevance of these groups of tens";*
2. *"difficulty in working with these groups of tens, in terms of constructing and deconstructing them";*
3. *"difficulty in working with two different groups of tens at the same time";*
4. *"difficulty in the interpretation of the calculus procedures in relation to the mathematical operations (additions, subtractions, multiplications, divisions), in terms of groups of tens that leads to classic errors on the operations"* (Bednarz and Janvier, 1984).

The analysis of pupils' errors carried out by Parouty (2005) reinforces the fourth of these difficulties on the relation between the numeration system and the calculus. The activities observation proposed in the mathematics classroom explains why students have these type of difficulties. Actually, these activities mostly concern the positional principle of the numeration system. This is why pupils' learning is principally based on this aspect. Bednarz and Janvier (1984) make the same observation regarding the activities choice. For example, "the number representation appears according to the alignment related to the canonical form of the written-number". 26 years later, Tempier (2010) finds the same difficulties in students. For these reasons, a lot of researches focuses on how to take into account the decimal principle of the numeration system in the current teaching. In particular, different research studies have pointed out several working hypotheses on which constructing didactical situations highlighting the decimal principle (Tempier, 2010; Chaachoua 2016).

Didactical hypotheses

In this paper, we will consider three of these didactical hypotheses developed in Chaachoua (2016).

The relations among numbers units

According to Chambris (2008), it is very important to consider the different relations among the numbers units for mobilising the decimal principle, favouring the conversion tasks, for example converting 23 hundreds into tens. That is why our first working hypothesis is:

(HT1) *"The relations among numbers units": to work on the numeration system, we have to consider tasks that mobilise the relations among numbers units.*

Big numbers

Big numbers allow us to explicitly work on numeration system and, particularly, on the decimal principle. Actually, the introduction of a new numbers unit produces different relations among numbers units. This way, the repetition of grouping of tens and the exchanges allow us to better understand the decimal principle of the numeration system. Hence, our second working hypothesis is the following:

(HT2) *"Big numbers": The introduction of big numbers increases and reinforces the understanding of numeration system and, particularly, its decimal principle.*

The objects' manipulation

For teaching numbers and the numeration system, the objects' manipulation constitutes a very important phase in sense-making.

According to Raoul-Bellanger and Bellanger (2010), the manipulation in mathematics allows pupils to construct a mental image and to improve their abstraction capacity (iconic or symbolic system). This becomes really true for pupils with learning difficulties where the manipulation could be used in the remediation phase. Hence, our third working hypothesis is:

(HT3) "The objects' manipulation": the objects' manipulation is important for practising rules concerning grouping of tens and exchanges for giving sense to the decimal principle of the numeration system.

Drawing on these different didactical hypotheses, we will focus on the type of task "Counting a big collection of objects", through which we can take into in account the decimal principle after having chosen a relevant collection configuration.

The scientific challenges

In general, within the tangible manipulation activities, the time for accomplishing some actions (i.e., grouping of tens) increases when the collection size is big. For this reason, for discouraging the employing of some techniques not adapted to fit with big numbers, it is necessary to repeat tasks and the manipulation becomes time consuming.

Monitoring pupils individually during their manipulation actions is a difficult task for the teacher. For this reason, the implementation of teaching and learning situation based on this manipulation encounters three obstacles: (1) the manipulation of big collections demands a lot of time, (2) the tangible objects don't produce relevant retro-actions with respect to pupils' learning and (3) the teacher cannot observe different pupils at the same time. The last point will not be discussed in this paper, but it is the research theme of the Ph.D. thesis of Brasset (2016).

Our research question is two-fold: *does this technological device preserve all of the characteristics of the tangible manipulation and how this simulating device can overtake the challenges (1) and (2) related to the type of task "Counting a big collection" based on the manipulation of tangibles objects?*

In this paper, we will focus on the first part of our research question.

THE SIMULATION DEVICE "SIMBÛCHETTES"

With respect to the challenges presented above, a research project is carrying out by the MeTAH team of the University of Grenoble Alpes and it concerns the design of a simulating device "Simbûchettes" and of an orchestration device for monitoring all of the students (Wang et al., 2017). In this paper, we will focus on the simulating device "Simbûchettes". Our hypothesis is that this device can take into account the decimal principle of the numeration system, according to the previous part. In the frame of this project a simulation on a tablet (Fig. 1) has been developed. It allows us to manipulate virtual objects, to move small sticks, to put them into boxes, to group and ungroup them, to duplicate them, etc. All of the actions made on the tablet can be recorded. The treatment of these actions constitutes a retroaction for the pupils and they can also inform the teacher. The touch screen interface conception, the variables choice and the treatment of the actions are based on the didactical computing model T4TEL (Chaachoua and Bessot, 2016). This theoretical framework refers to the Anthropological Theory of Didactic (ATD) (Chevallard, 1992, 1998, 1999) and in particular to the praxeological approach.

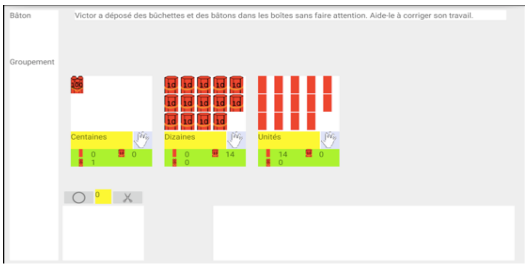


Fig 1. Example of the interface of the simulating device

This device gives us the possibility of choosing and defining the parameters which allow us to create different didactical situations. These parameters can concern the displaying of the constitutive elements of the interface (boxes, duplication zone, action buttons, etc.), the elements available for pupils and the actions that are authorised or not. We can configure the device forbidding some specific type of action. For example, we can forbid the introduction of a tenth element after having already put in a box 9 small sticks or 9 packs of small sticks. The device

"Simbûchettes" gives the possibility to easily make and unmake groups of ten. It also gives the opportunity of producing exchanges among different numbers units focusing on their relations. Moreover, we can manage collections with a big number of elements. Actually, in terms of the time and equipment management, it allows us to make repetitions: this is an important condition to avoid costly techniques. This makes the teacher able to know all of the actions of pupils during the manipulation without coaching them one by one.

METHODOLOGY

This research involves 30 pupils in two third grade classes of a primary school of Grenoble. 7 of them have worked with the tangible material and the others 23 with the device "SimBûchettes" on the same activities concerning the type of task "Counting a collection", in according to the didactical hypotheses explained in the theoretical part of the paper. This type of task has been studied in Chaachoua (2016) in which the author has developed an epistemological model of reference according to the theoretical framework T4TEL. We have relied on this study for conceiving our activities. In particular, we have focused on three different problems and many different activities that we resume in the table below. The question was always the same: "How many sticks?"

Problem	Example of an activity of the experimentation
1. Counting a collection "in bulk"	80 sticks "in bulk"
2. Counting a homogeneous collection	9 tens of sticks, 12 tens of sticks, 67 tens of sticks
3. Counting a completely ordered collection	2 hundreds of sticks, 23 tens of sticks, 15 sticks

Table 1. General description of the different tasks proposed to the pupils with examples

Pupils' actions were video recorded during the activities with the camera facing their hands and the material on the table in the case of the tangible experiment or the tablet in the other one. All voice and hands movement during the activity were recorded. The videos were transcribed for data analysis.

In the following, we will describe pupils' techniques we have observed for solving the different tasks listed above. Concerning the structure of the analysis, in the first part, we will show that in the virtual environment, we have observed the same techniques appeared in the tangible one, even if the implementation of a same technique is deeply different in the two cases. In the second part, we will go further showing how "Simbûchettes" produces other techniques that strongly mobilise the decimal principle.

DATA ANALYSIS

In the first part of the data analysis, for each problem, we are going to present the different techniques we have identified during the teaching experiments with "Simbûchettes".

Problem 1:

- **Technique 1.1**: Grouping by tens, counting by tens (or by 20 or by 30).
- **Technique 1.2**: Grouping by tens, counting by numbers units, converting numbers units to ones.
- **Technique 1.3**: Counting by *n*, where *n* is 1, 2, 3…

Problem 2:

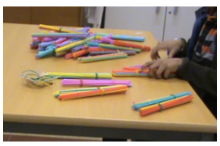
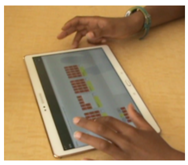
- **Technique 2.1**: Grouping by tens, counting by numbers units, converting numbers units to ones**.**
- **Technique 2.2**: Grouping and counting by X, where X is a power of ten (1, 10, 100…).
- **Technique 2.3**: Counting by *n*, where *n* is 1, 2, 3…

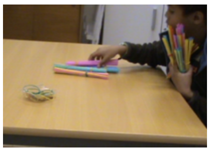
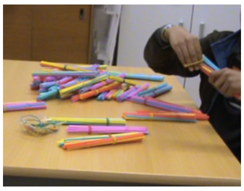
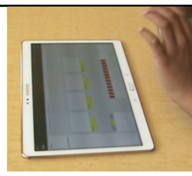
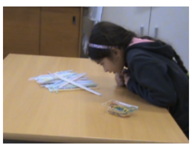
Problem 3:

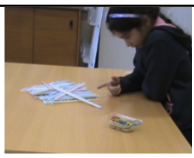
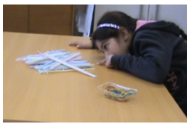
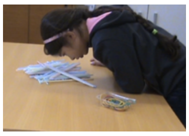
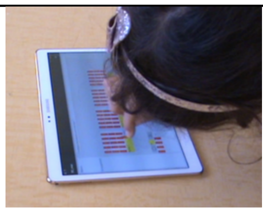
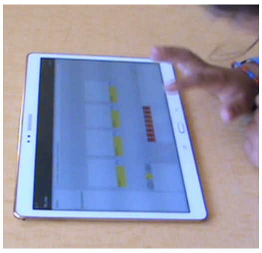
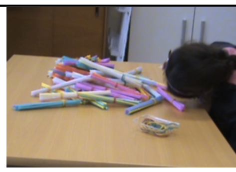
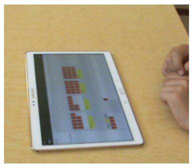
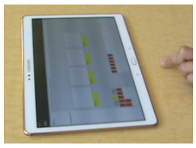
- **Technique 3.1**: Counting separately each rank by number units, converting the number units to ones and adding them.
- **Technique 3.2**: Counting separately each rank by number units, converting the number units to ones and thinking to the configuration of the written-number.

As shown above, the simulating device doesn't avoid the techniques related to the tangible manipulation presented in Chaachoua (2016). This point answers to the first part of our research question: from an ecological point of view, the simulating device preserves all of the characteristics of the tangible manipulation. Moreover, we can highlight that the simulating device enhances the variety of techniques: for example, we have observed that the technique of "grouping and counting by hundreds" is present in the virtual manipulation, while it doesn't appear in the tangible one. Moreover, in general, all of the pupils employed the technics of "grouping by tens" in the virtual environment, while in the tangible one they used very often other technics without making groupings. These observations lead us to hypothesize that with "SimBûchettes" it is more natural for pupils to make grouping of tens or of hundreds with respect to the tangible material.

In the second part of the data analysis, we will show the mathematical activity of two pupils (S1 and S2 in the following) who employ different technics for the same activities in the two environments. Three months have passed between the tangible teaching experiment and the virtual one. During the experimentations, before starting the activities, the researchers (R in the following) gives some preliminary information about the material pupils had at disposal. For the tangible material, the researchers said that each package of sticks contains exactly 10 sticks and each package has the same cardinality and that there were some elastics on the table that pupils can use if they wish. For the virtual environment, the researchers explains the different elements constituting the interface of "SimBûchettes".

S1	**Activity: 12 tens of sticks**		**Activity: 67 tens of sticks**	
	Tangible	**"SimBûchettes"**	**Tangible**	**"SimBûchettes"**
	First of all, S1 counts the number of sticks in a package (see Fig.1)  Fig. 1: S1 counts the sticks in one package After, S1 mentally counts the number of packages, taking in his hands the	S1 immediately constructs one grouping of tens for making the hundreds (see Fig.3)  Fig. 3: On the table zone there are the package of hundreds and two packages	S1 makes groupings of two tens without using the elastic and he puts them on one of the corners of the table (see Fig.4).  Fig. 4: S1 makes the groupings of two tens without using the elastic	S1 makes immediately groupings of hundreds (see Fig. 6 and Fig. 7)  Fig. 6 S1 is moving the packages of tens in the construction zone of the interface

	packages one by one (see Fig.2).  Fig. 2 S1 counts the packages	of tens	Then, he decides to take an elastic for encircling the packages of two tens he made (see Fig. 5).  Fig. 5: S1 takes an elastic for encircling the grouping of two tens he made	 Fig. 7 At the end, S1 has on the table 6 groupings of hundreds and 7 groupings of tens
Transcript	S1: 120 R: How did you do? S1: 10 by 10	R: How did you do? S1: I have found that there were 12 packages of tens. I have put 10	S1: 680 R: How did you do? S1: I started to count 20	S1: 670 R: Can you count aloud?
Analysis	S1 counts the sticks 10 by 10 (without making grouping of hundreds), after having verified that in one package there were 10 sticks.	In this case, S1 counts the number of packages and he makes a grouping of hundreds. Then, he makes 100 plus 20.	S1 started to make the grouping of two tens for counting 20 by 20, probably because he wanted to save time. But, then, he continued to count 10 by 10, probably because it was simpler for him to count 10 by 10 with respect to 20 by 20. At the end, he made an error in the calculus.	S1 makes groupings of hundreds and he counts 100 by 100.
S2	**Activity: 80 sticks "in bulk"**		**Activity: 67 tens of sticks**	
	Tangible	**"SimBûchettes"**	**Tangible**	**"SimBûchettes"**
	S2 counts without moving the sticks (see the sequence of figures below), even if many times the researcher said to her that she could move them. 	S2 makes groupings of tens in the construction zone of the interface (see Fig. 8) and, then, she counts the packages she made (see Fig. 9).	S2 counts without moving the sticks (see the sequence of figures below), even if many times the researcher said to her that she could move them.	S2 makes groupings of hundreds in the construction zone of the interface (see Fig. 10) until she makes 6 packages of hundreds. After she tries to make another grouping of hundreds but she becomes aware of the fact that there are 7 grouping of tens instead

	   	 Fig. 8 S2 moves the sticks in the grouping zone for making packages of 10  Fig. 9 She counts the number of packages she has	  	of 10. She decides to leave the 7 packages of 10 in the construction zone (see Fig. 11) and the she counts.  Fig. 10 She moves the last 7 packages of 10 she had for trying to make another package of 100, but she becomes aware of the fact that the number of packages is not sufficient for making 100  Fig. 11 She left the 7 packages in the grouping zone and she begins to count.
Transcript	R: Can you count this way? S2: uhm	S2: I'm done R: How many sticks are there?	S2: 500 R: 500? S2: yes	S2: I'm done R: How many sticks are there?
Analysis	S2 counts the sticks one by one without moving them, just pointing them while she counted and	In this case, S2 is more self-confident of her result with respect to the other case: in fact,	S2 probably counts 10 by 10 without moving the packages.	S2, after making grouping of hundreds, makes an addition, founding the correct

DISCUSSION AND CONCLUSION

As we discussed in the data analysis, the technological device allows to produce the same techniques identifying with the tangible material and it enhances others techniques (i.e., making grouping of hundred) deeply linked to the decimal principle of the numeration system that we have rarely observed in experimentations with the tangible material. From an ergonomic point of view, probably, it is not easy for pupils to make, for example, grouping of hundred with tangible sticks. Moreover, as we shown above, even a same pupil who immediately makes grouping of hundreds with "Simbûchettes", three months earlier, for the same activity, he preferred to count 10 by 10 without making packages of 100. This element gives us some first information about the potential of “Simbûchettes” with respect to the tangible material. In the future research, we will try to investigate more deeply this potential answering to the second part of the research question *“how this simulating device can overtake the challenges (1) and (2) related to the type of task “Counting a big collection” based on the manipulation of tangibles objects?* We will study if “Simbûchettes” could reduce the time of manipulation with respect to what happens in the tangible environment. Anyway, we can already state that, with “Simbûchettes”, the teacher reduces her time regarding the preparation of the different configurations of collections. Moreover, even if the time will not be different, we could investigate the different implications that the two environments have on students’ learning and how the role of the retro-actions offered by the simulation device could enhance them.

REFERENCES

Bednarz, N., Janvier, B. (1984). La numération : les difficultés suscitées par son apprentissage, *Grand N*, *(33)*, 5-31

Bosch, M. et Chevallard, Y. (1999). La sensibilité de l’activité mathématique aux ostensifs. Objetnd’étude et problématique. *Recherches en didactique des mathématiques*, *19(1)*, 77-124.

Brasset, N. (2016). Simulation du matériel de numération « Bûchettes ». *Actes du 43ième Colloque Copirelem*, 14-16 Juin 2016, Le-Puy-En-Velay, France.

Chaachoua Y. (2016), *Praxéologie de référence de l’aspect décimal de la numération par la manipulation selon le modèle T4TEL*, Mémoire de Master de Didactique des Sciences, Université de Grenoble Alpes.

Chaachoua, H., Bessot A. (à paraître) Introduction de la notion de variable dans le modèle praxéologique. *Actes du 5e congrès pour la Théorie Anthropologique du Didactique*. Castro-Urdiales, Espagne, 2016.

Chambris, C. (2008). *Relations entre les grandeurs et les nombres dans les mathématiques de l'école primaire. Évolution de l'enseignement au cours du 20e siècle. Connaissances des élèves actuels*. Thèse de l’Université Paris Diderot, Laboratoire de didactique André Revuz.

Chevallard, Y. (1992). Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique. *Recherches en Didactique des Mathématiques*, *12(1)*, 73–112.

Chevallard, Y. (1998). Analyse des pratiques enseignantes et didactique des mathématiques : l’approche anthropologique. *In R. Noirefalise (Ed.), Actes de l’École d’été de la Rochelle*, 91-120.

Chevallard Y. (1999) L’analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques*, *19(2)*, 221–266.

Chevallard, Y. (2001). Organiser l’étude 1. Structures et Fonctions. *In J-L. Dorier & al. (eds) Actes de la 11ième Ecole d’été de didactique des mathématiques*, *Recherches en Didactique des Mathématiques*, pp. 3-22.

Parouty, V. (2005). Compter sur les erreurs pour compter sans erreurs: état des lieux sur l'enseignement de la numération décimale de position au cycle 3; *Actes du XXXIème colloque sur la formation des maîtres*.

Raoul-Bellanger A., Bellanger F. (2010) *Construire les notions mathématiques, 50 activités de manipulation*, Retz

Tempier, F. (2010). Une étude des programmes et manuels sur la numération décimale au CE2. *Grand N*, *(86)*, 59-90.

Serfati, M. (2005). La révolution symbolique : la constitution de l'écriture symbolique mathématique,
Paris : Pétra.

Wang P., Tchounikine P., Quignard M. (à paraître), Chao: a framework for the development of orchestration technologies for Technology-Enhanced Learning activities using tablets in classroom. *In International Journal of Technology Enhanced Learning*.

HANDWAVER: A GESTURE-BASED VIRTUAL MATHEMATICAL MAKING ENVIRONMENT

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We report on the design and development of HandWaver, *a gesture-based mathematical making environment for use with immersive, room-scale virtual reality. A beta version of* HandWaver *was developed at the* IMRE Lab *at the University of Maine and released in the spring of 2017. Our goal in developing* HandWaver *was to harness the modes of representation and interaction available in virtual environments and use them to create experiences where learners use their hands to make and modify mathematical objects. In what follows, we describe the sandbox construction environment, an experience within* HandWaver *where learners construct geometric figures using a series of gesture-based operators, such as stretching figures to bring them up into higher dimensions, or revolving figures around axes that learners can position by dragging and locking. We describe plans for research and future development.*

Keywords: Geometry, Virtual Reality, Technology

OVERVIEW OF HANDWAVER

HandWaver is a gesture-based virtual mathematical making environment, currently optimized for in-room (as opposed to seated) immersive virtual reality platforms (such as the HTC Vive) that support gesture recognition. From points in space, users can construct uni-, two-, and three-dimensional mathematical objects through iterations of gesture-based operators. Figure 1 shows iterations of the *stretch* operator: a point is stretched into a line segment; the line segment is stretched into a plane figure; the plane figure is stretched into a prism. The hands that are shown in the images are virtual renderings of a user's actual hands, tracked via a *Leap Motion* sensor that is mounted to the virtual reality headset (see Figure 2).

Figure 1. Different cases of the *stretch* operator: a point is stretched into a line segment, the segment is stretched into a plane figure, and the plane figure is stretched into a prism.

Figure 2. A user (red sweatshirt) in the virtual space. The large monitor displays a 2D view of the user's first-person view of the virtual world. The device that tracks the user's hand movements is mounted to the front of the headset he is wearing.

A second gesture-based operator is *revolve*. Users can position an axis in space, select objects to rotate around the axis, and then spin a wheel to revolve the selected objects around the axis. Revolving objects in this way creates surfaces of revolution. Figures 3 and 4 show different cases of the *revolve* operator. In Figure 3, a point is revolved to create a circle; the circle is then revolved around itself to create a sphere; and the circle is revolved around an axis to create a torus.

Figure 3. Different cases of the *revolve* operator. The ship's wheel is a spindle that users turn to revolve figures. The line through the ship's wheel is the axis of rotation.

In Figure 4, a segment is revolved parallel to an axis of rotation to create a cylinder; a segment is revolved perpendicular to an axis of rotation to create an annulus; the annulus is revolved around itself to create a sphere with a hole in its center.

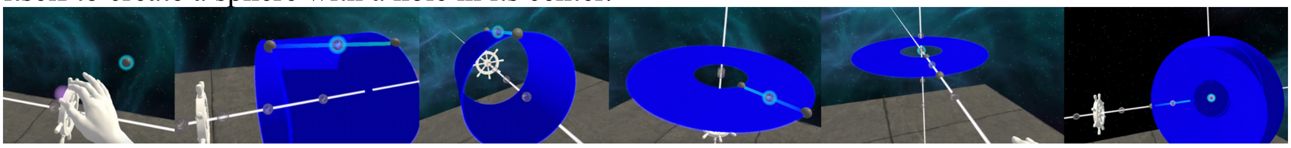

Figure 4. Different cases of revolving a segment. When the segment is parallel to the axis of rotation, the result is a cylinder. When the segment is perpendicular, the result is an annulus. The last image shows an annulus revolved around itself to create a sphere with a hole in its center (note: the hole is visible in the image by slicing the sphere).

We organized the sandbox environment around the *stretch* and *revolve* operators to help learners train their dimensional deconstruction skills (Duval, 2014). Dimensional deconstruction is the process of resolving geometric figures into lower-dimensional components, rather than seeing them as whole, fixed shapes. In the *HandWaver* sandbox, learners can fluidly move from lower-dimensional shapes (e.g., circles) to their higher dimensional analogs (e.g., spheres) and vice versa. The environment brings plane and solid geometry together—subjects that have been separated from each other in the usual presentation of geometry in K-12 schools.

The solid analogs of plane figures, in particular sphere-and-plane constructions, are "seldom developed" or "slighted...owing to their theoretic nature" (Franklin, 1919, p. 147). Three-dimensional dynamic geometry software (e.g., GeoGebra or Cabri 3D) has made it possible to engage in such constructions, however the limitations of two-dimensional screens has constrained their practicability. But for users immersed in a three-dimensional space—where the user has natural control over the angle at which an object is viewed, is able to move and manipulate the object in space, and can readily select the components of a figure to be incorporated into a new construction—three-dimensional constructive geometry becomes more feasible.

Thus, a final feature of the sandbox environment is three-dimensional analogs of classic construction tools. The *arctus* tool (Figure 5) allows users to make a sphere centered at a point, through any other point. The size of the arc shown in the figure is variable, and the midpoint of the *arctus* tool can be locked to any point in the display. *Arctus* is a spatial compass that creates spheres. The user sets the arc to have the desired radius and then generates a sphere by spinning the arc through space.

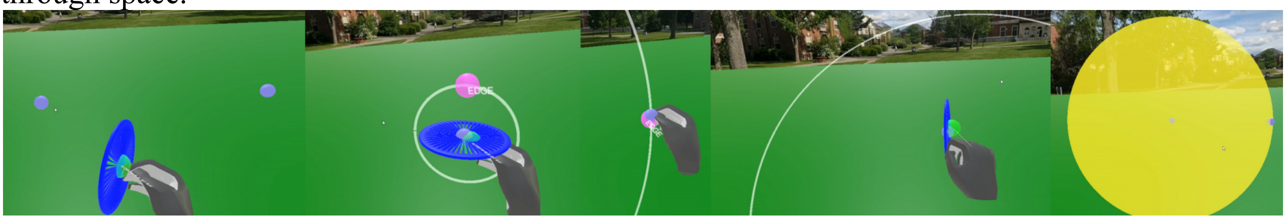

Figure 5. The *arctus* tool being used to inscribe a sphere. Users position the tool on a center point and on a point on the surface of the sphere . To generate the sphere, one turns the circle through space by spinning the blue wheel.

The *flatface* tool (Figure 6) allows users to define a plane through any three points. A user sets one of the lines of the *flatface* to coincide with two of the three points. Once in place, the user sets the second line so that it is collinear with the third point. To generate the plane, one acts with the *stretch* gesture on one of the lines of the flatface. We implemented plane-and-sphere constructions via gesture- (and motion-) based virtual tools to mimic the physical actions of spinning a compass or drawing a line with a straightedge. Our goal in doing so was to highlight the manual history of making geometric figures.

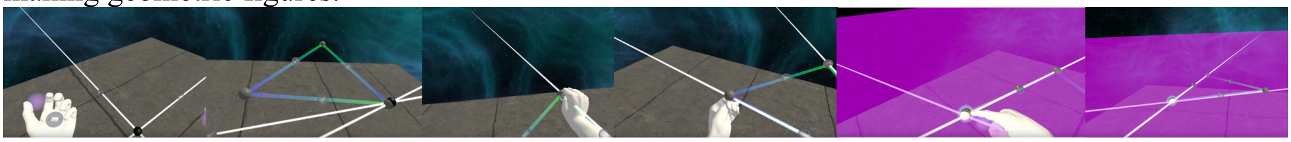

Figure 6. Series of images showing the *flatface* tool being used to spawn a plane.

With *arctus* and *flatface*, learners can complete solid geometry construction tasks that are inherently virtual, such as constructing a tetrahedron from three spheres (see Figure 7).

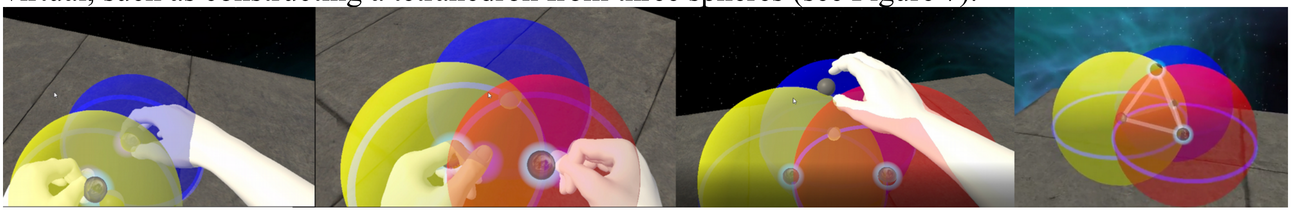

Figure 7. Constructing a tetrahedron from three-spheres in the *HandWaver* sandbox.

These tools provide an occasion for learners to explore how plane geometry construction protocols can be extended to higher dimensions. Other experiences within the *HandWaver* environment include a volume lab, an operator lab, and *LatticeLand*, which is a spatial analog of the geoboard (Kennedy & McDowell, 1998). Users can define the edges or faces of polyhedra by selecting a circuit of lattice points with a virtual pin (see Figure 8).

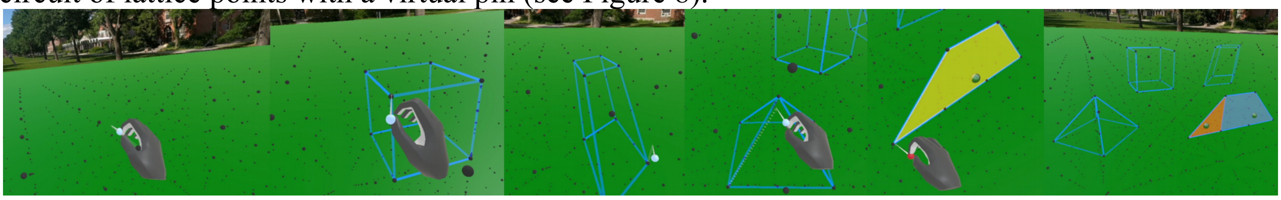

Figure 8. Connecting the dots in *LatticeLand* to define a the edges of a cube (second frame), a parallelepiped (third frame), a pyramid (fourth frame), and a trapezoid (fifth frame); the sixth frame shows the trapezoid cut into components (the orange triangle, the blue trapezoid).

MOTIVATION AND DESIGN CONSIDERATIONS

Our primary goal in developing *HandWaver* was to provide a space where learners at all levels could use their hands to act directly on mathematical objects, without the need to mediate intuitions through equations, symbol systems, keyboards, or mouse clicks (Sinclair, 2014). We designed the environment around natural movements of user's hands to foreground the connection between diagrams and gestures (de Freitas & Sinclair, 2012; Chen & Herbst, 2013). As one example of how the environment realizes this connection, the *stretch* operator multiplies (Davis, 2015) single points into many to form a segment, or multiples single segments into many to form a plane figure, or multiplies a single plane figure into many to form a solid. The notion that *n*-dimensional figures consist of adjoined (*n*-1)-dimensional figures is foregrounded by the generative use of the stretching gesture.

Gestural interfaces (Zuckerman & Gal-Oz, 2013), where objects can be manipulated in natural, intuitive ways by movements of one's hands, allow a degree of direct access to virtual objects that have been shown to facilitate learning (Abrahamson & Sánchez–García, 2016) while minimizing cognitive barriers (Sinclair & Bruce, 2015; Barrett, Stull, Hsu, Hegarty, 2015). Virtual environments with gestural interfaces have affordances for translating multimodal cues—e.g., head or hand movements—into mathematical operations, such as projecting a plane figure into three dimensions by pulling it up into space. The name of the environment, *HandWaver*, is an attempt to reposition "hand waving"—a term used to criticize mathematics that is insufficiently rigorous—as a means for doing mathematical work.

A further motivation for developing a construction environment with a gesture-based interface was to make it accessible to younger learners. Soon, children will routinely and increasingly incorporate virtual reality environments into their leisure activities. They will be playing games that require spatial reasoning and problem solving skills—imagine, for example, an immersive first-person version of *Monument Valley* (Ustwo, 2014)—but what will they be doing in schools?

Currently, children's encounters with geometry in elementary schools are limited to shape recognition and naming tasks (Bruce & Hawes, 2015). Yet a growing body of research indicates that children have the interest and capacity to train their spatial reasoning skills (Hallowell, Okomato, Romo, La Joy, 2015; Whiteley, Sinclair, & Davis, 2015; Taylor & Hutton, 2013) and study meaningful mathematics (Newton & Alexander, 2013; Sinclair & Bruce, 2015) from the moment they enter the schoolroom door. New modes of interacting with virtual mathematical objects (Hwang & Hu, 2013; Kaufman 2011) have the potential to expand children's access to deep geometric ideas. For all of its educational promise, however, virtual reality is on a track to follow the slow, complex process of technology acceptance and adoption that is standard in schools and that falls short of true integration (Ertmer, 1999; Inan & Lowther, 2010). Given how difficult it has been, historically, to incorporate promising technologies into classrooms at scale, there is every reason to believe that the educational potential of virtual reality will remain unfulfilled.

Our final reason for developing *HandWaver* is thus perhaps the most important: We developed the environment so that we would be able to critically investigate the disparity between what is and what could be in using virtual reality to enhance mathematics education. There is a "scarcity of bold research on interactive mathematics learning" that "impedes the formulation of empirically based progressive policies concerning the integration of technological environments into educational institutions" (Abrahamson & Sánchez–García, 2016, p. 204). In addition to investigating how students explore mathematical structures within an immersive virtual mathematics laboratory (Bock & Dimmel, *in press*), we are convening study groups to investigate (1) how practicing teachers would manage the challenges and opportunities of incorporating virtual reality technology into their instruction, and (2) how pre-service teachers could be adequately prepared for teaching with such technology.

DEVELOPMENT PROCESS

The environment is built in room-scale virtual reality, with a 4 meters by 8 meters activity space. This provides affordances of consistent head tracking and perspectives from varied physical heights and locations, which are not available with seated virtual reality (e.g., GearVR) and 360-video hardware (e.g., Google Cardboard). Recent advances in hardware have made significant improvements in performance and cost. The HTC Vive and Oculus Touch head mounted devices (HMDs) both provide room-scale virtual reality with consumer-grade hardware and cost similar to other classroom technology (e.g., Interactive White Boards). We chose the HTC Vive for it's larger activity space, early room-scale availability and local multiplayer in a shared activity space. Recent advances in consumer GPUs have expanded access to the processing power required to drive these HMDs to consumer workstations. The combination of improvements in processing and in the

HMDs has minimized previous issues with motion sickness. Room-scale optimizes problems with posture and fatigue in the environment, and also allows for more advanced image processing to improve immersion. Finally, the LeapMotion Orion SDK allows for reliable hand tracking integrated across the HTC and Oculus platforms.

RESEARCH PLANS

We are engaged in parallel lines of research using *HandWaver*. One line of research concerns documenting student encounters with mathematical objects in the virtual space. The immersive nature of the environment, combined with the gestural interface, provides a level of control over perspective, orientation, and position relative to mathematical objects that is difficult to replicate with other display technologies. Even the relatively straightforward means for rotating the graphics view in the 3D version of GeoGebra is complicated when compared to moving one's head, walking around a figure, or examining it from several different angles in quick succession. How do students use the angle of their gaze, the position of their bodies relative to virtual mathematical figures, or the ability to quickly change the scale of figures—from something that one could hold in one's hands, to something that one could fit inside—to explore mathematical structures?

This line of research frames activity within *HandWaver* (e.g., the volume laboratory) using the *conceptions-knowing-concept* (cK¢) model of conceptions (Balacheff & Gaudin, 2010; Balacheff, 2013): the virtual environment creates a *milieu* where students encounter problems that they explore using a suite of virtual operators, such as the ability to compare solid figures by superposition (see Figure 9).

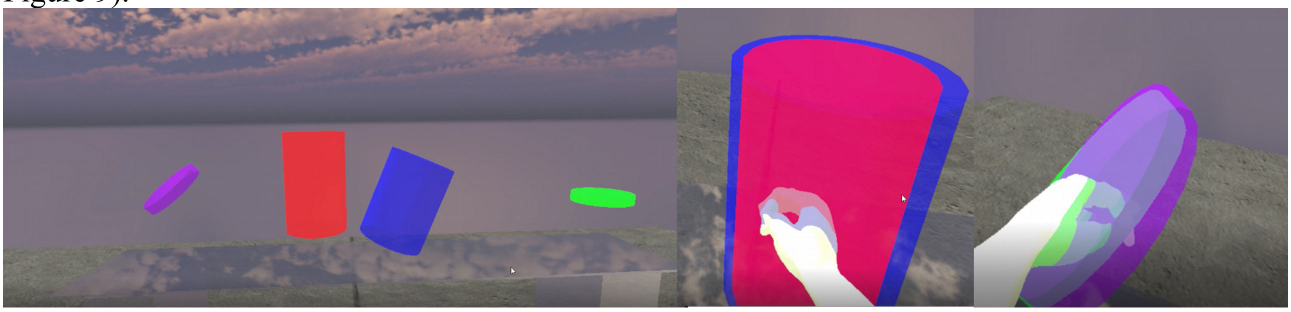

Figure 9. Comparing volumes of virtual solids by superpositioning.

In one study (Bock & Dimmel, *in press*), we used semi-structured interviews where participants—three master's students pursuing certification as science teachers—were asked to think-aloud as they explored the volume of a pyramid. One of the operators available to participants was the ability to dynamically change the pyramid by pinching and dragging its apex in space. Participants could lock the apex in the *z*-direction (shearing) or *xy*-directions (elongating) to control how the apex moved. Other operators included the ability to enclose the pyramid in a unit cube and add additional pyramids to it (see Figure 10).

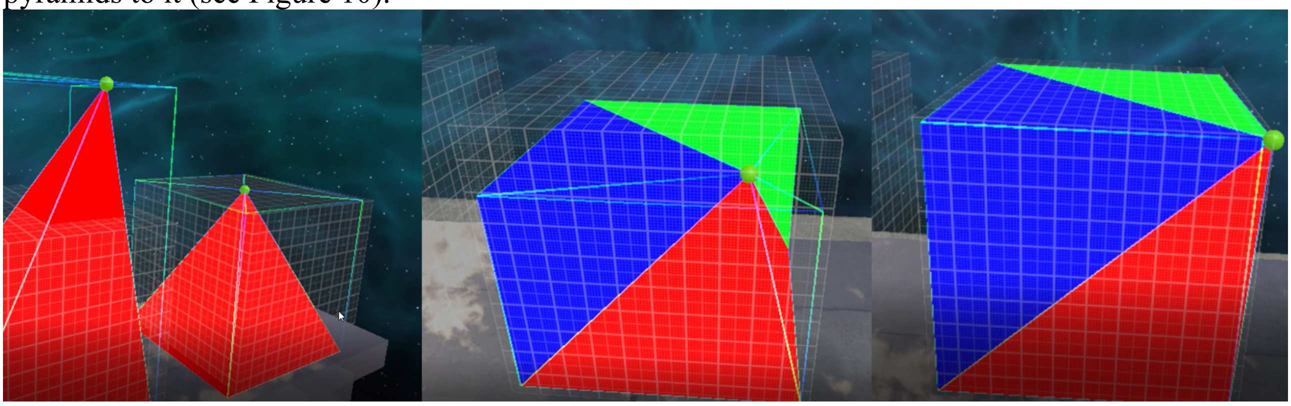

Figure 10. Enclosing the pyramids in a unit cube, adding additional pyramids, and adjusting the pyramids by moving the apex.

Participants could then explore how the volumes of the added pyramids were affected by movements of the apex. One strategy used by participants in this study was to reason about the volume of a pyramid by analyzing how the surface area of its faces was affected when the apex was moved in different ways. We are planning an interview study that would investigate how participants use the gesture-based operators available in the sandbox to construct different geometric figures.

A parallel line of research pertains to issues of instructional implementation: How do practicing and preservice teachers imagine incorporating virtual reality technology into their teaching? What support do they need? What barriers do they anticipate? For this research, we are developing multiplayer and partial immersion modes so that *HandWaver* could be used by a teacher with a whole classroom. The multiplayer mode will allow more than one user to be in the same virtual world at one time. The partial immersion mode will allow other users to view what is happening in the virtual world through a tablet. The partially immersed users will also be able to have some limited interactions with the virtual world, such as using gestures to control their angle of view, their position within the environment, or to construct figures. We are anticipating a time in the not-too-distant future when it will become feasible for a classroom to have multiple VR consoles that will allow students to work on problems in groups. In such configurations of virtual reality enhanced mathematical explorations—what we call *virtual mathematics laboratory experiences*—some students would be fully immersed in a virtual world and others would access the environment via a gesture-tracking tablet. We have a dedicated laboratory classroom space at the University of Maine where we will convene groups of teachers to study the instructional potential of teaching in a virtual reality-enabled classroom. Groups of participating teachers will explore and critique the *HandWaver* environment. They will work with each other to devise plans for how such an environment could be used in their teaching and anticipate obstacles they would expect to encounter. The first study group will be convened during the 2017-2018 academic year.

FUTURE DEVELOPMENT

The development of *HandWaver* is ongoing. We are planning a second release that will have new experiences, new modes of interacting within the environment, and new tools for use within existing experiences. A new experience that we are developing is a spherical trigonometry and nautical science lab, where users would be able to investigate properties of triangles that are inscribed on the surfaces of spheres. We are also developing a suite of measurement tools for use in the sandbox and volume labs, such as a paint roller that has different shaped heads (e.g., triangular, rectilinear, circular) that can be varied in size. Users would be able to "roll on" various area units to cover plane figures. The purpose of such a tool would be to provide a visual representation that units for measuring area are two-dimensional.

The advent of consumer grade virtual reality consoles (e.g., Oculus, HTC Vive) is likely to usher a frenzy of development of commercial, virtual reality educational content. If such development follows the path of educational apps, a preponderance of the mathematics education content that is developed for virtual reality consoles will amount to little more than immersive, visually engaging flashcards (Davis, 2015). By designing and developing the *HandWaver* environment, we are attempting to ensure that research-based ideas about the nature of productive mathematical activity are represented in this next generation of virtual learning environments.

REFERENCES

Abrahamson, D., & Sánchez–García, R. (2016). Learning is Moving in New Ways: The Ecological Dynamics of Mathematics Education. *Journal of the Learning Sciences*, (online first edition).

Barrett, T. J., Stull, A. T., Hsu, T. M., & Hegarty, M. (2015). Constrained interactivity for relating multiple representations in science: When virtual is better than real. *Computers & Education*, *81*, 69-81.

Bruce, C. D., & Hawes, Z. (2015). The role of 2D and 3D mental rotation in mathematics for young children: what is it? Why does it matter? And what can we do about it?. *ZDM*, *47*(3), 331-343.

de Freitas, E., & Sinclair, N. (2012). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics*, *80*(1-2), 133-152.

Chen, C. L., & Herbst, P. (2013). The interplay among gestures, discourse, and diagrams in students' geometrical reasoning. *Educational Studies in Mathematics*, *83*(2), 285-307.

Davis, B. (2015). Gumm(i)ing up the works? Lessons learned through designing a research-based "app-tutor". *Proceedings of the 12th International Conference on Technology if Mathematics Teaching,* Faro, Portugal.

Dimmel, J. K., & Herbst, P. G. (2015). The Semiotic Structure of Geometry Diagrams: How Textbook Diagrams Convey Meaning. *Journal for Research in Mathematics Education*, *46*(2), 147-195.

Duval, R. (2014). Commentary: Linking epistemology and semio-cognitive modeling in visualization. *ZDM*, *46*(1), 159-170.

Ertmer, P. A. (1999). Addressing first-and second-order barriers to change: Strategies for technology integration. *Educational Technology Research and Development*, *47*(4), 47-61.

Franklin, P. (1919). Some Geometrical Relations of the Plane, Sphere, and Tetrahedron. *The American Mathematical Monthly*, *26*(4), 146-151.

Hallowell, D. A., Okamoto, Y., Romo, L. F., & La Joy, J. R. (2015). First-graders' spatial-mathematical reasoning about plane and solid shapes and their representations. *ZDM*, *47*(3), 363-375.

Hwang, W. Y., & Hu, S. S. (2013). Analysis of peer learning behaviors using multiple representations in virtual reality and their impacts on geometry problem solving. *Computers & Education*, *62*, 308-319.

Inan, F. A., & Lowther, D. L. (2010). Factors affecting technology integration in K-12 classrooms: A path model. *Educational Technology Research and Development*, *58*(2), 137-154.
Kennedy, J., & McDowell, E. (1998). Geoboard Quadrilaterals. *The Mathematics Teacher, 91*(4), 288-290.
Kaufmann, H. (2011). Virtual Environments for Mathematics and Geometry Education. *Themes in Science and Technology Education*, *2*(1-2), 131-152.
Moss, J., Bruce, C. D., & Bobis, J. (2015). 5 Young Children's Access to Powerful Mathematics Ideas. *Handbook of International Research in Mathematics Education*, 153.
Murphy, C. (2012). The role of subject knowledge in primary prospective teachers' approaches to teaching the topic of area. *Journal of Mathematics Teacher Education*, *15*(3), 187-206.
Newton, K. J., & Alexander, P. A. (2013). Early Mathematics Learning in Perspective: Eras and Forces of Change. In *Reconceptualizing Early Mathematics Learning* (pp. 5-28). Springer Netherlands.
Sinclair, N. (2014). "Generations of research on new technologies in mathematics education." *Teaching mathematics and its applications*, *3*, 166-178.
Sinclair, N., & Bruce, C. D. (2015). New opportunities in geometry education at the primary school. *ZDM*, *47*(3), 319-329.
Taylor, H. A., & Hutton, A. (2013). Think3d!: training spatial thinking fundamental to STEM education. *Cognition and Instruction*, *31*(4), 434-455.
Ustwo (2014). Monument Valley [computer game]. Available at: https://www.monumentvalleygame.com/mv1
Whiteley, W., Sinclair, N., & Davis, B. (2015). What is spatial reasoning? In Davis, B. (ed.) *Spatial reasoning in the early years: Principle, assertions, and speculations.* New York, NY: Routledge.
Zuckerman, O., & Gal-Oz, A. (2013). To TUI or not to TUI: Evaluating performance and preference in tangible vs. graphical user interfaces. *International Journal of Human-Computer Studies*, *71*(7), 803-820.

MONITORING A TECHNOLOGICAL BASED APPROACH IN MATHEMATICS IN PORTUGAL — THE CASE OF KHAN ACADEMY

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This paper intends to present a project to monitor the implementation of the Khan Academy Platform (KAP), in mathematics classrooms of 1st to 9th grade in Portugal. Based on a partnership between EDUCOM, Portugal Telecom Foundation and the Ministry of Education, a project is underway that involves the training of teachers in the use of the platform (KAP) and its implementation with students from five schools in the outskirts of Lisbon. We present here the theoretical and methodological assumptions that underlie this monitoring, with the objective of characterizing the training of the teachers involved, the students' learning and the role played by the platform in the process of teaching and learning mathematics.

Keywords*: Khan Academy, teacher training, learning platforms*

INTRODUCTION

The use of technologies is a recurring theme in the Portuguese curriculum. Since the 80's of the last century, several methodologies have been proposed that involve the use of technology as a learning tool. For the implementation of these methodologies it is necessary, on the one hand, to prepare teachers, equipping them with skills and knowledge that allow them to integrate the computational tools into their professional practice. On the other hand, it is necessary to plan the activities to be developed in order to integrate the computational tools in the mathematics class, giving visibility to the *modeled curriculum* and *curriculum in action* (Gimeno, 2000). It is expected that actions focused between these curriculum levels and the use of technology lead to more solid and lasting learning by the students.

Based on these assumptions, a pilot training project for teachers using the Khan Academy Platform is underway in a partnership between three institutions: EDUCOM, Portugal Telecom Foundation and the Ministry of Education. This project involves the realization of two Training Workshops, with a total of 30 teachers of Basic Education (1st, 2nd and 3rd Cycles). These workshops have a total of 50 hours of training, distributed throughout a school year, where teachers mobilize half of this time for in-person training to work with the platform (KAP). The remaining hours are designed to work in class with their students. This project also provides for a second year for the implementation and improvement of practices initiated with the training process.

The Platform (KAP), translated into Portuguese, presents a vast set of functionalities where it is possible to visualize videos about specific topics, perform exercises and tasks proposed by the teacher, while it is possible to monitor all the actions carried out by students enrolled in this environment. The fact that this environment have a game character has shown a strong support by the students involved in it.

In this paper we discuss the underlying theoretical assumptions, as well as the research methodology that allows us to monitor the actions of the different actors (trainers, trainees and students) in the different interactions with the platform (KAP) and its integration in the curriculum and the teaching and learning process.

THEORETICAL FRAMEWORK

The theoretical constructs called for the foundation of the work that is intended to be carried out in the monitorization of this project, are essentially based on three dimensions: a) activity theory, b) professional knowledge of teachers and c) students' learning. With activity theory (Engeström, 2001), we intend to broadly frame the actions of the various stakeholders in the project. In this way we can characterize the actions of the various actors in the integration of the technological tool in use. We thus seek support for the processes of instrumentation and instrumentalization (Rabardel 1995) which will help us to interpret how teachers and students relate to technology in general and the platform in particular, reinforcing the semiotic power of the artefact (Bartolini Bussi & Mariotti, 2008).

In order to clarify the way in which teachers take ownership of the artifact, we use the notions of professional knowledge of teachers, where knowledge of content, pedagogical knowledge and technological knowledge are framed (Ball, Thames, & Phelps, 2008). In order to characterize students' learning, we will also use the activity theory, establishing and comparing the different systems of student activity when involved in working with the technological tool.

Activity Theory

Activity Theory initiated by Vygotsky and developed by Leont'ev, assuming its system of collective activity (object oriented and mediated by artifacts) as the unit of analysis, has been developed over three generations. Was initially based on the idea of mediation introduced by Vygotsky in his triangular model that becoming the triad subject - object - mediator artifact, leaving behind the separation between the person and the social environment (Engeström, 2001). In the second generation, centered Leont'ev the unit of analysis is no longer individual and now includes links to other areas involved in a collective activity system, focusing now on the interrelationships between individual objects and communities. The third generation of activity theory could be summarized by Engeström (2001) seeing the object of activity as a moving target for an expansive transformation in activity systems supported by the contradictions as a source of development.

These contradictions are not conflicts since it evolves a dialectic and multi-directional relation supported by Marx and Hegel in the contradictions of the dialectic relation (Figure 1).

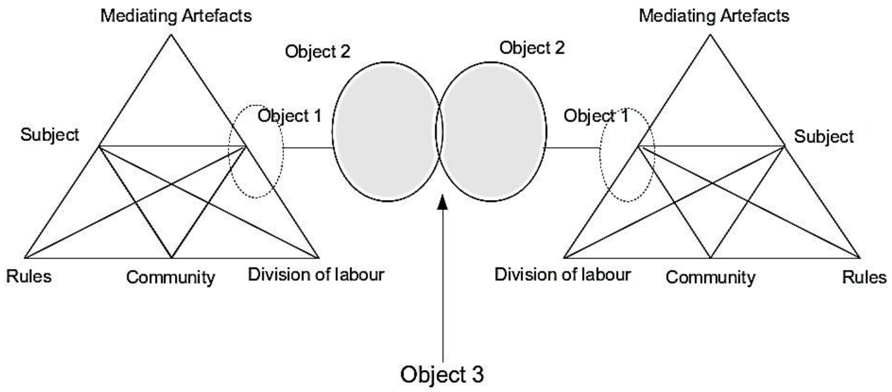

Figure 1 – Interaction between two systems of activity (Adapted from Engeström, 2001, p. 136)

Human activity is mediated by cultural artifacts, which are culturally, historically and socially produced and reproduced, by means of complex and multidimensional relationships (Engeström, 1999). Artifacts have possibilities for action that the user may or may not use. We are concerned with the ways in which teachers appropriate these artifacts, or, following Drijvers and Trouche's (2008), terminology how they become instruments. Instrumental genesis (Rabardel 1995), therefore, is the progressive construction of schemas of use for an artifact by an actor for a given purpose, which was adapted to the study of teaching and learning mathematics by Artigue (2002), Ruthven (2002), particularly in technology-mediated learning. The instrumental genesis will be deepened here in order to understand in depth how the different actors relate to technology and the platform. Although this relation is present in the Activity Theory, it is intended to give a special emphasis to this relation because it represents a very important aspect of the relation between the subject and the artifact.

With the use of technological artifacts it seems to be crucial consider the notion of semiotic mediation (Bartolini Bussi & Mariotti, 2008) to enhance mathematics teaching and learning. In this context, it is important take into account the semiotic potential of the artifact that involves two semiotic links, one between the artefact and the personal signs that emerging from its use and the second between the artifact and the mathematical signs evoked by its use and recognizable as mathematics by an expert.

Professional knowledge of the teacher

The current education systems are organized around a set of dimensions which give it a structure and a coherent organization, which believes it can boost its development and impact on the preparation of future generations. The role of the teacher is considered as one of these dimensions, occupying a central position throughout the process. Given this premise the professional development of teachers becomes a fundamental element so that the process of teaching and learning has the desired impact on students and the educational community in general.

Several studies have been addressing this issue, focusing sometimes on the curricular dimension as a way to promote success, namely success in mathematics. Ball (2003) considers that this intervention is only effective if it is focused on the way teachers teach "In curriculum teaches itself, and standards do not operate independently of professionals' interpretations of them" (p.1). The mathematical knowledge for teaching has thus been a concern of many researchers seeking to identify and discuss the various domains that this knowledge involves.

Ball, Thames and Phelps (2008) use the notion of knowledge of pedagogical content of Shulman (1986), which refers to the existence of knowledge of unique content to teach, trying to identify the competences of teaching, starting from an empirical approach, to understand the knowledge of the content necessary to teach (figure1).

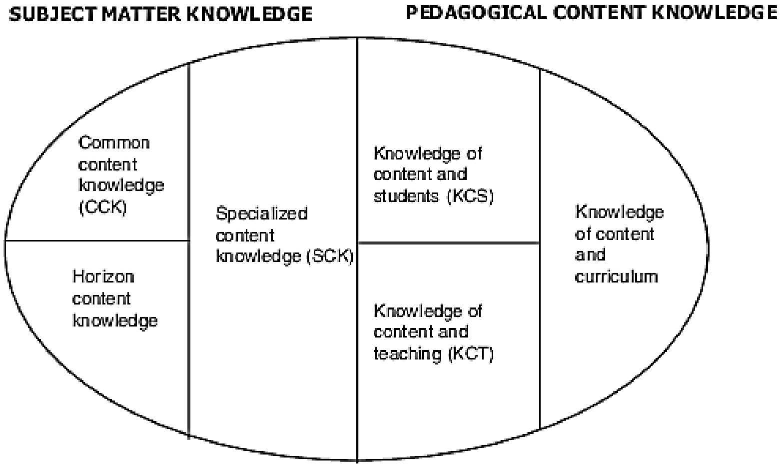

Figure 1: Mathematical knowledge domains to teach (Ball, Thames, & Phelps, 2008)

Given that we live in an age marked by technology and the role of computational tools, Koehler and Mishra (2009) extend TPACK (technological pedagogical content knowledge), which they consider to be teachers' pedagogical knowledge to integrate technology. Koehler, Mishra and Cain (2013) add that the interaction of these forms of knowledge, both theoretical and practical, yields the types of flexible knowledge needed to successfully introduce / integrate technology into teaching. The TPACK results from the intersection of content knowledge, with pedagogical knowledge and technological knowledge (Figure 2)

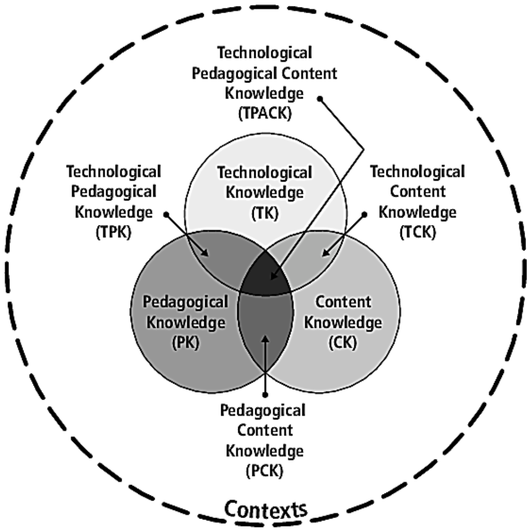

Figure 2: Domains of technological knowledge (Koehler, Mishra, & Cain, 2013)

The identification of these dimensions becomes an asset to create learning opportunities for teachers, since one can not expect that they know or do what they had no opportunity to learn. In this sense, the careful development of courses, workshops and well-designed and managed materials is fundamental (Ball, 2003). It is in this sense that the work with the trainees has been developed, involving them in the use of the platform, both as learners and as teachers of their students in the computational environment.

The theoretical framework presented here establishes and maintains the main constructs to mobilize in monitoring the implementation of the Platform (KAP) in basic education schools involved in the project. This is still a macro perspective who will be detailed as the study progress on the ground. The students' learning will be subject to a deeper analysis taking into account the specificities of the Platform and classroom orchestration conducted by the teacher.

METHODOLOGY

The development of this monitoring study can be considered as a mixed study from the methodological point of view. The qualitative dimension is mostly present in monitoring actions. It involves a descriptive and interpretive analysis of teacher training processes, their appropriation of the technological tool (KAP), the integration of this tool in their pedagogical practice and the students' learning when using the platform. Due to the intrinsic characteristics of the platform, it is possible to monitor student performance in solving the tasks and challenges proposed. It is thus possible to quantify the evolution of the students, from the time it takes to solve each task, the number of hits and errors committed, the working time devoted to each task or subject, among others. The triangulation of these two approaches will allow a better understanding of both the formative process of the teachers and the learnings carried out by the students. It is also intended to carry out some case studies, both with teachers and with students, in order to deepen the different types of knowledge developed.

To develop this monitoring work we used essentially on three analysis tools. One of these tools intends to synthesize an inventory to analyze a task. This inventory involves the following categories: content, process and task type (Pepin, 2012). In the corresponding category of content are taken into account the content domain and connections with math. In the category of processes, in addition to the processes of representation, analysis, interpretation and communication, are taken into account the connections with mathematics. In the category referring to the type of task, procedural fluency, familiarity, context, conceptual understanding, cognitive requirement, mathematical representation and the tools used are taken into account.

The second tool is related to the type of feedback and relationship with the activity, which involves the following phases: Literature review, Development of task analysis scheduling, Task analysis, Assessment, task analysis and national curriculum and Learning steps (Pepin, 2012). These phases involve the following types of feedback: reflexive and diagnostic.

The third tool involves an inventory for the analysis of an artifact produced with a technological resource for the actions in class. In this inventory are highlighted different types of task and their relation with the work to be developed by the student (Teixeira, 2015).

The techniques of data collection are varied and serve different purposes. Teacher training is accompanied by a non-participant observation, where teachers are followed in all training sessions. Field notes taken during the training sessions, the semi-structured interviews with trainers and later with teachers, seek to realize some of the dimensions of their professional development. The observation of teachers' classes when using PKA and the participation of students in these same classes help to understand the process of instrumental genesis in teachers and students. Conducting interviews with students, who will be involved in specific case studies will assess on their learning at the same time that these qualitative data are being crossed with quantitative data provided by KAP.

Participants in the study are teachers from five schools in the west of Lisbon, a total of 30 teachers and 700 students of classes belonging to these same teachers (1st through 9th grade). Teachers are involved in a 50-hour Training Session (25 classroom and 25 at distance, in class work with their students). The training process involves introducing teachers to the platform (PKA) and its use with students.

After this process of instrumentation teachers are invited to develop learning paths based on KAP that will later implement with the students. All students were enrolled on the platform (KAP) and are followed in the course of two processes: the implementation of learning pathways previously designed by teachers and performing specific tasks on the platform that are suggested by teachers as a training supplement taking into account the performance that students demonstrate on the platform.

IN SUMMARY

The project is currently in an early stage with the teachers finishing a phase of appropriation in the use of the platform. The development of lesson planning using KAP is ongoing, depending on the curriculum topics that each teacher intends to implement. Along with this approach, students have already been introduced to the platform (KAP) where they are developing some concepts review tasks, with teacher supervision.

Throughout this process it is possible to identify a general satisfaction of the teachers because they belong to the privileged group that integrates this pilot project. It is possible to see that a large part of the teachers involved are taking the first steps in introducing the technology in their classes, showing a dynamics and involvement that was not observable at the beginning of the training process. The students involved are also very motivated. It is possible to verify that most of them have already used the platform to make their first experiences, and some are already reaching levels of excellence in the field of some elementary concepts. The fact that the platform (PKA) has a strategy game has been singled out by teachers and students as an asset to the strong interaction that comes to check.

The lack of equipment in schools, to ensure fair access to the platform, has been the main problem detected. Many of the student accesses are made from home, out of school hours. Accesses from school are still unsatisfactory, with some students expressing dissatisfaction with the orchestration of classes, where only part of the class can access the platform, while other students are invited to develop paper and pencil tasks.

REFERENCES

Artigue, M. (2002). Learning Mathematics in a CAS environment: the genesis of a re ection about instrumentation and the dialectic between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, *7*(3), 245–274.

Ball, D. L. (2003). *Mathematics in the 21st century: What mathematical knowledge is needed for teaching mathematics*. Paper presented at the Secretary's Summit on Mathematics, U.S.

Department of Education, Washington, DC.
Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching what makes it special? *Journal of Teacher Education*, *59*(5), 389-407.
Bartolini Bussi, M. G., e Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 750–787). Mahwah, NJ: LEA.
Drijvers, P., e Trouche, L. (2008). From artefacts to instruments: A theoretical framework behind the orchestra metaphor. In G. W. Blume e M. K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Vol. 2. Cases and perspectives* (pp. 363–392). Charlotte, NC: Information Age.
Engeström, Y. (1999). Activity theory and individual and social transformation. In Y.Engeström, R. Miettinen, & R.-L. Punamäki (Eds.), *Perspectives on activity theory* (pp. 19-38). Cambridge: Cambridge University Press.
Engeström, Y. (2001). Expansive learning at work: toward an activity theoretical reconceptualization. *Journal of Education and Work*, 14 (1), 133-156.
Gimeno, J. (2000). *O currículo: uma re exão sobre a prática* (3.ª ed.). Porto Alegre: Artmed.
Koehler, M. J., & Mishra, P. (2009). What is technological pedagogical content knowledge? *Contemporary Issues in Technology and Teacher Education*, *9*(1). Retrieved from http://www.citejournal.org/volume-9/issue-1-09/general/what-is-technological-pedagogicalcontent-knowledge.
Koehler, M. J., Mishra, P., & Cain, W. (2013). What is technological pedagogical content knowledge (TPAK)? *Journal of Education, 193*(3), 13-19. Recuperado de http://www.bu.edu/journalofeducation/current-issues/journal-of-education-2000-2010/volume-193/
Rabardel, P. (1995). *Les hommes et les technologies*. Paris: Armand Colin.
Ruthven, K. (2002). Instrumenting mathematical activity: Re ections on key studies of the educational use of computer algebra systems. *International Journal of Computers for Mathematical Learning*, *7*(3), 275–291.
Shulman, L.S. (1986). Paradigms and research programs for the study of teaching. In M.C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed.). New York: Macmillan.
Pepin, B. (2012). Task Analysis as “Catalytic Tool” for Feedback and Teacher Learning: Working with Teachers on Mathematics Curriculum Materials. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to 'lived' resources: mathematics curriculum materials and teacher development* (pp. 123-142). Berlin: Springer.
Teixeira, P. (2015*). Construindo novas ferramentas didáticas em matemática: professores, aula e recursos tecnológicos*. Doctoral Thesis. Lisbon: Universidade Nova de Lisboa, Faculdade de Ciências e Tecnologia.

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Revisão do paper

Notation Grid

- Focus and rationale : 6 - Research question to be precised
  The paper presents a project of in-training sessions aiming at using the Khan academy platform in the classes.

- Theoretical and methodological (TMF) : 6 - TMF might be further improved
  What I miss is the articulation of the different frameworks that are presented. For example, is it necessary to have references to the instrumental genesis and what does it brings that is not studied through the activity theory?

- Statement and discussion of results : 3 - Missing or unfounded results
  The weaknes of this paper in my sense lies in the absence of significative results both regarding the in-training and the effective interest of the use of the platform in the maths classes.

- Clarity and relevance to ICTMT 13 : 6 - Relevant but needs further improvments

› Comment author

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THE DESIGN AND USE OF OPEN ONLINE MODULES FOR BLENDED LEARNING IN STEM TEACHER EDUCATION

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Blended learning, a teaching format in which face-to-face and online learning is integrated, nowadays is an important development in education. Little is known, however, about its affordances for teacher education, and for domain specific didactical courses in particular. To investigate this topic, we carried out a design research project in which teacher educators engaged in a co-design process of developing and field-testing open online learning units for mathematics and science didactics. The preliminary results concern descriptions of the work processes by the design teams, of design heuristics, and of typical ways of collaborating. These findings are illustrated for the case of two of the designed online units on statistics didactics and mathematical thinking, respectively.

Keywords: blended learning, design teams, mathematics education, STEM, teacher education

INTRODUCTION

In august, 2015, a new curriculum for mathematics in upper secondary education (grades 9-12) was introduced in the Netherlands. Among others, this curriculum has a stronger focus on mathematical thinking and on new approaches to statistics education, based on data sets made available through the use of ict. The crucial factor in curriculum innovation, however, is to make these innovations impact on classroom practice (Anderson, 1997; Fullan, 2007).

Obviously, there is a responsibility for in-service teachers, as well as for teacher education institutes. The former have to ensure that they are capable of teaching the new curricula adequately; the latter have to make sure that their students, being prospective mathematics teachers, are not only familiar with the new curricula, but also with ways of *how to* teach them. Exploiting the potential of information and communication technology still deserves special attention (Hegedus et al., 2017).

When addressing this 'how to', blended learning comes into play. Roughly speaking, blended learning means *blending* face-to-face education with online learning activities. Nowadays, more than twenty-five years after the introduction of the worldwide web as part of the internet (Berners-Lee, 1989), a staggering amount of digital resources for the teaching and learning of mathematics is available online. This leads educational designers and teachers to selecting, re-designing and arranging resources in order to orchestrate their students' learning (Drijvers et al., 2010). For the case of teacher education, however, and for courses on domain-specific didactics in particular, the affordances of blended learning remain largely unexplored.

In this paper we describe how online learning units for pre-service teacher education for secondary mathematics in a blended learning context are designed, implemented and evaluated by design teams. We describe the design of two specific units, one for mathematical thinking, the other for exploratory data analysis utilizing ICT. The results of using these learning units in pre-service teacher training will be available later this spring.

THEORETICAL FRAMEWORK

In higher education, blended learning has been on the rise since the early 2000's. With respect to terminology, quite a few buzz words came along. In fact, one might wonder if educational goals have fundamentally changed since researchers from the University of Illinois in 1960 utilized a mainframe computer with work stations for their students for computer assisted learning, which they called Programmed Logic for Automatic Teaching Operations (PLATO, see Woolley, 1994). Terminology evolved from computer-assisted (or -based or -supported) learning to intelligent tutoring systems (Anderson, 1995), E-learning (Clark & Mayer, 2008), with blended learning as a popular teaching approach nowadays (Bonk & Graham, 2006). In retrospective, all terminology boils down to roughly the same issue, i.e., how to arrange the educational resources -including information and communication technology- into an educational design that optimizes learning? What we appreciate in the term 'blended learning' is that it explicitly points at the fact that there is more than one medium to be addressed when designing instruction.

From the perspective of learning theory, scientific insights have evolved as well: from the behaviourist view on human learning (Skinner, 1954), suitable for computer assisted mastery learning (Skinner, 1958), to the nowadays accepted social constructivist view, as initiated by Vygotsky (1962), which can be supported by a more open learning environment. Blended learning is a technological paradigm that suits this view on learning and teaching.

A major didactical issue with respect to blended learning is how to arrange the interplay between online and face-to-face mathematical activities, and how to co-design such arrangements. In this paper, we address this issue for the case of domain-specific didactics courses within pre-service mathematics teacher training. In this way, we address the following research question:

> How to collaboratively design, evaluate and disseminate digital blended learning units for mathematics teacher education?

METHOD

The context of this study is a small, one-year project granted by the Dutch ministry of education and supervised by SURFnet, the collaborative ICT organisation for Dutch education and research [1]. The aim of the project is to co-design, evaluate and disseminate four blended modules for pre- and in-service teacher training, and for domain specific STEM didactics in particular. In this paper we focus on the design of two of these units, one on the topic of mathematical thinking, and one on statistics didactics.

For each of the modules, a design team was set up. Each design teams consisted of three teacher educators: one from the HU University of Applied Sciences, one from Utrecht University, and one from another teacher training institute in the Netherlands. The latter would facilitate dissemination and bring in a wider view. Most of the designers were experienced teacher educators, but had limited experience with (the design of) blended learning resources.

As each of the designers had limited time for the project (like 40 hours over the whole period of one year), the coordinating team -this paper's authors- decided to organize short, intensive collaborative "boot camp" design sessions. During the fall of 2016, three of such one-day boot camps were organized, during which the design teams engaged in their co-design, but informal exchange between teams was also possible. Camera teams were available, as well as tools such as light boards for the production of video clips.

A collaborative online design environment was set up, so that the designers could continue their co-design activities between the boot camp sessions. The ICT environment was provided by Kennisnet, a Dutch semi-governmental organisation for ICT in education [2]. In this way, a blended design approach was made possible.

The different teams met during boot camp days to discuss overarching topics, such as module layout and structure, the guidelines for use that a teacher educator might use. During the design process, design heuristics and decisions were monitored. After the design period, the use of the blended modules will be field-tested in didactics courses by teacher educators all over the country, including co-designers and educators not involved in the design. To evaluate the experiences, the field tests will be monitored through pre- and post-interviews with the educators.

RESULTS

Unfortunately, as the monitoring process of the field tests is currently ongoing during spring 2017, its results will only be available in July. Therefore, we now focus on the design process, which we will describe subsequently for the two units, one on statistics didactics and the other on mathematical thinking.

Case 1: A learning unit on statistics didactics

As a first case of designing an open online learning unit for a blended course on mathematics didactics, we now briefly describe the design process of a unit on statistics didactics for pre-service teacher education. Based on general ideas on exploratory data analysis (Tukey, 1977) and the analysis of large data sets through the use of ICT, the statistics curricula have been reformed recently. Therefore, statistics didactics is an issue in teacher education and this explains the choice for this topic.

The design team consisted of two pre-service teacher educators and one professor in mathematics education. As the team members had not collaborated before so closely, the first day of the three-day design process was spent on getting to know each other and exploring the unit's theme. A joint dropbox folder had been created to exchange ideas and existing materials. It was noticed that many mathematics teachers, due to their education, only have limited knowledge about statistics and the new approach to it, so that some content knowledge should be intertwined with a pedagogical and didactical approach. As a consequence, the team decided to focus on core aspects of statistics, namely (1) Describe data, and (2) Beyond data.

During the second design boot camp, the outline was elaborated. In the Describe data part, particular attention is given to data visualization, levels of measurement, and statistical literacy. The Beyond data part focuses on correlation and causality, the interpretation of p-values, and of confidence intervals. These topics were selected because on the one hand, we expected them to be beneficial to teachers' content knowledge, and on the other hand we identified them as didactical challenges while teaching.

During the third and final design day, special attention was paid to design tasks for the teacher-students. Also, the team worked on the comments provided by an external review committee.

In the design process, a mix was made of existing resources such as video clips, text books, research papers, and newly designed resources such as tasks for teacher-students and guidelines for the teacher educator, and dedicated video clips. On the one hand, it made sense to make use as much as possible from existing resources. On the other hand, the need was felt to have dedicated resources that fit well to the specific Dutch situation and curriculum. Figure 1 shows a still of a new clip made with light board technology. Figure 2 shows an extract of a dialog between Dutch mathematics teachers' Facebook group on a particular problem, which is used in the online learning unit to enhance discussion between students during the face-to-face part of the blended course.

The results of the design are available online [3]. As part of the ongoing design process, input from other teacher educators is expected to further improve and extend the unit in a collaborative way.

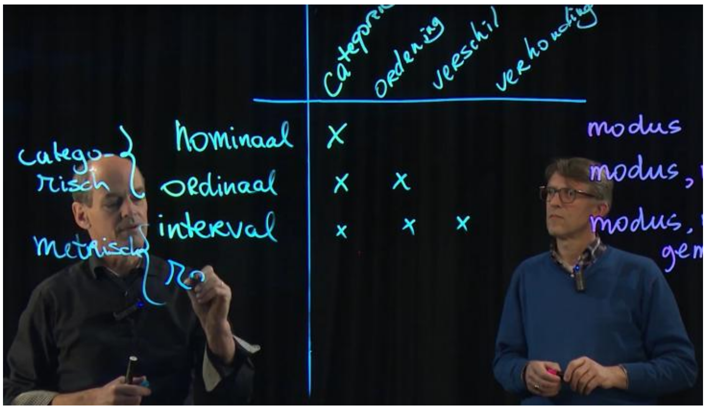

Figure 1. Still from a video made with light board technology

Figure 2. Copy of a dialog on the Dutch mathematics teachers' Facebook Group

Case 2: A learning unit on mathematical thinking

The second case we describe concerns an open online unit about didactics for fostering mathematical thinking. Attention to this topic is evident in the international research community (e.g., see Devlin, 2012; Schoenfeld, 1992) and was invigorated in the Netherlands by recent curriculum developments in Dutch secondary education (Drijvers, De Haan, & Doorman, submitted).

The team that designed the unit about mathematical thinking consisted of three teacher educators, all of whom had some prior experience with this topic. One of them also developed and taught a course on mathematical thinking as in-service training for teachers.

The outlines of the online unit were quickly decided on. One reason for this was that the members of the design team already had collaborated in different projects (although not about mathematical thinking) and also had shared ideas about the topic under concern. The unit was planned to consist of several self-contained student activities divided into three categories: (i) designing classroom tasks that stimulate mathematical thinking, (ii) supporting such classroom tasks in the classroom, (iii) assessing proficiency in mathematical thinking. Having established an outline, the ideas for the individual student activities emerged organically during the three boot camp days. Here follows an impression of this process.

During the first day the team focused on a key article (Swan, 2005) as inspiration for the design of the first student activity. This resulted in a small set of materials, including a video clip, and a guide for teacher educators how the material could be used. Figure 3 gives an impression. The day was further spent in selecting appropriate study materials from a large set that the design team had collected in advance.

During the second boot camp day, the idea emerged of using materials produced by some of the experienced teachers that had participated in the in-service training of one of the design team's members. Those materials consisted of a classroom exercise aimed at stimulating mathematical thinking and a description about how it worked out in practice. The team planned a filmed interview with one of those teachers and worked out ideas for four film projects. These film projects shared the same set-up: each focused on an exercise explicitly designed to stimulate mathematical thinking; and each consisted of a sequence of three clips A, B and C. Clip A showed two team members discussing the exercise before it was used in practice (see Figure 3). They tried to predict what kind of thought processes the question would evoke in pupils. Clip B was filmed inside a school building. A pupil was asked to work on the set question, and was then interviewed about the strategies he or she had used. Clip C showed the team members again, but now they reflected on their experiences with the pupils. The film projects were placed on the website together with suggestions for use in teacher education. The suggestions involved a choice for the teacher educator. He could either just use the clips B together with digital copies of the exercises, or use the whole series of clips modelling how to discuss potential thought provoking questions. In the former case, his students can predict and reflect on the quality of the exercises in a whole-class discussion. In the latter case, students can be given the task to try it out themselves with other (e.g., self-designed) exercises in their own classrooms.

Between the second and third boot camp day an external reviewing committee provided feedback on the site. Besides useful comments on usability, the remark was made that the important aspect of 'the culture of answer getting in mathematical classrooms' had been ignored. During the third boot camp, the team tried to deal with this by adding materials about experiences of working teachers. They also made a film interview with an expert on the subject.

The results of the design process are available online [4]. Input from other teacher educators is expected to further improve and extend the unit in a collaborative way.

Figure 3. An impression of the web site (in Dutch)

CONCLUSION

The research question is how to collaboratively design, evaluate and disseminate digital blended learning units for mathematics teacher education. At present, we can only draw conclusions with respect to the design phase. Results on the evaluation are expected in July, followed by results about dissemination in October. Concerning the design aspect, the preliminary conclusions fall into three categories: (1) the composition of design teams, (2) design heuristics, and (3) ways of collaborating.

The design teams were each composed of three experienced teacher educators from different institutes. In consequence, the team members could share experiences and materials. This resulted in a shared collection of existing materials that were already, but unknown or inaccessible outside individual institutes. The project was also instructive for the team members – both with regard to their personal subject knowledge (e.g., statistics didactics) as to blended learning skills (e.g., camera experience). A drawback of using mixed teams is that people need time to getting to know each other and to form a joint vision on the subject at hand. Although this is important for a fruitful collaboration, care must be taken that teams dwell too long in this phase. This leads to the first conclusion.

1. Small design teams of experienced teacher educators from different institutes leads to boundary crossing between institutes, resulting in (i) rich material and (ii) professional development of the educators themselves, although a pitfall is that (iii) too much time may be spent on discussion rather than on the actual design.

The most important design heuristics from the start were that we aimed at learning units which were open online and blended. ‘Open online’ implied that the materials would eventually be published on the web under a creative commons license [5]. Without any difficulty this turned out to be a tenable mind set, although extra care was given in using materials from others that could have copyrights on them. In practice, ‘blended’ meant that materials should at least encompass texts, film clips and descriptions for student activities. These student activities involved both classroom tasks supervised by a teacher educator and online tasks.

Other design heuristics emerged in the course of the process: we used mid-session intervals during the boot camps to discuss ideas and explicate some shared heuristics. The most important one that emerged in this process concerned the target audience. In fact, two target groups were recognized: the student teachers and the teacher trainers. This was apparent in the materials: film clips, worksheets, etc. aimed at the former, guidelines and suggestions for use at the latter. But design teams had the teacher trainer in mind also in another way. A comparison was made with the way an educator uses a handbook, selecting exercises from it, skipping or supplementing content, etc. It was felt that this level of autonomy should also be provided in the learning units that were designed.

2. Digital blended learning units should be designed with both student teacher and teacher educators in mind. Toward teacher educators, a balance need be found between guidance on the one hand and autonomy on the other.

As explained above, we stimulated collaboration by organizing boot camps. These boot camps were intensive days of working in the design teams, apart from a plenary meeting at the start of the project and the aforementioned mid-session discussions during lunch. We facilitated the teams by setting a time and place and organizing technical support for the film clips. Teams sat together and worked in ways they could decide for themselves. This was successful: participants on the one hand experienced a large measure of autonomy – which, we belief, had a positive effect on their motivation – while on the other hand were encouraged to reserve three full days outside their usual working habitats. This last aspect could be the most difficult one to generalize, since we experienced that it is difficult to schedule days where everyone is available – especially in an extra-institutional context.

3. Scheduling design sessions were teams can collaborate for several hours with full focus on producing materials makes it feasible to construct digital blended learning units in a short time span. Readily available technical assistance during these sessions lowers the barrier for producing film clips.

These preliminary conclusions concern the design process. This spring, teacher educators throughout the Netherlands will field-test (parts of) the designed learning units. To monitor this, they fill in an online questionnaire beforehand, to assess their intentions and ideas. After the field test, they will receive a second questionnaire to assess their appreciation of the units as well as the ways in which they used them in practice. This will be followed by a limited number of interviews with some of these educators. Through the analysis of these data, we hope to be able to answer questions on the evaluation and dissemination of the learning units, and to extrapolate them to more general recommendations on the process of co-designing blended learning for teacher education.

NOTES

1. See https://www.surf.nl/en/innovationprojects/customised-education.html
2. See https://www.wikiwijsleermiddelenplein.nl/
3. For the current state of the unit (in Dutch) see http://maken.wikiwijs.nl/86112/Didactiek_van_statistiek
4. For the current state of the unit (in Dutch) see http://maken.wikiwijs.nl/85927/Wiskundige_denkactiviteiten
5. See https://creativecommons.org/licenses

REFERENCES

Anderson, J. R., Corbett, A. T., Koedinger, K. R., & Pelletier, R. (1995). Cognitive tutors: lessons learned. *The Journal of the Learning Sciences, 4*(2), 167-207.

Anderson, S. E. (1997). Understanding Teacher Change: Revisiting the Concerns Based Adoption Model. *Curriculum Inquiry, 27*(3), 331-367

Berners-Lee, T. (1989). *Information management: a proposal*. Geneva: CERN.

Bonk, C. J., & Graham, C. R. (2006). *The Handbook of Blended Learning: Global Perspectives, Local designs*. Hoboken, NY: John Wiley & Sons.

Clark, R. C., & Mayer, R. E. (2008). *e-Learning and the Science of Instruction: Proven Guidelines for Consumers and Designers of Multimedia Learning*. San Francisco, CA, USA: Pfeiffer.

Devlin, K. (2012). *Introduction to Mathematical Thinking*. Palo Alto, CA: Keith Devlin.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75*(2), 213-234.

Drijvers, P., De Haan, D., & Doorman, M. (submitted). *Scaling up mathematical thinking; a case study on the implementation of curriculum reform in the Netherlands.*

Fullan, M. (2007). *The new meaning of educational change (Fourth Edition)*. New York: Teachers College Press.

Hegedus, S., Laborde, C., Brady, C., Dalton, S., Siller, H.-S., Tabach, M., et al. (2017). *Uses of Technology in Upper Secondary Mathematics Education. In Uses of Technology in Upper Secondary Mathematics Education* (pp. 1-36). Cham: Springer International Publishing.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning: A Project of the National Council of Teachers of Mathematics* (pp. 334-370). New York: MacMillan.

Skinner, B. F. (1954). The science of learning and the art of teaching. *Harvard Educational Review, 24*, 86-97.

Skinner, B. F. (1958). Teaching machines. *Science, 128*, 969-977.

Swan, M. (2005). *Improving learning in mathematics: challenges and strategies*. London: Department for Education and Skills Standards Unit.

Tukey, J. W. (1977). *Exploratory Data Analysis*. Reading, MA: Addison-Wesley.

Vygotsky, L. S. (1962). *Thought and Language*. Cambridge, MA: The MIT Press.

Woolley, D. R. (1994). The emergence of online community. *Computer-Mediated Communication Magazine*, *1*(3), 5-5.

TEACHING LOCUS AT UNDERGRADUATE LEVEL: A CREATIVITY APPROACH

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In this paper, we present our experience, while working in the MC Squared project, with the design and evaluation of educational digital resources aiming at promoting creative mathematical thinking among undergraduate students. The resources, called "c-books" (c for creative), are produced within an innovative socio-technological environment by a community gathering together computer scientists and researchers in mathematics and mathematics education. The paper highlights the importance of teaching loci curves at undergraduate *level as an introduction to implicit equations. It presents the design choices providing the c-book with affordances to promote creativity in mathematics in terms of personalized non-linear path, constructivist approach, and meta-cognition based activities, among others. The paper also presents experts' a priori evaluation of the c-book mathematical creativity potential.*

Keywords: Creative mathematical thinking, socio-technology environment, locus, implicit equations, experimental geometry.

INTRODUCTION

Historically, mathematical curves essentially occurred as loci, e.g. a parabola as the locus of all points having the same distance from a given fixed line and a given fixed point outside the line. The concept of a curve within a coordinate system was only quite recently developed by Descartes, Fermat, and others in the 17th century – about 2000 years after the Greek had performed quite detailed studies of various curves, and also had created interesting and important new curves for specific purposes, purely via their description as loci (Boyer, 2004, pp. 74-102).

The curves in Descartes' "La Geometrie" (Descartes, 1637, see also Boyer, 2004, pp. 74-102) then arise naturally as implicit curves as a result of solving systems of implicit equations where each equation represents a condition on the geometric objects involved. In the following centuries, curves and in particular the special cases of graphs of functions in one variable were being studied deeply. But at the beginning of the 20th century when Felix Klein was working a lot on the question of how to teach mathematics, implicit equations still played a very important role for him. E.g., in the first section on algebra in his Mathematics from a Higher Standpoint (Klein, 1924, part I, pp. 93-109) when he discusses simple examples such as graphs of quadratic functions, he immediately uses implicit equations as well, namely some discriminants describing the reality of the roots of the functions. It was only later that some others misinterpreted his ideas to focus on the importance of functions in a too narrow way, namely only to functions from R to R.

Besides the historical importance of loci and their often implicit equations there are many reasons for using them at school level. The following is an important one: The description of a curve as a locus gives a more intrinsic description and also a more operational description than an equation. So, even for curves appearing as graphs of functions, looking at them as geometric loci often deepens their understanding. Moreover, using loci and implicit equations early in teaching is a good preparation for studying implicit equations later in linear algebra, e.g. planes and the classification of quadrics. It is also a good companion to the implicit equation of a circle which is otherwise a quite isolated example of such an equation in many cases (sometimes, ellipses are also mentioned, at least in their standard form $x^2/a^2+y^2/b^2=1$, but the fact that implicitly described curves are a most natural thing to consider is often not mentioned).

The importance of implicit loci arises even more at the undergraduate level: for example, a curve might be associated not only with an explicit equation, a function graph, a parametrized curve, or an implicit algebraic equation but as well with solutions of differential equations. It is important to understand function graphs, implicit equations and parametrized curves as loci in order to be able to fully grasp differential geometry tools such as tangent line or plane, curvature, osculating circles or ellipsoids. In real life mathematics or engineering, most objects are loci of some sort. Control theory for example deals with trying to keep a mobile position not far from a target trajectory, with the help of integral and differential calculus. The investigation of soft loci with a dynamic geometry system (Healy, 2000; Laborde, 2005) is very helpful in building this picture in the mind of students.

The flexible production of loci is of paramount interest in industry and design to define curves and surfaces used in computer aided design, such as Bézier curves and their variants (see Piegl, 2013). The main feature of those is the fact that they can be described in many different ways: as parametrized curves, as implicit curves, and also as loci. All those descriptions have their advantages for the application at hand such as: Through the parameterization, many properties of the curves can be studied easily; with the implicit equation, it is straightforward to decide if a given point is on the curve, or on one side or the other side; and the description as a locus provides a numerically robust and quick way to compute points on the curve, draw it or 3D-print it.

Promoting creative mathematical thinking (CMT) is a central aim of the European Union by being connected to personal and social empowerment for future citizens (EC, 2006). It is also considered as a highly valued asset in industry and as a prerequisite for meeting economic challenges. Exploratory digital media provide users with potential for developing CMT in unprecedented ways (Hoyles & Noss, 2003; Healy & Kynigos, 2010). Yet, new designs are needed to support learners' engagement with CMT in collectives using dynamic digital media.

The MC Squared project, briefly presented in the next section, looks for new methodologies that would assist designers of digital educational media to explore, identify and bring forth resources stimulating more creative ways of mathematical thinking. The paper focuses on the design of one such resource, the “Experimental Geometry” c-book, highlighting the design choices and the resource affordances to foster CMT in its users. Concluding remarks bringing forward factors stimulating creativity in the collaborative design of digital educational resources are proposed in the final section. The research work presented in this paper is related to (Trgalova, El-Demerdash, Labs, &Nicaud, 2016), submitted to ICME13, with more emphasis on the locus theoretical background and its importance to introduce implicit equations at the undergraduate level. Design choices to foster CMT are as well detailed here.

THE MC SQUARED PROJECT

MC Squared project (http://mc2-project.eu/) aims at designing a software system, the “c-book environment”, to support stakeholders from creative industries producing educational content to engage in collective forms of creative design of appropriate digital media. The c-book environment provides an authorable tool including diverse dynamic widgets, an authorable data analytics engine and a tool supporting collaborative design of resources called “c-books”. The project studies the processes of collaborative design of c-books intended to enhance CMT.

CREATIVE MATHEMATICAL THINKING

Based on a literature review studying creativity (El-Demerdash, 2010; El-Demerdash & Kortenkamp, 2009; Haylock 1997; Weth 1998, among others), CMT has been defined in the project as an intellectual activity generating new mathematical ideas in a non-routine mathematical situation. Drawing on Guilford's (1950) model, the generation of new ideas shows fluency (the ability to generate quantities of ideas), flexibility (the ability to create different categories of ideas), originality (the ability to generate new and unique ideas that others are not likely to generate), and elaboration (the ability to redefine a problem to create others by changing one or more aspects).

THE "EXPERIMENTAL GEOMETRY" C-BOOK

The notion of geometric loci of points is the topic of the "Experimental Geometry" c-book presented in this section. According to Jare and Pech (2013), this notion is difficult to grasp and technology can be an appropriate media to facilitate its learning. The authors suggest one way is to use dynamic geometry software to "find the searched locus and state a conjecture" and a computer algebra system to "identify the locus equation".

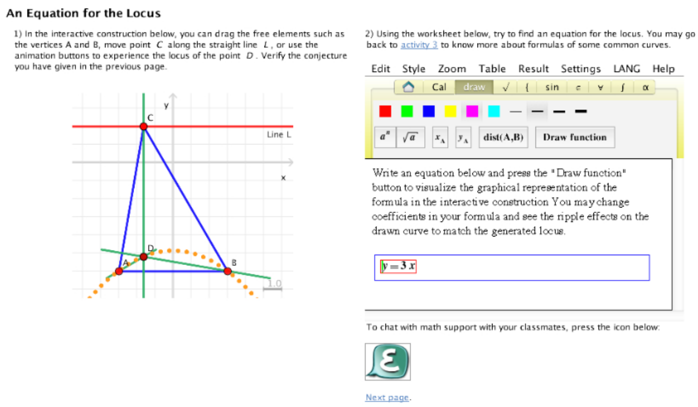


Fig. 1: A screenshot of a c-book page showing three widgets: Cinderella, EpsilonWriter and EpsilonChat.

The challenge in designing this c-book was to exploit c-book technology affordances to propose a comprehensive study of geometric and algebraic characterization of some loci within the c-book. We decided to create activities aiming at studying loci of important points in a triangle. These loci (for example locus of the orthocenter) are generated by the movement of one vertex of a triangle along a line parallel to the opposite side (see Fig. 1). These are classical problems from the field of geometry of movement that were proposed for teaching purposes even before the advent of dynamic geometry (Botsch, 1956). Elschenbroich (2001) revisits the problem of locus of the orthocenter in a triangle with a new media, dynamic geometry software. El-Demerdash (2010) uses this example to promote CMT among mathematically gifted students at high schools.

The c-book invites students to experiment geometric loci generated by intersection points of special lines of a triangle while one of its vertices moves along a line parallel to the opposite side (see Fig. 2b). The activity can give rise to a number of various configurations, which makes it a rich situation for exploring, conjecturing, experimenting, and proving.

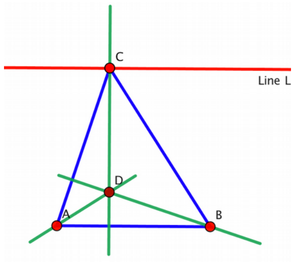


Fig. 2a: Geometrical situation proposed with Cinderella (Act. 1, page 1).

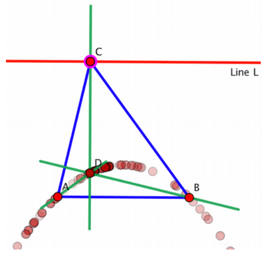
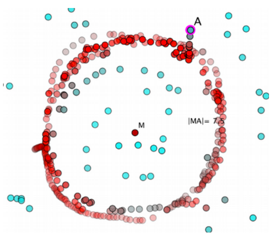


Fig. 2b: Visualizing the trace of D while C moves on the red line (Act. 1, page 2).

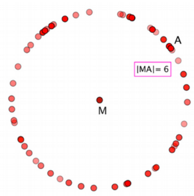
The c-book is organized in three sections to which the student can refer in case he needs or wants it. The first section proposes the main activity called "Loci of special points of a triangle". It starts by inviting the students to explore, with Cinderella (http://www.cinderella.de) dynamic geometry software, the geometric locus of the orthocenter of a triangle while one of its vertices moves along a line parallel to the opposite side (Fig. 2a). The students are asked to explore the situation, formulate a conjecture about the geometrical locus of the point D (page 1), and test the conjecture (page 2) by visualizing the trace of the point D (Fig. 2b).

On page 3, the students are asked to find an algebraic formula of the locus, which is a parabola. The formula is to be written with the EpsilonWriter software (http://epsilonwriter.com/en/) and the interoperability between this widget and Cinderella allows the students to check whether the provided formula fits the locus or not.

The next pages invite the students to think of, explore, and experiment the geometrical loci in other similar situations, such as the locus of the circumcenter (intersection of the perpendicular bisectors), the incenter (intersection of the angle bisectors) or the centroid (intersection of the medians). Other situations can be generated by considering the intersection of two different lines, for example a height and a perpendicular bisector. Twelve such situations can be generated. For each case, one page is devoted offering to the students:



a



b

Fig. 3: Circle as a locus of points that are at a given distance from a given point: (a) "soft" locus, and (b) "robust" locus.

- a Cinderella widget with a triangle ABC such that the vertex C moves along a line parallel to [AB] and a collection of tools for constructing intersection point, midpoint, line, perpendicular line, angle bisector, locus, as well as the tool for visualizing the trace of a point;
- an EpsilonWriter widget enabling a communication with Cinderella;
- EpsilonChat widget enabling remote communication among students.

The second section called “The concept of geometric locus” introduces the concept of locus of points. It starts by a baby example, leading the students to “discover” the fact that a circle can be characterized as a locus of points that are at the same distance from a given point (page 1). The students first experiment a “soft” locus (Healy, 2000; Laborde, 2005) of a point A placed at the distance 6 cm from a given point M (Fig. 3a), and then they verify their conjectures by observing a “robust” construction of the circle centered at A with a radius 6 cm (Fig. 3b).

The next page is constructed in a similar way in order to allow the students to explore perpendicular bisector as a geometric locus of points that are at a same distance from two given points. Finally, the page 3 proposes a synthesis of these two activities and provides a definition of the concept of geometrical locus of points.

The third activity, “Algebraic representation of loci”, proposes a guided discovery of algebraic characterization of the main curves that can be generated as loci of points as those in section 2.

Design choices and rationale

Personalized non-linear path: The c-book is designed to allow students to go through it according to their knowledge and interest. They are invited to enter by the main activity (section 1). However, the concept of geometric locus is a prerequisite. In case this knowledge is not acquired yet, or the students need revising it, they can reach the section 2 by an internal hyperlink from various places of the main activity. Similarly, section 3, which allows the students to develop knowledge about the algebraic characterization of some common curves that is useful in the main activity, is reachable from the main activity. Thus the students can “read” the c-book autonomously, in a non-linear personalized way, deepening their knowledge about geometric or algebraic aspects of loci of points according to their needs.

Promoting creative mathematical thinking: The c-book is designed in a way to support the development of creative mathematical thinking through promoting its four components (fluency, flexibility, originality, and elaboration) among undergraduate students. First, the main activity is designed in a way to call for students’ elaboration: they are invited to modify the initial situation by considering various combinations of special lines in a triangle, whose intersection point generates a locus to explore. Fluency and flexibility are fostered by providing the students a rich environment in which they can explore geometric configurations and related algebraic formulas while benefitting from a feedback allowing them to control their actions and verify their conjectures (see feedback and learning analytics section). Specific feedback is implemented toward directing students to produce different and varied situations and help them to break down their mind fixation by considering yet different configurations, such as two different kinds of special lines in a triangle passing through the movable vertex (e.g. a height intersecting with an angle bisector), and then the intersection of two different lines that do not pass through the movable vertex. The c-book provides the students not only with digital tools enabling them to explore geometric and algebraic aspects of the studied loci separately, but also with a so-called “cross-widget communication” affordances between Cinderella, a dynamic geometry environment, and EpsilonWriter, a dynamic algebra environment, which makes it possible to experimentally discover the algebraic formula that matches the generated locus in a unique way; this feature may contribute to the development of original approaches by the students.

Constructivist approach – Learning by doing – Guided discovery: The c-book activities in both section 2 and section 3 are developed based on the constructivist learning theory practices through guided discovery approach in order to enable students to create new experiences and link them to their prior cognitive structure supported with learning opportunities for conjecturing, exploration, explanation, and mathematics communication.

Meta-cognition - Learning by reflecting and promoting mathematics communication skills: All c-book sections end up with a meta-cognitive activity that has been designed to encourage students to reflect about their learning and enable them further understanding, analysis, and control of their cognitive processes. These activities have been also designed aimed at the development of students' written mathematical communication skills through the use of EpsilonChat mathematical chat engine.

Technological development: An outstanding feature of the c-book environment is the fact that it does not only come with a large number of existing widgets in the mathematical context from several different European developer teams, but it also comes with so-called widget factories, one from each of the developer teams allowing authors to generate their own specialized widgets, if they want. The interesting point of this is that all these diverse widgets work perfectly together with the back-end of the environment and they can even collaborate with each other within pages. For example, the dynamic algebra system EpsilonWriter is an interesting tool for manipulating formulas and equations via a unique drag and drop interface (right part of Fig. 1). But it neither has a built-in function graphing tool nor geometric construction capabilities. These aspects are some of the specialties of the programmable dynamic geometry system Cinderella (left part of Fig. 1).

Later, when working with the c-book, a student may have produced a reasonable equation for a function within EpsilonWriter, and she can visualize it by using the 'draw' tab. The graph of the function will be shown in the Cinderella construction at the right. For the student, this is visually clear and intuitive; but technically a lot is happening in the background: First, the equation will be sent from the EpsilonWriter software via a standardized protocol to the c-book environment and from there to the Cinderella software which finally visualizes it as a part of the interactive construction. All this is possible within the c-book player running in a web-browser.

As the example above illustrates, cross-widget communication is a quite powerful feature. In this case, it opens the opportunity for the c-book author to make explicit connections between different representations of a mathematical object: a curve represented as a geometrical locus, its formula or equation with the ability to modify it dynamically, and a geometric figure combining both the construction as a locus and the visualization of the curve given by the equation. Within the c-book environment, such opportunities exist in other branches of mathematics as well: e.g., via this mechanism statistics and probability widgets may be connected to geometry, algebra, a number theory widget or even to a logo programming widget, to name just a few more use cases.

Learning analytics and feedback: Another advantage of the c-book environment and the widget factories working with it is that it is easy for a c-book is to decide which of the student's actions should be logged to a database while she is studying the c-book. There have been many different types of logs implemented in this c-book that enable the teacher to capture the student's path in studying the c-book. Two types of feedback are provided to students, while they are studying the c-book to guarantee their smooth move from page to page and switch between the c-book activities: technical feedback and mathematical or educational feedback for CMT, breaking down mind fixations.

A PRIORI EVALUATION OF THE C-BOOK CMT AFFORDANCES

The c-book CMT affordances were evaluated by "experts" (researchers involved in the MC Squared project). This a priori evaluation was guided by the following two research questions:

R.Q1 - Which of the four cognitive components of CMT: fluency, flexibility, originality and elaboration, and social and affective aspects have been better integrated and promoted through the design of the c-book units? That is, what affordances are perceived by the evaluators as enhancers of these components?

R.Q2 - Is there any correlation among the cognitive components of CMT, as perceived by the evaluators?

In order to answer the above research questions we used an evaluation tool called "CMT Affordances Grid" (Appendix A). This tool was developed and refined within the MC Squared project. The grid contains three sections. The 13 first items evaluate the c-book affordances towards the development of mathematical creativity in users/students. These items address the c-book affordances such as nature of the activities or variety of representations of mathematical concepts at stake and ask the evaluators to what extent these affordances are likely to enhance the user's cognitive processes (fluency, flexibility, originality, elaboration). The second and third sections deal with social and affective aspects of the c-book that are likely to enhance mathematical creativity in its users.

As for the first aspect, the responders were asked to evaluate the items in relation to each one of the four cognitive components of mathematical creativity in a scale from 1 (weak affordance) up to 4 (strong affordance). There was an extra option called N/A in case the affordance was not applicable for the specific item.

The evaluation of the mathematical creativity affordances of this c-book was done by three experts in the field of mathematics education, a senior researcher, a post-doctoral researcher, and a Ph. D. student who were not involved in its design. It was organized in three steps. First, the evaluators had to use the c-book and be acquainted with the affordances. Second, a teleconference was organized by the main designer of the c-book to address evaluators' needs for understanding and clarification. Third, the evaluators evaluated the c-book affordances based on the grid using an online form prepared for this purpose.

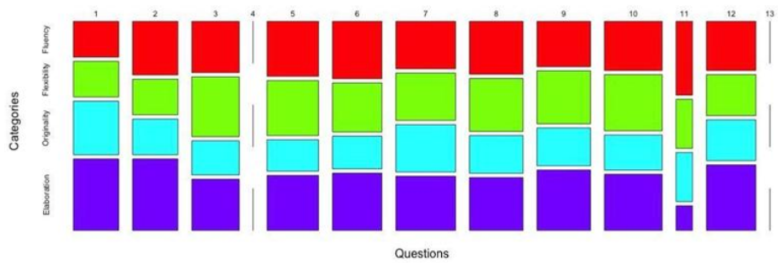


Fig. 4: Evaluation of CMT Cognitive from the experts' point of view on CMT affordances Grid.

The chart, shown in Fig. 4, represents the evaluation of the cognitive components of CMT from the experts' point of view. The height of the bars represents the mean value of each component (fluency, flexibility, originality and elaboration), while the thickness represents the mean between the four aspects for each question. From the evaluators' point of view, there are no affordances on the items 4 and 13, which means that the c-book does not establish connections between different knowledge areas and mathematics (item 4) and it does not include half-baked constructs that call for intervention (item 13). On the other hand, the evaluators consider that the c-book encourages exploratory activity and user experimentations (item 7) and encourages also generalizing mathematical phenomena, going from concrete cases to general ones or generalizing real world phenomena through the use of mathematics (item 10).

In Table 1, we present the quantitative data for each component computing the mean from No Affordance (scored 1) to Strong Affordance (scored 4). From the scale defined to evaluate the c-book we got the following values for each component, as shown in the Table 1 above: Fluency = 2.53, Flexibility = 2.46, Originality = 1.96 and Elaboration = 2.92. Except the originality component, all other components are in the range of "weak to possible" affordances. The originality got a value of 1.96 which means "no affordance". However, the value is quite close to "weak" affordance.

Fluency	Flexibility	Originality	Elaboration	Social	Affective
2.53	2.46	1.96	2.92	2.3	1.6

Table 1. CMT Evaluation Summary

The highest value for this c-book in terms of cognitive aspects was elaboration for which the value achieved the rank of "good affordance". It means that, in general, the c-book is judged to have a potential to boost the students' development of their ability to provide many responses or to come up with many strategies to solve a mathematical problem or challenge. Fluency and flexibility are the components with lower values of good affordance.

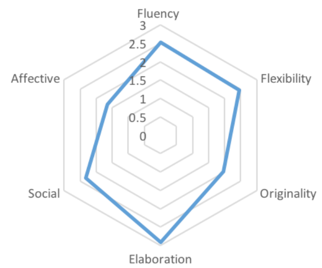


Fig. 5: Radar Distribution of CMT aspects.

The radar chart (Fig. 5) shows the distribution of the evaluation among the evaluated categories. This chart shows which component of CMT is most likely to be enhanced by the use of the c-book. In the case of this c-book it is the elaboration aspect, followed by fluency and flexibility.

	Fluency	Flexibility	Originality	Elaboration
Fluency	1.00	0.94	0.83	0.89
Flexibility	0.94	1.00	0.82	0.84
Originality	0.83	0.82	1.00	0.92
Elaboration	0.89	0.84	0.92	1.00

Table 2. Correlation Values of CMT Components

Table 2, collating the 13 questionnaire items, shows correlations among the four cognitive components of CMT. We can notice that the correlations are strong between some cognitive aspects. It means that, considering a significant value of $r > 0.80$ ($p = 0.05$), we may conclude that fluency, flexibility and elaboration can be fostered at the same time. In the case of originality, there is no statistical evidence that supports the hypothesis that this component can be fostered by the other ones.

We can conclude that even though the c-book main activity is designed to call for students' elaboration (they are invited to modify the initial situation by considering various combinations of special lines in a triangle, whose intersection point generates a locus to explore), fluency and flexibility are fostered by providing the students a rich environment in which they can explore geometric situations and try out algebraic formulas whereas benefitting from a feedback system allowing them to control their actions and verify their conjectures. Specific feedback is implemented toward directing students to produce different and varied situations and help them to break down their mind fixation by considering yet different configurations.

The c-book provides the students not only with digital tools enabling them to explore geometric and algebraic aspects of the studied loci separately, but also with a so-called "cross-widget communication" of Cinderella and EpsilonWriter, which makes it possible to experimentally discover the algebraic formula that matches the generated locus in a unique way; this feature may contribute to the development of original approaches by the students.

CONCLUSION

The c-book presented in this paper is the result of a collaborative work of a group of designers coming from various professional backgrounds, as the group comprises researchers in mathematics, mathematics education and computer science, as well as educational software developers. Without the synergy among those group members, a number of design choices would have remained in a hypothetical state, namely the technological advances in terms of cross-widget communication and learning analytics features. The design of the c-book has thus become a driving force in the c-book technology development, and in return, the unique c-book technology features enabled the creation of a resource with affordances promoting creative mathematical thinking.

This experience brings to the fore factors stimulating creativity in the collaborative design of digital educational resources. Among these are the following two:

- A variety of designers' profiles, as pointed out by Fischer (2005), as it encourages the search for novel information and perspectives;
- A close collaboration with software developers which is critical for the design and implementation of unique features of the c-book technology resulting in a creative resource. Thus the development of the technology and the educational resources designed with this technology feeds each other.

REFERENCES

Botsch, O. (1956). Bewegungsgeometrie. Reinhardt-Zeisberg, Band 4b. Moritz Diesterweg.

Boyer, C. B. (2012). History of analytic geometry. Courier Corporation Publications.

Descartes, R. (1637). La Géométrie. Appendix to Discours de la méthode.

EC (2006). Recommendation 2006/962/EC of the European Parliament and of the Council of 18 December 2006 on key competences for lifelong learning. Official Journal of the European Union, L 394, 10–18.

El-Demerdash, M. (2010). The Effectiveness of an Enrichment Program Using Dynamic Geometry Software in Developing Mathematically Gifted Students' Geometric Creativity in High Schools. Doctoral dissertation, University of Education Schwaebisch Gmünd, Germany.

El-Demerdash, M. & Kortenkamp, U. (2009). The effectiveness of an enrichment program using dynamic geometry software in developing mathematically gifted students' geometric creativity.

Paper Presented at the 9th International Conference on Technology in Mathematics Teaching – ICTMT 9, Metz, France, July 6-9, 2009.

Elschenbroich, H. J. (2001). Dem Hoehenschnittpunk auf der Spur. In Herget, W. et al; (Eds.), Medien verbreiten Mathematik (Proceedings) (pp. 86-91), Berlin: Verlag Franzbecker, Hildesheim.

Guilford, J. P. (1950). Creativity. American Psychologist, 5, 444–454.

Fischer, G. (2005). Distances and diversity: sources for social creativity. In Proceedings of the 5th conference on Creativity & cognition (pp. 128-136), ACM New York, NY, USA.

Guilford, J. P. (1950). Creativity. American Psychologist, 5, 444–454.

Haylock, D. (1997). Recognizing mathematical creativity in schoolchildren. Zentralblatt für Didaktik der Mathematik, 27(2), 68-74.

Healy, L. (2000) Identifying and explaining geometrical relationship: interactions with robust and soft Cabri constructions In: Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education, T. Nakahara and M. Koyama (Eds.) (Vol.1, pp. 103-117) Hiroshima: Hiroshima University

Healy, L., & Kynigos, C. (2010). Charting the microworld territory over time: design and construction in learning, teaching and developing mathematics. Zentralblatt für Didaktik der Mathematik, 42(1), 63-76.

Hoyles, C., & Noss, R. (2003). What can digital technologies take from and bring to research in mathematics education? In A. J. Bishop et al. (Eds.), Second International Handbook of Mathematics Education. Dordrecht: Kluwer Academic Publishers

Trgalova, J., El-Demerdash, M., Labs, O., & Nicaud, J. (2016). Collaborative design of educational digital resources for promoting creative mathematical thinking. Paper Presented at the 13th International Congress on Mathematical Education – ICME 13, Hamburg, Germany, July 24-31, 2016

Jares, J., & Pech, P. (2013). Exploring loci of points by DGS and CAS in teaching geometry. Electronic Journal of Mathematics and Technology, 7(2), 143-154.

Klein, F. (1924). Elementarmathematik vom h?heren Standpunkte aus, Band I. Dritte Auflage. Springer.

Laborde, C. (2005). Robust and soft constructions: two sides of the use of dynamic geometry environments. In Proceedings of the 10th Asian Technology Conference in Mathematics, Korea National University of Education, pp. 22-35.

Peigl, L. A., & Wayne, T. (2013). The NURBS book: monographs in visual communication. Springer, New York.

APPENDICES

Appendix A: CMT Affordances Grid

MATHEMATICS FOR GRAPHICS COMPUTING: students learn Algebra and program Python to create a project where they make Algebra create a scenario's photo.

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This paper presents and evaluates an innovative multidisciplinary approach to teach algebra. Students learn, at the same course, algebra and object-oriented programming in Python (a programming language). Both contents are widely used to create a final project. The final project is a python program that, using the usual algebra sylabus simulates a camera; Given, as input, a 3D scene created using triangles (three 3Dpoints), some light sources (3Dpoints) and a point of view (3Dpoint) from where the "photo" is taken it gives, as output, an image file with the photo of that scene. Many students revealed that to create an immediate application of algebra enhances their engagement. Other input (other scene, other lights or other point of view) produces as output a new photo. Many of the participants: teachers, students and researchers, evaluate positively this multi contents course besides the hard work it demands from all.

Keywords: Algebra; Python; Informatics Undergraduate Students; Applications; Engagement

INTRODUCTION

The integration of mathematics and computers is spread all over research. For example, in the *Principles and Standards of School Mathematics the National Council of Teachers of Mathematics* (NCTM, 2000) one of six principles for high quality mathematics education is the "Technology principle". The act of programming a mathematical concept makes school practice not essentially repetitive and foster reflective teaching experiences (Teixeira, Matos, & Domingos, 2015). Michele Artigue (2016) argues that dramatic changes come with the technological evolution in the ways that teachers and students access information and resources, learn, communicate, interact, work and produce with others. Using programation to teach mathematics is certainly one of those dramatic changes.

In a review of research on project-based learning, J. W. Thomas (2000) found many studies involving project based learning and concluded that although it has some limitations, its evaluation was positive among students and teachers. Mills and Treagust (2003) states that "the use of project-based learning as a key component of engineering programs should be promulgated as widely as possible, because it is certainly clear that any improvement to the existing lecture-centric programs that dominate engineering would be welcomed by students, industry and accreditors alike."

To contrast programming languages, Fangohr (2004) made a comparison of C, MATLAB and Python as teaching languages for engineering students. His study comprised two phases: to make an algorithm to solve a problem and to translate it to a programming language. He found Python as the best choice in terms of clarity and functionality. Python was also used to teach mathematics (Schliep & Hochstättler, 2002) since teaching algorithms is one of the natural applications of multimedia in mathematics. The mathematics objects that they considered were of a highly dynamic nature and require an adequate dynamic visualization using Python. Students also prefered Python as the first programming language to learn when compared to Java and other commercial languages (Radenski & Atanas, 2006).

CONTEXT

In the semester of 2011/12, at Instituto Superior de Engenharia de Lisboa, Instituto Politécnico de Lisboa, was developed a new course, Multimedia and Computer Science Engineering Graduation, and a course of Algebra for graphic computation conceived by a mathematician (Carlos Leandro) and then itteratively improved by another mathematician (Lucía Suárez) both working together with a computer scientist (João Beleza de Sousa). The syllabus was conceived from the beginning and all Algebra concepts taught were programmed in Python and used to create the project. This is a second semester curricular unit, students already know the programming basics, and are beginning object oriented programming. The course has weekly classes of 1h30m of Python and 3h of Algebra, all taught by the same professor (some are mathematicians, others are computer scientists). The classification is obtained 60% from exam, 20% from online homework and 20% from individual final project with required discussion (students must grade higher than 50% in every one of the 3 parts).

The course goal, as illustrated in the final project, is to develop a program in Python that when the user inserts (input) a 3D scenario made of triangles (three 3D points) with an associated RGB colour (triple/3D point), including the position (3D point) and colour of some lights (RGB colour-triple) and the position of the camera (3D point), the output is a photo of that scenario. If the input are different triangles, lights or position of the camera it immediately produces a different photo as a ".ppm" file.

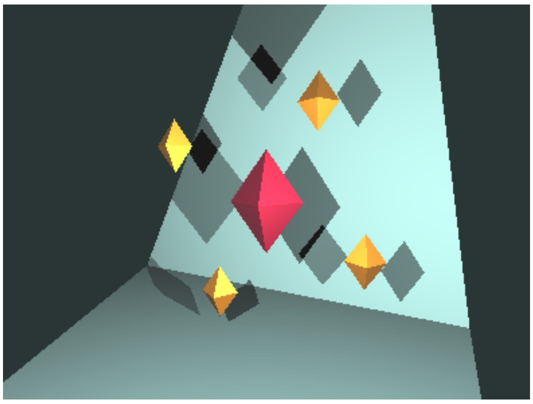

Figure 1. Example of the output of a student's project (not totally correct but very illustrative). The student made a scenario with a background big blue trinangle and five pyramids: four yellow pyramids and one red. Two light focuses give rise to two shadows for each pyramid and the different face colours dependent of the light incidence.

To create that program, students work with 3D points, vectors and matrices. The whole syllabus was designed to be nearly all applied in this project. The syllabus is:

1. Matrices operations; Inverse matrix; Determinants; Linear equation systems; Proper values and vectors.
2. 2D and 3D: Vectors and points; Referential and coordinates; Lines and plans; Internal and external product; Angles; Barycentric coordinates.
3. Geometric transformations (rotations, translations, scaling, …); Homogeneous coordinates and matricial representation; Perspective and parallel projections.
4. Surfaces: Intersection of lines and planes; normal vectors, reflection and refraction.

DETAILED PROJECT

In "Algebra" classes, which are theorethical/practical, students are taught traditional Algebra.

In "Python" classes, typically every week, students program a class/object into a file, by themselves with the natural support of teachers, using a guide provided before by teachers. Those students are Computer Science students and all take their own labtop to class, all have one – this is not an issue. The first week there is an example, out of context, to teach students the basics of object oriented programming, for example, students create a class named Circle with the data "radius", its "constructor" and some operations as: area, perimeter; double_perimeter (whose output is the double of the perimeter) and n_perimeter (analogous). In the following weeks, student program every class (part of the program) that are needed for the final project.

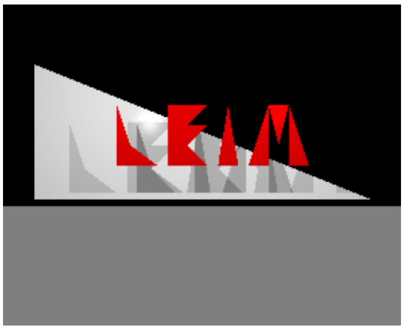

Figure 2 . Example of an image produced by a Ray Tracer project (as output)

For example, to produce the image in figure 2, the input is:

Colors

red = ColorRGB(1.0, 0.0, 0.0)

green = ColorRGB(1.0, 1.0, 1.0)

black = ColorRGB(0.0, 0.0, 0.0)

grey = ColorRGB(0.25, 0.25, 0.25)

bright = 100.0

Triangles

letter L - triangule 1

l1-v1 = Point3D(-4.25, 0.0, 0.0)

l1-v2 = Point3D(-3.25, 0.0, 0.0)

l1-v3 = Point3D(-4.25, 3.0, 0.0)

l1 = TriangleFace(l1-v1, l1-v2, l1-v3, letters-color)

letter L - triangule 2

l2-v1 = Point3D(-4.25, 0.0, 0.0)

l2-v2 = Point3D(-2.25, 0.0, 0.0)

l2-v3 = Point3D(-4.25, 1.5, 0.0)

l2 = TriangleFace(l2-v1, l2-v2, l2-v3, letters-color)

letter E - triangle 1

e1-v1 = Point3D(-1.75, 1.0, 0.0)

e1-v2 = Point3D(0.25, 3.0, 0.0)

e1-v3 = Point3D(-1.75, 3.0, 0.0)

e1 = TriangleFace(e1-v1, e1-v2, e1-v3, letters-color)

letter E - triangle 2

e2-v1 = Point3D(-1.75, 0.0, 0.0)

e2-v2 = Point3D(0.25, 2.0, 0.0)

e2-v3 = Point3D(-1.75, 2.0, 0.0)

e2 = TriangleFace(e2-v1, e2-v2, e2-v3, letters-color)

letter E - triangle 3

e3-v1 = Point3D(-1.75, 0.0, 0.0) e3-v2 = Point3D(0.25, 0.0, 0.0)

…

letter M - triangle 1

m3-v1 = Point3D(4.25, 0.0, 0.0)

m3-v2 = Point3D(5.25, 0.0, 0.0)

m3-v3 = Point3D(4.25, 3.0, 0.0)

m3 = TriangleFace(m3-v1, m3-v2, m3-v3, letters-color)

Background

background-v1 = Point3D(-10.0, -2.0, -2.0)

background-v2 = Point3D(10.0, -2.0, -2.0)

background-v3 = Point3D(-10.0, 6.0, -2.0)

background = TriangleFace(background-v1, background-v2, background-v3, background-color)

Floor

floor-v1 = Point3D(-12.0, -2.0, 0.0)

floor-v2 = Point3D(12.0, -2.0, 0.0)

floor-v3 = Point3D(0.0, -2.0, 1.0*10**4)

floor = TriangleFace(floor-v1, floor-v2, floor-v3, floor-color)

faces-list = [l1, l2, e1, e2, e3, i1, m1, m2, m3, background, floor]

Lights list

light1-position = Point3D(-5.0, 4.0, 5.0)

light2-position = Point3D(5.0, 4.0, 5.0)

light1 = PontualLight(light1-position,white, white ,white)

light2 = PontualLight(light2-position, white, white, white)

lights-list = [light1, light2]

The camera

camera-position = Point3D(0.0, 0.0, 10.0)

camera-looking-to = Point3D(0.0, 0.0, 0.0)

camera-vertical = Vetor3D(0.0, 1.0, 0.0)

camera-distance-eye-to-projection-plane= 5.0

camera-large-projection-rectangle = 10.0

camera-high-projection-rectangle = 8.0

camera-resolution-horizontal = 300

camera-vertical-resolution = 240

Main body

camera =. . .

background-color = black

ray-tracer = RayTracer(faces-list, lights-list, camera, background-color)

Roughly, we have a World Coordinates System where the scene (set of coloured triangles) is implemented; the Camera

Coordinates System and a plane (projection plane) where the scene is projected and where the image is produced.

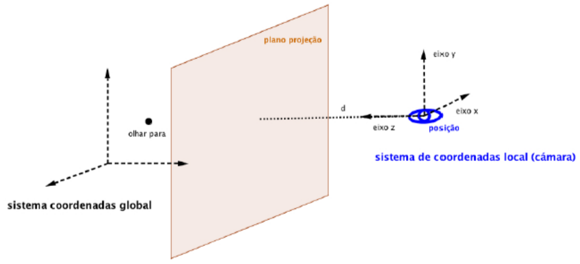

Figure 3. The two coordinate systems in use and the projection plan.

The “Ray Tracing” model will be explained now in detail. In order to produce an image from a 3D scene, a virtual camera is placed on the scene. The virtual camera projection surface is a rectangle placed at some distance from the camera position, along the view direction. This projection rectangle is divided into pixels. Since the image resolution is finite only a finite number of rays that came from each light must be followed.

Given that the virtual camera properties, such as the distance from the camera position to the projection rectangle, are specified in the Camera Coordinates System, and the scene, composed by triangles, is defined in the World Coordinates System, a simple ray tracing algorithm would be:

~~1)~~ for each pixel in the projection rectangle create a ray (a line) that start at the camera position (the eye position-a 3D point) and passes through the pixel position (a 3D point).

2) convert the scene coordinates (3D points) from the World Coordinates System to the Camera Coordinates System (Referential change). This is because the ray constructed in the previous step (defined in the Camera Coordinates System) will be intercepted with the scene triangles defined in the World Coordinates System. This conversion is done using a matrix.

3) intercept each ray (line) with each triangle in the scene. Each triangle is defined by a plane equation. The interception of ray/triangle (equations system) is solved by the Crammer method (matrix determinants).

   a) if the ray does not intersect any triangle, use the scene background color as the pixel color.

   b) if the ray intercepts a set of triangles, choose the nearest one, as the visible one (this assumes that the triangles are opaque). Determining the nearest triangle is done by computing vectors (defined between the eye position and the interception) length.

      i) create new rays (lines) starting at the interception point and ending at each light source. If some other triangle in the scene intercepts this new ray, that means that the point is in shadow. In either case, in shadow or not, use a color model such as the Phong model, to get the color of the interception point.

      ii) add the contribution of all light sources to get the pixel color (sum of 3Dpoints).

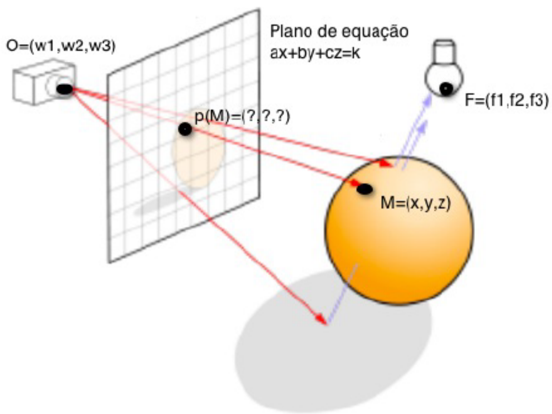

Figure 4. A scheme illustrating the Ray Tracer procedure.

The classes and operations created during the semester are below (names are in English and in Portuguese to be the most natural possible).

- **CorRGB**(r,g,b): __init__; __repr__ ; soma ; __add__ ; set_hsv ; multiplica ; multiplica_escalar ;__mul__ .
- **Imagem**(numero_linhas, numero_colunas, linhas): __init__; __repr__ ;set_cor; get_cor; guardar_como_ppm.
- **Matriz**(numero_linhas, numero_colunas, linhas):__init__; __repr__; set_entrada; get_entrada; adiciona; __add__; transposta; multiplica; multiplica_escalar; __mul__; det_2x2; det_3x3; sub_matriz; det; copia; set_linha; set_coluna.
- **Ponto3D**(x,y,z):__init__; get_x; get_y; get_z; __repr__; adiciona_vetor; __add__; subtrai_ponto; __sub__.
- **LuzPontual**(posição; intensidade_ambiente; intensidade_difusa; intensidade_especular): __init__; __repr__; get_posicao; get_intensidade_ambiente; get_intensidade_difusa; get_intensidade_especular.
- **CorPhong**(k_ambiente; k_difusa; k_especular; brilho): __init__; __repr__; get_cor_rgb.
- **Reta**(origem, destino, vetor_diretor): __init__; __repr__; soma; __add__; set_hsv; multiplica; multiplica_escalar; __mul__.
- **Plano**(ponto1; ponto2; ponto3; normal): __init__; __repr__; interceta_triangulo.
- **FaceTriangular**(ponto1; ponto2; ponto3; normal; cor_phong): __init__; __repr__; get_cor_phong.
- **Camara**(posição; olhar_para; vertical; distancia_olho_plano_projecao; largura_retangulo_projecao ; altura_retangulo_projecao; resolucao_horizontal; resolucao_vertical; eixo_x; eixo_y; eixo_z; incremento_horizontal; incremento_vertical; canto_superior_esquerdo_x; canto_superior_esquerdo_y; canto_superior_esquerdo_z; matriz):

And the code of the final project is below, students must have all the previous classes completed and tested. And students should create the remaining methods, the scenario and test their program.

Code for ray_tracer_xxxxx.py

```
from ponto_xxxxx import Ponto3D
from cor_rgb_xxxxx import CorRGB
from cor_phong_xxxxx import CorPhong
from face_xxxxx import FaceTriangular
from luz_xxxxx import Luz
from vetor_xxxxx import Vetor3D
from camara_xxxxx import Camara
from reta_xxxxx import Reta
from imagem_xxxxx import Imagem
class RayTracer:
# miss constructeur
# miss the method __str__
# miss the method renderiza
# tests
if __name__ == "__main__":
# constructeur test
# constructeur test - cor da face
verde = CorRGB(0.0, 0.3, 0.0)
brilho = 100.0
cor = CorPhong(verde, verde, verde, brilho)
# constructeur test - face
p1 = Ponto3D(0.0, 0.0, 0.0)
p2 = Ponto3D(1.0, 0.0, 0.0)
p3 = Ponto3D(0.0, 1.0, 0.0)
face = FaceTriangular(p1, p2, p3, cor)
lista_faces = [face]
# constructeur test - luz
branco = CorRGB(1.0, 1.0, 1.0)
luz_posicao = Ponto3D(1.0, 0.0, 2.0)
luz = Luz(luz_posicao, branco, branco, branco)
lista_luzes = [luz]
# constructeur test - camara
camara_posicao = Ponto3D(0.0, 0.0, 2.0)
olhar_para = Ponto3D(0.0, 0.0, 0.0)
vertical = Vetor3D(0.0, 1.0, 0.0)
distancia_olho_plano_projecao = 1.5
largura_retangulo_projecao = 2.0
altura_retangulo_projecao = 2.0
resolucao_horizontal = 50
resolucao_vertical = 50
camara = Camara(camara_posicao,
olhar_para,
vertical,
distancia_olho_plano_projecao,
largura_retangulo_projecao,
altura_retangulo_projecao,
resolucao_horizontal,
resolucao_vertical)
# constructeur test - cor de fundo
cor_fundo = CorRGB(0.0, 0.0, 0.2)
# constructeur test - ray tracer
ray_tracer = RayTracer(lista_faces, lista_luzes, camara, cor_fundo)
# teste a __str__
print(ray_tracer)
# constructeur test - renderiza
imagem = ray_tracer.renderiza()
imagem.guardar_como_ppm("teste1.ppm")
# referency test
# miss camara definition
# miss lista de faces definition
# miss cor de fundo definition
# miss lista de luzes definition
# ray tracer
ray_tracer = RayTracer(lista_faces, lista_luzes, camara, cor_fundo)
# renderization
imagem = ray_tracer.renderiza()
# file with renderization
imagem.guardar_como_ppm("teste2.ppm")
```

Table 1. Code to drive students to create the final class: Ray Tracer

METHODOLOGY

This research is a design research (Reeves, Herrington, & Oliver, 2005). The research question is: Is it possible to create an innovative multidisciplinary course joining algebra and a programming language that is positively evaluated by students and teachers? This is by itself a significant educational problem, but also it is made using a real project wich makes it pedagogicaly even more relevant.

This approach has been taught for seven years, reaching around 700 students. In the last two of those years an anonymous survey was presented to the students to get their feedback from the course. The approval rates of the last three years were monitorized. Iterative teachers' and researcher's reflection lead to changes and corrections to the project until to arrive to this stable status. For example, in the beginning teachers give the correct programing code to students before all Python classes, but now students program the code (following a guide) by themselves mostly in class with teacher's support. In the beginning the program was slower, it was improved to became faster, however, is still slow; in the following semester, we will study the implementation of a slightly different program wich is not a Ray Tracer but a Rendering Pipeline which produces images with lower quality but much faster.

DATA ANALYSIS

In 2016, the survey was compound of three questions:

- General evaluation of the course (0 to 20).
- Positive aspects of the course, to maintain.
- Negative aspects of the course, to change.

It was presented to the 45 students who completed the project immediately after getting their final mark and was answered anonimously. The mean grade given to the course was 15,5. And 35 over 45 gave 15 or more as the grade to the course. Many different aspects were approached as positive and negative. The most relevant to this research was:

- 14 students refer the interest of the course and final project, for example: "extremely positive the connection between mathematics and Python"; "final project strongly interesting"; "positive: connect mathematics and Python", "interesting subject"; "abstract subject that makes connection to reality".
- 9 students refer as negative that need more classes/support in Python while 14 refer that there is a high/enought support from teachers.
- 9 students had nothing to refer as negative while only two had nothing to refer as positive.

In 2017, the anonymous survey was presented online to all subscribed students and it was answered by 43 over the 113 subscribed. Beyond the questions of the previous semester some were added. One of them allows us to know that 72% of respondents were approved students, which as in the previous survey introduces some biase on results.

The mean grade given to the course was 16,1. And 17 over 31 gave 15 or more as the grade to the course. Many different aspects were approached as positive and negative. The most relevant to this research was: As positive: "The fact that mathematics complements programming and vice versa"; "Programming certain functions helped me to better understand Algebra"; "The creation of RayTracer, I found interesting and a good application to mathematics", "The project"; "I like the work of Python and Mathematics"; "the interconnection between the given subject and the final work is excellent and helps to consolidate all the knowledge and to test it."; "There should be MCG2"; "increases motivation";… As negtive: little time to finish the project (5 students reffered it).

All the four teachers of the course unanimously evaluate that approach as highly positive and that showing an immediate utilization of Algebra, using programming languages to create an image makes mathematics more interesting and engaging to those students. Moreover, Graphics computing, it's not just an example, is a central issue on a Multimedia Degree. Aldo, it creates a course with double difficulty since students must learn mathematics and programming. In 2015, the approval rate over assessed students was of 24/84=29% and over subscribed was 36/109=33%. In 2016, the approval rate over assessed students was of 36/72=50% and over subscribed was

36/110=33%. In 2017, the approval rate over assessed students was of 49/93=53% and over subscribed was 49/113=43%.

The approved students in fact learned all the parts since they grade more than 50%: in mathematic's final exam; in online individual (randomly generated) online homework (around 6 a semester) and also since they created and defended on a 30 minutes individual interview their project of object oriented programing in Python.

CONCLUSIONS AND FUTURE WORK

As conclusion, it is, in fact, possible to create a real multidisciplinary approach to teach algebra and a programming language together. Teachers and many students mostly found as positive that innovative approach that allows to experience the connection between mathematics and Python. However, some problems occurred, like mixing two difficult contents that makes the approval rate of students to be lower than desirable.

The Ray tracer program is a bit slow when renderizing the image, so we are studying the hypotesis of migrating to a Pipeline project that allows the image to be created faster but with lower quality.

REFERENCES

Artigue, M. (2016, March). Mathematics Education Research at University Level: Achievements and Challenges. In *First conference of International Network for Didactic Research in University Mathematics*.

Fangohr, H. (2004). A comparison of C, MATLAB, and Python as teaching languages in engineering. In *International Conference on Computational Science* (pp. 1210-1217). Springer Berlin Heidelberg.

Mills, J. E., & Treagust, D. F. (2003). Engineering education—Is problem-based or project-based learning the answer. *Australasian journal of engineering education*, *3*(2), 2-16.

NCTM. (2000). Principles and standards for school mathematics, Reston, VA.

Radenski, Atanas. "Python First: A lab-based digital introduction to computer science." *ACM SIGCSE Bulletin* 38.3 (2006): 197-201.

Reeves, T. C., Herrington, J., & Oliver, R. (2005). Design research: A socially responsible approach to instructional technology research in higher education. *Journal of Computing in Higher Education*, *16*(2), 96.

Schliep, A., & Hochstättler, W. (2002). Developing gato and catbox with python: Teaching graph algorithms through visualization and experimentation. In *Multimedia Tools for Communicating Mathematics* (pp. 291-309). Springer Berlin Heidelberg.

Teixeira, P., Matos, J. M., & Domingos, A. (2015). MATHEMATICS TEACHERS'INSTRUMENTAL GENESIS OF TECHNOLOGICAL MATERIALS. In 12th International Conference on Technology in Mathematics Teaching (p. 328).

Thomas, J. W. (2000). A review of research on project-based learning.

Chapter 7

SOFTWARE AND APPLICATIONS

An interactive book on axial symmetry and the synergic use with paper and pin

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This work presents results from a teaching experiment concerning the construction-conceptualization of axial symmetry at Primary School through an interactive book, developed in a Dynamic Geometry Environment (DGE), which embeds a set of tasks to be accomplished with selected DGE tools. The tasks are part of a teaching sequence, framed by the Theory of Semiotic Mediation (TSM), whose main characteristic is the synergic use of a "duo of artefact". The duo is made up of a digital artefact - the interactive book - and a manipulative artefact, constituted by paper and pin. Herein, we describe the design of the interactive book and we show how a cognitive synergy arises from its use combined with the use of the manipulative artefact within the sequence, thus leading to the conceptualization of mathematical meanings.

Keywords: Synergy between artefacts, Duo of artefacts, Digital Artefacts, Dynamic Geometry Environments, Theory of Semiotic Mediation

INTRODUCTION

Nowadays scholars generally agree that the use of tools, being manipulative or digital artefacts, can have potential to enhance mathematical understanding (Monaghan et al., 2016). In particular, many researchers have investigated on the use of Dynamic Geometry Environments (DGEs) in the mathematical teaching and learning processes. Leung (2008), for instance, underlies that a DGE has the ability to visually make explicit the implicit dynamism of "think about" mathematical concepts. The dragging function in DGE, indeed, allows to perceive patterns of variation and to discover invariant properties, thus playing a key role in the construction of mathematical meanings. However, different epistemological approaches to mathematical learning have different implications on designing tool-based teaching and learning activities (Leung and Bolite-Frant, 2015). For example, tools can be seen as mediators for mathematical discourse (Sfard, 2008) or as psychological tools in the context of social and cultural interaction, developed through the zone of proximal development and internalization processes (Vygotsky, 1978).

This work is part of a research project developed in a Vygotskian perspective and, more precisely, under the overarch of the Theory of Semiotic Mediation (TSM) (Bartolini Bussi and Mariotti, 2008), in which artefacts can be seen as tools of semiotic mediation. In order to design a teaching sequence, aimed at fostering the construction/conceptualization of axial symmetry at Primary School, we have considered a "duo of artefacts" (Maschietto and Soury-Lavregne, 2013), composed by a manipulative artefact and a digital one. The choice of the artefacts has been done with the aim to develop a synergy between their use, whereby the potential of the activities with the artefacts would be enhanced.

In this paper we describe, in particular, the design of the digital artefact, an Interactive Book (IB) developed in a DGE. The IB is presented focusing on the semiotic potential of its use, according to the TSM, within the designed sequence, in which it is combined with the use of the manipulative artefact.

Moreover, we present and discuss outcomes from the experimentation of the sequence, aiming to answer to the following research question: can the synergic use of our duo of artefacts develop a cognitive synergy fostering the conceptualization of axial symmetry?

THEORETICAL FRAMEWORK

The Theory of Semiotic Mediation, developed by Bartolini Bussi and Mariotti (2008), deals with the complex system of semiotic relations among: the artefact, the task, the mathematical knowledge that is the object of the activity, and the teaching/learning processes that take place in the class.

According to it, in semiotic activities various signs are produced: the "artefact signs", that often have a highly subjective nature and are linked to the learner's specific experience with the artefact and the task to be carried out; the "mathematical signs", in other words the knowledge of mathematics to which the "artefact signs" must evolve; and finally the "pivot signs", that illustrate the evolution between artefact signs and mathematical signs, through the linked meanings.

The role of the teacher is to foster, through Mathematical Discussions (Bartolini Bussi, 1998), the shared construction of mathematical signs, guiding the evolution of personal meanings toward mathematical meanings. In the design of our teaching sequence, we followed the general scheme of successive "didactic cycles", which organize the coordination between activities with the artefact and semiotic activities, finalized to make the expected evolution of signs occur.

Moreover, in the design process of the teaching activities we focused upon the "semiotic potential" of the artefact, that is the basis underlying, on the one hand, the design of the teaching activities and, on the other, the analyses of both the actions and production of signs and the evolution of meanings.

To complete the description of the theoretical framework of this research we need to refer to the notion of "duo of artefacts". Maschietto and Soury-Lavregne (2013) have designed a digital artefact corresponding to a given physical artefact in order to investigate if such a "duo of artefacts", can enlarge and improve the learning experience of the students. In our study, as in their duo, the two artefacts must have some common characteristics, enabling transfer and reinvestment from one to the other. For this reason, whilst our digital artefact is not a digital counterpart of the manipulative artefact, we do use this notion in our work as well.

RESEARCH METHODOLOGY

Following the teaching experiment methodology (Steffe and Thompson, 2000), a teaching sequence has been designed in conformity with the chosen theoretical framework and the formulated hypothesis. It constructs the environment where the data, on which to analyse the results of the experiment, are collected. The sequence is framed on the TSM taking into account a theoretical reflection on the meaning of axial symmetry, with its definition and its properties, and an a priori analysis of the semiotic potential of the artefacts. It has been implemented with fourth grade students in a pilot study, involving two groups of four pupils, and in a further study, involving a whole class of twenty pupils. The teaching experiments were videotaped and conversations were transcribed, that also took into account the specific actions taken with the artefacts. The videotapes and transcriptions were then used to analyse the teaching experiments.

Analysis of the pilot study results, not only showed that the sequence contributed to the emergence and evolution of signs – in line with what expected by the a priori analysis - but also demonstrated the development of a cognitive synergy, linked to the alternate use of the two artefacts that promoted the construction of meanings (Faggiano et. al, 2016). The need to examine any changes in order to develop the same path in a "real" class led to the design and implementation of the teaching experiment with the class. The results presented here are based on this last study.

THE INTERACTIVE BOOK AND THE SYNERGIC USE WITH PAPER AND PIN

The artefacts of our duo address the same mathematical content and have been chosen for their semiotic potential, in terms of meanings that can be evoked when carrying out suitable tasks involving their use.

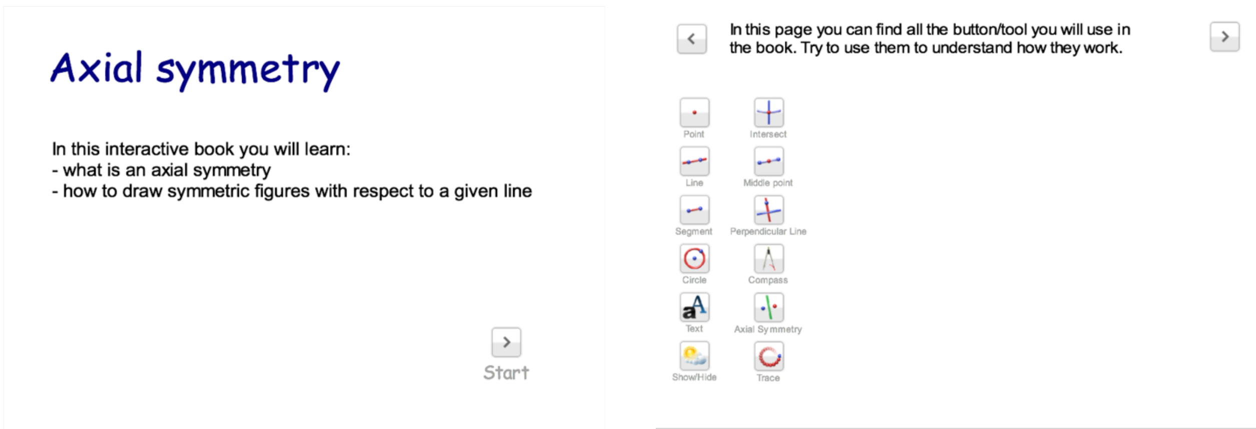
The components of the manipulative artefact are a sheet of paper, with a straight line drawn on it marking where to fold it, and a pin to be used to pierce the paper at a point in order to construct its symmetrical point. This artefact allows an axial symmetry to be created in a direct fashion, because the sheet naturally models the plane and the fold allows the production of two symmetrical points using the pin.

The components of the digital artefact, that appears as an Interactive Book (IB), originate from the components of a specific Dynamic Geometry Authoring Environment (New Cabri - Cabrilog), in which learning activities, involving objects and tools of a DGE, can be created. The IB Book is described with more details in the following section.

The design of the Interactive Book and an overview of the teaching sequence

The Interactive Book has been designed, in order to exploit the potential offered by the dragging function. Below we present, how the design of the tasks embedded in the IB has been developed for the digital artefact and the manipulative one. The main hypothesis inspiring the design concerns the potential synergy between the use of one artefact with respect to the other.

The IB contains a title page (Fig. 1.a) and a page created with the aim to introduce the buttons/tools involved in the activities of the IB (Fig. 1.b). The chosen tools are: those that allow the construction of some geometric objects (Point, Straight Line, Segment, Middle Point, Perpendicular Line, Intersection Point); the “Compass”; the “Symmetry”, which gives back a symmetric figure, provided that a figure and a line/axis have been chosen; and the “Trace” tool which, allowing the observation of the relations among the trajectories, makes more evident the effects of the dragging. The next pages of the IB have been integrated in the sequence as it will be explained below.



1.a 1.b

Figure 1. The first two pages of the Interactive Book

In the teaching sequence, in accordance with the study hypothesis, it was decided to alternate activities involving the use of one or the other artefact, formulating tasks that could exploit the complementarity of their semiotic potentials. The sequence, made up of six didactic cycles, begin with the use of the manipulative artefact. It continues with the use of the digital artefact in the second cycle and alternating the use of the artefacts in the third and the fourth cycle, while the order of the artefacts in the last two cycles is inverted.

In the first cycle, pupils are asked to construct the symmetric figure of a given figure with respect to a given line, by folding the paper along the line and piercing with the pin on the necessary points. The acts to fold the paper along the line and to pierce on a point with a pin, in order to obtain a couple of overlapping points, is a first possible way to concretely realise a symmetric configuration.

Such a manipulative experience, can foster the emergence of the idea that an axial symmetry is a one-to-one correspondence between points in the plane, defined by a line, locus of fixed points. In addition, joining the points, obtained with the pin, is the process that yields as product the symmetrical figure, provided that the correspondence between the segments is preserved. This evokes the idea that axial symmetry transforms segments into congruent segments. In the following task of the cycle, pupils are asked to compare what changes and what remains unchanged when drawing two symmetrical figures of the same figure, with respect to two distinct axes. This task has been conceived to evoke the dependence of the symmetrical figure on the axial symmetry.

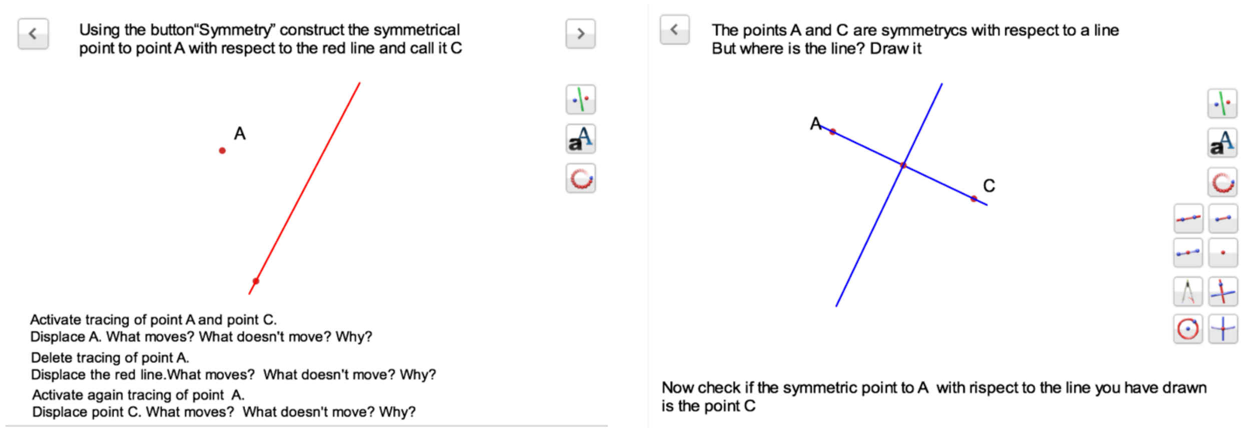
The first activity page of the IB (Fig. 2.a) presents the tasks of the second cycle. They have been designed with the aim to make two key meanings emerge: the dependence of a symmetric point from the point of origin and the role of the line to define an axial symmetry. The pupils are asked to construct the symmetric point of a given point A with respect to a given line, using the "Symmetry" button/tool, and call it C. Then pupils are invited to activate the "Trace" tool on point A and point C, move A and see what moves and what doesn't, and explain why. In the next two steps, in the same way, the pupils are invited to move the line and the symmetric point and to observe what happens during the dragging.

We emphasize that, in the DGE used, unlike for example in Cabrì Géometrè, it is possible to drag the symmetric point obtained, and this in fact allows the whole paper to be "shifted".

In this activity, dragging the point of origin and observing the resulting movement of the symmetrical point evokes the idea of the dependence of the symmetrical point on the point of origin; dragging the axis and observing the resulting movement, only of the symmetrical point, evokes the idea of dependence of the symmetrical point on the axial symmetry; dragging the symmetrical point and observing the resulting rigid movement of the entire configuration evokes the idea of the dual dependence of the symmetrical point both on the point of origin and on the axis. The difference in the movements between the symmetrical point and the point of origin can be compared to the distinction between dependent and independent variable.

The tasks of the third cycle aim at: observing that the line joining two symmetrical points is perpendicular to the axis and that the two points are equidistant from the axis; recognizing that these two properties are reversible and that they characterize axial symmetry. With this purpose, pupils are asked to construct the symmetric point without the use of the pin.

The tasks of the fourth cycle are embedded in the next page of the IB. Similarly to the third cycle, pupils are asked to construct the symmetric point without the use of the "Symmetry" button/tool. In order to make this construction, the two properties that characterise axial symmetry, already emerged in the previous cycle, need to be properly used. Pupils, indeed, have to: draw the perpendicular line to the axis, passing through the point of origin; draw the circumference with centre in the intersection point between the axis and the perpendicular line; and finally find the symmetric point as the intersection point between the circumference and the perpendicular line.



2.a 2.b

Figure 2. Examples of activity pages of the Interactive Book

In the fifth and sixth cycles the tasks are the same: there is a pair of points A and C that must be interpreted as symmetrical points with respect to a symmetry where the axis is hidden. Pupils are asked to identify and draw the axis (Fig 2.b). Finally, they are asked to check, using the "Symmetry" button/tool or with the pin, whether the symmetrical point of A with respect to the draw line is really C.

A priori analysis of the potential synergy

The hypothesis formulated is that a reciprocal boosting process will occur, in the form of a synergic process of mediation through the different types of artefacts.

For example, at the second cycle we can expect that the meanings that have already emerged thanks to the use of the manipulative artefact may be extended and completed by the specific meanings that should emerge using the digital artefact.

In other words, the images on the screen can be better interpreted in the light of the previous acts of folding and piercing. In this sense, after having constructed the symmetrical point using the digital artefact, the relation between the two points can be interpreted through the actions of folding, so the two points can be seen as two holes generated by the pin. While the meaning of the relation can be enhanced by the distinction between the original point and the corresponding point, which in the IB corresponds respectively to the direct movement and the indirect movement.

In this way, such a combined interpretation, may contribute to the development of the mathematical meaning of a functional relation between a point (independent) and its symmetrical point (dependent).

Conversely, at the third cycle, we can expect that, the interpretation of the actions and the configurations with the manipulative artefact might be related to the experiences within the digital environment.

For example, we can expect that two different points, of which to construct the symmetric points, can be interpreted as different positions adopted by a point that has been dragged, thereby contributing to the generalization of the two properties (perpendicularity and equidistance) and to the evolution of the status of these properties from being seen as contingent to being seen as characterizing.

Further similar considerations can be done concerning the expected synergy between the artefacts in the next cycles. More details can be found in (Montone et a., 2017).

THE TEACHING EXPERIMENT: RESULTS AND DISCUSSION

In this section we present the analysis of some episodes. In the analysis we attempted to figure out how the use of these two artefacts and their synergy are involved in the construction of the mathematical meanings and the interactions during the discussions.

The first episode refers to the discussion, held in the classroom, at the end of the second cycle after children had used the IB on computers.

During this discussion, one of the children has constructed the symmetric point of a given point with respect to a line, using the IB on the Interactive Whiteboard (IWB). In order to make pupils focus on the diverse movements of the objects on the screen, the teacher asked them:

- to predict what happens if the objects on the screen are moved;
- to verify what happens when they move point A, point C and the line/axis;
- to verify objects' behaviour resulting by the dragging.

In particular, at a certain point the discussion focused on the reason why point C moves and the line doesn't move when dragging point A. The excerpt (Tab.1) concerning M.'s reasoning and V.'s conclusion, is particularly interesting due to the gestures which M. made when speaking.

The importance to refers to the gestures lies in the fact that these signs together with words reveals that the emerging and synergically evolving meanings originate from and remain tied with actions carried out with both the artefacts.

<table>
<tr><th>Transcription and gestures</th><th>Comments</th></tr>
<tr><td>M. if you move point A only, point C has to move with point A because they must be symmetrical
M. has her elbows on the desk and moves her hands ahead of her while speaking
</td><td>The objects of M.'s representation move on a virtual space, that is vertical as the screen of the laptop she used, or as the IWB, which is in front of her during the discussion.</td></tr>
<tr><td>like, if you move point A higher…
she raises her left hand to indicate point A moving higher and looks towards her left hand
</td><td>M. accompanies her discourse gesticulating in the space in front of her. These gestures can be considered as pivot signs, because, on the one hand they are related to actions done with the artefact in order to accomplish the task (drag A… in this case "higher"), on the other hand, they are connected, through the feedback of the artefact (point C moves… in this case "lower"), with M.'s sign "the same space", combined with the gesture.</td></tr>
</table>

point C moves lower… so it is the same… *she puts her hands in front of her face, to simulate, with the thumb and index of each hand, two identical segments, she moves her right hand lower to show that, in this case, point C moves lower and looks towards her right hand*  …because there must be… the same space… between the two points *with a fast coordinated movement of her hands, she simulates two segments having the same length, using the thumb and index and bending the other fingers* 	The equidistance of the points A and C from the axis, thus, is evoked simultaneously by the verbal sign with the gesture. This sign is again a pivot sign.
V. because there must be the same distance between the line… there must be always the same distance between the two points and the line	The pivot sign “the same space” is evolving into the mathematical sign “same distance”.
M. …between the line and the point A and, between the line and the point C	M. recalls what V. said, as to further explain that the distance to be considered is exactly that between each of the points and the axis.
Teacher: Why?	
V. and M. (together) because otherwise they aren’t symmetrical!	The equidistance between each of the points and the axis is recognised as a necessary

	condition for the points to be symmetrical.

Table 1. From "the same space" to "the same distance"

As expected, this episode shows the unfolding of the semiotic potential of the dynamic environment, but also illustrates how the elements used by pupils to support their claims are not limited to refer to the dragging process visualized in the digital artefact. The manipulative artefact appears to be essential to construct the symmetric point and to give rise to a starting conceptualization. However, it gives a static vision because, for instance, after finding a symmetric point of a given point, making a hole on a sheet of paper by piercing it with a pin, the two points cannot move at all. In the previous transcription, instead, M. refers to the dynamic process visualized with the digital artefact: "if you move it", "it moves" and matches words with hand gestures that simulate what she saw on the computer.

The discussion followed-up and the role of synergy emerges: in order to indicate what a symmetric point is, pupils refer to the activity carried out with the sheet of paper and the pin and their initial conceptualization depends on the direct experience of piercing made at the beginning.

Moreover, a further interesting episode which underlines the need to mentally go back to the digital artefact as for G. and the reference in synergy of both artefacts as for V.. The teacher restarts and asks again how they know that the distance is always the same, and G. says:

G.: We figured it out because when [the pupil acting on the IWB] moved point A, point C moved too, but when they were very far away from the red line it was always the distance from the red line… from point C to the red line there was the same distance as… from point A to the red line.

G. matches his speech gesticulating in the space ahead of him. In fact, he looks towards the IWB screen, points his finger towards a hypothetical point A in front of him, with his right hand, while he symmetrically raises his left hand at the same height. He leans back with his body and spreads his arms outwards simulating the two points moving and keeping the same distance from the axis. Here, it shows how the interaction with the digital artefact allowed G. to perceive the invariant element, the distance, thanks to the variation on the screen of the position of point A and consequently of point C, which depends on A. He visually perceives and anticipates the generalization of the invariance of the distance of these two points from the line. In other words, it is as if the pupil visually analysed the variation of an aspect of the whole configuration, keeping another aspect constant, hence anticipating the surfacing of invariant schemes.

Then V., in order to analyse the relationships, explicitly re-calls the manipulative artefact, synergically joins the two activities, and says:

V.: If we have available a sheet of paper that can be folded….

She receives from the teacher a sheet of paper and a pin and makes a symmetric point folding and piercing the paper with the pin, reopens the paper, looks at it, and, simultaneously looking at the IWB adds:

V.: It is more visible there and it is easier… because there you can move the point and so I easily realise that if I move the point… the already created figure… it is easier to realize that there is the same distance because just by moving, you can understand, especially when we distance a lot from the line, that also point C moves… and so there is always the same distance. But I was able to understand it on the paper, also.

V.'s words confirm the hypothesis that the digital artefact is acting in synergy with the manipulative one. However, it is also clear that the modality with which these two artefacts operate is different. The manipulative artefact allows the direct action of the pupil. The pupil's body learns while acting and, in order to describe what a symmetric point is, pupils simulate the folding and piercing of a sheet of paper. When they refer to the digital artefact, instead, pupils simulate with their own body the objects of the actions that they perform with the artefacts: they move the harms as if they were lines and the hands as if they were points, drawing the tracing seen on the screen in the space ahead.

The dragging function, combined with the trace, after allowed pupils to mentally move the objects, and the previous visualization of what happened made explicit the implicit dynamism of thinking mathematical objects.

The next steps show the difference in the way pupils understand that the distance between A and C from the line is always the same: with the manipulative artefact, folding the sheet of paper and observing the superimposition of the two holes; with the digital artefact, animating/moving point A and observing how consequently point C moves. The underlined difference is at the base of the synergic use of the two artefacts since they operate on cognitive processes and different operative and non-superimposable modalities.

CONCLUSIONS

In this paper we have presented the design of a digital artefact, an interactive book developed in a DGE, and of a teaching sequence involving it together with a manipulative artefact. The use of this duo in the teaching sequence was framed by the TSM. The related teaching experiment, conducted with fourth grade students, has been analysed from a semiotic mediation point of view.

The analysis of the results showed, not only the unfolding of the semiotic potential of the artefacts, but also the development of a cognitive synergy, linked to the alternate use of the duo that fostered the construction/conceptualization of axial symmetry and its properties.

REFERENCES

Bartolini Bussi, M. G. (1998). Verbal interaction in mathematics classroom: A Vygotskian analysis. In H. Steinbring, M. G. Bartolini Bussi, & A. Sierpinska (Eds.), *Language and communication in mathematics classroom* (pp. 65–84). Reston, VA: NCTM.

Bartolini Bussi M. G. & Mariotti M. A. (2008), Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective, in L. English (ed.), *Handbook of International Research in Mathematics Education*, (second edition), Routledge.

Faggiano, E., Montone, A. and Mariotti, M. A., (2016) Creating a synergy between manipulatives and virtual artefacts to conceputalize axial symmetry at primary school, In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 235–242. Szeged, Hungary: PME

Leung, A. (2008). Dragging in a dynamic geometry environment through the lens of variation. *International Journal of Computers for Mathematical Learning*, 13, 135–157.

Leung, A. & Bolite-Frant, J. (2015). *Designing Mathematics Tasks: The Role of Tools.* In A. Watson and M. Ohtani (Eds.). Task Design in Mathematics Education: The 22nd ICMI Study (New ICMI Study Series) (pp. 191-225). New York: Springer

Maschietto M., Soury-Lavergne S. (2013) Designing a Duo of Material and Digital Artifacts: the Pascaline and Cabri Elem e-books in Primary School Mathematics. ZDM – *The International Journal on Mathematics Education* 45(7), 959-971

Monaghan, J., Trouche, L., & Borwein, J. (2016). *Tools and mathematics*. Dordrecht: Springer International Publishing.

Montone, A., Faggiano E., Mariotti, M. A. (2017). *The design of a teaching sequence on axial symmetry, involving a duo of artefacts and exploiting the synergy resulting from alternate use of these artefacts,* Proceedings of the 10th CERME Conference, to appear

Sfard, A. (2008). *Thinking as communicating*. Cambridge: Cambridge University Press.

Steffe L. P. & Thompson, P. W. (2000). Teaching experiment methodology: underlying principles and essential elements. In R. Lesh & A. E. Kelly (eds.), *Research design in mathematics and science education.* Hillsdale, NJ: Erlbaum, 267-307

Vygotskij, L. S. (1978), *Mind in Society. The Development of Higher Psychological Processes*, Harvard University Press.

STUDENTS' COVARIATIONAL REASONING: A CASE STUDY USING FUNCTION STUDIUM SOFTWARE

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This paper presents a case study on students' development of covariational reasoning, while using Function Studium software to perform activities about rate of change of linear and quadratic functions. This software was designed by LEMATEC-EDUMATEC/UFPE, a research group, and its development was guided by a model of software process based on Informatic-Didactic Engineering. The results of the case study pointed out some contribution of activities designed within the software to support students' covariational reasoning, such as: dynamic and simultaneous connections of the different representations of "rate of change" allowed the students to infer patterns of variation of these types of functions; and to coordinate average of rate of change to instantaneous rate of change.

Keywords: function, covariational reasoning, educational software

INTRODUCTION

This paper presents a case study on students' covariational reasoning using an *adhoc* software for the study of functions developed by LEMATEC-UFPE [1] (a Brazilian research group), as part of two master's dissertations (Tibúrcio, 2016; Silva, 2017).

The development of *Function Studium* software (Bellemain, Gitirana, Silva, & Tibúrcio, 2016) was guided by a model of software process that aims to combine aspects of teaching and learning mathematical concepts to computational aspects, contributing to a framework in both areas. The process model (Tibúrcio, 2016) is based on the idea of Didactic-Informatic Engineering (Bellemain, Ramos, & dos Santos, 2015), an object of study in LEMATEC - the research group.

Regarding *Function Studium* characteristics, the concept of rate of change and the variational perspective of functions underpin its tools. The ideas of covariational reasoning discussed in Carlson, Jacobs, Coe, Larsen, and Hsu (2002) contributed both to design the software tools and to build a framework for assessing their contribution on the development of students' covariational reasoning after undertaken some activities within *Function Studium*.

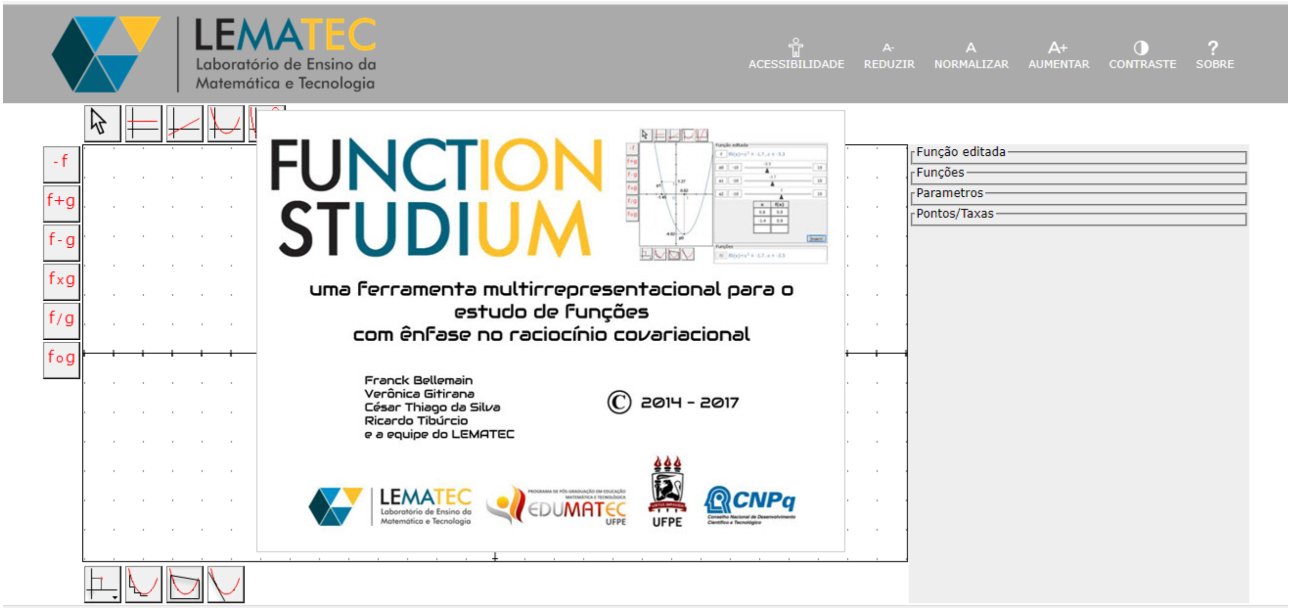

Figure 1. First screen of *Function Studium* Software

In this paper, we discuss some results of the case study, which was undertaken with a pair of preservice mathematics teachers of a Brazilian university. It is structured by: a brief discussion of the concept of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), the development of *Function Studium*, the methodology used in the case study and the analyse of some results obtained.

COVARIATIONAL REASONING

Covariation approach of function privileges the relationship between variables and how variation of one variable affects the variation of the other. Covariational reasoning is defined as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002, p.354). The researchers designed a framework to analyse students' covariational reasoning while exploring dynamic situations of functions: five mental actions. According them, as students develop such reasoning, they advance in levels of covariational reasoning, as shown in Table 1

Mental action	**Description of the mental action**
MA1	*Coordinating the value* of one variable with changes in the other
MA2	*Coordinating the direction of change* of one variable with changes in the other variable
MA3	*Coordinating the amount of change* of one variable with changes in the other variable
MA4	*Coordinating the average rate of change* of a function with uniform increments of change in the input variable
MA5	*Coordinating the instantaneous rate of change* of the function with continuous changes in the independent variable for the entire domain of the function

Table 1. Mental Actions of Covariational Framework (Adapted from Carlson, Jacobs, Coe, Larsen, & Hsu, 2002, p.357)

The concept of rate of change assumes such an important role in the study of real functions under the variational approach. It is explored from elementary school to advanced courses for some areas, in which derivative is a central concept. Rate of change gains even more importance within the covariational perspective. Instead of emphasizing the correspondence among values, it focuses on how the variation of one variable affects the variation of the other, and so, it is expressed by the rate of change.

FUNCTIONS STUDIUM SOFTWARE DEVELOPMENT

To develop *Function Studium* software, a methodology that integrates principles from Didactic of Mathematics to Software Engineering process was built. This methodology is a proposal of implementing the educational software engineering principles (ESE) (Tchounikine, 2011) in the conception-development of a microworld for math teaching. More specifically, we worked in the integration of the firsts stages of Didactic Engineering (Artigue, 1996). The methodology takes into consideration methods of requirement, from software engineering, integrated with Didactic Engineering, which can be defined as an Engineering (Bellemain, Ramos, & dos Santos, 2015), in which we contemplate specifically theoretical potentialities (from teaching and learning the knowledge) and technologies (of computation).

The engineering phase (Bellemain, Ramos, & dos Santos, 2015) comprises essentially of four steps: delimitation of the field, theoretical, experimental, validation.

The step of delimitating the field aims to select the field of knowledge the software will exploit. In it, some questions were focused:

> […] which mathematics knowledge will be exploit within the software, what are the correlated knowledge that will also be needed to exploit, and son on and what kinds of professional can help in this development.

(Tibúrcio, 2016, p.57, our translation).

The theoretical step comprised of a review of literature which aims to reach the state of the knowledge (didact, epistemological, cognitive and technological) regarding the selected field. It is the starting point of the process of requirements gathering. In the theoretical step, it is also important to address some questions which regards the didactic transposition (Balacheff, 1994): how computer potentialities will be used to digitally represents the knowledge domain; the way objects, relationships and operations are “internally” coded; and how they “dynamically” behave at the interface. In this step, it also starts software prototyping, in which: situations of use were designed, problems that could be raised while using the software were predicted; user’s answers were hypothesized; and the software prototype is developed to start some tests.

The experimental step comprises specific moments to test and to analyse the software prototype: interface, commands, bottoms, and so on, within the validation regarding of teaching and learning objectives traced within the situations of use.

The last step, validation, comprises an analyses a-posteriori of the results reached during the experimentation and the confrontation of this analysis with the theoretical one. This confrontation is made within the student's results for each activity, it gives us elements to improve the software as well as the situation of use. The step of validation can be done within different experiments, such as in a case study with pairs of students, and with the whole class. In this paper, we are still in the test with a pair of students.

Technologies used

Function Studium is a web-based software developed using HTML (HTML5), CSS and JAVASCRIPT. These languages, interpreted by any browser, can be edited with a simple text treatment and have already also innumerous object libraries what allows to shorten the period of development and to dedicate, almost exclusively, to implement the software codes regarding *the* didactic-informatic transposition of the concept of function and of rate of change, as well as the didactic proposal for these concepts.

The choice to use web platform also facilitates: to share it with the team involved in the Project, what helped to reconfigure the process of educational software engineering, mainly between the software engineering who conceive and technically develop it and the requirement engineering who worked in the conception/development regarding teaching and learning aspects. The use of an agile method and/or a methodology to facilitate a robust and quick interaction among the team was facilitated by this choice.

Function Studium Software

Function Studium presents a main window (Figure 2) rounded by icons which represent tools or configurations for the graphic representation, and others secondary Windows, in which others representations of function, such as tabular, algebraic, where implemented to interact with the graph.

Figure 2. Main Screen of *Function Studium*

Function Studium starts with one prototype of each type of function (constant, affine, quadratic and cubic), obtained by menu (2). It is based on the idea of an algebra of functions, which starts with some basic functions and operations defined (area 1). Other polynomial functions can be obtained as the result of an operation with these functions, as well as, the rational functions. In area 3, the window “Função editada” (Edited function) shows the algebraic representation of the function, while it is being inserted. In this window, it is possible to define the value of the coefficients of the function, using sliders. In area 4 (Functions), it shows the algebraic model of already defined functions. About area 5 (Parameters), it is exhibiting the coefficient of the already defined functions, what allows dynamic changes by sliders. Area 6 (Point/rates) shows the variable values, their variations and the rate of change of the functions in the select input values. It is possible to define such values both, directly through the window and through the graph. In area 7, there is tools to define a point in the graph. Regarding area 8, there is the tool "Rate of change", which, when activated, allows to calculate the rate of change of a selected function, both, between two points of the graph and between a sequence of successive intervals of x with the same length. Areas 9 and 10 refer to the tools "secant line" and "tangent line", in which it is possible to define tangent or secant lines to the graph of the function at the selected points, to articulate these lines with the rate of change of the function, contextualizing this concept with its geometrical meaning.

A detailed presentation and analyse which do not fit in this paper is necessary to clearly understand how the software works and to justify its conception, the chosen behaviour and articulations of the various representation registries of functions used in it. We suggest Silva (2016) and Tibúrcio (2016) to dive deeper in *Function Studium* elaboration. Concerning the functionalities to support and investigate the covariational reasoning of the students developed in the artefact, the conception followed the specifications elaborated during the requirements engineering process which basically specifies:

a) A covariational perspective: tools and characteristics that support students in the coordination of variable variation;

b) Dynamism, interactivity and different notations connected simultaneously: such characteristics are based on Kaput (1992), which synthetises possibilities of computational environments to instantiate variables and represent functions;

c) Tools and characteristics based on the results of the preliminary analysis on the concept of rate of change: these analyses pointed out elements of epistemology, teaching and learning of the concept, therefore, they should also support the software tools construction.

METHODOLOGY OF THE CASE STUDY

A Case Study was specially designed as a first validation of *Function Studium*. The study was undertaken with a pair of mathematics preservice teachers, who used the software to perform two activities about rate of change of linear and quadratic functions. The data were analysed to search contributions and/or limitations to the students' covariational reasoning derived from the use of the software.

The activities took place in a session of two and a half hours, in a classroom with the pair of students and the researcher, who had an observer role, except in moments in which it was necessary to interact with the students to solve doubts about the technical aspects of using the software. The students explored *Function Studium*, in a computer with internet access, and had to discuss to solve de problems proposed in a worksheet.

The activities focused on the last two levels of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), that is, on the coordination of the average rate of change and the instantaneous rate of change, although aspects of initial levels were naturally included in the questions. Some questions will be explicit in the analysis.

The data collected comprises a screen capture video with students' interactions in the software, a video with students' interactions between themselves, the notes on worksheets and a researcher diary. The data were analysed by the researcher, searching the students' answers and the dialogues associated with their actions on the computer screen, to identify extracts which explicit how the software interfered to the students' covariational reasoning.

ANALYSE OF THE RESULTS

As regard the last two levels of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), that is, on the coordination of average of rate of change and the instantaneous rate of change, the following results could be stated.

First, the dynamic variation of the independent variable in the graph, simultaneously connected to the dependent variable variation, allowed the students to coordinate the variation of the rate of change continuously (MA4). This allowed inferences about the rate of change behaviour in each type of functions addressed. An example is the students' discussion about question 1.4 (Figure 3) when exploring a linear function.

Use the tool "rate of change" and in some interval of x, make the Δx decreasing in the graph or directly in the "points / rates" window (for example, 1; 0.5; 0.1; 0.05; 0.01; 0.001; 0.0001), observe the behavior of the rate of change. In this software, when Δx is small enough, for example Δx = 0.0001, it is possible to simulate the limit of the function at point x_0 when x tends to x_0.

1.4) When simulating the process described above, take different linear functions and vary the variable x, observing the value of Δy / Δx. How does this value change as a function of x? What does this suggest about the rate of change in linear function?

Figure 3. Question 1.4

After the students varied x in the graph and observed, for a fixed Δx, and observed the behaviour of Δy, they inferred the behaviour of the related function.

Student 1: [...] "Delta x" is not changing correctly? Only x ... (He varies x in the graph and observes the invariance of the value of the rate of change for all the points reached). Do you understand? (He questioned whether his colleague had the same conclusion.). The value of "delta y" over "delta x" ... It doesn't vary. When you make delta x always smaller and then you change the value of x, it (Refers to Δy/Δx) will not vary.

Student 2: Ok…

Student 1: What does this suggest about variation in linear function? ... Humm ... Ah, that its variation is constant... What do you think?

Student 2: Thus, the rate of change of it is constant. It is always equal to the coefficient.

Second, the "rate of change" tool, which calculates the rate of change at successive intervals of the function domain and displays them both in graph and in points windows, was an important resource for students to coordinate the variation of average rate of change (MA3 and MA4) and to observe patterns of variation in quadratic functions. This can be seen in their discussion regarding question 2.3 while exploiting a quadratic function.

2.3) Still in the simulation of the previous item, vary the variable x in the graph and note the variation of the variation rate Δ (Δy/Δx) in the "Points/Rates" window. How does Δ (Δy/Δx) behave with the variation of x? Test other quadratic functions, describe what you perceive and what this suggests in relation to the variation in quadratic functions.

Figure 4. Question 2.3

Student 1 varies x in the graph and observes the "points window", in which the successive differences between the rates of change in the intervals of the graph are shown. Thus, he argued.

Student 1: In relation of the rate of change, when you change x, the rate of change will vary ... But, there, the difference between "delta y - two" over "delta x - two" and "delta y - one " over "delta x – one" is constant. The difference between the rates of change will be constant. That's interesting.

Third, their explorations of the "points window" helped them to coordinate the variation of average rate of change (MA4), since it exhibited in the same area the variation of Δx, Δy and Δy/Δx, simultaneously with the variation of x in the graph. Their discussion regarding question 2.1 shows that. Student 1 chooses the function $x^2–2x+2$ and uses the tool "rate of change" to change x in the graph while observes the variation of Δy in the points window.

Student 1: Look, I'm changing x here, and look at Δy ... Δx is the same, it's "one", and then Δy increases. Consequently, the rate of change, is that right? Because if Δx is always equal to "one" and Δy is increasing, then the rate of change will increase as well. Write there!

Student 2: How? What did you mean?

Student 1: Δy is increasing as you increase the values of x (...)

Fourth, as regard to the coordination of the instantaneous rate of change (MA5), as can be seen, students were able to vary the value of Δx in the "points window", making it closer to 0, while observing the variation of Δy approaching a specific value. In doing so, they could coordinate the transition from the average rate of change to the instantaneous rate of change, through smaller and smaller refinements of Δx, obtained in the "point window" and in the graph.

Fifth, the simultaneous connection of actions in different representations enabled within *Function Studium* supported them to coordinate the variation of the instantaneous rate of change, while varying x in the graph. This connection allowed them to explore aspects such as the signal and the variation of the rate of change as a function of x, as well as aspects of graph such as concavity and inflection points, from a variational perspective, relating these aspects to the behaviour of the rate of change.

CONCLUSIONS

This paper presented some of the results of a case study on covariational reasoning of a pair of students, who exploited *Function Studium* software within activities about rate of change of linear and quadratic functions. The results reveled that some characteristics and tools of the software supported the students' covariational reasoning till the last level (MA5), with emphasis on the simultaneous connection of different notations, which contributed to coordinate aspects such as the sign of the rate of change and patterns of variation of each type of function, as well as, it contributed to a variational interpretation of inflection points and concavity in the graph. Moreover, the dynamism within simultaneous variation of Δx, Δy and $\Delta y/\Delta x$ allowed them to coordinate both the average and instantaneous rates of change, by means of smaller and smaller refinements in Δx.

NOTES

1. LEMATEC – Laboratory of Studies in Mathematics and Technology – of EDUMATEC - the Program of High Studies in Mathematics and Technologic Education – at UFPE - University Federal of Pernambuco - Brazil.

REFERENCES

Artigue, M. (1996). Engenharia Didática. In J. Brun (Eds.), *Didática das Matemáticas* (193-217). Lisboa: Instituto Piaget.

Balacheff N. (1994) La transposition informatique. Note surun nouveau problème pour la didactique. In M. Artigue, M., R. Gras, C. Laborde, P. Tavignot, & N. Balacheff (eds) *Vingt ans de didactique des mathématiques en France*. (pp.364-370). Grenoble: La pensée sauvage éditions.

Bellemain, F.G.R., Gitirana, V., Silva, C., & Tibúrcio, R. (2016). *Function Studium Software* © , [web-based software]. Recife: LEMATEC – EDUMATEC/UFPE. Retrieved from http://lematecstudium.net.br/function/

Bellemain, F.G.R., Ramos, C.S., & dos Santos, R.T. (2015). Engenharia de Software Educativos, o caso do Bingo dos Racionais. In *Anais do VI SIPEM* - Seminário Internacional de Pesquisa em Educação Matemática (pp. 1-12). Pirenópolis, Brasília: Sociedade Brasileira de Educação Matemática.

Carlson, M.P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33, 352-378.

Kaput, J. (1992). Technology and Mathematics Education. In D.A. Grows (Ed.) *Handbook of Research on Mathematics Teaching and Learning* (515-556). NY: Macmillan.

Silva, C.T.J. (2016) *A Engenharia Didático-Informática na prototipação de um software para abordar o conceito de taxa de variação*. (Master dissertation. Universidade Federal de Pernambuco, Brazil). Retrived from http://repositorio.ufpe.br/handle/123456789/19687.

Tchounikine, P. (2011). *Computer Science and Educational Software Design*. Berlin Heidelberg: Springer-Verlag. doi:10.1007/978-3-642-20003-8_1.

Tibúrcio, R.S. (2016) *Processo de desenvolvimento de software educativo: Um estudo da prototipação de um software para o ensino de função*. (Master dissertation, Universidade Federal de Pernambuco, Brazil) Retrieved from http://repositorio.ufpe.br/handle/123456789/17425.

CATO-ANDROID: THE GUIDED USER INTERFACE FOR CAS ON ANDROID SMARTPHONES

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CATO is the self-explanatory user interface for several CAS on Windows, Linux/Unix and Mac/Ox. The principles of CATO are for example: packages realized as selection menus for the commands and extra windows for the input of multi-parameter commands.

The author has developed an application, CATO-Android, with the principles of CATO for smartphones with the operating system android. It is a user interface for guided input for CAS, at the moment only for Symja.

In the article below, the author describes this app and how he realized the principles of guided input.

Keywords: GUI, CAS, android, smartphone, small display

THE REASONS FOR A GOOD USER INTERFACE

The demand for better designs of user interfaces for computer algebra systems is almost as old as the systems themselves. Kajler has described and developed his ideas for a perfect user interface in various works (Kajler, 1992) and (Kajler, 1993). He has postulated that well-designed computer algebra interfaces should afford intuitive access. As such, users should be able to enter commands with more than one parameter in a two-dimensional fashion. This prevents syntactic and structural errors. In addition, all templates and masks should follow the convention of operating from left to right.

Many of his reflections and wishes are not realized by the computer algebra systems running on Windows, Linus or MacOSx. So the author has developed the GUI CATO, (Janetzko, 2015) for intuitive usage of CAS. CATO aims at users who want to use the CAS only sporadically, e.g., one, two or three times a week.

The author believes the usage of a CAS on a smartphone (Fujimoto, 2014) will be sporadically, too. Therefore, a guided input is very important.

THE PRINCIPLES OF CATO

CATO, the **C**omputer **A**lgebra **T**aschenrechner (calculator) **O**berfläche (surface), is a realization of some principles for a guided input: The commands of CATO are structured into packages in the usual mathematical kind: "analysis I, analysis II, linear algebra, numeric, …". Alternatively, the names of the packages clearly identify the content: "solving equations, trigonometric functions, integral transformations, …". Because there are only twenty-seven packages in CATO, some packages will have sub-packages. The user can select a package with a drop-down menu and then the correct command (or the sub-package) in a second one, commands of a sub-packages in a third one. The menus are collocated from left to right. All multi-parameter commands have their own input window with one input row for each parameter and a short description for it. Consequently, the windows of the multi-parameter commands are uniform, independent of the CAS. Furthermore, the absence of abbreviations is very crucial for an effective sporadic usage of CAS and consequently, CATO does not abbreviate commands.

Other possibilities of CATO reflect state-of-the-art settings of other CAS: the kind of output, numerical or accurate, is a global option, also the user defined precision of numeric output. The design guidelines of CATO are thoroughly maintained as for example options can be selected and set like commands. Furthermore, there exists the package “chronicle”, which collects all used commands (if they are selected by a menu). Additionally, another global option is the choice of the selected CAS. CATO provides a log for the re-usage of recent inputs.

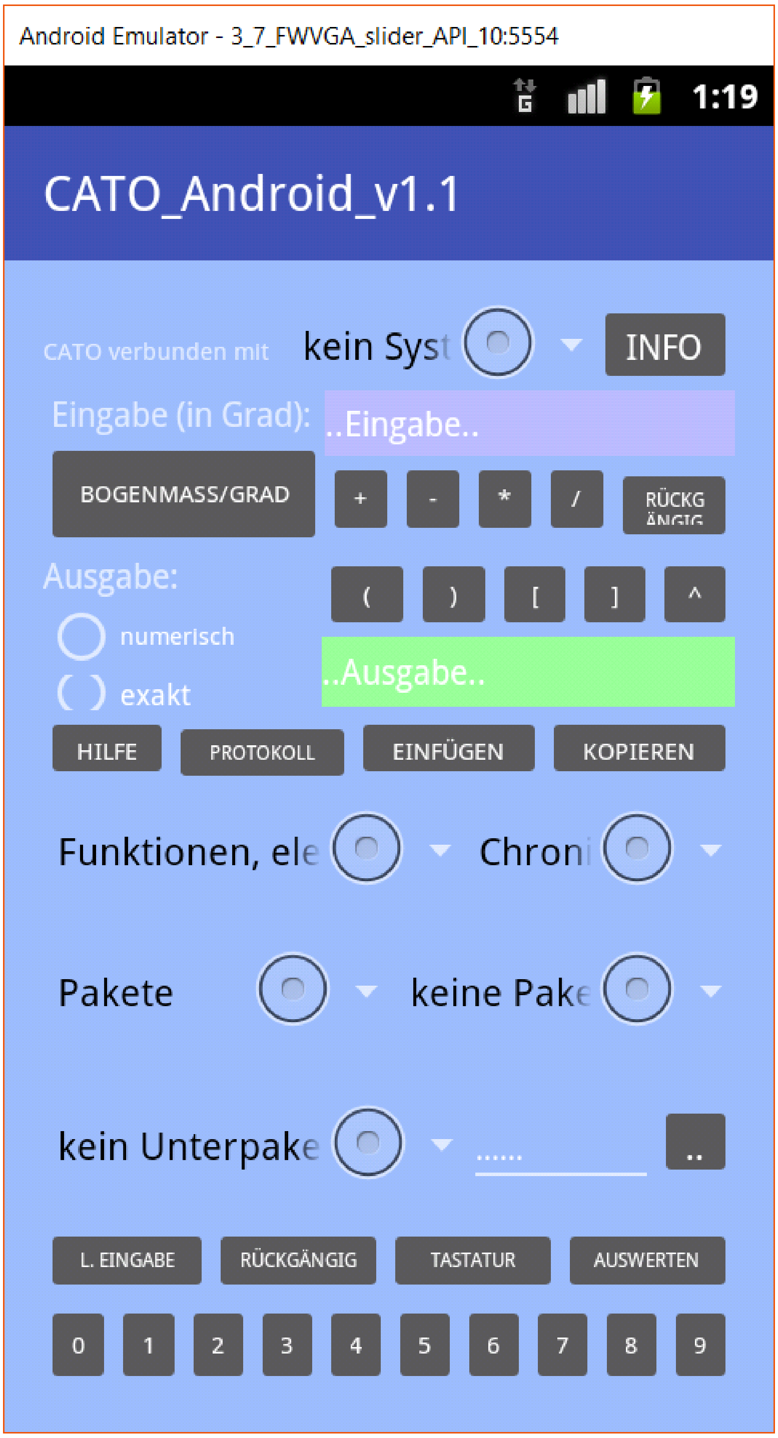

Figure 1. The graphical user interface on Android.

THE APP CATO-ANDROID

Starting the app, see Figure 1, two text areas, reserved for input and output, several buttons and six menus for selecting, spinners, are visible. To use CATO, the user should first select at the head of the app the computer algebra system of his choice; at the moment only Symja is available. CATO-Android then instantiates a connection to the chosen computer algebra system. Now the user can select commands. For example, for calculating “sin(3.4)”, he has to select in the menu “Pakete” (packages) “Analysis I” (analysis I), see Figure 2.

Fig. 2: Selecting Analysis I in the spinner packages.

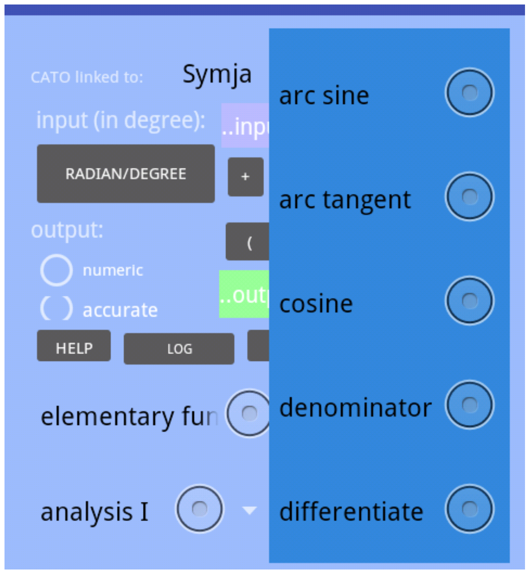

Fig. 3: Selecting a command in the spinner right next.

In the menu right next, the commands of this package are now available. The user can select the command he wants, see Figure 3. In the input text area he reads “Sinus(“ and he can use the buttons of the app for the remaining part of the input.

For the input of more complicated terms he gets with the button “tastatur” (keyboard) a window with several buttons for many signs.

THE EXTRA WINDOWS FOR MULTI-PARAMETER COMMANDS

The user can also select a command with more than one input, for example “differentiate”. He selects at first “analysis I” in the menu “packages” and the command itself in the menu right next. The extra window will appear, see figure 4. The user can see at first the name of the command, then a short description of it, and two input rows for the parameters with a short description “fonction” (function) and “variable”. Like in the Desktop version of CATO, the user does not need to know the right order of the parameters, the correct brackets or separators. But it is an android window for dialog, so the user can not apply the functionality of the CATO-Android surface itself: Therefore several buttons of the surface are part of this window useable for the input, also the button “tastatur” (“clavier”) for a keyboard. For the same reason the selecting menus spinner can not be part of this window. The author has solved this problem with the application of scrollviews, vertical tables of buttons. The user can use them like the spinners for selecting packages and commands.

Also the log of the version 1.1 is an android window for dialog, where the user can select an old input by typing its number in a field for input.

Figure 4: The extra window of the command "differentiate".

OUTLOCK

English and French versions of CATO-Android like the English or French versions of CATO are published. Also the connection of CATO-Android with Symja will be a first step, other computer algebra systems will follow.

REFERENCES

Fujimoto, M. (2014). How to develop a mobile computer algebra system. Retrieved from http://www.inftyproject.org/issac2014/ISSAC2014_Tutorial_fujimoto.pdf

Janetzko, H.-D. (2015). CATO – a guided user interface for different CAS. In N´elia Amado (Ed.) *Proceedings of the 12th international conference on technology in mathematics teaching, ICTMT 12*, (pp 429–437). Faro: University of Algarve, Retrieved from http://sapientia.ualg.pt/handle/10400.1/6081

Kajler, N. (1992). CAS/PI: a portable and extensible interface for computer algebra systems. *International Conference on Symbolic and Algebraic Computation* (pp. 376-386). ACM New York, NY, USA.

Kajler, N. (1993). Building a computer algebra environment by composition of collaborative tools. *Lecture notes in computer science, 1993* (pp. 85-85).

WIMS: AN INTERACTIVE EXERCISE SOFTWARE 20 YEARS OLD AND STILL AT THE TOP

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WIMS (Web Interactive Multipurpose Server) is a collaborative, open source e-learning platform hosting online interactive exercises in many different fields such as mathematics, chemistry, physics, biology, French and English among others. It is widely used in France mainly in mathematics from secondary school and up to the first university years. Using it in a proper way can bring advantage both to students and teachers.

Keywords: e-learning platform, grading, motivation, pedagogical alignment, community of practice.

INTRODUCTION

WIMS (Web Interactive Multipurpose Server) is a collaborative, open source e-learning platform, under the GNU (general public license), hosting online interactive exercises in many different fields such as mathematics, chemistry, physics, biology, French and English among others. It is used mostly in France [1] and mostly in mathematics in high school or during the first years of higher education [2]. It provides real learning advantages for students and for teachers. In this paper we will first present a quick history of its first development and describe the communities of developers and users. In a second part we present the way the software is built and how it interacts with other softwares. Then we present the advantages using such an exercise software can provide to students in terms of learning, skills development in a specific subject area but also in cognitive skills development. We will then discuss some observation we have obtained during our implementation of WIMS in UPEM. In the last part we give some hints to start with WIMS and conclude.

A BRIEF HISTORY OF WIMS AND THE COMMUNITY AROUND IT

WIMS has been first created by Xiao Gang (1951-2014) [3]. The initial version is released in 1997. Professor XIAO Gang was a brilliant mind, born in China, professor of mathematics at the University of Nice (France), one of the best world specialists of algebraic surfaces. Ten years after the first release, the association WIMS EDU is founded, and Xiao Gang leaves the software in the hands a small community of developers. The development is then taken over, mainly by Bernadette Perrin-Rioux in a mastery way. The association WIMS EDU [4], whose main goal is to support the diffusion of WIMS, organizes, among other things, once every two year, a colloquium attended by more than a hundred persons.

WIMS was created at about the same time as Google (registered on September 15, 1997) and long before Wikipedia (created in January 2001) or Moodle (first release in 2002). Comparing WIMS to these giants may seem to lower it's impact. But the mere fact that it continues to exists after 20 years, is still growing, has a wide community of users both teachers and students, an important community of exercise developers, and a very active and effective community maintaining and developing the software itself, is significant and remarkable.

It appears, from different statics and surveys, that classes in WIMS are almost only created by teachers, mostly french, proficient and very involved in teaching. We have no evidence showing students using WIMS by themselves, without belonging to a WIMS class created by a teacher. Whence, in terms of teaching theory, WIMS appears to be class depended with the teacher as a key-stone.

WHAT IS WIMS

In this part we will first explain the real specificity of WIMS: the WIMS exercise, and then explain the whole environment it provides to student, to teacher, and to exercise developer.

WIMS's interactive random exercise

The main specificity of WIMS is it interactive exercises. Most LMS develop interactive exercises such us MCQs, matching, drag-and-drop exercises. WIMS can do much more. It provides a framework and a specific language that allows to use very powerful softwares as MuPAD, PARI/GP, Octave, Gnuplot, POV-ray, Coq proof assistant, GeoGebra among others. Thus formal calculus, drawing figures can be made quiet simply. And truly new and original exercises can be proposed to students.

The most original feature of the WIMS exercise is its random feature. To make it simple, let us consider a very simple example and look what is necessary in order to check whether a student can add small numbers. A naive way to achieve it is to program a question *2+2= Answer Field*. The student's answer is stored in the variable *StudentAnswer*. The *TrueAnswer*, here "*4*", is compared to the *StudentAnswer*. If these two variables are equal the student receives the notice of success. If the variables are different, *TrueAnswer* can be displayed and more feed-back can be programed.

In a WIMS exercise one doesn't program the question *2+2* but *a+b* where *a* and *b* will be variables randomly chosen by WIMS between values determined by the programmer, for instance integers between 1 and 10. When the student calls a cession of a WIMS exercise the software presents a random draw of *a*, *b*. Thus in several lines of programming one can obtain an exercise that will have a great and maybe almost infinite versions. For examples of WIMS exercises we advise the reader to explore a WIMS server [5].

WIMS allows a to have a direct access to mistake. Once the answer's given in one clic WIMS displays if the answer's right or not. In most cases the right answer can be given. However, when there is more than one solution for instance, it can be more tricky to program. Consider the following WIMS exercise: *Let A be a subset of the real numbers defined as follows: A= .... – Is A bounded (yes or no) if yes give an upper-bound of A.* Here, the programer decides to chose randomly the set *A* between union or intersections of intervals, set of the values taken by types of sequences, values taken by a quiet simple function. The programmer may calculate for each type of examples the supremum *s* of the set *A*, and test if *s* is less and equal than the answer *m*. When the student is wrong, his answer *m* is strictly smaller than the supremum *s,* programming the feed-back of the exercise in order to get an element *a* in *A* which realizes *a>m* can be a much greater brainteaser. Of course the feed-back can just propose a content where a similar example is solved.

WIMS as an LMS

WIMS belongs to the class of LMS (Learning Management System). Anyone, with web access and an e-mail address, can create a virtual class and become a teacher on any WIMS server around the world in less than 2 minutes. Then the new teacher has two tasks: first build the resources of her class, and then enroll the students. Building the resources consists mostly in choosing or creating and organizing the exercises in sheets of exercises. There are short ways for doing it. For instance if a so called *Classe ouverte* corresponds to the teachers aim, she can in several clicks copy it and have it privatize for her own students.

Of course, it can be customized, exercises can be changed, added, taken away. One can also (it takes two minutes) restore a class built previously and backed-up. A research engine can also propose full exercise sheets corresponding to the key-word and to the level, and thus import parts of the class structure in a fast way. The research engine can also help you to select exercises one by one. An important task, which has many pedagogical implications, is the choice of evaluation and rating. Several parameters have to be set. The *severity* (if one mistake is heavily paid or not), the *rating scale*, *i.e.* the way the grade obtained in several repetitions of an exercise will be taken into account (will the final grade be an average of all trials, of the last *n* trials, of the best *n* trials, the worst of the best *n* trials…). Let us underline that the choices made by the teacher can influence motivation of the students, sustainability of the training. One has to be aware that sometimes good grades does not mean good work, if for instance a gambling strategy allows to obtain the maximal grade in a short amount of time. An interesting modality of parametrizing exercises is to make *strings*. A string is a pool of exercises made of several steps, each step can be one item of the exercise or an item of a different exercise. For instance the second step of the string adds a new kind of difficulty to the exercise proposed in the first step. The grade is given at the end of the string. Counting on gambling to fulfill a string of exercise is no longer a time saving option. And taking the time for deep understanding becomes a time gaining strategy. The teacher can also define the *weight of an exercise* in its exercise sheet, the *weight of exercise sheets* in the global average. The task of choosing the exercises and the parametrization of a class is the occasion of a didactical reflection. Sometimes it is time consuming. After all, composing a classical exercise sheet can also be long. Of course, if there is no resources corresponding to your curriculum in WIMS, you always have the choice of developing them. And this involves even more time.

The second task is to enroll the students. Several modalities are possible. One of them consists simply to provide them with the address of the server and the name and the code of the class (the teacher chooses this code while creating the class). They can then enroll by creating their private user name and password. The teacher may also registers the class students, creating user names and passwords. It is also possible to use directly a CAS identification.

WIMS' Analytics

The student, when she enters her class sees the sheets (you can have a very precise overview by entering in a *classe ouverte*) organized by chapters. At the bottom of the sheets one sees a tool-bar composed of little squares, each corresponding to an exercise (or a string of exercises) that will be green once she has succeeded. Thus in the glimpse of an eye, she can see where she stands, what she has achieved, what has to be done. By clicking on one sheet she has access to the list of exercises composing the sheet. A last click and she is confronted to the exercise and thats where the work begins. Of course at any moment she can consult *mes notes* that is *my grades*.

The teacher, can also see in his class the results of each students first by global average, by average on each sheet, or detailed in one sheet exercise by exercise. Other statistics of the class can be found. One of the very meaningful is the *indice de difficulté d'un exercice*. It indicates the average number of times necessary to get the exercise done. Clearly if this indicator is between 1 and 2, the exercise isn't difficult. Experience shows that when this indicator is above 3, the teacher should consider explaining the solution to the exercise to the class.

Let us mention that the teacher can also set groups by defining "variables techniques", say group A, B, C. A group can be given specific exercises and the day set to open or close a sheet can be specified depending on groups. Of course, analytics can also be sorted by these variables.

An LMS which favours sharing

We want to underline two strong specificities of WIMS which make this server so unique. First, the very rich typology of exercises, enabled by the use of powerful software. It allows, once one has mastered the programming language, to create in several lines a very large set of versions of an exercise. Second, it is one of the rare existing LMS which allows and encourages sharing exercises. Indeed everyone who creates his exercise is then invited to publish them. After some review on the code and of the content, the exercise enters in the common base and is published under a free license. Then anyone will be able not only to use it but also can register the code of the exercise in his own class and change or modify it. This specificity is very rare, and users of WIMS are very attached to it. Of course the procedure of edition could still be more fluid and the searching engine improved. It is an important demand of the community of users and work is being currently done.

The economical model

WIMS is under GNU license. It is thus a free and open-source software. The resources published in WIMS are also published under a free license. WIMS during his 20 years long life, has not receive directly any founding from public authorities. The software developers are volunteers. So are, in a major part, the exercises developers. Though one can nevertheless consider that there is some public funding when a university attributes a server to WIMS and maintains it, or if some institution pays out the creation of exercises.

There are some wishes to evolve on this model, to be able to benefit from a IT provider to develop some aspects of the software. It appears though [14] that, for developers and users as well, a crucial value of WIMS relies on its sharing potentiality. The free and open source model relying on benevolent contributors (as wikipedia) is certainly the best model to support this value.

WHAT ADVANTAGES WIMS PROVIDES TO STUDENTS – OUR THEORY

Learning an unknown area of knowledge can be hard time. To grasp it, one relies on previous knowledge and skills, and on indications from the teacher, not always well understood. The faster one has to handle and manipulate the new material, the better. Yet first time can be hard. F. Garnier shows an experiment where more than a hundred persons were asked to open, for the first time of their life, a pressure cooker, with the help of a drawn instruction. The video shows a young woman fighting for opening it and achieving it in 104 seconds [6] . And then she does it again in 2 seconds, as anyone. WIMS allows to take the necessary time to achieve a task. Then, in the way to expertise, rehearsal is needed. Again WIMS meets this demand.

Let us underline that the human brain is much more analogical than deductive by nature. And the mathematical activity is clearly an activity which requires both ways of thinking, deductive of course, but also analogical during the creative steps. For an analogy to be made, one has first to experiment a number of different cases.

The necessity of multiplying examples of different type of representation in order to allow students to form a good representation of a problem is very well explained in Cordier & Cordier. It appears clearly that students only exposed to the use of Thales theorem when the parallels are from the same side of the intersection of the two scant lines, have a lesser possibility to understand the whole generality of Thales theorem than those exposed to a greater generality of cases. WIMS clearly meets this purpose. It's random features pushes the creator of an exercise to imagine a variety of examples and to encode them.

Let us also recall the celebrated lack of differentiation between the concepts of length and quantity for small children. Piaget's experience consists in showing two sets of 5 balls on a line, but the second set of balls are closer to each over than in the first set. Children are asked which of the sets has more balls. Up to six years, children answer that there are more balls on the line where the balls take more place. It appears that small children have some difficulty in distinguishing between length and number. Mixing up two concepts does not require to be under six, and a good discrimination comes not only with age but also with the occasion of seeing multiple situations and understanding their common features and differences. Again, as WIMS allows to offer the students an occasion to work and a wild diversity of cases, it can be used to train their discrimination skills.

If the deductive way of proving is certainly the most used in school especially in teaching mathematics, let us underline that the analogical thinking seems strikingly effective and deeply wired in human brains. To learn what is a cat or a dog no needs to explain. Present several items is enough. And then the animals will be recognized with no cognitive expensive. Achieve this type of quick recognition can be an important skill in learning. Again WIMS allows to enforce such type of learning thanks to the diversity of exercises it can generate.

Another celebrated experiment by Kahneman is the *Linda experiment:*

> Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations. – Which is more probable? (A) Linda is a bank teller. (B) Linda is a bank teller and is active in the feminist movement.

It appears that 89% of the persons asked answer (B), which is in fact false because (B) is clearly a subset of (A) and therefore has a smaller probability. This example illustrates the existence of two modes of thinking, fast and slow. The fast way of thinking, or system 1 – which works by analogy and is closer to feelings – jumps to the conclusion that Linda has to be at least active in the feminist movement and cannot be only a bank teller and thus chooses (B). When the emotional context is very vivid, the mathematical way of thinking, which is slower, has almost no chance to take over. And indeed, even when the persons interviewed are specialists, the number of wrong answers is surprisingly high. Olivier Houdé argues that the only way to get out of this dilemma is to work on what he calls system 3. In order to be able to let system 2 to work, one has first to inhibit the fast system 1. This can only be done by using an emotional key. This key is the shame of having been mistaken a first time. Thus doing mistakes is a fundamental step in constructing knowledge. The uncomfortable moment when a mistake happens generates thus the possibility of updating the thinking process which has caused this mistake. Moreover making this mistakes appears to be a necessary step. And one has to suffer its bitterness to be able avoiding it later. Better to experience it while training on WIMS.

If training is very central to learn, the influence of grading should not to be neglected in order to motivate students and push them to repeat. Experience shows that repetition is not always the clue to successful learning. Indeed if you choose for rating scale "*the best of trials*" or "*the mean of the three best trials*" the result is an increasing function of the answers' number. Letting chance decide, a wining strategy requiring no endeavor, might be chosen by some students. With no profit in learning. On the other side, choosing a rating scale involving *quality* (Q is calculated as a mean of the grade of all trials with a decreasing influence of the first ones), fosters another deviant behavior.

Some students are ready to pass hours on an exercise to improve the quality indicator repeating over and over an exercise they have understood. With no profit in learning again. At UPEM we are found of *strings* with very *high severity* index and we take the best of the trials. To obtain full grade on a string of four or five items with a high severity requires some mastership.

Research shows that memory, in particular *working memory*, is one of the best predicator of success in studies. And research shows that it can be trained by proper exercise. The main impact of mental arithmetics, to which generations of children where submitted together with the learning of dead languages, was perhaps the development of working memory. Thankfully, it can be trained at all ages and has similar characteristics as physical training: it is particularly efficient if it fits to the level of the training person. WIMS allows, to some extent, to individualize training and to enforce working memory. WIMS exercises, proposed with proper grading, can also enforce *attention*: students are particularly cautious at the last item of a string of four questions, and analyze the question with mindful attention. The diversity of types of questions which allow to ask questions of different types enforce *processing*. Sequencing as well can be strengthen. A possible scenario is to propose an exercise whose solution needs *sequencing*. A first the student is asked directly for the final result. If she does not succeed, the exercise is proposed in a sequenced form, step by step. Once it has been done with success the question is asked again directly.

EXPERIMENTING WIMS AT UPE: AN IDEA PROJECT

The community of Universities Paris Est (UPE) has won a call for project called (IDEA) in the context of « initiative d'Excellence en Formations Innovantes » (IDEFI) and of the Programme Investissements d'Avenir (PIA) financed by the Research National Agency (ANR). The project allowed working groups to get support in order to set and experiment pedagogical disposals. It appeared to be an opportunity to develop and test the use of WIMS during the first year of mathematics at university. This project started in November 2014. There are four mathematical courses, each one corresponding to 6 UCTS and to 2 hours of lecture and 3 hours of tutorials. The principle aim of the project was to build pathways of exercises corresponding to each of the learning module.

There exists huge numbers of exercises in the common base of WIMS corresponding to the first year after bachelor degree and we could rely on this resources. In addition hundreds of exercises have been developed, especially basic ones. For each chapter of the courses two sheets of exercises have been created, a basic and a standard one. Each sheet is composed of 8 to 15 strings of exercises. Within the basic sheet, exercises give the occasion to manipulate directly elementary notions of chapter. The standard sheet aims to propose exercises which correspond to the recourse's level. With one big difference: WIMS does not train writing a proof. Yet with a part of the training made on WIMS, more time can be spent in the classroom to practice this competency.

We choose an evaluation in WIMS with exercises's strings, high severity, and the best of the string success for grade. The sheets are opened for two weeks. After closure, students can continue to practice, but the grade is frozen. We use WIMS as a formative assessment. Its grade counts for a part of the continuous assessment. The final grade is given by the maximum between the exam from one hand and the average between the continuous assessment and the exam from the second hand. The work on WIMS has to be done outside of the classroom. If asked, teachers answer questions and use video projector to address some examples. A tutorship system is organized. First compulsory during the two pre-entry weeks, a daily permanence is then opened at noon along the whole academic year. The mentorship is provided by the top students of the previous years.

It appears from WIMS statistics that students work on WIMS for 2 to 3 hours on average. The amount of time does not depend of the student's level. Good student finish all with the maximum grade whereas average students may have trouble to succeed the full standard sheet.

Anonymous inquiry has been proposed. In the first semester of year 2016-2017 we obtained 82 answers on a promotion of 250 students. The inquiry dealt not only on WIMS but on all the aspects of the course. Concerning WIMS, it turns out that it has convinced users. More specifically 84,2% answered that WIMS' goals have been reached. A large majority thinks that the evaluation was clear and just. There was some complains about the time during which the sheets were opened. Indeed at the end the semester this information hasn't been provided clearly enough due to some overflow of the teaching staff. But a large majority (70,7%) declared to be satisfied by WIMS. Concerning learning methods, we are faced with students lacking of method. This is made clear by the way they engage in learning: only 7% answer that they open the course notes or the lecture notes shortly after the course, about 54% open it while preparing to an evaluation, and 49% read there notes while working on WIMS. This enables us to think that WIMS may be a tool that fosters working on the course itself. 78% used a scrap paper while doing WIMS exercises. Hence WIMS invites students to mobilize appropriate tools to build their thinking paths and answers. This remark has to be set against the fact that only 36,6% declare paying sustained attention to the reading of the statement of the exercises. Half of the students did not appreciate the feedback given by WIMS. More investigation has to be made to understand why.

According to 83,3% of students, WIMS helped developing competencies in mathematics. Some give testimony from which it seems to appear that it is through WIMS that they understood the principal course concepts and have begun to construct their mathematical thinking. According to 43% of students, WIMS has also helped to develop meta-competencies. This inquiry seems to establish first that WIMS was a truly effective tool in order to structure time during which students had the occasion to mobilize the resources of the course. Second WIMS invites the students to be rigorous in calculous or in reasoning, this fact has been often underlined. Third the exercise paths could still be optimized as students stipulate that sometimes the exercises are repetitive and the time required to achieve the exercise sheet is sometimes too long.

SOME HINTS TO START

Let us emphasize that organizing complete pathways of exercises is time consuming. It usually takes 3 years of work unless one can find resources that correspond to the wishes. If nothing corresponds, you've got to start programing.

Of course the main task is to align the pedagogical objectives and work on the coherence between lectures, classical exercises, class practice, WIMS exercises and exam. With striking efficiency, as shown in Berland 2017.

Here some pragmatic points. First, students have to be paid for their work: the grade obtained in WIMS has to be taken in account. Second, it is important to propose easy exercises which allow to work on basic notion of the course. Third, the sheets have to be apparently not to long and having one easier sheet to begin a chapter is a plus.

To preserve engagement of students, and to foster the efficiency of training on WIMS, teacher should follow students' work, speak of WIMS in class, and address some examples, especially when an exercise has a difficulty index greater than 3.

To preserve teacher's engagement, having contacts with WIMS EDU in order to enter a community of practice is a true help. Yet, WIMS gains to be part of the school or university project, to appear in the institutional brochure. Moreover, teacher promoting its use, by developing exercises for instance, should receive some kind of compensation.

To conclude, we underline again that WIMS is 20 years old. In 1997 it was really a visionary tool. And even if some parts would gain to be updated, it still is at the top. One of the greatest strength is its community and its free and open model. This allows to use and share exercises and possibly whole classes of exercises. We underline that there is something in the values shared that fosters a great engagement from the volunteers developing exercises and the software itself. Is the heroical time where the entire development is sustained by volunteers finishing? Will the model be updated in order to be able to sustain the payment of providers, needed to develop the software? Its future is not yet written. But whatever happens, these fundamental values, that foster engagement, should be carefully preserved. As for the pedagogical advantages of WIMS, to support student's learning, recall Von Neuman's quote: *Young man, in mathematics you don't understand things. You just get used to them.* WIMS, when well used, appears to be a tool that can make the *getting used to* much easier.

[1] Interactive map showing where WIMS is used. http://downloadcenter.wimsedu.info/download/map/map2.html

[2] Enquête auprès des utilisateurs WIMS http://moin.irem.univ-mrs.fr/groupe-wims/Enquete [consulted 2017/09/14]

[3] Le professeur Xiao Gang, créateur du logiciel WIMS s'est éteint le vendredi 27 juin 2014 http://unice.fr/fil/service-communication/actualites/le-professeur-xiao-gang-createur-du-logiciel-wims-s2019est-eteint-le-27-juin-2014

[4] WIMS EDU site http://wimsedu.info

[5] To see some examples of WIMS, connect to a WIMS-server, for instance http://wims.auto.u-psud.fr/wims/?lang=fr. You can change the language but the french version is the richest one. Then explore some *classes ouvertes* as a *visiteur anonyme*.

REFERENCES

Berland, C. (2017) *E-learning platform WIMS for bachelor first year courses on electric circuit.* Presented at EAEEIE Annual conference 2017, Grenoble, Fr.

Cordier, F. & Cordier, J. (1991). L'application du théorème de Thalès. Un exemple du rôle des représentations typiques comme biais cognitifs. *Recherche en didactique des Mathématiques*, 11(1), 45-64.

Ganier, F (2016). *C'était pourtant écrit !!! Pourquoi est-il si difficile de comprendre des instructions?* Presented at WIMS EDU colloque 2016, Annecy, Fr. http://webcast.in2p3.fr/videos-pourquoi_est_il_si_difficile_de_comprendre_des_instructions

Giner, E. & Kobylanski, M. (2017). *Retour sur l'expérimentation WIMS*. Research Report. UPEM.

Houdé, O. (2014). *Le raisonnement.* Que Sais-je ? PUF.

Jacquemin, L. (2017). *Enquête sociologique auprès d'utilisateurs enseignants de WIMS sur leurs usages et pratiques effectives*. Research Report. UPEM.

Khaneman, D. (2011). *"Linda: Less is More". Thinking, Fast and Slow*. New York: Farrar, Straus and Giroux. pp. 156-165.

Kobylanski, M. (2015). Plates-formes d'apprentissage et WIMS. *Gazette des Mathematiciens*, (145), 47-51.

GEOGEBRA AUTOMATED REASONING TOOLS: A TUTORIAL WITH EXAMPLES

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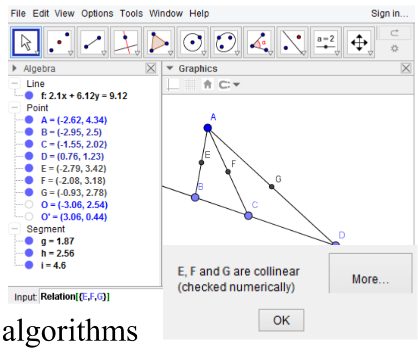
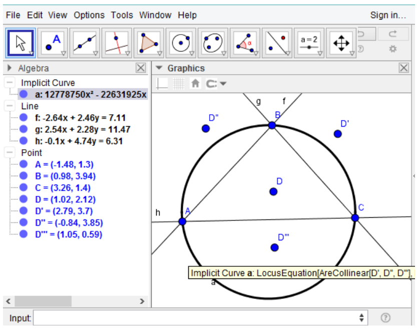
GeoGebra Automated Reasoning Tools (GGB-ART) are a collection of GeoGebra tools and commands ready to automatically derive, discover and/or prove geometric statements in a dynamic geometric construction. The aim of this workshop is to present, through examples, the use of GGB-ART and to argue about its potential impact in the classroom.

Keywords: Automated theorem proving and discovery, GeoGebra, Dynamic geometry software, Elementary geometry in education.

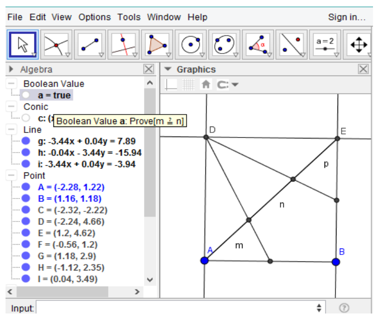
AUTOMATED REASONING IN ELEMENTARY GEOMETRY …



By “automated derivation of geometry statements” we refer to tools that, rigorously, output some/all geometric relations verified by a collection of selected elements within a geometric construction. For instance[1]: given a free point *A* and three points *B, C, D* on a line, consider *E, F, G*, the midpoints of segments *AB*, *AC* and *AD*. Then, the automatic derivation tool should output some property relating *E, F* and *G*.



By “automated discovery of geometry statements” we refer to algorithms that systematically find complementary, necessary, hypotheses for the truth of a conjectured geometric statement. For example[2], given a triangle *ABC* and a point *X*, let *M, N, P*, be the symmetric images of *X* with respect to the sides of the triangle. Then *M, N, P* are aligned. Obviously, this conjecture is false but…the automatic discovery algorithm should be able to output the necessary (and sufficient) location for *X* in order to have the alignment of *M, N, P*.



Finally, by “automated proving of geometry statements” we refer to algorithms that accept as input a geometric statement, such as[3]: “If two lines are drawn from one vertex of a square to the midpoints of the two non-adjacent sides, then they divide the diagonal into three equal segments”. Then, the algorithm performs an exact computation (i.e. not using floating point numbers) and outputs a mathematically rigorous (e.g. not based upon a probabilistic proof) yes/no answer to the truth of the given statement.

The community of mathematicians and computer scientists has been working on these issues along the past 50 years, with a variety of approaches, outcomes and popularization results. See, for instance, the pioneer work of Gelertner (1959) in the A.I. context, or the algebraic geometry framework for automated reasoning in geometry, disseminated by the book of Chou (1988), that is behind our current implementation). Moreover, it is clear that the didactic perspective on proof (with or without technology) has been a research topic for over 40 years in the world of mathematics education (Richard, Oller & Meavilla, 2016).

… ITS GEOGEBRA IMPLEMENTATION

Hence, we consider quite relevant to present in this workshop a tutorial describing in detail the very recent implementation (2016) of tools and commands for the automatic deriving, discovery and proving of geometric theorems over the free dynamic geometry software GeoGebra, with a great impact in mathematics education. See: Abánades et al. (2016), Hohenwarter et al. (2016).

To begin with GGB-ART we have to draw in GeoGebra a geometric construction. Then we will exhibit the many possibilities that GeoGebra offers to enhance investigating and conjecturing about geometric properties of our construction. Say: investigating visually; using the `Relation` tool to compare objects and to obtain relations; or using the `Locus` tool to learn about the trace of a point subject to some constraints. These methods are usually well known by the GeoGebra community and well documented at the GeoGebra Materials web (https://www.geogebra.org/materials/). But these methods are mostly numerical, i.e. not mathematically rigorous, they only work on the specific construction with concrete coordinates, so they do not allow to deal with general statements.

GGB-ART brings to GeoGebra new capabilities for automatic reasoning in Euclidean plane geometry in an exact way, by using symbolic computations behind the concrete construction: the `Relation` tool and command can be now used to re-compute the results symbolically; the `LocusEquation` command refines the result of the `Locus` command by displaying the algebraic equation of the graphical output, allowing to investigate and conjecture statements; the `Prove` and `ProveDetails` commands decide in an exact way if a statement is true (i.e. checking the mathematical correctness of some previously found relation).

… ITS EDUCATIONAL IMPACT

Our final goal is to share these tools with the community of math teachers and math education researchers, aiming to improve, after suitably addressing the necessary changes and approaches in the educational context, geometry education (Botana, Recio & Vélez, 2017). This is an involved didactical issue, dealing with human reasoning with technology and with the validation modes available in the classroom (i.e. deductive, inductive and instrumental), so that the student can accomplish his/her mathematical work (Richard, Oller & Meavilla, 2016). It is not a new issue: in fact, it was already 30 years ago when educators started reflecting about the potential role in education of software programs dealing with automatic theorem proving (automatic discovery and derivation were inexistent at that time). See, for instance, the visionary ICMI Study "School Mathematics in the 1990's" (Howson and Wilson, 1986) or the inspiring paper by P. Davis (1995), with a section that refers to the "transfiguration" power of computer-based proofs of geometry statements. But these reflections were formulated rather as considerations about the future than as proposals for the present time of their authors…

Currently, although there already are some studies concerning the development of intelligent tutorial systems designed to assist students to construct proofs in geometry, such as GRAMY (Matsuda and Vanlehn, 2004), GeoGebraTutor or QED-Tutrix (Tessier-Baillargeon, Richard, Leduc and Gagnon, 2014) –as detailed in the most comprehensive review of existing tutorial systems, available in the comparative study of Tessier-Baillargeon, Leduc, Richard and Gagnon (2017)– it is fair to say that, up to now, the dissemination, use and impact of these achievements in the educational context is very limited. For example, another recent survey by Sinclair et al. (2016), on geometry in education, although it includes a full section on the role of technologies and another one on "Advances in the understanding of the teaching and learning of the proving process", does not refer at all about automated reasoning tools.

Thus, since the program over which we have implemented our automatic reasoning tools (ART) is currently available over computers, tablets and smartphones, with and without internet connection, i.e. on a well spread, dynamic geometry program, we think the time has arrived to consider the following question: what could be the role, in mathematics instruction, of the ample availability of such tools? In this direction, our final goal is to make an open call to the community of math teachers and math education researchers, to join us preparing a research project to address the following issues: Are ART in geometry education good for anything? If yes, what are they good for? What should be the necessary changes and requirements in the educational context, if ART are to be considered good for anything?

…AND DIDACTIC FRAMEWORK

It is easy to consider the ART as an authentic geometric calculator. First, because they determine equations, even measures, and above all because they link different effects to help discovering new properties or to produce valid reasoning, like propositional calculus. We can consider the benefits or drawbacks of geometric calculators from a user perspective, here the teacher or the student who exploits them in school. In the same way that conventional or graphical calculators do not reveal the models on which the algorithms are based, the ordinary user of geometric calculators does not have access to the models that run them and produce answers. However, from a behavioral perspective, GeoGebra ART is not merely a black box that produces effects or reactions to actions determined by a waiting user. In fact, just as the ancients were questioning an oracle to predict what would happen in a given context, the user employs an ART as a guiding stick in the geometric environment.

Indeed, with regard to the theory of didactical situations in mathematics of Brousseau[4], we can see the ART as belonging to the *milieu*, that is to say, as being a playing partner of the student in the construction of knowledge. Of course, the *milieu* conveys knowledge and it is the model implemented in the tool that determines the need for it. However, the need for the student in interaction with the *milieu* can be quite diverse. In the case of ART we regard this interaction as follows: the student works in a situation (context, problem or task), questions the *milieu* in the particular logic of the situation and in a more general logic of the didactic contract that binds him or her to the knowledge at stake. He or she wants answers to fit the context, to solve the problem or to accomplish the task; he or she probably does not need to mobilize all logical artillery of mathematical proofs with its particular mode of expression and its high epistemic value. In terms of reasoning, ART helps producing genuine abductions, in the sense of Pierce, which facilitates the student inquiry into the situation, even when he or she was trying to solve a problem of geometric formal proof.

Several works have already dealt with the merging of mathematical proofs, visualization and dynamic geometry, but surprisingly, references to other natural links with geometry are often missing in the literature. If we consider the work carried out in the working groups on geometrical thinking, as in the CERME (for details, see Kuzniak, Richard & Michael-Chrysanthou, 2017), we can mention that few research works focus on modelling of physical phenomena using geometrical tools, or deal with solving problems in geometry that are not problems of proof, or go beyond the mere discovery of some characteristic properties well defined and known in advance by the teacher and by the student. However, the very constitution of the geometric model by the student is certainly an incarnation of what modelling of form, shape and space is. Unfortunately, modelling activity is generally not widely practiced in compulsory education, and problem solving in geometry classes is often limited to those based on well-defined tasks. Moreover, few studies concern the solving of open problems or those that require a problematization which is not already linked to a geometric model known in advance. In this context, we believe that the functionalities of ART are particularly useful in supporting the development of mathematical competencies through the development of a geometrical culture, building on mathematical discovery and modelling approaches.

References

Abánades, M.A., Botana, F., Kovács, Z., Recio T. & Sólyom-Gecse, C. (2016). *Development of automatic reasoning tools in GeoGebra*. Software Demonstration Award at the ISSAC 2016 Conference. Retrieved from http://www.issac-conference.org/2016/awards.php)

Botana, F., Recio, T. & Vélez, M.P. (2017). *The role of automated reasoning of geometry statements in mathematics instruction.* Poster CERME-10, TWG 15. Retrieved from (https://keynote.conference-services.net/resources/444/5118/pdf/CERME10_0181.pdf)

Chou, S.C. (1988). *Mechanical Geometry Theorem Proving*. D. Reidel Pub. Co. Dordrecht, Holland.

Davis, P. (1995). The rise, fall, and possible transfiguration of triangle geometry: a mini-history. *The American Mathematical Monthly*, 102(3), pp. 204-214.

Gelernter, H. (1959): *Realization of a geometry theorem proving machine.* Proc. Int. Conf. on Information Processing. Paris, UNESCO house. pp. 273-282.

Hohenwarter, M., Kovács, Z. & Recio, T. (2016). *Deciding geometric properties symbolically in GeoGebra*. ICME 13, TSG 18: Reasoning and Proof in Mathematics Education.

Howson, G. and Wilson, B. (1986). *ICMI Study series: School mathematics in the 1990's*. Kuwait: Cambridge University Press.

Kuzniak, A., Richard, P.R. & Michael-Chrysanthou, P. (2017). From geometrical thinking to geometrical working competencies. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger & K. Ruthven (Eds.), *The ERME-Book: Development of European Research in Mathematics Education - Twenty Years of Communication, Cooperation and Collaboration* (in press).

Matsuda, M., Vanlehn, K. (2004). GRAMY: A Geometry Theorem Prover Capable of Construction. *Journal of Automated Reasoning*, 32, pp. 3-33.

Richard, P.R., Oller, A.M. & Meavilla, V. (2016). The Concept of Proof in the Light of Mathematical Work. *ZDM-The International Journal on Mathematics Education, 48*(6), 843–859. DOI: 10.1007/s11858-016-0805-9

Sinclair, N., Bartolini Bussi, M.G., de Villiers, M. & Owens, K. (2016). Recent research on geometry education: an ICME-13 survey team report. *ZDM Mathematics Education*, 48(5), pp. 691-719.

Tessier-Baillargeon, M., Leduc, N., Richard, P.R. & Gagnon, M. (2017). Étude comparative de systèmes tutoriels pour l'exercice de la démonstration en géométrie. *Annales de didactique et de sciences cognitives,* pp. *22*, 91-117.

Tessier-Baillargeon, M., Richard PR., Leduc N. & Gagnon M. (2014). Conception et analyse de geogebratutor, un système tutoriel intelligent: Genèse d'un espace de travail géométrique idoine. *Revista Iberoamericana de Investigación en Matemática Educativa*, 17(4-II), pp. 303-326.

1 Reference question N°046 https://www.emis.de/data/projects/reference-levels/EMS_RQ_BUNDLE_ENGLISH.pdf

2 Recio, T. and Vélez, M.P. (1999). *Automatic Discovery of Theorems in Elementary Geometry*. Journal of Automated Reasoning 23, 63-82. DOI: 10.1023/A:1006135322108

3 Howson, G. and Wilson, B. (1986). ICMI Study: School mathematics in the 1990's. Cambridge University Press.

4 See, for some of his key works, translated to English, http://www.springer.com/gp/book/9780792345268

FUNCTION HERO: AN EDUCATIONAL GAME TO AFFORD CREATIVE MATHEMATICAL THINKING

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The enhancement of educational processes at all levels of education can be achieved by implementing Game Based Learning (GBL), engaging students to the educational objectives affording the "Flow" mind state in which individuals optimize their actions scaffolded by intrinsic motivation. In mathematics, most of the games are based on arithmetical or logical thinking due the software's limitation in assessing user's inputs. Aiming to develop a game about functions affording Creative Mathematical Thinking (CMT), we used a Dynamic Cinderella Software (DCS) called Cinderella, the Game Development Environment, Unity and the Kinect Sensor from Microsoft. In this paper, we presentdesign elements of the game Function Hero that affords CMT.

Keywords: game based learning; technology in education; mathematics; functions;

INTRODUCTION

The emergence of motion-controlled technologies within the increasing usage of embodied cognition and augmented reality environments open new doors for students to use their different senses into their learning process. Nowadays, students can *experiment* mathematics more than just listening and watching what happened on the blackboard overcoming the traditional approaches in teaching mathematics, moreover, having fun during the learning activity.

Creating fun in learning environment is not only to make up the content or the activity. Having fun means that the individual entered in a engagement state known as "Flow", optimizing its motivation and relaxation. Relaxation enables a learner to take things more easily, and motivation enables them to put forth effort without resentment (Prensky, 2007). This mental state also contributes to creativity. When in flow, the creator ignores the external environment, apart from the action which is performed, the distractions don't effect the individual and one's mind is fully open and attuned to the act of creating.

In the flow state, the challenges presented and the ability to solve them are almost perfectly matched, and the individual often accomplish things that they thought they couldn't, along a great deal of pleasure. To keep a person in the flow state the designer must consider that making things too easy, the players become bored and stop. Making thing to hard, the players stop because they become frustrated. The same rule can be applied at the educational tasks. The representation of the concept of flow is shown on the figure below.

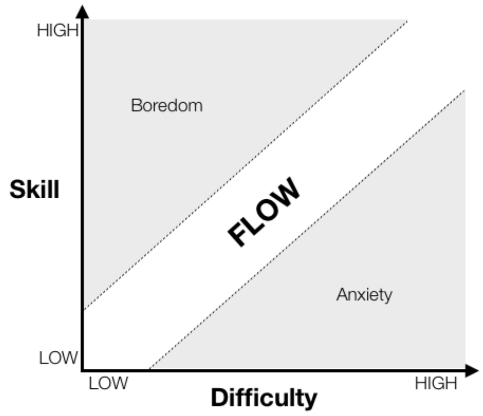

Figure 1: Flow, boredom, and anxiety as they relate to task difficulty and user skill level. Csikszentmihalyi, 1990.

Mathematicians profess that performing mathematics is a creative activity (Hadamard, 1954). Technology supported inquiry based learning is a possible way to put students in situations where their creativity is needed and can be expressed (Blumenfeld et al., 1991). In this article, we first introduce the concept of Creative Mathematical Thinking. We then present the educational resource under consideration in the reported experiment, the game "Function Hero".

CREATIVE MATHEMATICAL THINKING

Based on the literature review on creativity (Guilford, 1950; Kaufman & Sternberg, 2010), mathematical creativity (Sriraman, 2004; Leikin & Lev, 2007; El-Demerdash, M. & Kortenkamp, U, 2009) and mathematical thinking (Tall, 2002; Blinder, 2013), the CMT (Creative Mathematical Thinking) can be understood as the combination of divergent and convergent thinking in mathematics. Starting from this principle we created the concept of "Creativity's Diamond" (Lealdino F, et al, 2015) (Figure 2).

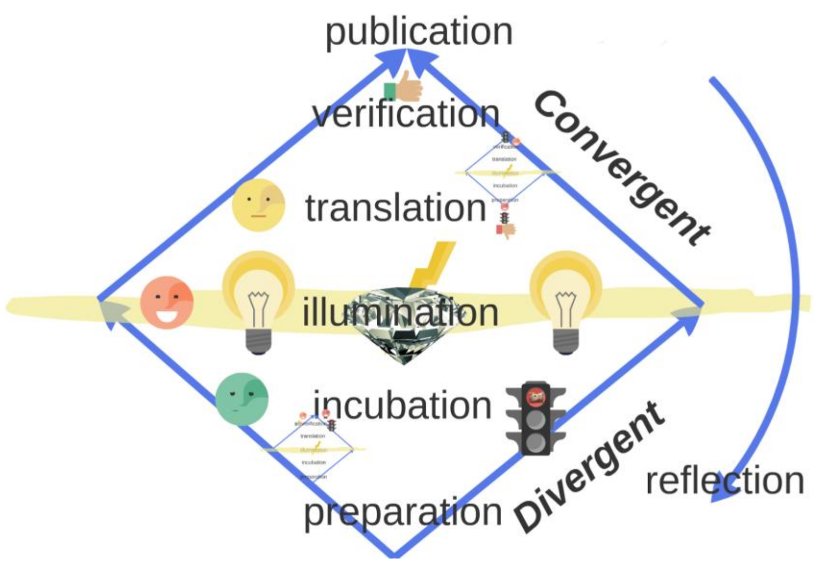

Figure 2: Creative Process

This idea is based mainly on Guilford's and Walla's models of creativity. Guilford emphasized the distinction between convergent and divergent thinking. In 1976 he introduced the model of Divergent Thinking as the main ingredient of creativity. Guilford appointed the following characteristics for creativity:

- Fluency: The students' ability to provide many responses or to come up with many strategies to solve a mathematical problem or challenge.
- Flexibility: The students' ability to provide different/varied responses or to come up with different/varied strategies to solve a mathematical problem or challenge.
- Originality: The students' ability to come up with unique (original) responses (solutions, strategies, representations, etc.) to a mathematical problem or challenge.
- Elaboration: The students' ability to describe, substitute, combine, adapt, modify, magnify, extend the usability, eliminate or rearrange mathematical situations.

Wallasoutlines four stages of the creative process - preparation, incubation, illumination, and verification - dancing in a delicate osmosis of conscious and unconscious work. These phases go as follows:

- Preparation: The problem is investigated in all directions as the thinker readies the mental soil for the sowing of the seeds. It's the accumulation of intellectual resources out of which to construct new ideas. It is fully conscious and entails part research, part planning, part entering the right frame of mind and attention.
- Incubation: Next comes a period of unconscious processing, during which no direct effort is exerted upon the problem at hand - this is where the combinatory play that marked Einstein's thought takes place. Wallas notes that the stage has two divergent elements - the "negative fact" that during Incubation we don't consciously deliberate on a particular problem, and the "positive fact" of a series of unconscious, involuntary mental events taking place. "*Voluntary abstention from conscious thought on any problem may, itself, take two forms: the period of abstention may be spent either in conscious mental work on other problems, or in a relaxation from all conscious mental work.*
- Illumination: Following Incubation is the Illumination stage, which Wallas based on French polymath Henri Poincaré's concept of "sudden illumination" - that flash of insight that the conscious self can't will and the subliminal self can only welcome once all elements gathered during the preparation stage have floated freely around during incubation and are now ready to click into an illumination new formation.
- Verification: The last stage, unlike the second and the third, shares with the first a conscious and deliberate effort in the way of testing the validity of the idea and reducing the idea itself to an exact form.

THE GAME FUNCTION HERO

To develop the artifact used in this study, the Kinect sensor was used. Kinect is a motion detection device, equipped with RGB camera, infrared depth detection sensor, microphone and a dedicated processor. Originally designed to be a gaming accessory for the Microsoft Xbox 360 gained popularity within developers and a windows compatible version was released.

To integrate the affordances from kinect sensor we use two main softwares: Unity and Cinderella. Unity is a game development environment where is possible to create 2D and 3D experiences and afterwards exploit to many platforms, whether android, iOS, Windows, Linux or macOS. Providing a wide range of possibilities in creating digital content to be used in education.

The other software, Cinderella, is a dynamic geometry software developed to provide an environment to develop high-end educational applications to teach geometry. It has its own programming language, called CindyScript, which afford the possibility to create with considerable freedom, interactive digital content, either for geometry or physical simulations. This software was used in the study to translate the data received from Kinect in function graphs.

Therefore, using the technology available and taking into account the kinesthetic learning approach, where students use their bodies to perform and react in accordance with what is being demanded by the game. We expect to enrich the repository of digital tools to teach mathematics and at same time, to enhance student's motivation towards learning mathematical functions using game based learning approach, promoting efficient learning and fun for those who play the game "Function Hero"

The game was played by various students and exposed in some science or mathematics fairs. Following the same gameplay of games like Guitar Hero, Just Dance, Rock Band and more, the player must perform the graph of the functions, given in their algebraic expression, with its body. The choreographies are created by the rival players and sent via a web page. Then, the game shows the expressions on the screen as goals to be performed by the user. See on the figure below the game in action.

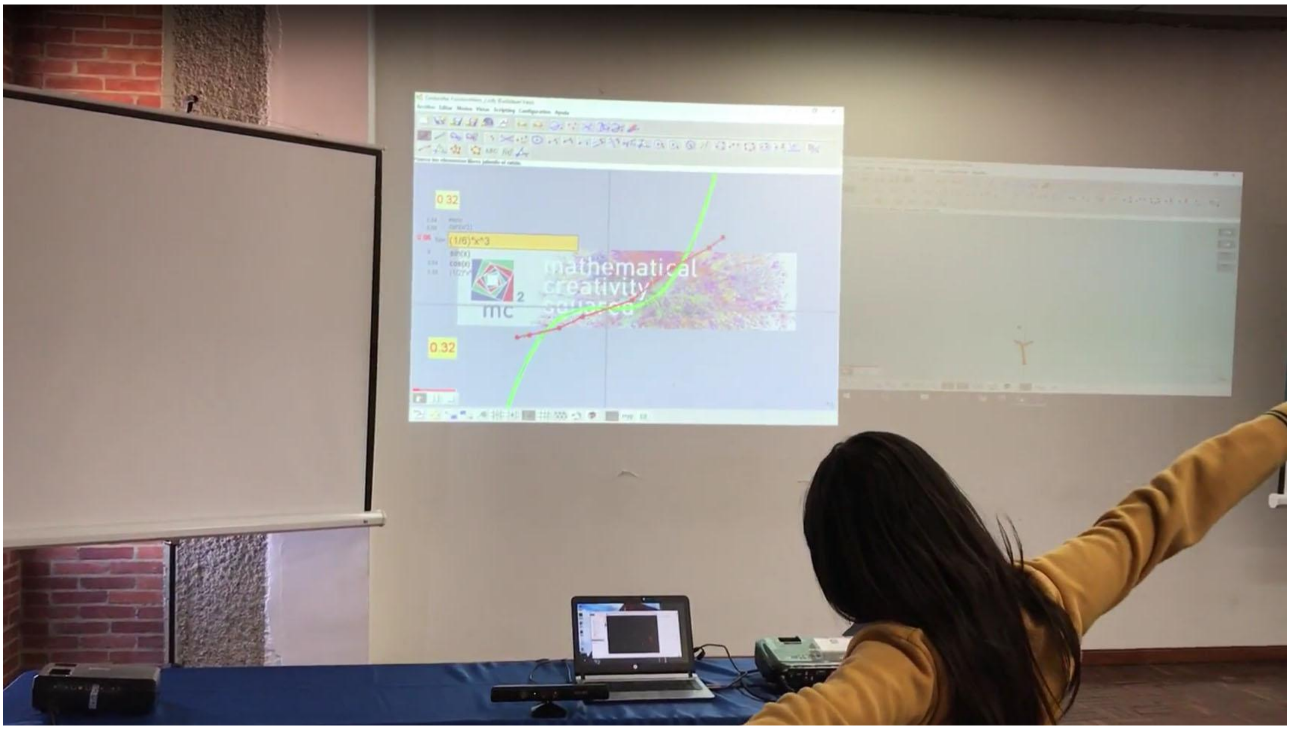

Figure 3: Student playing the Function Hero

DESIGN ELEMENTS THAT AFFORDCMT

For CMT affordances we started from Gibson's (1979) theory of affordances. Gibson considers affordances as properties in the environment that present possibilities for action or as cues in environment such as substances, surfaces, objects, that hold possibilities for action. In a technological environment, and according to Akrich and Latour (1992), affordance is a legislator within a sociotechnical system, e.g., what a device allows or forbids from the actors, what it prescribes and what permits. Therefore, CMT affordances of a technology are about its properties, features, its structure or organization, its inherent conditions or qualities, which:

- Mathematical content: Consist of open and/or non-standard problems that connect (i) multiple representations of the same concept, (ii) different mathematical fields, and (iii) different knowledge areas and mathematics.
- Mathematical processes: Offer interaction with the technology that allow engaging with and making sense of mathematics by exploration of, experimentation within, and formulation of mathematics problems.
- Creative Mathematical Thinking skills: Foster the users' cognitive processes of Fluency, Flexibility, Originality and Elaboration by stimulating/encouraging students to make and check conjectures, find multiple solutions and/or strategies for the same problem, think and reflect on their mathematical work, generalize mathematical phenomena.
- Social Aspect: Value the mathematical communicative skills.
- Affective aspect: Promote engagement by generating a feeling of (aesthetic) pleasure because of the narrative, some game features of the flow of the mathematical activities.

RESULTS

After the game was introduced in the classes, the students had one week to change their choreographies on the web site. The set of functions created by them where played in a sort of tournament. The engagement of students was more than expected, they created the choreographies and played along the tournament supporting their colleagues and vibrating with the scores.

Team	Choreography
0	abs(sin(x)), cos(x)+2, x^(-2), log(x), exp(2*log(x)) , exp(-x) , 3*x , -x^(10)
1	abs(x),log(-x+4),log(x+6),cos(x),tan(-x),0,sin(x-2),exp(x^2),cos(x^1/2),cos(-x),(x-3)(x+5)
2	5*x^3,sin(86)+arctan(1515),abs(sin(x)),arccos(x^69)),1/(1+6*x^2),arccos(x^1664)
3	ln((-3+2*x)^(-1)),heavyside(x),-x ,x^3,cos(x),log(exp(sqrt(x))),tan(x),sin(exp(x)),log(x^2)
4	(x+2)^1, abs(x)+2,(-(x)+1)^1,-abs(x)+1^3,sin(-x),x^3+1^4,-(x^3)+1^2, 2+(17^12)x*0, x^-2*log(x)
5	2*x,log(x+2),x^(-2),exp(x), ch(x+1/2)
6	log((-3+2*x)^(-1)),-x^3,-x ,x^3,cos(x),log(exp(sqrt(x))),tan(x),sqrt(-x)
7	(x+2)^1, abs(x)+2,(-(x)+1)^1,-abs(x)+1^3,sin(-x),x^3+1^4,-(x^3)+1^2, 2+(17^12)x*0

Table 1: Teams' choreographies for the Function Hero Tournament

CONCLUSION

The game Function Hero was created with the intention to provide another technology to the mathematics education scenario allowing to promote motivation and engagement using an educational game. It differs from the other game in the aspect of its design, which doesn't ask user closed answers.

Following the CMT affordances elements (Table 1), we developed a game which doesn't present standard problems since the goal is to create sets of functions representing choreographies to be "danced" by others. The multiple representation of the same subject, functions, is seen when students must think about the algebraic expression, translate it into machine expression and imagine the graphical result in accordance body movement that fits well the function thought by them.

The social aspect with other players was encouraged promoting the Function Hero Tournament, which teams faced each other dancing the choreographies created by themselves and by other teams.

The affective aspect is present in the game itself, providing scores which students can verify their movements and evaluation of the algebraic expression in real time.

Analysing the choreographies, it's possible to trace some elements of divergent thinking listed as Fluency, Flexibility, Originality and Elaboration. For instance, we notice that the Team 2 shows more fluency in trigonometric functions while the Team 4 in quadratic and cubic functions. The Team 6 elaborated their choreography using some redundant expression to make harder to the other teams to recognize but easy to themselves (log(exp(sqrt(x))).

Thus, the game allows the evaluation of any gesture of the student giving points in accordance to it. Some further studies will be conducted to verify if there is a learning gain on the recognition and understanding of functions using the game.

REFERENCES

Akrich, M., & Latour, B. (1992). A summary of a convenient vocabulary for the semiotics of human and nonhuman assemblies. In W. Bijker & J. Law (Eds.), Shaping technology / building society: Studies in sociotechnical change (pp. 259–264). Cambridge: MIT Press

Blinder, S. M. (2013). Guide to Essential Math. Elsevier: New York.

El-Demerdash, M. (2010). The effectiveness of an enrichment program using dynamic geometry software in developing mathematically gifted students' geometric creativity in high schools Doctoral dissertation, University of Education Schwäbisch Gmünd, Germany.

Gibson, J. J. (1979). *An ecological approach to visual perception.* (Boston, MA, Houghton Mifflin)

Guilford, J.P. (1950). Creativity. American Psychologist, 5, 444-454.

Hadamard, J. (1954). The psychology of invention in the mathematical field. New York: Dover Publications.

Kaufman, J. C., & Sternberg, R.J. (Eds.) (2010). The Cambridge Handbook of Creativity. Cambridge: Cambridge University Press.

Lealdino, P. F., Mercat, C., Emin, V., Trgalova, J., Essonnier, N.(2015). Assigned to creativity: didactical contract negotiation and technology. In K. Krainer & N. Vondrová (Eds), Proceedings of the Ninth Conference of the European Society for Research in Mathematics Education (CERME9, 4-8 February 2015) (pp. 992-1006). Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME.

Leikin, R., & Lev, M. (2007). Multiple solution task as a magnifying glass for observation of mathematical creativity. In J. H. Woo et al. (Eds.), Proceeding of the 31st International Conference of the Psychology of Mathematics Education, 3 (pp. 161 – 168), Seoul, Korea.

Prensky, 2007. Digital Game-Based Learning: pratical ideas for the application of digital game-based learning. McGraw-Hill Pub.

Sriraman, B. (2004). The characteristics of mathematical creativity. The Mathematics Educator, 14(1), 19-34.

Tall, D. (2002). Advanced Mathematical Thinking. Dordrecht: Kluwer.

Wallas, G. (1926). The art of thought. New York, NY: Harcourt Brace Jovanovich. The Stages of Creativity. In Rothenberg, A. & Hausman, C. (Eds) The creativity question. (1976) . (pp. 69-73).

(Un)intended representations in dynamic geometry software: pedagogical considerations

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In recent years, dynamic geometry software (DGS) has become common in classrooms for teaching and learning of mathematics. In this paper, I address some representational issues with which students and teachers may encounter while using DGS. Unintended representations may stem from the design principles for DGS, tasks that involve constructions with a limitation and representations of mathematical objects in DGS. Pedagogical considerations about using those representations as an opportunity for mathematical investigation are discussed.

Keywords: dynamic geometry software, (un)intended representations, pedagogical considerations

INTRODUCTION

Mathematics educators call for using technology in mathematics classrooms such as dynamic interactive mathematics technologies (Association of Mathematics Teacher Educators (AMTE), 2006; National Council of Teachers of Mathematics (NCTM), 2000). An issue for teachers may include finding the right tool to use in the mathematics classroom to enhance students' mathematical learning (Smith, Shin, & Kim 2016). Smith et al. (2016) emphasize that a quick search on the Internet for a mathematical topic yields in a number of commercial and free-of-charge tools.

Dick (2008) provides some criteria for selection of technologies teachers may take into consideration. For example, a technological tool should stay true in mathematics – that is known as *mathematical fidelity* (Dick, 2008; Dick & Hollebrands, 2011). Also, a digital tool should not trigger a mismatch between students thinking and intended mathematics learning – that is known as *cognitive fidelity* (Dick, 2008; Dick & Hollebrands, 2011). For example, the angle between two perpendicular lines is perceived as an acute or obtuse angle in an unequal scale of coordinate system (see Dick & Burrill, 2016; Dick & Hollebrands, 2011). Moreover, a technological tool should be pedagogically faithful, in that "the student should perceive the tool as (a) facilitating the creation of mathematical objects, (b) allowing mathematical actions on those objects, and (c) providing clear evidence of the consequences of those actions" (Dick, 2008, p.334). Smith et al. (2016) found that pedagogical and mathematical fidelity for selecting a digital tool to use in classrooms was important for in-service and prospective mathematics teachers value.

Leung and Bolite-Frant (2015) emphasize that a technological tool with a limitation or uncertainty has a *discrepancy potential*. The researchers state that unintended mathematical representations open a pedagogical space for teachers. For example, teachers may capitalize an unintended mathematical concept with a focus on technological representations. A pedagogical space may take place by means of "feedback due to the nature of the tool or design of the task that possibly deviates from the intended mathematical concept or (ii) uncertainty created due to the nature of the tool or design of the task that requires the tool users to make decisions" (p.212). In this paper, representational issues stemming from constructions with a limitation, the design of DGS and representations of mathematical objects in DGS are discussed.

CONSTRUCTION WITH A LIMITATION

DGS allows for manipulating primitive elements (e.g., points, line segments) and exploring the invariant attributes of geometric objects. Properties of geometric objects remain invariant when a point or object is dragged in a properly constructed shape (Laborde, Kynigos, Hollebrands, & Strässer, 2006). Students or teachers may use a DGS *drawing* – that is “a process that involves the use of "freehand" tools to create a geometrical object”, and focus on its perceptual characteristics (Hollebrands & Smith, 2009, p.221). Also, teachers may provide a construction with a limitation for students.

Drawings or constructions with a limitation trigger unintended mathematical representations (see Mariotti, 2013; Ruthven, Hennessy, & Deaney, 2008). Such technological representations stem from how the tools in DGS are utilized. For example, in Figure 1a, the *Parallel line* tool is utilized to create a trapezoid with one pair of parallel sides. Students may notice a trapezoid can also have two pairs of parallel sides dragging point *C* towards point *D* (Figure 1b) and conclude that “a trapezoid is sometimes a parallelogram.” When points *C* and *D* coincide as shown in Figure 1c, the trapezoid becomes a triangle. Moreover, if point *C* crosses point *D*, a crossed quadrilateral is created (de Villiers, 1994). This construction does not preserve the invariant properties of the trapezoid and has a limitation. However, this construction may allow for a mathematical discussion about the counterexamples of the trapezoid. Researchers point out that constructions with a limitation or drawings give an opportunity for students to reason about geometric objects with the supervision of mathematics teachers. For example, Ruthven et al. (2008) stress that teachers capitalize drawings or unintended mathematical constructions. On the other hand, they observe a teacher who concealed anomaly constructed geometric objects.

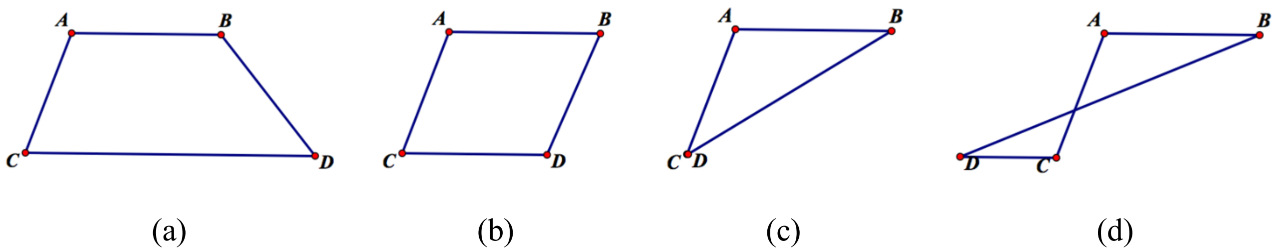


Figure 1. (a) A trapezoid, (b) The trapezoid becomes a parallelogram, (c) The trapezoid becomes a triangle, (d) The trapezoid becomes a crossed quadrilateral

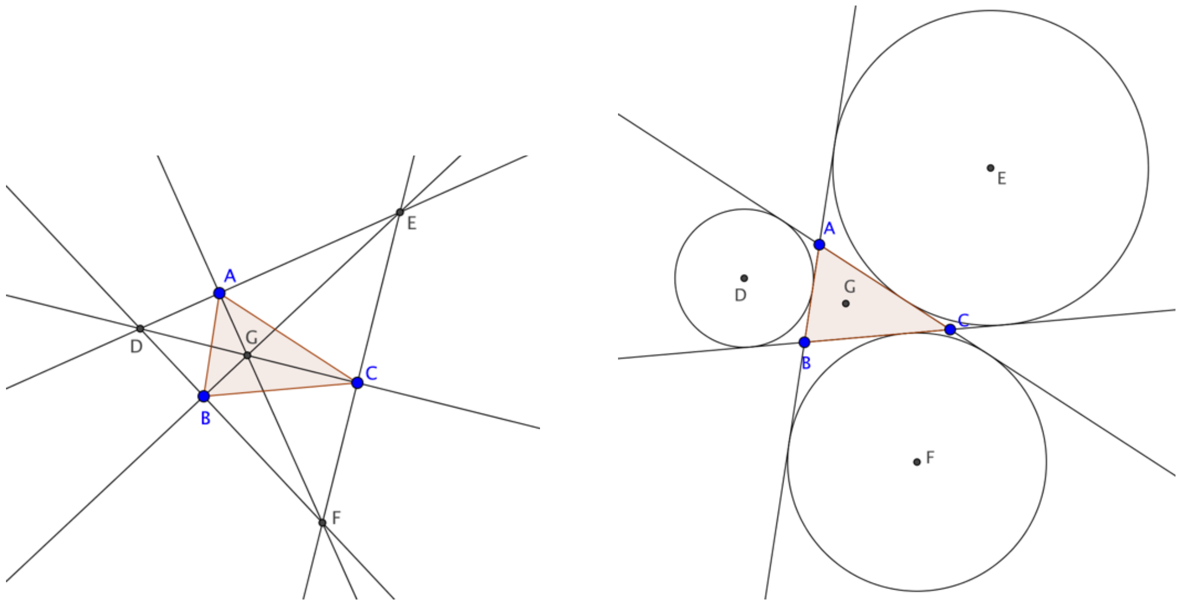
Unintended representations may result in interruptions in the flow of a lesson and teachers may make ad hoc decisions about how to respond in these moments (Cayton, Hollebrands, Okumuş, & Boehm, 2017). For example, teachers may emphasize counterexamples of geometric objects using an unintended representation with a focus on mathematics and technological representation. On the other hand, they may eliminate unintended representations. Teachers’ pedagogical dispositions determine if they conceal, capitalize or eliminate an unintended mathematical representation (Dick & Burrill, 2016; Mariotti, 2013; Ruthven et al., 2008). For the elimination of unintended representations, the teacher should use his or her technological and mathematical knowledge to construct objects that stay true in mathematics (Dick & Burrill, 2016). For example, the restriction of point *D* on a ray that is parallel to $\overleftrightarrow{AB}$ as shown in Figure 2a eliminates the counterexamples of the trapezoid. Then, point *C* does not meet at or cross point *D* (Figure 2b). Teachers’ mathematical and technological knowledge should be in action to construct a geometric sketch that preserves the critical attributes of a geometric shape (Dick & Burrill, 2016).

(a) (b)

Figure 2. (a) Point *D* bounded on a ray, (b) Points *D* and *C* do not coincide

DESIGN OF DGS

Developers of DGS make design decisions and users (e.g., teachers) most often have no freedom to change the interface for the tool. The interface for a tool may violate mathematical fidelity and provide incorrect feedback (Dick, 2008; Dick & Burrill, 2016; Dick & Hollebrands, 2011). In GeoGebra (a free dynamic geometry program), the *Angle Bisector* tool creates two angle bisector lines when two lines/line segments are selected (Steketee, 2010). Then, angle bisectors of a triangle meet at four points as shown in Figure 3a. This representation may be confusing for students and teachers because three angle bisectors of a triangle should meet at a point – that is called *incenter*. However, the design decision on the *Angle Bisector* tool results in demonstrating the excenters of a triangle [the center of a circle that is tangent to a side of a triangle and the extension lines of the other two sides]. Teachers may prefer to use DGS that provides more transparent feedback for students. For example, the Geometer's Sketchpad (a commercial dynamic geometry program) creates a ray as an angle bisector and the angle bisectors of a triangle meet at a point (see Steketee, 2010).

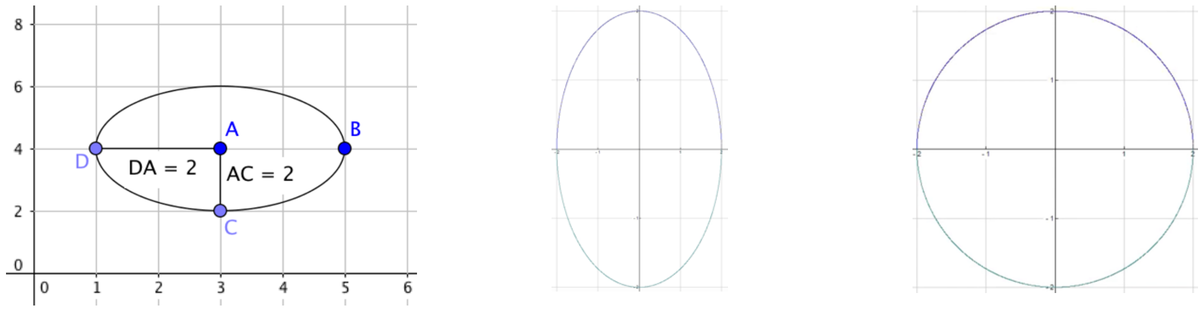


(a) (b)

Figure 3. (a) Angle bisectors of a triangle in GeoGebra, (b) The excenters (Points *D*, *E*, *F*) of a triangle

Teachers should be able to make sense of an unintended technological representation to determine if the tool provides a correct mathematical representation. This skill requires establishing a link between mathematical and content knowledge to reason about unintended representations (Dick & Burrill, 2016). Teachers may utilize different techniques to use the tools in DGS to eliminate unintended representations. For example, the *Angle bisector* tool in GeoGebra does not demonstrate the excenters if the three vertex points of a triangle are selected. Knowledge of an alternative utilization of a tool in DGS or about different dynamic geometry programs may assist teachers in making a decision about identifying the right DGS.

Smith et al. (2016) found that in-service and student teachers were not concerned about cognitive fidelity. However, tools that violate cognitive fidelity result in giving misleading information (Dick & Burrill, 2016; Dick & Hollebrands, 2011). For example, in GeoGebra, one may change the scale of coordinate system. As shown in Figure 4a, a circle in an unequal scale of coordinate system looks like an ellipse (see Steketee, 2010). On the other hand, some programs (e.g., Graphic Calculus) create graphs on an unequal scale of system as default when a graph is plotted (Figure 4b). Familiarity with the tool may eliminate unintended representations. For example, the *Square* tool in Graphic Calculus equalizes the axes (Figure 4c) (van Blokland, van de Giessen, & Tall, 2006).



(a) (b) (c)

Figure 4. (a) A circle on an unequal scale of coordinate system in GeoGebra, (b) a circle in an unequal scale of coordinate system in Graphic Calculus, (c) the circle in an equalized coordinate system

REPRESENTATIONS OF MATHEMATICAL OBJECTS IN DGS

According to Laborde (1993), "*drawing* refers to the material entity while *figure* refers to a theoretical object" (p.49). *Material drawings* (e.g., diagrammatic representation of a circle in DGS or on a sheet of paper) have flaws, for example, "marks have a width, straight lines are not really straight" (p.50). She refers to the abstraction of material drawings as *idealized drawings*. A material drawing may result in a confusion for students/teachers and they may have difficulty identifying its corresponding figure (theoretical object). Similarly, a dual relationship between mathematical and technological representations should be established because DGS may not provide an accurate representation for a figure. For example, a quadrilateral signifies a plane in Cabri 3D as shown in Figure 5. Then, students may think of a plane as a quadrilateral or a bounded object because it does not extend in all directions forever. The *Sector* tool that extends the plane as shown in Figure 6 may be utilized to demonstrate the unboundedness of plane. Knowledge of tools in DGS assists teachers in providing a more accurate representation of plane. Teachers may consider using different dynamic geometry programs to develop their understanding of figures. For example, some dynamic geometry programs (e.g., GeoGebra) do not allow students/teachers to extend the plane in all directions. Google SketchUp provides a more accurate representation of plane as default (see Panorkou & Pratt, 2016). Then, dynamic geometry programs have different discrepancy potentials.

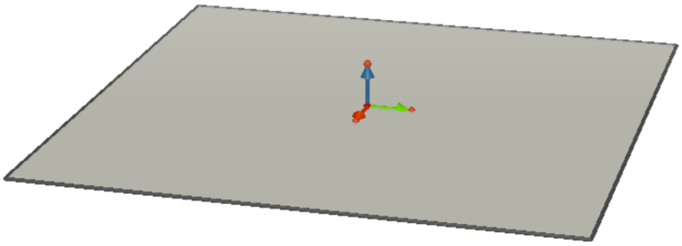


Figure 5. Representation of a plane in Cabri 3D

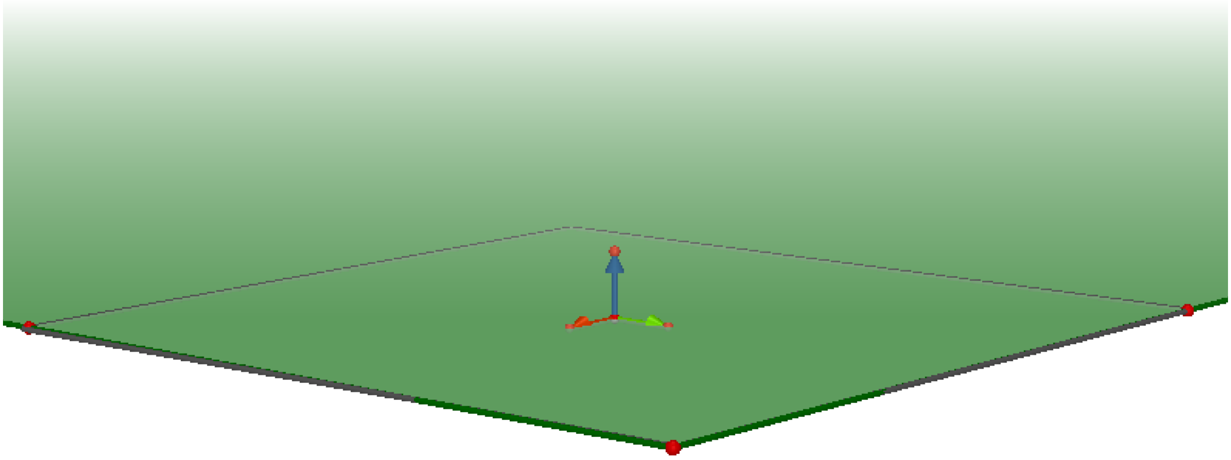


Figure 6. Extended plane in Cabri 3D

PEDAGOGICAL CONSIDERATIONS

Dynamic geometry programs that violate mathematical, cognitive and pedagogical fidelity may disrupt the flow of a lesson if teachers do not pre-plan to use them. Representational issues may stem from the design principles for DGS, tasks that involve constructions with a limitation and representations of mathematical objects in DGS. Researchers emphasize the importance of teachers' abilities in identifying affordances and constraints of a tool with a focus on how a tool may help or hinder students' thinking (Bartolini Bussi & Mariotti, 2008, Leung, & Bolite-Frant, 2015).

Dick and Burrill (2016) address that technological content knowledge "is important for teachers employing technology in the classroom, for it can help them anticipate what issues and phenomena students may encounter while using technology for a mathematical problem solving task or exploration" (p.44). For example, teachers may eliminate a representational issue and provide students accurate representations or constructions using their technological content knowledge. Also, knowledge of different dynamic geometry programs may guide teachers through technologies that have the best potential to enhance students' learning. Accordingly, they may prefer to DGS that is pedagogically, cognitively and mathematically faithful.

On the other hand, how a teacher makes an ad hoc decision when they encounter with an unintended representation is related to their technological pedagogical content knowledge. On the one hand, they may conceal unintended representations (Ruthven et al., 2008). On the other hand, they capitalize unintended representations with a focus on technology and mathematics (Mariotti, 2013; Ruthven et al., 2008). For example, Mariotti (2013) found that a student's drawing that did not preserve the invariant properties of a square gave an opportunity for students to construct a square using the function tools of DGS (e.g., the *Perpendicular Line* tool). In other words, the teacher used the student's drawing as an opportunity for mathematical investigation and generated a whole-class discussion. Also, inaccurate representations may allow students and teachers to revisit the definitions of a geometrical object and discuss about its counterexamples. Leung and Bolite-Frant (2015) emphasize "task design can intentionally make use of a tool's discrepancy potential to create uncertainties and cognitive conflicts which are conducive to student learning" (p.221).

REFERENCES

Association of Mathematics Teacher Educators (AMTE). (2006). Preparing teachers to use technology to enhance the learning of mathematics: A position of the Association of Mathematics Teacher Educators. Retrieved from http://www.amte.net

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. D. English & D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education, second edition* (pp. 746-783). New York and London, Routledge.

Cayton, C., Hollebrands, K., Okumuş, S., & Boehm, E. (2017). Pivotal teaching moments in technology-intensive secondary geometry classrooms. *Journal of Mathematics Teacher Education*, *20*(1), 75-100.

de Villiers, M. (1994). The role and function of a hierarchical classification of quadrilaterals. *For the Learning of Mathematics*, *14*(1), 11-18.

Dick, T. P. (2008). Keeping the faith, fidelity in technological tools for mathematics education. In G. W. Blume & M. K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Vol. 2. Cases and perspectives* (pp. 333–339). Charlotte, NC: Information Age.

Dick, T. P., & Burrill, G. F. (2016). Design and Implementation Principles for Dynamic Interactive Mathematics Technologies. In M. Niess, S. Driskell, & K. Hollebrands (Eds.), *Handbook of research on transforming mathematics teacher education in the digital age* (pp. 23-51). Hershey, PA: IGI Global.

Dick, T P. & Hollebrands, K. F. (2011). *Focus in High School Mathematics: Technology to Support Reasoning and Sense Making* (pp. xi-xvii). Reston, VA: National Council of Teachers of Mathematics.

Hollebrands, K., & Smith, R. (2009). The impact of dynamic geometry software on secondary students' learning of geometry: Implications from research. In T. Craine, & R. Rubenstein (Eds.), *Understanding geometry for a changing world*. (pp. 221-232) Reston, VA: NCTM.

Laborde, C. (1993). The computer as part of the learning environment: The case of geometry. In C. Keitel & K. Ruthven (Eds.), *Learning from Computers: Mathematics Education and Technology* (pp. 48-67), Springer-Verlag, Berlin.

Laborde, C., Kynigos, C., Hollebrands, K., & Strässer, R. (2006). Teaching and learning geometry with technology. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 275–304). Rotterdam: Sense.

Leung, A., & Bolite-Frant, J. (2015). Designing mathematics tasks: the role of tools. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education: The 22nd ICMI study* (pp. 191– 225)., New ICMI study series Cham: Springer.

Mariotti, M. A. (2013). Introducing students to geometric theorems: How the teacher can exploit the semiotic potential of a DGS. *ZDM*, *45*(3), 441-452.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Panorkou, N., & Pratt, D. (2016). Using Google SketchUp to Develop Students' Experiencesof Dimension in Geometry. *Digital Experiences in Mathematics Education*, *2*(3), 199-227.

Ruthven, K., Hennessy, S., & Deaney, R. (2008). Constructions of dynamic geometry: A study of the interpretative flexibility of educational software in classroom practice. *Computers and Education*, *51*(1), 297-317.

Smith, R. C., Shin, D., & Kim, S. (2016). Prospective and current secondary mathematics teachers' criteria for evaluating mathematical cognitive technologies. *International Journal of Mathematical Education in Science and Technology*, 1-23.

Steketee, S. (2010). *Comparison of Sketchpad and GeoGebra*. Retrieved from http://math.kendallhunt.com/documents/GSP5/Detailed_Comparison_of_Sketchpad_and_GeoGebra.pdf

van Blokland, P., van de Giessen, C., & Tall, D. (2006). *Graphic Calculus Software*. VUsoft.

INCORPORATING LyX AS STANDARD TOOL FOR WRITING MATHEMATICS - EFFECTS ON TEACHING AND LEARNING

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We report on the experience of half a decade of teaching mathematics in the South Pacific region, incorporating LyX as a standard tool for the students in the preparation of their submitted assignments. LyX is a What-You-See-Is-What-You-Mean graphical frontend to LaTeX, the most widely used mathematics markup tool for publishing mathematics documents. We briefly survey the current state of affairs of software used in common practices relevant to the teaching of mathematics, and then concentrate on the advantages offered by LyX. We describe the practicalities of adopting LyX as part and parcel of the course tools and aims,and we then discuss the immediate and longer term effects thereof, and contemplate on the pedagogical efficacy and relevance of LyX as a communication tool.

Keywords: LyX, LaTeX, mathematics writing, mathematics communication.

INTRODUCTION

Communication in writing is at the heart of learning and teaching – a statement so patently obvious that one may find curious the need to state it at all. But indeed, there is a curiosity to be resolved. Students in all taught subjects expect all written communication from the lecturer to be readable, clearly typed, and professionally presented. Similarly, teachers expect students to hand in written assignments which are well-prepared, with attention and care given not just to the content but also to the presentation thereof. In fact, the ability to present ideas and results in a fashion conforming to the subject standards is generally considered part of the competency buildup of a study program. In mathematics students still expect no less than professionally typed material from the lecturer, while the students themselves are allowed to submit handwritten solutions which are far removed from being professionally typed and are often barely readable.

This asymmetry, unique to mathematics teaching, is unfortunate from several perspectives. The written material presented to the students serves as a beacon of mastery. A standard of presentation to appreciate, enjoy, and to strive to achieve. A failure to guide and nurture such a vital communication competency should be viewed as suboptimal design. Further, particularly in mathematics, typing up one's thoughts into a readable, coherent, and beautifully presented document, even if consisting of just a few lines of text, significantly heightens one's understanding of the material due to mathematics' unique feature of being communicated as a mixture of a natural language, typically English, and a formal language, typically set theory formalized to a certain degree of comfort. Students often find making the distinction between the formal and the natural components very difficult, especially when writing their own solutions on a piece of paper. The pen-and-paper's permitting nature, giving the student complete freedom, serves to further blur the line between the formal and the natural aspects in their answers. Lastly, we mention an important psychological effect related to this issue. It is quite disheartening if after solving a difficult problem, all that the student has to show for it is a few sheets of scribbled paper which, even if marked as a full 10/10, cannot be considered anything even remotely close to a document. It simply does not look impressive. If, instead, part and parcel of obtaining full marks is to present the solution as a professionally typed document, the end result becomes truly something to strive for; a readable piece of work, elegant in content and in form. Further work, opinions, and discussion on the importance of communication competencies along these lines can be found in Pugalee, D. K. (2001), Quinn, R. J., & Wilson, M. M. (1997), Baxter, J. A., Woodwar, J., & Olson, D. (2005), and Bicer, A., Capraro, R. M., & Capraro, M. M. (2013).

Various software solutions exist for typesetting mathematics formulas within a document. Perhaps the two most worthy of mentioning are MS Word's equation editor and LaTeX. The former is mentioned due to the widespread use of MS

Word, while the latter is without a shadow of a doubt the publishing standard for professional mathematicians. Most mathematicians probably never even once invoked MS Word's equation editor, and are thus reluctant to prescribe usage of it as part of their teaching. On the other hand, LaTeX is not a word processor, and using it requires quite a bit of preparation with some non-trivial hurdles to surmount. For that reason, lecturers are reluctant to introduce LaTeX early on, resorting to allowing handwritten submissions.

The aim of this paper is to report on the experience of using LyX as an alternative solution while the author taught mathematics courses at the main campus of the University of the South Pacific in Fiji from 2011 to 2016. The plan of the paper is to first give the reader a quick taste of LaTeX, not shying away from its unpleasant features, in order to appreciate what it does and why it is not the case that students can be expected to simply start using it without significant guidance. There follows a glance survey of LyX, emphasizing its key aspects for the purposes of this report. Then the author's experience is recounted, including a brief description of the common practice in sufficient detail to allow mimicry for those interested. The observed effects are reviewed, followed by a discussion and concluding remarks.

THE LaTeX FEAR FACTOR

LaTeX, unlike Word, is a document processor rather than a word processor. It is used to produce professional looking documents by means of a markup language typed in an editor which then compiles to produce the end result. All formatting decisions are made by LaTeX during compilation, leaving the writer to concentrate on the content. There are numerous LaTeX editors and compilers, each with its own set of extras and special features. The examples below were all created using Valletta Venture's TeXpad. Figure 1 is a snapshot of typical work in progress, cycling through editing and compiling phases. Already the brief description above is sufficient to send tremors of anxiety down the spines of many brave souls. There are many books dedicated to imparting the mysteries of LaTeX, e.g., Gratzer, G. (2016) or Mittelbach, F., Goossens, M., Braams, J., Carlisle, D., & Rowley, C. (2004), but often such texts serve to further deter curious newcomers, making one seek the comfort of the familiarity of a word processor rather than all this business with weird looking commands, editors, and compilers.

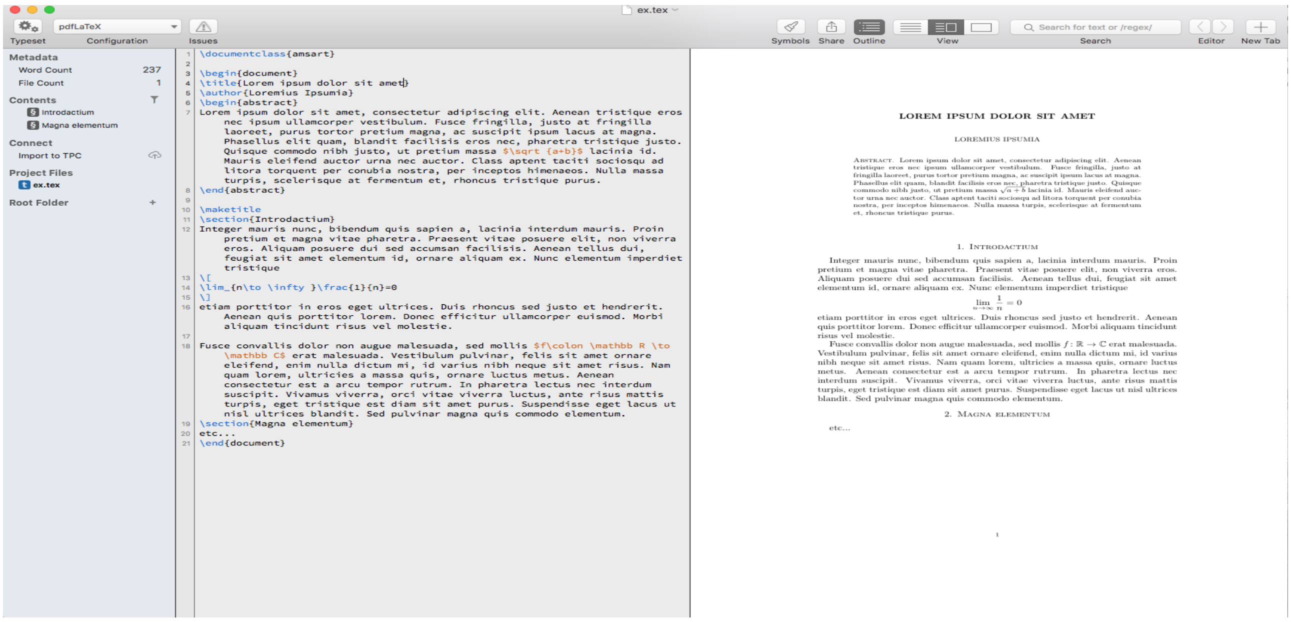

Figure 1: TeXpad session showing LaTeX code on the left and the compiled result on the right.

On the left-hand-side of Figure 1 is the LaTeX code one must type in in order to produce what is displayed on the right-hand-side. A particularly deterring feature of LaTeX is its unforgiving nature to the slightest of mistakes, coupled with its tendency to deliver most cryptic errors when one, unfortunately inevitably, types incorrectly. For instance, if in the

LaTeX code presented in Figure 1, the line reading “\begin{abstract}” were to be replaced by “\begin{abstrat}”, an innocent enough typo, the LaTeX compiler is known to respond quite harshly and unintelligibly, producing output as shown in Figure 2 and errors as shown in Figure 3.

The fact that virtually all professional mathematicians use LaTeX to communicate their research, given the non-

Figure 2: Output due to a simple typo.

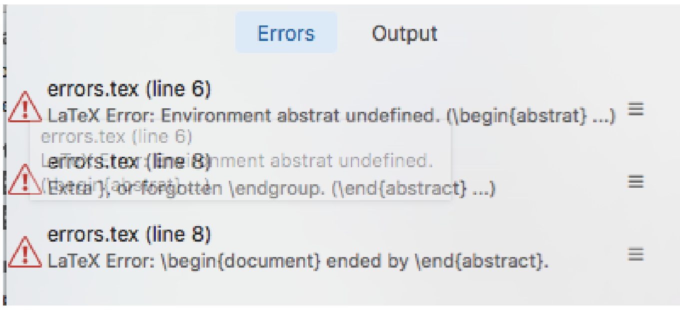

Figure 3: Errors reported due to a simple typo.

triviality of getting used to using it, should indicate that there is sufficient gain to using LaTeX rather than a word processor. Here is not the place to further expand on that point, but it is certainly the place to remark that it should now be obvious that teaching LaTeX, especially early on, is challenging.

A further difficulty that one meets just a few steps down the road to using LaTeX is the inevitable consequence of writing code, namely the need to debug. A significant difference between an experienced LaTeX user and a novice is that that former mastered working techniques that minimize the risk of bugs, while the latter spends much time searching one's code for elusive bugs.

Other than the typographical advantages offered by LaTeX, a pedagogically important feature is the need (lest dreaded errors will be reported) to clearly separate text from mathematics content, as can be seen in Figure 1. The unforgiving nature of LaTeX is here a blessing since only 100% correct code will be accepted, forcing the writer to first very clearly figure out on her own precisely what she means to say. Suffice it to say that LaTeX has much to offer, but it is not easy to get going. A more user-friendly solution is desirable.

THE LyX SOLUTION

LyX is an open source project providing a freely distributed graphical frontend for LaTeX. It must be emphasized that under the hood of a document produced using LyX is LaTeX code, but that code is automatically produced and compiled by LyX. The LyX interface resembles that of a word processor, in fact so much so that a casual observer is likely to mistake it for just that. Figure 4 illustrates that feature in a snapshot of a LyX session, with the LyX interface on the left-hand-side and the output on the right-hand-side. The resemblance to a word processor immediately dispels any anxiety as one automatically feels at home when working with LyX. It is a familiar environment, there are no odd looking commands, no visible compiling, and no unwieldy errors to cope with. This is all true, but just on the surface. What LyX does very well is hide the code and the compilation, providing convenient graphical features to give one the illusion that the whole thing has nothing to do with LaTeX. But the illusion is easily broken by LyX's ability to export any LyX document as a LaTeX document, a feature the uninterested user, and certainly newcomers, can safely disregard. Unbeknown to the user, LyX constantly compiles whatever the user is typing, a feature that pretty much guarantees there will never be a need to debug a LyX document. It is worthwhile noticing that some LaTeX editors resort to a similar technique in order to reduce the risk of bugs.

Still just on the surface, LyX differs from a word processor in its use of environments (a pull-down menu at the top left corner) used to specify different sections or portions of text, and in its highly sophisticated math mode. The latter is LyX's support for typing mathematical symbols, equations, formulas, and so on. It is here that LyX shines when compared to LaTeX. Much thought went into the design of LyX's math mode in order to provide the most novice of users with immediate capabilities.

In short, LyX is a document processor offering all the advantages of LaTeX with little to no disadvantages. It is user-friendly, freely distributed, highly fine-tuned, and constantly developed and improved. It cleverly and efficiently hides all of the mess of LaTeX under the surface, leaving a clean working environment devoid of any scary bits. Such a tool is an optimal choice to be presented to students on their very first day of an introductory class in mathematics.

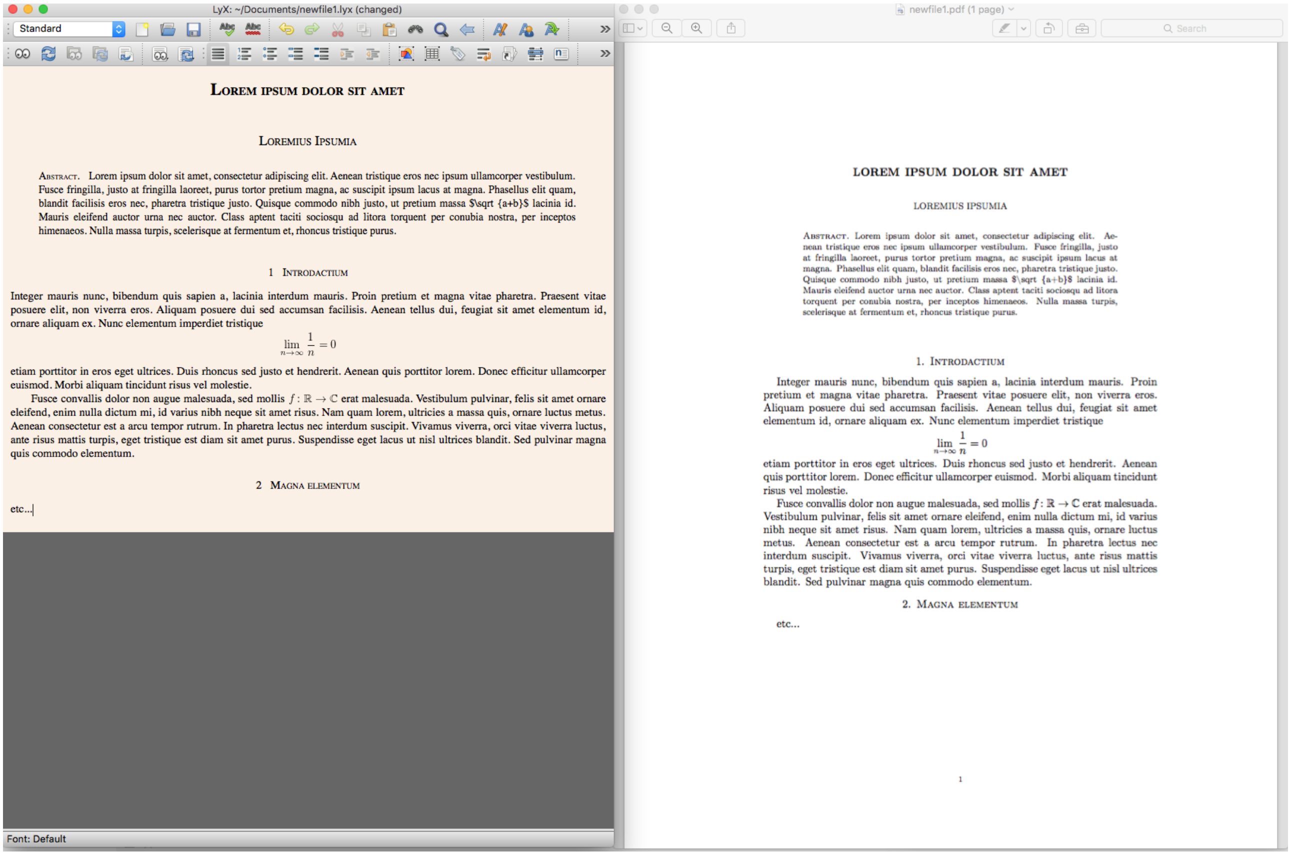

Figure 4: Screen caption of a LyX session showing the LyX environment on the left and the end-result on the right.

HALF A DECADE OF IMPLEMENTATION

I joined the University of the South Pacific as a mathematics lecturer in 2011 and worked there until 2016. During those years, I taught mathematics courses across the undergraduate curriculum, including calculus, linear algebra, abstract algebra, advanced calculus, and discrete mathematics. Some of these courses are proof based (e.g., abstract algebra) and some are of a more calculation based nature (e.g., calculus).

Each course taught, be it a first-year course or a third-year course, would dedicate a one-hour lab session in the first week to teaching the basics of LyX and bringing the students to a sufficient level of mastery to continue using LyX on their own. The visual similarity of LyX to an ordinary text editor was exploited to quickly get the students to produce a simple document. Then the unique features of LyX were discussed, namely the use of environments and LyX's math mode. Experience shows that the environment pull-down menu of LyX is intuitive enough to allow most students a very smooth transition. The more serious obstacle is typesetting mathematics symbols and formulae, due to the need to find the LaTeX commands for the symbols one requires. Here LyX offers much assistance in the form of automatically suggesting symbols and capabilities in a pane that opens up as soon as one engages math mode. Further, the online tool Detexify (made available by Kirelabs at http://detexify.kirelabs.org/classify.html) is a website allowing one to draw any desired symbol and obtain the LaTeX command for it. All in all, after this opening one-hour session over 90% of the students were able to produce a simple looking document, like the one shown above, without any difficulties. Struggling students were typically helped out of their confusion on an individual basis, allowing competency in using LyX to a degree that permits students to immediately start using it to be achieved very early on.

In the first year of the experiment I provided the students with a lengthy hand-in assignment to work on. The assignment consisted of several problems with a preparation time of about six weeks. The instructions were to work on the problems progressively and to use a weekly one-hour lab session to type-up their work as a LyX document. I was then still reluctant to require only a PDF submission, and so declared the lab sessions as highly recommended but not mandatory. When the submission deadline arrived 80% of submissions were printouts of professional looking documents, clearly prepared with LyX. Encouraged by the outcome, in year two I declared that only PDF submissions prepared using LyX will be accepted. The deadline was met largely without any issues, save for a few students who ran into technical problems preventing them from obtaining a PDF. All of these issues were resolved on an individual basis, resulting in a 100% PDF submission of type-set documents.

From year three of the experiment onwards I adopted LyX as a standard tool for students to use in the preparation of all of their mathematics related work. I kept holding a first week induction phase, quickly introducing students to LyX, though this quickly became relevant only for the first-year students. I declared that only submissions of professional looking typeset documents will be accepted for relevant coursework. Consistently, deadlines were met according to the set guidelines with typically 2-3% of students reporting difficulties, all of which were solved on an individual basis. The main source of problems for those students facing difficulties was an inability to export their work as a PDF file, primarily due to installation issues.

To conclude, from the very beginning of the experiment students showed no signs of distress or discomfort with the new technology. Using LyX so naturally builds upon existing word editing competencies shared by virtually all students that a single one-hour lab session is all that was required to bring the students to a level of competency granting them independence in the typesetting of their work. Students were able to immediately start typesetting their hand-in work, requiring very little further support. Technical issues related to installation on students' private computers sometimes led to inability to produce a PDF, a problem usually discovered shortly before a submission deadline. Such problems were typically solved individually by allowing the student to submit the LyX file directly, or suggesting the student re-installs LyX.

EFFECTS ON LEARNING AND TEACHING

Consistently throughout the experiment, feedback from students was very positive, with statements such as "we learned how to produce professional mathematics documents" and "I now know how to produce beautiful worksheets for my own students" appearing often in student evaluations. Retention of LyX capabilities was also very high, and in fact it was often reported to me by other lecturers that they see a significant increase in typeset submissions in their courses too, even though they do not make any efforts to encourage that, indicating that students see the added value of using LyX and choose to do so even when not instructed to. As a by-product, other colleagues' feedback is also very positive since it is much more pleasant to grade a typeset document rather than hard to decipher scribbles on a piece of paper.

Other than these appreciative responses from students and pleasant side-effects for the teachers, positive effects of a pedagogical nature were also observed. The need to enter math mode in LyX in order to type-set a symbol or an equation forces the student to make a clear distinction in her mind between the language and the mathematical content. It becomes much clearer how the surrounding language supports the mathematical content and that the two are truly very different in nature. The use of LaTeX commands, with their alien look, all starting with a backslash, serves to accentuate that difference even more, generally leading to better understanding and better performance, and since LyX immediately converts the commands to the symbols they stand for, the student is not distracted away from the content she is producing. Particularly in the proof-intensive courses (e.g., abstract algebra) a marked improvement was observed in the students' ability to produce correct proofs, primarily since simply following LyX's mode of operation forces one to pay a great deal of attention precisely to those aspects of one's solution which are crucial to a reasonable flow of ideas and presentation in a proof.

In all of the courses taking part of this experiment students were quite satisfied to be working with LyX, clearly appreciative of the relative ease with which they produced professional documents. Many of my students who were themselves high-school teachers were appreciative of that new capability and reported on considerable time reduction in their own preparation of worksheets for their students.

To conclude, with very minimal adaptation and preparation I was able to successfully and efficiently incorporate LyX as a convenient and powerful tool to empower students in their written mathematics communication, resulting in increased competency when dealing with the material, appreciation of the ability to produce elegant documents, and happier colleagues who now know they too, just like in any other taught subject, have an alternative to allowing handwritten submissions out of inertia.

DISCUSSION

The importance of being able to express oneself clearly, elegantly, and with relative ease as a contributing factor to effective learning is probably widely accepted. However, in mathematics teaching, due to significant initial technical hurdles one must overcome before one can use the most prevalent software solution used by experts, the development of mastery of exposition is deferred to later stages of the study program, and often deferred completely out of existence. The negative aspects of this situation include pedagogical issues, such as increased difficulty in distinguishing between the formal and natural use of language in one's solutions, as well as psychological issues, namely a lack of a neatly looking 'finished product' to be proud of and refer to once an assignment is completed, leading to lack of interest in one's solutions past the immediate need to fulfill a course requirement.

A further complication is the disconnect between software solutions used by expert mathematicians and software used by educators. The latter typically use MS Word, perhaps with the aid of its equation editor, while the former exclusively use LaTeX. This disparity leads to a poor exchange of practices and a general avoidance of introduction of any software solution as a standard tool for students. The solution discussed in this report is using LyX, an open-access software providing a graphical frontend to LaTeX. Among professional mathematicians LyX is very seldom used, primarily due to inertia and the fact that the vast majority of journals require LaTeX submissions (though it is crucial to remember that LyX can export any document to a perfectly acceptable LaTeX file). Thus, LyX emerges as an obvious candidate to bridge the chasm between handwritten solutions and LaTeX typeset solutions by providing a sufficiently familiar working environment allowing students to quickly produce a satisfactory first document and become self-sufficient with minimal time investment in the beginning of a course.

Half a decade of teaching mathematics at the University of the South Pacific demonstrates the efficacy of incorporating LyX as a powerful and much appreciated learning aid, contributing to student success and engagement. Particularly in developing countries, the fact that LyX is distributed completely free of charge is of great importance, and since pricy licenses are an issue for any university, the no extra cost involved with adopting LyX is significant everywhere. Moreover, there is already a well-established dedicated online community of LyX users who are generally very eager to offer assistance to newcomers, all in all manifesting LyX as a very robust and friendly solution to a long-standing problem in the teaching of mathematics. The author intends to follow a similar path introducing LyX in all first-year mathematics courses at the University of Portsmouth, England, in September 2017.

REFERENCES

Gratzer, G. (2016). More Math Into LaTeX, *Springer*, xxx-609.

Mittelbach, F., Goossens, M., Braams, J., Carlisle, D., & Rowley, C. (2004). The LaTeX Companion (Second Edition), *Addison Wesley*, xxx-1120.

Pugalee, D. K. (2001). Using communication to develop students' mathematical literacy. *Mathematics Teaching in the Middle School*, 6(5), 296.

Quinn, R. J., & Wilson, M. M. (1997). Writing in the mathematics classroom: Teacher beliefs and practices. *The Clearing House*, 71(1), 14-20.

Baxter, J. A., Woodwar, J., & Olson, D. (2005). Writing in mathematics: an alternative form of communication for academically low-achieving students. *Learning Disabilities Research & Practice*, 20(2), 119-135.

Bicer, A., Capraro, R. M., & Capraro, M. M. (2013). Integrating writing into mathematics classroom to increase students' problem solving skills. *International Online Journal of Educational Sciences*, 5(2), 361-369.

Workshops

PRACTICING WIMS: HANDS-ON TRAINING

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WIMS (Web Interactive Multipurpose Server) is a collaborative, open source e-learning platform hosting online interactive exercises in many different fields such as mathematics, chemistry, physics, biology, French and English among others. The pedagogical specificities of WIMS were shown, such as a bank of exercises readily available, the deeply embedded random feature, a wide variety of exercise types, including formal answers and personalized student monitoring tools. We also presented how to make custom exercises with an interactive editor.

INTRODUCTION

WIMS, Web Interactive Multipurpose Server, described extensively elsewhere in these proceedings (Kobylanski,~2017), is a collaborative, open source e-learning platform hosting online interactive exercises in many different fields such as mathematics, chemistry, physics, biology, French and English among others. A bank of exercises is readily available on each server.

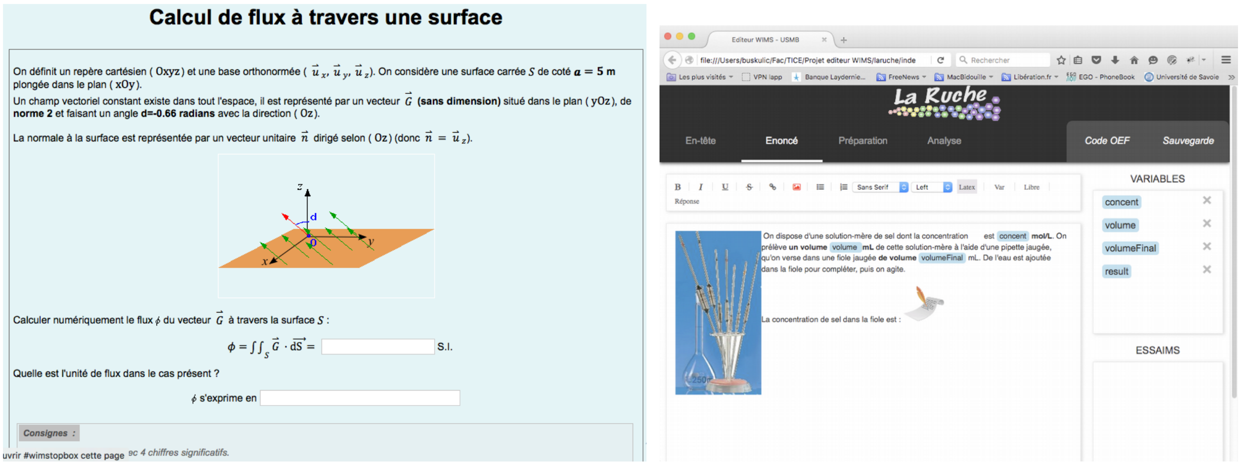

Figure 1: Examples of WIMS interfaces.

WIMS has some unique pedagogical specificities, such as

- An extensive bank of interactive exercises in various fields (around 4000 exercises, mostly in French, but also in English)
- WIMS is open source. Custom exercises produced by the users can be shared with all the WIMS servers in the world
- A deeply embedded random feature, providing the possibility to do an exercise in different ways and to test a large number of students simultaneously
- The possibility to configure a wide variety of exercise types such as multiple choice questions, drag and drop exercises, selecting a graph, drawing interactive curves, exercises requesting a numerical value or a formal answer, and many others.
- Support for many languages
- Support for external packages / software (JSMol, JSXGraph Povray, maxima, Pari/GP, Octave…)

- Support for LaTeX, HTML, Javascript, MathML
- Personalized monitoring tools for filtering student results: success rate, quality of work, results by skill, …

The WIMS platform complements various other platforms and Learning Management Systems such as Moodle or Chamilo.

GOALS OF THE WORKSHOP

The goals of the workshop were to learn how to create a virtual class on the WIMS platform by selecting exercises readily available from the exercise bank, learn how to create custom exercises and show how some of the specific pedagogical aspects of WIMS can be integrated in the exercises.

TOPICS AND ACTIVITIES DURING THE WORKSHOP

After a general presentation of the interface, we presented the following features:

Creation of a class with worksheets and documents

A class is the basic frame in which a teacher puts exercise sheets, students do their exercises, read documents and answer questions. A teacher is able to view the student activity, marks, difficulties and can process the overall results of all the students.

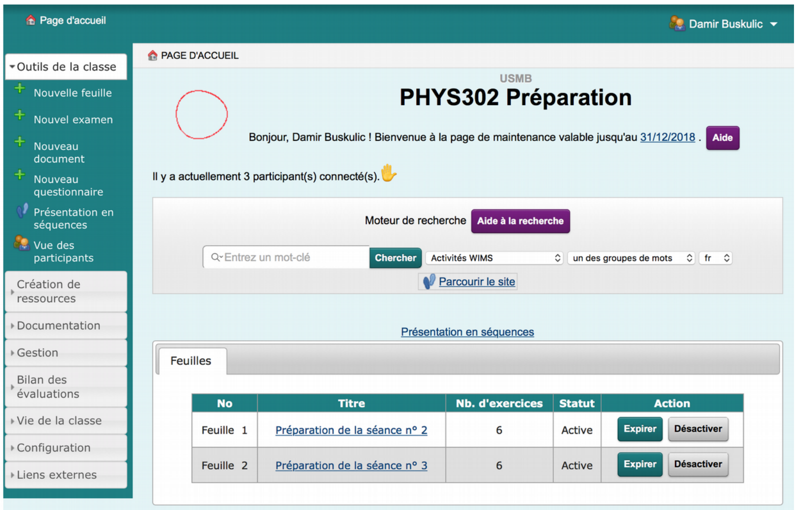

Figure 2: The class preparation interface

Finding an exercise or activity

WIMS integrates a search engine that allows to find exercises and activities by name or field. There is also available a taxonomic classification (Fig. 3).

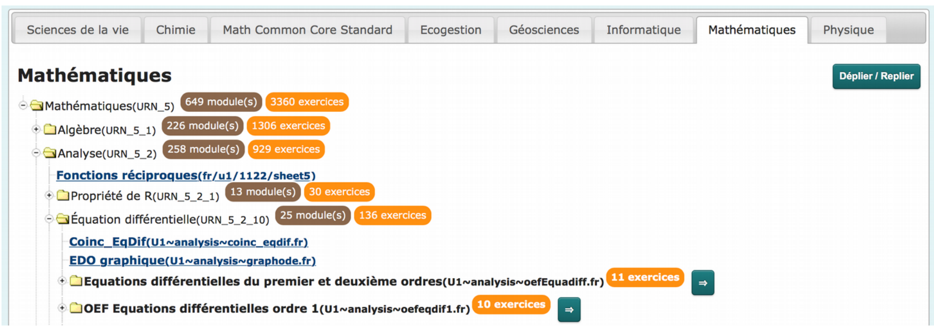

Figure 3: An interface to a taxonomy of mathematical subjects

Integrating existing exercises in the class.

Creating an exercise or a pedagogical sequence is time consuming. The WIMS embedded mechanism for sharing the teacher's creations, exercises and activities allows for a reduction of the exercise development time. The simplest way to create an exercise sheet is to gather the exercises among the existing ones in the available database. This is the way used by newcomers to quickly understand all the potentialities of the WIMS platform.

Creation of an exercise using an interactive editor in WIMS.

The creation of an exercise may be done in three different ways. The first is a set of simple exercise models which can be modified according to several variables. The second way is by means of programming in the Open Exercise Format (OEF) language. This allows for the building of almost any kind of exercise. When a user wants complete control over the appearance and functionalities of an exercise, he can use the Modtool mode which gives him complete control over the html code.

However, except for the prepared models, there is still a need for programing knowledge, which all teachers are not able or willing to learn. The WIMS developers are in the last stages of the development of an interactive editor which should greatly simplify the programming of OEF exercises. The user will be able to use a web word editor as well as a simplified graphical programing interface. This is shown in figure 4.

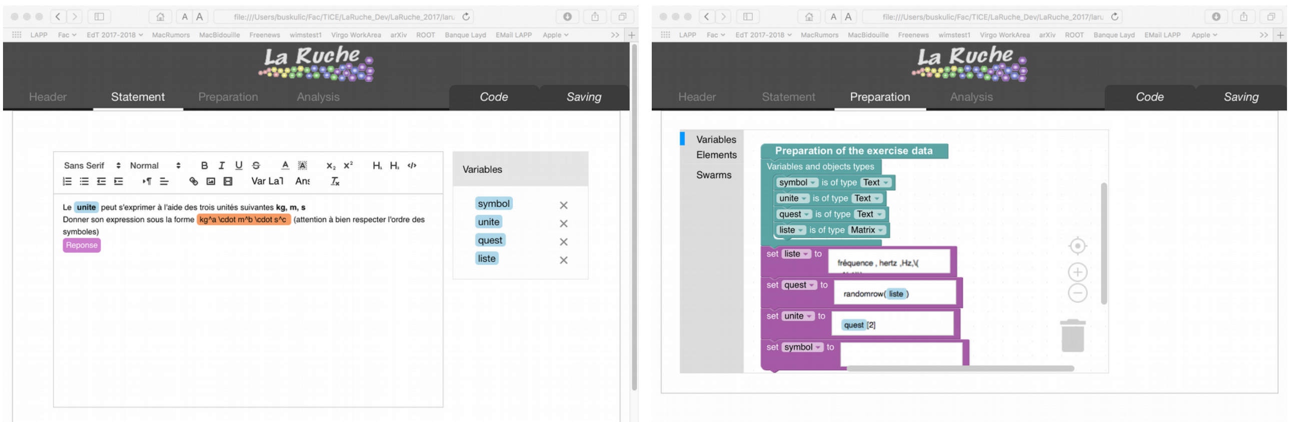

Figure 4: The interface of the exercise editor

RESOURCES

Server at the Université Savoie Mont Blanc: https://wims.univ-savoie.fr/

Server at the Université Paris Sud: https://wims.auto.u-psud.fr/wims/

CNRS Server: https://wims.math.cnrs.fr/wims/

Tutorial for the creation of classes: https://wims.di.u-psud.fr/wims/wims.cgi?module=help/teacher/docbeginner.fr

Site of the WIMSEDU NGO: http://wimsedu.info

REFERENCES

Kobylansky, M. (2017). WIMS Www Interactive Multipurpose Server an interactive exercise software that has 20 years and still is at the top, *Communication at the ICTMT13 conference, Lyon, France.*

Chapter 8

POSTERS

AUGMENTED LOG: USING AR TECHNOLOGY TO CONSTRUCT LEARNING ABOUT LOGARITHMS AND EXPONENTIALS

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This poster describes how an augmented reality learning experience effectively motivated and supported students in organising mathematical content related to a usually painful subject in high school mathematics. The outcomes of this experience, monitored through classroom observation, were aligned with the theoretical framework of AR and proved that the use of technology reinforced students' understanding and fostered long-term memory about the topic.

Keywords: Augmented Reality, high school math, constructivist learning

INTRODUCTION AND THEORETICAL FRAMEWORK

Augmented Reality is rather a new entry within educational technologies and has huge potential power as a learning tool that has yet to be explored.

This project had a double goal: from a research point of view the aim was evaluating whether "as a cognitive tool or pedagogical approach, AR aligns with situated and constructivist learning theory as it positions the learner within a real-world physical and social context, while guiding, scaffolding and facilitating participatory and metacognitive learning processes such as authentic inquiry, active observation, peer coaching, reciprocal teaching and legitimate peripheral participation with multiple modes of representation", as suggested by Dunleavy & Dede (2013).

From a pedagogical point of view the purpose was to investigate the educational dimension of a vision-based AR technology (that is to say the triggering of a superimposed computer generated layer pointing a GPS-enabled device to a precise spot), enhanced by the fact that students themselves have been authors of the digital content, with the goal of evaluating how technology could foster the development of significant mathematical literacy and assess how students could use their day to day technological skills to support their mathematical learning.

METHOD AND ACTIVITY

The activity, which involved a class group of 28 16 y.o. students attending the 3rd year of high school (Liceo Scientifico), was divided into three phases: introduction and synthesis took place at school, while the actual production of the digital media (trigger image and video overlay) was assigned as homework.

The results were gathered by the teacher through the assessment of students' homework and the observation of the resulting classroom discussion.

The three phases were organised as follows:

1. At school: after completing the module on logarithm and exponential functions, the teacher selected and assigned to each student a specific segment of the subject, ranging from practical topics as the properties of logarithm and the techniques for solving exponential equations to historical themes such as the number *e* and the legend of the chessboard.

The teacher then had students download the *Aurasma* AR free app for iOS- and Android-based mobile devices, log in with the class account – augmentedlog – which had been created for that purpose and point it to the image in figure 1, which triggered an instruction video for the task.

Figure 1

2. At home: each student produced a 2/3 minutes video on the given topic with a related trigger image and, using the class account, uploaded and connected them through the Aurasma Studio desktop site https://studio.aurasma.com/landing (figure 2).

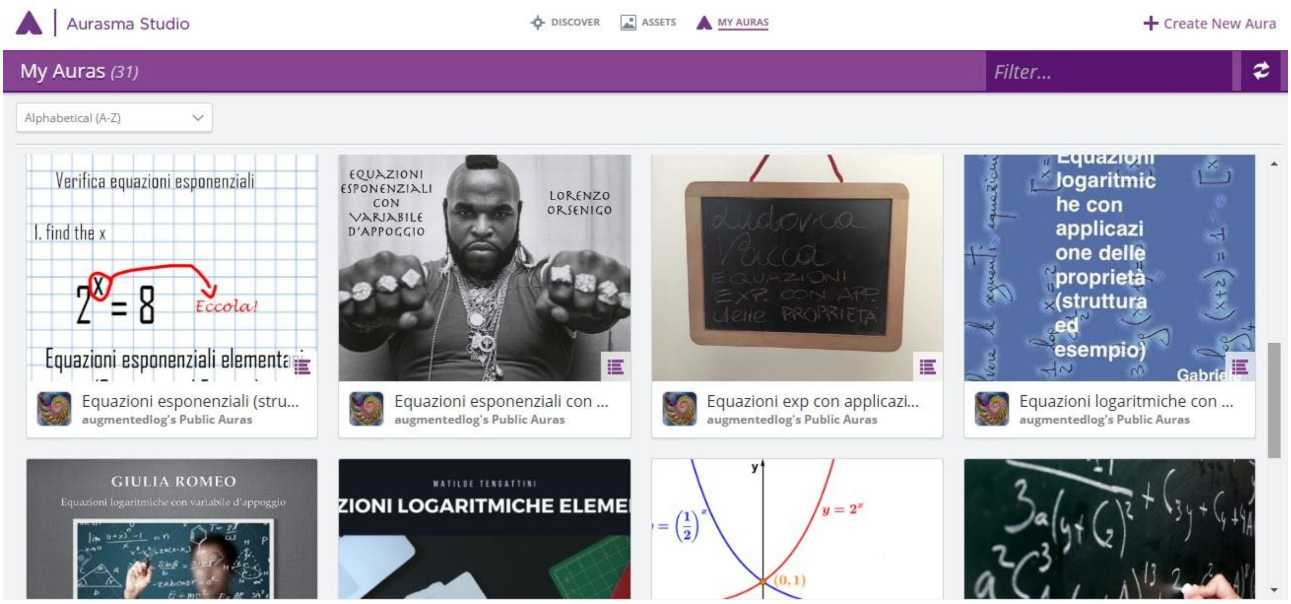

Figure 2

Once the production of the videos has been completed (students had two weeks to finish the task), the teacher collected all the trigger images in a poster (figure 3) and had it printed.

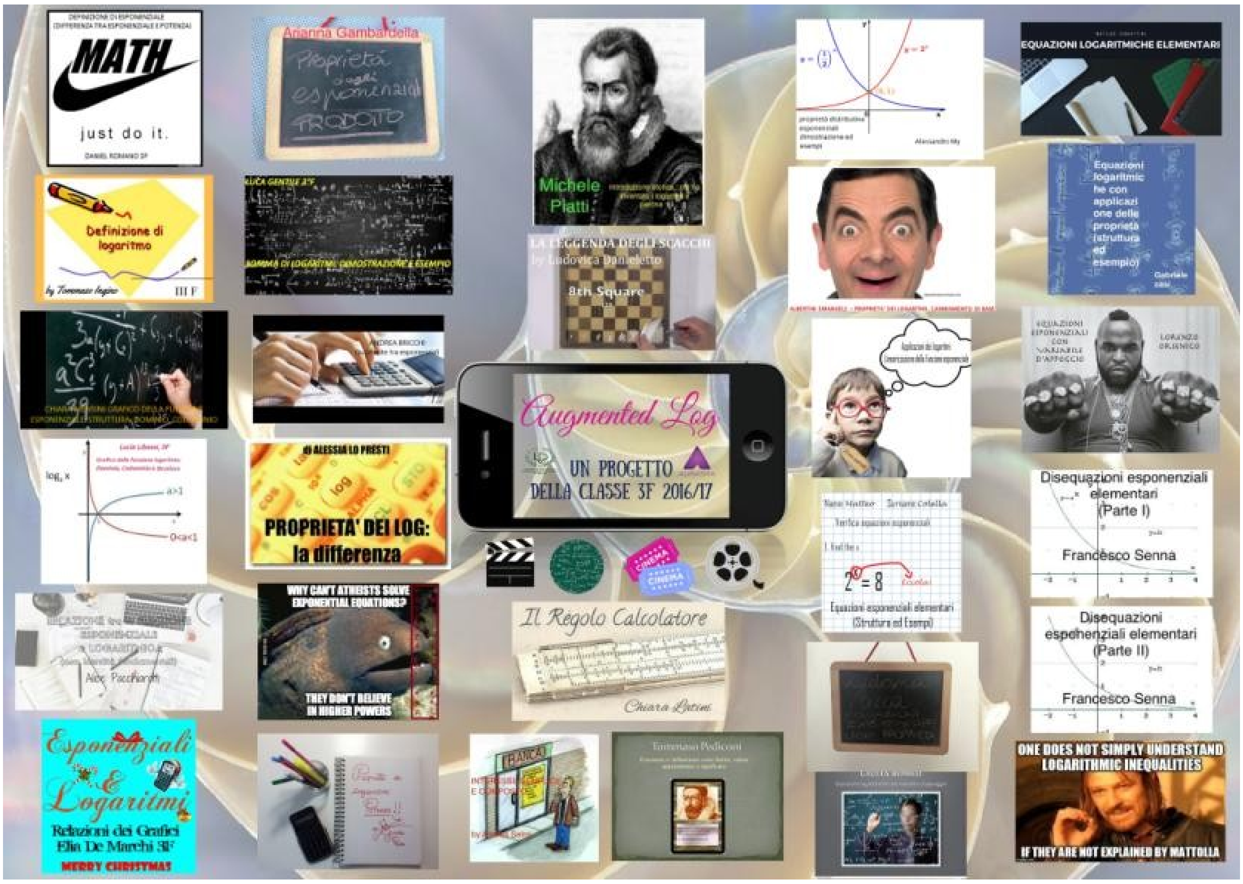

Figure 3

3. At school: the poster was brought to school and hung in the classroom so that students could view and share their productions pointing their devices to the trigger images (figure 4) and discuss the results.

Figure 4

To sum up, the AR experience was designed as an *interactive storytelling* learning activity which each student contributed to by creating a specific piece of the story.

The role of the teacher was to identify and assign the single pieces of the story to each student, organise the virtual Aurasma Studio learning environment through which students could connect their digital content and eventually collect and assess the final products as well as build a synthesised whole in order to give back the complete view of the topic.

RESULTS AND CONCLUSION

The skills and attention showed by the students in creating their products and commenting those of the classmates proved that working with AR has been a powerful strategy: this technology captivates students more than other digital means, increasing their yearning to participate and fixing the activity and its mathematical content in their memories for good.

The described jigsaw design approach added value to this experience, making learning "a co-constructed, participatory process" (Dunleavy & Dede) and encouraging students to share their products.

The "interplay between competition and collaboration" (Dunleavy & Dede) turned this AR activity into a deep and successful learning experience, combining traditional and non-traditional settings and interactions, in which the appreciation of students' technological skills acted as a strong motivational drive and empowered the students' willingness to focus and discuss mathematical concepts.

References

Bower, M. Howe, C. McCredie, N. Robinson A. Grover, D. (2014) *Augmented Reality in education – cases, places and potentials*, Educational Media International, 51:1, 1-15.

Dunleavy, M., and Dede, C. (2013). *Augmented reality teaching and learning*. In J.M. Spector, M.D Merrill, J. Elen, & M.J. Bishop (Eds.), The Handbook of Research on Educational Communications and Technology (4th ed.), pp. 735-745. New York: Springer.

Petrucco, C., Agostini. D., *Augmented Reality Learning: Pedagogical aspects and Technologies for a future Methodological Framework*. University of Padua Department of Philosophy, Sociology, Education and Applied Psychology.

SUPPORTING PROBLEM SOLVING THROUGH HEURISTIC TREES IN AN INTELLIGENT TUTORING SYSTEM

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In this article we address how to teach the use of heuristics in problem solving. We present an approach using a new support model within an online tutoring system. The outcomes of a pilot study conducted in the Netherlands are discussed. On the basis of these outcomes we make suggestions to improve the model and its implementation.

Keywords: problem solving, heuristics, heuristic tree, digital learning environment

THEORETICAL BACKGROUND

To solve a mathematical problem, one needs to combine mathematical skills and activities that have already been mastered. So one needs to make strategic decisions on a cognitive level that transcends the procedural. A way to guide a student in such strategic decisions is by providing heuristics. A heuristic (Pólya, 1945) is a general strategy to address a problem, e.g., *investigate special cases*. Heuristics are a form of support on the high end of the cognitive spectrum. On the other end are concrete hints that reveal steps towards a solution.

In this research we study how the delivery of heuristics and hints to learners should be structured. Schoenfeld claims (1985, p. 73): “many heuristic labels subsume half a dozen strategies or more. Each of these more precisely defined strategies needs to be fully explicated before it can be used reliably by students”.

So how should the support using heuristics be structured? The phases suggested by Pólya (1945) and elaborated by Schoenfeld (1985) provide structure. Additionaly, several studies suggest a role for fading in various ways, in particular using Intelligent Tutoring Systems. Renkl et al. (2004) begin instruction with worked examples of multi-step problems. In each subsequent problem they remove more solution steps from the example and end up with problems to be solved unguided. This strategy proves to be effective in their experiments. Bokhove and Drijvers (2012) report that even though gradually fading the available feedback on a task causes the performance on that task to gradually decrease as well, the overall effect of the course seems to increase. Roll, Baker, Aleven and Koedinger (2014) discuss an intelligent tutoring system that influences help-seeking patterns. They find that “overusing help is associated with lower learning gains” and “on steps for which students lack basic knowledge, failed attempts are more productive than seeking help”.

SUPPORT FOR PROBLEM SOLVING THROUGH HEURISTIC TREES

For this study we designed a series of problem solving tasks in an online tutoring system (the Digital Mathematics Environment, recently renamed Numworx[1]). All the problems needed Pythagoras’ Theorem in some way.

[1] http://www.numworx.nl/en/

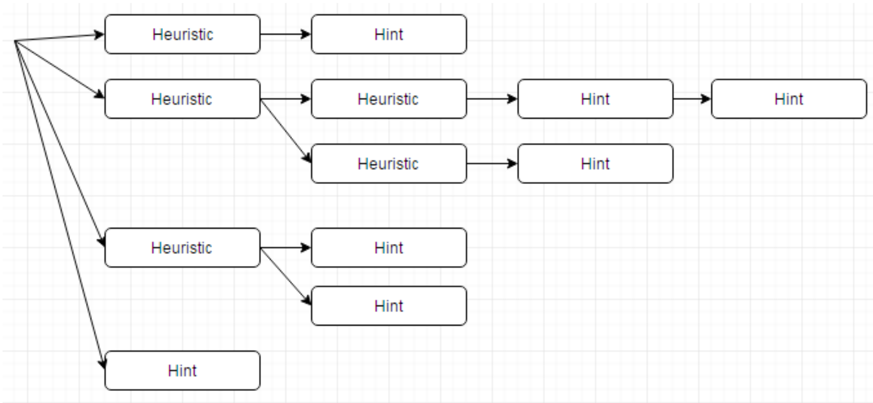

Figure 1: the structure of a heuristic tree

Each problem comes with a *heuristic tree* (see Figure 1). Along each branch a heuristic is node-by-node translated into concrete hints. Each branch represents either a phase or stepping stone in the problem solving process. The learner can choose to access the next node along a branch, but, to avoid overuse, is discouraged to do so by forfeiting a point for each step. This way the fading effect is in the hands of the learner. The goal is a self-regulated transition from procedural thinking to conceptual thinking about the problems involving Pythagoras theorem.

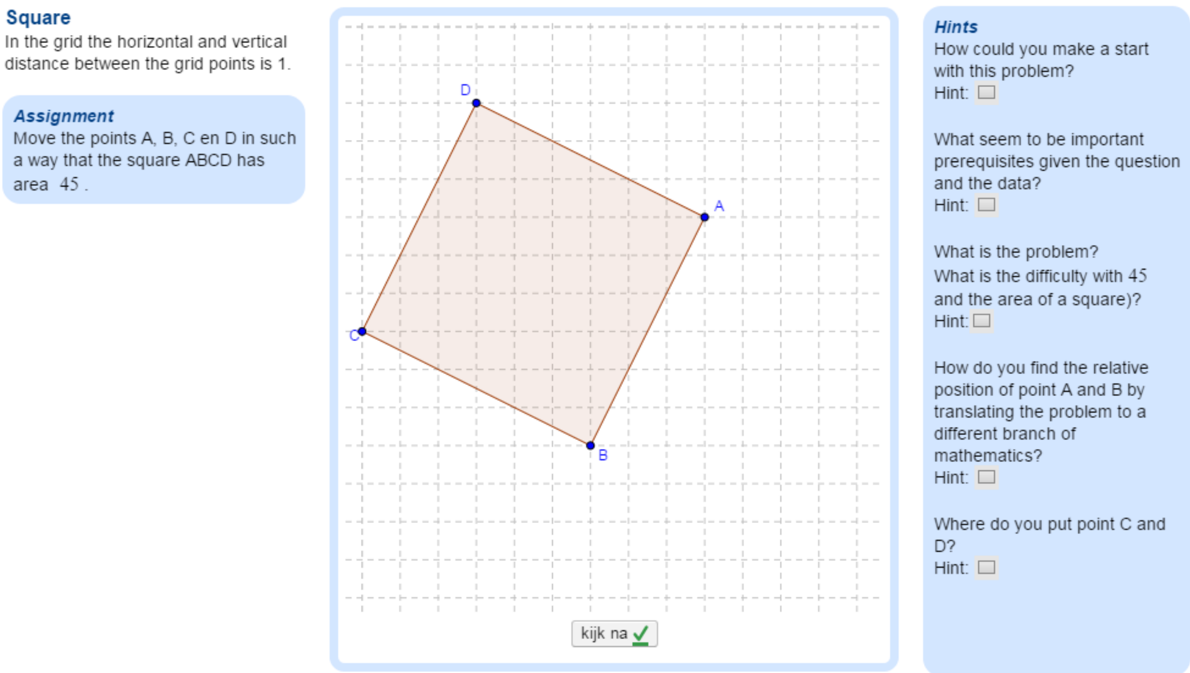

Figure 2: screen shot of DME's problem course

Our hypothesis is that learners' progress along branches gradually decreases as they learn to employ the strategies indicated by the heuristics without further explication. The research question is whether this self-regulated top down approach of teaching heuristics in a digital learning environment improves students' ability to apply heuristics in problem solving and improves their problem solving results.

DESIGN PRINCIPLES FOR HEURISTIC TREES

We would like to design a heuristic tree structure that both supports learners in problem solving and teaches them to use heuristics. Navigating the heuristic tree should be intuitive and logical for the learners. To this purpose we used the following design principles:

1. The structure of the tree should represent the logical order of reasoning within a solution model.
2. The various branches should also be ordered following the various stages of problem solving. In general: orientation, planning and acting, reflection.
3. The structure of the tree should also match with the intuitive approach of the problem taken by learners.
4. The order along a branch should be *from* heuristics *to* more concrete hints, thereby explaining the use of the heuristic.
5. The help offered in different branches should be independent stepping stones, in the sense that for the help offered in one branch no information in any of the other branches should be needed.
6. Each click should not give more help than asked for.
7. The formulation before the click should not yet give away the heuristic or hint, but give an indication of what can be obtained.

It is a challenge to simultaneously satisfy principles 1,2 and 3, because they not always agree. Principles 5, 6 and 7 are meant to ensure that learners do not receive more help than desired. Designing a heuristic tree that satisfies all these principles is a challenge.

THE ROLE OF HEURISTIC TREES

The ideal heuristic tree should offer help that is as well suited to the learner as the help offered by a real life teacher. The student must self-diagnose what help is needed, whereas in traditional classroom situations it is the teacher who makes that diagnosis

The concept of the heuristic tree was conceived for the implementation of heuristic problem solving training in digital learning environments. A defining characteristic of problems is that students working on them get stuck. Providing help during a problem solving session with a big group of students can therefore be very demanding for a teacher. The digital environment can provide relief and the possibility to monitor the students' progress in the use of heuristics. In our implementation the teacher can track which heuristics and hints have been used by individual learners.

This suggests that designing a heuristic tree for a problem is useful preparation for a teacher who wants to use the problem even in a classroom without a digital learning environment. In two recent workshops teachers were given the task of designing a heuristic tree for a given problem. It sparked engaged didactical discussions on how to support the students in their problem solving. The heuristic tree provides a structure for the teachers' thoughts and discussions and highlights both the phasing of problem solving and the tension between hints and heuristics.

PILOT STUDY OUTCOMES AND OUTLOOK

Pilot studies took place in one grade 8 and one grade 11 class at a secondary school in the Netherlands. The students had about 45 minutes to work on the problems. They first received a short instruction in how to navigate the Digital Mathematics Environment and the problem solving tutor.

The general conclusion is that, in focusing on the structure of heuristic trees, this first version did not pay enough attention to important factors of problem solving as discussed in, for example, Schoenfeld (1985): control (self-regulation) and belief (motivation). These aspects will have to be implemented in version 2.0. Before conclusions can be drawn regarding the hypothesis, these issues and the problems with navigation the heuristic tree and help abuse will have to be addressed.

Self-regulation. Learners either used no hints or used them all at once. They either wanted to solve the problem without help or search the whole tree for the golden hint. This fits in with two main forms of help abuse as described by Aleven et al. (2006): clicking through the hints, and help avoidance. They suggest that the learner should actively be taught how to use (digital) assistance. To this purpose they designed a "help-seeking tutor", but they also presented the learner with a video explaining what they consider the ideal way to seek help.

Motivation. Many learners struggled with the difficulty of the problems. This caused loss in motivation for some. In version 2.0 the problems will have a wider range of levels, beginning with easier ones and building up from there. Different avenues within problems or a choice of problems should be offered.

Another issue: navigation. Learners find it hard to navigate through the heuristic tree. They do not realize that it is structured on the phases of the solving process or that the heuristics precede the hints. Learning to use it properly should be part of the lesson, as well as learning about heuristics. 45 minutes was not long enough: one would probably need a short series of lessons. The phases should perhaps be based less on Pòlya's phases or the steps of the solution and more on the intuitive approaches of students; on the questions that come to them naturally while working on the problem.

The next step in this research project will be to improve the online course on the points mentioned above. This will be followed by a larger scaled field test over a longer period of time, hopefully giving us data to draw conclusions on our main hypothesis concerning the use of heuristic trees.

REFERENCES

Aleven, V., McLaren, B., Roll, I. & Koedinger, K. R. (2006). Toward Meta-cognitive Tutoring: A Model of Help-Seeking with a Cognitive Tutor. *International Journal of Artificial Intelligence in Education*, *16*, 101-130.

Bokhove, C. & Drijvers, P. (2012). Effects of feedback in an online algebra intervention. *Technology, Knowledge and Learning, 17*(1-2), 43-59.

Pólya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.

Renkl, A., Atkinson, R. K. & Große, C. S. (2004). How fading worked solution steps works – a cognitive load perspective. *Instructional Science*, *32*(1), 59-82.

Roll, I., Baker, R. S. J. D., Aleven, V., Koedinger, K. R. (2014). On the benefits of seeking (and avoiding) help in online problem solving environment. *Journal of the Learning Sciences, 23*(4), 537-560.

Schoenfeld, A. H. (1985). *Mathematical Problem Solving*. Orlando, FLA: Academic.

TECHNOLOGY AS A RESOURCE TO PROMOTE INTERDISCIPLINARITY IN PRIMARY SCHOOLS

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This poster aims to show how technology may be used as a resource to promote interdisciplinarity, namely using mobile technologies to engage children to learn mathematics, applied to the sound subject, according to1st grades of primary school syllabus. A preliminary study shows that the proposed tasks resultant from technology resources are efficient to catch the attention of the students and can engage them to learn mathematics and science.

Keywords: Technology, interdisciplinarity, hands-on, mobile technologies, primary school.

INTRODUCTION, LITERATURE REVIEW AND THE PRELIMINARY STUDY

The great lack of professionals in the STEM areas must be countered with an early intervention at the level of the early years of schooling, (DeJarnette, 2012; Rocard et al, 2007). The incorporation of hands-on experimental activities into the classroom, leads to significant improvements in performance and produce positive attitudes towards science (Mody, 2015; Johnston, 2005).

Kim e Bolger (2016) sustain the creation of a curriculum that integrates Mathematics, Science and Technology, being crucial to involve teachers into interdisciplinarity lessons adequate to this approach. Kermani e Aldemir (2015) defend the integration of Mathematics, Science and Technology in the first years of school, through teachers' professional development, as well as the creation of well-designed materials to implement hands-on experimental activities.

Technologies in primary education can promote children's attention, socialization, development of language and learning (Gimbert, & Cristol, 2004). Technology leads to a positive impact on student's motivation and meaningful learning, provides hands-on learning oportunities and can integrate school subjects like mathematics (Costley, 2014).

This study is part of a bigger pedagogical intervention project, in first grades of elementary schools, aiming to introduce cross-cutting methodologies, focused on learning and teaching mathematics, science and technology, within a cluster of schools in Portugal. This poster aims to show how technology may be used as a resource to promote interdisciplinarity, namely using mobile technologies to engage primary school children to learn mathematics, applied to the sound subject, according to primary school syllabus. In order to achieve this purpose, a team of university teachers, in the areas of electrical engineer and mathematics, designed sound artefacts to explore mathematical tasks with technology.

A preliminary study occurred with 3rd and 4th grade students of local primary schools who worked the sound with technology. With a design research methodology, we intend to present how children engaged on the proposed tasks. At the classroom, students were introduced to sound contents and performed hands-on activities exploring the day to day sound and how to measure it, with technology. After this presentation, children organized in groups with a tablet or mobile phone per group, were invited to play a game, called "SonicPaper". First, they installed on their tablets/smartphones, the Sound Meter application that allows sound intensity measurements and the

QR code reading application. To perform the game, questions to be answered and clues, together with the campus map, allows participants to find the key points, previously defined, where they had to register the sound intensity. Key points had signs alluding to the sound and QR codes giving answers to some questions and clues to the next location.

After finishing the game, children return to the classroom, to find out if their answers are correct and to present the registered sound measurements. Organization and processing of data of the measurement results was performed, in order to promote interdisciplinarity with mathematics.

FINAL CONSIDERATIONS AND FUTURE WORK

Data analysis, from participant observation and semi-structured interviews, lead to the conclusion that the proposed technology resources are efficient to catch the attention of the participants and can engage students to learn mathematics and science, according to the school syllabus. In particular, it permits to work school subjects like “space orientation” and “organization and processing of data”.

In the course of the hands-on activities and the SonicPaper game, children were very participatory, showing a great interest in the tasks performed. Because this strategy promotes students’ motivation and attention, we propose to use these resources in the context of teachers’ professional development (Costa & Domingos, 2017), to adapt them to be implemented at primary school.

REFERENCES

Costa, M. C.; & Domingos, A. (2017). *Innovating teachers’ practices: potentiate the teaching of mathematics through experimental activities*. Dublin City University: CERME 10.

Costley, K. C. (2014). The Positive Effects of Technology on Teaching and Student Learning. *Online Submission*.

DeJarnette, N. K. (2012). America's children: Providing early exposure to STEM (science, technology, engineering, and math) initiatives. *Education*, *133 (1),* 77-85

Gimbert, B., & Cristol, D. (2004). Teaching curriculum with technology: enhancing children's technological competence during early childhood. *Early Childhood Education Journal, 31*(3), 207-216.

Johnston, J. (2005). *Early explorations in science: Exploring primary science and technology*. 2nd Edition. England: Open University Press

Kermani, H., & Aldemir, J. (2015). Preparing children for success: Integrating science, math, and technology in early childhood classroom. *Early Child Development and Care, 185*(9), 1504-1527. doi:10.1080/03004430.2015.1007371

Mody, C. C. M. (2015). Scientific practice and science education. *Science Education, 99*(6), 1026-1032. doi:10.1002/sce.21190

Rocard, M., Csermely, P., Jorde, D., Lenzen, D., Walberg-Henriksson, H., & Hemmo, V. (2007). *Science education now: A renewed pedagogy for the future of Europe*. Bruxelas: Comissão Europeia.

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DESIGNING TASKS THAT FOSTER MATHEMATICALLY BASED EXPLANATIONS IN A DYNAMIC SOFTWARE ENVIRONMENT

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This poster introduces the main ideas behind a study conducted during the spring 2017. The aim was to investigate how different formulations of tasks, where students are expected to provide mathematically based explanations, might influence their responses. Preliminary results from the first stage in the analysis process indicate that there are some interesting differences due to small differences in task formulation.

Keywords: dynamic mathematics software, task design, mathematically based explanation

BACKGROUND

The increased availability of different kinds of technology in mathematics classrooms offers new possibilities, but it requires change in teaching and learning practice. For example there is a need for different kinds of task to utilize the affordances provided by new technology (Hegedus et al., 2017). Recently, the issue of designing tasks suitable in the digital mathematics classroom had an entire book devoted to it: *Digital Technologies in Designing Mathematics Education Tasks* edited by Leung and Baccaglini-Frank (2017). The literature suggests various task design principles to promote mathematical reasoning in a Dynamic Mathematics Software (DMS) environment, the particular technology used in this study.

This study builds on our previous work on developing new types of task environment to foster students' mathematical reasoning (Brunström & Fahlgren, 2015; Fahlgren & Brunström, 2014). One result from a design-based research project, conducted in a DMS environment by the authors of this paper in collaboration with four upper-secondary school teachers, showed that students' explanations tended to be superficial and more descriptive than explanatory. These results are in line with results from other studies showing that there is a risk that students do not reflect on the mathematics involved when using DMS to explore and conjecture (e.g. Drijvers, 2003; Healy & Hoyles, 1999; Joubert, 2013).

It is important for task designer to be aware that small differences in the formulation of tasks might have significant impact on students' responses (Sierpinska, 2004). We found that the wording is crucial in the formulation of questions where students are asked for explanations (Brunström & Fahlgren, 2015). So far, however, there are few studies that have investigated how small changes in wording might influence students' explanatory responses in a DMS environment.

This study compares two different ways of formulating explanation tasks in a DMS environment. The explanation tasks are embedded in a task sequence with the aim of developing students' awareness of some of the connections between the standard form of quadratic function $f(x) = ax^2 + bx + c$ and the corresponding graphical representation and quadratic equation. In total, the task sequence includes three explanation tasks formulated in the following two versions: (A) "Explain why…" and (B) "Give a mathematical explanation why…". The aim with the study is to investigate if this small difference in task formulation has any impact on student responses.

THE STUDY

The study involves seven 10th grade upper-secondary classes in which half of each class received the A-version and the other half received the B-version of the task sequence. The students worked in pairs with *one* computer per pair. The purpose of this is that the computer screen should provide a shared object for discussions between students (Brunström & Fahlgren, 2015; Paiva, Amado, & Carreira, 2015). The empirical data consists of the written responses from 229 students; 121 version A, and 108 version B.

PRELIMINARY RESULTS

So far, a preliminary analysis of the first explanation task, with focus on parameter *c*, has been made. In this task, the students are asked: (1) to investigate and find out how the value of *c* affects the graph; then (2) how the value of *c* can be found in the coordinate system; and finally (3) to (A) Explain why/(B) Give a mathematical explanation why the value of *c* can be found in this way. When answering the second subtask, almost all students described that the value of *c* can be found where the graph intersects the *y*-axis. Our focus in the analysis was on student responses on the third subtask. The tables below indicate differences in student responses, both in terms of types of explanation (Table 1) and forms of representation (Table 2).

TYPE OF EXPLANATION	**VERSION A**	**VERSION B**
Correct and complete, i.e. explains that $c = f(0)$.	4 %	13 %
Refers to the *b*-value in the straight line equation $y = mx + b$	44 %	57 %
Describes that *c* can be found where the graph intersects the *y*-axis (i.e. repeats the answer to the previous subtask)	21 %	13 %
Provides more than one explanation.	16 %	29 %

Table 1. Some differences in student responses concerning types of explanation

The preliminary results indicate that the B-version that includes the words "mathematical explanation" prompts student responses based on mathematical properties and relations to a higher degree than the A-version does. Even if not many students gave a correct and complete explanation this was more frequent among students responding to the B-version. We also found it interesting that these students more often referred to their previous knowledge concerning the straight line equation, and also gave more than one explanation to a greater extent.

FORM OF	**VERSION**	**VERSION**

REPRESENTATION	A	B
Verbal only	75 %	42 %
Algebraic Symbols only	1 %	16 %
Verbal and Algebraic Symbols	4 %	17 %
Verbal with Elements of Algebraic Symbols	11 %	18 %
No answer	9 %	7 %

Table 2. Some differences in student responses concerning types of representation

The preliminary results also indicate that the task formulation including "mathematical" prompts more students to use algebraic symbols in their explanations, and fewer to use solely verbal explanations. In student responses classified as "Verbal with Elements of Algebraic Symbols" formulas or other algebraic symbols are just mentioned without being used. Hence, the categories "Algebraic Symbols only" and "Verbal and Algebraic Symbols" are the only categories where students really use algebraic symbols (even if not always in an appropriate way). When merging these two categories the tendency becomes clear, 33 % of the students answering the B-version used algebraic symbols while the corresponding value for those responding to the A-version was 5 %.

The next step in the analysis process, is to develop a more general framework to use in the analysis of all three explanation tasks.

REFERENCES

Brunström, M., & Fahlgren, M. (2015). Designing prediction tasks in a mathematics software environment. *International Journal for Technology in Mathematics Education, 22*(1), 3-18.

Drijvers, P. (2003). *Learning algebra in a computer algebra environment: Design research on the understanding of the concept of parameter*. Dissertation, Utrecht: Freudenthal Institute, Utrecht University.

Fahlgren, M., & Brunström, M. (2014). A Model for Task Design with Focus on Exploration, Explanation, and Generalization in a Dynamic Geometry Environment. *Technology, Knowledge and Learning, 19*(3), 287-315.

Healy, L., & Hoyles, C. (1999). Visual and symbolic reasoning in mathematics: making connections with computers? *Mathematical Thinking and Learning, 1*(1), 59-84.

Hegedus, S., Laborde, C., Brady, C., Dalton, S., Siller, H.-S., Tabach, M., . . . Moreno-Armella, L. (2017). Uses of Technology in Upper Secondary Mathematics Education *ICME-13 Topical Surveys* (pp. 1-36). New York, NY: Springer.

Joubert, M. (2013). Using computers in classroom mathematical tasks: revisiting theory to develop recommendations for the design of tasks. In C. Margolinas (Ed.), *Task design in Mathematics Education. Proceedings of ICMI Study 22* (pp. 69-77). Oxford, United Kingdom: University of Oxford.

Leung, A., Baccaglini-Frank, A., & (Eds.). (2017). *Digital Technologies in Designing Mathematics Education Tasks Potential and Pitfalls* (Vol. 8). New York, NY: Springer.

Paiva, J., Amado, N., & Carreira, S. (2015). The role of peer and computer feedback in students learning. In N. Amado & S. Carreira (Eds.), *Proceedings of the 12th International Conference on Technology in Mathematics Teaching* (pp. 491-499). Faro, Portugal: University of Algarve.

Sierpinska, A. (2004). Research in mathematics education through a keyhole: Task problematization. *For the learning of mathematics, 24*(2), 7-15.

BOUNDARY OBJECTS IN INTERDISCIPLINARY RESEARCH ON MULTIMODAL ALGEBRA LEARNING

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We present a set of boundary objects we indentified in an ongoing interdisciplinary research and development project, where mathematics didacticians collaborate with human-computer interaction experts to develop digitally enhanced versions of manipulatives for algebra learning.

Keywords: interdisciplinary research, multimodality, tangible interaction, technology design

Developing truly novel technologies for learning requires interdisciplinary teams that consist of experts from different domains, e.g., didactitians and computer scientists. This is also the case in the research project MAL: Mathematics educators and human-computer interaction (HCI) researchers jointly develop interactive and digitally enhanced versions of manipulatives for algebra learning.

BOUNDARY OBJECTS AT THE ELECTRONIC FRONTIER

While both mathematics education and HCI can be seen as interdisciplinary in themselves, conferences, journals, professorships, etc. have been established for each, thus exemplifying disciplinarity "(a) [as] a phenomenon of the social world marked by increasing specialization and differentiation of (material and discursive) practices and (b) [as] a form of discourse making the specialization thematic" (Williams et al., 2016, p. 4). For interdisciplinary research, common objects are a condition: "they coordinate the activities involved even though the practices surrounding these objects differ. These objects are known as *boundary objects*" (p. 11). Here, we share experiences about boundary objects we encountered in our particular collaboration.

Design cycles. The widespread implementation of design research gives didacticians common ground with engineering and design disciplines, where the term originated. However, what is investigated in such research can vary. Computer scientists and HCI researchers, in order to build systems, have to focus on details (colour schemes, sizes, single modes of feedback) and sometimes consider these isolated from each other. In contrast, the design of mathematics tasks is more dominated by basic assumptions that both guide the design and are subject to testing in each cycle.

Users. HCI research focuses heavily on usability and user experience. Doing so, the user is defined the person who deals with the technology, *either* a (group of) student(s) *or* the teacher or possibly another person with a defined role. From a didactical perspective, however, the users of new learning environments are (a) not limited to teachers and learners as individuals and may encompass larger institutions (up to the society as a whole) and (b) are often seen as interconnected.

Embodiment and Modalities. In both disciplines theories on *embodied cognition* have been taken up. In this context, the involved *modalities* are central. The term, however, can be used differently. In HCI, modalities are usually either defined by the sensory channels (e.g. Obrist et al., 2016) or by the input and output modes (Oviatt, 2012) used for interaction with a system. In mathematics education, modalities are used as an analytic term whose definition is often dependent on the mathematical context: A picture of a situation may be seen as a different modality than a graph in a coordinate system, although both would be "graphical output" in HCI terminology. **Gesture** is one particular modality that has received much attention in both fields. Again, both disciplines refer to the same background theory (e.g., McNeill, 1992). But in practice, the HCI discourse is dominated by the gestures possible to track with current technology, while mathematics educators are more open towards all the gestures that may occur in the classroom (de Freitas and Sinclair, 2017).

Feedback. In HCI, feedback refers to communication from the system to the user as a direct result of a user's action (Shneiderman, 1987). While there is awareness that feedback is not an end in itself, this definition may lead to oversimplified interactions. Research from mathematics education can be helpful here, e.g. by identifying different levels of feedback (Hattie & Timperley, 2007) or by working out adequate feedback in specific learning situations referring to concepts like scaffolding (e.g., Sharma & Hannafin, 2007).

OUTLOOK

Although often a pragmatic approach will suffice to fulfil the goals of a particular collaboration, the identification and deeper reflection of boundary objects can help to understand each other and the roles of both sides. If interdisciplinary working groups manage to understand their partners' fields, the transfer of discourses and practices from one field to the other is facilitated. Mathematics educators could help HCI researchers broaden their view on users and feedback, for example. HCI researchers could challenge their mathematics education partners to identify gaps in existing technology, and then proceed together to improve it. Furthermore, what each side learns when working with the other may also help shaping the self-understanding of the two juvenile disciplines, and possibly prevent their boundaries from becoming incrusted.

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REFERENCES

de Freitas, E. & Sinclair, N. (2017). *Mathematical gestures: multitouch technology and the indexical trace*. Paper presented at CERME10. Retrieved from https://keynote.conference-services.net/resources/444/5118/pdf/CERME10_0525.pdf

Hattie, J. & Timperley, H. (2007). The power of feedback. *Review of Educational Research, 77*(1), 81–112.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago, IL: University of Chicago Press.

Obrist, M., Velasco, C., Vi, C., Ranasinghe, N., Israr, A., Cheok, A., … (2016). Sensing the future of HCI: Touch, taste, and smell user interfaces. *interactions, 23*(5), 40–49.

Oviatt, S. Multimodal interfaces. (2012) In J. A. Jacko (Ed.), *The human-computer interaction handbook: Fundamentals, evolving technologies and emerging applications* (3rd ed.) (pp. 405–429). Boca Raton, FL: CRC Press.

Sharma, P. & Hannafin, M. J. (2007). Scaffolding in technology-enhanced learning environments. *Interactive Learning Environments, 15*(1), 27–46.

Shneiderman, B. (1997). *Designing the user interface: Strategies for effective human-computer interaction* (3rd ed.). Boston, MA: Addison-Wesley Longman.

Williams, J., Roth, W.-M., Swanson, D., Doig, B., Groves, S., Omuvwie, M., … (2016). *Interdisciplinary mathematics education. A state of the art*. Cham: Springer.

LEARNING WITH INTERACTIVE VIRTUAL MATH IN THE CLASSROOM

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Interactive Virtual Math (IVM) is a visualization tool to support secondary school students' learning of dynamic functions situations graphs. The logbook-function allows teachers to get continuous and real-time assessment on classroom progress and of individual students' learning process. In a teaching experiment involving four mathematics teachers and their students, we investigated how the tool was used by the students and by the teachers.

Keywords: visualization, Virtual Reality, interactive tool, secondary education, learning analytics

AIM AND RESEARCH BACKGROUND

Students' difficulties with tasks involving dynamical situations are well documented in the literature. And there is also a body of knowledge that shows that conventional curricula have not been effective in promoting covariational reasoning in students (Carlson, Larsen, & Lesh, 2003). New technologies can allow for studying dynamic events and therefore be valuable for students to analyse and interpret dynamic function situations. The aims of the Interactive Virtual Math-project are to design and develop a digital tool for learning covariation graphs at high school (14-17 years old students) and to explore the use of new technologies for learning in classroom. The project started in 2016 as a proof of concept in which a prototype tool was developed and tried out with 14-15 years old students (Palha and Koopman, 2016). In the present stage we explore how the tool is used in classroom by teachers.

INTERACTIVE VIRTUAL MATH

Research provides some directions to develop instruction that supports the learning of covariational reasoning. Thompson (2011) states that it is critical that students first engage in mental activity to visualize a situation and construct relevant quantitative relationships prior to determining formulas or graphs. Also, learners should be helped to focus on quantities and generalizations about relationships, connections between situations, and dynamic phenomena. Digital tools can be valuable for students to analyse and interpret dynamic functional situations. These experiences, when connected to proper curriculum materials and teacher support, can become rich opportunities for students to learn covariational reasoning (Carlson et al, 2003). Tools that include Educational Data Mining (or learning analytics) also have the possibility to generate new understandings of how students learn and how to adapt our environments to those new understandings (Berland, Baker, & Blikstein, 2014). Following these ideas, the IVM-tool was designed and developed to (i) help learners to focus on the relevant quantitative relationships and engage them in the mental activity of visualizing these relationships; (ii) help teachers to get more data about students processes while solving covariation problems. The tool and an instructional video about how it works can be respectively found at https://virtualmath.hva.nl (select EN for English) and https://youtu.be/lc7mNUcZ8CQ.

Students' visualizing relationships

When entering the tool the students are given a task that encourage them to imagine two variables changing simultaneously. The tool requests the students to construct the graphical representation and the verbal explanation for this relation on themselves within the application. That is, it requires students to represent their concept image graphically and verbally (Vinner, 1983). Through hints and feedback the student is challenged to improve his own construction. The tool also includes the use of Virtual Reality (VR), which is still very limited. The use of VR (sound, movement, interaction) is expected to improve the experience of the graphic situation.

Teacher's use of data about students' processes

Another feature of the tool is the logbook-function, which is only available for teachers. Students' attempts to solve the tasks and whether they view the help-features are recorded and summarized in the logbook. This function allows teachers to get continuous and real-time assessment on the classroom progress and on individual student's learning process, which can be used by the teacher to provide individual feedback and to orchestrate classroom discussions. It also provides more data about students' processes while solving covariation problems.

METHOD

Two versions of the prototype have been developed so far. The first version of the prototype was tested with four students. The four students improved their original graphical representation through relating representations and using quantitative reasoning. In the present study we investigate the second prototype version of the tool use in classroom. We conducted a small scale experiment at secondary and tertiary education involving four classes and their students and teachers that used IVM during one lesson (45-50 minutes). Because we wanted to explore how students use the tool in the regular classroom practice the teachers were encouraged to setup the lesson from themselves. This paper reports part of the whole study (Palha, 2017). It concerns students' experiences with the tool and the corpus data consists of students' responses to questionnaires.

The participants were seventy nine students and four teachers from four classrooms in different schools in The Netherlands: nine students from the first year of the bachelor mathematics teacher, twenty-eight students from 11th grade with, pre-university stream with mathematics B; twenty one students from 10th grade with pre-university stream with mathematics B and twenty one students from 10th grade with, vocational stream with mathematics A. The four classes vary in their mathematical knowledge and ability. It is expected that the 10th grade vocational is the class with less pre-knowledge. No student had, as far as we know, worked before with the tool before the experiment.

The four teachers were invited to take part of the study; they knew about the tool but they were not used to work with it. The teachers are two men and two women with ages varying between 28 and 40 years and with teaching experience varying between 5 to 15 years. The teachers were selected by their teaching experience (we wanted to have a different range of experience since this is a factor that influences classroom performance). And, because they had previously showed interest in using the tool with their students. Not all teachers dare to experiment new approaches especially technological tools that are still in development. We should therefore be careful with the generalization of the results of the experiences of these teachers as they are not representative for the Dutch teachers. More details about the study can be found in Palha (2017).

MAIN FINDINGS

About half of students in all classes reported that the tool have helped them to create, to improve or correct a graph. The way students felt supported by the tool varied per class. Students at 10th and 11th grade with mathematics B reported to have already an idea about the shape of the graph and the tool helped them to work it out and consolidate this idea. Students-teachers at the bachelor have a good idea about the graph and the tool helped them to correct some mistake. The half of the students at the 10th grade following vocational stream also felt support of the tool but they did not have previously any idea about how the graph would be or they had vague idea. The tool helped them in the construction of the graph and to improve their vague initial image.

Specifically, all classes reported that seeing the result of the form of the jar at the end and the self-construction graph were the most helping to them (with exception of one class, in which a slightly higher percentage pointed the help 3D animation as more helpful than the self-construction). Also the comparison feature was considered by the four classes helpful. The help-features were not often mentioned

All students in the four classes (with one exception in one class) reported that they could work independently with the tool. In three of the four classes a great percentage of the students (81%-89%) reported that they haven't needed help at all and a small percentage reported that they felt the need of some help (11%-19%). In the fourth class (10^{th} grade vocational) about the halve (48%) didn't need help and the other halve (48%) needed some help

The findings suggest that the students can work independently with the tool in the classroom and without much help. The tool can create opportunities for students to produce and try to improve a mathematical representation of a dynamic event. However, we do not provide much information about the process of coming to generate the graph representations and verbal explanations and its transformation. This study invites further research on this matter. Our research also calls for an extension of the tool and improvement of some features, students provided insightful suggestions that can help us in this direction.

REFERENCES

Berland, M., Baker, R. S., & Blikstein, P. (2014). Educational data mining and learning analytics: Applications to constructionist research. Technology, Knowledge and Learning, 19(*1-2*), 205-220.

Carlson, M., Larsen, S., & Lesh, R. (2003). Integrating a models and modeling perspective with existing research and practice. Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching, 465-478..

Palha, S., & Koopman, S. (2016). Interactive Virtual Math: a tool to support self-construction graphs by dynamical relations. Proceedings of CERME10 will be available at HAL archives website: https://hal.archives-ouvertes.fr/ (see more info about CERME10 at www.cerme10.org).

Palha, S. (2017). Students learning with Interactive Virtual Math: an exploratory study in the classroom. *Ensino Tecnologia em Revista*, 1(*1*), 80-102. https://periodicos.utfpr.edu.br/etr/issue/view/333

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), New perspectives and directions for collaborative research in mathematics education. WISDOMe Mongraphs (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming"

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*, 14(3), 293-305.

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DIGITAL MATHEMATICS TEXTBOOKS: ANALYZING STRUCTURE AND STUDENT USES

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The use of digital tools in the Mathematics classroom is an important focus of research in the field of mathematics teaching. In the context of digital textbooks, however, there is still a high need for research and development. Although there are first versions of digital textbooks in the German language, it is unclear what structure and elements digital Mathematics textbooks generally offer or how learners work with them. This contribution addresses this research topic.

Keywords: digital mathematics textbooks, student uses, structure of digital textbooks

THEORETICAL BACKGROUND

When working with (traditional) textbooks, students use the textbook to engage in the field of the book's subject, e.g. mathematics. By doing that, the use of the textbook is affected by its content and structure. Both aspects define the artefact *textbook* and, therefore, influence student uses of the textbook as Chazan and Yerushalmy (2014) pointed out:

> [T]extbooks give teachers guidance on both what and how students should learn. On the one hand, especially initially, textbooks organized the content of what students were to learn and indicated what students needed to know at what age, grade level, or institutional track within schooling. On the other hand, by presenting instructional tasks, textbooks attempt to organize the knowledge that they present in ways that will help make this content learnable. (Chazan & Yerushalmy, 2014, S. 67)

When talking about student uses of textbooks and to see how students work with textbooks in order to engage in the field of mathematics, Rabardel's theory of *instrumental genesis* (Béguin & Rabardel, 2000) is helpful. According to Rabardel, the user turns the artefact textbook into an instrument for learning mathematics during the two intertwined processes (a) *instrumentalization* and (b) *instrumentation*. In the course of *instrumentalization*, the user individually attributes functions to (parts of) the artefact that can be fulfilled, while the *instrumentation* process is concerned with how students select relevant content within the textbook. Rezat (2011) applied this framework to students' paper textbook uses focusing on elements students select for certain learning activities and thus describing student uses of elements within the textbook. He pointed out that students mostly engage in the activities of *practicing* and *solving tasks and problems* and that they usually work with *exercises* and *boxes containing basic knowledge* (cf. Rezat, 2011, p. 171).

In the context of digital mathematics textbooks, the question arises whether the same learning activities and elements can be identified or whether these categories need to be extended.

RESEARCH QUESTIONS AND METHODOLOGY

Transferring and adapting research findings to the context of digital mathematics textbooks involves knowledge on a) the structure and elements of a range of existing digital textbooks concepts and b) what roles/functions students ascribe to certain elements of the digital textbook during a variety of learning activities. These two research perspectives can be framed into the following research questions a) What kind of structural characteristics and elements can be identified in digital mathematics textbooks? and b) What kind of structural elements do students *instrumentalize* when working with digital mathematics textbooks?

Analysing digital mathematics textbooks within the frame of Mayring's *Qualitative Content Analysis* (2008) offers a valid category-developing method in order to identify relevant structural elements. This serves as a basis for the second focus – analysing student uses of the textbook within the frame of *Instrumental Genesis (Rabardel, 2002).*

Approaching the first research question involves analysing several digital mathematics textbooks (normative analysis). Concerning the second research question (empirical analysis), student uses of the analysed structural elements were videotaped, transcribed and studied based on two levels – *concept-related* and *structural-element-related*.

RESULTS AND DISCUSSION

The main outcome in terms of the normative analysis is that several structural elements could be identified that had not been identified for traditional textbooks (cf. Rezat 2009). This means that a broader variety of types of exercises in terms of their static or dynamic mode were characterised as well as different kinds of feedback modes on how well (or not) exercises were completed. More precisely, we could identify dynamic exercises with a *drag-and-drop* mode where students can select and move small elements within the exercise, *notification exercises* where the user can write down notes, calculation methods or ideas and download the notes in the end, *calculation exercises* where the solution (of that exercise) can be entered in a predefined field and checked for its correctness, or *interactive exercises* which allow students to access mathematical ideas in a dynamic and visualised way. Furthermore, the digital nature of textbooks allows a variety of different feedback modes, i.e. *solution*, *solution process*, and *check solution*. While the first feedback mode gives the learner the correct answer to the task and the second one reveals the solution process and the necessary calculations, only the structural element *check solution* allows a dynamic feedback on the student solution displaying whether the entered solution is right or wrong. The following table lists all structural elements that could be identified for digital textbooks:

Structural Element	Information
Additional information	Tip for specific task/exercise
Animation	Visualisation of mathematic content, not modifiable through user
Box with basic knowledge	Formula, definition
Box with hints	General information on current topic
Check solution	Dynamic feedback on the student solution displaying whether the entered solution is right or wrong
Drag-and-drop exercise	Dynamically select and move small elements within the exercise
Dynamic calculation exercise	Solution (of that exercise) can be entered in a predefined field and checked for its correctness
Exercise	Static exercise
Interactive exercises	Access mathematical ideas in a dynamic and visualised way
Notification exercise	Notes, calculation methods or ideas can be entered and

	downloaded
Picture	Static
Solution	Correct answer to the task
Solution process	Solution process and the necessary calculations
Table	Static
Text	Continuous text

Table 1: Overview of structural elements identified for digital mathematics textbooks

Based on the normative analysis or rather the identified structural elements, student uses of the analysed structural elements were videotaped, transcribed and studied. In order to see how new structural elements in digital textbooks effect student uses, the empirical analysis concentrated on how different kinds of feedback options influence student uses of these structural elements. The main outcome of this focus was that individual uses of structural elements on both the *concept-related* level as well as on the *structural-element-related* level could be identified. For example, students referred to their mathematical concept images (in German: "*Grundvorstellungen*") when comparing their results (*concept-related*) before using the structural elements *solution*, *solution process*, or *check solution* as the following example shows:

Student 1 Your book … is definitely smaller than your room.

Student 3 (…) I compared the book with ehh with a small coin.

We can see that although the students compared the book referring to different objects they applied the same underlying concept image, i.e. the concept of 'comparing' (in German: "Vergleichsaspekt") (Weigand et al. 2014, S. 160). Furthermore, student 3 demonstrates the concept image of 'filling out' (in German: "Ausfüllaspekt") (Weigand et al. 2014, S. 161) as the following transcript extract shows:

Student 3 (…) Let's say the book is as big as this [takes a sheet of paper in A4]. Maybe a little bit smaller. Eh … let's say (…) it is up to here [draws a line on the sheet] (…) If I put coins down, then everything will be full of coins. For me, the coin is small. The pinhead is very small. (…) And if you take that to the book, for me, it results in medium-sized.

Student 3 does not only compare the coin to the pinhead and the book, but also fills out the book with a lot of coins whereby the argument refers to the concept image of 'filling out'.

After the learners have used the technological check and correct or wrong assignments are displayed, the statements from the users show reactions on a *structural-element-related level*. For example, students rejected the textbook solution when they did not understand the textbook solution. Another reaction can be seen in the following example as the student is comparing the textbook solution to her own solution:

Student 3 [M]y reasoning was better [than your reasoning].

Here, the student extrapolates from the textbook's feedback to her reasoning and to that of her classmates. Therefore, her argumentation refers to the displayed solutions – hence, to the textbook. A third reaction which could be observed was that of student 2 as he – on the basis of the textbook's feedback – is trying to make sense of the computer solution:

Student 2	[points on the screen] Yes, but … the cent coin is small. Then … the stamp is bigger … then, the stamp must be (…) *smaller than medium* but that is the size of the book so that does not make sense.

All in all, these reactions can be differentiated into the four categories. For a more detailed analysis, see Pohl & Schacht (in press).

Structural element-related Categories
Rejecting the computer solution
Comparison of right or rather wrong solutions
Reconstructing the computer solution based on the computer solution
Reconstructing the own solution based on the computer solution

Table 2: Overview on the effect of the structural elements providing feedback on student solutions on structural-element-related level (cf. Pohl & Schacht in press)

Overall, this contribution shows, besides the student argumentation on a *concept-related level*, the multiplicity of user reactions on the *structural-element-related level* based on the use of individual structural elements with a verifying function. Further empirical analyses of student uses of digital mathematics textbooks should therefore be examined to see whether *concept-related* and *structural-element-related* arguments can collaborate. The two categories "reconstructing the computer solution based on the computer solution" or "reconstructing the own solution based on the computer solution" already suggest a collaboration on both argumentation levels, since the process of reasoning based on the concept images with reference to the structural element becomes apparent.

REFERENCES

Béguin, P. & Rabardel, P. (2000): Designing for Instrument-Mediated Activity. In: Scandinavian Journal of Information Systems, 12(2000), 173-190.

Chazan, D., & Yerushalmy, M. (2014). The Future of Mathematics Textbooks: Ramifications of Technological Change. In M. Stocchetti (Ed.), Media and Education in the Digital Age. Concepts, Assessments, Subversions (pp. 63-76). Frankfurt: Peter Lang GmbH Internationaler Verlag der Wissenschaften.

Mayring, P. (2008): Qualitative Inhaltsanalyse. Weinheim, Basel: Beltz.

Rabardel, P. (2002). People and Technology: A Cognitive Approach to Contemporary Instruments.

Rezat, S. (2009). Das Mathematikbuch als Instrument des Schülers: Eine Studie zur Schulbuchnutzung in den Sekundarstufen. Wiesbaden: Vieweg+Teubner.

Rezat, S. (2011). Wozu verwenden Schüler ihre Mathematikschulbücher? Ein Vergleich von erwarteter und tatsächlicher Nutzung. Journal für Mathematik-Didaktik, 32(2), 153-177.

Weigand, H.-G., Filler, A., Hölzl, R., Kuntze, S., Ludwig, M., Roth, J., Schmidt-Thieme, B., Wittmann, G. (2014). Didaktik der Geometrie für die Sekundarstufe I. Mathematik Primarstufe und Sekundarstufe I + II. Berlin: Springer Spektrum.

Pohl & Schacht (in press). *Digitale Mathematikschulbücher hands-on. Eine Analyse digitaler Mathematikschulbücher und den Schülernutzungen in der Sekundarstufe I exemplarisch an einem Lehrwerk*. In M. Schuhen, & M. Froitzheim. (Ed.), Das Elektronische Schulbuch 2017. Fachdidaktische Anforderungen und Ideen treffen auf Lösungsvorschläge der Informatik. Münster: LIT Verlag.

BRINGING APPROPRIATE MENTAL IMAGES TO THE FOREGROUND USING DYNAMIC GEOMETRY AS A SEMIOTIC MEDIATOR: WHEN IS A RECTANGLE A RECTANGLE?

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Abstract: We claim that a Dynamic Geometry Environment could be an effective semiotic mediator to bring a correct mental image to the foreground, despite the prevalence of a primeval and incorrect image.

Keywords: Dynamic Geometry Environments; mental images; mental models; semiotic mediators.

A GEOMETRIC PROBLEM: WHAT IS A RECTANGLE?

Elisa Gallo (1994, 1989) studied experimental settings to recognize formed or not formed, appropriate or not appropriate models (Ackermann-Valladao et al. 1983). In particular, pupils aged 14–15 were asked to perform the following activity: "Draw a rectangle ABCD, with the side AB lying on the line *r* and the points A, C as given" —where the line *r*, a point A of *r* and a point C outside of the line are drawn on the text of the activity with—this is paramount! — the line *r* not parallel to the sides of the sheet. (Fig. 1)

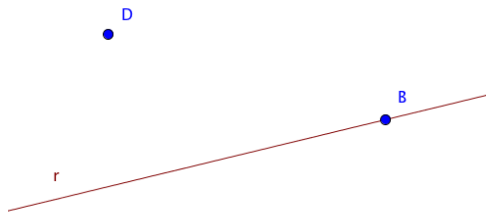

Fig. 1. Elisa Gallo's activity

401 answers were collected and classified in 37 different models (some are shown in Fig. 2), only one of them being correct: the correct answer appears 100 times, 60 more answers are rectangles, 206 answers are about parallelograms and 35 answers are other figures (right trapetia, triangles, etc.). Notice that pupils were asked not to erase any attempt.
An informal observation while displaying the present work at ICTMT showed about a third of the observees drawing a parallelogram (the third item in Fig. 2) with their finger before giving the correct answer.

THE CONCEPTUAL FRAMEWORK

Following D'Amore (1999) we define a *mental model* of a phenomenon as a collection of mental images that arose from the different manifestations of that phenomenon. Thus, many of the results (Fig. 2) of E. Gallo's experiment could be read as the emergence of a primeval mental image, branded in primary school: *a rectangle is the part of the plane limited by a couple of horizontal lines and by a couple of vertical lines*. A *formed but not appropriate image*, in Ackermann-Valladao's framework.

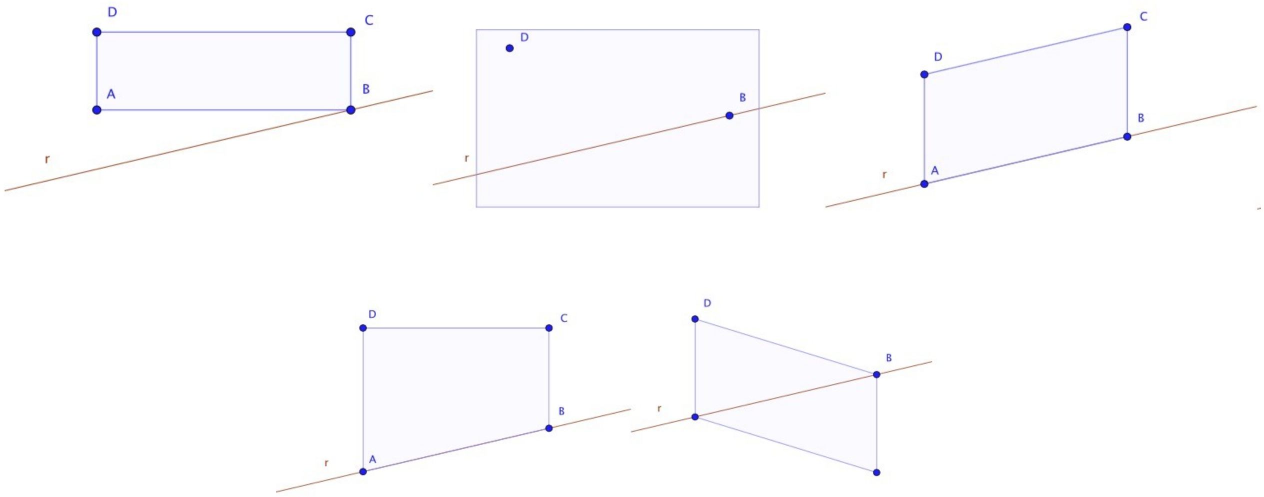

Fig. 2. Some of the results

Rabardel and Samurçay (2001) define an *instrument* as a mixed entity "made up of both artifact-type components and schematic components that we call utilization schemes." A double semiotic link appears between such an instrument, a task and a piece of knowledge (Bartolini Bussi & Mariotti, 2008).

EXPERIMENTAL HYPOTHESIS

Mariotti & Bartolini Bussi (1998) and Mariotti (2002) show that dragging in a Dynamic Geometry Environment (DGE) carries the pupil—through an "expected although not simple and spontaneous" process—to internalize the construction of a geometric figure (a square, in their case).
We expect that, in a similar way, the use of a DGE perpendicular line *instrument* (as defined by Rabardel) through its capability to manage and facilitate (Bu, Spector e Haciomeroglu 2011) would effectively mediate in bringing to the foreground the mental image of a rectangle as an equiangular quadrilateral.

PLANNING THE EXPERIMENT

The experimentation will be carried out during the 2017/2018 school year, in eighth and ninth grade classes of teachers in the Milan area with whom we are collaborating.

DUO OF ARTIFACTS

We intend to extend the proposed experimentation to allow the use of a duo of material and digital instruments (Maschietto & Soury-Lavergne, 2013), too; developing—similarly to Faggiano, Montone, Rossi (2017)—a teaching plan based on the interplay of material (viz. a geometry set) and digital (viz. perpendicular line tool) instruments.

REFERENCES

Ackermann-Valladao E, et al. (1983). Formation et actualisation des modèles du sujet en situation de résolution de problème. *Archives de Psychologie* 51: 61-70.

Bartolini Bussi, MG & Mariotti, MA (2008). Semiotic mediation in the mathematics classroom. In L English, et al. (Eds.), *Handbook of International Research in Mathematics Education* (2nd ed., pp. 746–805). Mahwah: Lawrence Erlbaum.

Bu, L, Spector JM & Haciomeroglu ES (2011). Toward model-centered mathematics. In: Model-Centered Learning. Rotterdam: SensePublishers. 13-40.
D'Amore B (1999). Elementi di didattica della matematica. Bologna: Pitagora.
Faggiano E, Montone A & Rossi PG (2017). Tecnologie per la didattica ed educazione matematica con le tecnologie. *Atti del XXXIV seminario nazionale di didattica della matematica*. Rimini.
Mariotti, MA & Bartolini Bussi, MG (1998). From drawing to construction. In: Proceedings of the 22nd PME Conference, Stellenbosh, vol 3, pp. 247–254.
Maschietto M., & Soury-Lavergne S. (2013). Designing a Duo of Material and Digital Artifacts: the Pascaline and Cabri Elem e-books in Primary School Mathematics. *ZDM Mathematics Education* 45(7), 959-971.
Gallo E (1989). "Sul ruolo dei modelli nella risoluzione di problemi di geometria: controllo ascendente e discendente." Ricerche di didattica della matematica 7. Torino: Nucleo di ricerca didattica CNR.
Gallo E (1994). Elaboration of models for problem resolution in interaction with 14/15-year-old pupils. In: Proceedings of the Second Italian-German Bilateral Symposium on Didactics of Mathematics: Haus Ohrbeck (Georgsmarienhütte) 21-26 April 1992. Inst. für Didaktik der Mathematik der Univ. Bielefeld.
Rabardel P & Samurçay R (2001). From artifact to instrument-mediated learning. In: Symposium on New challenges to research on Learning. University of Helsinki.

THE USE OF COMPUTER BASED ASSESSMENT PISA 2012 ITEMS IN MATHEMATICS CLASS: STUDENTS' ACTIVITIES AND TEACHERS' PRACTICES

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Until few years ago, the PISA assessment was paper based. Recently, the computer based test administration modality has been chosen for PISA' next cycles. How student mathematical performance measurement could be affected by this modality? WouldFrench teacherspractices,particularly assessment practices,change by consideringthe PISA's new framework, and if so, how? This poster presents the first exploratory steps of a thesis research project through analysing the mathematical task and students'activity at stake in a PISA 2012 CBA released items.

Assessment; TeachersPractices; Technology; Mathematics; PISA

PISA 2012 CBA ITEM « CD PRODUCTION » ANALYSIS RATIONALE

In 2012, OECD'S PISA proposed an optional assessment in Mathematics in a computer based environment (OECD, 2013). New itemswere specifically developed at this occasion. The "CD Production" is an example of such an item. Using Activity Theory and its development in the French sphere of didactics (Robert & Rogalski, 2005; Abboud-Blanchard & Vandebrouck, 2012) as theoretical background, we analysedthe mathematical task in this item by taking also into account the levels of mathematical knowledge operation (Roditi&Salles, 2015). Additionally, we identified how the item potentially explores students' Instrumental Genesis (Folcher, Rabardel, 2004) and how this affects the item's task performing. We will show how confronting the*a priori* analysis to actual students' responses, transcripts and mathematical work on the item,recorded during cognitive laboratory run with two 9 graders, guide us in our investigation of such items' affordances.

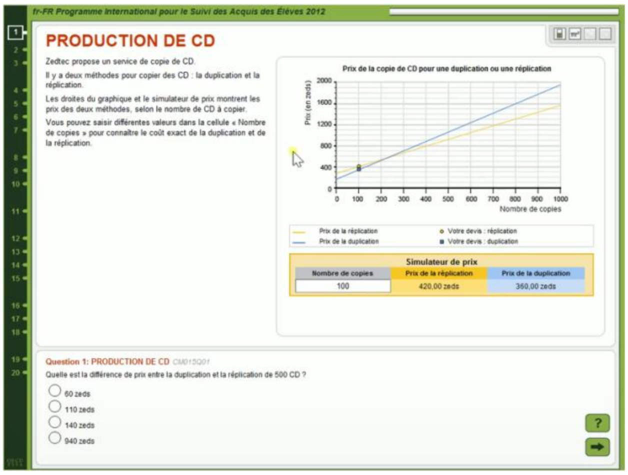

Figure 1. CD Production Question 01, French National version,MENESR, DEPP, OECD, PISA 2012

APRIORI MATHEMATICAL TASK ANALYSIS

This « real life » situation compares two different techniques used to copy CDs, more specifically their cost by the number of copies. Information is given in “hotlinked” numerical and graphical representations. Students can put in numbers of copies in a price calculator that outputs prices for both techniques. The task consists in working out the difference in prices for 500 copies. Response format is multiple choices. The operation at stake is a subtraction. However one has first to adapt the information given to find the values to operate. At this end, students can either use the graphical representation and work out an approximate difference or work in the numerical representation and use the price calculator to find exact costs for 500 copies before subtracting. The choice depends on the level of accuracy needed by the subject. The distractor 110 is close enough to the correct response (140) to allow approximate graphical values lead to a wrong answer, whereas the correct use of the price simulator leads to the correct answer only.

ABOUT THE TECHNOLOGY RICH SITUATION

The technological tools available to students in this item, a ready to use calculator as well as the embedded price simulator, release the “burden of computation”. Hence, students can focus on the strategy and the structure of the given information. More specifically, dealing with both a graphical and a numerical representation of the relation between number of copies and price, is allowed and eased by the fact that representations are “hotlinked” (Stacey, Wiliam, 2013), as the graph displays the points from the coordinates entered in the price calculator. Besides, the price simulator can be instrumented(Folcher, Rabardel, 2004) by students to the finding of exact values.

AN EXAMPLE OF STUDENTS’ ACTIVITIES WITH COGNITIVE LABORATORY RECORDINGS METHODOLOGY

Two grade 9 students have been audio and video recorded when performing this task in a collaborative way. The objective of such a cognitive laboratoryconsists in gathering as much information as possible regarding students’ activity. Students are encouraged to collaborate and speak aloud during the process and a short interview is administered when finished.Results of observation give information on the time spent to solve, student’s pointer moves on the screen, numbers entered and chosen responses, as well as an audio recording of their collaboration and of the interview with the researcher. Results show that the simulator has been instrumented, but this instrumentation was not straightforward for one of the two students. The graphical representation has quickly been abandoned to the profit of the numerical one. One of the students was familiar with a price simulator and she was at the initiative of using it which reveals the availability of using schemes (Folcher, Rabardel, 2004) at the service of the mathematical activity.

REFERENCES

Abboud-Blanchard, M. & Vandebrouck, F. (2012). Analysing teachers’ practices in technology environments from an Activity Theoretical approach. *The International Journal for Technology in Mathematics Education,* vol 19.4, 159-164.

OECD (2013), PISA 2012 Assessment and Analytical Framework: Mathematics, Reading, Science, Problem Solving and Financial Literacy, pp. 43-44. OECD Publishing.

Robert, A. & Rogalski, J. (2005). A cross-analysis of the mathematics teacher’s activity. An example in a French 10th grade class. *Educational Studies in Mathematics,59,* 269-298.

Roditi, E. & Salles, F. (2015) Nouvelles analyses de l'enquête PISA 2012 en mathématiques. Un autre regard sur les résultats In Education & Formations n° 86-87, pp.235-257, MENESR, DEPP

Folcher V., Rabardel P. (2004) Hommes-Artefacts- Activités : perspective instrumentale In P. Falzon (Eds) L 'ergonomie, (pp. 251-268) PUF.

Stacey, K.&WiliamD. (2013), "Technology and Assessment in Mathematics", in Clements M. A. (Ken), A. Bishop, C. Keitel, J. Kilpatrick and F. Leung (eds.), Third International Handbook of Mathematics Education, (pp. 721 – 752) Springer.

THE GRAPHING CALCULATOR IN THE DEVELOPMENT OF THE MATHEMATICS CURRICULUM IN THE 7TH GRADE OF BASIC EDUCATION

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This poster refers to the first data of a project that involves an educational teaching experiment that seeks to integrate technology in the curriculum of the 7th grade of basic education. This study is supported by the Activity Theory and seeks understand the instrumental genesis and semiotic potential played by technology in student's activity system, developing the process of semiotic mediation.

Keywords: Graphing calculator, curriculum development, Activity Theory, Semiotic Mediation, Instrumental Approach.

INTRODUCTION

Adopting several teaching strategies, in an essentially exploratory learning environment, based on several tasks that involve the use of the graphing calculator, it is intended to create an unusual curricular dynamics at this level of education. Some of the tasks are specific to the different topics of the curriculum (eg Algebra, Statistics or Geometry) and others are intended to relate various domains of mathematics, using mathematical modeling.

It seeks to understand how the student builds mathematical knowledge in solving specific tasks with the support of the graphing calculator as a member of a learning community. The Activity Theory (Engeström, 2001) is used to understand how the teacher, faced as a representative of a cultural community of reference, taking into account the semiotic potential of the graphing calculator, orchestrated didactic interventions with this mediating artifact, developing the process of semiotic mediation which is increased through the instrumented activity of the student.

The purpose of this study is to investigate, in the development of the curriculum, how the use of technology, namely the graphing calculator, promotes the processes of instrumental genesis and semiotic mediation, in the student's system of activity in interaction with other systems of activity, through the orchestration of the teacher. In this sense, the central questions inherent to the study are presented: *What are the schemes of instrumented action created by students when they use a graphing calculator? How does the graphing calculator act as semiotic mediation tool?*

THEORICAL FRAMEWORK

Benefits of the implementation of technology in the teaching of mathematics

There are several benefits that emphasize the incorporation of technology into didactic environments, namely increased motivation, involvement, cooperation, hands-on learning opportunities, confidence, and students' technological skills (Costley, 2014). For Schwartz (1999) there are five aspects of mathematical activity: (a) conjectures and exploration; (b) acquisition,

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evaluation and analysis of data; (c) modeling; (d) conceptual foundation of manipulative skills; (e) deepening and broadening understanding; that if they are explored through a weighted software, it will contribute to the development of abilities in the students and teachers, as well as influence the achievement of the educational objectives of the society, present in the mathematics curriculum.

The inclusion of the graphing calculator by students at different levels of school performance, shows that all have positive benefits, when a teaching approach is done with this artifact and with more significant emphasis in students with special educational needs (Li, 2010).

Activity Theory

Being the unit of analysis, the activity system within the classroom, the third generation of Activity Theory (Engeström, 2001) allows us to understand what happens when different systems of activity interact.

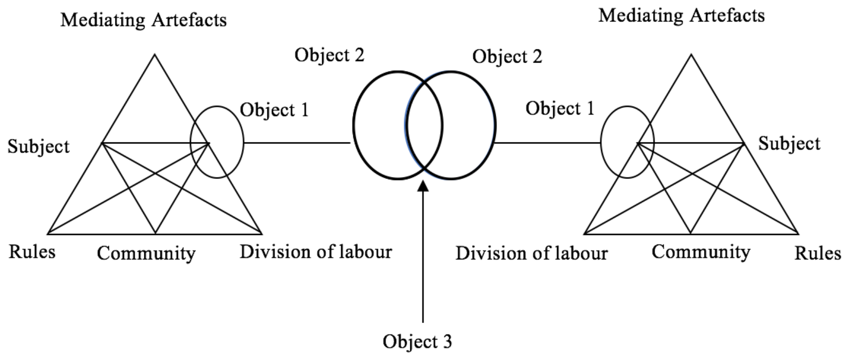

Figure1 - Two activity systems in interaction (Adapted from Engeström, 2001, p. 136)

Instrumental Genesis

The construction of an instrument is not spontaneous and occurs according to a process called instrumental genesis. An instrument is seen as a mixed entity, as it results from the appropriation of an artifact, material or symbolic, by the subject, through associated schemes (Rabardel, 1995). The schemes of use are directed to the management of the artifact and the schemes of instrumented action are entities directed to the accomplishment of the task (Drijvers & Trouche, 2008).

Semiotic Mediation

The Semiotic Mediation is a theoretical approach that at a didactic level approaches the teaching and learning of mathematics through the integration of technology, with the objective to analyze the different types of signs included in activities oriented by artifacts. In a classroom environment, in activities performed with artifacts, several signs emerge that can be used intentionally by the teacher to explore semiotic processes, aiming to guide the evolution of meanings within the class community. From the individual point of view, there are personal meanings that are related to the use of the artifact arise, namely as regards to the objective of accomplishing the task, on the other hand, from the social point of view, the mathematical meanings that may be related to the artefact and its use. In this sense, there is a double semiotic relationship articulated by the artifact, called by the semiotic potential of the artefact that is characterized by the easiness it has in associating culturally determined mathematical meanings, with individual meanings that each subject develops in the use of the same or in the accomplishment of tasks with their support. The artifact plays a dual role, both as a means of performing a task, and as a semiotic mediation tool to fulfill a didactic goal (Bussi & Mariotti, 2008).

METHODOLOGY

The techniques used to collect data for the research problem were based on the planning of the study units, writing of reports by pupils resulting from the completion of the tasks and reports from the participant observation of the teacher, insofar as investigator and mediator. It also consolidated rigorous, attentive and structured observation of classes, using the logbook and photographs of graphic representations of the graphing calculator (Creswell, 2012).

DATA ANALYSIS

Taking into account task 1 and task 2, the first one was given at the beginning of the experiment and the second given one month later:

Task 1 - Use the graphing calculator to represent the following functions: $y = 2x$; $y = 3x$; $y = 5x$; $y = - 2x$; $y = - 3x$; $y = - 5x$. What do you conclude?

Task 2 - What is the relationship between the amplitude of inscribed angle and the amplitude of angle to the center of a circle? What is the mathematical model that fits the situation?

In task 1 it was noticed that the students were still appropriating the graphing calculator artifact, using schemes of use, in situations such as:

Student1: How do I enter another function?

Student2: Press control **tab**!

As time went by the students were more comfortable in manipulating the graphing calculator and developed instrumented schemes of action, solving the tasks. On the other hand, the students, in task 1 when analyzing the graphs of the functions, taking into account their personal meanings, arrived at the properties of the linear function.

In task 2 the students have easily understood that the amplitude of an angle to the center is double the amplitude of an inscribed angle and they also have easily transited among various representations (geometric, tabular, graphical, algebraic). The worst performing student was able to first arrive at the modeling function of the relationship.

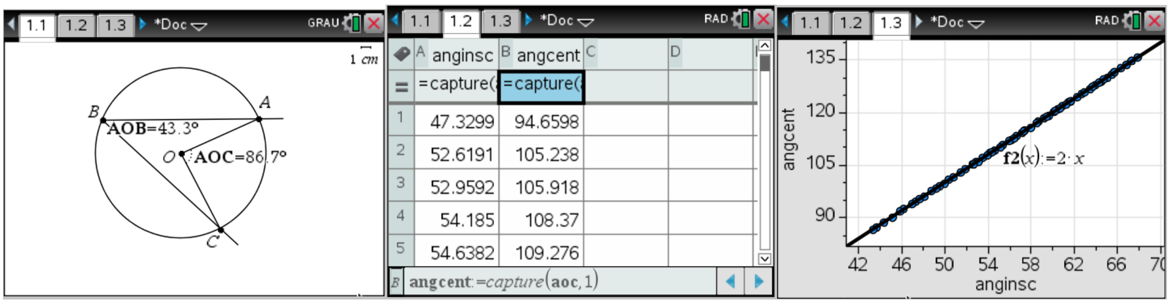

Figure2 - Records of the various representations of the graphing calculator in the resolution of the task2

CONCLUSIONS

The students developed schemes of instrumented action (mental schemes) through schemes of use. Being difficult to directly observe the mental schemes, the observations was limited to the techniques that the students accomplished with the artifact and also as they said in their oral reports. The transformation of the artifact (graphing calculator) into an instrument is still being done.

Given the semiotic potential of the artifact the teacher acted as a mediator and used the artifact as a semiotic mediation tool in solving the tasks in social environment where several activity systems interact. The students produced personal signs, related to the meanings that emerge from the accomplishment of the task and the use of the artifact, developing the collective production of common signs related to the use of the artifact and the mathematical contents to be learned. In this sense, in both tasks the students managed to articulate the personal meanings with the mathematical meanings, operating a process of semiotic mediation.

This project will continue in the school year 2017/2018.

REFERENCES

Bussi, M.G.B., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: artifacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of International Research in Mathematics Education, second revised edition* (pp. 746-783). Mahwah, NJ: Lawrence Erlbaum.

Costley, K. C. (2014). *The positive effects of technology on teaching and student learning*. Reports-Evaluative. (Document ERIC nº ED554557). Retrievied from http://www.eric.ed.gov/contentdelivery/ servelet/ERICServelet?accno=ED554557

Creswell, J. W. (2012). *Educacional Research, Planning, Conducting, and Evaluating Quantitative and Qualitative Research*. (4 th ed.). Boston: Pearson.

Drijvers, P., & Trouche, L. (2008). From artefacts to instruments: A theoretical framework behind the orchestra metaphor. In G. W. Blume & M. K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics*: *Vol. 2. Cases and perspective*s (pp. 363–392). Charlotte, NC: Information Age.

Engeström, Y. (2001). Expansive Learning at Work: Toward an activity theoretical reconceptualization. *Journal of Education and Work, 14*(1), 133-156.

Li, Q., & Ma, X. (2010). A meta-analysis of the effects of computer technology on school students' mathematics learning. *Educational Psychology Review, 22*, 215-243.

Rabardel, P. (1995). *Les hommes et les technologies: une approche cognitive des instruments contemporains.* Paris: Armand Colin.

Schwartz, J. L. (1999). Can technology help us make the mathematics curriculum intellectually stimulating and socially responsible?. *International Journal of Computers for Mathematical Learning*, *4*(2-3), 99-119.

WeDRAW: USING MULTISENSORY SERIOUS GAMES TO EXPLORE CONCEPTS IN PRIMARY MATHEMATICS

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ABSTRACT

In this paper we introduce weDRAW, a project to support primary school children in the exploration of mathematical concepts, through the design, development and evaluation of multisensory serious games, using a combination of sensory interactive technologies. Working closely with schools, using participatory design techniques, the games will be embedded into the school curricula, and configurable by teachers. Besides application to typically developing children, a major goal is to explore the benefits of this multisensory approach with visually impaired and dyslexic children.

Keywords: Mathematics, Multisensory, Serious Games, Geometry, Arithmetic.

INTRODUCTION

weDRAW (http://www.wedraw.eu) is a two-year project which aims to mediate the teaching of primary school mathematical concepts, such as geometry and arithmetic, through the design, development and evaluation of multisensory serious games, using a combination of sensory interactive technologies, taking into account developmental psychology and classroom interaction. The project proposes an embodied and enactive approach to learning. Enactive knowledge is not simply multisensory mediated knowledge, but knowledge stored in the form of motor responses and acquired by the act of doing. The games will integrate visual, sound and haptic feedback, in response to whole body movement. In this paper we will introduce the project and discuss work carried out to date.

MULTISENSORY LEARNING

The past two decades have seen increased exploration of technology to supporting teaching and learning (Laurillard, 2012, p. 2; Roschelle, Pea, Hoadley, Gordin, & Means, 2001), with a recent emphasis on multimodal and multisensory interaction. While concepts of embodiment are not new, the growth of ubiquitous computing and the possibility to enhance physical environments and interaction have brought discussions around embodiment to the forefront, emphasising the role of experience, the sensory body, emotion and social interaction for cognition and learning (Barsalou, 2008; Shaun Gallagher, 2005; Smith & Gasser, 2005; Wilson, 2002). There is evidence that mathematical cognition is embodied (Lakoff & Núñez, 2000), since it is grounded in the physical environment, and based in perception and action (Alibali & Nathan, 2012). That mathematical understanding arises from physical experiences suggests that learning environments need to introduce concepts through physical means, such as action or gestures. The importance of engaging with other modalities besides the visual for learning is not new. According to Kalogirou, Elia and Gagatsis (2013, in Jones & Tzekaki 2016), in the context of geometry, visual perception provides "direct access to the shape and never gives a complete apprehension of it" (p.129-130). Hall & Nemirovsky (2012) highlight the value in experiencing the difference between looking down on a geometric figure on paper or being inside it, or the tactile experience of that same figure. Several recent studies show the benefits of embodied learning approaches in primary mathematics (e.g. Goldin-Meadow, Wagner Cook, & Mitchell, 2009; Manches & O'Malley, 2016).

The use of di erent modalities can reduce cognitive load and improve learning (Moreno & Mayer, 1999), as well as o er new opportunities. Multimodal feedback has been shown to support skills development in children with dyslexia, for example, a musical training programme including cross-modal activities such as rhythm production, which has been shown to improve the reading problems experienced by dyslexic children (Habib et al., 2016). Another opportunity is to support learning for visually impaired children through the provision of additional stimuli. People who have never had any visual experience (congenitally blind), or who have lost their vision in early infancy (early blind) are seriously impaired when performing spatial tasks compared to blind participants who have lost their vision after becoming an adult (late blind) or sighted participants. As a result, complex computations that rely on such types of representations are more di cult (Thinus-Blanc & Gaunet, 1997), which impacts the ability to understand concepts of geometry. However, research from psychophysics and developmental psychology suggests that children have a preferential sensory channel for learning, and that vision is not always the dominant channel, especially for children under 8-10 years of age (Cappagli & Gori, 2016; Gori, Del Viva, Sandini, & Burr, 2008). For example, auditory feedback has been shown to improve spatial cognition in visually impaired children (Finocchietti, Cappagli, & Gori, 2017; Gori, Sandini, Martinoli, & Burr, 2014). The use of body movement has been shown to deepen and strengthen learning, retention, and engagement (Klemmer, Hartmann, & Takayama, 2006). Body movement is naturally associated with space and could be used to reinforce the understanding of spatial concepts which is weakened in visually impaired individuals.

TECHNOLOGY AND SERIOUS GAMES

Games and play are an important part of the social and cognitive development of young children (Nicolopoulou, 1993). 'Serious games' are (digital) games with a purpose beyond pure entertainment. There are related, and sometimes overlapping domains, such as e-learning, edutainment, and game-based learning, but the goals of serious games go much further (Susi, Johannesson, & Backlund, 2007) and they can motivate learners in new ways (Prensky, 2005). Consideration of serious games is often limited to video games, played on a desktop computer, however this reduces the affordances available for multisensory learning. Digital technology has the potential to create new educational materials which exploit different sensory modalities, offering opportunities for new ways of thinking and processing information, and opening new avenues for creativity. The goals of most serious games are to facilitate learning higher order thinking skills through characteristics of gameplay. However, a serious game will not succeed just because it is a game with educational content. To be effective, instructional designers and video game designers need to understand how game characteristics such as competition and goals, rules, challenges, choices, and fantasy can influence motivation and facilitate learning (Charsky, 2010). Serious games have previously been applied to the learning of STEM subjects, but largely focused on teenage children, and as a result lack a developmental perspective (Berta, Bellotti, van der Spek, & Winkler, 2015; Ritterfeld, Cody, & Vorderer, 2009, Chapter 10,11). Renewed neuroscienti c understanding about how sensory modalities interact, and are integrated during development, need to be taken into account during game design. Educational research has also found that working in pairs or small groups can have beneficial effects on learning and development, particularly in early years and primary education (Benford et al., 2000), hence weDRAW games will aim to foster collaboration and interaction between children, as well as with the teacher, in the classroom.

RESEARCH PROGRESS

Working closely with primary school teachers, weDRAW makes use of observations of everyday classroom activity and practice, interviews and ongoing workshops to inform design requirements. In order to identify the most appropriate mathematical concepts to support through digital multi-sensory activities, the project team has collected data from teachers in UK and Italy, through teacher workshops and questionnaires (completed by over 100 teachers). The key areas of the primary mathematics curriculum that children find challenging, and where multimodal and multisensory engagement hold particular promise were found to include isometric transformations, symmetry, adding and multiplying fractions, measurement and estimation, and making the link between fractions, percentages and decimals. Interestingly, the concepts described as most challenging were not consistent across the levels or ages of children. This is partly because some concepts are not introduced to children until a certain age, and partly because the complexity of a concept increases as the children progress from year to year of the national curriculum (Department for Education, 2013). This has implications for the design of the games, to ensure that stretch and challenge is appropriate for all children in the target age range (6-10 years old), whilst providing an accessible entry point that develops knowledge through exploration (Price, Duffy, & Gori, 2017). For visually impaired students specifically, young children were described as finding it difficult to conceptualise arithmetic magnitude, in particular in the use of number lines where negative numbers were included. A geometric challenge was understanding the beginning and end of a shape; visually impaired students requiring reference points on a spatial and temporal continuum.

During teacher workshops, participants were supported to imagine what a future solution might look like (Rosson & Carroll, 2002), using participatory design techniques. These were then developed into design scenarios. Four workshops were undertaken with teachers in UK and Italy: three involved teachers from mainstream schools, and one involved teachers from a school for visually impaired children. Some common classroom activities described by the teachers in brainstorming sessions, such as constructing physical shapes using paper to demonstrate nets, or folding paper to explore symmetry, were thought to lend themselves more naturally to multimodal approaches than others, such as number lines. Colour was commonly recognised as a useful visual resource, but audio or tactile resources were perceived as less commonly used in the classroom. However, some activities described by the teachers, such as manipulating paper into 3D shapes, folding to find lines of symmetry, or using a trundle wheel suggest that there may be an unrecognised tactile or audio aspect that can be exploited by the weDRAW project.

CONCLUSION

Working with the concepts identified, we will encourage the creative capacities of children, as well as support the role of the teacher in the learning process. The suite of games created will be flexible and modular, allowing teachers to customise content to best suit their students' preferred mode of learning (i.e. audio, tactile, motor and visual), whilst unified by an overarching game story and narrative. A hardware and software platform will be developed to support this approach, and three serious games designed to evaluate it. The adoption of an embodied and enactive learning paradigm will allow motoric behaviour to be mapped onto the preferential sensory modality for typically developed children, or onto an alternative modality for impaired children. As a result, the same learning paradigm can be applied to all children interacting together in the classroom, reducing differences and social barriers.

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REFERENCES

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in Mathematics Teaching and Learning: Evidence From Learners' and Teachers' Gestures. *Journal of the Learning Sciences*, *21*(2), 247–286.

Barsalou, L. W. (2008). Grounded Cognition. *Annual Review of Psychology*, *59*, 617–645.

Benford, S., Bederson, B. B., Åkesson, K.-P., Bayon, V., Druin, A., Hansson, P., … Taxén, G. (2000). Designing storytelling technologies to encourage collaboration detween young children. In *Proceedings of the SIGCHI conference on Human factors in computing systems - CHI '00* (pp. 556–563). New York, New York, USA: ACM

Berta, R., Bellotti, F., van der Spek, E., & Winkler, T. (2015). A tangible serious game approach to science, technology, engineering, and mathematics (STEM) education. In *Handbook of Digital Games and Entertainment Technologies* (pp. 571–592).

Cappagli, G., & Gori, M. (2016). Auditory spatial localization: Developmental delay in children with visual impairments. *Research in Developmental Disabilities*, *53–54*, 391–398.

Charsky, D. (2010). From Edutainment to Serious Games: A Change in the Use of Game Characteristics. *Games and Culture*, *5*(2), 177–198. Department for Education. (2013). *Mathematics programmes of study: key stages 1 and 2 National curriculum in England*.

Finocchietti, S., Cappagli, G., & Gori, M. (2017). Auditory spatial recalibration in congenital blind individuals. *Frontiers in Neuroscience*, *11*(76), 1–7.

Goldin-Meadow, S., Wagner Cook, S., & Mitchell, Z. A. (2009). Gesturing Gives Children New Ideas About Math. *Psychological Science*, *20*(3), 267–272.

Gori, M., Del Viva, M., Sandini, G., & Burr, D. C. (2008). Young Children Do Not Integrate Visual and Haptic Form Information. *Current Biology*, *18*(9), 694–698.

Gori, M., Sandini, G., Martinoli, C., & Burr, D. C. (2014). Impairment of auditory spatial localization in congenitally blind human subjects. *Brain*, *137*(1), 288–293.

Habib, M., Lardy, C., Desiles, T., Commeiras, C., Chobert, J., & Besson, M. (2016). Music and Dyslexia: A New Musical Training Method to Improve Reading and Related Disorders. *Frontiers in Psychology*, *7*(26).

Hall, R., & Nemirovsky, R. (2012). Modalities of Body Engagement in Mathematical Activity and Learning. *Journal of the Learning Sciences*, *21*(2), 207–215.

Jones, K., & Tzekaki, M. (2016). Research on the Teaching and Learning of Geometry. In Ángel Gutiérrez, G. C. Leder, & P. Boero (Eds.), *The Second Handbook of Research on the Psychology of Mathematics Education* (pp. 109–149). Rotterdam: SensePublishers.

Klemmer, S. R., Hartmann, B., & Takayama, L. (2006). How Bodies Matter: Five Themes for Interaction Design. In *Proceedings of the 6th ACM conference on Designing Interactive systems - DIS '06* (pp. 140–149). Pennsylvania: ACM Press.

Lakoff, G., & Núñez, R. E. (2000). Preface. In *Where mathematics comes from: How the embodied mind brings mathematics into being* (pp. XI–XVII). New York: Basic Books.

Laurillard, D. (2012). *Teaching as a Design Science: Building Pedagogical patterns for learning and technology. London: Routledge.* Routledge.

Manches, A., & O'Malley, C. (2016). Cognition and Instruction The Effects of Physical Manipulatives on Children's Numerical Strategies. *Cognition and Instruction*, *34*(1), 27–50.

Moreno, R., & Mayer, R. E. (1999). Cognitive principles of multimedia learning: The role of modality and contiguity. *Journal of Educational Psychology*, *91*(2), 358–368.

Nicolopoulou, A. (1993). Play, cognitive development, and the social world: Piaget, vygotsky, and beyond. *Human Development*, *36*(1), 1–23.

Prensky, M. (2005). Computer games and learning: Digital Game-Based Learning. *Handbook of Computer Game Studies*, *18*, 97–122.

Price, S., Duffy, S., & Gori, M. (2017). Developing a Pedagogical Framework for Designing a Multisensory Serious Gaming Environment. In *1st ACM SIGCHI International Workshop on Multimodal Interaction* (p. In press). Glasgow: Association for Computing Machinery.

Ritterfeld, U., Cody, M., & Vorderer, P. (2009). *Serious games: Mechanisms and effects*. (Ute Ritterfeld, Michael Cody, & Peter Vorderer, Eds.). Routledge.

Roschelle, J., Pea, R., Hoadley, C., Gordin, D., & Means, B. (2001). Changing how and what children learn in school with collaborative cognitive technologies. *The Future of Children*, *10*(2), 76–101.

Rosson, M. B., & Carroll, J. M. (2002). Scenario-Based Design. In *The human-computer interaction handbook: fundamentals, evolving technologies and emerging applications* (pp. 1032–1050).

Shaun Gallagher. (2005). *How the Body Shapes the Mind*. Oxford: Clarendon Press.

Smith, L., & Gasser, M. (2005). The Development of Embodied Cognition: Six Lessons from Babies. *Artificial Life*, *11*(1–2), 13–29.

Susi, T., Johannesson, M., & Backlund, P. (2007). *Serious Games – An Overview*. *Technical Report HS-IKI-TR-07-001*.

Thinus-Blanc, C., & Gaunet, F. (1997). Representation of space in blind persons: vision as a spatial sense? *Psychological Bulletin*, *121*(1), 20–42.

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic Bulletin & Review*, *9*(4), 625–36.

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Chapter 9

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