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Direct derivation of the stochastic CRB of DOA estimation for rectilinear sources

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Abstract—Several direction of arrival (DOA) estimation algorithms have been proposed to exploit the structure of rectilinear or strictly second-order noncircular signals. But until now, only the compact closed-form expressions of the corresponding deterministic Cramér Rao bound (DCRB) have been derived because it is much easier to derive than the stochastic CRB (SCRB). As this latter bound is asymptotically achievable by the maximum likelihood (ML) estimator, while the DCRB is unattainable, it is important to have a compact closed-form expression for this SCRB to assess the performance of DOA estimation algorithms for rectilinear signals. The aim of this paper is to derive this expression directly from the Slepian-Bangs formula including in particular the case of prior knowledge of uncorrelated or coherent sources. Some properties of these SCRBs are proved and numerical illustrations are given.

Index Terms—Deterministic and stochastic Cramér Rao bound (CRB), direction of arrival (DOA), circular, noncircular, rectilinear, strictly noncircular, Slepian-Bangs formula.

I. INTRODUCTION

Various DOA estimation algorithms such as MUSIC [1], [2], root-MUSIC [3], standard ESPRIT [4] and unitary ESPRIT [5], [6] have been adapted to exploit the structure of rectilinearity or strictly second-order noncircularity of signals, which include commonly used digital modulation schemes such as BPSK and ASK. These algorithms are known to achieve a higher estimation accuracy and can resolve up to twice as many sources compared to the traditional DOA algorithms. To assess the performance of these algorithms, it is necessary to derive the SCRB for rectilinear sources. Nonetheless, only the SCRB for arbitrary noncircular sources [7], [8] and the DCRB for rectilinear sources [9]–[11] are available, among many other bounds (e.g., [12] and references therein). But the first bound does not take into account the prior knowledge of rectilinearity and the second bound, although providing valuable engineering insight is unattainable.

As generally the exploitation of prior knowledge usually reduces the estimation error, this paper derives closed-form expressions of the SCRB for arbitrary rectilinear sources and for the specific prior knowledge of uncorrelated, and fully correlated (referred to coherent) sources. Note that explicit expressions of circular and noncircular SCRBs for DOA parameter alone have been derived by two different methods for arbitrary sources. The first one consists of computing the asymptotic covariance matrix of the concentrated ML estimator [13], which is asymptotically efficient and the other one is obtained directly from the Slepian-Bangs formula [14], [15]. Similarly to [16], we present here a direct derivation of the different rectilinear SCRBs from the extended Slepian-Bangs formula [7] for noncircular Gaussian distributions. Finally some properties of these SCRBs are proved and numerical illustrations are given.

II. DATA MODEL AND PROBLEM FORMULATION

Consider $K$ zero-mean narrowband signals ($x_{t,k}$)_{k=1,...,K} impinging on an arbitrary array of $M$ sensors. These signals are supposed rectilinear (also called strictly second-order noncircular), i.e., described by the following model:

$$x_{t,k} = t_{t,k} e^{i \phi_k} \text{ with } s_{t,k} \text{ real-valued},$$

where the phases $\phi_k$ associated with different propagation delays are assumed fixed, but unknown during the array observation. The array output at time $t$ is modeled as

$$y_t = A_\theta \Delta_s s_t + n_t, \quad t = 1, \ldots, T,$$

where $(y_t)_{t=1,...,T}$ are independent. $A_\theta \overset{\text{def}}{=} [a(\theta_1), ..., a(\theta_K)]$ denotes the conventional steering matrix, $\Delta_s \overset{\text{def}}{=} \text{Diag}(e^{i \phi_1}, ..., e^{i \phi_K})$ and $s_t \overset{\text{def}}{=} (s_{t,1}, ..., s_{t,K})^T$. $n_t$ is the additive noise, which is assumed zero-mean circular complex Gaussian, spatially uncorrelated with $E(n_t n_t^H) = \sigma_n^2 I$ and independent from $s_{t,k}$. $(s_{t,k})_{k=1,...,K,t=1,...,T}$ are either real-valued deterministic unknown parameters (in the so-called conditional or deterministic model), or zero-mean real-valued Gaussian distributed with covariance $E(s_t s_t^H) = R_s$ (in the so-called unconditional or stochastic model).

To derive the CRB from the Slepian-Bangs formula, we have to carefully specify the parameters of the Gaussian distribution of $(y_t)_{t=1,...,T}$. Under the deterministic assumption, $y_t$ are circularly Gaussian distributed with mean $(A_\theta \Delta_s s_t)_{t=1,...,T}$ and covariance $\sigma_n^2 I$, which are parameterized by the real-valued parameter:

$$\alpha = (\theta^T, \phi^T, \rho^T, \sigma_n^2)^T.$$
where \( \theta \) is usually parameterized by (3), but where \( \rho \) is now the \( K(K+1)/2 \) vector made from \( [\mathbf{R}_s]_{:i} \) for \( 1 \leq i \leq K \). In the particular case, where prior knowledge of uncorrelated or full coherent sources \( s_{i,k} \) are incorporated, the parameter \( \rho \) reduces to \( \rho = (\sigma_1^2, \ldots, \sigma_K^2)^T \) where \( \sigma_k^2 \) is the covariance of the extended signal \( \tilde{y}_{i,k} \), and \( \mathbf{R}_s \) is the covariance of \( \tilde{y}_{i,k} \) and \( \mathbf{R}_r = \mathbf{cc}^T \), respectively.

Under the stochastic assumption, \( (y_i)_{i=1,\ldots,T} \) are independent and noncircular Gaussian distributed and therefore the Fisher information matrix (FIM) for the parameter \( \alpha \) is given (elementwise) by [7]:

\[
\text{FIM}_{i,j} = \frac{T}{2} \text{Tr} \left[ \frac{\partial \mathbf{R}_s}{\partial \phi_i} \mathbf{R}_y^{-1} \frac{\partial \mathbf{R}_r}{\partial \phi_j} \mathbf{R}_y^{-1} \right],
\]

where \( \mathbf{R}_y \) is the covariance of the extended signal \( \tilde{y}_i = [y_i^T, \tilde{y}_i^T]^T \) given by:

\[
\mathbf{R}_y = \mathbb{E}(\tilde{y}_i \tilde{y}_i^T) = \mathbf{A} \mathbf{R}_s \mathbf{A}^H + \sigma_n^2 \mathbf{I},
\]

where \( \mathbf{A} \) is a matrix whose columns are \( \mathbf{a}_k = [a_{1,k}, \ldots, a_{K,k}]^T \) with \( a_{k} \) defined as:

\[
[\tilde{y}_{i,k}] = [\tilde{y}_{i,k}] = [a_{1,k}, \ldots, a_{K,k}]^T.
\]

The purpose of the next section is to directly derive the SCRB of the parameter \( \theta \) alone from the FIM (6). Noting that the parameters \( \theta_k \) and \( \phi_k \) are non-linearly related in the extended steering vector \( \mathbf{a}_k \), closed-form expressions of the SCRB of the couple \( \mathbf{w} = [\theta^T, \phi^T]^T \) are first derived through its inverse \( \text{CRB}^{\text{sto}}(\mathbf{w})^{-1} \) defined as:

\[
\begin{align*}
\text{CRB}^{\text{sto}}(\theta) &= \begin{pmatrix} I_{\theta,\theta} & I_{\theta,\phi} \\ I_{\theta,\phi}^T & I_{\theta,\phi} \end{pmatrix}^{-1}.
\end{align*}
\]

Thus the SCRB of \( \theta \) alone is deduced by:

\[
\text{CRB}^{\text{sto}}(\theta) = \begin{pmatrix} I_{\theta,\theta} - I_{\theta,\phi} I_{\phi,\phi}^{-1} I_{\phi,\theta}^T \end{pmatrix}^{-1}.
\]

III. DERIVATION OF THE DIFFERENT CURB

Writing the FIM (6) in compact matrix form as:

\[
\text{FIM} = \frac{T}{2} \left( \frac{\partial \mathbf{R}_y}{\partial \theta} \right)^H \left( \mathbf{R}_y^{-1} \frac{\partial \mathbf{R}_y}{\partial \theta} \right)^{-1},
\]

where \( \mathbf{R}_y \) is the covariance of \( \tilde{y}_i = [\tilde{y}_{i,k}] \) and \( \sigma_n^2 \) is the variance of \( \mathbf{R}_s \), the following expression is deduced:

\[
\text{CRB}^{\text{sto}}(\mathbf{w})^{-1} = \mathbf{G}^H \Pi_\Delta \mathbf{G}.
\]
sources for \( K < M \):
\[
\text{CRB}_{\text{sto}}^{\text{rec}}(\theta) = \frac{\sigma^2}{2T} \left( \text{Re}\left[ (D_\theta^T \Pi_{\tilde{\lambda}_\theta} D_\theta) (a_1^2, \ldots, a_K^2) \right] \right),
\]
where \( D_\theta \) is defined as the matrix formed by the elements \( \frac{\partial a_k}{\partial \theta_1}, \ldots, \frac{\partial a_K}{\partial \theta_1} \).

### IV. ANALYTICAL AND NUMERICAL COMPARISONS

Considering the comparison of the previously introduced closed-form expressions of the CRB, the main steps of the proof of the following theorem are given in the Appendix:

**Theorem 4:** Under the general rectilinear assumption, the DCRB (for \( T \to \infty \), i.e., replacing \( R_s T \) by \( R_s ) and SCR have the relationships:
\[
\text{CRB}_{\text{sto}}^{\text{rec}}(\theta) \leq \text{CRB}_{\text{sto}}^{\text{reci}}(\theta) \leq \text{CRB}_{\text{sto}}^{\text{nci}}(\theta).
\]
Note that for a finite value of \( T \), we cannot be sure that \( \text{CRB}_{\text{sto}}^{\text{rec}}(\theta) \leq \text{CRB}_{\text{sto}}^{\text{reci}}(\theta) \). In fact, for low \( T \) and high SNRs, the inequality reverses.

If we consider now the exploitation of prior knowledge, the following theorem is proved in the Appendix:

**Theorem 5:** Under the prior knowledge that the sources are rectilinearly correlated, \( \text{CRB}_{\text{sto}}^{\text{reci}}(\theta) \) is reduced w.r.t. \( \text{CRB}_{\text{sto}}^{\text{nci}}(\theta) \). In contrast, the exploitation of fully coherency of the sources does not reduce \( \text{CRB}_{\text{sto}}^{\text{nci}}(\theta) \):
\[
\text{CRB}_{\text{sto}}^{\text{rec}}(\theta) \leq \text{CRB}_{\text{sto}}^{\text{reci}}(\theta), \quad \text{CRB}_{\text{sto}}^{\text{nci}}(\theta) = \text{CRB}_{\text{sto}}^{\text{nci}}(\theta).
\]
Note that similar properties have been proved for circular sources in [17] for uncorrelated sources and in [18] for fully coherent sources.

Finally, in the case of a single rectilinear source, we have proved after tedious algebraic manipulations that the SCR of \( \theta_1 \) alone deduced from (11), (13) and (14) reduce to:
\[
\text{CRB}_{\text{sto}}^{\text{nci}}(\theta_1) = \frac{1}{2Ta_1^2} \frac{\sigma^2}{\sigma^2} \left( 1 + \frac{\sigma^2}{2\sigma^2 ||a_1||^2} \right),
\]
where \( a_1 = a(\theta_1) \) and \( a' = \frac{\partial a(\theta_1)}{\partial \theta_1} \). Comparing (19) to the SCR derived in [7] under the general noncircular assumption, we see that \( \text{CRB}_{\text{sto}}^{\text{nci}}(\theta_1) = \text{CRB}_{\text{sto}}^{\text{nci}}(\theta_1) \), i.e., the SCR is not reduced by exploiting the rectilinear prior knowledge.

To illustrate the difference between the different CRBs, we consider now the case of two equal-power rectilinear sources of signal-to-noise ratio \( 10 \log_{10}(\sigma^2/\sigma^2) = 10 \) dB and correlation \( \rho \), impinging on an ULA of \( M = 6 \) sensors with half-wavelength spacing. Figs 1-3 exhibit different ratios of CRB. Fig.1 shows that there are significant gaps between the DSCRB and SCR for closely spaced and strongly correlated rectilinear sources. More generally, extensive numerical comparisons have shown that this gap increases for low SNRs, low phase and DOA separations and high source correlation, but this gap will always vanish for high SNR. This proves that the conclusions based on the DCRB may be very optimistic for not too high SNR. Fig. 2 highlights that the exploitation of the prior of rectilinearity greatly reduces the estimation error for closely spaced and uncorrelated sources. Finally, Fig. 3 proves that the joint exploitation of the prior of uncorrelatedness and rectilinearity greatly reduces the estimation error for closely spaced sources with different phases.

### V. APPENDIX

**Proof of theorem 1:** (Detailed proofs of Theorem 1 are available at [23]) Since \( R_s \) is a \( (K \times K) \) real symmetric matrix, it then follows from [19, rel.(7.18)] that \( \text{vec}(R_s) = D_K \rho \)
where \( D_K \) is a so-called duplication matrix, and hence [16, rel.(19)] becomes

\[
V = (R_{\theta}^{-T/2}A^* R_{\theta}^{-1/2}A) D_K \overset{\text{def}}{=} WD_K.
\]

Then it follows from [19, Theorem 7.38], and some simple algebraic manipulations using [19, Theorem 7.34, rel.(b)] that [16, rel.(20)] becomes

\[
\Pi_V = I - W(U \cup U) N_K W^H,
\]

where \( U \overset{\text{def}}{=} \bar{A}^H R_{\bar{A}}^{-1} \bar{A} \) and \( N_K \) is an \((K \times K)\) matrix defined in [19, Theorem 7.34]. By evaluating the derivatives in \( G \) and \( u \), and through some further algebra, one finds

\[
u^H \Pi_V g_k = 0,
\]

where \( g_k \) is the \( k \)th column of \( G \) given by \( g_k = \text{vec}(Z_k + \tilde{Z}_k^H) \) and where \( Z_k \overset{\text{def}}{=} R_{\theta}^{-1/2} \bar{A} r_k \tilde{A}^H R_{\theta}^{-1/2}, \tilde{a}_k \overset{\text{def}}{=} \partial a_k / \partial u_k \) and \( r_k \) is the \( k \)th column of \( r \). This identity allows us to rewrite the individual elements of (10) as

\[
\frac{2}{T} \left[ \text{CRB}_{\text{sto}}(\omega) \right]_{k,i} = g^H_k \Pi_V g_i = \frac{2\kappa}{\sigma^2} \Omega_k \Pi_V \Omega_k.
\]

Finally, we can write (20) in matrix form as in (11).

**Proof of theorem 2:** (Detailed proofs of Theorem 1 are available at [23]) We follow the steps similar to those in the proof of Theorem 1. Since \( R_{\bar{A}} = \bar{c} \bar{c}^T \) and its derivative w.r.t. \( c \) is given by \( D_c \overset{\text{def}}{=} c \in I + I \otimes c = 2N_K(c \otimes I) \), it follows that \( V \) has the form \( V = WD_c \). After some algebraic manipulation using [19, Theorem 7.34, rel.(d)], it follows that

\[
\Pi_V \overset{\text{def}}{=} I - V(V^H V)^{-1} V^H = I - V_1 V_1^H,
\]

with \( V_1 \overset{\text{def}}{=} WN_K(c \otimes I) \) and \( V \overset{\text{def}}{=} \frac{1}{2}(\kappa_c U + u_c u_c^T) \) where \( u_c \) and \( \kappa_c \overset{\text{def}}{=} c^T U c \). Thanks to the matrix inversion lemma, we have \( V^{-1} = \frac{1}{\kappa_c} (U^H - \frac{1}{2\kappa_c} U c^T) \).

Through some further algebra using \( u = \text{vec}(Z_k^{-1}) \) and \( g_k = \text{vec}(Z_k + \tilde{Z}_k^H) \) where \( Z_k \overset{\text{def}}{=} R_{\theta}^{-1/2} \bar{A} C_k \bar{A}^H R_{\theta}^{-1/2} \), one finds that \( u^H V_1 = c^T \tilde{U} \) where \( \tilde{U} \overset{\text{def}}{=} \bar{A}^H R_{\bar{A}}^{-2} \bar{A} \) and \( g_k \) is the \( k \)th entry of \( \bar{A}^H R_{\bar{A}}^{-2} \tilde{A} \), and \( V_1^H g_k \) is the \( k \)th entry of \( \bar{A}^H R_{\bar{A}}^{-1} \tilde{A} \), where \( \bar{a} \) is the \( \bar{a}^H \) of \( \bar{A} \). By further calculations we arrive at \( u^H \Pi_V g_k = 0 \). This identity allows us to rewrite the individual elements of (10) as

\[
\frac{2}{T} \left[ \text{CRB}_{\text{sto}}(\omega) \right]_{k,i} = g^H_k \Pi_V g_i = 2\kappa \Omega_k \Pi_V \Omega_k,
\]

which can also be written in the matrix form (13).

**Proof of theorem 3:** Noting that (7) becomes \( R_{\theta} = \sum_{k=1}^K \kappa^2 \tilde{a} a_k^H + \sigma^2 \), all the steps of the proof given in [17, Appendix A] apply to the parameter \( \omega \) with the FIM (6) associated with the noncircular zero-mean Gaussian distribution of \( \psi \).
REFERENCES