Sobolev multipliers, maximal functions and parabolic equations with a quadratic nonlinearity

Pierre Gilles Lemarié-Rieusset

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Abstract

We develop a general framework to describe global mild solutions to a Cauchy problem with small initial values concerning a general class of semilinear parabolic equations with a quadratic nonlinearity. This class includes the Navier–Stokes equations, the subcritical dissipative quasi-geostrophic equation and the parabolic–elliptic Keller-Segel system.

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1 Presentation of the results.

In this paper, we shall study parabolic semi-linear equations on $(0, +\infty) \times \mathbb{R}^n$ of the type :

$$\partial_t u + (-\Delta)^{\alpha/2} u = (-\Delta)^{\beta/2} u^2 \tag{1}$$

with $0 < \alpha < n + 2\beta$ and $0 < \beta < \alpha$.

More generally, we consider the following Cauchy problem : given $\vec{u}_0 \in (\mathcal{S}'(\mathbb{R}^n))^d$, find a vector distribution $\vec{u}$ on $(0, +\infty) \times \mathbb{R}^n$ (or on $(0, T) \times \mathbb{R}^n$) such that, for $i = 1, \ldots, d$, we have

$$\partial_t u_i = -(-\Delta)^{\alpha/2} u_i + \sum_{j=1}^{d} \sum_{k=1}^{d} \sigma_{i,j,k}(D)(u_j u_k) \tag{2}$$

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and

$$\lim_{t\to 0} u_i(t,x) = u_{i,0}. \tag{3}$$

We assume that $\sigma_{i,j,k}(D)$ is a homogeneous pseudo-differential operator of degree β with $0<\beta<\alpha<n+2\beta$: for $f\in\mathcal{S}(\mathbb{R}^n)$ with Fourier transform $\mathcal{F}f$, we have :

$$\sigma_{i,j,k}(D)f = \mathcal{F}^{-1}\big(\sigma_{i,j,k}(\xi)\mathcal{F}f(\xi)\big) \tag{4}$$

where $\sigma_{i,j,k}$ is a smooth (positively) homogeneous function of degree β on $\mathbb{R}^n-\{0\}$:

$$\text{for } \lambda>0 \text{ and } \xi\neq 0, \quad \sigma_{i,j,k}(\lambda\xi)=\lambda^\beta\sigma_{i,j,k}(\xi). \tag{5}$$

We rewrite equation (2) in a vectorial form :

$$\partial_t\vec{u} = -(-\Delta)^{\alpha/2}\vec{u} + \sigma(D)(\vec{u}\otimes\vec{u}) \tag{6}$$

and use Duhamel's formula to transform the problem into an integral problem :

$$\vec{u} = e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{u}\otimes\vec{u})\, ds. \tag{7}$$

We shall use the classical estimate :

Lemma 1 *There exists a constant C_0 (depending on σ) such that, for two functions $\vec{u}$ and $\vec{v}$ on $\mathbb{R}^n$ (with values in $\mathbb{R}^d$) we have*

$$|e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{u}\otimes\vec{v})| \leq C_0\int_{\mathbb{R}^n}\frac{|\vec{u}(y)||\vec{v}(y)|}{(|t-s|^{1/\alpha}+|x-y|)^{n+\beta}}\, dy. \tag{8}$$

Due to homogeneity, this lemma is a direct consequence of Lemma 7 which will be proved in Appendix A.

The core of the paper is the discussion of the equation

$$U(t,x) = U_0(t,x) + C_0\iint_{\mathbb{R}\times\mathbb{R}^n} K_{\alpha,\beta}(t-s,x-y)\, U^2(s,y)\, ds\, dy \tag{9}$$

with

$$K_{\alpha,\beta}(t,x) = \frac{1}{(|t|^{1/\alpha}+|x|)^{n+\beta}} \tag{10}$$

and $U_0\geq 0$.

The link with our initial problem is easy to see. Indeed, due to Lemma 1, we have the following domination principle :

Theorem 1 *If there exists a function $W(t,s)$ such that*

$$\iint_{\mathbb{R}\times\mathbb{R}^n} \frac{1}{\Big(|t-s|^{\frac{1}{\alpha}}+|x-y|\Big)^{n+\beta}} W^2(s,y)\ ds\ dy \leq W(t,x) \tag{11}$$

and such that, for some $T\in(0,+\infty]$ we have

$$1_{0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0| \leq \frac{1}{4C_0}W \tag{12}$$

[where C_0 is the constant given in Lemma 1], then defining inductively $\vec{U}_k$ on $(0,T)\times\mathbb{R}^n$ and W_k on $\mathbb{R}\times\mathbb{R}^n$ as

- $\vec{U}_0(t,x) = e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0$ *for* $0<t<T$
- $W_0(t,x) = 1_{0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0|$
- $\vec{U}_{k+1} = \vec{U}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{U}_k\otimes\vec{U}_k)\ ds$
- $W_{k+1} = W_0 + \iint_{\mathbb{R}\times\mathbb{R}^n} \frac{C_0}{\Big(|t-s|^{\frac{1}{\alpha}}+|x-y|\Big)^{n+\beta}} W_k(s,y)^2\ ds\ dy$

we have the following results :

- *W_k converges monotonically to a function W_∞ such that $W_\infty \leq \frac{1}{2C_0}W$ and*

$$W_\infty = W_0 + \iint_{\mathbb{R}\times\mathbb{R}^n} \frac{C_0}{\Big(|t-s|^{\frac{1}{\alpha}}+|x-y|\Big)^{n+\beta}} W_\infty(s,y)^2\ ds\ dy \tag{13}$$

- $|\vec{U}_0| \leq W_0$ *and* $|\vec{U}_{k+1}-\vec{U}_k| \leq W_{k+1}-W_k$ *on* $(0,T)\times\mathbb{R}^n$
- *the sequence $(\vec{U}_k(t,x))_{k\in\mathbb{N}}$ converges pointwise to a solution $\vec{U}_\infty$ of*

$$\vec{U}_\infty = \vec{U}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{U}_\infty\otimes\vec{U}_\infty)\ ds. \tag{14}$$

Proof : Just use monotone convergence for W_k and dominated convergence for $\vec{U}_k$. ◇

The aim of this paper is to describe the class of initial values $\vec{u}_0$ to which this domination principle can be applied. To this end, we shall introduce the following sets of Lebesgue measurable functions on $\mathbb{R}\times\mathbb{R}^n$ and of distributions on $\mathbb{R}^n$:

Definition 1 *$\mathcal{C}_{\alpha,\beta}$ is the set of non-negative measurable functions W on $\mathbb{R}\times\mathbb{R}^n$ such that $W<+\infty$ (almost everywhere) and*

$$\iint_{\mathbb{R}\times\mathbb{R}^n} \frac{1}{\left(|t-s|^{\frac{1}{\alpha}}+|x-y|\right)^{n+\beta}} W^2(s,y)\, ds\, dy \le W(t,x) \tag{15}$$

Definition 2 *$\mathcal{V}^{\alpha,\beta}$ is the space of [classes of] measurable functions f on $\mathbb{R}\times\mathbb{R}^n$ such that there exists $\lambda\ge 0$ and $\Omega\in\mathcal{C}_{\alpha,\beta}$ such that $|f(x)|\le\lambda\Omega$ almost everywhere.*

Proposition 1 *The function $f\in\mathcal{V}^{\alpha,\beta}\mapsto \|f\|_{\mathcal{V}^{\alpha,\beta}}=\inf\{\lambda\ /\ \exists\Omega\in\mathcal{C}_{\alpha,\beta}\ |f|\le\lambda\Omega\}$ is a norm on $\mathcal{V}^{\alpha,\beta}$. Moreover, $\mathcal{V}^{\alpha,\beta}$ is a Banach space for this norm.*

This proposition is proved in Appendix B (Proposition 9).

Definition 3 *Let $T\in(0,+\infty]$. The space $X_{\alpha,\beta,T}(\mathbb{R}^n)$ is defined as the space of tempered distributions f such that $f\in B^{\beta-\alpha}_{\infty,\infty}$ (if $T<+\infty$) [or $f\in\dot B^{\beta-\alpha}_{\infty,\infty}$ if $T<+\infty$] and $1_{0<t<T}\,e^{-t(-\Delta)^{\alpha/2}}f\in\mathcal{V}^{\alpha,\beta}$. It is normed by $\|f\|_{X_{\alpha,\beta,T}(\mathbb{R}^n)}=\|1_{t0<t<T}\,e^{-t(-\Delta)^{\alpha/2}}f\|_{\mathcal{V}^{\alpha,\beta}}+\sup_{t0<t<T}t^{1-\frac{\beta}{\alpha}}\|e^{-t(-\Delta)^{\alpha/2}}f\|_\infty$.*

Proposition 2 *$X_{\alpha,\beta,T}(\mathbb{R}^n)$ is a Banach space. Moreover, if $T_1<+\infty$ and $T_2<+\infty$, then $X_{\alpha,\beta,T_1}(\mathbb{R}^n)=X_{\alpha,\beta,T_2}(\mathbb{R}^n)$ with equivalent norms.*

This proposition will be proved in Section 4. Theorem 1 may then be rewritten as :

Theorem 2 *If $\vec u_0\in X_{\alpha,\beta,T}(\mathbb{R}^n)$ and*

$$\|1_{0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec u_0|\|_{\mathcal{V}^{\alpha,\beta}}<\frac{1}{4C_0} \tag{16}$$

(where C_0 is the constant in Lemma 1), then the equation

$$\vec u=e^{-t(-\Delta)^{\alpha/2}}\vec u_0+\int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec u\otimes\vec u)\,ds. \tag{17}$$

has a solution $\vec u$ on $(0,T)\times\mathbb{R}^n$ such that $1_{0<t<T}\vec u\in(\mathcal{V}^{\alpha,\beta})^d$.

Theorem 1 is essentially obvious, and thus so is its restatement as Theorem 2. Thus, $X_{\alpha,\beta,T}$ appears as the maximal space where trivial arguments exhibit solutions for our equations. However, neither $\mathcal{V}^{\alpha,\beta}$ nor $X_{\alpha,\beta,T}$ are trivial spaces. The paper will thus be devoted to give easy criteria to check whether $\vec u_0$ belongs to $X_{\alpha,\beta,T}$. These criteria will be described in the following section, using the theory of parabolic Morrey spaces.

2 $\mathcal{V}^{\alpha,\beta}$ as a space of multipliers.

We shall use the parabolic (quasi)-distance

$$\delta_\alpha((t,x),(s,y)) = |t-s|^{1/\alpha} + |x-y| \tag{18}$$

on $\mathbb{R}\times\mathbb{R}^n$. The space $\mathbb{R}\times\mathbb{R}^n$ endowed with this metric and with the Lebesgue measure $dt\,dx$ is a space of homogeneous type, as described by Coifman and Weiss [7]. Its associated homogeneous dimension is $Q = n+\alpha$.

We may write the kernel $K_{\alpha,\beta}$ as

$$K_{\alpha,\beta}(t-s,x-y) = \frac{1}{\delta_\alpha(t-s,x-y)^{Q-(\alpha-\beta)}}. \tag{19}$$

This leads to the following definition :

Definition 4 *$\mathcal{I}_{\alpha,\alpha-\beta}$ is the Riesz potential operator asociated to $K_{\alpha,\beta}$:*

$$\mathcal{I}_{\alpha,\alpha-\beta} f(x) = \iint_{\mathbb{R}\times\mathbb{R}^n} K_\alpha(t-s,x-y) f(s,y)\ ds\,dy. \tag{20}$$

and $W^{\alpha,\beta}$ is the potential space defined by

$$g \in W^{\alpha,\beta} \Leftrightarrow \exists h \in L^2\ g = \mathcal{I}_{\alpha,\alpha-\beta} h. \tag{21}$$

A direct application of Kalton and Verbitsky's theorem [[14], Theorem 5.7] (see Appendix B, Theorems 9 and 10) on quadratic equations with symmetric kernels then gives the following characterization of $\mathcal{V}^{\alpha,\beta}$:

Theorem 3 *Let $0<\beta<\alpha<n+2\beta$. $\mathcal{V}^{\alpha,\beta}$ is the space $\mathcal{M}(W^{\alpha,\beta}\mapsto L^2)$ of pointwise multipliers from $W^{\alpha,\beta}$ to $L^2(\mathbb{R}\times\mathbb{R}^n)$, and the norm of $\mathcal{V}^{\alpha,\beta}$ is equivalent to the norm*

$$\|f\|_{\mathcal{M}(W^{\alpha,\beta}\mapsto L^2)} = \sup_{\|g\|_{W^{\alpha,\beta}}\le 1} \|fg\|_2.$$

This space of multipliers is not easy to handle (it can be characterized through capacitary inequalities, see [23] for the Euclidean case). Instead, we will use some spaces that are very close to $\mathcal{V}^{\alpha,\beta}$: the (homogeneous) Morrey–Campanato spaces.

Definition 5 *The (homogeneous) Morrey–Campanato space $\dot{M}^{p,q}_\alpha(\mathbb{R}\times\mathbb{R}^n)$ $(1<p\le q<+\infty)$ is the space of functions that are locally L^p and satisfy*

$$\|f\|_{\dot{M}^{p,q}_\alpha} = \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^n}\ \sup_{R>0} R^{Q(\frac{1}{q}-\frac{1}{p})}\Big(\int_{B((t,x),R)} |f(s,y)|^p\ ds\,dy\Big)^{1/p} < +\infty \tag{22}$$

where $B(x,R) = \{(s,y)\in\mathbb{R}\times\mathbb{R}^n\ /\ \delta_\alpha((t,x),(s,y))<R\}$.

We have the following generalization of the Fefferman–Phong inequality [9] (proved in Appendix C, Theorem 11) :

Theorem 4 *Let $0 < \beta < \alpha < n + 2\beta$ and $2 < p \le \frac{n+\alpha}{\alpha-\beta}$. Then we have :*

$$\dot{M}_{\alpha}^{p,\frac{n+\alpha}{\alpha-\beta}} \subset \mathcal{V}^{\alpha,\beta} = \mathcal{M}(W^{\alpha,\beta} \mapsto L^2) \subset \dot{M}_{\alpha}^{2,\frac{n+\alpha}{\alpha-\beta}}. \tag{23}$$

3 The case $\beta < 2$.

We now give another characterization of $\mathcal{V}^{\alpha,\beta}(\mathbb{R} \times \mathbb{R}^n)$:

Theorem 5 *If $\beta < 2$, we define $\mathbb{W}^{\alpha,\beta}(\mathbb{R} \times \mathbb{R}^n)$ as the Banach space of tempered distributions such that their Fourier transforms $\hat{f}$ are locally integrable and satisfy*

$$\int\int (|\xi|^{\alpha-\beta} + |\tau|^{1-\frac{\beta}{\alpha}})^2 |\hat{f}(\tau,\xi)|^2 \, d\xi \, d\tau < +\infty. \tag{24}$$

Equivalently, we have : $\mathbb{W}^{\alpha,\beta}(\mathbb{R} \times \mathbb{R}^n) = L^2_t \dot{H}^{\alpha-\beta}_x \cap L^2_x \dot{H}^{1-\frac{\beta}{\alpha}}_t$.

$\mathcal{V}^{\alpha,\beta}$ is the space $\mathcal{M}(\mathbb{W}^{\alpha,\beta} \mapsto L^2)$ of pointwise multipliers from $\mathbb{W}^{\alpha,\beta}$ to $L^2(\mathbb{R} \times \mathbb{R}^n)$, and the norm of $\mathcal{V}^{\alpha,\beta}$ is equivalent to the norm

$$\|f\|_{\mathcal{M}(\mathbb{W}^{\alpha,\beta} \mapsto L^2)} = \sup_{\|g\|_{\mathbb{W}^{\alpha,\beta}} \le 1} \|fg\|_2.$$

To prove this theorem, we shall use the theory of γ-stable processes on $\mathbb{R}^p$ for the cases $p = n$ and $\gamma = \beta$, and $p = 1$ and $\gamma = \frac{\alpha-\beta}{\alpha}$.

Let $W_{\gamma,p}(x)$ be defined, for $p \in \mathbb{N}^*$ and $0 < \gamma \le 2$, as

$$W_{\gamma,p}(x) = \frac{1}{(2\pi)^p} \int_{\mathbb{R}^p} e^{-|\xi|^\gamma} e^{i\,x.\xi} \, d\xi. \tag{25}$$

When $\gamma = 2$, we get the Gaussian function

$$W_{2,p}(x) = \frac{1}{(4\pi)^{p/2}} e^{-\frac{|x|^2}{4}}. \tag{26}$$

When $0 < \gamma < 2$, we have a subordination of $W_{\gamma,p}$ to $W_{2,p}$:

$$W_{\gamma,p}(x) = \int_0^{+\infty} \frac{1}{\sigma^{p/2}} W_{2,p}(\frac{x}{\sqrt{\sigma}}) \, d\mu_\gamma(\sigma) \tag{27}$$

where $d\mu_\gamma$ is a probability measure on $(0, +\infty)$[25].

We have the following important result of Blumenthal and Getoor [3] :

Lemma 2 *Let $0 < \gamma < 2$. There exists a positive constant $c_{\gamma,p}$ such that*

$$\lim_{|x|\to+\infty} W_{\gamma,p}(x)|x|^{p+\gamma} = c_{\gamma,p}. \tag{28}$$

Thus, we have

$$e^{-t(-\Delta)^{\gamma/2}} f = \int_{\mathbb{R}^p} \frac{1}{t^{\frac{p}{\gamma}}} W_{\gamma,p}(\frac{y}{t^{\frac{1}{\gamma}}}) f(x-y)\, dy \tag{29}$$

with

$$\frac{1}{t^{\frac{p}{\gamma}}} W_{\gamma,p}(\frac{y}{t^{\frac{1}{\gamma}}}) \approx \frac{t}{(t^{\frac{1}{\gamma}} + |y|)^{p+\gamma}} \tag{30}$$

(where the notation $F \approx G$ stands for the existence of two positive constants c_1 and c_2 such that $c_1 < F/G < c_2$).

In order to prove Theorem 5, let us remark that equation (9) involves a convolution on $\mathbb{R}\times\mathbb{R}^n$ with $K_{\alpha,\beta}$. It will be interesting to give an approximate Fourier transform of the convolution kernel $K_{\alpha,\beta}$.

Proposition 3 *Let $0 < \beta < \min(\alpha, 2)$. Let $\mathbb{K}_{\alpha,\beta}(t,x)$ be defined on $\mathbb{R}\times\mathbb{R}^n$ as*

$$\mathbb{K}_{\alpha,\beta}(t,x) = \frac{1}{|t|^{\frac{n+\beta}{\alpha}}} W_{\beta,n}\left(\frac{x}{|t|^{\frac{1}{\alpha}}}\right). \tag{31}$$

Then :

$$\mathbb{K}_{\alpha,\beta}(t,x) \approx K_{\alpha,\beta}(t,x). \tag{32}$$

Let $M_{\alpha,\beta}(\tau,\xi)$ be the Fourier transform of $\mathbb{K}_{\alpha,\beta}(t,x)$. Then

$$M_{\alpha,\beta}(\tau,\xi) \approx \frac{1}{|\xi|^{\alpha-\beta} + |\tau|^{1-\frac{\beta}{\alpha}}}. \tag{33}$$

Proof : Inequality (32) is a direct consequence of (30) with $\gamma = \beta$ and $p = n$. We then compute the Fourier transform $M_{\alpha,\beta}(\tau,\xi)$ as the Fourier transform in the time variable t of the Fourier transform $N(t,\xi)$ in the space variable x of $\mathbb{K}_{\alpha,\beta}$. We have

$$N(t,\xi) = \frac{1}{|t|^{\frac{\beta}{\alpha}}} e^{-|t|^{\frac{\beta}{\alpha}}|\xi|^{\beta}} \tag{34}$$

so that

$$M_{\alpha,\beta}(\tau,\xi) = C \int_{\mathbb{R}} \frac{1}{|\tau-\eta|^{1-\frac{\beta}{\alpha}}} \frac{1}{|\xi|^{\alpha}} W_{\frac{\beta}{\alpha},1}\left(\frac{\eta}{|\xi|^{\alpha}}\right)\, d\eta \tag{35}$$

Thus, we have

$$M_{\alpha,\beta}(\tau,\xi) \approx \int_{\mathbb{R}} \frac{1}{|\tau-\eta|^{1-\frac{\beta}{\alpha}}} \frac{|\xi|^\beta}{(|\xi|^\alpha+|\eta|)^{1+\frac{\beta}{\alpha}}}\, d\eta. \tag{36}$$

We may rewrite this estimate as

$$M_{\alpha,\beta}(\tau,\xi) \approx \frac{1}{|\xi|^{\alpha-\beta}} A_{\alpha,\beta}(\frac{\tau}{|\xi|^\alpha}) \tag{37}$$

with

$$A_{\alpha,\beta}(\tau) = \int_{\mathbb{R}} \frac{1}{|\tau-\eta|^{1-\frac{\beta}{\alpha}}} \frac{1}{(1+|\eta|)^{1+\frac{\beta}{\alpha}}}\, d\eta. \tag{38}$$

Let $G(\tau) = \frac{1}{|\tau|^{1-\frac{\beta}{\alpha}}}$ and $H(\tau) = \frac{1}{(1+|\tau|)^{1+\frac{\beta}{\alpha}}}$, so that $A_{\alpha,\beta} = G*H$. Since $G \in L^1 + L^\infty(\mathbb{R})$ and $H \in L^1 \cap L^\infty(\mathbb{R})$, we have that $G*H$ is continuous, positive and bounded, so that we have : for $|\tau| \leq 2$, $A_{\alpha,\beta}(\tau) \approx \Omega(1)$. For $|\tau| > 2$, we write :

- $G*H(\tau) \geq \left(\frac{2}{|\tau|}\right)^{1-\frac{\beta}{\alpha}} \int_{-1}^1 H(\eta)\, d\eta$
- $\int_{-|\tau|/2}^{|\tau|/2} G(\tau-\eta)H(\eta)\, d\eta \leq \left(\frac{2}{|\tau|}\right)^{1-\frac{\beta}{\alpha}} \|H\|_1$
- $\int_{|\eta|>|\tau|/2} G(\tau-\eta)H(\eta)\, d\eta \leq \int_{|\eta|>|\tau|/2} \frac{1}{|\tau-\eta|^{1-\frac{\beta}{\alpha}}} \frac{1}{|\eta|^{1+\frac{\beta}{\alpha}}}\, d\eta = C\frac{1}{|\tau|} \leq C\left(\frac{1}{|\tau|}\right)^{1-\frac{\beta}{\alpha}}$

so that $A_{\alpha,\beta}(\tau) \approx \Omega\left(\frac{1}{|\tau|^{1-\frac{\beta}{\alpha}}}\right)$. ◇

Now, Theorem 5 is a direct consequence of Proposition 3. Indeed, replacing the kernel $K_{\alpha,\beta}$ with the kernel $\mathbb{K}_{\alpha,\beta}$ will not change the space of resolution $\mathcal{V}^{\alpha,\beta}$, but only replace its norm with an equivalent one. We now endow $\mathbb{R}\times\mathbb{R}^n$ with the quasi-metric $\tilde{\rho}_{\alpha,\beta}((t,x),(s,y)) = (\mathbb{K}_{\alpha,\beta}(t-s,x-y))^{-\frac{1}{n+\beta}}$ and apply again Kalton and Verbitsky's theorem. We find that $\mathcal{V}^{\alpha,\beta} = \mathcal{M}(\tilde{W}^{\alpha,\beta} \mapsto L^2)$ whith $\tilde{W}^{\alpha,\beta} = \mathcal{J}_{\alpha,\alpha-\beta}L^2$ and $\mathcal{J}_{\alpha,\alpha-\beta}$ defined in the same way as $\mathcal{I}_{\alpha,\alpha-\beta}$ (replacing $K_{\alpha,\beta}$ with $\mathbb{K}_{\alpha,\beta}$). Taking the Fourier transform, we see that $\tilde{W}^{\alpha,\beta} = \mathbb{W}^{\alpha,\beta}$. ◇

4 The spaces $X_{\alpha,\beta,T}$.

We now study the spaces $X_{\alpha,\beta,T}$ for $T < +\infty$ and the space $X_{\alpha,\beta}. = X_{\alpha,\beta,\infty}$. We begin with some remarks on $\mathcal{V}^{\alpha,\beta}$:

- if $F \in \mathcal{V}^{\alpha,\beta}$ and $|G| \leq |F|$, then $G \in \mathcal{V}^{\alpha,\beta}$ and $\|G\|_{\mathcal{V}^{\alpha,\beta}} \leq \|F\|_{\mathcal{V}^{\alpha,\beta}}$.
- if $F \in \mathcal{V}^{\alpha,\beta}$ and $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, then $F(\cdot - t_0, \cdot - x_0) \in \mathcal{V}^{\alpha,\beta}$ with the same norm.
- if $A \in L^1(\mathbb{R})$ and $B \in L^1(\mathbb{R}^n)$, then, if $F \in \mathcal{V}^{\alpha,\beta}$, we have $A *_t F = \int A(s)F(\cdot - s, \cdot)\, ds \in \mathcal{V}^{\alpha,\beta}$ with $\|A *_t F\|_{\mathcal{V}^{\alpha,\beta}} \leq \|A\|_1 \|F\|_{\mathcal{V}^{\alpha,\beta}}$ and $B *_x F = \int B(y)F(\cdot, \cdot - y)\, dy \in \mathcal{V}^{\alpha,\beta}$ with $\|B *_x F\|_{\mathcal{V}^{\alpha,\beta}} \leq \|B\|_1 \|F\|_{\mathcal{V}^{\alpha,\beta}}$.
- if $F \in \mathcal{V}^{\alpha,\beta}$ and $\lambda > 0$, then $\lambda^{\alpha-\beta} F(\lambda^\alpha \cdot, \lambda \cdot) \in \mathcal{V}^{\alpha,\beta}$ with the same norm.

These properties are easily transferred to the spaces $X_{\alpha,\beta,T}$:

- if $f \in X_{\alpha,\beta,T}$ $(0 < T \leq +\infty)$ and $x_0 \in \mathbb{R}^n$, then $f(\cdot - x_0) \in X_{\alpha,\beta,T}$ with the same norm.
- if $B \in L^1(\mathbb{R}^n)$, then, if $f \in X_{\alpha,\beta,T}$ $(0 < T \leq +\infty)$, we have $B * f \in X_{\alpha,\beta,T}$ with $\|B * f\|_{X_{\alpha,\beta,T}} \leq \|B\|_1 \|f\|_{X_{\alpha,\beta,T}}$.
- if $f \in X_{\alpha,\beta,T}$ and $\lambda > 0$, then $\lambda^{\alpha-\beta} f(\lambda \cdot) \in X_{\alpha,\beta,\lambda^{-\alpha}T}$ with the same norm.

We may now prove Proposition 2. First, we recall some basic facts on the homogeneous Besov space $\dot{B}^{-\gamma}_{\infty,\infty}(\mathbb{R}^n)$ with $\gamma > 0$. It can be defined in two equivalent ways :

- thermic characterization : A tempered distribution f belongs to $\dot{B}^{-\gamma}_{\infty,\infty}(\mathbb{R}^n)$ with $\gamma > 0$ if and only if $\sup_{t>0} t^{\gamma/2} \|e^{t\Delta} f\|_\infty < +\infty$.
- Littlewood–Paley characterization : A tempered distribution f belongs to $\dot{B}^{-\gamma}_{\infty,\infty}(\mathbb{R}^n)$ with $\gamma > 0$ if and only if the homogeneous Littlewood–Paley decomposition of f into dyadic blocks $\Delta_j f$ $(j \in \mathbb{Z})$ satisfies $\sup_{j\in\mathbb{Z}} 2^{-j\gamma} \|\Delta_j f\|_\infty < +\infty$ and $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ (convergence in $\mathcal{S}'$).

As the kernels of $e^{-(-\Delta)^{\alpha/2}}$ and of $\Delta^N e^{-(-\Delta)^{\alpha/2}}$ are integrable functions (see Lemma 6), we have

$$\|e^{-t(-\Delta)^{\alpha/2}} \Delta_j f\|_\infty \leq C_{\alpha,N} \min(1, 2^{-j2N} t^{-\frac{2N}{\alpha}}) \|\Delta_j f\|_\infty.$$

Taking $N > \gamma/2$, we find that

$$\|e^{-t(-\Delta)^{\alpha/2}} f\|_\infty \leq C_{\alpha,\gamma} t^{-\frac{\gamma}{\alpha}} \|f\|_{\dot{B}^{-\gamma}_{\infty,\infty}}.$$

Conversely, using the fact that the kernel of $\Delta_0 e^{+(-\Delta)^{\alpha/2}}$ is integrable (see Lemma 5), we find that $\sup_{t>0} t^{\frac{\gamma}{\alpha}} \|e^{-t(-\Delta)^{\alpha/2}} f\|_\infty$ is a norm on $\dot{B}^{-\gamma}_{\infty,\infty}$ which is equivalent to the usual norm on $\dot{B}^{-\gamma}_{\infty,\infty}$.

Similarly, the quantities $\sup_{0<t<T} t^{\frac{\gamma}{\alpha}} \|e^{-t(-\Delta)^{\alpha/2}} f\|_\infty$ for $T < +\infty$ are norms on $B^{-\gamma}_{\infty,\infty}$ which are equivalent to the usual norm on $B^{-\gamma}_{\infty,\infty}$.

Now, it is clear that the spaces $X_{\alpha,\beta,T}(\mathbb{R}^n)$ are Banach spaces, since $\{(f,g) \in B^{\beta-\alpha}_{\infty,\infty} \times \mathcal{V}^{\alpha,\beta} \ / \ g = 1_{0<t<T} e^{-t(-\Delta)^{\alpha/2}} f\}$ is closed in $B^{\beta-\alpha}_{\infty,\infty} \times \mathcal{V}^{\alpha,\beta}$. The equality of $X_{\alpha,\beta,T_1}(\mathbb{R}^n)$ and $X_{\alpha,\beta,T_2}(\mathbb{R}^n)$ for $0 < T_1 < T_2 < +\infty$ is easy to check. We have obviously $X_{\alpha,\beta,T_2}(\mathbb{R}^n) \subset X_{\alpha,\beta,T_1}(\mathbb{R}^n)$ (continuous embedding). Conversely, if $f \in X_{\alpha,\beta,T_1}(\mathbb{R}^n)$, we shall prove that $f \in X_{\alpha,\beta,2^p T_1}(\mathbb{R}^n)$ for every $p \in \mathbb{N}$, hence $f \in X_{\alpha,\beta,T_2}(\mathbb{R}^n)$. Of course, it is enough to deal with the case $p = 1$ (and then go on by induction). Let $g = 1_{0<t<T_1} e^{-t(-\Delta)^{\alpha/2}} f$ and $G = 1_{0<t<2T_1} e^{-t(-\Delta)^{\alpha/2}} f$. We have $G = g + e^{-T_1(-\Delta)^{\alpha/2}} *_x g(\cdot - T_1, \cdot)$; as $g \in \mathcal{V}^{\alpha,\beta}$, we find that $G \in \mathcal{V}^{\alpha,\beta}$.
Thus Proposition 2 is proved. $\diamond$

For $\beta < 2$ (at least), we may simplify the norm of $X_{\alpha,\beta,T}(\mathbb{R}^n)$, as we have the following estimate :

Proposition 4 *Let $0 < \beta < \alpha < n + 2\beta$ and $\beta < 2$. Then, the norm of f in $X_{\alpha,\beta,T}(\mathbb{R}^n)$ ($0 < T \leq +\infty$) is equivalent to $\|1_{0<t<T} e^{-t(-\Delta)^{\alpha/2}} f\|_{\mathcal{V}^{\alpha,\beta}}$.*

Proof : We are going to show the inequality

$$\|e^{-t(-\Delta)^{\alpha/2}} f\|_\infty \leq C_{\alpha,\beta} t^{\beta-\alpha} \|1_{0<s<t} e^{-s(-\Delta)^{\alpha/2}} f\|_{\mathcal{V}^{\alpha,\beta}}.$$

We write

$$e^{-t(-\Delta)^{\alpha/2}} f = \int \frac{1}{t^{\frac{n}{\alpha}}} W_{\alpha,n}(\frac{x-y}{t^{1/\alpha}}) f(y)\, dy$$

where $W_{\alpha,n}$ is the inverse Fourier transform of $e^{-|\xi|^\alpha}$. As we have shift-invariance and the scaling property, it is enough to prove the inequality for $x = 0$ and $t = 1$.

For $1/4 < \tau < 1/2$, we may write as well

$$e^{-(-\Delta)^{\alpha/2}} f = \int \frac{1}{(1-\tau)^{\frac{n}{\alpha}}} W_{\alpha,n}(\frac{x-y}{(1-\tau)^{1/\alpha}}) e^{-\tau(-\Delta)^{\alpha/2}} f(y)\, dy.$$

Picking a non-negative function $\theta \in \mathcal{D}(\mathbb{R})$ which is supported within $(1/4, 1/2)$ and such that $\int \theta\, ds = 1$, we find

$$e^{-(-\Delta)^{\alpha/2}} f(0) = \iint H(\tau, y) 1_{0<\tau<1}(\tau) e^{-\tau(-\Delta)^{\alpha/2}} f(y)\, d\tau\, dy$$

with

$$H(\tau, y) = \theta(\tau) \frac{1}{(1-\tau)^{\frac{n}{\alpha}}} W_{\alpha,n}(\frac{y}{(1-\tau)^{1/\alpha}}).$$

From Lemma 6, we find that

$$|H(\tau,y)| \le C\theta(\tau)\frac{|1-\tau|}{(|1-\tau|^{1/\alpha}+|y|)^{n+\alpha}} \le C'\theta(t)\frac{1}{(1+|y|)^{n+\alpha}}.$$

Let $\delta \in \mathcal{D}(\mathbb{R})$ be supported in $(0,1)$ and equal to 1 on $[1/4, 1/2]$, and let

$$\gamma(\tau,y) = \delta(\tau)\frac{1}{(1+|y|)^{\frac{n+\alpha}{2}}} \text{ and } G = \frac{H}{\gamma}.$$

We have $|G| \le C'\theta(\tau)\frac{1}{(1+|y|)^{\frac{n+\alpha}{2}}}$, so that $G \in L^2(\mathbb{R}\times\mathbb{R}^n)$. On the other hand, it is easy to see that $(-\partial_t^2)^{\frac{1-\frac{\beta}{\alpha}}{2}}\gamma \in L^2(\mathbb{R}\times\mathbb{R}^n)$ and $(-\Delta_x)^{\frac{\alpha-\beta}{2}} f \in L^2(\mathbb{R}\times\mathbb{R}^n)$ (just check that all the derivatives in time and space variables of every order of γ belong to $L^2(\mathbb{R}\times\mathbb{R}^n)$ and then interpolate). Thus, γ belongs to $\mathbb{W}^{\alpha,\beta}(\mathbb{R}\times\mathbb{R}^n) = L^2_t\dot H^{\alpha-\beta}_x \cap L^2_x \dot H_t^{1-\frac{\beta}{\alpha}}$ and we find that

$$|e^{-(-\Delta)^{\alpha/2}}f(0)| \le C\|G\|_2\|\gamma\|_{\mathbb{W}^{\alpha,\beta}}\|1_{0<s<1}e^{-s(-\Delta)^{\alpha/2}}f\|_{\mathcal{V}^{\alpha,\beta}}.$$

The proposition is proved. $\diamond$

5 Cheap solutions for a semilinear parabolic equation.

In this section we go back to our Cauchy problem : given $\vec{u}_0 \in (\mathcal{S}'(\mathbb{R}^n))^d$, find a vector distribution $\vec{u}$ on $(0,+\infty)\times\mathbb{R}^n$ such that, for $i = 1,\dots,d$, we have

$$\partial_t u_i = -(-\Delta)^{\alpha/2}u_i + \sum_{j=1}^d\sum_{k=1}^d \sigma_{i,j,k}(D)(u_j u_k) \tag{39}$$

and

$$\lim_{t\to 0} u_i(t,x) = u_{i,0} \tag{40}$$

where $\sigma_{i,j,k}(D)$ is a homogeneous pseudo-differential operator of degree β with $0<\beta<\alpha<n+2\beta$.

Theorem 1 gives us a way to exhibit solutions through a domination principle. In this theorem, we are only interested in the pointwise convergence of the Picard iterates to some Lebesgue measurable solution of the equation. As we did not use any refined analysis of the coefficients $\sigma_{i,j,k}(D)$ (no maximum principle, no conservation of energy, and so on), but just controlled the integrals by the absolute values of the integrands, we shall call the solutions we found as *cheap solutions* : they do not provide much insight into the structure of the equation.

Theorem 2 restates the result as a result in terms of Banach spaces $X_{\alpha,\beta}$ and $\mathcal{V}^{\alpha,\beta}$. This theorem is a direct consequence of Theorem 1, but we could as well prove it through the classical formalism associated to the Banach contraction principle. Let us sketch this proof. We define an operator $\mathcal{B}$ on $(\mathcal{V}^{\alpha,\beta})^d$ by

$$B(\vec{u},\vec{v}) = \int_{-\infty}^{t} e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{u}\otimes\vec{v})\; ds \tag{41}$$

and we are going to solve $\vec{U} = \vec{U}_0 + \mathcal{B}(\vec{U},\vec{U})$ with $\vec{U}_0 = 1_{0<t<T}e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0$.

We have, from Lemma 1, that

$$|\mathcal{B}(\vec{U},\vec{V})| \le C_0 \int_{\mathbb{R}\times\mathbb{R}^n} K_{\alpha,\beta}(t-s,x-y)|\vec{U}(s,y)|\;|\vec{V}(s,y)|\; ds\; dy \tag{42}$$

so that

$$\|\mathcal{B}(\vec{U},\vec{V})\|_{\mathcal{V}^{\alpha,\beta}} \le C_0\|\vec{U}\|_{\mathcal{V}^{\alpha,\beta}}\|\vec{V}\|_{\mathcal{V}^{\alpha,\beta}}. \tag{43}$$

The Banach contraction principle gives that, when $\|\vec{U}_0\|_{\mathcal{V}^{\alpha,\beta}} < \frac{1}{4C_0}$, there exists a unique solution $\vec{U}$ such that $\|\vec{U}\|_{\mathcal{V}^{\alpha,\beta}} < \frac{1}{2C_0}$. For $\vec{u}_0$ satisfying the assumptions of Theorem 2, we can thus find a solution $\vec{U}$ of $\vec{U} = \vec{U}_0 + \mathcal{B}(\vec{U},\vec{U})$ with $\vec{U}_0 = 1_{t>0}\, e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0$; this solution, obtained by iteration, satisfies $\vec{U} = 0$ for $t<0$. The solution $\vec{u}$ of Theorem 2 is then given by $\vec{u} = 1_{0<t<T}\vec{U}$. $\diamond$

Remark that, for $T = +\infty$, we have found a global solution.

6 Regularity of the solutions.

In this section, we discuss the size and regularity of global cheap solutions. We begin with the following lemma :

Lemma 3 *There exists a constant C which depends only on n, α and β such that :*

$$\begin{aligned} &|t|^{1-\frac{\beta}{\alpha}}|\iint K_{\alpha,\beta}(t-s,x-y)W^2(s,y)\,ds\,dy| \\ &\le C\|W\|_{\mathcal{V}^{\alpha,\beta}}(\|W\|_{\mathcal{V}^{\alpha,\beta}} + \sup_{s\in\mathbb{R}}|s|^{1-\frac{\beta}{\alpha}}\|W(s,.)\|_\infty). \end{aligned} \tag{44}$$

Proof : The proof is based on the following remark : the function

$$J(t,x) = \iint_{|s|>|t|} \frac{1}{(|t-s|^{\frac{1}{\alpha}} + |x-y|)^{n+\beta}} \frac{1}{(|s|^{\frac{1}{\alpha}} + |y|)^{\alpha+\frac{n-\beta}{2}}}\, ds\, dy \tag{45}$$

is well-defined for $(t,x) \neq (0,0)$, as $\beta < \alpha$ (local integrability) and $\frac{n+\beta}{2} > 0$ (integrability at infinity). By Fatou's lemma, it is lower semi-continuous, hence, since $\{(t,x) \ / \ \rho_\alpha(t,x) = 1\}$ is a compact set, we have

$$\gamma = \inf_{\rho_\alpha(t,x)=1} J(t,x) > 0. \tag{46}$$

By homogeneity, we find

$$J(t,x) \geq \gamma \frac{1}{\rho_\alpha(t,x)^{(n+\beta)/2}}. \tag{47}$$

We may now estimate $I(t,x) = \iint K_{\alpha,\beta}(t-s, x-y) W^2(s,y)\, ds\, dy$. Let $\epsilon \in (0, 1/2)$ and let

$$A_\epsilon(t,x) = \iint_{|t-s|<\epsilon|t|} K_{\alpha,\beta}(t-s, x-y) W^2(s,y)\, ds\, dy \tag{48}$$

and $B_\epsilon(t,x) = I(t,x) - A_\epsilon(t,x)$. Let us define moreover $N_1 = \|W\|_{\mathcal{V}^{\alpha,\beta}}$ and $N_2 = \sup_{s\in\mathbb{R}} |s|^{1-\frac{\beta}{\alpha}} \|W(s,.)\|_\infty$. We have

$$A_\epsilon(t,x) \leq N_2^2 \left(\frac{2}{|t|}\right)^{2-\frac{2\beta}{\alpha}} \iint_{|t-s|<\epsilon t} K_{\alpha,\beta}(t-s, x-y)\, ds\, dy = C N_2^2 \left(\frac{\epsilon}{|t|}\right)^{1-\frac{\beta}{\alpha}}. \tag{49}$$

On the other hand, writing $J_\epsilon(t,x) = 1_{|t-s|>\epsilon|t|} J(t-s, x-y)$, we have

$$B_\epsilon(t,x) \leq \frac{1}{\gamma^2} \iint J_\epsilon(t-s, x-y)^2 W^2(s,y)\, ds\, dy \tag{50}$$

and

$$J_\epsilon(t-s, x-y) \leq \iint \frac{1}{(|s-\sigma|^{\frac{1}{\alpha}} + |y-z|)^{n+\beta}} \frac{1_{|t-\sigma|>\epsilon|t|}}{(|t-\sigma|^{\frac{1}{\alpha}} + |z-x|)^{\alpha+\frac{n-\beta}{2}}}\, d\sigma\, dz. \tag{51}$$

Let $F_{t,x,\epsilon}(\sigma, z) = \frac{1_{|t-\sigma|>\epsilon|t|}}{(|t-\sigma|^{\frac{1}{\alpha}} + |z-x|)^{\alpha+\frac{n-\beta}{2}}}$; we get

$$B_\epsilon(t,x) \leq \frac{1}{\gamma^2} N_1^2 \iint |F_{t,x,\epsilon}(\sigma,z)|^2\, d\sigma\, dz = C N_1^2 \frac{1}{(\epsilon|t|)^{1-\frac{\beta}{\alpha}}}. \tag{52}$$

We conclude the proof by taking $\epsilon^{1-\frac{\beta}{\alpha}} = \frac{1}{2}\frac{N_1}{N_1+N_2}$. $\diamond$

We now consider a solution $\vec{u}$ on $(0,+\infty)\times\mathbb{R}^n$ of the semi-linear heat equation

$$\vec{u} = e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{u}\otimes\vec{u})\,ds \tag{53}$$

obtained by the iteration algorithm :

$$\vec{U}_0 = e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0 \text{ and } \vec{U}_{k+1} = \vec{U}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{U}_k\otimes\vec{U}_k)\,ds. \tag{54}$$

We already know that, if $1_{0<t}\vec{U}_0$ is small enough in $(\mathcal{V}^{\alpha,\beta}(\mathbb{R}\times\mathbb{R}^n))^d$ (i.e. if $\vec{u}_0$ is small enough in $(X_{\alpha,\beta}(\mathbb{R}^n))^d$), then $\sum_{k=0}^{+\infty}\|1_{0<t}(\vec{U}_{k+1}-\vec{U}_k)\|_{\mathcal{V}^{\alpha,\beta}} < +\infty$. We get other estimates from this inequality :

Proposition 5 *If*

$$\|1_{0<t}\,\vec{U}_0\|_{\mathcal{V}^{\alpha,\beta}} + \sum_{k=0}^{+\infty}\|1_{0<t}(\vec{U}_{k+1}-\vec{U}_k)\|_{\mathcal{V}^{\alpha,\beta}} < +\infty, \tag{55}$$

then

$$\sup_{0<t} t^{1-\frac{\beta}{\alpha}}\|\vec{U}_0(t,.)\|_\infty + \sum_{k=0}^{+\infty}\sup_{0<t} t^{1-\frac{\beta}{\alpha}}\|\vec{U}_{k+1}(t,.)-\vec{U}_k(t,.)\|_\infty < +\infty. \tag{56}$$

Proof : Writing $\vec{U}_{-1}=0$, $A_k = |\vec{U}_k - \vec{U}_{k-1}|$ and $B_k = |\vec{U}_k|$, we have, for all $k\in\mathbb{N}$,

$$A_{k+1}(t,x) \le C_0\int_0^t\int \frac{A_k(s,y)(B_k(s,y)+B_{k-1}(s,y))}{(|t-s|^{1/\alpha}+|x-y|)^{n+\beta}}\,ds\,dy. \tag{57}$$

We define

$$\begin{aligned}
\alpha_k &= \sup_{0<t} t^{1-\frac{\beta}{\alpha}}\|A_k(t,.)\|_\infty + \|A_k\|_{\mathcal{V}^{\alpha,\beta}},\\
\beta_k &= \sup_{0<t} t^{1-\frac{\beta}{\alpha}}\|B_k(t,.)\|_\infty + \|B_k\|_{\mathcal{V}^{\alpha,\beta}},\\
\gamma_k &= \|A_k\|_{\mathcal{V}^{\alpha,\beta}},\\
\delta_k &= \|B_k\|_{\mathcal{V}^{\alpha,\beta}}.
\end{aligned} \tag{58}$$

We remark that $\|\sqrt{|FG|}\|_\infty \le \sqrt{\|F\|_\infty\|G\|_\infty}$ and $\|\sqrt{|FG|}\|_{\mathcal{V}^{\alpha,\beta}} \le \sqrt{\|F\|_{\mathcal{V}^{\alpha,\beta}}\|G\|_{\mathcal{V}^{\alpha,\beta}}}$, therefore we may apply Lemma 3 (with $W=\sqrt{A_k(B_k+B_{k-1})}$) and get :

$$\alpha_{k+1} \le C\sqrt{\alpha_k(\beta_k+\beta_{k-1})\gamma_k(\delta_k+\delta_{k-1})}. \tag{59}$$

Let

$$\epsilon_k = \sum_{j\le k} \alpha_j \text{ and } M = \sum_{k\in\mathbb{N}} \gamma_k. \tag{60}$$

From (59), we get the inequality

$$\alpha_{k+1} \le \frac{1}{2}\alpha_k + CM\gamma_k\epsilon_k \tag{61}$$

which gives

$$\epsilon_{k+1} \le 2CM \sum_{j\le k} \gamma_j\epsilon_j \tag{62}$$

hence

$$\sum_{j\le k+1} \gamma_j\epsilon_j \le (1+2CM\gamma_{k+1}) \sum_{j\le k} \gamma_j\epsilon_j \tag{63}$$

which gives

$$\sum_{j\le k+1} \gamma_j\epsilon_j \le \gamma_0\epsilon_0 \prod_{l=1}^{+\infty}(1+2CM\gamma_l) \tag{64}$$

and finally

$$\sup_{k\in\mathbb{N}} \epsilon_k \le 2CM\gamma_0\epsilon_0 \prod_{l=1}^{+\infty}(1+2CM\gamma_l). \tag{65}$$

Proposition 5 is proved. $\diamond$

Proposition 6 *Under the same assumptions as in Proposition 5, we have, for all positive γ, that*

$$\sup_{0<t} t^{\frac{\alpha-\beta+\gamma}{\alpha}} \|\vec{u}(t,.)\|_{\dot{B}^{\gamma}_{\infty,\infty}} < +\infty. \tag{66}$$

Hence the solution $\vec{u}$ is $\mathcal{C}^\infty$ on $(0,T)\times\mathbb{R}^n$.

Proof: Let $\gamma \ge 0$. Start from the information that $\sup_{0<t} t^{\frac{\alpha-\beta+\gamma}{\alpha}} \|\vec{u}(t,.)\|_{\dot{B}^{\gamma}_{\infty,\infty}} < +\infty$ if $\gamma > 0$ and that $\sup_{0<t} t^{\frac{\alpha-\beta}{\alpha}} \|\vec{u}(t,.)\|_\infty < +\infty$. We then have the estimate $\sup_{0<t} t^{\frac{\alpha-\beta+\gamma}{\alpha}} \|\vec{u}(t,.)\otimes\vec{u}(t,.)\|_{\dot{B}^{\gamma}_{\infty,\infty}} < +\infty$. Then write

$$\vec{u}(t,x) = e^{-\frac{t}{2}(-\Delta)^{\alpha/2}}\vec{u}(\frac{t}{2},x) + \int_{t/2}^{t} e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{u}(s,.)\otimes\vec{u}(s,.))\,ds \tag{67}$$

to control the norm of $\vec{u}$ in $\dot{B}^{\gamma+\alpha-\beta}_{\infty,\infty}$. $\diamond$

7 A Besov-space approach of cheap solutions.

Theorem 2 gives a criterion to grant existence of a solution : the initial value is required to satisfy $1_{0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0| \in \mathcal{V}^{\alpha,\beta}$. But the space of the distributions such that $1_{0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0| \in \mathcal{V}^{\alpha,\beta}$ is not a classical one and we might try to find some subspaces that are close enough to this maximal space but belong to a classical scale of spaces.

We shall thus describe Banach spaces X of measurable functions in time and space variables that lead to cheap solutions : one should have the following properties :

- if $f(t,x) \in X$ and if $|g(t,x)| \leq |f(t,x)|$, then $g \in X$ and $\|g\|_X \leq \|f\|_X$
- for $f, g \in X$, $F = \iint K_{\alpha,\beta}(t-s, x-y)|f(s,y)|\,|g(s,y)|\;ds\;dy \in X$ and $\|F\|_X \leq C_X \|f\|_X \|g\|_X$

From Proposition 10, we know that $X \subset \mathcal{V}^{\alpha,\beta}(\mathbb{R} \times \mathbb{R}^n)$ and from Lemma 1 we know that we may find a solution $\vec{u}$ to the equation

$$\vec{u} = e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{u}\otimes\vec{u})\;ds \tag{68}$$

on $(0,T)\times\mathbb{R}^n$ such that $1_{0<t<T}\vec{u} \in X^d$ as soon as $1_{0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0| \in X$ and $\|1_{t0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0|\|_X < \frac{1}{4C_0C_X}$ (where T might be a positive real number [local solution] or equal to $+\infty$ [global solution]).

The simplest way to find such a space X is to replace the kernel $K_{\alpha,\beta}$ by kernels whose action are well documented on functions in time variable or in space variable. For instance, if $\max(1/2, \beta/\alpha) < \gamma < \min(1, \frac{n+2\beta}{2\alpha})$, we may write

$$K_{\alpha,\beta}(t,x) \leq \frac{1}{|t|^\gamma}\frac{1}{|x|^{n+\beta-\alpha\gamma}}. \tag{69}$$

Let $I_{x,\alpha\gamma-\beta}$ be the convolution operator (in x variable) with $\frac{1}{|x|^{n+\beta-\alpha\gamma}}$ and $I_{t,1-\gamma}$ be the convolution operator (in t variable) with $\frac{1}{|t|^\gamma}$. We have :

$$\iint K_{\alpha,\beta}(t-s,x-y)|f(s,y)|\,|g(s,y)|\;ds\;dy \leq I_{t,1-\gamma}I_{x,\alpha\gamma-\beta}(|fg|)(x). \tag{70}$$

In this way, we have dissociated the action on the variable x from the action on the variable t.

Let E be a Banach space of measurable functions on $\mathbb{R}^n$ satisfying $\||f|\,\|_E \leq C_E\|f\|_E$. We see that $X_{E,\delta} = \{f \;/\; \sup_{t>0} t^{\delta/\alpha}\|f(t,.)\|_E < +\infty\}$ will be contained in $\mathcal{V}^{\alpha,\beta}$ if $(f,g) \mapsto I_{\alpha\gamma-\beta}(fg)$ is bounded from $E\times E$ to E and

$(f,g) \mapsto I_{1-\gamma}(fg)$ is bounded on $X_t = \{f \ / \ |t|^{\delta/\alpha} f \in L^\infty\}$. Using again the theory of multipliers, we find that the maximal Banach space E we can associate (in this way) to γ is $X_{E,\delta}$ with

$$E = \mathcal{V}^{\alpha\gamma-\beta}(\mathbb{R}^n) = \mathcal{M}(\dot{H}^{\alpha\gamma-\beta}(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n))$$

and $\gamma = 1 - \frac{\delta}{\alpha}$. Thus, we find that we can easily get cheap solutions when the initial value $\vec{u}_0$ belongs to (and is small in) X_0^d, where X_0 is the Besov space $X_0 = \dot{B}^{-\alpha+\beta+r}_{\mathcal{V}^r,\infty}$ with $\max(0, \frac{\alpha-2\beta}{2}) < r < \min(\alpha-\beta, \frac{n}{2})$ [18].

Due to the Fefferman–Phong inequality, we may replace the space $\mathcal{V}^r$ by a Morrey spce $\dot{M}^{s,n/r}$ with $2 < s \leq \frac{n}{r}$. The corresponding space X_0 will be a Besov-Morrey space $B^{-\frac{\alpha}{p}}_{\dot{M}^{s,q},\infty}$ (see Kozono and Yamazaki [17]) with Serrin's scaling relation $\frac{\alpha}{p} + \frac{n}{q} = \alpha - \beta$ (and with $2 < s \leq q$, $\frac{n}{\alpha-\beta} < q$ and, if $\alpha > 2\beta$, $q < \frac{2n}{\alpha-2\beta}$). If $s = q$, we find the classical Besov space $B^{-\frac{\alpha}{p}}_{q,\infty}$ (see Cannone [5]).

More precisely, we have the following result :

Theorem 6 *Let $0 < \beta < \alpha \leq n + 2\beta$. Let $X^{\alpha,\beta}$ be the Banach space of distributions such that $X^{\alpha,\beta} \subset \dot{B}^{\beta-\alpha}_{\infty,\infty}$ and $1_{0<t}\ e^{-t(-\Delta)^{\alpha/2}} u_0 \in \mathcal{V}^{\alpha,\beta}$. Then :*

- *if $\beta > \alpha/2$, then*
$$\frac{1}{|t|^{1-\frac{\beta}{\alpha}}} \in \mathcal{V}^{\alpha,\beta} \tag{71}$$
so that $X^{\alpha,\beta} = \dot{B}^{\beta-\alpha}_{\infty,\infty}$.

- *if $\beta \leq \alpha/2$, there exists $u_0 \in \dot{B}^{\beta-\alpha}_{\infty,\infty}$ such that $u_0 \notin X^{\alpha,\beta}$. More precisely :*
  - *if $\beta < \alpha/2$, then there exists $u_0 \in \dot{B}^{\beta-\alpha}_{\infty,1}$ such that $u_0 \notin X^{\alpha,\beta}$;*
  - *if $\beta \leq \alpha/2$, then $\dot{B}^{\beta-\alpha+\frac{n}{q}}_{q,\infty} \subset X^{\alpha,\beta} \Leftrightarrow q < \frac{2n}{\alpha-2\beta}$.*

Proof : If $\beta > \alpha/2$ and $2 < r < \frac{\alpha}{\alpha-\beta}$, then

$$\iint_{\rho_\alpha(t-s,x-y)<R} \frac{1}{\left(|s|^{1-\frac{\beta}{\alpha}}\right)^r}\, ds\, dy \leq C \int_0^{R^\alpha} \frac{R^n\, ds}{\left(|s|^{1-\frac{\beta}{\alpha}}\right)^r} = C' R^{n+\alpha-r(\alpha-\beta)}. \tag{72}$$

This inequality implies that $\frac{1}{|t|^{1-\frac{\beta}{\alpha}}} \in \dot{M}_\alpha^{r,\frac{n+\alpha}{\alpha-\beta}} \subset \mathcal{V}^{\alpha,\beta}$

We now consider the case $2\beta \leq \alpha$. We shall consider the cheap parabolic equation of Montgomery–Smith [24] :

$$\partial_t u + (-\Delta)^{\alpha/2} u = (-\Delta)^{\beta/2}(u^2) \tag{73}$$

and the associated bilinear operator

$$B_{\alpha,\beta}(u,v)=\int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}(-\Delta)^{\beta/2}(u(s,.)v(s,.))\,ds. \tag{74}$$

Let $\theta\in\mathcal{S}(\mathbb{R}^n)$ such that $1_{|\xi|<1}\leq\hat{\theta}(\xi)\leq 1_{|\xi|<2}$. For $\gamma\in\mathbb{R}$, we take $u_\gamma=2\sum_{j\geq 3}\theta(x)\cos(2^jx_1)2^{\gamma j}$. Then u_γ belongs to $\dot{B}^{-\gamma}_{q,\infty}$ for every $q\in[1,+\infty]$, and belongs to $\dot{B}^{\delta}_{\infty,1}$ for every $\delta<-\gamma$. Let $v_{\alpha,\gamma}=e^{-(t(-\Delta)^{\alpha/2}}u_\gamma$. If $B_{\alpha,\beta}(v_{\alpha,\gamma},v_{\alpha,\gamma})$ is well defined, we check it against a test function $\omega(t,x)$ which satisfies, in spatial Fourier variables,

$$1_{1/2<t<1}1_{|\xi|<1}\leq\hat{\omega}(t,\xi). \tag{75}$$

For $|\eta|=\Omega(2^j)$, $|\xi|<1$, $1/2<t<1$, we have

$$\begin{aligned}\int_0^t e^{-(t-s)|\xi|^\alpha-s|\eta|^\alpha-s|\xi-\eta|^\alpha}\,ds &\geq e^{-1}\int_0^t e^{-s|\eta|^\alpha-s|\xi-\eta|^\alpha}\,ds\\ &\geq e^{-1}\frac{1-e^{-t(|\eta|^\alpha+|\xi-\eta|^\alpha)}}{|\xi-\eta|^\alpha+|\eta|^\alpha}\\ &\geq c_\alpha 2^{-\alpha j}.\end{aligned} \tag{76}$$

We thus have (with $\epsilon_1=(1,0,\ldots,0)$)

$$\begin{aligned}&(2\pi)^n\langle B_{\alpha,\beta}(v_{\alpha,\gamma},v_{\alpha,\gamma})|\omega\rangle\geq\\ &c_\alpha\int_{1/2}^1\int_{|\xi|<1}|\xi|^\beta\sum_j 2^{(2\gamma-\alpha)j}\int\hat{\theta}(\xi-\eta-2^j\epsilon_1)\hat{\theta}(\eta+2^j\epsilon_1)\,d\xi\,dt\\ &\qquad\geq c'_\alpha\sum_{j=3}^{+\infty}2^{j(2\gamma-\alpha)}\end{aligned} \tag{77}$$

with $c'_\alpha>0$. Thus, $B_{\alpha,\beta}(v_{\alpha,\gamma},v_{\alpha,\gamma})$ cannot be well defined for $2\gamma\geq\alpha$.

Hence, we have $u_{\alpha/2}\notin X^{\alpha,\beta}$. On the other hand, we know that $u_{\alpha/2}\in\dot{B}^{\beta-\alpha}_{\infty,1}$ if $\beta-\alpha<-\alpha/2$, i.e. $\beta<\alpha/2$. Similarly, if $\beta\leq\alpha/2$ and $q=\frac{2n}{\alpha-2\beta}$, we know that $u_{\alpha/2}\in\dot{B}^{-\frac{\alpha}{2}}_{q,\infty}=\dot{B}^{\beta-\alpha+\frac{n}{q}}_{q,\infty}$. Theorem 6 is thus proved. $\diamond$

Remark : In this paper, we deal only with critical spaces and global existence. But it is easy to check that the same example of the cheap equation and of the initial value $u_{\alpha/2}$ shows that there is no local existence result for the subcritical spaces $\dot{B}^{-\delta}_{\infty,\infty}$ with $\alpha/2\leq\delta<\alpha-\beta$.

8 The case $\alpha = 2\beta$.

We have seen that for $\beta > \alpha/2$ we have $\dot{B}^{\beta-\alpha}_{\infty,\infty} \subset X^{\alpha,\beta}$, so that the Cauchy problem for our general parabolic equation with a small initial value in $(\dot{B}^{\beta-\alpha}_{\infty,\infty})^d$ will have a solution. For $\beta > \alpha/2$, we found an example $u_{\alpha/2} \in \dot{B}^{\beta-\alpha}_{\infty,1}$ such that, for every $\lambda > 0$, the Cauchy problem for the cheap equation with the initial value $\lambda u_{\alpha/2}$ will have no solution.

In the limit case $\beta = \alpha/2$, the counter-example $u_{\alpha/2}$ belongs to $\dot{B}^{-\alpha/2}_{\infty,\infty}$, so that $\dot{B}^{-\alpha/2}_{\infty,\infty}$ is not included in $X^{\alpha,\alpha/2}$. However, the Cauchy problem for the general parabolic equation with a small data in $(\dot{B}^{-\alpha/2}_{\infty,1})^d$ will have a global solution. As a matter of fact, the Koch and Tataru theorem [16] gives that this is true for a small initial value in $(BMO^{-\alpha/2})^d$ where we have $\dot{B}^{-\alpha/2}_{\infty,1} \subset \dot{B}^{-\alpha/2}_{\infty,2} \subset BMO^{-\alpha/2} = (-\Delta)^{\alpha/4} BMO = \dot{F}^{-\alpha/2}_{\infty,2} \subset \dot{B}^{-\alpha/2}_{\infty,\infty}$.

We don't detail the proof here, as it is exactly the same one as for the Koch and Tataru theorem (see [18] for details). Use the fixed-point theorem in the space of functions $u(t,x)$ which satisfy $\sup_{t>0} t^{1/2}\|u(t,.)\|_\infty < +\infty$ and

$$\sup_{t>0, x\in\mathbb{R}^n} t^{-\frac{n}{\alpha}} \int_0^t \int_{B(x,t^{\frac{1}{\alpha}})} |u(s,y)|^2\, ds\, dy < +\infty. \tag{78}$$

Note that the proof involves an integration by parts [using the fact that $(-\Delta)^{\alpha/2} e^{-t(-\Delta)^{\alpha/2}} f = -\partial_t (e^{-t(-\Delta)^{\alpha/2}} f)$, see Lemma 16.2 in [18]]. Thus, the proof does not involve domination by a positive kernel, and $BMO^{-\alpha/2}$ is not a subspace of $X^{\alpha,\alpha/2}$. But we have obviously (due to scaling invariance and local square integrability in $\mathcal{V}^{\alpha,\alpha/2}$) the embedding $X^{\alpha,\alpha/2} \subset BMO^{-\alpha/2}$.

9 Persistency.

When $\vec{u}_0$ is small in $(X^{\alpha,\beta})^d$ (or, when $\alpha = 2\beta$, in $(BMO^{-\alpha/2})^d$), we know that a solution $\vec{u}$ may be constructed through the iteration algorithm :

$$\vec{U}_0 = e^{-t(-\Delta)^{\alpha/2}} \vec{u}_0 \text{ and } \vec{U}_{k+1} = \vec{U}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}} \sigma(D)(\vec{U}_k \otimes \vec{U}_k)\, ds, \tag{79}$$

and that we have

$$\sup_{0<t} t^{1-\frac{\beta}{\alpha}} \|\vec{U}_0(t,.)\|_\infty + \sum_{k=0}^{+\infty} \sup_{0<t} t^{1-\frac{\beta}{\alpha}} \|\vec{U}_{k+1}(t,.) - \vec{U}_k(t,.)\|_\infty < +\infty. \tag{80}$$

This will allow us to use the persistency theory developed in [18]. Let us recall first the definition of a shift-invariant Banach space of local measures :

Definition 6 *A) A shift-invariant Banach space of test functions is a Banach space E such that we have the embeddings $\mathcal{D}(\mathbb{R}^n) \subset E \subset \mathcal{D}'(\mathbb{R}^n)$ are continuous and such that:*

- *shift-invariance : for all $x_0 \in \mathbb{R}^d$ and for all $f \in E$, $f(\cdot - x_0) \in E$ and $\|f\|_E = \|f(\cdot - x_0)\|_E$;*
- *scaling : for all $\lambda > 0$ there exists $C_\lambda > 0$ such that for all $f \in E$ $f(\lambda \cdot) \in E$ and $\|f(\lambda \cdot)\|_E \leq C_\lambda \|f\|_E$;*
- *$\mathcal{D}(\mathbb{R}^d)$ is dense in E.*

B) A shift-invariant Banach space of distributions is a Banach space E, which is the topological dual of a shift-invariant Banach space of test functions $E^{(*)}$. The space $E^{(0)}$ of smooth elements of E is defined as the closure of $\mathcal{D}(\mathbb{R}^d)$ in E.
C) A shift-invariant Banach space of local measures is a shift-invariant Banach space of distributions E such that for all $f \in E$ and for all $g \in \mathcal{S}(\mathbb{R}^d)$ we have $fg \in E$ and $\|fg\|_E \leq C_E \|f\|_E \|g\|_\infty$, where C_E is a positive constant (which depends neither on f nor on g).

An important property of shift–invariant Banach spaces E of distributions or of test functions is that convolution is a bounded bilinear operator from $L^1 \times E$ to E : $\|f * g\|_E \leq \|f\|_1 \|g\|_E$.

We measure regularity with semi–norms $\|f\|_{\dot{H}^\rho_E} = \|(-\Delta)^{\rho/2} f\|_E$, or $\|f\|_{\dot{B}^\rho_{E,q}} = \left(\sum_{j\in\mathbb{Z}} 2^{j\rho q} \|\Delta_j f\|_E^q\right)^{1/q}$. These are only semi–norms, but we shall work in spaces $L^\infty \cap \dot{H}^\rho$, or $L^\infty \cap \dot{B}^\rho_{E,q}$, so that we don't bother with the kernel of the semi–norms.

The persistency theory then tells us the following :

Theorem 7 *Let $\vec{u}_0$ be small enough in $(X^{\alpha,\beta})^d$ (or, when $\alpha = 2\beta$, in $(BMO^{-\alpha/2})^d$) to grant that*

$$\sup_{0<t} t^{1-\frac{\beta}{\alpha}} \|\vec{U}_0(t,.)\|_\infty + \sum_{k=0}^{+\infty} \sup_{0<t} t^{1-\frac{\beta}{\alpha}} \|\vec{U}_{k+1}(t,.) - \vec{U}_k(t,.)\|_\infty < +\infty \tag{81}$$

and

$$\sup_{0<t} \|\vec{U}_0(t,.)\|_{\dot{B}^{\beta-\alpha}_{\infty,\infty}} + \sum_{k=0}^{+\infty} \sup_{0<t} \|\vec{U}_{k+1}(t,.) - \vec{U}_k(t,.)\|_{\dot{B}^{\beta-\alpha}_{\infty,\infty}} < +\infty. \tag{82}$$

Let F be a shift-invariant Banach space of local measures.

- *If moreover $\vec{u}_0 \in F^d$, then the limit $\vec{u}$ of $\vec{U}_k$ satisfies $\vec{u} \in L^\infty((0,+\infty), F^d)$.*
- *Let E be a space of regular distributions over F: for some positive ρ and for some $q \in [1,+\infty]$, $E = \dot{H}^\rho_F$ or $E = \dot{B}^\rho_{F,q}$ (with $1 \le q \le \infty$). If $\vec{u}_0 \in E^d$ then $\vec{u} \in L^\infty((0,\infty), E^d)$*

Proof : If $\vec{u}_0 \in F^d$, then $\vec{U}_0 \in L^\infty((0,+\infty), F^d)$. We then write, for $\vec{W}_k = \vec{U}_k - \vec{U}_{k-1}$ and $\alpha_k = \sup_{t>0} t^{1-\frac{\beta}{\alpha}} \|\vec{W}_k(t,.)\|_\infty$, :

$$\begin{aligned} \|\vec{W}_{k+1}(t,.)\|_F &\le \int_0^t \frac{C}{|t-s|^{\frac{\beta}{\alpha}}} \|\vec{U}_k(s,.) \otimes \vec{W}_k(s,.) + \vec{W}_k(s,.) \otimes \vec{U}_{k-1}(s,.)\|_F \, ds \\ &\le (\|\vec{U}_k\|_{L^\infty_t F} + \|\vec{U}_{k-1}\|_{L^\infty_t F})\alpha_k \int_0^t \frac{C}{|t-s|^{\frac{\beta}{\alpha}}} \frac{1}{|s|^{1-\frac{\beta}{\alpha}}} \, ds \\ &= C'(\|\vec{U}_k\|_{L^\infty_t F} + \|\vec{U}_{k-1}\|_{L^\infty_t F})\alpha_k. \end{aligned} \tag{83}$$

If $A_k = \sum_{j=0}^k \|\vec{W}_j\|_{L^\infty_t F}$, we have $\|\vec{u}\|_{L^\infty_t F} \le \sup_{k \in \mathbb{R}^n} A_j$. Moreover, we have

$$A_{k+1} = A_k + \|\vec{W}_{k+1}\|_{L^\infty_t F} \le A_k(1 + 2C'\alpha_k) \tag{84}$$

so that $\vec{u} \in L^\infty((0,+\infty), F^d)$ with $\|\vec{u}\|_{L^\infty_t F} \le \|\vec{u}_0\|_F \prod_{k=0}^\infty (1 + 2C'\alpha_k)$

We now consider the case when $\vec{u}_0 \in E^d$. We find that $\vec{U}_0 \in L^\infty((0,+\infty), (\dot{B}^\rho_{F,\infty})^d)$. We write $\vec{W}_k = \vec{U}_k - \vec{U}_{k-1}$, $\alpha_k = \sup_{t>0} t^{1-\frac{\beta}{\alpha}} \|\vec{W}_k(t,.)\|_\infty$, $\Gamma = \sum_{k \in \mathbb{N}} \|\vec{W}_k\|_{L^\infty \dot{B}^{\beta-\alpha}_{\infty,\infty}}$ and $B_k = \sum_{j=0}^k \|\vec{W}_j\|_{L^\infty_t \dot{B}^\rho_{F,\infty}}$.

We begin by estimating fg when $f, g \in L^\infty \cap \dot{B}^{\beta-\alpha}_{\infty,\infty} \cap \dot{B}^\rho_{F,\infty}$. Using the Littlewood–Paley decomposition $f = \sum_{j \in \mathbb{Z}} \Delta_j f = S_k f + \sum_{j \ge k} \Delta_j f$ (see [18]), we write $fg = u + v$, where $u = \sum_k \sum_{j \le k+3} \Delta_j f \Delta_k g = \sum_{k \in \mathbb{Z}} S_{k+4} f \Delta_k g$ and $v = \sum_k \sum_{j \ge k+4} \Delta_j f \Delta_k g = \sum_j \Delta_j f S_{j-3} g$. We have $\|\Delta_l(S_{k+4} f \Delta_k g)\|_F \le C\|f\|_\infty \|\Delta_k g\|_F \le C\|f\|_\infty \|g\|_{\dot{B}^\rho_{F,\infty}} 2^{-k\rho}$ if $k \ge l-6$, and $= 0$ if $k < l-6$. Hence $u \in \dot{B}^\rho_{F,\infty}$ and

$$\|u\|_{\dot{B}^\rho_{F,\infty}} \le C\|f\|_\infty \|g\|_{\dot{B}^\rho_{F,\infty}}. \tag{85}$$

On the other hand, when $k \le j-4$, we have $\|\Delta_l(\Delta_j f \Delta_k g)\|_F = 0$ when $|l-j| \ge 3$; if $|l-j| \le 2$, we write $\|\Delta_l(\Delta_j f \Delta_k g)\|_F \le C2^{-k\rho} \|g\|_{\dot{B}^\rho_{F,\infty}} \|f\|_\infty$ and $\|\Delta_l(\Delta_k f \Delta_j g)\|_F \le 2^{k(\alpha-\beta)} \|g\|_{\dot{B}^{\beta-\alpha}_{\infty,\infty}} 2^{-j\rho} \|f\|_{\dot{B}^\rho_{F,\infty}}$ We then fix λ such that $\frac{\rho}{\rho+\alpha-\beta} < \lambda < 1$, and we find that

$$\|\Delta_l(\Delta_j f S_{j-3} g)\|_F \le C2^{j(-\rho+\lambda(\alpha-\beta))} (\|f\|_{\dot{B}^\rho_{F,\infty}} \|g\|_{\dot{B}^{\beta-\alpha}_{\infty,\infty}})^\lambda (\|g\|_{\dot{B}^\rho_{F,\infty}} \|f\|_\infty)^{1-\lambda} \tag{86}$$

and thus

$$\| v\|_{\dot B^{\rho-\lambda(\alpha-\beta)}_{\infty,\infty}} \le C(\|f\|_{\dot B^\rho_{F,\infty}}\|g\|_{\dot B^{\beta-\alpha}_{\infty,\infty}})^\lambda(\|g\|_{\dot B^\rho_{F,\infty}}\|f\|_\infty)^{1-\lambda}. \tag{87}$$

The second step is to check that $e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)$ maps $(\dot B^\delta_{F,\infty})^{d\times d}$ to $(\dot B^\rho_{F,1})^d$ for $\delta < \rho+\beta$:

$$\|e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)f\|_{\dot B^\rho_{F,1}} \le C(t-s)^{-\frac{\rho+\beta-\delta}{\alpha}}\|f\|_{\dot B^\delta_{F,\infty}}. \tag{88}$$

Combining these estimates, we find that

$$\begin{aligned}\|\vec W_{k+1}(t,.)\|_{\dot B^\rho_{F,1}} \le{}& \alpha_k B_k\int_0^t \frac{C}{|t-s|^{\frac\beta\alpha}}\frac{1}{s^{1-\frac\beta\alpha}}\,ds\\ &+(\alpha_kB_k)^\lambda(\Gamma\|\vec W_k(t,.)\|_{\dot B^\rho_{F,1}})^{1-\lambda}\int_0^t\frac{C}{|t-s|^{\frac\beta\alpha+(1-\lambda)(1-\frac\beta\alpha)}}\frac{1}{s^{\lambda(1-\frac\beta\alpha)}}\,ds.\end{aligned} \tag{89}$$

We take λ close enough to 1 to ensure that

$$\lambda(1-\frac\beta\alpha)<1 \text{ and } \frac\beta\alpha+(1-\lambda)(1-\frac\beta\alpha)<1. \tag{90}$$

We thus have

$$\|\vec W_{k+1}(t,.)\|_{\dot B^\rho_{F,1}} \le C\alpha_kB_k + C(\alpha_kB_k)^\lambda(\Gamma\|\vec W_k(t,.)\|_{\dot B^\rho_{F,1}})^{1-\lambda}. \tag{91}$$

We find that

$$\|\vec W_{k+1}(t,.)\|_{\dot B^\rho_{F,1}} \le \delta_kB_k+\frac12\|\vec W_k(t,.)\|_{\dot B^\rho_{F,1}} \tag{92}$$

with $\delta_k = C\alpha_k(1+\lambda(2(1-\lambda)\Gamma)^{\frac{1-\lambda}{\lambda}})$. For $1\le p\le k$, we also have $\|\vec W_p(t,.)\|_{\dot B^\rho_{F,1}} \le \delta_{p-1}B_k+\frac12\|\vec W_{p-1}(t,.)\|_{\dot B^\rho_{F,1}}$, while $\|\vec W_0(t,.)\|_{\dot B^\rho_{F,1}}\le\delta_{-1}B_k$ if we take $\delta_{-1}=1$. This gives

$$\|\vec W_{k+1}(t,.)\|_{\dot B^\rho_{F,1}} \le B_k(\sum_{j=-1}^k\delta_j2^{j-k}) \tag{93}$$

so that

$$B_{k+1}\le B_k(1+\sum_{j=-1}^k\delta_j2^{j-k}) \tag{94}$$

and finally

$$\sup_{k\in\mathbb N}B_k\le B_0\prod_{k=0}^{+\infty}(1+\sum_{j=-1}^k\delta_j2^{j-k})<+\infty. \tag{95}$$

The theorem is proved : for $E = \dot H^\sigma_F$ or $\dot B^\rho_{F,q}$, we have

$$\|\vec u\|_{L^\infty E} \le \|\vec u_0\|_E + \sum_{k=1}^\infty \|\vec W_k\|_{L^\infty \dot B^\rho_{\infty,1}} < +\infty \tag{96}$$

and we conclude since $\dot B^\rho_{\infty,1} \subset E$. ◇

10 A Triebel-space approach to cheap solutions.

Recall that $X^{\alpha,\beta}$ is defined by $u_0 \in X^{\alpha,\beta} \Leftrightarrow 1_{t>0} e^{-t(-\Delta)^{\alpha/2}} u_0 \in \mathcal{V}^{\alpha,\beta}$. In section 8, we tried to give an approximation of $X^{\alpha,\beta}$ by Besov spaces. Another way of approximating $X^{\alpha,\beta}$ is to approach $\mathcal{V}^{\alpha,\beta}$ with Morrey spaces, using the Fefferman–Phong inequality.

We thus define $\mathcal{F}^{\alpha,\beta}_p$ for $2 < p \le \frac{n+\alpha}{\alpha-\beta}$ by :

$$u_0 \in \mathcal{F}^{\alpha,\beta}_p \Leftrightarrow u_0 \in \dot B^{\beta-\alpha}_{\infty,\infty} \text{ and } 1_{t>0} e^{-t(-\Delta)^{\alpha/2}} u_0 \in \dot M_\alpha^{p,\frac{n+\alpha}{\alpha-\beta}}. \tag{97}$$

We have of course (for $2 < p \le \frac{n+\alpha}{\alpha-\beta}$)

$$\mathcal{F}^{\alpha,\beta}_p \subset X^{\alpha,\beta} \subset \dot B^{\beta-\alpha}_{\infty,\infty}. \tag{98}$$

Assume now that $p\frac{\alpha-\beta}{\beta} > 1$. For $R > 0$ and $x_0 \in \mathbb{R}^n$, we find that

$$\begin{aligned}\int_{B(x_0,R)} \int_0^{+\infty} |e^{-t(-\Delta)^{\alpha/2}} u_0|^p \, dt\, dy \le & \iint_{\rho_\alpha(t-0,y-x_0)<R} |1_{t>0} e^{-t(-\Delta)^{\alpha/2}} u_0|^p \, dt\, dy \\ & + \int_{B(x_0,y)} \int_{R^\alpha}^{+\infty} |e^{-t(-\Delta)^{\alpha/2}} u_0|^p \, dt\, dy \\ & \le C\|u_0\|^p_{\mathcal{F}^{\alpha,\beta}_p} \, R^{n+\alpha-p(\alpha-\beta)} \\ & + C\|u_0\|_{\dot B^{\beta-\alpha}_{\infty,\infty}} R^n R^{\alpha(1-p\frac{\alpha-\beta}{\alpha})}.\end{aligned} \tag{99}$$

Thus, we find that $\left(\int_0^{+\infty} |e^{-t(-\Delta)^{\alpha/2}} u_0|^p \, dt\right)^{1/p} \in \dot M^{p,q}(\mathbb{R}^n)$, where q satisfies the Serrin scaling relation $\frac{\alpha}{p} + \frac{n}{q} = \alpha - \beta$. We thus see that $\mathcal{F}^{\alpha,\beta}_p$ is aTriebel–Lizorkin–Morrey space, as studied by Sickel, Yang and Yuan [26] :

Theorem 8 *For $2 < p < \frac{n+\alpha}{\alpha-\beta}$ such that $p\frac{\alpha-\beta}{\beta} > 1$, the space $\mathcal{F}^{\alpha,\beta}_p$ is equal to the homogeneous Triebel–Lizorkin–Morrey space $\dot F^{-\frac{\alpha}{p},\frac{1}{p}-\frac{1}{q}}_{p,p}$.*

11 Examples

11.1 The Navier–Stokes equations.

The Navier–Stokes equations are given on $(0,+\infty)\times\mathbb{R}^3$ by

$$\begin{cases} \partial_t\vec{u}+(\vec{u}.\vec{\nabla})\vec{u}=\Delta\vec{u}-\vec{\nabla}p \\ \text{div }\vec{u}=0. \end{cases} \tag{100}$$

Using the Leray projection operator $\mathbb{P}$ on divergence-free vector fields and the fact that $\vec{u}$ is divergence free, we get rid of the pressure (on the assumption that p is small at infinity) and get

$$\partial_t\vec{u}=\Delta\vec{u}-\mathbb{P}\text{ div }(\vec{u}\otimes\vec{u})=0. \tag{101}$$

This is a system of equations analogous to (2) with $\alpha=2$ and $\beta=1$. Since 2001, from the Koch and Tataru theorem [16], we know that we may find a global solution as soon as the initial value $\vec{u}_0$ is small enough in BMO^{-1}.

Initially, in 1964 [10], the proof of existence of global solutions was given for an initial value in $H^s(\mathbb{R}^3)$ with $s\geq 1/2$ and with a small norm in $\dot{H}^{1/2}(\mathbb{R}^3)$. It is easy to see that $\dot{H}^{1/2}\subset X^{2,1}$ so that the existence of a global solution in $L^\infty_t H^s$ is then a combination of Theorems 7 and 9.

Later, in 1984 [15], Kato proved existence of global solutions in $L^\infty_t L^3$ for an initial value with a small norm in L^3. Again, this can be proved through a combination of Theorems 7 and 9, as $L^3\subset X^{2,1}$.

Then, in 1995 [5], Cannone considered the case of an initial value in L^3, with a small norm in $\dot{B}^{-1+\frac{3}{q}}_{q,\infty}$ with $3<q<+\infty$ and obtained existence of a global solution in $L^\infty_t L^3$. Again, this can be proved through a combination of Theorems 7 and 9, as $\dot{B}^{-1+\frac{3}{q}}_{q,\infty}\subset X^{2,1}$.

Let us recall that ill-posedness in the critical Besov space $\dot{B}^{-1}_{\infty,\infty}$ was established in 2008 by Bourgain and Pavlović [4], following the example given by Montgomery–Smith for the cheap equation [24].

11.2 The modified Navier–Stokes equations.

The diffusion term in the Navier–Stokes equations has been modified in some studies by a fractional diffusion :

$$\begin{cases} \partial_t\vec{u}+(\vec{u}.\vec{\nabla})\vec{u}=-(-\Delta)^{\alpha/2}\vec{u}-\vec{\nabla}p \\ \text{div }\vec{u}=0 \end{cases} \tag{102}$$

Initially, α was taken larger than 2 (it is the hyperdiffusive case). Indeed, when $\alpha > 5/2$, the problem is locally well posed in L^2, and, using the energy inequality that ensures that the norm in L^2 stays bounded, local existence is turned into global existence [21]. More recently, the case $1 < \alpha < 2$ has been considered, due to the increased use of α–stable processes in non-local diffusion models.

Using again the Leray projection operator $\mathbb{P}$, we get the system

$$\partial_t \vec{u} = -(-\Delta)^{\alpha/2}\vec{u} - \mathbb{P} \text{ div } (\vec{u} \otimes \vec{u}) = 0 \tag{103}$$

This is a system of equations analogous to (2) with $\alpha > 1$ and $\beta = 1$.

When $1 < \alpha < 2$, we know from Theorem 6 that we may find a global solution as soon as the initial value $\vec{u}_0$ is small enough in $\dot{B}^{1-\alpha}_{\infty,\infty}$ (this is the theorem of Yu and Zhai [28]).

When $\alpha > 2$, in accordance with Theorem 6 and the remark we made after the Theorem, Cheskidov and Shvydkoy [6] have shown illposedness in $\dot{B}^{\gamma}_{-\infty,\infty}$ for $1 - \alpha \leq \gamma \leq -\alpha/2$.

11.3 The subcritical quasi-geostrophic equation

The subcritical quasi-geostrophic equation is given by the system

$$\begin{cases} \partial_t \theta + (\vec{u}.\vec{\nabla})\theta = -(-\Delta)^{\alpha/2}\theta \\ (u_1, u_2) = (-\frac{\partial_2}{\sqrt{-\Delta}}\theta, \frac{\partial_1}{\sqrt{-\Delta}}\theta) \end{cases} \tag{104}$$

where $1 < \alpha < 2$.

If we use the unknowns (θ, u_1, u_2), we get the system

$$\begin{cases} \partial_t \theta = -(-\Delta)^{\alpha/2}\theta - \text{ div } (\theta\vec{u}) \\ \partial_t u_1 = -(-\Delta)^{\alpha/2}u_1 + \frac{1}{\sqrt{-\Delta}}\partial_2 \text{ div } (\theta\vec{u}) \\ \partial_t u_2 = -(-\Delta)^{\alpha/2}u_2 - \frac{1}{\sqrt{-\Delta}}\partial_1 \text{ div } (\theta\vec{u}). \end{cases} \tag{105}$$

This is a system of equations analogous to (2) with $1 < \alpha < 2$ and $\beta = 1$. We know from Theorem 6 that we may find a global solution as soon as the initial value θ_0 is small enough in $\dot{B}^{1-\alpha}_{\infty,\infty}$ (this is the theorem of May and Zahrouni [22]).

In particular, when $\theta_0 \in L^{\frac{2}{\alpha-1}} \subset \dot{B}^{1-\alpha}_{\infty,\infty}$ and is small in $\dot{B}^{1-\alpha}_{\infty,\infty}$, we know that the solution θ satisfies $\theta \in L^\infty L^{\frac{2}{\alpha-1}}$. If $\theta_0 \in L^q$ with $\frac{2}{\alpha-1} < q < +\infty$, we have local existence in $L^\infty L^q$; moreover θ satisfies a maximum principle : $\|\theta(t,.)\|_q \leq \|\theta_0\|_q$, and this implies that local existence is turned into global existence [27].

11.4 The parabolic-elliptic Keller–Segel system

The parabolic-elliptic Keller–Segel system is given on $(0,+\infty)\times\mathbb{R}^n$ by

$$\begin{cases} \partial_t u = \Delta u - \text{ div } (u\vec{\nabla}\chi) \\ \qquad\qquad -\Delta\chi = u. \end{cases} \tag{106}$$

If we use the unknowns $\vec{v} = \vec{\nabla}\chi = -\frac{1}{\Delta}\vec{\nabla}u$, we get the system

$$\partial_t\vec{v} = \Delta\vec{v} + \sum_{i=1}^n \frac{1}{\Delta}\vec{\nabla} \text{ div } \partial_i(v_i\vec{v}) - \frac{1}{2}\vec{\nabla}(\sum_{i=1}^n v_i^2). \tag{107}$$

This is a system of equations analogous to (2) with $\alpha = 2$ and $\beta = 1$. We thus know that we may find a global solution as soon as the initial value $\vec{v}_0$ is small enough in BMO^{-1}, i.e. u_0 is small enough in BMO^{-2}. This result seems to be new : in [13], the case $u_0 \in \dot{B}_{q,\infty}^{-2+\frac{2}{q}}$ is discussed.

Let us assume that $u_0 \in L^{n/2}\cap L^1$ (and $n \geq 2$), with the norm of u_0 small enough in BMO^{-2} (remark that $L^{n/2} \subset BMO^{-2}$). Then we know from Theorem 9 that the solution $\vec{v}$ will belong to $L^\infty \dot{H}^1_{L^{n/2}} \cap L^\infty \dot{H}^1_{L^1}$, and that $\vec{w} = \vec{v} - e^{t\Delta}\vec{v}_0 \in L^\infty \dot{B}^1_{n/2,1} \cap L^\infty \dot{B}^1_{1,1}$. Writing

$$u = e^{t\Delta}u_0 + \text{ div } \vec{w}, \tag{108}$$

we find that $u \in L^\infty L^{n/2} \cap L^\infty L^1$: this is the theorem of Corrias, Perthame and Zaag [8].

A final remark is that one usually deals with positive solutions (as u represents a density of cells). We have the inequalities

$$\|u_0\|_{\dot{B}^{-2}_{\infty,\infty}} \leq C\|u_0\|_{BMO^{-2}} \leq C'\|u_0\|_{\dot{M}^{1,n/2}} \tag{109}$$

when the space $\dot{M}^{1,n/2}$ is the space of locally bounded (signed) measures μ such that : $\sup_{x_0\in\mathbb{R}^n, R>0} R^{2-n}\int_{B(x_0,R)} d|\mu(y)| < +\infty$. When u_0 is a non-negative distribution (i.e. a non-negative locally bounded measure), we have the reverse inequality

$$\|u_0\|_{\dot{M}^{1,n/2}} \leq C''\|u_0\|_{\dot{B}^{-2}_{\infty,\infty}} \tag{110}$$

(see [20]). Thus, the critical norm to be controlled is indeed the norm in the Morrey space $\dot{M}^{1,n/2}$.

A Homogeneous distributions.

in this section, we recollect some more or less classical estimates on homogeneous distributions.

Lemma 4 *Let $\gamma > -n$ and σ be a smooth (positively) homogeneous function of degree γ on $\mathbb{R}^n - \{0\}$:*

$$\text{for } \lambda > 0 \text{ and } \xi \neq 0, \quad \sigma(\lambda\xi) = \lambda^\gamma \sigma(\xi). \tag{111}$$

Let $\theta \in \mathcal{D}$ be equal to 1 on a neighborhood of 0, and let T, T_0 and T_1 be the inverse Fourier transforms of σ, $\theta\sigma$ and $(1-\theta)\sigma$. Then :

- *the distribution T is positively homogeneous of degree $-n-\gamma$*
- *the restriction of T to $\mathbb{R}^n \setminus \{0\}$ is a smooth (positively) homogeneous function*
- *the distribution T_0 is a smooth function that satisfies*
$$\sup_{x\in\mathbb{R}^n} (1+|x|)^{n+\gamma} |T_0(x)| < +\infty$$
- *the restriction of T_1 to $\mathbb{R}^n \setminus \{0\}$ is a smooth function that satisfies for every $N \in \mathbb{N}$*
$$\sup_{|x|\geq 1} |x|^N |T_1(x)| < +\infty.$$

Proof : Just write that T is homogeneous, T_0 is smooth and that the Fourier transform of $|x|^{2N} T_1$ is integrable as soon as $2N > n+\gamma$. The size estimates are then obvious. (The derivatives of T, T_0 and T_1 are controlled in the same way, as the Fourier transforms of the derivatives of T are still homogeneous). ◇

Lemma 5 *Let $\gamma > 0$ and σ be a smooth (positively) homogeneous function of degree γ on $\mathbb{R}^n - \{0\}$. Let $\theta \in \mathcal{D}$ be equal to 1 on a neighborhood of 0, and let U be the inverse Fourier transform of θe^σ. Then the distribution U is a smooth function that satisfies*

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^{n+\gamma} |U(x)| < +\infty.$$

We have a similar decay for the derivatives of U : if $\delta \in \mathbb{N}^n$, then

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^{n+\gamma+|\delta|} |\partial^\delta U(x)| < +\infty.$$

Proof : Write $e^t = \sum_{k=0}^{N} \frac{t^k}{k!} + R_N(t)$. If $N\gamma > 2M > n + \gamma$, we find that $(-\Delta)^{2M}(\theta R_N(\sigma)))$ is integrable, so that the inverse Fourier transform of $\theta R_N(\sigma)$ is $O(|x|^{-2M})$ for $|x| \to +\infty$. On the other hand the inverse Fourier transform of $\theta\sigma^k$ is $O(|x|^{-n-k\gamma})$ for $k \geq 1$, and the inverse Fourier transform of $\theta\sigma^0 = \theta$ belongs to the Schwartz class.

The same proof holds for $\partial^\delta U$ (by differentiating $\xi^\delta\theta R_N(\sigma)$). ◇

Lemma 6 *Let $\gamma > 0$ and σ be a smooth (positively) homogeneous function of degree γ on $\mathbb{R}^n - \{0\}$. Assume that $\sigma(\xi) > 0$ for all $\xi \in \mathbb{R}^n - \{0\}$. Let V be the inverse Fourier transform of e^σ. Then the distribution V is a smooth function that satisfies*

$$\sup_{x\in\mathbb{R}^n}(1 + |x|)^{n+\gamma}|V(x)| < +\infty.$$

We have a similar decay for the derivatives of V : if $\delta \in \mathbb{N}^n$, then

$$\sup_{x\in\mathbb{R}^n}(1 + |x|)^{n+\gamma+|\delta|}|\partial^\delta V(x)| < +\infty.$$

Proof : With the notations of Lemma 5, we have $V - U \in \mathcal{S}(\mathbb{R}^n)$. ◇

Lemma 7 *Let $0 < \beta < \alpha$, σ_β and σ_α be smooth (positively) homogeneous functions (respectively, of degree β and α) on $\mathbb{R}^n - \{0\}$. Assume that $\sigma_\alpha(\xi) > 0$ for all $\xi \in \mathbb{R}^n - \{0\}$. Let V be the inverse Fourier transform of $e^{\sigma_\alpha}\sigma_\beta$. Then the distribution V is a smooth function that satisfies*

$$\sup_{x\in\mathbb{R}^n}(1 + |x|)^{n+\beta}|V(x)| < +\infty.$$

Proof : Let $\theta \in \mathcal{D}$ be equal to 1 on a neighborhood of 0, and let T_β be the inverse Fourier transform of $\theta\sigma_\beta$. Let V_α be the inverse Fourier transform of e^{σ_α}. Now, remark that $V - V_\alpha * T_\beta$ belongs to $\mathcal{S}$, while we control $V_\alpha * T_\beta$ with Lemmas 4 and 6. ◇

B Semilinear equation with a positive kernel.

In this section, we discuss the general integral equation

$$f(x) = f_0(x) + \int_X K(x, y)f^2(y)\, d\mu(y) \tag{112}$$

where μ is a non-negative σ-finite measure on a space X and K is a positive measurable function on $X \times X$: $K(x, y) > 0$ almost everywhere. We shall make a stronger assumption on K : there exists a sequence X_n of measurable subsets of X such that $X = \cup_{n\in\mathbb{N}} X_n$ and

$$\int_{X_n}\int_{X_n} \frac{d\mu(x)\ d\mu(y)}{K(x,y)} < +\infty. \tag{113}$$

We start with the following easy lemma :

Proposition 7 *Let f_0 be non-negative and measurable and let f_n be inductively defined as*

$$f_{n+1}(x) = f_0(x) + \int_X K(x,y) f_n^2(y)\ d\mu(y). \tag{114}$$

Let $f = \sup_{n\in\mathbb{N}} f_n(x)$. Then either $f = +\infty$ almost everywhere or $f < +\infty$ almost everywhere. If $f < +\infty$, then f is a solution to equation (112).

Proof : Due to the inequalities $f_0 \geq 0$ and $K \geq 0$, we find by induction that $0 \leq f_n$, so that f_{n+1} is well defined (with values in $[0, +\infty]$); we get moreover (by induction, as well) that $f_n \leq f_{n+1}$. We thus may apply the theorem of monotone convergence and get that $f(x) = f_0(x) + \int_X K(x,y) f^2(y)\ d\mu(y)$. If $f = +\infty$ on a set of positive measure, then $\int_X K(x,y) f^2(y)\ d\mu(y) = +\infty$ almost everywhere and $f = +\infty$ almost everywhere. $\diamond$

We see that if f_0 is such that equation (112) has a solution f which is finite almost everywhere, then we have $f_0 \leq f$ and $\int_X K(x,y) f^2(y)\ d\mu(y) \leq f(x)$. This is almost a characterization of such functions f_0 :

Proposition 8 *Let C_K be the set of non-negative measurable functions Ω such that $\Omega < +\infty$ (almost everywhere) and $\int_X K(x,y)\Omega^2(y)\ d\mu(y) \leq \Omega(x)$. Then, if $\Omega \in C_K$ and if f_0 is a non-negative measurable function such that $f_0 \leq \frac{1}{4}\Omega$, equation (112) has a solution f which is finite almost everywhere.*

Proof : Take the sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined in Proposition 7. By induction, we see that $f_n \leq \frac{1}{2}\Omega$, and thus $f = \sup_n f_n \leq \frac{1}{2}\Omega$. $\diamond$

This remark leads us to define a Banach space of measurable functions in which it is natural to solve equation (112) :

Proposition 9 *Let $\mathcal{E}_K$ be the space of measurable functions f on X such that there exists $\lambda \geq 0$ and $\Omega \in C_K$ such that $|f(x)| \leq \lambda\Omega$ almost everywhere. Then :*

- $\mathcal{E}_K$ *is a linear space*

- *the function* $f \in \mathcal{E}_K \mapsto \|f\|_K = \inf\{\lambda \ / \ \exists\Omega \in C_k \ |f| \leq \lambda\Omega\}$ *is a semi-norm on* $\mathcal{E}_K$

- $\|f\|_K = 0 \Leftrightarrow f = 0$ *almost everywhere*

- *The normed linear space* E_K *(obtained from* $\mathcal{E}_K$ *by quotienting with the relationship* $f \sim g \Leftrightarrow f = g$ *a.e.) is a Banach space.*

- *If* $f_0 \in \mathcal{E}_K$ *is non-negative and satisfies* $\|f_0\|_K < \frac{1}{4}$*, then equation (112) has a non-negative solution* $f \in \mathcal{E}_K$*.*

Proof : Since $t \mapsto t^2$ is a convex function, we find that C_K is a balanced convex set and thus that $\mathcal{E}_K$ is a linear space and $\| \ \|_K$ is a semi-norm on $\mathcal{E}_K$.

Next, we see that, for $\Omega \in C_K$ and $q \in \mathbb{N}$, we have

$$\int_{X_q} \Omega(x) \ d\mu(x) \leq \frac{\int_{X_q}\int_{X_q} \frac{d\mu(x) \ d\mu(y)}{K(x,y)}}{(\mu(X_q))^2}. \tag{115}$$

To prove (115), we recall that Ω is finite almost everywhere and that X is locally finite. Writing $X = X = \cup_{n\mathbb{N}} Y_n$ with $\mu(Y_n) < +\infty$, we introduce $Z_p = \{x \in \cup_{n=0}^p Y_n \ / \ \Omega(x) \leq p\}$ and $\Omega_p = 1_{Z_p}(x)\Omega(x)$. We have, by monotonous convergence, $\mu(X_q) = \lim_{p\to+\infty} \mu(Z_p \cap X_q)$, $\int_{X_q} \Omega(x) \, d\mu(x) = \lim_{p\to+\infty} \int_{Z_p\cap X_q} \Omega_p(x) \, d\mu(x)$; moreover, $\int_{Z_p\cap X_q} \Omega_p(x) \, d\mu(x) < +\infty$ and, for $x \in Z_p$, we have $\int_X K(x,y)\Omega^2(y) \, d\mu(y) \leq \Omega(x) = \Omega_p(x)$. Then, (115) is easily checked by writing that

$$\int\int_{(Z_p\cap X_q)^2} \Omega_p(y) \ d\mu(y) \ d\mu(x) \leq \sqrt{\int_{X_q}\int_{X_q} \frac{d\mu(x) \ d\mu(y)}{K(x,y)}} \sqrt{\int_{Z_p\cap X_q} [\int K(x,y)\Omega^2(y) \ d\mu(y)] \ d\mu(x)}. \tag{116}$$

Thus we find that, when $\|f\|_K = 0$, we have $\int_{X_q} |f(x)| \ d\mu(x) = 0$ for all q, so that $f = 0$ almost everywhere.

Similarly, we find that if $\lambda_n \geq 0$, $\Omega_n \in C_K$ and $\sum_{n\in\mathbb{N}} \lambda_n = 1$, then, if $\Omega = \sum_{n\in\mathbb{N}} \lambda_n \Omega_n$, we have (by dominated convergence),

$$\int_{X_q} \Omega(x) \ d\mu(x) \leq \frac{\int_{X_q}\int_{X_q} \frac{d\mu(x) \ d\mu(y)}{K(x,y)}}{(\mu(X_q))^2} \tag{117}$$

so that $\Omega < +\infty$ almost everywhere. Moreover (by dominated convergence) we have $\Omega \in C_K$. From that, we easily get that E_K is complete.

Finally, existence of a solution of (112) when $\|f_0\|_K < \frac{1}{4}$ is a consequence of Proposition 8. $\diamond$

An easy corollary of Proposition 9 is the following one :

Proposition 10 *If E is a Banach space of measurable functions such that :*

- $f \in E \Rightarrow |f| \in E$ *and* $\| \, |f| \, \|_E \le C_E \|f\|_E$
- $\| \int_X K(x,y) f^2(y) \, d\mu(y) \|_E \le C_E \|f\|_E^2$

then E is continuously embedded into E_K.

Now, we recall a result of Kalton and Verbitsky that characterizes the space E_K for a general class of kernels K.

Theorem 9 (Kalton and Verbitsky [14], Theorem 5.7) *Assume that the kernel K satisfies :*

- $\rho(x,y) = \frac{1}{K(x,y)}$ *is a quasi-metric :*
  1. $\rho(x,y) = \rho(y,x) \ge 0$.
  2. $\rho(x,y) = 0 \Leftrightarrow x = y$.
  3. $\rho(x,y) \le \kappa(\rho(x,z) + \rho(z,y))$.
- *K satisfies the following inequality : there exists a constant $C > 0$ such that, for all $x \in X$ and all $R > 0$, we have*

$$\int_0^R \int_{\rho(x,y)<t} d\mu(y) \frac{dt}{t^2} \le CR \int_R^{+\infty} \int_{\rho(x,y)<t} d\mu(y) \frac{dt}{t^3}. \tag{118}$$

Then the following assertions are equivalent for a measurable function f on X :

- *(A)* $f \in E_K$.
- *(B) There exists a constant C such that, for all $g \in L^2$, we have*

$$\int_X |f(x)|^2 \Big| \int_X K(x,y) g(y) \, d\mu(y) \Big|^2 \, d\mu(x) \le C \|g\|_2^2. \tag{119}$$

A direct consequence of this theorem is the following one :

Theorem 10 *Let (X, δ, μ) be a space of homogeneous type :*

- *for all $x, y \in X$, $\delta(x,y) \geq 0$*
- $\delta(x,y) = \delta(y,x)$
- $\delta(x,y) = 0 \Leftrightarrow x = y$
- *there is a positive constant κ such that :*
$$\text{for all } x,y,z \in X, \delta(x,y) \leq \kappa(\delta(x,z) + \delta(z,y)) \tag{120}$$
- *there exists postive A, B and Q which satisfy :*
$$\text{for all } x \in X,\ \text{for all } r > 0, Ar^Q \leq \int_{\delta(x,y)<r} d\mu(y) \leq Br^Q \tag{121}$$

Let

$$K_\alpha(x,y) = \frac{1}{\delta(x,y)^{Q-\alpha}} \tag{122}$$

(where $0 < \alpha < Q/2$) and E_{K_α} the associated Banach space (defined in Proposition 9). Let $\mathcal{I}_\alpha$ be the Riesz operator associated to K_α :

$$\mathcal{I}_\alpha f(x) = \int_X K_\alpha(x,y) f(y)\ d\mu(y). \tag{123}$$

We define two further linear spaces associated to K_α :

- *the potential space W^α defined by*
$$g \in W^\alpha \Leftrightarrow \exists h \in L^2\ g = \mathcal{I}_\alpha h \tag{124}$$
- *the multiplier space $\mathcal{V}^\alpha$ defined by*
$$f \in \mathcal{V}^\alpha \Leftrightarrow \|f\|_{\mathcal{V}^\alpha} = \big(\sup_{\|h\|_2 \leq 1} \int_X |f(x)|^2 |\mathcal{I}_\alpha h(x)|^2\ d\mu(x)\big)^{1/2} < +\infty \tag{125}$$
(so that pointwise multiplication by a function in $\mathcal{V}^\alpha$ maps boundedly W^α to L^2).

Then we have (with equivalence of norms) for $0 < \alpha < Q/2$:

$$E_{K_\alpha} = \mathcal{V}^\alpha. \tag{126}$$

Proof : It is enough to see that $At^{\frac{Q}{Q-\alpha}} \leq \int_{\rho(x,y)<t} d\mu(y) \leq Bt^{\frac{Q}{Q-\alpha}}$ (with $\rho(x,y) = \frac{1}{K(x,y)}$) and that $1 < \frac{Q}{Q-\alpha} < 2$, then use Theorem 9. $\diamond$

C Parabolic Morrey spaces and the Fefferman–Phong inequality.

We follow in this section the notations of Theorem 9 : (X, δ, μ) is a space of homogeneous type, with homogeneous dimension Q. For $0 < \alpha < Q/2$, $\mathcal{I}_\alpha$ is the Riesz potential associated to the kernel $K_\alpha = \frac{1}{\delta(x,y)^{Q-\alpha}}$, and $\mathcal{V}^\alpha$ is the space of functions that satisfy

$$\|f\|_{\mathcal{V}^\alpha} = \big(\sup_{\|h\|_2 \leq 1} \int_X |f(x)|^2 |\mathcal{I}_\alpha h(x)|^2 \, d\mu(x) \big)^{1/2} < +\infty. \tag{127}$$

Definition 7 *The (homogeneous) Morrey–Campanato space $\dot{M}^{p,q}(X)$ ($1 < p \leq q < +\infty$) is the space of the functions that are locally L^p and satisfy*

$$\|f\|_{\dot{M}^{p,q}} = \sup_{x \in X} \sup_{R > 0} R^{Q(\frac{1}{q} - \frac{1}{p})} \big(\int_{B(x,R)} |f(y)|^p \, d\mu(y) \big)^{1/p} < +\infty \tag{128}$$

where $B(x, R) = \{y \in X \ / \ \delta(x, y) < R\}$.

Remark that $L^q \subset \dot{M}^{p,q}(X)$, as it is easy to check by using Hölder inequality (since $\mu(B(x, R)) = CR^Q$).

We shall need two technical lemmas on Morrey–Campanato spaces. The first lemma deals with the Hardy–Littlewood maximal function :

Lemma 8 *Let $\mathcal{M}_f$ be the Hardy–Littlewood maximal function of f :*

$$\mathcal{M}_f(x) = \sup_{R>0} \frac{1}{\mu(B(x, R))} \int_{B(x,R)} |f(y)| \, d\mu(y). \tag{129}$$

Then there exists constants C_p and $C_{p,q}$ such that :

- *for every $f \in L^1$ and every $\lambda > 0$,*

$$\mu(\{x \in X \ / \ \mathcal{M}_f(x) > \lambda\}) \leq C_1 \frac{\|f\|_1}{\lambda}$$

- *for $1 < p \leq +\infty$ and for every $f \in L^p$*

$$\|\mathcal{M}_f\|_p \leq C_p \|f\|_p$$

- *for every $1 < p \leq q < +\infty$ and for every $f \in \dot{M}^{p,q}(X)$*

$$\|\mathcal{M}_f\|_{\dot{M}^{p,q}} \leq C_{p,q} \|f\|_{\dot{M}^{p,q}}.$$

Proof : The weak type (1,1) of the Hardy–Littlewood maximal function is a classical result (see Coifman and Weiss [7] for the spaces of homogeneous type). The boundedness of the maximal function on L^p for $1 < p \leq +\infty$ is then a direct consequence of the Marcinkiewicz interpolation theorem [11].

Thus, we shall be interested in the proof for $\dot{M}^{p,q}(X)$. Let $f \in \dot{M}^{p,q}(X)$. For $x \in X$ and $R > 0$, we need to estimate $\int_{B(x,R)} |\mathcal{M}_f(y)|^p \, d\mu(y)$. We write $f = f_1 + f_2$, where $f_1(y) = f(y) 1_{B(x,2\kappa R)}(y)$. We have $\mathcal{M}_f \leq \mathcal{M}_{f_1} + \mathcal{M}_{f_2}$. We have

$$\int_{B(x,R)} \mathcal{M}_{f_1}(y)^p \, d\mu(y) \leq (C_p \|f_1\|_p)^p \leq C_p^p \|f\|_{\dot{M}^{p,q}}^p (2\kappa R)^{Q(1-\frac{p}{q})}.$$

On the other hand, for $\delta(x,y) \leq R$,

$$\mathcal{M}_{f_2}(y) = \sup_{\rho > R} \frac{1}{\mu(B(y,\rho))} \int_{B(y,\rho)} |f_2(z)| \, d\mu(z) \leq \sup_{\rho > R} \frac{1}{A\rho^Q} \|f\|_{\dot{M}^{p,q}} \rho^{Q(1-\frac{1}{q})}$$

so that $1_{B(x,R)} \mathcal{M}_{f_2} \leq \frac{\|f\|_{\dot{M}^{p,q}}}{A R^{\frac{1}{q}}}$ and

$$\int_{B(x,R)} \mathcal{M}_{f_2}(y)^p \, d\mu(y) \leq \mu(B(x,R)) \|1_{B(x,R)} \mathcal{M}_{f_2}\|_\infty^p \leq \frac{B}{A^p} \|f\|_{\dot{M}^{p,q}}^p R^{Q(1-\frac{p}{q})}.$$

$\diamond$

The second lemma is a pointwise estimate for the Riesz potential, known as the *Hedberg inequality* [12, 1].

Lemma 9 *If $f \in \dot{M}^{p,q}(X)$ and if $0 < \alpha < \frac{Q}{q}$, then*

$$|\int_X \frac{1}{\delta(x,y)^{Q-\alpha}} f(y) \, d\mu(y)| \leq C_{p,q,\alpha} (\mathcal{M}_f(x))^{1-\frac{\alpha q}{Q}} \|f\|_{\dot{M}^{p,q}}^{\frac{\alpha q}{Q}}. \qquad (130)$$

Proof : Let $R > 0$. We have

$$\begin{aligned} |\int_{\rho(x,y)<R} \frac{f(y)}{\delta(x,y)^{Q-\alpha}} d\mu(y)| &\leq \sum_{j=0}^{+\infty} \int_{\frac{R}{2^{j+1}} \leq \rho(x,y) < \frac{R}{2^j}} \frac{|f(y)|}{\delta(x,y)^{Q-\alpha}} \, d\mu(y) \\ &\leq \sum_{j=0}^{+\infty} B 2^{-j\alpha} R^\alpha \frac{1}{\mu(B(x,2^{-j}R))} \int_{B(x,2^{-j}R)} |f(y)| \, d\mu(y) \\ &\leq B \frac{1}{1-2^{-\alpha}} R^\alpha \mathcal{M}_f(x) \end{aligned}$$

and

$$
\begin{aligned}
|\int_{\rho(x,y)\geq R} \frac{f(y)}{\delta(x,y)^{Q-\alpha}} d\mu(y)| &\leq \sum_{j=0}^{+\infty} \int_{2^j R\leq\rho(x,y)<2^{j+1}R} \frac{|f(y)|}{\delta(x,y)^{Q-\alpha}}\ d\mu(y) \\
&\leq \sum_{j=0}^{+\infty} \frac{1}{(2^j R)^{Q-\alpha}} B^{1-\frac{1}{p}} (2^{j+1}R)^{Q(1-\frac{1}{p})} (2^{j+1}R)^{Q(\frac{1}{p}-\frac{1}{q})} \|f\|_{\dot{M}^{p,q}} \\
&\leq B^{1-\frac{1}{p}} \frac{2^{Q(1-\frac{1}{q})}}{1-2^{\alpha-\frac{Q}{q}}} R^{\alpha-\frac{Q}{q}} \|f\|_{\dot{M}^{p,q}}.
\end{aligned}
$$

We then end the proof by taking $R^{\frac{Q}{q}} = \frac{\|f\|_{\dot{M}^{p,q}}}{\mathcal{M}_f(x)}$. $\diamond$

As a direct corollary of Lemma 9, we get the following result of Adams [2] on Riesz potentials :

Corollary 1 *For $0 < \alpha < \frac{Q}{q}$, the Riesz potential $\mathcal{I}_\alpha$ is bounded from $\dot{M}^{p,q}(X)$ to $\dot{M}^{\frac{p}{\lambda},\frac{q}{\lambda}}(X)$, with $\lambda = 1 - \frac{\alpha q}{Q}$.*

We may now state the comparison result between spaces of multipliers and Morrey–Campanato spaces, a result which is known as the *Fefferman–Phong inequality* [9] :

Theorem 11 *Let $0 < \alpha < Q/2$ and $2 < p \leq \frac{Q}{\alpha}$. Then we have :*

$$\dot{M}^{p,\frac{Q}{\alpha}}(X) \subset \mathcal{V}^\alpha = \mathcal{M}(W^\alpha \mapsto L^2) \subset \dot{M}^{2,\frac{Q}{\alpha}}(X). \tag{131}$$

Proof : For $f \in \dot{M}^{p,\frac{Q}{\alpha}}(X)$ and $g \in \dot{M}^{p,\frac{Q}{\alpha}}(X)$, we have $fg \in \dot{M}^{\frac{p}{2},\frac{Q}{2\alpha}}(X)$. We have $p/2 > 1$ and $\alpha < Q/q$ with $q = \frac{Q}{2\alpha}$, hence, since $\lambda = 1 - \frac{\alpha q}{Q} = 1/2$, $\mathcal{I}_\alpha(fg) \in \dot{M}^{p,q}(X)$. Thus, from Proposition 10, we see that $\dot{M}^{p,\frac{Q}{\alpha}}(X) \subset \mathcal{V}^\alpha$.

The embedding $\mathcal{V}^\alpha \subset \dot{M}^{2,\frac{Q}{\alpha}}(X)$ is easy to check. Indeed, if $F = 1_{B(x,2\kappa R)}$, we have for $y \in B(x,R)$

$$\mathcal{I}_\alpha F(y) \geq \int_{\rho(z,y)<R} \frac{d\mu(z)}{\rho(z,y)^{Q-\alpha}} \geq \frac{\mu(B(y,R))}{R^{Q-\alpha}} \geq AR^\alpha$$

hence, for $f \in \mathcal{V}^\alpha$,

$$\int_{B(x,R)} |f(y)|^2\ d\mu(y) \leq \frac{1}{A^2 R^{2\alpha}} \|f\|^2_{\mathcal{V}^\alpha} \|F\|_2^2 \leq \frac{B}{A^2} \|f\|^2_{\mathcal{V}^\alpha} R^{Q-2\alpha}.$$

$\diamond$

Remark : The embeddings are strict. For a proof in the case of the Euclidean space, see for instance [19].

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