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Iterative Decomposition Guided Variable Neighborhood Search for Graphical Model Energy Minimization

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Abstract

Graphical models factorize a global probability distribution/energy function as the product/sum of local functions. A major inference task, known as MAP in Markov Random Fields and MPE in Bayesian Networks, is to find a global assignment of all the variables with maximum a posteriori probability/minimum energy. A usual distinction on MAP solving methods is complete/incomplete, i.e., the ability to prove optimality or not. Most complete methods rely on tree search, while incomplete methods rely on local search. Among them, we study Variable Neighborhood Search (VNS) for graphical models. In this paper, we propose an iterative approach above VNS which uses (partial) tree search inside its local neighborhood exploration. The resulting hybrid method offers a good compromise between completeness and anytime behavior than existing tree search methods while still being competitive for proving optimality. We further propose a parallel version of our method improving its anytime behavior on difficult instances coming from a large graphical model benchmark. Last we experiment on the challenging minimum energy problem found in Computational Protein Design, showing the practical benefit of our parallel version. Solver at www.inra.fr/mia/T/toulbar2 v1.0.

1 INTRODUCTION

Probabilistic graphical models (Koller and Friedman, 2009) are formed by variables linked to each other by stochastic relationships. They enable to model complex systems with heterogeneous data and to capture uncertainty. Graphical models have been applied in a wide range of areas such as image analysis, speech recognition, bioinformatics, and ecology.

We focus on models with discrete variables like Markov Random Field and Bayesian Network. Our goal is to find a global assignment of all the variables with maximum a posteriori probability. This optimization task defines an NP-complete problem (Shimony, 1994). Solving methods can be categorized in two groups: exact and local search methods. Exact methods rely on tree search, variable elimination, linear programming, or a combination of them (Marinescu and Dechter, 2009; Otten and Dechter, 2012; Allouche et al., 2015). Graph-cut and message-passing algorithms like loopy belief propagation and variational approaches (Kolmogorov, 2006; Wainwright and Jordan, 2008; Sontag et al., 2008; 2012; Wang and Koller, 2013) are exact only in some particular cases (e.g., Potts model or tree structure). Local search methods are stochastic algorithms like Gibbs sampling, Guided Local Search (Park, 2002; Hutter et al., 2005), and Stochastic Greedy Search (Mengshoel et al., 2011). Some of them have theoretical asymptotic proof of convergence, i.e., the optimal solution is guaranteed to be found if infinite time is available. In practice, they may exhibit a better anytime behavior than exact methods on large and difficult problems (Marinescu et al., 2003; Hutter et al., 2005; Mengshoel et al., 2011), i.e., they produce better solutions in less time.

A few attempts have been done to combine exact and local search methods. A simple way is to run sequentially a local search algorithm then tree search, where solutions found by local search will be used as initial upper bounds for branch and bound exact methods. Another approach is to design a local search framework where the neighborhood exploration is performed by tree search in a systematic or non-systematic way. VNS/LDS+CP (Loudni and Boizumault, 2003) combines a metaheuristic, Variable Neighborhood Search (VNS) (Mladenović and Hansen, 1997) with Limited Discrepancy Search (LDS) (Harvey and Ginsberg, 1995),
a partial tree search method (Section 2). We propose in Section 3 an iterative variant of VNS/LDS+CP called UDGVNS, adapted to graphical models and able to prove optimality when the neighborhood size is maximal and with unbounded tree search. In Section 4, we describe a coarse-grained parallel version called UPDGVNS with asynchronous cooperative execution of UDGVNS processes with centralized information exchange as in (Davidovic and Crainic, 2012; Ouali et al., 2015). We report experimental results in Sections 5-6.

2 PRELIMINARIES
2.1 GRAPHICAL MODEL

Definition 1 A probabilistic graphical model (or Gibbs distribution) (Koller and Friedman, 2009) is a triplet \( \langle \mathcal{X}, \mathcal{D}, \mathcal{F} \rangle \) with \( \mathcal{X} = \{X_1, \ldots, X_n\} \), a set of \( n \) random variables, \( \mathcal{D} = \{D_1, \ldots, D_n\} \), a set of \( n \) finite domains of values of maximum size \( d = \max_{i=1}^n |D_i| \), and \( \mathcal{F} \), a set of potential functions. Each variable \( X_i \) takes values in \( D_i \). An assignment of \( \mathcal{X} \) is a set \( x = (x_1, \ldots, x_n) \), with \( x_i \in D_i \). The set of all possible assignments of \( \mathcal{X} \) is denoted \( \Delta = \prod_{i=1}^n D_i \). Let \( A = \{D'_1, \ldots, D'_n\} \) with \( D'_i \subseteq D_i \) represent a restricted set of \( \Delta \) called a partial assignment. If \( S \) is a subset of \( V = \{1, \ldots, n\} \), \( X_S \), \( x_S \) and \( \Delta_S \) are respectively the subset of random variables \( \{X_i, i \in S\} \), the assignment \( x_S \) obtained from \( x \), and the set of all possible assignments of \( X_S \). Given a set \( S \) of \( V \), the set \( \mathcal{F} = \{f_S\}_{S \in \mathcal{S}} \) of maps from \( \Delta_S \) to \( \mathbb{R}^+ \), called potential functions, is said to factorize a joint probability distribution \( \mathbb{P} \) iff:

\[
\mathbb{P}(x) = \frac{1}{Z} \prod_{f_S \in \mathcal{F}} f_S(x_S)
\]

where \( Z = \sum_{x \in \Delta} \prod_{f_S \in \mathcal{F}} f_S(x_S) \) is the normalizing constant, also called partition function.

Among the various tasks, the Most Probable Explanation (MPE) problem is to find the most likely assignment \( x \in \Delta \) to all the variables in \( \mathcal{X} \) maximizing \( \mathbb{P}(x) \).

By taking the opposite of the logarithm of \( \mathbb{P}(x) \), i.e.,

\[- \log \mathbb{P}(x) = \sum_{f_S \in \mathcal{F}} - \log f_S(x_S) - \log Z\]

we obtain an additive model with \( \varphi(x_S) = - \log f_S(x_S) \) called energy functions. Finding a solution of minimum energy is equivalent to MPE. In the rest of the paper, we consider energy minimization. When \( \varphi(x_S) \) maps to \( \mathbb{N}^+ \cup \{\infty\} \), the corresponding deterministic graphical model is called a Cost Function Network (CFN) (Meseguer et al., 2006). Finding a solution of minimum cost is the same as doing energy minimization on the equivalent probabilistic model (Hurley et al., 2016).

2.2 TREE DECOMPOSITION

Definition 2 A tree decomposition of a connected graphical model is a pair \( (C_T, T) \) where \( T = (I, A) \) is a tree with nodes set \( I \) and edges set \( A \) and \( C_T = \{C_i \mid i \in I\} \) is a family of subsets of \( \mathcal{X} \), called clusters, such that: (i) \( \bigcup_{i \in I} C_i = \mathcal{X} \), (ii) \( \forall S \in \mathcal{F}, \exists C_i \in C_T \) s.t. \( S \subseteq C_i \), (iii) \( \forall i, j, k \in I \) if \( j \) is on the path from \( i \) to \( k \) in \( T \), then \( C_i \cap C_k \subseteq C_j \).

Definition 3 The intersection of two clusters \( C_i \) and \( C_j \) is called a separator, and notated sep\((C_i, C_j)\).

Definition 4 A graph of clusters for a tree decomposition \( (C_T, T) \) is an undirected graph \( G = (C_T, E) \) that has a vertex for each cluster \( C_i \in C_T \), and there is an edge \( (C_i, C_j) \in E \) when \( \text{sep}(C_i, C_j) \neq \emptyset \).

As finding an optimal tree decomposition is NP–hard, we use fast approximate algorithms like min-fill heuristic.

2.3 DFBB & LIMITED DISCREPANCY SEARCH

Depth-First Branch and Bound (DFBB) methods explore a search tree in a systematic way by recursively choosing the next unassigned variable to assign and by choosing a value in its domain for the assignment (the branch part) until a better solution is found or it can be proved that the subtree rooted at the current search node has no better solutions and it can be pruned (the bound part). DFBB depends on its variable and value ordering heuristics for branching in order to find good solutions rapidly and to reduce the size of the search tree to be explored.

Limited Discrepancy Search (LDS) (Harvey and Ginsberg, 1995) is a heuristic method that explores the search tree in a non-systematic way by making a limited number of wrong decisions w.r.t. its value ordering heuristic. We assume a binary search tree where at each search node either the selected variable is assigned to its chosen preferred value (left branch) or the value is removed from the domain (right branch). Each value removal corresponds to a wrong decision made by the search, it is called a discrepancy. The number of discrepancies is limited by a parameter denoted \( \ell \). See Algorithm 1 where \( 1b(A) \) gives a lower bound on the minimum energy \( \min_{x \in \prod_{D_i \in A} D_i} - \log \mathbb{P}(x) \) of the partial assignment \( A \).

In order to produce good quality solutions as time passes, a simple strategy is to iterate LDS with an increasing number of discrepancies \( \ell \) going from \( \ell_{\text{min}} \) to \( \ell_{\text{max}} \). Each iteration does at most \( \ell \) discrepancies along the path from the root search node to a terminal node. Let \( h \) be the maximum height of the explored search tree, the number of terminals with \( \ell \) discrepancies is bounded by \( \binom{h}{\ell} \). Thus, LDS has a time complexity in \( O(h^{\ell+1}) \) (for \( \ell < h/2 \)) (Larrosa et al., 2016) and a linear space complexity (thanks to the depth-first search principle as in DFBB). Notice that \( h \leq n + \ell \) (right branches followed by \( n \) left branches). Without any discrepancy limit, we have at most \( n(d-1) \) value removals (right branches).
Algorithm 1: Pseudo-code of mono LDS

<table>
<thead>
<tr>
<th>Procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td>LDS (ℓ, A, ub : In/Out, x : In/Out, opt : In/Out)</td>
</tr>
<tr>
<td>if (∃D, r ∈ A,</td>
</tr>
<tr>
<td>Choose an unassigned variable X_i ∈ X,</td>
</tr>
<tr>
<td>Choose a value x_i ∈ D_i;</td>
</tr>
<tr>
<td>A’ ← (A \ {D_i}) ∪ {x_i}; // left branch</td>
</tr>
<tr>
<td>if (1b(A’) &lt; ub) then LDS (ℓ, A’, ub, x, opt);</td>
</tr>
<tr>
<td>if (ℓ &gt; 0) then</td>
</tr>
<tr>
<td>A” ← (A \ {D_i}) ∪ {D_i \ {x_i}}; // right branch</td>
</tr>
<tr>
<td>if (1b(A”) &lt; ub) then LDS (ℓ − 1, A”, ub, x, opt)</td>
</tr>
<tr>
<td>else</td>
</tr>
<tr>
<td>opt ← false;</td>
</tr>
<tr>
<td>end</td>
</tr>
<tr>
<td>ub ← 1b(A), x ← A; // new sol. found</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

1 to reach a terminal node, where d is the maximum domain size of all variables. In order to be able to prove optimality, we set ℓ_{max} = n(d − 1), i.e., the last iteration is complete. In practice, optimality may be proved with a much smaller ℓ, and we just have to check if the discrepancy limit was reached during the search (falsified condition at line 1 of Alg. 1) to detect an incomplete search (flag opt set to false at line 2).

2.4 VARIABLE NEIGHBORHOOD SEARCH

Variable Neighborhood Search (VNS) (Mladenović and Hansen, 1997) is a metaheuristic that uses a finite set of pre-selected neighborhood structures N_k, k = 1, 2, ..., k_{max} to escape from local minima by systematically changing the neighborhood structure if the current one does not improve the current incumbent solution. VNS repeatedly performs three major steps. In the first one, called shaking, a solution x’ is randomly generated in the neighborhoods of x denoted N_k(x). In the second one, a local search method is applied from x’ to obtain a local optimum x”’. In the third one, called neighborhood change, if x”’ is better than x, x is replaced with x”’ and k is set to 1; otherwise, k is increased by one.

2.5 VNS METHODS IN GRAPHICAL MODELS

The use of VNS scheme for solving deterministic graphical models started with VNS/LDS+CP (Loudni and Boizumault, 2003). This approach is related to LNS (Shaw, 1998), but it adjusts dynamically the neighborhood size when search seems to stagnate as in VNS.

VNS/LDS+CP algorithm. Let N_k be the neighborhood structure of size k; N_k denotes the set of all subsets of k variables among X. Let x be the current solution. First, x is partially destroyed by un-assigning a subset of k variables and an exploration of its (large) neighborhood is performed until the solution is repaired with a new one. The core of VNS/LDS+CP is its reconstruction phase. It runs LDS with a fixed discrepancy to explore the neighborhood of the solution. One advantage of this choice is its exploration speed that improves the quality profile and allows a more balanced exploration of the search tree.

Decomposition Guided VNS. More recently, Fontaine et al. (Fontaine et al., 2013) investigated the incorporation of tree decomposition within VNS/LDS+CP in order to efficiently guide the exploration of large neighborhoods. They proposed DGVNS, a first local search approach that exploits the graph of clusters provided by a tree decomposition to build relevant neighborhood structures. For both DGVNS and VNS/LDS+CP, the reconstruction phase is performed using LDS with a fixed discrepancy ℓ, but neighborhood structures are managed in a different way. Instead of the neighborhood structures N_k used by VNS/LDS+CP, DGVNS uses neighborhood structures N_{k,c}, where k is the neighborhood size and C is the cluster where the variables will be selected from. If (k > |C|), we complete the set of candidate variables to be unassigned by adding clusters adjacent to C. The neighborhood change in DGVNS is performed in the same way as in VNS. However, DGVNS considers successively all the clusters. This ensures a better diversification by covering a large number of different regions.

3 AN ITERATIVE DGVNS METHOD

We present UDGVNS (see Algorithm 2), unifying two complete and incomplete search methods. As done by iterative LDS, it restores the completeness of DGVNS by applying successive calls with an increasing discrepancy limit, controlling if tree search was partial using opt flag and the fact that the current neighborhood kept some variables assigned in A (test at line 9). In UDGVNS, optimality can be proven in two cases: (i) when the current neighborhood corresponds to the whole problem and the discrepancy value is greater than or equal to the maximum number of right branches or (ii) by examining the bounds at the root node (ub = 1b(D) at lines 5 and 8). In this case, the search space is implicitly explored by the algorithm, therefore optimality is proven.

The initial solution of UDGVNS is obtained at line 4 by a modified version of LDS, denoted LDS”, that stops immediately after a first solution is found. Thus LDS” (∞, D, ub, x, opt) will return a solution with non-zero probability (i.e., satisfying the constraints).

UDGVNS has two main parameters for tuning its compromise between optimality proof and anytime behavior: the discrepancy limit (ℓ) and the local search neighbor-
Algorithm 2: Unified DGVNS algorithm.

Function $\text{UDGVNS}(\ell_{\min}, \ell_{\max}, +\ell, k_{\min}, k_{\max}, +k, ub; In/Out, x; In/Out)$ : boolean

let $(C_T, T)$ be a tree decomposition of $(X, D, F)$;

if $(ub = 1b(D))$ then opt $\leftarrow$ true;
c $\leftarrow 1$; // current cluster index
$r \leftarrow 0$; // number of iterations
$\ell \leftarrow \ell_{\min}$; // initial discrepancy limit
while $(\neg opt \land \ell \leq \ell_{\max})$

$i \leftarrow 0$; // nb. of failed neighbor.
$k \leftarrow k_{\min}$; // init. neighbo. size
while $(\neg opt \land k \leq k_{\max})$

$A \leftarrow \text{getNeighborhood}(x, C_u, k)$;
$ub' \leftarrow ub$, opt $\leftarrow$ true;
$LDS^r(\ell, A, ub', x', opt)$; // nei. search
if $(ub' = 1b(D))$ then opt $\leftarrow$ true;
else if $(A \neq F)$ then opt $\leftarrow$ false;
if $(ub' < ub)$ then

$x \leftarrow x'$, $ub \leftarrow ub'$; // new best sol
$i \leftarrow 0$, $k \leftarrow k_{\min}$;
$r \leftarrow 0$, $\ell \leftarrow \ell_{\min}$;
else

$i \leftarrow i + 1$;
if $(k < k_{\max})$ then

$k \leftarrow \min(k_{\max}, k_{\min} + k i)$;
else $k \leftarrow \infty$;
$c \leftarrow 1 + c \mod |C_T|$; // get next clus.
$r \leftarrow r + 1$;
if $(\ell < \ell_{\max})$

$\ell \leftarrow \min(\ell_{\max}, \ell_{\min} + \ell r)$;
else $\ell \leftarrow \infty$;

return opt;

4 A PARALLEL VERSION OF UDGVNS

This section describes how UDGVNS can be parallelized. We denote the resulting algorithm $\text{UPDGVS}$ for Unified Parallel DGVNS. It relies on asynchronous cooperative execution of several UDGVNS (i.e. workers) processes with centralized information exchange. In the proposed parallelization one global best solution is shared among the worker processes. Initially, the master initiates the search by launching $npr$ worker processes in parallel with the same initial solution (line 4 done in the master). Each worker process obtains from the master a copy of the current best solution $x$, the index $c$ of the cluster to be processed and performs destroy and repair operations on its local copy. As soon as a new solution is found by a worker, it is sent to the master and a new cluster is requested (line 13 done in the master) to restart a new exploration starting from the best available overall solution. Like UDGVNS, each worker process controls locally how the discrepancy limit $\ell$ and the neighborhood size $k$ evolve during successive explorations (except that $ub$ comes from the master at line 10). The whole process stops as soon as a worker terminates (line 14) with an optimality proof or it reaches $\ell_{\max}$ and $k_{\max}$. 
Figure 1: Comparative evaluation on probabilistic and deterministic graphical models

5 UAI EVALUATION RESULTS

Benchmarks description. We performed experiments on probabilistic and deterministic graphical models coming from a large multi-paradigm benchmark repository (Hurley et al., 2016)\(^4\). Among the 3016 available instances, we selected all the instances that were used in previous UAI competitions, in image analysis, or in Cost Function Network optimization. It includes 319 instances from UAI 2008 Evaluation and Probabilistic Inference Challenge (PIC 2011)\(^5\), 1461 instances from the Computer Vision and Pattern Recognition (CVPR) OpenGM2 benchmark\(^6\) (Kappes et al., 2015), and 281 instances from the Cost Function Library\(^7\). In order to have a fair comparison between solvers, we preprocessed all the instances by polynomial-time problem reformulations and simplifications that remove variables (using bounded and functional variable elimination (Favier et al., 2011)), values (using dead-end elimination (de Givry et al., 2013)), and fixed-value potentials, after an initial lower bound computation by Equivalence Preserving Transformations (Cooper et al., 2010) (enforcing Virtual Arc Consistency (VAC) as a message-passing algorithm). The resulting instances are smaller while preserving the same optimum. We used TOULBAR2 with options -A -z=2 for this preprocessing step. We kept 1669 non-trivial instances (with more than one variable) for the experiments. The number of variables \(n\) ranges from 4 (CVPR-GeomSurf-3-gm13) to 48,566 (CVPR-ColorSeg-8-crops-small) with mean value \(n \approx 403\,4\) (instead of 1,316.5 before preprocessing). For DGVNS methods, we built tree decompositions using min-fill heuristic. Because the number of clusters \(m\) can be very large (\(m \approx 256\,7\)), we merged any pair of connected clusters \((C_i, C_j)\) when the separator size \(|\text{sep}(C_i, C_j)|\) is too large compared to the individual cluster sizes \(|C_i|, |C_j|\) (instead of 49.6 without merging). In order to experiment sequential and parallel methods on the most difficult instances, we selected a subset of instances unsolved in 1 hour by all our DFBB, LDS, and VNS algorithms. We took at most twenty instances per problem category (avoiding over-representation issues by some categories), resulting

\(^4\)genoweb.toulouse.inra.fr/~degivry/evalgm
\(^6\)hci.iwr.uni-heidelberg.de/opengm2
\(^7\)costfunction.org/benchmark
in a selection of 114 difficult instances\(^8\).

**Experimental protocol.** LDS\(^7\) employs a randomized (for breaking ties) dynamic variable ordering heuristic\(^9\). Its value ordering heuristic chooses the EAC value as the preferred value and lower bounds are deduced by enforcing EDAC, as explained in \(\text{(Larrosa et al., 2005).} \)

In the following, we set \(k_{\text{min}} = 4\), \(k_{\text{max}} = n = |\mathcal{X}|\), \(l_{\text{min}} = 1\), and \(l_{\text{max}} = n(d - 1)\). Here, DFBB corresponds to UDGVNS \((\infty, \infty, +, +, \infty, \infty, +)\), LDS to UDGVNS \((1, \infty, \text{mult2}, |\mathcal{X}|, |\mathcal{X}|, +)\), and DGVNS to UDGVNS \((3, 3, +, k_{\text{min}}, k_{\text{max}}, +)\).

We compared with state-of-the-art exact solvers. DAOPT\(^10\) won PIC 2011. It has a time-bounded initial phase of lower bound computation based on Message Passing Linear Programming algorithm \(\text{(Sontag et al., 2008; 2012; Ihler et al., 2012)}\) and mini-bucket elimination \(\text{(Dechter and Rish, 2003)}\) with iterative min-fill heuristic. It also finds good initial upper bounds using stochastic local search GLS\(^+\) \(\text{(Hutter et al., 2005)}\). We tested three parameter settings as suggested in \(\text{(Otten et al., 2012)}\), controlling the time spent to compute initial lower and upper bounds. In the 3600sec setting, SLS is run 20 times with 10 seconds per run. The best solution found is used as an initial bound for an AND/OR exhaustive tree search. INCOP+TOULBAR\(^2\) \(\text{(Hurley et al., 2016)}\) won the UAI 2010 Evaluation at 20 min. time limit. TOULBAR2 takes a starting solution from the best result of three runs of the IDWalk \(\text{(Neveu et al., 2004)}\) local search algorithm \(\text{(100,000 local moves per run)}\). It is followed by an exhaustive hybrid best-first search \(\text{(Allouche et al., 2015)}\). IBM ILOG CPLEX 12.7.0.0 \(\text{(using parameters EPAGAP, EPGAP, and EPINT set to zero to avoid premature stop)}\) was reported as being very competitive on some image analysis \(\text{(Kappes et al., 2015)}\) and Markov Random Field problems \(\text{(Hurley et al., 2016)}\). CPLEX explores its search tree using best-first search. It applies several heuristics methods to find good solutions before and during the search. We also compared with message-passing algorithms: LBDAI\(^12\) \(\text{(Mooij, 2010)}\), winner of the UAI 2010 Evaluation at 20sec. and 1hour time limits, MLP\(^13\) \(\text{(Sontag et al., 2008; 2012)}\), and TRW-S\(^14\) \(\text{(Kolmogorov, 2006)}\). Note that LBDAI and TRW-S are applied on the original instances rather than the preprocessed ones as we found they produced better results without applying VAC first. All solvers read problems in uai tabular format, except CPLEX which uses the local polytope formulation \(\text{(called support encoding in (Hurley et al., 2016))}\). All computations were performed on a cluster of 48-core AMD Opteron 6176 at 2.3 GHz and 384 GB of RAM with a 1-hour CPU time limit\(^15\).

**Optimality results.** The efficiency of DFBB, LDS, and VNS methods to prove optimality is shown in the cactus plot of Figure 1a. DFBB was slightly more efficient than LDS and solved 1442 \(\text{(resp. 1433)}\) instances among 1669 in 1-hour time limit. They are followed by three UDGVNS strategies with \((k \ \text{mult2}, \ell \ \text{mult2})\) \(\text{(1430 solved)}\), \((k \ \text{Luby}, \ell \ \text{mult2})\) \(\text{(1425 solved)}\) and \((k \ \text{add1/jump}, \ell \ \text{mult2})\) \(\text{(1421 solved)}\), remaining very close in terms of performance. Next, a set of three less-and-less efficient UDGVNS strategies rise: \((k \ \text{add1}, \ell \ \text{mult2})\) \(\text{(1384 solved)}\), \((k \ \text{add1}, \ell \ \text{Luby})\) \(\text{(1361 solved)}\) and \((k \ \text{add1}, \ell \ \text{add1})\) \(\text{(1333 solved)}\), showing the importance of faster discrepancy increase to speed up optimality proofs. The worst strategy here was using a fixed discrepancy level \(\ell = 3\) as originally proposed in \(\text{(Fontaine et al., 2013)}\) which solved 1128 instances. For comparison, DAOPT \(\text{(3600sec setting)}\) solved 1409 instances, CPLEX solved 1423, and INCOP+TOULBAR2 1440 instances \(\text{(results in Supp. material)}\).

**Anytime upper bound profiles.** In order to summarize the evolution of upper bounds as time passes, we took a subset of 114 difficult instances, unsolved in 1 hour by our DFBB, LDS, and VNS methods \(\text{(whereas CPLEX could solve 17 instances)}\). Specifically, for each instance \(I\) we normalize all energies as follows: the best, potentially suboptimal solution found by any algorithm is 1, theworst solution is 2. This normalization is invariant to translation and scaling. Figure 1b shows the upper bound behavior for different VNS strategies compared to DFBB and LDS \(\text{(see Supp. material for a zoom)}\).

The ranking of best methods is the opposite of the cactus plot order, except for \((k \ \text{add1/jump}, \ell \ \text{mult2})\) which comes in second position. The \(\ell = 3\) strategy got the best upper bounds in average, but still very close to the other VNS strategies, except may-be \((k \ \text{Luby}, \ell \ \text{mult2})\) and \((k \ \text{mult2}, \ell \ \text{mult2})\). We conclude that our new iterative UDGVNS method \(\text{(especially} \ (k \ \text{add1/jump}, \ell \ \text{mult2})\) offers a good compromise between anytime behavior and optimality proof. These results also show that variable neighborhood search is by far superior to classical systematic (DFBB) or non-systematic (LDS) tree search methods, improving by more than 20% \(\text{(resp. 10%)}\) the quality of the solu-

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\(^8\)UAI DBN, Grid, Linkage, ObjectDetection, CVPR ChineseChars, ColorSeg-8, InPainting-4, InPainting-8, ProteinInteraction, and CFN CELAR, ProteinDesign, SPOTS, Warehouse categories.

\(^9\)Weighted degree heuristic \(\text{(Bousserram et al., 2004)}\).

\(^10\)graphmodics.ics.ucl.ac.be/group/DAOPT-UAI12

\(^11\)www.inria.fr/maia/T/toubar2 version 0.9.8 with parameters \(-\text{dec}\ \text{add}\).

\(^12\)bitbucket.org/jorism/libdai.git version 0.3.2 using UAI 2010 settings.

\(^13\)cs.nyu.edu/\~dsontag/code/READMR_v2.html using \(2.10^{-7}\) int gap thres.

\(^14\)github.com/opengm/opengm v2.3.5 with TRW-S v1.3 stopping after 100, 000 iterations or \(10^{-5}\) gap thres.

\(^15\)Using parameter \(-\text{pe parallel\_lump min}(2x, 30)\) on a SUN Grid Engine for a method exploiting \(x\) core(s) to ensure half-load of the nodes on the cluster.
tions. In the following figures, we assume UDGVS with \( (k\ add1/jump,\ell\ mult2) \) strategy.

Figure 1c compare UDGVS with state-of-the-art methods. Message-passing algorithms like TRW-S and LIB-DAI gave the worst results. They could not find any solution for 20 (resp. 19) instances (mostly in UAI/Linkage and CFN/SPOT5 categories, both containing hard constraints). The same problem occurred for MPLP2 on 5 instances (SPOT5), but it obtained much better results on the remaining instances. CPLEX ran out of memory on two instances without producing any solution due to the heavy local polytope encoding (CFN/Warehouse/capb,capmq5). All other methods got better results in average and produced at least one solution per instance. According to its initial phase setting, DAOOPT provides different anytime behaviors, very close to the best solutions in 1 hour. UDGVS performed the best, slightly better than INCOP+TOULBAR2, improving by 1.7% (resp. 2.3%) on average after 1 hour (20 min).

**Parallelization.** Finally, in order to evaluate the impact of core numbers, we consider the anytime upper bound behavior of the parallel release: UPDGVS using \( (k\ add1/jump,\ell\ mult2) \) with \( \ell_{\text{min}} = 3 \), taken from the best strategies enlightened by UDGVS. We made a comparison with CPLEX using 10 and 30 cores (results in Supp. material), improving its anytime behavior, but still being far from the other competitors (30 cores gave solutions 10% higher than UPDGVS after 1 hour). We could not compare with the parallel version of DAOOPT as it is based on a different cluster engine (condor) and it does not parallelize its initial phase.

Figure 1d shows that UPDGVS (with 10 or 30 cores) provides slightly better upper bounds than UDGVS (using 1 core) in less than 20 min. The results seems to be, in average, poorly sensitive to the core number, due to the fact than the 10-core curve is extremely close from the 30-core one. The 10 and 30-core UPDGVS curves converge quickly in less than 2min. INCOP+TOULBAR2 quickly drops out around 1 min and never reaches the same quality level. UDGVS converges slower but still going down after 20 min. DAOOPT (1200sec setting) gave results 10% in average worse than UPDGVS with either 10 and 30 processors.

6 **COMPUTATIONAL PROTEIN DESIGN**

**Benchmarks description.** One main challenge in Computational Protein Design (CPD) lies in the exploration of the amino-acid sequence space, while considering, to some extent, side chain flexibility. The exorbitant size of the search space urges for the development of efficient exact deterministic search methods enabling identification of low-energy sequence-configuration models, corresponding to the Global Minimum Energy Conformation (GMEC). The GMEC corresponds to a maximum probability mass due to the Boltzmann relation between molecular energy and probability, which is, for that matter, equivalent in the Markov Random Field modeling with Maximum A Posteriori probability (MAP-MRF) estimation.

CPD has been recently modeled as CFN, and various intensive benchmarking had demonstrated that in most cases CFN techniques are more efficient than other discrete optimization methods (Allouche et al., 2014), especially for full design of instances lesser than 100 Amino acid (Simoncini et al., 2015). In this paper we would like to evaluate VNS methods capability to solve difficult instances. Accordingly, we tried to generate new larger ones supposed to be more difficult to solve than those generated in (Simoncini et al., 2015). For this aims, in a first stage, we selected proteins structure in the PDB databases with size between 100 and 300 amino-acid. On the basis of a database query described in the supplementary material, we obtained a first set including 438 PDB references, which has been sorted according to the average volume per variable calculated using the radius of gyration of each instance (see Supp. material). At last, we extracted for benchmarking only the 20th first elements and used as benchmarking set. Each protein structure was fully redesigned according to (Simoncini et al., 2015) protocol. On the basis of energy matrix generated with a modified release of PYROSETTA.4 script (Simoncini et al., 2015) in order to use the last released Rosetta force-field (aka Beta November 2016) (Alford and others, 2017). The instances (see Supp. material) contain from 130 up to \( n = 282 \) variables with maximum domain size from 383 to 438, and between 1706 and 6208 cost/energy functions, and besides the tree width ranges from 21 to 68 and from 0.16 to 0.34 for a normalized tree width.

In this section, we evaluate the effectiveness of UDGVS and UPDGVS on CPD instances in terms of optimality proof (cf. part (a)) and solution quality vs. CPU time (cf. part (b)). All computations were performed on a cluster of 96-core AMD Opteron 6174 at 2.2 GHz and 256 GB of RAM. The parallelization has been done in MPI.

**Experimental Protocol.** We compared UDGVS and UPDGVS with VNS/LDS+CP and FIXBB (CPD dedicated algorithm provided by Rosetta package). We also compared with TOULBAR2. We tested TOULBAR2 using Virtual Arc Consistency (Cooper et al., 2010) as preprocessing combined with hybrid best-first search with
tree decomposition (BTD-HBFS) using min-fill heuristic. Dead end elimination is turned off, according to (Simoncini et al., 2015) (the resulting command line is: toulbar2-b=1-A-dee:-O=-3). The TOLBAR2 experiments use a 24-hour CPU time limit.

Concerning the VNS methods, in order to evaluate variability due to the random selection of neighborhoods, a set of 10 runs per instance with different seeds has been performed with a time limit of 1-hour per run. For the parallel strategy, the number of processes npr is set to 96 (i.e. maximum number of available processes). For UDGVNS and UPDGVNS, following the results observed in Section 5, we considered the following two settings for operators +k and +ℓ: (k add1, ℓ = 3) which yields the best anytime performances and (k add1/jump, ℓ mult2) which offers the best compromise between both anytime performances and optimality proof. kmin, kmax and ℓmin have been respectively set to 4, n (the total number of variables) and 3 (they correspond to the same parameter settings as those described in (Fontaine et al., 2013)).

(a) Optimality results. Our first set of experiments aims at evaluating the efficiency of UDGVNS and UPDGVNS in terms of optimality proof comparing with TOLBAR2. As indicated before, we used the setting (k add1/jump, ℓ mult2). For all methods (see Supp. material), the CPU-time is the time to find and prove an optimal solution. TOLBAR2 clearly outperforms the two VNS methods: UDGVNS and UPDGVNS were able to prove optimality only on 3 instances among the 10 ones closed by TOLBAR2. However, the VNS methods are faster (except on 5dbl), despite the fact that BTD-HBFS benefits from the lower bounds reported by HBFS in individual clusters to improve the anytime behavior and from the global pruning lower bounds of BTD. This greatly improves the overall performance of TOLBAR2 compared to VNS methods.

(b) CPU time and solution quality results. Our second set of experiments aims at evaluating VNS capability with respect to finding the optimal solution or a solution of better quality on instances for which optimal solutions are unavailable. For this aims, we selected (k add1, ℓ = 3) as setting for operators +k and +ℓ.

Table 1 shows a comparative evaluation of VNS methods with FIXBB and TOLBAR2. For each instance and each VNS method, we report the number of successful runs to reach the optimum (or the best found solution for unsolved instances) within a 3600-second time limit, the average CPU times (in seconds) over the 10 runs (for unsuccessful runs, the CPU time is set to the time limit) ± the standard deviation. The Energy gap ∆E between the best VNS solution and the two external references FIXBB and TOLBAR2 are also given. For TOLBAR2, reported CPU times correspond to times to find an optimal solution (for solved instances) or a best one (for unsolved ones) within the 24-hour time limit.

VNS methods vs. FIXBB. Rosetta Modeling suite is one of the most popular software package used in CPD field. It provides a Monte Carlo based Simulated Annealing algorithm called FIXBB. In this work, the best solutions exhibited by 1000 FIXBB cycles performed on each CPD instance have been used as base-line to compare solution quality of the solution provided by VNS methods when TOLBAR2 BTD-HBFS fails to solve the instance.

The FIXBB CPU times are two orders of magnitude higher than the 1h time limit imposed for VNS evaluation. They are not reported as they exceed 100-hours in sequential mode, even if FIXBB cycles are independent and thus be easy to parallelize. As we can see in Table 1 (see column ∆E of (4)), the solution quality of FIXBB is in all cases inferior to the best solution found by the VNS methods. The Energetic gap ∆E between FIXBB solution and the best VNS overall solution ranges between ±0.16 and ±5.20 Rosetta Energy Unit (R.E.U). As shown in (Simoncini et al., 2015) such a level of energy difference can strongly impact the designed protein solution (i.e. the corresponding sequences of the two methods can be far in terms of hamming distance).

VNS methods vs. TOLBAR2. The comparison between best solutions found by VNS methods and TOLBAR2 shows that, excepted VNS/LDS+CP, both UDGVNS and UPDGVNS provide in all case the same or even better solution than TOLBAR2 (see column ∆E of (5) in Table 1). The Energetic gap ∆E in the worst case reaches 17.86 R.U.E. Concerning the number of successful runs reported over the 10 runs, VNS/LDS+CP seems less robust respectively than UDGVNS and the parallel release UPDGVNS. This last one provides in all cases the best solution over all. Table 1 also compares the VNS methods in terms of speedups. We observe that speedup values are fluctuating from one instance to another, very likely due to the tree decomposition resulting from the 3D shape of each instance. For VNS/LDS+CP and UDGVNS, it range between 0.62 and 2.86 over the 14 instances solved by both methods. As expected, when tree decomposition and parallelization are used, not only the speed of resolution increases but the reliability too (speedup values between 4.1 and 18.5). Moreover, the comparison between UDGVNS and UPDGVNS shows significant accelerations (between 2.06 and 9.22), thus confirming the practical interest of parallelization in addition to the exploitation of problem decomposition. We can observe (in Supp. material) that both UDGVNS and UPDGVNS find optimal solutions more quickly than TOLBAR2 within the one hour time limit (with the exception of UDGVNS
The speedup values are significantly improved, in particular with UPDGVNS (between 32.3 and 354.6 with a mean of 188.48 over all the solved instances). For unsolved instances, we measured the CPU times spent by the three VNS methods (within the 3600-second time limit) to reach the best solution produced by TOULBAR2. Results (in Supplementary material) show a clear ordering in terms of CPU times across different solvers, from TOULBAR2, VNS/LDS+CP, UDGVS, and UPDGVNS. The speedup values are significantly improved, in particular with UPDGVNS (between 32.3 and 354.6 with a mean of 188.48 over all the unsolved instances). These results confirm the superiority of VNS methods in terms of anytime performance as compared to TOULBAR2.

CONCLUSION

In this paper we proposed a unified view of VNS methods including various LDS and neighborhood evolution strategies. Experiments performed on difficult instances, coming from a large graphical model benchmark, showed that our hybrid method has a much better anytime behavior than existing tree search methods and still being competitive for proving optimality. We further proposed a parallel version of our method improving its anytime behavior. It remains as future work to automatically choose the best parameter settings per instance.

Acknowledgements This work has been partially funded by the French “Agence nationale de la Recherche”, reference ANR-16-C40-0028. We are gratefully to the genoulot bioinformatics platform Toulouse Midi-Pyrenees (Bioinfo Genotoul) and the HPC HAYTHAM of university of Oran 1 for providing computing and storage resources. We thank Mathieu Fontaine for his contribution to the code of DGVNS.

References


1 Cactus plot of UDGVNS versus CPLEX, DAOOPT, and INCOP+TOULBAR2

![Cactus plot](image)

Figure 1: Number of instances solved by each method as time passes (UDGVNS = UDGVNS(k add1/jump,l mult2)).

2 Anytime upper bound zoom for UDGVNS versus LDS

![Anytime zoom](image)

Figure 2: Anytime upper bound zoom for UDGVNS versus LDS.
3 Anytime upper bound of UDGVNS and UPDGVNS versus CPLEX

Figure 3: Anytime upper bound with 1, 10 and 30 processors respectively for cplex and UPDGVNS (UPDGVNS = UDGVNS(k add1/jump, ℓ mult2)).

4 Solving time and anytime behavior of U(P)DGVNS versus CPLEX and DAOOPT on UAI-Linkage category

Table 1: Best CPU time (in seconds) for sequential versions and best wall-clock time for multiple-core ones to find and prove optimality on Pedigree instances. A “-” means no proof of optimality in 1 hour (except DAOOPT with no time limit) (in parenthesis, unnormalized upper bound founds after 1 hour).

```
<table>
<thead>
<tr>
<th>Linkage</th>
<th>pedigree19 (optimum / worst solution)</th>
<th>pedigree31 (optimum / worst solution)</th>
<th>pedigree44 (optimum / worst solution)</th>
<th>pedigree51 (optimum / worst solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPLEX (1 core)</td>
<td>750 / 59.3</td>
<td>191 / 9.00</td>
<td>2724 / 6.35</td>
<td>4625 / 36.23</td>
</tr>
<tr>
<td>CPLEX (10 cores)</td>
<td>191 / 9.00</td>
<td>191 / 9.00</td>
<td>2383 / 4.92</td>
<td>2958 / 32.00</td>
</tr>
<tr>
<td>CPLEX (30 cores)</td>
<td>375/11.0</td>
<td>375/11.0</td>
<td>3303 / 8.35</td>
<td>4625 / 36.23</td>
</tr>
<tr>
<td>DAOOPT (1 core)</td>
<td>75 / 7.17</td>
<td>75 / 7.17</td>
<td>2724 / 6.35</td>
<td>4625 / 36.23</td>
</tr>
<tr>
<td>DAOOPT (20 cores)</td>
<td>2724 / 6.35</td>
<td>2724 / 6.35</td>
<td>3303 / 8.35</td>
<td>4625 / 36.23</td>
</tr>
<tr>
<td>DAOOPT (100 cores)</td>
<td>4626 / 17.75</td>
<td>4626 / 17.75</td>
<td>3303 / 8.35</td>
<td>4625 / 36.23</td>
</tr>
<tr>
<td>UDGVNS (1 core)</td>
<td>(4549) / 4.94</td>
<td>(4549) / 4.94</td>
<td>(4762) / 3.34</td>
<td>(6651) / 6.40</td>
</tr>
<tr>
<td>UDGVNS (10 cores)</td>
<td>- (4549) / 4.94</td>
<td>- (4549) / 4.94</td>
<td>- (4762) / 3.34</td>
<td>- (6651) / 6.40</td>
</tr>
<tr>
<td>UDGVNS (30 cores)</td>
<td>- (6128) / 17.75</td>
<td>- (6128) / 17.75</td>
<td>- (4762) / 3.34</td>
<td>- (6651) / 6.40</td>
</tr>
</tbody>
</table>
```

We report DAOOPT time from (OD17), obtained on a cluster of dual 2.67 GHz Intel Xeon X5650 6-core CPUs and 24 GB of RAM.

5 CPD instances filtering

In this paper we would like to evaluate VNS methods capability to solve difficult instances. Accordingly, we tried to generate new larger ones supposed to be more difficult to solve than those generated in (SAdG + 15). For this aims, in a first stage, we selected protein structures in the PDB databases (http://www.rcsb.org/pdb/home/home.do) with sizes range between 100 and 300 amino-acid. The resulting query has been filtered with the following criteria: resolution has to be lower than 2.5 A, membrane proteins, proteins complex, as well as proteins with disulfur bright have been removed, in addition, with proteins including non natural amino acid. We also discarded proteins with missing residues, out of the N and C terminal part of the sequence in order to select
a protein subset without any hole. Proteins with identity sequences higher than 90% are unselected. The corresponding remaining set includes 438 PDB references. Therefore, we sorted the 438 putative instances according to the average volume per variable and we extracted for benchmarking only the 20th first elements and used as benchmarking set. Each protein structure was fully redesigned according to (SAdG+15) protocol. On the basis of energy matrix generated with a modified release of pyrosetta script (SAdG+15) in order to use the last released Rosetta force-field (aka Beta November 2016) (A+17). The instances characteristic contain from 130 up to n = 282 variables width maximum domain size from 383 to 438, and between 1706 and 6208 cost functions. The tree width ranges from 21 to 68, and from 0.16 to 0.34 for a normalized tree width.

In order to select well decomposable instances, after preliminary reorientation of each protein according to his inertial moment, we filtered the resulting PDB set by 3D geometrical criteria. By construction, due to the cutoff distance used for pairwise energy calculation and his related constraints in the model, Globular protein (i.e spherical one) will correspond to CFN very nearby as click. In practice, the spherical shape which can be detected by a symmetric repartition of the Radius of gyration (Rg) component Rg(x), Rg(y), Rg(z). Indeed, Rg and his subcomponent are defined as the root mean square distance from each atom of the protein to their centroid or in the orthogonal plan to the related axis (x,y,z). Thus, for selecting non spherical protein, we first calculated the Rg, respectively around x, y and z axis as and we filtered the proteins characterized by a ratio:

\[
\min (Rg(x)/Rg + Rg(y)/Rg + Rg(z)/Rg)
\]

The \(\min(Rg(i)/Rg)\) is one figure to detect putative well structured instances.

Another structural criterion that can be used for structured instances filtering is a decreasing sort according to the approximated means space volume occupied associated to each variable. This volume is also related to the Gyration radius by the following equation:

\[
\bar{V} = \frac{\pi \ast Rg^3}{|X|}
\]

This criterion does not give at first sight a direct information about fold shape, but interestingly, both structural criteria produce the same 15th first instances set, with only few re-ranking, very likely because protein compactness got maximum bound and indeed a minimum average volume per amino-acid when the protein fold is close to sphere or vice versa for linear protein fold. After the 15th instances, the two critters remains less correlated. As a matter of fact, in the 5 next instances, the covering between the two sorted list is 2 over 5. On Account of computer power for benchmarking, we can not bench the full protein set.

| pdbid | X | d | e | tw | tw/|X| | min(Rg/Rg) | V(⊂/|X|) |
|-------|---|---|---|----|-----|-----|----------------|---------|
| 5dsl | 130 | 384 | 1,706 | 21 | 0.16 | 0.150 | 1,212.49 |
| 1l5p3 | 263 | 406 | 5,220 | 41 | 0.16 | 0.239 | 655.58 |
| 3q4d | 271 | 418 | 5,518 | 43 | 0.16 | 0.341 | 472.88 |
| 1om7 | 170 | 439 | 2,636 | 33 | 0.19 | 0.316 | 457.81 |
| 4086 | 262 | 430 | 6,208 | 51 | 0.18 | 0.269 | 439.28 |
| 2css | 235 | 407 | 4,745 | 44 | 0.19 | 0.354 | 404.42 |
| 1kaw | 107 | 412 | 1,623 | 28 | 0.26 | 0.308 | 378.04 |
| 1e10 | 133 | 400 | 2,296 | 34 | 0.26 | 0.294 | 344.68 |
| 1h0v | 152 | 389 | 2,587 | 51 | 0.34 | 0.420 | 343.38 |
| 1r9q | 181 | 415 | 3,449 | 54 | 0.30 | 0.392 | 352.69 |
| 2ax5 | 292 | 410 | 5,693 | 68 | 0.25 | 0.427 | 350.23 |
| 1nq2 | 176 | 397 | 3,135 | 60 | 0.34 | 0.473 | 309.16 |
| 5e2z | 151 | 456 | 2,865 | 49 | 0.32 | 0.403 | 304.87 |
| 2ge2 | 188 | 397 | 3,715 | 38 | 0.20 | 0.367 | 293.25 |
| 5eoz | 136 | 420 | 2,367 | 36 | 0.26 | 0.362 | 279.00 |
| 1lgy7 | 176 | 418 | 3,538 | 49 | 0.28 | 0.414 | 276.35 |
| 3k96 | 154 | 436 | 2,976 | 37 | 0.28 | 0.367 | 251.97 |
| 3w00 | 120 | 416 | 2,133 | 31 | 0.24 | 0.323 | 251.51 |
| 1is1 | 185 | 431 | 3,740 | 48 | 0.26 | 0.459 | 245.58 |
| 5eqp | 138 | 434 | 2,967 | 33 | 0.24 | 0.338 | 241.93 |
| 3lace | 188 | 363 | 4,163 | 44 | 0.25 | 0.347 | 234.11 |

Table 2: Characteristics of PDB instances: pdbid is the code reference in PDB database, |X| is the number of variable, d is the maximum domain size, e is number of cost functions, tw is the Min-fill tree width and tw/|X| a normalized tree width by |X|. The two last columns correspond to structural criteria respectively defined in (1) and (2). Grey color corresponds to the re-ranking when one \(\min(Rg/Rg)\) or \(V\) is used for PDB list sorting.
### 6 Comparing best solutions of TOULBAR2 vs VNS methods

<table>
<thead>
<tr>
<th>Instance</th>
<th>Time (s)</th>
<th>Speed-up</th>
</tr>
</thead>
<tbody>
<tr>
<td>5jdd</td>
<td>3,248±162</td>
<td>639±76</td>
</tr>
<tr>
<td>2x8c</td>
<td>1,012±44</td>
<td>191±25</td>
</tr>
<tr>
<td>1yq</td>
<td>1,205±38</td>
<td>277±26</td>
</tr>
<tr>
<td>2b1</td>
<td>2,175±67</td>
<td>866±246</td>
</tr>
<tr>
<td>1ng2</td>
<td>400±12</td>
<td>211±26</td>
</tr>
<tr>
<td>3e27</td>
<td>1,948±84</td>
<td>277±48</td>
</tr>
<tr>
<td>1y7t</td>
<td>1,575±16</td>
<td>428±83</td>
</tr>
<tr>
<td>3e5r</td>
<td>1,197±37</td>
<td>250±28</td>
</tr>
<tr>
<td>1st</td>
<td>2,158±76</td>
<td>317±53</td>
</tr>
<tr>
<td>4nns</td>
<td>467±9</td>
<td>427±83</td>
</tr>
</tbody>
</table>

(a) Un solved CPD instances.

<table>
<thead>
<tr>
<th>Instance</th>
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<th>Speed-up</th>
</tr>
</thead>
<tbody>
<tr>
<td>5dbl</td>
<td>1,828.27</td>
<td>791.16</td>
</tr>
<tr>
<td>3r8q</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4bxp</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1f00</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1xaw</td>
<td>2,917.04</td>
<td>1,171.98</td>
</tr>
<tr>
<td>5e10</td>
<td>839.52</td>
<td>196.43</td>
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<tr>
<td>2gee</td>
<td>-</td>
<td>9,705.59</td>
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<td>5e5x</td>
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<tr>
<td>3eqg</td>
<td>-</td>
<td>41,813</td>
</tr>
</tbody>
</table>

(b) Solved CPD instances.

(c) Optimality proof.

Table 3: Tables (3a) and (3b) report, for each instance, the CPU times spent by VNS methods (within the 3600-seconds time limit) to obtain the best solution computed by TOULBAR2 (5). A "-" indicates that the corresponding solver was not able to compute a solution of equal/better quality than TOULBAR2. (1): VNS(\(k_{\text{add1}}, \ell = 3\)) (2): UDGVNS(\(k_{\text{add1}}, \ell = 3\)) (3): UPDGVNS(\(npr, k_{\text{add1}}, \ell = 3\)). Table (3c) reports the CPU-times required by UDGVNS, UPDGVNS and TOULBAR2 to prove the optimum within the 24-hours time limit. A "-" indicates that the corresponding solver failed to prove optimality. (2\(^*\)): UDGVNS(\(k_{\text{add1}}/\text{jump}, \ell \text{ mult2}\)) (3\(^*\)): UPDGVNS(\(npr, k_{\text{add1}}/\text{jump}, \ell \text{ mult2}\)).

### References

