Poster summarizing ”The abc conjecture and some of its consequences”

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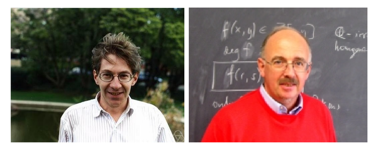
The abc conjecture and some of its consequences

The abc conjecture
Oesterlé and Masser (1985)

For any $\varepsilon > 0$, there exists $\kappa(\varepsilon)$ such that, if a, b and c are relatively prime positive integers which satisfy $a+b=c$, then

$$\mathrm{Rad}(abc) > \kappa(\varepsilon)c^{1-\varepsilon},$$

where for any positive integer n, $\mathrm{Rad}(n)$ is the product of its distinct prime factors.



Remarks

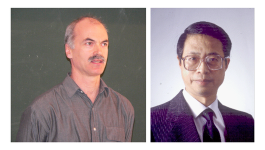
- The set $\{(a,b,c) = (1, 3^{2^k}-1, 3^{2^k}) \mid k \in \mathbb{N}\}$ shows that one cannot put $\varepsilon = 0$.
- If $u^p + v^q = w$ then the triple $(a,b,c) = ((uw^q)^p, (vw^p)^q, w^{pq+1})$ shows than one cannot drop the condition that $\gcd(a,b,c) = 1$.

Best unconditional result
Stewart and Kunrui Yu (1991, 2001)

For all a, b, c triple of coprime positive integers such that $a+b=c$ we have

$$e^{\kappa R^{1/3}(\log R)^3} \geq c,$$

where $R = \mathrm{Rad}(abc)$ and κ is an absolute constant.



Pillai's conjecture (1945)

Let k be a positive integer. The equation

$$x^p - y^q = k,$$

where the unknowns x, y, p and q take integer values, all ≥ 2, has only finitely many solutions (x,y,p,q).

The case $k=1$
Cassels, Tijdeman, Langevin, Mignotte

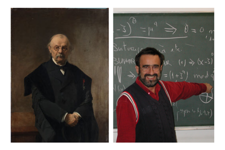
The equation $|x^p - y^q| = 1$ has no integer solution (x,y,p,q) with $p, q > 1$ and $\max(x^p, y^q) > \exp\exp\exp\exp(730)$.

The Catalan-Mihăilescu theorem (1844, 2002)

The only solution to the equation

$$x^p - y^q = 1$$

with $x, y > 0$ and $p, q > 1$ is $3^2 - 2^3 = 1$.

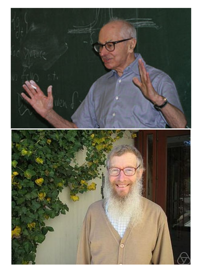


The Lang-Waldschmidt conjecture (1978)

Let $\varepsilon > 0$. There exists a constant $c(\varepsilon) > 0$ with the following property. If $x^p \neq y^q$, then

$$|x^p - y^q| \geq c(\varepsilon)\max\{x^p, y^q\}^{\kappa-\varepsilon}$$

with $\kappa = 1 - \frac{1}{p} - \frac{1}{q}$.



The abc conjecture implies Lang-Waldschmidt and therefore Pillai's conjecture

Hall's conjecture (1971)

The case $p=3$, $q=2$: If $x^3 \neq y^2$, then

$$|x^3 - y^2| \geq c\max\{x^3, y^2\}^{1/6}.$$

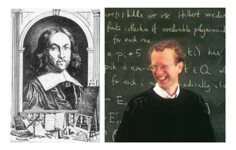
To deduce Hall's conjecture from the abc conjecture, one would need to take $\varepsilon = 0$, which is not allowed.

The Fermat-Wiles theorem (1621, 1994)

The equation

$$x^n + y^n = z^n$$

has no integer solutions (x,y,z,n) with $x, y, z > 0$ and $n > 2$.



The abc conjecture implies asymptotic Fermat-Wiles

Assume $x^n + y^n = z^n$ with $\gcd(x,y,z) = 1$. Then abc applied to (x^n, y^n, z^n) implies

$$z^3 > xyz = \mathrm{Rad}(x^n y^n z^n) > \kappa(\varepsilon)z^{n(1-\varepsilon)}.$$

When $n \geq 4$ we set $\varepsilon = \frac{1}{5}$ and obtain a bound on z^n.

The Fermat-Catalan conjecture
Brun (1914)

The equation

$$x^p + y^q = z^r$$

has a finite set of solutions (x,y,z,p,q,r) in positive integers such that $\gcd(x,y,z) = 1$ and

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

Beal's Prize (1M\$), supported by the AMS, will be given for a proof or a disproof of the statement that there is no solution to the Fermat-Catalan equation with coprime integers (x,y,z) and p, q, r all ≥ 3. The 10 known solutions are

$$1 + 2^3 = 3^2, \quad 2^5 + 7^2 = 3^4, \quad 7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2,$$
$$3^5 + 11^4 = 122^2, \quad 33^8 + 1\,549\,034^2 = 15\,613^3,$$
$$1414^3 + 2213459^2 = 65^7, \quad 9262^3 + 15312\,283^2 = 113^7,$$
$$17^7 + 76271^3 = 21063928^2, \quad 43^8 + 96222^3 = 30042907^2.$$

The abc conjecture implies asymptotic Fermat-Catalan conjecture
Tijdeman (1988)

An elementary study shows that the condition on (p,q,r) actually implies

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{41}{42}.$$

The abc conjecture applied to $\varepsilon = \frac{1}{84}$ and $(a,b,c) = (x^p, y^q, z^r)$ implies

$$z^{r(1-2\varepsilon)} > xyz \geq \mathrm{Rad}(x^p y^q z^r) > \kappa(\varepsilon)z^{r(1-\varepsilon)}.$$

The case of fixed (p,q,r)
Darmon and Granville (1995)

For each triple (p,q,r) with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ there exist only finitely many coprime solutions (x,y,z) to the Fermat-Catalan equation.

The cases $(p,p,2)$ and $(p,p,3)$
Darmon and Merel (1997)

The Fermat-Catalan equation has no solution in relatively prime positive integers for $p = q \geq 4$ and $r = 2$ and also for $p = q \geq 3$ and $r = 3$.

Szpiro's conjecture (1983)

Given any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that, for any elliptic curve over $\mathbf{Q}$ with minimal discriminant Δ and conductor N,

$$|\Delta| < C(\varepsilon)N^{6+\varepsilon}.$$

The abc conjecture implies Szpiro's conjecture
Oesterlé (1988)

Conversely, Szpiro's conjecture implies a weak form of the abc conjecture, with $1-\varepsilon$ replaced by $5/6 - \varepsilon$.

Wieferich's theorem (1909)

Let p be a prime and x, y, z positive integers such that $x^p + y^p = z^p$ and p doesn't divide xyz. Then p has the property that

$$p^2 \text{ divides } 2^{p-1} - 1.$$

Such a prime is called a Wieferich prime. An effective bound on the set of Wieferich primes would yield a new proof of the Fermat-Wiles theorem in the first case (p does not divide xyz).

Infinitely many non-Wieferich primes
Silverman (1988)

The abc conjecture implies that there are infinitely many non-Wieferich primes.

Nothing is known about the finitness of the set of Wieferich primes, the only two known examples being 1093 and 3511.

The Erdős-Woods conjecture (1981)

There exists an absolute constant k such that, if x and y are positive integers satisfying

$$\mathrm{Rad}(x+i) = \mathrm{Rad}(y+i)$$

for $i = 0, 1, \ldots, k-1$, then $x = y$.

The abc conjecture implies Erdős-Woods
Langevin (1996)

Already in 1975, Langevin studied the radical of $n(n+k)$ (with $\gcd(n,k) = 1$) using lower bounds for linear forms in logarithms of algebraic numbers (Baker's method).

Dirichlet's approximation theorem ($\approx$1830)

For any irrational α there exist infinitely many relatively prime pairs (p,q) such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

This theorem implies that the Pell-Fermat equation

$$x^2 - dy^2 = 1$$

has non-trivial solutions for any squarefree $d > 1$, a result which was previously proved by Lagrange (1766) and extended in a work on quadratic forms by Gauss (1801).

The Thue-Siegel-Roth's theorem (1909, 1921, 1955)

For any irrational algebraic number α and any positive ε the set of relatively prime integers p, q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}$$

is finite.

The number fields abc conjecture implies a refinement
Bombieri (1994)

The abc conjecture implies that in the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}$$

of the Thue-Siegel-Roth theorem one can replace the exponent ε with

$$\kappa(\log q)^{-1/2}(\log\log q)^{-1},$$

where κ depends only on α.

The Waring-Hilbert theorem (1770, 1909)

For any k there exists $g(k)$ such that each positive integer is a sum of at most $g(k)$ kth powers.

A conjecture on $g(k)$

J. A. Euler (1772): For all $k \geq 1$, $g(k) \geq I(k)$ where $I(k) = 2^k + \lfloor (3/2)^k \rfloor - 2$. Indeed, the integer $2\lfloor (3/2)^k \rfloor - 1$ is less than 3^k so it must be written so that only powers of 2 and 1 occur, and the most economic expression uses $I(k)$ terms.

Bretschneider's conjecture (1853): $g(k) = I(k)$ for any $k \geq 2$.

Evaluations of $g(k)$ for $k = 2, 3, 4, \ldots$

g(2)=4	Lagrange	1770
g(3)=9	Kempner	1912
g(4)=19	Balusubramanian, Dress, Deshouillers	1986
g(5)=37	Chen Jingrun	1964
g(6)=73	Pillai	1940
g(7)=143	Dickson	1936

A sufficient condition
Dickson, Pillai (1936)

If k is such that $2^k\{(3/2)^k\} + \lfloor (3/2)^k \rfloor \leq 2^k - 2$ then Bretschneider's conjecture holds for k.

Mahler's theorem (1957)

The condition of Dickson and Pillai is true for all but a finite set of integers k.

Kubina and Wunderlich (1990) created a fast algorithm to test the conjecture up to large values of k.

Effective bound assuming abc (2011)

A discussion between David and Waldschmidt lead to a proof of Mahler's result as a consequence of abc. Laishram proved that Bretschneider's conjecture follows from the explicit version of abc due to Baker. The same author proved a series of explicit results in a joint work with Shorey.

Baker's explicit version of the abc conjecture (2004)

Let (a,b,c) be three integers such that $\gcd(a,b) = 1$ and $c = a + b$. Then

$$c \leq \frac{6}{5}R\frac{(\log R)^\omega}{\omega!}$$

with $R = \mathrm{Rad}(abc)$ the radical of abc and $\omega = \omega(abc)$ the number of distinct prime factors of abc.

Siegel's theorem (1929)

Let g be the genus of a smooth algebraic curve in a given coordinate system, with coefficients in a number field K. If $g \geq 1$, then the curve has only finitely many integer points.

The effective abc conjecture implies effective Siegel
Surroca (2004)

In the proof she uses a theorem of Belyï.

Vojta's height conjecture (1987)

Vojta stated a conjectural inequality on the height which implies the abc conjecture. Another consequence of this inequality is the following. Let K be a number field and S a finite set of absolute values of K. If X is a variety with trivial canonical bundle and D is an effective ample normal crossing divisor, then the S-integral points on the affine variety $X \backslash D$ are not Zariski dense.

The Lang-Faltings theorem (1991)

If X is an abelian variety then the above statement holds.

Further consequences of the abc conjecture

- Lang's conjectures: lower bounds for heights, number of integral points on elliptic curves. (Frey 1987) (Hindry, Silverman 1988)
- Squarefree and powerfree values of polynomials. (Browkin, Filaseta, Greaves, Schinzel 1995)
- Bounds for the order of the Tate–Shafarevich group. (Goldfeld, Szpiro 1995)
- Frey proved in 1987 the equivalence between the height conjecture and the abc conjecture, while Mai and Murty in 1996 proved the equivalence with the degree conjecture.
- The abc conjecture for a cyclotomic number field K implies Greenberg's conjecture for infinitely many primes p: the Iwasawa invariants $\lambda_p(K)$ and $\mu_p(K)$ vanish. (Ichimura using a lemma of Sumida 1998)
- Effective abc implies Dressler's conjecture: between two positive integers having the same prime factors, there is always a prime. (Cochrane and Dressler 1999)
- The uniform abc conjecture for number fields implies a lower bound for the class number of imaginary quadratic fields (Granville and Stark 2000), and Mahler has shown that this implies that the associated L-function has no Siegel zeros.

Is the abc conjecture optimal?
Theorems by Stewart and Tijdeman and later by van Frankenhuijsen (1986, 2012)

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Stewart-Tijdeman (1986): For any $\delta > 0$ there are infinitely many triples (a,b,c) with $\gcd(a,b) = 1$ and $c = a + b$ for which

$$c > R\exp\left((4-\delta)\frac{(\log R)^{1/2}}{\log\log R}\right),$$

where $R = \mathrm{Rad}(abc)$.
In 2012 van Frankenhuijsen showed that $4 - \delta$ can be replaced in the inequality above by by 6.008.

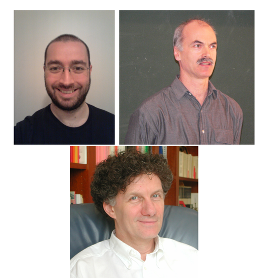
Heuristic:
$\mathrm{Rad}(a)$, $\mathrm{Rad}(b)$ and $\mathrm{Rad}(a+b)$ are independent
Robert, Stewart and Tenenbaum (2014)

For any $\delta > 0$ there exists $\kappa(\delta) > 0$ such that for any abc triple with $R = \mathrm{Rad}(abc) > 8$,

$$c < \kappa(\delta)R\exp\left((4\sqrt{3}+\delta)\left(\frac{\log R}{\log\log R}\right)^{1/2}\right).$$

Further, there exist infinitely many triples (a,b,c) such that $\gcd(a,b) = 1$ and $c = a + b$ for which

$$c > R\exp\left((4\sqrt{3}-\delta)\left(\frac{\log R}{\log\log R}\right)^{1/2}\right).$$



Mordell-Faltings theorem (1922, 1983)

Let g be the genus of an equation $P(x,y) = 0$ of coefficients in $\mathbf{Q}$. If $g \geq 2$ then the equation has only finitely many solutions with $(x,y) \in \mathbf{Q}^2$.

The effective abc implies effective Mordell
Elkies (1991)

The effective version of Mordell's conjectures amounts to giving bounds on the heights of rational points.

Under the (effective) abc conjecture for a number field K, the (effective) conjecture of Mordell holds for the same K.

In the quest for examples

Bosman, Broberg, Browkin, Brzezinski, Dokchitser, Elkies, Kanapka, Frey, Gang, Hegner, Nitaj, Reyssat, te Riele, P. Montgomery, Schulmeiss, Rosenheinrich, Visser, de Weger.
For any relatively prime positive integers a, b, c such that $a + b = c$ we set

$$\lambda(a,b,c) = \frac{\log c}{\log(\mathrm{Rad}(abc))}.$$

Baker's effective abc implies that $\lambda < 1,75$. The largest known examples are

$a+b$	$=$	c	$\lambda(a,b,c)$	author
$2 + 3^{10} \cdot 109$	$=$	23^5	$1.6299\ldots$	Reyssat
$11^2 + 3^2 \cdot 5^6 \cdot 7^3$	$=$	$2^{21} \cdot 23$	$1.6259\ldots$	de Weger
$19 \cdot 1307 + 7 \cdot 29^2 \cdot 31^8$	$=$	$2^8 \cdot 3^{22}.$	$1.6234\ldots$	Browkin, Brzezinski

Demeyer, Nitaj, de Weger, de Smit, H. Lenstra, Palenstijn, Rubin, Calvo, Wrobenski,
For any relatively prime positive integers a, b, c such that $a + b = c$ we set

$$\rho(a,b,c) = \frac{\log(abc)}{\log(\mathrm{Rad}(abc))}.$$

The largest known examples are

$a+b$	$=$	c	$\varrho(a,b,c)$	author
$13 \cdot 19^6 + 2^{30} \cdot 5$	$=$	$3^{13} \cdot 11^2 \cdot 31$	$4.4190\ldots$	Nitaj
$2^5 \cdot 11^2 \cdot 19^9 + 5^{15} \cdot 37^2 \cdot 47$	$=$	$3^7 \cdot 7^{11} \cdot 743$	$4.2680\ldots$	Nitaj
$2^{19} \cdot 13 \cdot 103 + 7^{11}$	$=$	$3^{11} \cdot 5^3 \cdot 11^2$	$4.2678\ldots$	de Weger.

The ABC conjecture for polynomials
Hurwitz, Stothers and Mason ($\approx$1900, 1981, 1984)

Let K be an algebraically closed field of characteristic zero. For any polynomial $P = \gamma\prod_i (x - \alpha_i)^{a_i}$ call the radical of P the polynomial $\mathrm{Rad}(P) = \prod_i (x - \alpha_i)$. Then for any three relatively prime polynomials A, B, C such that $A + B = C$ we have

$$\max(\deg(A), \deg(B), \deg(C)) \leq \deg(\mathrm{Rad}(ABC)) - 1.$$

An elementary proof

- Since $A + B = C$ we have $A' + B' = C'$ and then $W(A,B) = W(C,B) = W(A,C)$, where $W(A,B) = AB' - A'B$.
- Since A, B, C are relatively prime $W(A,B) \neq 0$. Indeed $AB' = A'B$ would imply that A divides A'.
- Clearly each of $G_A := \gcd(A, A')$, $G_B := \gcd(B, B')$ and $G_C := \gcd(C, C')$ divides $W(A,B)$. Since A, B and C are relatively prime, $G_A G_B G_C$ divides $W(A,B)$. Then
$$\begin{aligned}\deg(G_A) + \deg(G_B) + \deg(G_C) &\leq \deg(W(A,B)) \\ &\leq \deg(A) + \deg(B) - 1.\end{aligned}$$
- Since for all P, $\mathrm{Rad}(P) = P/\gcd(P, P')$ the theorem follows.

The abc conjecture for meromorphic function fields

The value distribution theory was introduced by Nevanlinna. The abc conjecture was extended to this context by Pei-Chu and Chung-Chun and later by Vojta.

Mochizuki announces a proof (2012)

Inter-universal Teichmüler IV: log-volume computations and set theoretic foundations.

The submitted proof is more than 500 pages long and is currently (November 2017) in the process of being peer-reviewed.

References

- An extensive litterature on the subject is available on the abc home page created and maintained by Abderrahmane Nitaj: https://nitaj.users.lmno.cnrs.fr/abc.html.
- This poster is a summary of the article *"The abc conjecture and some of its consequences". Michel Waldschmidt. 2015*, available online at https://webusers.imj-prg.fr/~michel.waldschmidt/articles/pdf/abcLahoreProceedings.pdf.

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