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Abstract

By methods of stochastic analysis on Riemannian manifolds, we derive explicit constants $c_1(D)$ and $c_2(D)$ for a d-dimensional compact Riemannian manifold D with boundary such that

$$c_1(D)\sqrt{\lambda}\|\phi\|_\infty \leqslant \|\nabla\phi\|_\infty \leqslant c_2(D)\sqrt{\lambda}\|\phi\|_\infty$$

holds for any Dirichlet eigenfunction ϕ of $-\Delta$ with eigenvalue λ. In particular, when D is convex with nonnegative Ricci curvature, this estimate holds for

$$c_1(D) = \frac{1}{d\mathrm{e}}, \quad c_2(D) = \sqrt{\mathrm{e}} + \frac{\mathrm{e}\sqrt{2}}{\sqrt{\pi}}.$$

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1 Introduction

Let D be a d-dimensional compact Riemannian manifold with boundary ∂D. We write $(\phi, \lambda) \in \mathrm{Eig}(\Delta)$ if ϕ is a Dirichlet eigenfunction of $-\Delta$ in D with eigenvalue $\lambda > 0$. According to [6], there exist two constants $c_1(D), c_2(D) > 0$ such that

$$c_1(D)\sqrt{\lambda}\|\phi\|_\infty \leqslant \|\nabla\phi\|_\infty \leqslant c_2(D)\sqrt{\lambda}\|\phi\|_\infty, \quad (\phi,\lambda) \in \mathrm{Eig}(\Delta). \tag{1.1}$$

In this paper, by using stochastic analysis of the Brownian motion on D, we present explicit expressions of these two constants in terms of the lower bounds of Ric_D and $\mathbb{I}_{\partial D}$ where Ric_D is the Ricci curvature on D and $\mathbb{I}_{\partial D}$ the second fundamental form of ∂D.

Theorem 1.1. *Let $K, \theta \geqslant 0$ be two constants such that*

$$\mathrm{Ric}_D \geqslant -K, \quad \mathbb{I}_{\partial D} \geqslant -\theta.$$

Let

$$\alpha_0 = \frac{1}{2}\Big((d-1)\theta + \sqrt{(d-1)K}\Big).$$

Then, for any nontrivial $(\phi,\lambda) \in \mathrm{Eig}(\Delta)$,

$$\frac{\lambda}{\sqrt{d\mathrm{e}(\lambda+K)}} \leqslant \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leqslant \sqrt{\mathrm{e}(\lambda+K)} + \mathrm{e}\left(\alpha_0 + \frac{\sqrt{2\lambda}}{\sqrt{\pi}} + \alpha_0 \wedge \frac{\alpha_0^2}{\sqrt{2\pi\lambda}}\right).$$

In particular, when Ric_D, $\mathbb{I}_{\partial D} \geqslant 0$,

$$\frac{\sqrt{\lambda}}{\sqrt{d\mathrm{e}}} \leqslant \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leqslant \sqrt{\lambda}\left(\sqrt{\mathrm{e}} + \frac{\mathrm{e}\sqrt{2}}{\sqrt{\pi}}\right), \quad (\phi,\lambda) \in \mathrm{Eig}(\Delta). \tag{1.2}$$

Proof. This result follows from Theorem 2.1 and Theorem 3.1 below for the special case $V = 0$. In this case, $\mathrm{Ric}_D^V = \mathrm{Ric}_D \geqslant -K$ is equivalent to (2.1) with $n = d$. □

Remark 1.1. Various other estimates can be obtained from our method. For instance, let $\alpha \in \mathbb{R}$ be such that $\frac{1}{2}\Delta\rho_{\partial D} \leqslant \alpha$ outside the focal set, where $\rho_{\partial D}$ denotes the distance to boundary ∂D. Then for $\lambda < \alpha^2/4$,

$$\frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leqslant \sqrt{\mathrm{e}(\lambda+K)} + \mathrm{e}\left(2\max(\alpha,0) + \frac{\alpha^2\mathrm{e}^2}{4}\sqrt{\frac{2\pi}{\lambda}}\mathrm{e}^{-\frac{\alpha^2}{2\lambda}}\right). \tag{1.3}$$

This relies on Remark 3.1 where another estimate of the right hand side of (3.13) is given. It improves the estimate in Theorem 1.1 in the case when $\alpha < 0$ and $|\alpha|$ is large. See Theorem 3.5 below for case that $k = 0$ and $\theta < 0$.

By (1.2), when D is convex with nonnegative Ricci curvature, (1.1) holds with

$$c_1(D) = \frac{1}{\sqrt{d\mathrm{e}}}, \quad c_2(D) = \sqrt{\mathrm{e}} + \frac{\mathrm{e}\sqrt{2}}{\sqrt{\pi}}.$$

To estimate $c_1(D)$ and $c_2(D)$ for positive K or θ, let $\lambda_1 > 0$ be the first Dirichlet eigenvalue of $-\Delta$ on D. Then Theorem 1.1 implies that the inequalities (1.1) hold for

$$c_1(D) = \frac{\sqrt{\lambda_1}}{\sqrt{de(\lambda_1 + K)}},$$
$$c_2(D) = \frac{\sqrt{e(\lambda_1 + K)}}{\sqrt{\lambda_1}} + e\bigg(\frac{(d-1)\theta + \sqrt{K(d-1)}}{2\sqrt{\lambda_1}} + \frac{\sqrt{2}}{\sqrt{\pi}} + \frac{\big((d-1)\theta + \sqrt{K(d-1)}\big)^2}{4\lambda_1\sqrt{2\pi}}\bigg).$$

This is due to the fact that the first expression is an increasing function of λ and the second one is a decreasing function of λ. Since there exist explicit lower bound estimates on λ_1 (see [8] and references within), this gives explicit lower bounds of $c_1(D)$ and upper bounds of $c_2(D)$.

The lower bound estimate of $\|\nabla\phi\|_\infty$ will be derived by using Itô's formula for $|\nabla\phi|^2(X_t)$ where X_t is a Brownian motion (with drift) on D, see Section 2 for details. A powerful probabilistic tool for establishing upper bound gradient estimates is the use of Bismut type formulas for the Dirichlet semigroup P_t^D on D, which gives

$$|\nabla P_t^D f(x)| \leqslant \frac{c(t)}{\rho_{\partial D}} \|f\|_\infty, \quad t > 0,\ f \in \mathscr{B}_b(D),$$

where $\rho_{\partial D}$ is the Riemannian distance to ∂D and $c(t)$ an explicit quantity depending on the geometry of D, see [7] for details. However, as this estimate blows up at the boundary ∂D, it does not give the wanted upper bound estimate of $\|\nabla\phi\|_\infty$ near the boundary. To achieve the goal of a uniform upper bound on D, we will construct some martingales to reduce $\|\nabla\phi\|_\infty$ to $\|\nabla\phi\|_{\partial D,\infty} := \sup_{\partial D} |\nabla\phi|$, and to estimate the latter using $\|\phi\|_\infty$, see Section 3 for details.

In general, we will consider Dirichlet eigenfunctions for the symmetric operator $L := \Delta + \nabla V$ on D where $V \in C^2(D)$. We denote by $\mathrm{Eig}(L)$ the set of pairs (ϕ, λ) where ϕ is a Dirichlet eigenfunction of $-L$ on D with eigenvalue λ.

2 Lower bound estimate

In this Section we will estimate $\|\nabla\phi\|_\infty$ from below using the following Bakry-Émery curvature-dimension condition:

$$\frac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f\rangle \geqslant -K|\nabla f|^2 + \frac{(Lf)^2}{n}, \quad f \in C^\infty(D), \tag{2.1}$$

where $K \in \mathbb{R}, n \geqslant d$ are two constants. When $V = 0$, this condition with $n = d$ is equivalent to $\mathrm{Ric}_D \geqslant -K$.

Theorem 2.1 (Lower bound estimate)**.** *Assume that* (2.1) *holds. Then*

$$\|\nabla\phi\|_\infty^2 \geqslant \|\phi\|_\infty^2 \sup_{t>0} \frac{\lambda^2(e^{Kt} - 1)}{nKe^{(\lambda+K)^+t}}, \quad (\phi, \lambda) \in \mathrm{Eig}(L). \tag{2.2}$$

Consequently, for $K^+ := \max\{0, K\}$ there holds

$$\|\nabla\phi\|_\infty^2 \geqslant \frac{\lambda^2}{ne(\lambda + K^+)}\|\phi\|_\infty^2, \quad (\phi, \lambda) \in \mathrm{Eig}(L). \tag{2.3}$$

Proof. Let X_t be the diffusion process generated by $\frac{1}{2}L$ in D, and let

$$\tau_D := \inf\{t \geqslant 0 : X_t \in \partial D\}.$$

By Itô's formula, we have

$$\mathrm{d}|\nabla\phi|^2(X_t) = \frac{1}{2}L|\nabla\phi|^2(X_t)\,\mathrm{d}t + \mathrm{d}M_t, \quad t \leqslant \tau_D, \tag{2.4}$$

for some martingale M_t. By the curvature dimension condition (2.1) and $L\phi = -\lambda\phi$, we obtain

$$\frac{1}{2}L|\nabla\phi|^2 = \frac{1}{2}L|\nabla\phi|^2 - \langle\nabla L\phi, \nabla\phi\rangle - \lambda|\nabla\phi|^2 \geqslant -(K+\lambda)|\nabla\phi|^2 + \frac{\lambda^2}{n}\phi^2.$$

Therefore, (2.4) gives

$$\mathrm{d}|\nabla\phi|^2(X_t) \geqslant \Big(\frac{\lambda^2}{n}\phi^2 - (K+\lambda)|\nabla\phi|^2\Big)(X_t)\,\mathrm{d}t + \mathrm{d}M_t, \quad t \leqslant \tau_D.$$

Hence, for any $t > 0$,

$$\begin{aligned} \mathrm{e}^{(K+\lambda)^+t}\,\|\nabla\phi\|_\infty^2 &\geqslant \mathbb{E}\Big[|\nabla\phi|^2(X_{t\wedge\tau_D})\mathrm{e}^{(K+\lambda)(t\wedge\tau_D)}\Big] \\ &\geqslant \frac{\lambda^2}{n}\mathbb{E}\left[\int_0^{t\wedge\tau_D} \mathrm{e}^{(K+\lambda)s}\phi(X_s)^2\,\mathrm{d}s\right] \\ &= \frac{\lambda^2}{n}\mathbb{E}\left[\int_0^t 1_{\{s<\tau_D\}}\mathrm{e}^{(K+\lambda)s}\phi(X_s)^2\,\mathrm{d}s\right]. \end{aligned}$$

Since $\phi|_{\partial D} = 0$ and $L\phi = -\lambda\phi$, by Jensen's inequality we have

$$\mathbb{E}\left[1_{\{s<\tau_D\}}\phi(X_s)^2\right] \geqslant \big(\mathbb{E}[\phi(X_{s\wedge\tau_D})]\big)^2 = \mathrm{e}^{-\lambda s}\phi(x)^2,$$

where $x = X_0 \in D$ is the starting point of X_t. Then, by taking x such that $\phi(x)^2 = \|\phi\|_\infty^2$, we arrive at

$$\begin{aligned} \mathrm{e}^{(K+\lambda)^+t}\,\|\nabla\phi\|_\infty^2 &\geqslant \frac{\lambda^2}{n}\int_0^t \mathrm{e}^{(K+\lambda)s}\mathrm{e}^{-\lambda s}\phi(x)^2\,\mathrm{d}s \\ &= \frac{\lambda^2\|\phi\|_\infty^2}{n}\int_0^t \mathrm{e}^{Ks}\,\mathrm{d}s = \frac{\lambda^2(\mathrm{e}^{Kt}-1)}{nK}\|\phi\|_\infty^2. \end{aligned}$$

This completes the proof of (2.2).

Since (2.1) holds for K^+ replacing K, we may and do assume that $K \geqslant 0$. By taking $t = \frac{1}{\lambda+K}$ in (2.2), we obtain

$$\|\nabla\phi\|_\infty^2 \geqslant \frac{\lambda^2(\mathrm{e}^{\frac{K}{\lambda+K}}-1)}{nK\mathrm{e}}\,\|\phi\|_\infty^2 \geqslant \frac{\lambda^2}{n\mathrm{e}(\lambda+K)}\,\|\phi\|_\infty^2.$$

Hence (2.3) holds. □

3 Upper bound estimate

Let $\mathrm{Ric}_D^V = \mathrm{Ric}_D - \mathrm{Hess}_V$.

Theorem 3.1 (Upper bound estimate)**.** *Let $K_V, K_0, \theta \geqslant 0$ be constants such that*

$$\mathrm{Ric}_D^V \geqslant -K_V, \quad \mathrm{Ric}_D \geqslant -K_0, \quad \mathbb{I}_{\partial D} \geqslant -\theta.$$

Let

$$\alpha = \frac{1}{2}\Big((d-1)\theta + \sqrt{(d-1)K_0} + \|\nabla V\|_\infty\Big). \tag{3.1}$$

Then, for any $(\phi, \lambda) \in \mathrm{Eig}(L)$,

$$\|\nabla\phi\|_\infty \leqslant \|\phi\|_\infty \bigg\{\sqrt{\mathrm{e}(\lambda + K_V)} + \mathrm{e}\Big(\alpha + \frac{\sqrt{2\lambda}}{\sqrt{\pi}} + \alpha \wedge \frac{\alpha^2}{\sqrt{2\pi\lambda}}\Big)\bigg\}.$$

To prove this result, we first estimate $\|\nabla\phi\|_\infty$ in terms of $\|\phi\|_\infty$ and $\|\nabla\phi\|_{\partial D,\infty}$ where $\|f\|_{\partial D,\infty} := \|1_{\partial D} f\|_\infty$ for a function f on D.

Lemma 3.2. *Assume $\mathrm{Ric}_D^V \geqslant -K_V$ for some constant $K_V \in \mathbb{R}$. Then, for any $(\phi, \lambda) \in \mathrm{Eig}(L)$,*

$$\|\nabla\phi\|_\infty \leqslant \mathrm{e}^{\frac{(\lambda+K_V)^+}{2}t}\|\nabla\phi\|_{\partial D,\infty} + \|\phi\|_\infty\, \mathrm{e}^{\frac{\lambda}{2}t}\left(\frac{K_V}{1-\mathrm{e}^{-K_V t}}\right)^{1/2}, \quad t > 0. \tag{3.2}$$

Consequently,

$$\|\nabla\phi\|_\infty \leqslant \mathrm{e}^{1/2}\left(\|\nabla\phi\|_{\partial D,\infty} + \sqrt{\lambda + K_V^+}\,\|\phi\|_\infty\right), \quad (\phi,\lambda) \in \mathrm{Eig}(L). \tag{3.3}$$

Proof. We first recall some facts concerning the diffusion process generated by $\frac{1}{2}L$, see for instance [1, 3]. For any $x \in D$, the diffusion X_t solves the SDE

$$\mathrm{d}X_t = \frac{1}{2}\nabla V(X_t)\,\mathrm{d}t + u_t \circ \mathrm{d}B_t, \quad X_0 = x,\ t \leqslant \tau_D, \tag{3.4}$$

where B_t is a d-dimensional Brownian motion, u_t is the horizontal lift of X_t onto the orthonormal frame bundle $\mathrm{O}(D)$ with initial value $u_0 \in \mathrm{O}_x(D)$, and

$$\tau_D := \inf\{t \geqslant 0 : X_t \in \partial D\}$$

is the hitting time of X_t to the boundary ∂D. Setting $Z := \nabla V$, we have

$$\mathrm{d}u_t = \frac{1}{2}Z^*(u_t)\,\mathrm{d}t + \sum_{i=1}^d H_i(u_t) \circ \mathrm{d}B_t^i \tag{3.5}$$

where $Z^*(u) := h_u(Z_{\pi(u)})$ and $H_i(u) := h_u(ue_i)$ are defined by means of the horizontal lift $h_u\colon T_{\pi(u)}D \to T_u\,\mathrm{O}(D)$ at $u \in \mathrm{O}(D)$. Note that formally $h_{u_t}(u_t \circ \mathrm{d}B_t) = \sum_i h_{u_t}(u_t e_i) \circ \mathrm{d}B_t^i = \sum_i H_i(u_t) \circ \mathrm{d}B_t^i$.

For $f \in C^\infty(D)$, let $a := \mathrm{d}f \in \Gamma(T^*D)$. Setting $m_t := u_t^{-1} a(X_t)$, we see by Itô's formula that

$$\mathrm{d}m_t \overset{\mathrm{m}}{=} \frac{1}{2} u_t^{-1} (\Box a + \nabla_Z a)(X_t)\,\mathrm{d}t \tag{3.6}$$

where $\Box a = \operatorname{tr} \nabla^2 a$ denotes the so-called connection (or rough) Laplacian on 1-forms and $\overset{\mathrm{m}}{=}$ equality modulo the differential of a local martingale.

Denote by $Q_t \colon T_x D \to T_{X_t} D$ the solution, along the paths of X_t, to the covariant ordinary differential equation

$$DQ_t = -\frac{1}{2} (\mathrm{Ric}_D^V)^\sharp Q_t\,\mathrm{d}t, \quad Q_0 = \mathrm{id}_{T_x D},\ t \leqslant \tau_D,$$

where $D := u_t \mathrm{d} u_t^{-1}$ and where by definition

$$(\mathrm{Ric}_D^V)^\sharp v = \mathrm{Ric}_D^V(\cdot, v)^\sharp, \quad v \in T_x D.$$

Thus, condition $\mathrm{Ric}_D^V \geqslant -K_V$ implies

$$|Q_t v| \leqslant \mathrm{e}^{\frac{K_V}{2} t}\,|v|, \quad t \leqslant \tau_D. \tag{3.7}$$

Finally, note that for any smooth function f on D, we have by the Weitzenböck formula:

$$\begin{aligned} \mathrm{d}\big(\Delta + Z\big) f &= \mathrm{d}\big(-\mathrm{d}^*\mathrm{d}f + (\mathrm{d}f)Z\big) \\ &= \Delta^{(1)}\mathrm{d}f + \nabla_Z \mathrm{d}f + \langle \nabla_{\boldsymbol{\cdot}} Z, \nabla f\rangle \\ &= (\Box + \nabla_Z)(\mathrm{d}f) - \mathrm{Ric}_D^V(\cdot, \nabla f) \\ &= \big(\Box - \mathrm{Ric}_D^V + \nabla_Z\big)(\mathrm{d}f) \end{aligned} \tag{3.8}$$

where $\Delta^{(1)}$ denotes the Hodge-deRham Laplacian on 1-forms.

Now let $(\phi, \lambda) \in \mathrm{Eig}(L)$, i.e. $L\phi = -\lambda\phi$, where $L = \Delta + Z$. For $v \in T_x D$, consider the process

$$n_t(v) := (\mathrm{d}\phi)(Q_t v).$$

Then

$$n_t(v) = \langle \nabla\phi(X_t), Q_t v\rangle = \langle u_t^{-1} (\nabla\phi)(X_t), u_t^{-1} Q_t v\rangle.$$

Using (3.6), we see by Itô's formula and formula (3.8) that

$$\mathrm{d}n_t(v) \overset{\mathrm{m}}{=} \frac{1}{2} (\Box \mathrm{d}\phi + \nabla_Z \mathrm{d}\phi)(X_t)\, Q_t v\,\mathrm{d}t + \mathrm{d}\phi(X_t)(DQ_t v)\,\mathrm{d}t = -\frac{\lambda}{2} n_t(v)\,\mathrm{d}t.$$

It follows that

$$\mathrm{e}^{\lambda t/2}\, n_t(v) = \mathrm{e}^{\lambda t/2}\, \langle \nabla\phi(X_t), Q_t v\rangle, \quad t \leqslant \tau_D,$$

is a martingale, and consequently, for any function $h \in C^1([0,\infty); \mathbb{R})$,

$$h_t\, \mathrm{e}^{\lambda t/2}\, \langle \nabla\phi(X_t), Q_t v\rangle - \int_0^t \dot{h}_s\, \mathrm{e}^{\lambda s/2}\, \langle \nabla\phi(X_s), Q_s v\rangle\,\mathrm{d}s, \quad t \leqslant \tau_D,$$

is a martingale as well. By the formula

$$e^{\lambda t/2}\,\phi(X_t)=\phi(X_0)+\int_0^t e^{\lambda s/2}\,\langle\nabla\phi(X_s),u_s\mathrm{d}B_s\rangle$$

we see then that

$$N_t(v):=h_t\,e^{\lambda t/2}\,\langle\nabla\phi(X_t),Q_tv\rangle-e^{\lambda t/2}\,\phi(X_t)\int_0^t\langle\dot h_sQ_sv,u_s\mathrm{d}B_s\rangle,\quad t\leqslant\tau_D,$$

is a martingale.

Now, for fixed $t>0$, we take $h\in C^1([0,t];[0,1])$ such that $h_0=1$ and $h_t=0$. Then, by the martingale property of $\{N_{s\wedge\tau_D}(v)\}_{s\in[0,t]}$ we obtain

$$\begin{aligned}&|\nabla_v\phi|(x)=|N_0(v)|=|\mathbb{E}N_{t\wedge\tau_D}(v)|\\&=\left|\mathbb{E}\left[1_{\{t>\tau_D\}}\,e^{\lambda\tau_D/2}\,h_{\tau_D}\langle\nabla\phi(X_{\tau_D}),Q_{\tau_D}v\rangle-1_{\{t\leqslant\tau_D\}}e^{\lambda t/2}\phi(X_t)\int_0^t\langle\dot h_s\,Q_sv,u_s\mathrm{d}B_s\rangle\right]\right|.\end{aligned}$$

This together with (3.7) yields

$$|\nabla\phi(x)|\leqslant e^{(\lambda+K_V)^+t/2}\,\|\nabla\phi\|_{\partial D,\infty}+e^{\lambda t/2}\,\|\phi\|_\infty\bigg(\int_0^t(\dot h_s)^2e^{K_Vs}\,\mathrm{d}s\bigg)^{1/2}.$$

Taking

$$h_s=\frac{e^{-K_Vt}-e^{-K_Vs}}{e^{-K_Vt}-1},\quad s\in[0,t],$$

we obtain (3.2). Finally, noting that

$$\frac{K_V}{1-e^{-K_Vt}}\leqslant\frac{K_V^+}{1-e^{-K_V^+t}}\leqslant t^{-1}e^{K_V^+t},$$

and taking $t=(K_V^++\lambda)^{-1}$ in (3.2), we prove (3.3). □

To estimate the term $\|\nabla\phi\|_{\partial D,\infty}$, we shall compare $\phi(x)$ and

$$\psi(t,x):=\mathbb{P}(\tau_D^x>t),\quad t>0,$$

for small $\rho_{\partial D}(x):=\mathrm{dist}(x,\partial D)$. Let P_t^D be the Dirichlet semigroup generated by $\frac{1}{2}L$. Then $\psi(t,x)=P_t^D1_D(x)$, so that

$$\partial_t\psi(t,x)=\frac{1}{2}L\psi(t,\cdot)(x),\quad t>0.\tag{3.9}$$

Lemma 3.3. *For any $(\phi,\lambda)\in\mathrm{Eig}(L)$,*

$$\|\nabla\phi\|_{\partial D,\infty}\leqslant\|\phi\|_\infty\inf_{t>0}e^{\lambda t/2}\,\|\nabla\psi(t,\cdot)\|_{\partial D,\infty}.\tag{3.10}$$

Proof. To prove (3.10), we fix $x \in \partial D$. For small $\varepsilon > 0$, let $x^\varepsilon = \exp_x(\varepsilon N)$, where N is the inward unit normal vector field of ∂D. Since $\phi|_{\partial D} = 0$ and $\psi(t,\cdot)|_{\partial D} = 0$, we have

$$|\nabla\phi(x)| = |N\phi(x)| = \lim_{\varepsilon\to 0} \frac{|\phi(x^\varepsilon)|}{\varepsilon}, \quad |\nabla\psi(t,\cdot)(x)| = \lim_{\varepsilon\to 0} \frac{|\psi(t,x^\varepsilon)|}{\varepsilon}. \tag{3.11}$$

Let X_t^ε be the L-diffusion starting at x^ε and τ_D^ε its first hitting time of ∂D. Note that

$$N_t := \phi(X^\varepsilon_{t\wedge\tau_D^\varepsilon})\, e^{\lambda(t\wedge\tau_D^\varepsilon)/2}, \quad t \geqslant 0,$$

is a martingale. Thus, for each fixed $t > 0$, we can estimate as follows:

$$\begin{aligned}
|\nabla\phi(x)| &= \lim_{\varepsilon\to 0} \frac{|\phi(x^\varepsilon)|}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\big|\mathbb{E}[\phi(X_t^\varepsilon)\,\mathbf{1}_{\{t<\tau_D^\varepsilon\}}]\, e^{\lambda(t\wedge\tau_D^\varepsilon)/2}\big|}{\varepsilon} \\
&\leqslant \|\phi\|_\infty\, \mathrm{e}^{\lambda t/2} \lim_{\varepsilon\to 0} \frac{\mathbb{E}[\mathbf{1}_{\{t<\tau_D^\varepsilon\}}]}{\varepsilon} \\
&\leqslant \|\phi\|_\infty\, \mathrm{e}^{\lambda t/2} \lim_{\varepsilon\to 0} \frac{\psi(t,x^\varepsilon)}{\varepsilon} \\
&= \|\phi\|_\infty\, \mathrm{e}^{\lambda t/2}\, |\nabla\psi(t,\cdot)|(x).
\end{aligned}$$

Taking the infimum over t gives the claim. □

We now estimate $\|\nabla\psi(t,\cdot)\|_\infty$. Let $\mathrm{cut}(D)$ be the cut-locus of ∂D, which is a zero-volume closed subset of D such that $\rho_{\partial D} := \mathrm{dist}(\cdot,\partial D)$ is smooth in $D \setminus \mathrm{cut}(D)$.

Proposition 3.4. *Let $\alpha \in \mathbb{R}$ be such that*

$$\frac{1}{2} L\rho_{\partial D}(x) \leqslant \alpha, \qquad x \in D \setminus \mathrm{cut}(D). \tag{3.12}$$

Then

$$\begin{aligned}
\|\nabla\psi(t,\cdot)\|_{\partial D,\infty} &\leqslant \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1-\mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}}\,\mathrm{d}s \\
&\leqslant \frac{\sqrt{2}}{\sqrt{\pi t}} + \min\left\{2\alpha^+,\ \alpha + \frac{\alpha^2\sqrt{t}}{\sqrt{2\pi}}\right\}.
\end{aligned} \tag{3.13}$$

Proof. Let $x \in D$ and let X_t solve SDE (3.4). As shown in [5], $(\rho_{\partial D}(X_t))_{t\leqslant\tau_D}$ is a semi-martingale satisfying

$$\rho_{\partial D}(X_t) = \rho_{\partial D}(x) + b_t + \frac{1}{2}\int_0^t L\rho_{\partial D}(X_s)\,\mathrm{d}s - l_t, \quad t \leqslant \tau_D, \tag{3.14}$$

where b_t is a real-valued Brownian motion starting at 0, and l_t a non-decreasing process which increases only when $X_t^x \in \mathrm{cut}(D)$. Hence, setting $\varepsilon = \rho_{\partial D}(x)$, we deduce from (3.12) and (3.14) that

$$\rho_{\partial D}(X_t(x)) \leqslant Y_t^\alpha(\varepsilon) := \varepsilon + b_t + \alpha t, \quad t \leqslant \tau_D. \tag{3.15}$$

Consequently, letting $T^\alpha(\varepsilon)$ be the first hitting time of 0 by $Y^\alpha_t(\varepsilon)$, we obtain

$$\psi(t,x) \leqslant \mathbb{P}(t < T^\alpha(\varepsilon)). \tag{3.16}$$

On the other hand, since $\psi(t,\cdot)$ vanishes on the boundary and is positive in D, we have for all $y \in \partial D$

$$|\nabla \psi(t,y)| = \lim_{x\in D,\, x\to y} \frac{\psi(t,x)}{\rho_{\partial D}(x)}. \tag{3.17}$$

Hence, by (3.16), to prove the first inequality in (3.13) it is enough to establish that

$$\limsup_{\varepsilon\downarrow 0} \frac{\mathbb{P}(t < T^\alpha(\varepsilon))}{\varepsilon} \leqslant \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1-\mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}}\,\mathrm{d}s. \tag{3.18}$$

It is well known that the (sub-probability) density $f_{\alpha,\varepsilon}$ of $T^\alpha(\varepsilon)$ is

$$f_{\alpha,\varepsilon}(s) = \frac{\varepsilon \exp\left(\frac{-(\varepsilon+\alpha s)^2}{2s}\right)}{\sqrt{2\pi s^3}}, \tag{3.19}$$

which can be obtained by the reflection principle for $\alpha = 0$ and the Girsanov transform for $\alpha \in \mathbb{R}$. Thus

$$\begin{aligned}\mathbb{P}(t \geqslant T^\alpha(\varepsilon)) &= \varepsilon \int_0^t \frac{\exp\left(\frac{-(\varepsilon+\alpha s)^2}{2s}\right)}{\sqrt{2\pi s^3}}\,\mathrm{d}s \\ &= \varepsilon \exp(-\alpha\varepsilon) \int_0^t \frac{\mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} \exp\left(-\frac{\varepsilon^2}{2s}\right)\mathrm{d}s \\ &= \exp(-\alpha\varepsilon)\int_0^{2t/\varepsilon^2} \frac{\mathrm{e}^{-1/r}}{\sqrt{\pi r^3}} \exp\left(-\frac{\alpha^2\varepsilon^2 r}{4}\right)\mathrm{d}r,\end{aligned} \tag{3.20}$$

where we have made the change of variable $r = 2s/\varepsilon^2$. With the change of variable $v = 1/r$ we easily check that

$$\int_0^\infty r^{-3/2}\mathrm{e}^{-1/r}\,\mathrm{d}r = \Gamma(1/2) = \sqrt{\pi}, \tag{3.21}$$

and this allows to write

$$\mathbb{P}(t \geqslant T^\alpha(\varepsilon)) = \exp(-\alpha\varepsilon)\left(1 - \int_{2t/\varepsilon^2}^\infty \frac{\mathrm{e}^{-1/r}}{\sqrt{\pi r^3}}\,\mathrm{d}r - \int_0^{2t/\varepsilon^2} \frac{\mathrm{e}^{-1/r}}{\sqrt{\pi r^3}}\left(1-\mathrm{e}^{\frac{-\alpha^2\varepsilon^2 r}{4}}\right)\mathrm{d}r\right). \tag{3.22}$$

As $\varepsilon \to 0$,

$$\int_{2t/\varepsilon^2}^\infty \frac{\mathrm{e}^{-1/r}}{\sqrt{r^3}}\,\mathrm{d}r = \int_{2t/\varepsilon^2}^\infty \frac{1}{\sqrt{r^3}}\,\mathrm{d}r + \mathrm{o}(\varepsilon) = \frac{\varepsilon\sqrt{2}}{\sqrt{t}} + \mathrm{o}(\varepsilon),$$

and with change of variable $s = \frac{1}{2}\varepsilon^2 r$

$$\int_0^{2t/\varepsilon^2} \frac{\mathrm{e}^{-1/r}}{\sqrt{\pi r^3}} \left(1 - \mathrm{e}^{-\frac{\alpha^2\varepsilon^2 r}{4}}\right) \mathrm{d}r = \varepsilon \int_0^t \frac{\mathrm{e}^{-\frac{\varepsilon^2}{2s}}}{\sqrt{2\pi s^3}} \left(1 - \mathrm{e}^{-\frac{\alpha^2 s}{2}}\right) \mathrm{d}s$$
$$= \varepsilon \int_0^t \frac{1 - \mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} \mathrm{d}s + \mathrm{o}(\varepsilon)$$

by monotone convergence. Combining these with $\mathrm{e}^{-\alpha\varepsilon} = 1 - \alpha\varepsilon + \mathrm{o}(\varepsilon)$, we deduce from (3.22) that

$$\mathbb{P}(t \geqslant T^\alpha(\varepsilon)) = 1 - \varepsilon\left(\alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - \mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds\right) + \mathrm{o}(\varepsilon) \tag{3.23}$$

which yields (3.18).

Obviously, the inequality $1 - \mathrm{e}^{-s} \leqslant s$ for $s \geqslant 0$ implies

$$\int_0^t \frac{1 - \mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} \mathrm{d}s \leqslant \int_0^t \frac{\alpha^2}{2\sqrt{2\pi s}} \mathrm{d}s = \frac{\alpha^2\sqrt{t}}{\sqrt{2\pi}}. \tag{3.24}$$

Moreover, we we will show that

$$\int_0^t \frac{1 - \mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} \mathrm{d}s = |\alpha| - \frac{\sqrt{2}}{\sqrt{\pi t}} \int_0^{|\alpha|\sqrt{t}} \mathrm{d}r \int_r^\infty \mathrm{e}^{-\frac{s^2}{2}} \mathrm{d}s \leqslant |\alpha|, \tag{3.25}$$

which then together with (3.24) gives the second inequality in (3.13).

Thus, to finish the proof, it remains to establish (3.25). Indeed, noting that

$$\sqrt{\frac{2}{\pi t}} \int_0^{|\alpha|\sqrt{t}} \left(\int_0^\infty \mathrm{e}^{-s^2/2} \mathrm{d}s\right) \mathrm{d}r = |\alpha|,$$

we see that (3.25) is equivalent to

$$\sqrt{\frac{2}{\pi t}} \int_0^{|\alpha|\sqrt{t}} \left(\int_0^r \mathrm{e}^{-s^2/2} \mathrm{d}s\right) \mathrm{d}r = \int_0^t \frac{1 - \mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} \mathrm{d}s. \tag{3.26}$$

Two changes of variables give

$$\sqrt{\frac{2}{\pi t}} \int_0^{|\alpha|\sqrt{t}} \left(\int_0^r \mathrm{e}^{-s^2/2} \mathrm{d}s\right) \mathrm{d}r = \frac{\alpha^2}{2\sqrt{2\pi t}} \int_0^t \frac{1}{\sqrt{u}} \left(\int_0^u \frac{1}{\sqrt{v}} \mathrm{e}^{-\frac{\alpha^2 v}{2}} \mathrm{d}v\right) \mathrm{d}u.$$

A first integration by parts choosing $2(\sqrt{u} - \sqrt{t})$ as primitive of $1/\sqrt{u}$ yields

$$\frac{\alpha^2}{2\sqrt{2\pi t}} \int_0^t \frac{1}{\sqrt{u}} \left(\int_0^u \frac{1}{\sqrt{v}} \mathrm{e}^{-\frac{\alpha^2 v}{2}} \mathrm{d}v\right) \mathrm{d}u = \frac{\alpha^2}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{u}} \mathrm{e}^{-\frac{\alpha^2 u}{2}} \mathrm{d}u - \frac{\alpha^2}{\sqrt{2\pi t}} \int_0^t \mathrm{e}^{-\frac{\alpha^2 u}{2}} \mathrm{d}u$$
$$= \frac{\alpha^2}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{u}} \mathrm{e}^{-\frac{\alpha^2 u}{2}} \mathrm{d}u - \frac{\sqrt{2}}{\sqrt{\pi t}} \left(1 - \mathrm{e}^{-\frac{\alpha^2 t}{2}}\right).$$

A second integration by parts shows that the right hand side is equal to $\displaystyle\int_0^t \frac{1 - \mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} \mathrm{d}s$. Therefore, (3.26) holds and hence (3.25) as well. □

Remark 3.1. We proved that

$$
\begin{aligned}
\|\nabla\psi(t,\cdot)\|_{\partial D,\infty} &\leqslant \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1-\mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}}\,\mathrm{d}s \\
&= 2\max(0,\alpha) + \sqrt{\frac{2}{\pi t}} \int_{|\alpha|\sqrt{t}}^{\infty} \left(\int_r^\infty \mathrm{e}^{-s^2/2}\,\mathrm{d}s \right) \mathrm{d}r
\end{aligned}
\tag{3.27}
$$

where the last equality follows from (3.25) and the observation that $\int_0^\infty \int_r^\infty \mathrm{e}^{-s^2/2}\,\mathrm{d}s\mathrm{d}r = 1$. This implies that for all $a > 1$

$$
\begin{aligned}
\|\nabla\psi(t,\cdot)\|_{\partial D,\infty} &\leqslant 2\max(0,\alpha) + \sqrt{\frac{2}{\pi t}}\,\mathrm{e}^{-\frac{t\alpha^2}{2a}} \int_{|\alpha|\sqrt{t}}^{\infty} e^{-\frac{r^2(a-1)}{4a}} \left(\int_r^\infty \mathrm{e}^{-\frac{s^2(a-1)}{4a}}\,\mathrm{d}s \right) \mathrm{d}r \\
&\leqslant 2\max(0,\alpha) + \sqrt{\frac{2}{\pi t}}\,\mathrm{e}^{-\frac{t\alpha^2}{2a}} \int_0^\infty \mathrm{e}^{-\frac{r^2(a-1)}{4a}} \left(\int_0^\infty \mathrm{e}^{-\frac{s^2(a-1)}{4a}}\,\mathrm{d}s \right) \mathrm{d}r \\
&\leqslant 2\max(0,\alpha) + \frac{a}{a-1}\sqrt{\frac{2\pi}{t}}\,\mathrm{e}^{-\frac{t\alpha^2}{2a}}.
\end{aligned}
$$

In the case when $4 < t\alpha^2$, taking $a = \dfrac{\alpha^2 t}{\alpha^2 t - 4}$ yields

$$
\|\nabla\psi(t,\cdot)\|_{\partial D,\infty} \leqslant 2\max(0,\alpha) + \frac{\mathrm{e}^2\alpha^2 t}{4}\sqrt{\frac{2\pi}{t}}\,\mathrm{e}^{-\frac{\alpha^2 t}{2}}
$$

and in particular

$$
\|\nabla\psi(1/\lambda,\cdot)\|_{\partial D,\infty} \leqslant 2\max(0,\alpha) + \frac{\mathrm{e}^2\alpha^2}{4\lambda}\sqrt{2\pi\lambda}\,\mathrm{e}^{-\frac{\alpha^2}{2\lambda}}.
$$

Combined with the Lemmas 3.2 and 3.3, this gives estimate (1.3) of Remark 1.1.

Finally, to estimate the constant α in (3.12), we shall use the Laplacian comparison theorem to bound $\Delta\rho_{\partial D}$ from above. See [4, 9] for the corresponding lower bound estimate.

Theorem 3.5. *Let $\theta, k \in \mathbb{R}$ be such that $\mathbb{I}_{\partial D} \geqslant -\theta$ and $\mathrm{Ric}_D \geqslant -(d-1)k$. For $t \geqslant 0$ let*

$$
h(t) = \begin{cases} \cos(\sqrt{-k}\,t) + \frac{\theta}{\sqrt{-k}}\sin(\sqrt{-k}\,t), & \text{if } k < 0, \\ 1 + \theta t, & \text{if } k = 0, \\ \cosh(\sqrt{k}\,t) + \frac{\theta}{\sqrt{k}}\sinh(\sqrt{k}\,t), & \text{if } k > 0. \end{cases}
$$

Let $h^{-1}(0)$ be the first zero of h (where $h^{-1}(0) := \infty$ if $h(t) > 0$ for all $t \geqslant 0$). Then for any $x \in D \setminus \mathrm{cut}(D)$ such that $\rho_{\partial D}(x) < h^{-1}(0)$, there holds

$$
\Delta\rho_{\partial D}(x) \leqslant (d-1)\frac{h'}{h}(\rho_{\partial D}(x)). \tag{3.28}
$$

In particular, if $\theta, k \geqslant 0$ we have

$$
\Delta\rho_{\partial D}(x) \leqslant (d-1)\big(\theta + \sqrt{k}\big), \quad x \in D \setminus \mathrm{cut}(D). \tag{3.29}
$$

Proof. The proof of (3.28) is adapted from [10, Theorem 1.2.2] where the corresponding Hessian upper bound is presented. For fixed $x \in D \setminus \mathrm{cut}(D)$, let p be the orthogonal projection of x on ∂M, which is the unique point on ∂D such that $\mathrm{dist}(x,p) = \rho := \rho_{\partial D}(x)$. Then

$$\gamma(s) := \exp_p(sN), \quad s \in [0,\rho],$$

is the minimal geodesic in D linking p and x. Let $X_0(0) = N(p)$, and $\{X_i(0)\}_{1 \leqslant i \leqslant d-1}$ be an orthonormal basis of $T_p\partial D$. For $0 \leqslant i \leqslant d-1$, let

$$X_i(s) = /\!/_{p \to \gamma(s)} X_i(0), \quad s \in [0,\rho],$$

be the parallel transport of $X_i(0)$ along the geodesic γ. Moreover, for any $1 \leqslant i \leqslant d-1$, let $\{J_i(s)\}_{s \in [0,\rho]}$ be the Jacobi field along γ such that $J(\rho) = X_i(\rho)$ and

$$\langle \dot{J}_i(0), U\rangle = -\,\mathbb{I}_{\partial D}(J_i(0), U), \quad U \in T_p\partial D.$$

By the second variational formula (see e.g. page 321 in [2]), we have

$$\begin{aligned}
\Delta\rho_{\partial D}(x) &= \sum_{i=1}^{d-1} \mathrm{Hess}_{\rho_{\partial D}}(X_i, X_i)(\rho_{\partial D}(x)) \\
&= -\sum_{i=1}^{d-1} \mathbb{I}_{\partial D}(J_i(0), J_i(0)) \\
&\quad + \sum_{i=0}^{d-1} \int_0^{\rho_{\partial D}(x)} \Big(|\dot{J}_i(s)|^2 - \langle \mathcal{R}(X_0(s), J_i(s))X_0(s), J_i(s)\rangle \Big)\, \mathrm{d}s
\end{aligned} \tag{3.30}$$

where $\mathcal{R}$ is the curvature tensor. Define

$$\tilde{J}_i(s) = \frac{h(s)}{h(\rho_{\partial D}(x))} X_i(s), \quad s \in [0, \rho_{\partial D}(x)], \quad 0 \leqslant i \leqslant d-1.$$

Then $\tilde{J}_i(\rho_{\partial D}(x)) = J_i(\rho_{\partial D}(x)) = X_i$, and by $\mathbb{I}_{\partial D} \geqslant -\theta$,

$$\langle \dot{\tilde{J}}_i(0), \tilde{J}_i(0)\rangle = \frac{\theta}{h(\rho_{\partial D}(x))^2} \geqslant -\,\mathbb{I}_{\partial D}(\tilde{J}_i(0), \tilde{J}_i(0)), \quad 1 \leqslant i \leqslant d-1.$$

Hence, by the index lemma (see the first displayed formula on page 322 in [2]), and using the lower bound conditions on $\mathbb{I}_{\partial D}$ and Ric_D, we deduce from (3.30) that

$$\begin{aligned}
\Delta\rho_{\partial D}(x) &\leqslant -\sum_{i=1}^{d-1} \mathbb{I}_{\partial D}(\tilde{J}_i(0), \tilde{J}_i(0)) \\
&\quad + \sum_{i=0}^{d-1} \int_0^{\rho_{\partial D}(x)} \Big(|\dot{\tilde{J}}_i(s)|^2 - \langle \mathcal{R}(X_0(s), \tilde{J}_i(s))X_0(s), \tilde{J}_i(s)\rangle \Big)\, \mathrm{d}s \\
&\leqslant \frac{1}{h(\rho_{\partial D}(x))^2} \bigg(-(\mathrm{tr}\, \mathbb{I}_{\partial D})(p) + \int_0^{\rho_{\partial D}(x)} \big\{ h'(s)^2 - h(s)^2 \,\mathrm{Ric}(X_0(s), X_0(s)) \big\}\, \mathrm{d}s \bigg) \\
&\leqslant \frac{d-1}{h(\rho_{\partial D}(x))^2} \bigg(-\theta + \int_0^{\rho_{\partial D}(x)} \big\{ h'(s)^2 + k h(s)^2 \big\}\, \mathrm{d}s \bigg) \\
&= (d-1)\frac{h'}{h}(\rho_{\partial D}(x)),
\end{aligned}$$

where in the last step we used the facts that $(hh')(0) = \theta$ and $h'' = kh$, and the latter implies

$$(h')^2 + kh^2 = (hh')' - hh'' + kh^2 = (hh')'.$$

Thus (3.28) holds. When $\theta, k \geqslant 0$, we have $h^{-1}(0) = \infty$ and

$$\begin{aligned} \frac{h'(t)}{h(t)} &= \frac{\sqrt{k}\sinh(\sqrt{k}t) + \theta\cosh(\sqrt{k}t)}{\cosh(\sqrt{k}t) + \frac{\theta}{\sqrt{k}}\sinh(\sqrt{k}t)} \\ &\leqslant \frac{\sqrt{k}\cosh(\sqrt{k}t) + \theta\cosh(\sqrt{k}t)}{\cosh(\sqrt{k}t)} = \sqrt{k} + \theta. \end{aligned}$$

Then (3.29) follows from (3.28). □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. By Theorem 3.5 with $k = \frac{K_0}{d-1}$, condition (3.12) holds for α as given in (3.1). Applying Lemmas 3.2, 3.3 and Proposition 3.4 with $t = s = \frac{1}{\lambda}$, we obtain

$$\begin{aligned} \|\nabla\phi\|_\infty &\leqslant e^{1/2}\Big(\sqrt{\lambda + K_V}\,\|\phi\|_\infty + \|\nabla\phi\|_{\partial D,\infty}\Big) \\ &\leqslant \|\phi\|_\infty\bigg\{\sqrt{e(\lambda + K_V)} + e\Big(\alpha + \frac{\sqrt{2\lambda}}{\sqrt{\pi}} + \alpha \wedge \frac{\alpha^2}{\sqrt{2\pi\lambda}}\Big)\bigg\}. \end{aligned}$$

The result follows by substitution. □

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