Gradient Estimates on Dirichlet and Neumann Eigenfunctions

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Abstract

By methods of stochastic analysis on Riemannian manifolds, we derive explicit constants $c_1(D)$ and $c_2(D)$ for a d-dimensional compact Riemannian manifold D with boundary such that

$$c_1(D)\sqrt{\lambda}\|\phi\|_\infty \leqslant \|\nabla\phi\|_\infty \leqslant c_2(D)\sqrt{\lambda}\|\phi\|_\infty$$

holds for any Dirichlet eigenfunction ϕ of $-\Delta$ with eigenvalue λ. In particular, when D is convex with non-negative Ricci curvature, the estimate holds for

$$c_1(D) = \frac{1}{d\mathrm{e}}, \quad c_2(D) = \sqrt{\mathrm{e}}\left(\frac{\sqrt{2}}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{4\sqrt{2}}\right).$$

Corresponding two-sided gradient estimates for Neumann eigenfunctions are derived in the second part of the paper.

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1 Introduction

Let D be a d-dimensional compact Riemannian manifold with boundary ∂D. We write $(\phi,\lambda) \in \mathrm{Eig}(\Delta)$ if ϕ is a Dirichlet eigenfunction of $-\Delta$ in D with eigenvalue $\lambda > 0$. According to [7], there exist two constants $c_1(D), c_2(D) > 0$ such that

$$c_1(D)\sqrt{\lambda}\|\phi\|_\infty \leqslant \|\nabla\phi\|_\infty \leqslant c_2(D)\sqrt{\lambda}\|\phi\|_\infty, \quad (\phi,\lambda) \in \mathrm{Eig}(\Delta). \tag{1.1}$$

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An analogous statement for Neumann eigenfunctions has been derived in [5].

Concerning Dirichlet eigenfunctions, an explicit upper constant $c_2(D)$ can be derived from the uniform gradient estimate of the Dirichlet semigroup in an earlier paper [10] of the third named author. More precisely, let $K, \theta \geqslant 0$ be two constants such that

$$\mathrm{Ric}_D \geqslant -K, \quad H_{\partial D} \geqslant -\theta, \tag{1.2}$$

where Ric_D is the Ricci curvature on D and $H_{\partial D}$ the mean curvature of ∂D. Let

$$\alpha_0 = \frac{1}{2} \max \{\theta, \sqrt{(d-1)K}\}. \tag{1.3}$$

Consider the semigroup $P_t = \mathrm{e}^{t\Delta}$ for the Dirichlet Laplacian Δ. According to [10, Theorem 1.1] where $c = 2\alpha_0$, for any nontrivial $f \in \mathscr{B}_b(D)$ and $t > 0$, the following estimate holds:

$$\frac{\|\nabla P_t f\|_\infty}{\|f\|_\infty} \leqslant 9.5\alpha_0 + \frac{2\sqrt{\alpha_0}(1+4^{2/3})^{1/4}\,(1+5\times 2^{-1/3})}{(t\pi)^{1/4}} + \frac{\sqrt{1+2^{1/3}}\,(1+4^{2/3})}{2\sqrt{t\pi}} =: c(t).$$

Consequently, for any $(\phi, \lambda) \in \mathrm{Eig}(\Delta)$,

$$\|\nabla\phi\|_\infty \leqslant \|\phi\|_\infty \inf_{t>0} c(t)\mathrm{e}^{\lambda t}.$$

In particular, when $\mathrm{Ric}_D \geqslant 0$, $H_{\partial D} \geqslant 0$,

$$\|\nabla\phi\|_\infty \leqslant \frac{\sqrt{\mathrm{e}\,(1+2^{1/3})}\,(1+4^{2/3})}{\sqrt{2\pi}} \sqrt{\lambda}\,\|\phi\|_\infty, \quad (\phi,\lambda) \in \mathrm{Eig}(\Delta). \tag{1.4}$$

In this paper, by using stochastic analysis of the Brownian motion on D, we develop two-sided gradient estimates; the upper bound given below in (1.8) improves the one in (1.4). Our result will also be valid for $\alpha_0 \in \mathbb{R}$ satisfying

$$\frac{1}{2}\Delta\rho_{\partial D} \leqslant \alpha_0 \quad \text{outside the focal set,} \tag{1.5}$$

where $\rho_{\partial D}$ is the distance to boundary. The case $\alpha_0 < 0$ appears naturally in many situations, for instance when D is a closed ball with convex distance to the origin. Note that by [10, Lemma 2.3], if under (1.2) we define α_0 by (1.3) then condition (1.5) holds as a consequence.

For $x \geqslant 0$, in what follows in the limiting case $x = 0$ we use the convention

$$\Big(\frac{1}{1+x}\Big)^{1/x} := \lim_{r\downarrow 0}\Big(\frac{1}{1+r}\Big)^{1/r} = \frac{1}{\mathrm{e}}.$$

Theorem 1.1. *Let $K, \theta \geqslant 0$ be two constants such that* (1.2) *holds and let α_0 be given by* (1.3) *or more generally satisfy* (1.5)*. Then, for any nontrivial $(\phi, \lambda) \in \mathrm{Eig}(\Delta)$,*

$$\frac{\lambda}{\sqrt{d\mathrm{e}(\lambda+K)}} \leqslant \frac{\lambda}{\sqrt{d(\lambda+K)}}\Big(\frac{\lambda}{\lambda+K}\Big)^{\lambda/(2K)} \leqslant \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \tag{1.6}$$

and

$$\frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leqslant \begin{cases} \sqrt{\mathrm{e}(\lambda+K)} & \textit{if} \quad \sqrt{\lambda+K} \geqslant 2A \\ \sqrt{\mathrm{e}}\,\big(A + \frac{\lambda+K}{4A}\big) & \textit{if} \quad \sqrt{\lambda+K} \leqslant 2A, \end{cases} \tag{1.7}$$

where

$$A := 2\alpha_0^+ + \frac{\sqrt{2(\lambda+K)}}{\sqrt{\pi}}\exp\left(-\frac{\alpha_0^2}{2(\lambda+K)}\right).$$

In particular, when $\mathrm{Ric}_D \geqslant 0$*,* $H_{\partial D}\geqslant 0$*,*

$$\frac{\sqrt{\lambda}}{\sqrt{d\mathrm{e}}} \leqslant \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leqslant \sqrt{\lambda}\left(\frac{\sqrt{2\mathrm{e}}}{\sqrt{\pi}} + \frac{\sqrt{\pi\mathrm{e}}}{4\sqrt{2}}\right), \quad (\phi,\lambda)\in \mathrm{Eig}(\Delta). \tag{1.8}$$

Proof. This result follows from Theorem 2.1 and Theorem 2.2 below in the special case $V=0$. In this case, $\mathrm{Ric}_D^V = \mathrm{Ric}_D \geqslant -K$ is equivalent to (2.1) with $n=d$. More sophisticated upper bounds are given below in Theorem 2.2. □

By (1.8), if D is convex with non-negative Ricci curvature then (1.1) holds with

$$c_1(D) = \frac{1}{\sqrt{d\mathrm{e}}}, \quad c_2(D) = \frac{\sqrt{2\mathrm{e}}}{\sqrt{\pi}} + \frac{\sqrt{\pi\mathrm{e}}}{4\sqrt{2}}.$$

To give explicit values of $c_1(D)$ and $c_2(D)$ for positive K or θ, let $\lambda_1>0$ be the first Dirichlet eigenvalue of $-\Delta$ on D. Then Theorem 1.1 implies that (1.1) holds for

$$\begin{aligned}
c_1(D) &= \frac{\sqrt{\lambda_1}}{\sqrt{d\mathrm{e}(\lambda_1+K)}},\\
c_2(D) &= \frac{\sqrt{\mathrm{e}(\lambda_1+K)}}{\sqrt{\lambda_1}}\mathbb{1}_{\{B>2A\}} + \frac{\sqrt{\mathrm{e}}}{\sqrt{\lambda_1}}\left(2\alpha_0^+ + \sqrt{\frac{2(\lambda_1+K)}{\pi}} + \frac{\lambda_1+K}{4\big(2\alpha_0^+ + \sqrt{2(\lambda_1+K)/\pi}\big)}\right)\mathbb{1}_{\{B\leqslant 2A\}}
\end{aligned}$$

with

$$B = \sqrt{\lambda_1+K} \quad \text{and} \quad A = 2\alpha_0^+ + \sqrt{\frac{2(\lambda_1+K)}{\pi}}.$$

This is due to the fact that the expression for $c_1(D)$ is an increasing function of λ and the expression for $c_2(D)$ a decreasing function of λ. Since there exist explicit lower bound estimates on λ_1 (see [9] and references within), this gives explicit lower bounds of $c_1(D)$ and explicit upper bounds of $c_2(D)$.

The lower bound for $\|\nabla\phi\|_\infty$ will be derived by using Itô's formula for $|\nabla\phi|^2(X_t)$ where X_t is a Brownian motion (with drift) on D, see Subsection 2.1 for details. To derive the upper bound estimate, we will construct some martingales to reduce $\|\nabla\phi\|_\infty$ to $\|\nabla\phi\|_{\partial D,\infty} := \sup_{\partial D}|\nabla\phi|$, and to estimate the latter in terms of $\|\phi\|_\infty$, see Subsection 2.2 for details.

Next, we consider the Neumann problem. Let $\mathrm{Eig}_N(\Delta)$ be the set of non-trivial eigenpairs (ϕ,λ) for the Neumann eigenproblem, i.e. ϕ is non-constant, $\Delta\phi=-\lambda\phi$ with $N\phi|_{\partial D}=0$ for the unit inward normal vector field N of ∂D. Let $\mathbb{I}_{\partial D}$ be the second fundamental form of ∂D,

$$\mathbb{I}_{\partial D}(X,Y) = -\langle \nabla_X N, Y\rangle, \quad X,Y\in T_x\partial M,\ x\in\partial M.$$

With a concrete choice of the function f, the next theorem implies (1.1) for $(\phi,\lambda)\in\mathrm{Eig}_N(\Delta)$ together with explicit constants $c_1(D), c_2(D)$.

Theorem 1.2. *Let* $K,\delta\in\mathbb{R}$ *be constants such that*

$$\mathrm{Ric}_D \geqslant -K, \quad \mathbb{I}_{\partial D}\geqslant -\delta. \tag{1.9}$$

For $f\in C_b^2(\bar D)$ *with* $\inf_D f = 1$ *and* $N\log f|_{\partial D}\geqslant \delta$*, let*

$$c_\varepsilon(f) = \sup_D\left\{\frac{4\varepsilon|\nabla\log f|^2}{1-\varepsilon} + K - 2\Delta\log f\right\}, \quad \varepsilon\in(0,1),$$

$$K(f) = \sup_D \big\{2|\nabla \log f|^2 + K - \Delta \log f\big\}.$$

Then for any non-trivial $(\phi, \lambda) \in \mathrm{Eig}_N(\Delta)$*, we have* $\lambda + c_\varepsilon(f) > 0$ *and*

$$\begin{aligned}\sup_{\varepsilon\in(0,1)} \frac{\varepsilon\lambda^2}{d\mathrm{e}(\lambda + c_\varepsilon(f))\|f\|_\infty^2} &\leqslant \sup_{\varepsilon\in(0,1)} \frac{\varepsilon\lambda^2}{d(\lambda + c_\varepsilon(f))\|f\|_\infty^2}\Big(\frac{\lambda}{\lambda + c_\varepsilon(f)}\Big)^{\lambda/c_\varepsilon(f)} \\ &\leqslant \frac{\|\nabla\phi\|_\infty^2}{\|\phi\|_\infty^2} \leqslant \frac{2\|f\|_\infty^2(\lambda + K(f))}{\pi}\Big(1 + \frac{K(f)}{\lambda}\Big)^{\lambda/K(f)} \\ &\leqslant 2\mathrm{e}\,\|f\|_\infty^2 \frac{\lambda + K(f)}{\pi}.\end{aligned}$$

Proof. Under the conditions (1.2), Theorem 3.3 below applies with $L = \Delta$, $K_V = K$ and $n = d$. The desired estimates are immediate consequences. □

When ∂D is convex, i.e. $\mathbb{I}_{\partial D} \geqslant 0$, we may take $f \equiv 1$ in Theorem 1.2 to derive the following result. According to Theorem 3.2 below, this result also holds for $\partial D = \varnothing$ where $\mathrm{Eig}(\Delta)$ is the set of eigenpairs for the closed eigenproblem.

Corollary 1.3. *Let* ∂D *be convex or empty. If* $\mathrm{Ric}_D^V \geqslant -K$ *for some constant* K*, then for any non-trivial* $(\phi, \lambda) \in \mathrm{Eig}_N(\Delta)$*, we have* $\lambda + K > 0$ *and*

$$\frac{\lambda^2}{d\mathrm{e}(\lambda + K^+)} \leqslant \frac{\lambda^2}{d(\lambda + K)}\Big(\frac{\lambda}{\lambda + K}\Big)^{\lambda/K} \leqslant \frac{\|\nabla\phi\|_\infty^2}{\|\phi\|_\infty^2} \leqslant \frac{2(\lambda + K)}{\pi}\Big(1 + \frac{K}{\lambda}\Big)^{\lambda/K} \leqslant \frac{2\mathrm{e}(\lambda + K^+)}{\pi}.$$

2 Proof of Theorem 1.1

In general, we will consider Dirichlet eigenfunctions for the symmetric operator $L := \Delta + \nabla V$ on D where $V \in C^2(D)$. We denote by $\mathrm{Eig}(L)$ the set of pairs (ϕ, λ) where ϕ is a Dirichlet eigenfunction of $-L$ on D with eigenvalue λ.

In the following two subsections, we consider the lower bound and upper bound estimates respectively.

2.1 Lower bound estimate

In this subsection we will estimate $\|\nabla\phi\|_\infty$ from below using the following Bakry-Émery curvature-dimension condition:

$$\frac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f\rangle \geqslant -K|\nabla f|^2 + \frac{(Lf)^2}{n}, \quad f \in C^\infty(D), \tag{2.1}$$

where $K \in \mathbb{R}$, $n \geqslant d$ are two constants. When $V = 0$, this condition with $n = d$ is equivalent to $\mathrm{Ric}_D \geqslant -K$.

Theorem 2.1 (Lower bound estimate)**.** *Assume that* (2.1) *holds. Then*

$$\|\nabla\phi\|_\infty^2 \geqslant \|\phi\|_\infty^2 \sup_{t>0} \frac{\lambda^2(\mathrm{e}^{Kt} - 1)}{nK\mathrm{e}^{(\lambda+K)^+t}}, \quad (\phi, \lambda) \in \mathrm{Eig}(L). \tag{2.2}$$

Consequently, for $K^+ := \max\{0, K\}$ *there holds*

$$\|\nabla\phi\|_\infty^2 \geqslant \frac{\lambda^2\|\phi\|_\infty^2}{n(\lambda + K^+)}\Big(\frac{\lambda}{\lambda + K^+}\Big)^{\lambda/K^+} \geqslant \frac{\lambda^2\|\phi\|_\infty^2}{n\mathrm{e}(\lambda + K^+)}, \quad (\phi, \lambda) \in \mathrm{Eig}(L). \tag{2.3}$$

Proof. Let X_t be the diffusion process generated by $\frac{1}{2}L$ in D, and let

$$\tau_D := \inf\{t \geqslant 0 : X_t \in \partial D\}.$$

By Itô's formula, we have

$$\mathrm{d}|\nabla\phi|^2(X_t) = \frac{1}{2}L|\nabla\phi|^2(X_t)\,\mathrm{d}t + \mathrm{d}M_t, \quad t \leqslant \tau_D, \tag{2.4}$$

for some martingale M_t. By the curvature dimension condition (2.1) and $L\phi = -\lambda\phi$, we obtain

$$\frac{1}{2}L|\nabla\phi|^2 = \frac{1}{2}L|\nabla\phi|^2 - \langle\nabla L\phi, \nabla\phi\rangle - \lambda|\nabla\phi|^2 \geqslant -(K+\lambda)|\nabla\phi|^2 + \frac{\lambda^2}{n}\phi^2. \tag{2.5}$$

Therefore, (2.4) gives

$$\mathrm{d}|\nabla\phi|^2(X_t) \geqslant \Big(\frac{\lambda^2}{n}\phi^2 - (K+\lambda)|\nabla\phi|^2\Big)(X_t)\,\mathrm{d}t + \mathrm{d}M_t, \quad t \leqslant \tau_D.$$

Hence, for any $t > 0$,

$$\begin{aligned}
\mathrm{e}^{(K+\lambda)^+t}\,\|\nabla\phi\|_\infty^2 &\geqslant \mathbb{E}\Big[|\nabla\phi|^2(X_{t\wedge\tau_D})\mathrm{e}^{(K+\lambda)(t\wedge\tau_D)}\Big] \\
&\geqslant \frac{\lambda^2}{n}\mathbb{E}\left[\int_0^{t\wedge\tau_D} \mathrm{e}^{(K+\lambda)s}\phi(X_s)^2\,\mathrm{d}s\right] \\
&= \frac{\lambda^2}{n}\mathbb{E}\left[\int_0^t 1_{\{s<\tau_D\}}\mathrm{e}^{(K+\lambda)s}\phi(X_s)^2\,\mathrm{d}s\right].
\end{aligned}$$

Since $\phi|_{\partial D} = 0$ and $L\phi = -\lambda\phi$, by Jensen's inequality we have

$$\mathbb{E}\left[1_{\{s<\tau_D\}}\phi(X_s)^2\right] \geqslant \big(\mathbb{E}[\phi(X_{s\wedge\tau_D})]\big)^2 = \mathrm{e}^{-\lambda s}\phi(x)^2,$$

where $x = X_0 \in D$ is the starting point of X_t. Then, by taking x such that $\phi(x)^2 = \|\phi\|_\infty^2$, we arrive at

$$\begin{aligned}
\mathrm{e}^{(K+\lambda)^+t}\,\|\nabla\phi\|_\infty^2 &\geqslant \frac{\lambda^2}{n}\int_0^t \mathrm{e}^{(K+\lambda)s}\mathrm{e}^{-\lambda s}\phi(x)^2\,\mathrm{d}s \\
&= \frac{\lambda^2\|\phi\|_\infty^2}{n}\int_0^t \mathrm{e}^{Ks}\,\mathrm{d}s = \frac{\lambda^2(\mathrm{e}^{Kt}-1)}{nK}\|\phi\|_\infty^2.
\end{aligned}$$

This completes the proof of (2.2).

Since (2.1) holds for K^+ replacing K, we may and do assume that $K \geqslant 0$. By taking the optimal choice $t = \frac{1}{K}\log(1+\frac{K}{\lambda})$ (by convention $t = \lambda^{-1}$ if $K = 0$) in (2.2), we obtain

$$\|\nabla\phi\|_\infty^2 \geqslant \frac{\lambda^2\|\phi\|_\infty^2}{\lambda+K}\Big(\frac{\lambda}{\lambda+K}\Big)^{\lambda/K} \geqslant \frac{\lambda^2\|\phi\|_\infty^2}{n\mathrm{e}(\lambda+K)}.$$

Hence (2.3) holds. ∎

2.2 Upper bound estimate

Let $\mathrm{Ric}_D^V = \mathrm{Ric}_D - \mathrm{Hess}_V$. For $K_0, \theta \geqslant 0$ such that $\mathrm{Ric}_D \geqslant -K_0$ and $H_{\partial D} \geqslant -\theta$, let

$$\alpha = \frac{1}{2}\left(\max\left\{\theta, \sqrt{(d-1)K_0}\right\} + \|\nabla V\|_\infty\right) \tag{2.6}$$

We note that $\frac{1}{2}L\rho_{\partial D} \leqslant \alpha$ by [10, Lemma 2.3].

Theorem 2.2 (Upper bound estimate)**.** *Let $K_V, \theta \geqslant 0$ be constants such that*

$$\mathrm{Ric}_D^V \geqslant -K_V, \quad H_{\partial D} \geqslant -\theta.$$

Let $\alpha \in \mathbb{R}$ be such that

$$\frac{1}{2}L\rho_{\partial D} \leqslant \alpha. \tag{2.7}$$

1. *Assume $\alpha \geqslant 0$. Then, for any nontrivial $(\phi, \lambda) \in \mathrm{Eig}(L)$,*

$$\frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leqslant \begin{cases} \sqrt{\mathrm{e}(\lambda + K_V)} & \textit{if} \quad \sqrt{\lambda + K_V} \geqslant 2A \\ \sqrt{\mathrm{e}}\left(A + \frac{\lambda + K_V}{4A}\right) & \textit{if} \quad \sqrt{\lambda + K_V} \leqslant 2A, \end{cases} \tag{2.8}$$

where

$$A := \alpha + \frac{\sqrt{2(\lambda + K_V)}}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{2(\lambda + K_V)}\right) + |\alpha| \wedge \frac{\sqrt{2}\alpha^2}{\sqrt{\pi(\lambda + K_V)}}. \tag{2.9}$$

In particular, (2.8) *holds with A replaced by*

$$A' := 2\alpha + \frac{\sqrt{2(\lambda + K_V)}}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{2(\lambda + K_V)}\right). \tag{2.10}$$

We also have

$$\frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leqslant \sqrt{\mathrm{e}}\left(\frac{2\alpha + \sqrt{2(\lambda + K_V)}}{\sqrt{\pi}} + \frac{\lambda + K_V}{4}\frac{\sqrt{\pi}}{2\alpha + \sqrt{2(\lambda + K_V)}}\right). \tag{2.11}$$

2. *Assume $\alpha \leqslant 0$. Then, for any nontrivial $(\phi, \lambda) \in \mathrm{Eig}(L)$,*

$$\frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leqslant \begin{cases} \sqrt{\mathrm{e}(\lambda + K_V)} & \textit{if} \quad \sqrt{\lambda + K_V} \geqslant 2A^* \\ \sqrt{\mathrm{e}}\left(A^* + \frac{\lambda + K_V}{4A^*}\right) & \textit{if} \quad \sqrt{\lambda + K_V} \leqslant 2A^*, \end{cases} \tag{2.12}$$

where

$$A^* := \frac{\sqrt{2(\lambda + K_V)}}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{2(\lambda + K_V)}\right). \tag{2.13}$$

In particular,

$$\frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leqslant \sqrt{\lambda + K_V}\left(\sqrt{\frac{2}{\pi}} + \frac{1}{4}\sqrt{\frac{\pi}{2}}\right)\sqrt{\mathrm{e}}. \tag{2.14}$$

In addition, the following estimate holds:

$$\frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leqslant \begin{cases} \sqrt{\mathrm{e}(\lambda+K_V)} & \text{if} \quad \sqrt{\lambda+K_V} \geqslant 2\sqrt{\mathrm{e}}\hat{A} \\ \mathrm{e}\hat{A} + \frac{\lambda+K_V}{4\hat{A}} & \text{if} \quad \sqrt{\lambda+K_V} < 2\sqrt{\mathrm{e}}\hat{A}, \end{cases} \tag{2.15}$$

where

$$\hat{A} := \alpha + \frac{\sqrt{2\lambda}}{\sqrt{\pi}}\mathrm{e}^{-\frac{\alpha^2}{2\lambda}} + |\alpha| \wedge \frac{\sqrt{2}\alpha^2}{\sqrt{\pi\lambda}}. \tag{2.16}$$

The strategy to prove Theorem 2.2 will be to first estimate $\|\nabla\phi\|_\infty$ in terms of $\|\phi\|_\infty$ and $\|\nabla\phi\|_{\partial D,\infty}$ (see estimate (2.24) below) where $\|f\|_{\partial D,\infty} := \|1_{\partial D} f\|_\infty$ for a function f on D. The this end we construct appropriate martingales in terms of ϕ and $\nabla\phi$.

We start by recalling the necessary facts about the diffusion process generated by $\frac{1}{2}L$, see for instance [1, 3]. For any $x \in D$, the diffusion X_t solves the SDE

$$\mathrm{d}X_t = \frac{1}{2}\nabla V(X_t)\,\mathrm{d}t + u_t \circ \mathrm{d}B_t, \quad X_0 = x,\ t \leqslant \tau_D, \tag{2.17}$$

where B_t is a d-dimensional Brownian motion, u_t is the horizontal lift of X_t onto the orthonormal frame bundle $\mathrm{O}(D)$ with initial value $u_0 \in \mathrm{O}_x(D)$, and

$$\tau_D := \inf\{t \geqslant 0 : X_t \in \partial D\}$$

is the hitting time of X_t to the boundary ∂D. Setting $Z := \nabla V$, we have

$$\mathrm{d}u_t = \frac{1}{2}Z^*(u_t)\,\mathrm{d}t + \sum_{i=1}^d H_i(u_t) \circ \mathrm{d}B_t^i \tag{2.18}$$

where $Z^*(u) := h_u(Z_{\pi(u)})$ and $H_i(u) := h_u(ue_i)$ are defined by means of the horizontal lift $h_u\colon T_{\pi(u)}D \to T_u\,\mathrm{O}(D)$ at $u \in \mathrm{O}(D)$. Note that formally $h_{u_t}(u_t \circ \mathrm{d}B_t) = \sum_i h_{u_t}(u_t e_i) \circ \mathrm{d}B_t^i = \sum_i H_i(u_t) \circ \mathrm{d}B_t^i$.

For $f \in C^\infty(D)$, let $a := \mathrm{d}f \in \Gamma(T^*D)$. Setting $m_t := u_t^{-1}a(X_t)$, we see by Itô's formula that

$$\mathrm{d}m_t \overset{\mathrm{m}}{=} \frac{1}{2}u_t^{-1}(\Box a + \nabla_Z a)(X_t)\,\mathrm{d}t \tag{2.19}$$

where $\Box a = \operatorname{tr}\nabla^2 a$ denotes the so-called connection (or rough) Laplacian on 1-forms and $\overset{\mathrm{m}}{=}$ equality modulo the differential of a local martingale.

Denote by $Q_t\colon T_xD \to T_{X_t}D$ the solution, along the paths of X_t, to the covariant ordinary differential equation

$$DQ_t = -\frac{1}{2}(\mathrm{Ric}_D^V)^\sharp Q_t\,\mathrm{d}t, \quad Q_0 = \mathrm{id}_{T_xD},\ t \leqslant \tau_D,$$

where $D := u_t\mathrm{d}u_t^{-1}$ and where by definition

$$(\mathrm{Ric}_D^V)^\sharp v = \mathrm{Ric}_D^V(\cdot, v)^\sharp, \quad v \in T_xD.$$

Thus, condition $\mathrm{Ric}_D^V \geqslant -K_V$ implies

$$|Q_t v| \leqslant \mathrm{e}^{\frac{K_V}{2}t}\,|v|, \quad t \leqslant \tau_D. \tag{2.20}$$

Finally, note that for any smooth function f on D, we have by the Weitzenböck formula:

$$\begin{aligned} \mathrm{d}\big(\Delta + Z\big) f &= \mathrm{d}\big(- \mathrm{d}^*\mathrm{d} f + (\mathrm{d} f) Z\big) \\ &= \Delta^{(1)} \mathrm{d} f + \nabla_Z \mathrm{d} f + \langle \nabla_{\bullet} Z, \nabla f\rangle \\ &= (\square + \nabla_Z)(\mathrm{d} f) - \mathrm{Ric}_D^V(\cdot, \nabla f) \\ &= \big(\square - \mathrm{Ric}_D^V + \nabla_Z\big)(\mathrm{d} f) \end{aligned} \tag{2.21}$$

where $\Delta^{(1)}$ denotes the Hodge-deRham Laplacian on 1-forms.

Now let $(\phi, \lambda) \in \mathrm{Eig}(L)$, i.e. $L\phi = -\lambda\phi$, where $L = \Delta + Z$. For $v \in T_x D$, consider the process

$$n_t(v) := (\mathrm{d}\phi)(Q_t v).$$

Then

$$n_t(v) = \langle \nabla\phi(X_t), Q_t v\rangle = \langle u_t^{-1}(\nabla\phi)(X_t), u_t^{-1} Q_t v\rangle.$$

Using (2.19), we see by Itô's formula and formula (2.21) that

$$\mathrm{d} n_t(v) \overset{\mathrm{m}}{=} \frac{1}{2}(\square \mathrm{d}\phi + \nabla_Z \mathrm{d}\phi)(X_t)\, Q_t v\, \mathrm{d}t + \mathrm{d}\phi(X_t)(D Q_t v)\, \mathrm{d}t = -\frac{\lambda}{2} n_t(v)\, \mathrm{d}t.$$

It follows that

$$\mathrm{e}^{\lambda t/2}\, n_t(v) = \mathrm{e}^{\lambda t/2} \,\langle \nabla\phi(X_t), Q_t v\rangle, \quad t \leqslant \tau_D, \tag{2.22}$$

is a martingale.

Lemma 2.1. Let $(\phi, \lambda) \in \mathrm{Eig}(L)$. We keep the notation from above. Then, for any function $h \in C^1([0,\infty); \mathbb{R})$, the process

$$N_t(v) := h_t\, \mathrm{e}^{\lambda t/2} \,\langle \nabla\phi(X_t), Q_t v\rangle - \mathrm{e}^{\lambda t/2}\, \phi(X_t) \int_0^t \langle \dot h_s Q_s v, u_s \mathrm{d}B_s\rangle, \quad t \leqslant \tau_D, \tag{2.23}$$

is a martingale. In particular, for fixed $t > 0$ and $h \in C^1([0,t]; [0,1])$ monotone such that $h_0 = 1$ and $h_t = 0$, we have

$$\begin{aligned} \|\nabla\phi\|_\infty \leqslant\ & \|\nabla\phi\|_{\partial D, \infty}\, \mathbb{P}\{t > \tau_D\}\, \mathrm{e}^{(\lambda + K_V)^+ t/2} \\ & + \|\phi\|_\infty\, \mathrm{e}^{\lambda t/2}\, \mathbb{P}\{t \leqslant \tau_D\}^{1/2} \left(\int_0^t |\dot h_s|^2 \mathrm{e}^{K_V s}\, \mathrm{d}s\right)^{1/2}. \end{aligned} \tag{2.24}$$

Proof. Indeed, from (2.22) we deduce that

$$h_t\, \mathrm{e}^{\lambda t/2} \,\langle \nabla\phi(X_t), Q_t v\rangle - \int_0^t \dot h_s\, \mathrm{e}^{\lambda s/2} \,\langle \nabla\phi(X_s), Q_s v\rangle\, \mathrm{d}s, \quad t \leqslant \tau_D,$$

is a martingale as well. By the formula

$$\mathrm{e}^{\lambda t/2}\, \phi(X_t) = \phi(X_0) + \int_0^t \mathrm{e}^{\lambda s/2} \,\langle \nabla\phi(X_s), u_s \mathrm{d}B_s\rangle$$

we see then that $N_t(v)$ is a martingale. To check inequality (2.24), we deduce from the martingale property of $\{N_{s\wedge\tau_D}(v)\}_{s\in[0,t]}$ that

$$\begin{aligned} \|\nabla\phi\|_\infty \leqslant\ & \|\nabla\phi\|_{\partial D, \infty}\, \mathbb{E}\left[1_{\{t > \tau_D\}}\, \mathrm{e}^{\lambda \tau_D/2}\, |h_{\tau_D}|\, |Q_{\tau_D}|\right] \\ & + \|\phi\|_\infty\, \mathrm{e}^{\lambda t/2}\, \mathbb{E}\left[1_{\{t \leqslant \tau_D\}} \sup_{|v|\leqslant 1} \left(\int_0^t \langle \dot h_s\, Q_s v, u_s \mathrm{d}B_s\rangle\right)^2\right]^{1/2}. \end{aligned}$$

The claim follows by using (2.20). $\square$

To estimate the boundary norm $\|\nabla\phi\|_{\partial D,\infty}$, we shall compare $\phi(x)$ and

$$\psi(t,x) := \mathbb{P}(\tau_D^x > t), \quad t > 0,$$

for small $\rho_{\partial D}(x) := \mathrm{dist}(x, \partial D)$. Let P_t^D be the Dirichlet semigroup generated by $\frac{1}{2}L$. Then

$$\psi(t,x) = P_t^D 1_D(x),$$

so that

$$\partial_t \psi(t,x) = \frac{1}{2} L\psi(t,\cdot)(x), \quad t > 0. \tag{2.25}$$

Lemma 2.3. *For any* $(\phi, \lambda) \in \mathrm{Eig}(L)$,

$$\|\nabla\phi\|_{\partial D,\infty} \leqslant \|\phi\|_\infty \inf_{t>0} \mathrm{e}^{\lambda t/2} \|\nabla\psi(t,\cdot)\|_{\partial D,\infty}. \tag{2.26}$$

Proof. To prove (2.26), we fix $x \in \partial D$. For small $\varepsilon > 0$, let $x^\varepsilon = \exp_x(\varepsilon N)$, where N is the inward unit normal vector field of ∂D. Since $\phi|_{\partial D} = 0$ and $\psi(t,\cdot)|_{\partial D} = 0$, we have

$$|\nabla\phi(x)| = |N\phi(x)| = \lim_{\varepsilon\to 0} \frac{|\phi(x^\varepsilon)|}{\varepsilon}, \quad |\nabla\psi(t,\cdot)(x)| = \lim_{\varepsilon\to 0} \frac{|\psi(t,x^\varepsilon)|}{\varepsilon}. \tag{2.27}$$

Let X_t^ε be the L-diffusion starting at x^ε and τ_D^ε its first hitting time of ∂D. Note that

$$N_t := \phi(X^\varepsilon_{t\wedge\tau_D^\varepsilon})\, \mathrm{e}^{\lambda(t\wedge\tau_D^\varepsilon)/2}, \quad t \geqslant 0,$$

is a martingale. Thus, for each fixed $t > 0$, we can estimate as follows:

$$\begin{aligned}
|\nabla\phi(x)| &= \lim_{\varepsilon\to 0} \frac{|\phi(x^\varepsilon)|}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\left|\mathbb{E}[\phi(X_t^\varepsilon)\, \mathbb{1}_{\{t<\tau_D^\varepsilon\}}]\, \mathrm{e}^{\lambda(t\wedge\tau_D^\varepsilon)/2}\right|}{\varepsilon} \\
&\leqslant \|\phi\|_\infty\, \mathrm{e}^{\lambda t/2} \lim_{\varepsilon\to 0} \frac{\mathbb{E}[\mathbb{1}_{\{t<\tau_D^\varepsilon\}}]}{\varepsilon} \\
&\leqslant \|\phi\|_\infty\, \mathrm{e}^{\lambda t/2} \lim_{\varepsilon\to 0} \frac{\psi(t,x^\varepsilon)}{\varepsilon} \\
&= \|\phi\|_\infty\, \mathrm{e}^{\lambda t/2}\, |\nabla\psi(t,\cdot)|(x).
\end{aligned}$$

Taking the infimum over t gives the claim. $\square$

We now work out an explicit estimate for $\|\nabla\psi(t,\cdot)\|_{\partial D,\infty}$. Let $\mathrm{cut}(D)$ be the cut-locus of ∂D, which is a zero-volume closed subset of D such that $\rho_{\partial D} := \mathrm{dist}(\cdot, \partial D)$ is smooth in $D \setminus \mathrm{cut}(D)$.

Proposition 2.4. *Let* $\alpha \in \mathbb{R}$ *such that*

$$\frac{1}{2} L\rho_{\partial_D} \leqslant \alpha. \tag{2.28}$$

Then

$$\|\nabla\psi(t,\cdot)\|_{\partial D,\infty} \leqslant \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - \mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}}\, \mathrm{d}s$$

$$\leqslant \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\alpha^2 t}{2}} + \min\left\{|\alpha|,\ \frac{\alpha^2\sqrt{2t}}{\sqrt{\pi}}\right\}, \tag{2.29}$$

and

$$\|\nabla\psi(t,\cdot)\|_{\partial D,\infty} \leqslant \frac{\sqrt{2}}{\sqrt{\pi t}} + \alpha + \frac{\sqrt{t}}{\sqrt{2\pi}}\alpha^2 \tag{2.30}$$

Notice that by [10, Lemma 2.3] the condition $\frac{1}{2}L\rho_{\partial D} \leqslant \alpha$ holds for α defined by (2.6).

Proof. Let $x \in D$ and let X_t solve SDE (2.17). As shown in [6], $(\rho_{\partial D}(X_t))_{t\leqslant \tau_D}$ is a semimartingale satisfying

$$\rho_{\partial D}(X_t) = \rho_{\partial D}(x) + b_t + \frac{1}{2}\int_0^t L\rho_{\partial D}(X_s)\,\mathrm{d}s - l_t, \quad t \leqslant \tau_D, \tag{2.31}$$

where b_t is a real-valued Brownian motion starting at 0, and l_t a non-decreasing process which increases only when $X_t^x \in \mathrm{cut}(D)$. Setting $\varepsilon = \rho_{\partial D}(x)$, we deduce from (2.31) together with $\dfrac{1}{2}L\rho_{\partial D} \leqslant \alpha$, that

$$\rho_{\partial D}(X_t(x)) \leqslant Y_t^\alpha(\varepsilon) := \varepsilon + b_t + \alpha t, \quad t \leqslant \tau_D. \tag{2.32}$$

Consequently, letting $T^\alpha(\varepsilon)$ be the first hitting time of 0 by $Y_t^\alpha(\varepsilon)$, we obtain

$$\psi(t,x) \leqslant \mathbb{P}(t < T^\alpha(\varepsilon)). \tag{2.33}$$

On the other hand, since $\psi(t,\cdot)$ vanishes on the boundary and is positive in D, we have for all $y \in \partial D$

$$|\nabla\psi(t,y)| = \lim_{x\in D,\, x\to y} \frac{\psi(t,x)}{\rho_{\partial D}(x)}. \tag{2.34}$$

Hence, by (2.33), to prove the first inequality in (2.29) it is enough to establish that

$$\limsup_{\varepsilon\downarrow 0} \frac{\mathbb{P}(t < T^\alpha(\varepsilon))}{\varepsilon} \leqslant \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - \mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}}\,\mathrm{d}s. \tag{2.35}$$

It is well known that the (sub-probability) density $f_{\alpha,\varepsilon}$ of $T^\alpha(\varepsilon)$ is

$$f_{\alpha,\varepsilon}(s) = \frac{\varepsilon\exp\left(-(\varepsilon+\alpha s)^2/(2s)\right)}{\sqrt{2\pi s^3}}, \tag{2.36}$$

which can be obtained by the reflection principle for $\alpha = 0$ and the Girsanov transform for $\alpha \neq 0$. Thus

$$\begin{aligned}
\mathbb{P}(t \geqslant T^\alpha(\varepsilon)) &= \varepsilon\int_0^t \frac{\exp\left(-(\varepsilon+\alpha s)^2/(2s)\right)}{\sqrt{2\pi s^3}}\,\mathrm{d}s \\
&= \varepsilon\exp(-\alpha\varepsilon)\int_0^t \frac{\mathrm{e}^{-\alpha^2 s/2}}{\sqrt{2\pi s^3}}\exp\left(-\frac{\varepsilon^2}{2s}\right)\mathrm{d}s \\
&= \exp(-\alpha\varepsilon)\int_0^{2t/\varepsilon^2} \frac{\mathrm{e}^{-1/r}}{\sqrt{\pi r^3}}\exp\left(-\frac{\alpha^2\varepsilon^2 r}{4}\right)\mathrm{d}r,
\end{aligned} \tag{2.37}$$

where we have made the change of variable $r = 2s/\varepsilon^2$. With the change of variable $v = 1/r$ we easily check that

$$\int_0^\infty r^{-3/2} \mathrm{e}^{-1/r} \, \mathrm{d}r = \Gamma(1/2) = \sqrt{\pi}, \tag{2.38}$$

and this allows to write

$$\mathbb{P}(t \geqslant T^\alpha(\varepsilon)) = \exp(-\alpha\varepsilon) \left(1 - \int_{2t/\varepsilon^2}^\infty \frac{\mathrm{e}^{-1/r}}{\sqrt{\pi r^3}} \, \mathrm{d}r - \int_0^{2t/\varepsilon^2} \frac{\mathrm{e}^{-1/r}}{\sqrt{\pi r^3}} \left(1 - \mathrm{e}^{-\alpha^2 \varepsilon^2 r/4} \right) \mathrm{d}r \right). \tag{2.39}$$

As $\varepsilon \to 0$,

$$\int_{2t/\varepsilon^2}^\infty \frac{\mathrm{e}^{-1/r}}{\sqrt{r^3}} \, \mathrm{d}r = \int_{2t/\varepsilon^2}^\infty \frac{1}{\sqrt{r^3}} \, \mathrm{d}r + \mathrm{o}(\varepsilon) = \frac{\varepsilon\sqrt{2}}{\sqrt{t}} + \mathrm{o}(\varepsilon),$$

and with change of variable $s = \frac{1}{2}\varepsilon^2 r$

$$\begin{aligned}
\int_0^{2t/\varepsilon^2} \frac{\mathrm{e}^{-1/r}}{\sqrt{\pi r^3}} \left(1 - \mathrm{e}^{-\frac{\alpha^2 \varepsilon^2 r}{4}} \right) \mathrm{d}r &= \varepsilon \int_0^t \frac{\mathrm{e}^{-\frac{\varepsilon^2}{2s}}}{\sqrt{2\pi s^3}} \left(1 - \mathrm{e}^{-\frac{\alpha^2 s}{2}} \right) \mathrm{d}s \\
&= \varepsilon \int_0^t \frac{1 - \mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} \, \mathrm{d}s + \mathrm{o}(\varepsilon)
\end{aligned}$$

by monotone convergence. Combining these with $\mathrm{e}^{-\alpha\varepsilon} = 1 - \alpha\varepsilon + \mathrm{o}(\varepsilon)$, we deduce from (2.39) that

$$\mathbb{P}(t \geqslant T^\alpha(\varepsilon)) = 1 - \varepsilon \left(\alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - \mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} \, ds \right) + \mathrm{o}(\varepsilon) \tag{2.40}$$

which yields (2.35).

Next, an integration by parts yields

$$\int_0^t \frac{1 - \mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} \, ds = \frac{\alpha^2}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{u}} \mathrm{e}^{-\frac{\alpha^2 u}{2}} \, \mathrm{d}u - \frac{\sqrt{2}}{\sqrt{\pi t}} \left(1 - \mathrm{e}^{-\frac{\alpha^2 t}{2}} \right). \tag{2.41}$$

With the change of variable $s = |\alpha| \sqrt{\dfrac{u}{t}}$ in the first term in the right we obtain

$$\frac{\alpha^2}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{u}} \mathrm{e}^{-\frac{\alpha^2 u}{2}} \, \mathrm{d}u = |\alpha| \sqrt{\frac{2t}{\pi}} \int_0^{|\alpha|} \mathrm{e}^{-\frac{s^2 t}{2}} \, \mathrm{d}s. \tag{2.42}$$

We arrive at

$$f(\alpha) := \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - \mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} \, \mathrm{d}s = \frac{\sqrt{2}}{\sqrt{\pi t}} \mathrm{e}^{-\frac{\alpha^2 t}{2}} + \alpha + |\alpha| \sqrt{\frac{2t}{\pi}} \int_0^{|\alpha|} \mathrm{e}^{-\frac{s^2 t}{2}} \, \mathrm{d}s. \tag{2.43}$$

Bounding $\sqrt{\dfrac{2t}{\pi}} \displaystyle\int_0^{|\alpha|} \mathrm{e}^{-\frac{s^2 t}{2}} \, \mathrm{d}s$ by $\sqrt{\dfrac{2t}{\pi}} \displaystyle\int_0^\infty \mathrm{e}^{-\frac{s^2 t}{2}} \, \mathrm{d}s = 1$, respectively bounding $\mathrm{e}^{-\frac{s^2 t}{2}}$ by 1 in the integral yield (2.29).

The function

$$f(\alpha) = \frac{\sqrt{2}}{\sqrt{\pi t}} \mathrm{e}^{-\frac{\alpha^2 t}{2}} + \alpha + |\alpha| \sqrt{\frac{2t}{\pi}} \int_0^{|\alpha|} \mathrm{e}^{-\frac{s^2 t}{2}} \, \mathrm{d}s$$

is smooth and an easy computation shows that

$$f(0) = \frac{\sqrt{2}}{\sqrt{\pi t}}, \quad f'(0) = 1, \quad f''(\alpha) = \frac{\sqrt{2t}}{\sqrt{\pi}} e^{-\frac{\alpha^2 t}{2}} \tag{2.44}$$

Using the fact that $f(\alpha) - \alpha$ is even, we also get

$$f(\alpha) = \frac{\sqrt{2}}{\sqrt{\pi t}} + \alpha + \int_0^{|\alpha|} \frac{\sqrt{2t}}{\sqrt{\pi}} e^{-\frac{s^2 t}{2}} s\, ds \leqslant \frac{\sqrt{2}}{\sqrt{\pi t}} + \alpha + \frac{\sqrt{t}}{\sqrt{2\pi}} \alpha^2. \tag{2.45}$$

which yields (2.30). □

Remark 2.2. One could use estimate (2.24) (optimizing the right-hand side with respect to t) together with Lemma 2.3 (again optimizing with respect to t) to estimate $\|\nabla\phi\|_\infty$ in terms of $\|\phi\|_\infty$. We prefer to combine the two steps.

Lemma 2.5. *Assume* $\mathrm{Ric}_D^V \geqslant -K_V$ *for some constant* $K_V \in \mathbb{R}$. *Let* α *be determined by* (2.28).

(a) *If* $\alpha \geqslant 0$, *then for any* $(\phi, \lambda) \in \mathrm{Eig}(L)$,

$$\|\nabla\phi\|_\infty \leqslant \inf_{t>0} \max_{\varepsilon\in[0,1]} e^{\frac{(\lambda+K_V^+)t}{2}} \left\{ \varepsilon \left(\alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\alpha^2 t}{2}} + \min\left(|\alpha|, \ \frac{\alpha^2 \sqrt{2t}}{\sqrt{\pi}} \right) \right) + \sqrt{\frac{1-\varepsilon}{t}} \right\} \|\phi\|_\infty,$$

as well as

$$\|\nabla\phi\|_\infty \leqslant \inf_{t>0} \max_{\varepsilon\in[0,1]} e^{(\lambda+K_V^+)t/2} \left\{ \varepsilon \left(\alpha + \sqrt{\frac{2}{\pi t}} + \frac{\sqrt{t}}{\sqrt{2\pi}} \alpha^2 \right) + \sqrt{\frac{1-\varepsilon}{t}} \right\} \|\phi\|_\infty$$

and

$$\|\nabla\phi\|_\infty \leqslant \inf_{t>0} \max_{\varepsilon\in[0,1]} e^{(\lambda+K_V^+)t/2} \left\{ \varepsilon \left(2\alpha + \sqrt{\frac{2}{\pi t}} \right) + \sqrt{\frac{1-\varepsilon}{t}} \right\} \|\phi\|_\infty.$$

(b) *If* $\alpha \leqslant 0$, *then*

$$\|\nabla\phi\|_\infty \leqslant \inf_{t>0} \max_{\varepsilon\in[0,1]} e^{(\lambda+K_V^+)t/2} \left\{ \varepsilon \sqrt{\frac{2}{\pi t}}\, e^{-\frac{\alpha^2 t}{2}} + \sqrt{\frac{1-\varepsilon}{t}} \right\} \|\phi\|_\infty.$$

In particular,

$$\|\nabla\phi\|_\infty \leqslant \inf_{t>0} \max_{\varepsilon\in[0,1]} e^{(\lambda+K_V^+)t/2} \left\{ \varepsilon \sqrt{\frac{2}{\pi t}} + \sqrt{\frac{1-\varepsilon}{t}} \right\} \|\phi\|_\infty.$$

Proof. For fixed $t > 0$ in (2.23), we take $h \in C^1([0,t];[0,1])$ such that $h_0 = 1$ and $h_t = 0$. Then, by the martingale property of $\{N_{s\wedge\tau_D}(v)\}_{s\in[0,t]}$, we obtain

$$\begin{aligned} |\nabla_v\phi|(x) &= |N_0(v)| = |\mathbb{E} N_{t\wedge\tau_D}(v)| \\ &= \left| \mathbb{E}\left[1_{\{t>\tau_D\}} e^{\lambda\tau_D/2} h_{\tau_D} \langle \nabla\phi(X_{\tau_D}), Q_{\tau_D} v\rangle - 1_{\{t\leqslant\tau_D\}} e^{\lambda t/2} \phi(X_t) \int_0^t \langle \dot{h}_s Q_s v, u_s \mathrm{d}B_s\rangle \right] \right|. \end{aligned} \tag{2.46}$$

Note that using (2.20) along with Lemma 2.3 we may estimate

$$\left| \mathbb{E}\left[1_{\{t>\tau_D\}} e^{\lambda\tau_D/2} h_{\tau_D} \langle \nabla\phi(X_{\tau_D}), Q_{\tau_D} v\rangle \right] \right|$$

$$
\begin{aligned}
&\leqslant \mathbb{E}\left[1_{\{t>\tau_D\}}\, \mathrm{e}^{\lambda\tau_D/2}\, |h_{\tau_D}|\, \|\nabla\phi\|_{\partial D,\infty}\, \mathrm{e}^{K_V\tau_D/2}|v|\right]\\
&\leqslant \mathbb{E}\left[1_{\{t>\tau_D\}}\, \mathrm{e}^{\lambda\tau_D/2}\, |h_{\tau_D}|\, \|\phi\|_\infty\, \|\nabla\psi(t-\tau_D,\cdot)\|_{\partial D,\infty}\, \mathrm{e}^{\lambda(t-\tau_D)/2}\, \mathrm{e}^{K_V\tau_D/2}\, |v|\right]\\
&= \mathbb{E}\left[1_{\{t>\tau_D\}}\, |h_{\tau_D}|\, \|\phi\|_\infty\, \|\nabla\psi(t-\tau_D,\cdot)\|_{\partial D,\infty}\, \mathrm{e}^{\lambda t/2}\, \mathrm{e}^{K_V\tau_D/2}\, |v|\right]\\
&\leqslant \mathrm{e}^{(\lambda+K_V^+)t/2}\, \|\phi\|_\infty\, \mathbb{E}\left[1_{\{t>\tau_D\}}\, |h_{\tau_D}|\, \|\nabla\psi(t-\tau_D,\cdot)\|_{\partial D,\infty}\, |v|\right],
\end{aligned}
$$

as well as

$$
\mathbb{E}\left[1_{\{t\leqslant\tau_D\}}\, \mathrm{e}^{\lambda t/2}\phi(X_t)\int_0^t \langle \dot h_s\, Q_s v, u_s \mathrm{d}B_s\rangle\right] \leqslant \mathrm{e}^{\lambda t/2}\, \|\phi\|_\infty\, \mathbb{P}\{t\leqslant\tau_D\}^{1/2}\left(\int_0^t |\dot h_s|^2 \mathrm{e}^{K_V s}\,\mathrm{d}s\right)^{1/2}.
$$

Taking

$$
h_s = \frac{t-s}{t}, \quad s\in[0,t],
$$

we obtain thus from (2.46)

$$
\begin{aligned}
|\nabla\phi(x)| \leqslant{}& \frac{\mathrm{e}^{(\lambda+K_V^+)t/2}}{t}\, \|\phi\|_\infty\, \mathbb{E}\left[1_{\{t>\tau_D\}}\,(t-\tau_D)\, \|\nabla\psi(t-\tau_D,\cdot)\|_{\partial D,\infty}\right]\\
&+ \mathrm{e}^{\lambda t/2}\, \|\phi\|_\infty\, \mathbb{P}\{t\leqslant\tau_D\}^{1/2}\, \frac{1}{t}\left(\frac{\mathrm{e}^{K_V^+ t}-1}{K_V^+}\right)^{1/2}.
\end{aligned}
$$

Note that

$$
\frac{\mathrm{e}^{K_V^+ t}-1}{K_V^+} \leqslant t\mathrm{e}^{K_V^+ t}.
$$

(i) By (2.29), assuming that $\alpha\geqslant 0$, we have on $\{t>\tau_D\}$:

$$
\begin{aligned}
\frac{t-\tau_D}{t}\, \|\nabla\psi(t-\tau_D,\cdot)\|_{\partial D,\infty} &\leqslant \alpha\frac{t-\tau_D}{t} + \frac{\sqrt{2}}{\sqrt{\pi}}\frac{\sqrt{t-\tau_D}}{t} + \frac{t-\tau_D}{t}\int_0^{t-\tau_D} \frac{1-\mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}}\,\mathrm{d}s\\
&\leqslant \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1-\mathrm{e}^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}}\,\mathrm{d}s\\
&\leqslant \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\alpha^2 t}{2}} + \min\left\{\alpha,\ \frac{\alpha^2\sqrt{2t}}{\sqrt{\pi}}\right\}.
\end{aligned}
$$

Thus, letting $\varepsilon = \mathbb{P}(t>\tau_D)$, we obtain

$$
|\nabla\phi(x)| \leqslant \mathrm{e}^{(\lambda+K_V^+)t/2}\, \|\phi\|_\infty \left[\varepsilon\left(\alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\alpha^2 t}{2}} + \min\left\{\alpha,\ \frac{\alpha^2\sqrt{2t}}{\sqrt{\pi}}\right\}\right) + \sqrt{\frac{1-\varepsilon}{t}}\right].
$$

(ii) Still under the assumption $\alpha\geqslant 0$, this time using estimate (2.30), we have on $\{t>\tau_D\}$:

$$
\|\nabla\psi(t-\tau_D,\cdot)\|_{\partial D,\infty} \leqslant \frac{\sqrt{2}}{\sqrt{\pi(t-\tau_D)}} + \alpha + \frac{\sqrt{t-\tau_D}}{\sqrt{2\pi}}\alpha^2,
$$

and thus letting $\varepsilon = \mathbb{P}(t>\tau_D)$, we get

$$
|\nabla\phi(x)| \leqslant \frac{\mathrm{e}^{(\lambda+K_V^+)t/2}}{t}\, \|\phi\|_\infty\, \mathbb{E}\left[1_{\{t>\tau_D\}}\left(\sqrt{\frac{2}{\pi}}\sqrt{t-\tau_D} + \alpha(t-\tau_D) + \frac{(t-\tau_D)^{3/2}}{\sqrt{2\pi}}\alpha^2\right)\right]
$$

$$
+ \mathrm{e}^{\lambda t/2} \, \|\phi\|_\infty \, \mathbb{P}\{t \leqslant \tau_D\}^{1/2} \, \frac{1}{t} \left(\frac{\mathrm{e}^{K_V^+ t} - 1}{K_V^+} \right)^{1/2}
$$

$$
\leqslant \mathrm{e}^{(\lambda + K_V^+)t/2} \, \|\phi\|_\infty \, \left[\varepsilon \left(\sqrt{\frac{2}{\pi t}} + \alpha + \frac{\sqrt{t}}{\sqrt{2\pi}} \alpha^2 \right) + \sqrt{\frac{1-\varepsilon}{t}} \right].
$$

(iii) In the case $\alpha \leqslant 0$, we get from (2.29) in a similar way:

$$
|\nabla \phi(x)| \leqslant \mathrm{e}^{(\lambda + K_V^+)t/2} \, \|\phi\|_\infty \, \left\{ \varepsilon \, \frac{\sqrt{2}}{\sqrt{\pi t}} \, e^{-\frac{\alpha^2 t}{2}} + \sqrt{\frac{1-\varepsilon}{t}} \right\}.
$$

This concludes the proof of Lemma 2.5. $\square$

Proposition 2.3. We keep the assumptions of Lemma 2.5.
(a) If $\alpha \geqslant 0$, then for any $(\phi, \lambda) \in \mathrm{Eig}(L)$,

$$
\|\nabla \phi\|_\infty \leqslant \sqrt{\mathrm{e}} \max_{\varepsilon \in [0,1]} \left\{ \varepsilon \left(\alpha + \frac{\sqrt{2(\lambda + K_V^+)}}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{2(\lambda + K_V^+)} \right) + \min\left(|\alpha|, \, \frac{\sqrt{2}\alpha^2}{\sqrt{\pi(\lambda + K_V^+)}} \right) \right) \right.
$$

$$
\left. + \sqrt{1-\varepsilon} \sqrt{(\lambda + K_V^+)} \right\} \|\phi\|_\infty,
$$

as well as

$$
\|\nabla \phi\|_\infty \leqslant \sqrt{\mathrm{e}} \max_{\varepsilon \in [0,1]} \left\{ \varepsilon \left(\alpha + \frac{\sqrt{2(\lambda + K_V^+)}}{\sqrt{\pi}} + \frac{\alpha^2}{\sqrt{2\pi(\lambda + K_V^+)}} \right) + \sqrt{1-\varepsilon} \sqrt{(\lambda + K_V^+)} \right\} \|\phi\|_\infty
$$

and

$$
\|\nabla \phi\|_\infty \leqslant \sqrt{\mathrm{e}} \max_{\varepsilon \in [0,1]} \left\{ \varepsilon \left(2\alpha + \frac{\sqrt{2(\lambda + K_V^+)}}{\sqrt{\pi}} \right) + \sqrt{1-\varepsilon} \sqrt{(\lambda + K_V^+)} \right\} \|\phi\|_\infty
$$

(b) If $\alpha \leqslant 0$, then

$$
\|\nabla \phi\|_\infty \leqslant \sqrt{\mathrm{e}} \max_{\varepsilon \in [0,1]} \left\{ \varepsilon \frac{\sqrt{2(\lambda + K_V^+)}}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{2(\lambda + K_V^+)} \right) + \sqrt{1-\varepsilon} \sqrt{(\lambda + K_V^+)} \right\} \|\phi\|_\infty.
$$

Proof. Take $t = 1/(\lambda + K_V^+)$ in Lemma 2.5. $\square$

We are now ready to complete the proof of Theorem 2.2.

Proof of Theorem 2.2. The claims of Theorem 2.2 (with the exception of estimate (2.15)) follow directly from the inequalities in Proposition 2.3 together with the fact that for any $A, B \geqslant 0$,

$$
\max_{\varepsilon \in [0,1]} \left\{ \varepsilon A + \sqrt{1-\varepsilon} B \right\} = B \mathbb{1}_{\{B > 2A\}} + \left(A + \frac{B^2}{4A} \right) \mathbb{1}_{\{B \leqslant 2A\}}. \tag{2.47}
$$
$\square$

Finally, to check (2.15) we may go back to (2.24) from where we have

$$
\|\nabla \phi\|_\infty \leqslant \varepsilon \mathrm{e}^{(\lambda + K_V)^+ t/2} \, \|\nabla \phi\|_{\partial D, \infty} + \sqrt{1-\varepsilon} \, \mathrm{e}^{\lambda t/2} \, \|\phi\|_\infty \left(\int_0^t |\dot{h}_s|^2 \mathrm{e}^{K_V s} \, \mathrm{d}s \right)^{1/2}.
$$

Taking

$$h_s = \frac{\mathrm{e}^{-K_V t} - \mathrm{e}^{-K_V s}}{\mathrm{e}^{-K_V t} - 1}, \quad s \in [0, t],$$

we obtain

$$\|\nabla\phi\|_\infty \leqslant \inf_{t>0} \max_{\varepsilon\in[0,1]} \left\{ \varepsilon \mathrm{e}^{(\lambda+K_V)^+ t/2} \|\nabla\phi\|_{\partial D,\infty} + \|\phi\|_\infty \mathrm{e}^{\lambda t/2}\sqrt{1-\varepsilon} \left(\frac{K_V}{1-\mathrm{e}^{-K_V t}}\right)^{1/2} \right\}.$$

Noting that

$$\frac{K_V}{1-\mathrm{e}^{-K_V t}} \leqslant \frac{K_V^+}{1-\mathrm{e}^{-K_V^+ t}} \leqslant t^{-1}\mathrm{e}^{K_V^+ t},$$

and taking $t = (K_V^+ + \lambda)^{-1}$ we obtain

$$\|\nabla\phi\|_\infty \leqslant \sqrt{\mathrm{e}} \max_{\varepsilon\in[0,1]} \left\{ \varepsilon \|\nabla\phi\|_{\partial D,\infty} + \sqrt{(1-\varepsilon)(\lambda+K_V^+)}\, \|\phi\|_\infty \right\}.$$

Applying Lemma 2.3 and Proposition 2.4 with $t = 1/\lambda$, we arrive at

$$\|\nabla\phi\|_\infty \leqslant \|\phi\|_\infty \max_{\varepsilon\in[0,1]} \left\{ \mathrm{e}\varepsilon \left(\alpha + \frac{\sqrt{2\lambda}}{\sqrt{\pi}} \mathrm{e}^{-\frac{\alpha^2}{2\lambda}} + |\alpha| \wedge \frac{\alpha^2\sqrt{2}}{\sqrt{\pi\lambda}} \right) + \sqrt{\mathrm{e}(1-\varepsilon)(\lambda+K_V^+)} \right\}.$$

The proof is then finished as above with observation (2.47). □

3 Proof of Theorem 1.2

As in Section 2, we consider $L = \Delta + \nabla V$ and let $\mathrm{Eig}_N(L)$ be the set of corresponding non-trivial eigenpairs for the Neumann problem of L. We also allow $\partial D = \varnothing$, then we consider the eigenproblem without boundary. We first consider the convex case, then extend to the general situation. In this section, P_t denotes the (Neumann if $\partial D \neq \varnothing$) semigroup generated by $L/2$ on D. Let X_t be the corresponding (reflecting) diffusion process which solves the SDE

$$\mathrm{d}X_t = u_t \circ \mathrm{d}B_t + \frac{1}{2}\nabla V(X_t)\,\mathrm{d}t + N(X_t)\,\mathrm{d}\ell_t, \tag{3.1}$$

where B_t is a d-dimensional Euclidean Brownian motion, u_t the horizontal lift of X_t onto the orthonormal frame bundle, and ℓ_t the local time of X_t on ∂D.

We will apply the following Bismut type formula for the Neumann semigroup P_t, see [15, Theorem 3.2.1], where the multiplicative functional process Q_s was introduced in [4].

Theorem 3.1 ([15]). *Let $\mathrm{Ric}_D^V \geqslant -K_V$ and $\mathbb{I}_{\partial D} \geqslant -\delta$ for some $K_V \in C(\bar{D})$ and $\delta \in C(\partial D)$. Then there exists a $\mathbb{R}^d \otimes \mathbb{R}^d$-valued adapted continuous process Q_s with*

$$\|Q_t\| \leqslant \exp\left(\frac{1}{2}\int_0^t K_V(X_s)\mathrm{d}s + \int_0^t \delta(X_s)\mathrm{d}\ell_s\right), \quad s \geqslant 0, \tag{3.2}$$

such that for any $t > 0$ and $h \in C^1([0,t])$ with $h(0) = 0$, $h(t) = 1$, there holds

$$\nabla P_t f = \mathbb{E}\left[f(X_t) \int_0^t h'(s) Q_s \mathrm{d}B_s \right], \quad f \in \mathscr{B}_b(D). \tag{3.3}$$

3.1 The case with convex or empty boundary

In this part we assume that ∂D is either convex or empty. When ∂D is empty, D is a Riemannian manifold without boundary and $\mathrm{Eig}_N(L)$ denotes the set of eigenpairs for the eigenproblem without boundary. In this case, if $\mathrm{Ric}^V \geqslant K_V$ for some constant $K_V \in \mathbb{R}$, then $\lambda + K_V \geqslant 0$ for $(\phi, \lambda) \in \mathrm{Eig}_N(L)$, see for instance [8].

Theorem 3.2. *Assume that ∂D is either convex or empty.*

(1) *If the curvature-dimension condition* (2.1) *holds, then for any $(\phi, \lambda) \in \mathrm{Eig}_N(L)$,*

$$\|\nabla\phi\|_\infty^2 \geqslant \frac{\lambda^2\|\phi\|_\infty^2}{n(\lambda+K)}\Big(\frac{\lambda}{\lambda+K}\Big)^{\lambda/K} \geqslant \frac{\lambda^2\|\phi\|_\infty^2}{n\mathrm{e}(\lambda+K^+)}.$$

(2) *If $\mathrm{Ric}_D^V \geqslant -K_V$ for some constant $K_V \in \mathbb{R}$, then for any $(\phi, \lambda) \in \mathrm{Eig}_N(L)$,*

$$\frac{\|\nabla\phi\|_\infty^2}{\|\phi\|_\infty^2} \leqslant \frac{2(\lambda+K_V)}{\pi}\Big(1+\frac{K_V}{\lambda}\Big)^{\lambda/K_V} \leqslant \frac{2\mathrm{e}(\lambda+K_V^+)}{\pi}.$$

Proof. (a) We start establishing the lower bound estimate. By Itô's formula, for any $(\phi, \lambda) \in \mathrm{Eig}_N(L)$ we have

$$\mathrm{d}|\nabla\phi|^2(X_t) = \frac{1}{2}L|\nabla\phi|^2(X_t)\,\mathrm{d}t + 2\,\mathbb{I}_{\partial D}(\nabla\phi,\nabla\phi)(X_t)\,\mathrm{d}\ell_t + \mathrm{d}M_t, \quad t \geqslant 0, \tag{3.4}$$

where ℓ_t is the local time of X_t at ∂D, which is an increasing process. Since $\mathbb{I}_{\partial D} \geqslant 0$, and since (2.1) and $L\phi = -\lambda\phi$ imply

$$\frac{1}{2}L|\nabla\phi|^2 \geqslant -(K+\lambda)|\nabla\phi|^2 + \frac{\lambda^2}{n}\phi^2,$$

we obtain

$$\mathrm{d}|\nabla\phi|^2(X_t) \geqslant \Big(\frac{\lambda^2}{n}\phi^2 - (\lambda+K)|\nabla\phi|^2\Big)(X_t)\,\mathrm{d}t + \mathrm{d}M_t, \quad t \geqslant 0.$$

Noting that for $X_0 = x \in D$ we have

$$\mathbb{E}[\phi(X_s)^2] \geqslant (\mathbb{E}[\phi(X_s)])^2 = \mathrm{e}^{-\lambda s}\phi(x)^2,$$

we arrive at

$$\begin{aligned}\mathrm{e}^{(\lambda+K)t}\,\|\nabla\phi\|_\infty^2 \geqslant \mathrm{e}^{(\lambda+K)t}\,\mathbb{E}[|\nabla\phi|^2(X_t)] &\geqslant \frac{\lambda^2}{n}\int_0^t \mathrm{e}^{(\lambda+K)s}\,\mathbb{E}[\phi^2(X_s)]\,\mathrm{d}s\\ &\geqslant \frac{\lambda^2}{n}\int_0^t \mathrm{e}^{Ks}\phi(x)^2\,\mathrm{d}s = \frac{\lambda^2(\mathrm{e}^{Kt}-1)}{nK}\phi(x)^2.\end{aligned}$$

Multiplying by $\mathrm{e}^{-(\lambda+K)t}$, choosing $t = \frac{1}{K}\log(1+\frac{K}{\lambda})$ (noting that $\lambda + K \geqslant 0$, in case $\lambda + K = 0$ taking $t \to \infty$), and taking the supremum over $x \in D$, we finish the proof of (1).

(b) Let ∂D be convex and $\mathrm{Ric}_D^V \geqslant -K_V$ for some constant K_V. Then Theorem 3.1 holds for $\delta = 0$, so that

$$\sigma_t := \Big(\mathbb{E}\int_0^t |h'(s)|^2\|Q_s\|^2\,\mathrm{d}s\Big)^{1/2} \leqslant \Big(\int_0^t |h'(s)|^2\,\mathrm{e}^{K_V s}\,\mathrm{d}s\Big)^{1/2}.$$

Taking

$$h(s) = \frac{\int_0^s \mathrm{e}^{-K_V r}\,\mathrm{d}r}{\int_0^t \mathrm{e}^{-K_V r}\,\mathrm{d}r}$$

we obtain

$$\sigma_t \leqslant \Big(\frac{K_V}{1-\mathrm{e}^{-K_V t}}\Big)^{1/2}.$$

Therefore,

$$\begin{aligned}
\|\nabla P_t f\|_\infty &\leqslant \|f\|_\infty\,\mathbb{E}\left|\int_0^t h'(s) Q_s \mathrm{d}B_s\right| \\
&\leqslant \|f\|_\infty \frac{2}{\sqrt{2\pi}\,\sigma_t}\int_0^\infty s\exp\left(-\frac{s^2}{2\sigma_t^2}\right)\mathrm{d}s \qquad (3.5)\\
&= \|f\|_\infty \frac{\sigma_t\sqrt{2}}{\sqrt{\pi}}, \quad t>0,\ f\in \mathscr{B}_b(D).
\end{aligned}$$

Applying this to $(\phi,\lambda)\in \mathrm{Eig}_N(L)$, we obtain

$$\mathrm{e}^{-\lambda t/2}|\nabla\phi| \leqslant \|\phi\|_\infty \frac{\sigma_t\sqrt{2}}{\sqrt{\pi}} \leqslant \|\phi\|_\infty\left(\frac{2K_V}{\pi(1-\mathrm{e}^{-2K_V t})}\right)^{1/2}, \quad t>0.$$

Consequently, $\lambda + K_V \geqslant 0$. Taking $t = \frac{1}{K_V}\log(1+\frac{K_V}{\lambda})$ as above, we arrive at

$$\frac{\|\nabla\phi\|_\infty^2}{\|\phi\|_\infty^2} \leqslant \frac{2(\lambda+K_V)}{\pi}\Big(1+\frac{K_V}{\lambda}\Big)^{\lambda/K_V}. \qquad \square$$

3.2 The non-convex case

When ∂D is non-convex, a conformal change of metric may be performed to make ∂M convex under the new metric; this strategy has been used in [2, 12, 13, 14] for the study of functional inequalities on non-convex manifolds. According to [15, Theorem 1.2.5], for a strictly positive function $f \in C^\infty(\bar D)$ with $\mathbb{I}_{\partial D} + N\log f|_{\partial D} \geqslant 0$, the boundary ∂D is convex under the metric $f^{-2}\langle\cdot,\cdot\rangle$. For simplicity, we will assume that $f\geqslant 1$. Hence, we take as class of reference functions

$$\mathscr{D} := \big\{f\in C^2(\bar D)\colon\ \inf f = 1,\ \mathbb{I}_{\partial D} + N\log f \geqslant 0\big\}.$$

Assume (2.1) and $\mathrm{Ric}_D^V \geqslant -K_V$ for some constants $n\geqslant d$ and $K, K_V\in\mathbb{R}$. For any $f\in\mathscr{D}$ and $\varepsilon\in(0,1)$, define

$$c_\varepsilon(f) := \sup_D\left\{\frac{4\varepsilon|\nabla\log f|^2}{1-\varepsilon} + \varepsilon K + (1-\varepsilon)K_V - 2L\log f\right\}.$$

We let λ_1^N be the smallest non-trivial Neumann eigenvalue of $-L$. The following result implies $\lambda_1 \geqslant -c_\varepsilon(f)$.

Theorem 3.3. *Let $f\in\mathscr{D}$.*

(1) *If (2.1) and $\mathrm{Ric}_D^V \geqslant -K_V$ hold for some constants $n\geqslant d$ and $K,K_V\in\mathbb{R}$. Then for any non-trivial $(\phi,\lambda)\in\mathrm{Eig}_N(L)$, we have $\lambda + c_\varepsilon(f)\geqslant 0$ and*

$$\frac{\|f\|_\infty^2\|\nabla\phi\|_\infty^2}{\|\phi\|_\infty^2} \geqslant \sup_{\varepsilon\in(0,1)}\frac{\varepsilon\lambda^2}{n(\lambda+c_\varepsilon(f))}\Big(\frac{\lambda}{\lambda+c_\varepsilon(f)}\Big)^{\lambda/c_\varepsilon(f)} \geqslant \sup_{\varepsilon\in(0,1)}\frac{\varepsilon\lambda^2}{n\mathrm{e}(\lambda+c_\varepsilon(f)^+)}.$$

(2) *Let* $\mathrm{Ric}_D^V \geqslant -K_V$ *for some* $K_V \in C(\bar{D})$*, and*

$$K(f) = \sup_D \left\{ 2|\nabla \log f|^2 + K_V - L \log f \right\}.$$

Then for any non-trivial $(\phi, \lambda) \in \mathrm{Eig}_N(L)$*, we have* $\lambda + K(f) \geqslant 0$ *and*

$$\frac{\|\nabla \phi\|_\infty^2}{\|\phi\|_\infty^2 \|f\|_\infty^2} \leqslant \frac{2(\lambda + K(f))}{\pi} \Big(1 + \frac{K(f)}{\lambda} \Big)^{\lambda/K(f)} \leqslant \frac{2\mathrm{e}(\lambda + K(f)^+)}{\pi}.$$

Proof. Let $f \in \mathscr{D}$ and $(\phi, \lambda) \in \mathrm{Eig}_N(L)$.

(1) On ∂D we have

$$\begin{aligned} N(f^2 |\nabla \phi|^2) &= (N f^2) |\nabla \phi|^2 + f^2 N |\nabla \phi|^2 \\ &= f^2 \big((N \log f^2) |\nabla \phi|^2 + 2\, \mathbb{I}_{\partial D}(\nabla \phi, \nabla \phi) \big) \\ &= 2 f^2 \big((N \log f) |\nabla \phi|^2 + \mathbb{I}_{\partial D}(\nabla \phi, \nabla \phi) \big) \geqslant 0. \end{aligned} \tag{3.6}$$

Next, by the Bochner-Weitzenböck formula, using that $\mathrm{Ric}_D^V \geqslant -K_V$ and $L\phi = -\lambda \phi$, we observe

$$\begin{aligned} \frac{1}{2} L |\nabla \phi|^2 &= \frac{1}{2} L |\nabla \phi|^2 - \langle \nabla L \phi, \nabla \phi \rangle - \lambda |\nabla \phi|^2 \\ &\geqslant \|\mathrm{Hess}_\phi\|_{\mathrm{HS}}^2 - (K_V + \lambda) |\nabla \phi|^2. \end{aligned}$$

Combining this with (2.5), for any $\varepsilon \in (0,1)$, we obtain

$$\begin{aligned} &\frac{f^2}{2} L |\nabla \phi|^2 + \langle \nabla f^2, \nabla |\nabla \phi|^2 \rangle \\ &\quad \geqslant -f^2 (\varepsilon K + (1-\varepsilon) K_V + \lambda) |\nabla \phi|^2 + \frac{\varepsilon \lambda^2}{n} f^2 \phi^2 \\ &\qquad + (1-\varepsilon) f^2 \|\mathrm{Hess}_\phi\|_{\mathrm{HS}}^2 - 2 \|\mathrm{Hess}_\phi\|_{\mathrm{HS}} \times |\nabla f^2| \times |\nabla \phi| \\ &\quad \geqslant - \left\{ \frac{|\nabla \log f^2|^2}{1-\varepsilon} + \varepsilon K + (1-\varepsilon) K_V + \lambda \right\} f^2 |\nabla \phi|^2 + \frac{\varepsilon \lambda^2}{n} f^2 \phi^2. \end{aligned}$$

Combining this with (3.6) and applying Itô's formula, we obtain

$$\begin{aligned} \mathrm{d}(f^2 |\nabla \phi|^2)(X_t) &\overset{\mathrm{m}}{=} \frac{1}{2} L (f^2 |\nabla \phi|^2)(X_t)\, \mathrm{d}t + N (f^2 |\nabla \phi|^2)(X_t)\, \mathrm{d}\ell_t \\ &\geqslant -\frac{1}{2} \Big(f^2 L |\nabla \phi|^2 + 2 \langle \nabla f^2, \nabla |\nabla \phi|^2 \rangle + |\nabla \phi|^2 L f^2 \Big)(X_t)\, \mathrm{d}t \\ &\geqslant \left\{ \frac{\varepsilon \lambda^2}{n} f^2 \phi^2 - \left(\frac{|\nabla \log f^2|^2}{1-\varepsilon} + \varepsilon K + (1-\varepsilon) K_V + \lambda - f^{-2} L f^2 \right) f^2 |\nabla \phi|^2 \right\} (X_t)\, \mathrm{d}t \\ &\geqslant \left(\frac{\varepsilon \lambda^2}{n} \phi^2 - \big(\lambda + c_\varepsilon(f) \big) f^2 |\nabla \phi|^2 \right) (X_t)\, \mathrm{d}t. \end{aligned}$$

Hence, for $X_0 = x \in D$,

$$\begin{aligned} \|f\|_\infty^2 \, \|\nabla \phi\|_\infty^2 \, \mathrm{e}^{(\lambda + c_\varepsilon(f))t} &\geqslant \mathbb{E} \left[\mathrm{e}^{c_\varepsilon(f) t} (f^2 |\nabla \phi|^2)(X_t) \right] \\ &\geqslant \frac{\varepsilon \lambda^2}{n} \int_0^t \mathrm{e}^{(\lambda + c_\varepsilon(f))s} \, \mathbb{E}[\phi(X_s)^2] \, \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{\varepsilon\lambda^2}{n}\int_0^t \mathrm{e}^{c_\varepsilon(f)s}\phi(x)^2\,\mathrm{d}s\\
&= \frac{\varepsilon\lambda^2(\mathrm{e}^{c_\varepsilon(f)t}-1)}{nc_\varepsilon(f)}\,\phi(x)^2.
\end{aligned}
$$

This implies $\lambda + c_\varepsilon(f) \geqslant 0$ and

$$
\begin{aligned}
\frac{\|f\|_\infty^2\|\nabla\phi\|_\infty^2}{\|\phi\|_\infty^2} &\geqslant \sup_{t>0}\frac{\varepsilon\lambda^2\left(\mathrm{e}^{-\lambda t}-\mathrm{e}^{-(\lambda+c_\varepsilon(f))t}\right)}{nc_\varepsilon(f)}\\
&= \frac{\varepsilon\lambda^2}{n(\lambda+c_\varepsilon(f))}\Big(\frac{\lambda}{\lambda+c_\varepsilon(f)}\Big)^{\lambda/c_\varepsilon(f)} \geqslant \frac{\varepsilon\lambda^2}{n\mathrm{e}(\lambda+c_\varepsilon(f)^+)}.
\end{aligned}
$$

(2) The claim could be derived from [2, inequality (2.12)]. For the sake of completeness we include a sketch of the proof. For any $p>1$, let

$$
K_p(f) = \sup_D\left\{K_V + p|\nabla\log f|^2 - L\log f\right\}.
$$

Note that $p|\nabla\log f|^2 - L\log f = p^{-1}f^pLf^{-p}$. Since $f\in\mathscr{D}$ implies $\mathbb{I}_{\partial D}\geqslant -N\log f$, we have

$$
\begin{aligned}
\|Q_t\|^2 &\leqslant \exp\left(\int_0^t K_V(X_s)\,\mathrm{d}s + 2\int_0^t N\log f(X_s)\,\mathrm{d}\ell_s\right)\\
&\leqslant \exp\big(K_p(f)t\big)\exp\left(-\frac{1}{p}\int_0^t(f^pLf^{-p})(X_s)\,\mathrm{d}s + 2\int_0^t N\log f(X_s)\,\mathrm{d}\ell_s\right).
\end{aligned}
$$

As

$$
\begin{aligned}
\mathrm{d}f^{-p}(X_t) &\overset{\mathrm{m}}{=} \frac{1}{2}Lf^{-p}(X_t)\,\mathrm{d}t + Nf^{-p}(X_t)\,\mathrm{d}\ell_t\\
&= -f^{-p}(X_t)\left(-\frac{1}{2}f^pLf^{-p}(X_t)\,\mathrm{d}t + pN\log f(X_t)\,\mathrm{d}\ell_t\right),
\end{aligned}
$$

we obtain that

$$
M_t := f^{-p}(X_t)\exp\left(-\frac{1}{2}\int_0^t f^p(X_s)Lf^{-p}(X_s)\,\mathrm{d}s + p\int_0^t N\log f(X_s)\,\mathrm{d}\ell_s\right)
$$

is a (local) martingale. Proceeding as in the proof of [15, Corollary 3.2.8] or [2, Theorem 2.4], we get

$$
\begin{aligned}
&\|f\|_\infty^{-p}\,\mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^t f^p(X_s)Lf^{-p}(X_s)\,\mathrm{d}s + p\int_0^t N\log f(X_s)\,\mathrm{d}\ell_s\right)\right]\\
&\quad\leqslant \mathbb{E}\left[f^{-p}(X_t)\exp\left(-\frac{1}{2}\int_0^t f^p(X_s)Lf^{-p}(X_s)\,\mathrm{d}s + p\int_0^t N\log f(X_s)\,\mathrm{d}\ell_s\right)\right]\\
&\quad= f^{-p}(x)\leqslant 1,
\end{aligned}
$$

since $f\geqslant 1$ by assumption. This shows that

$$
\|Q_t\|^2 \leqslant \mathrm{e}^{pK_p(f)t}\,\|f\|_\infty^p, \quad t\geqslant 0.
$$

Combining this for $p=2$ with Theorem 3.1 and denoting $K(f)=K_2(f)$, we obtain

$$
\sigma_t^2 := \mathbb{E}\int_0^t |h'(s)|^2\|Q_s\|^2\,\mathrm{d}s \leqslant \|f\|_\infty^2\int_0^t |h'(s)|^2\mathrm{e}^{K(f)s}\,\mathrm{d}s.
$$

Therefore, repeating step (b) in the proof of Theorem 3.2 with $K(f)$ replacing K_V, we finish the proof of (2). □

References

[1] M. Arnaudon, A. Thalmaier, F.-Y. Wang, *Gradient estimates and Harnack inequalities on non-compact Riemannian manifolds*, Stoch. Proc. Appl. **119** (2009), 3653–3670.

[2] L. Cheng, J. Thompson, A. Thalmaier, *Functional inequalities on manifolds with non-convex boundary*, arXiv:1711.04307 (2017).

[3] E. P. Hsu, *Stochastic Analysis on Manifolds*, Graduate Studies in Mathematics 38. American Mathematical Society, 2002.

[4] E. P. Hsu, *Multiplicative functional for the heat equation on manifolds with boundary*, Michigan Math. J. **50** (2002), no. 2, 351–367.

[5] J. Hu, Y. Shi, B. Xu, *The gradient estimate of a Neumann eigenfunction on a compact manifold with boundary,* Chinese Ann. Math. Ser. B **36** (2-15), 991–1000.

[6] W. S. Kendall, *The radial part of Brownian motion on a manifold: a semimartingale property,* Ann. Probab. **15** (1987), 1491–1500.

[7] Y. Shi, B. Xin, *Gradient estimate of a Dirichlet eigenfunction on a compact manifold with boundary*, Forum Math. **25** (2013), 229–240.

[8] F.-Y. Wang, *Application of coupling method to the Neumann eigenvalue problem*, Probab. Theory Related Fields **98** (1994), 299–306.

[9] F.-Y. Wang, *Estimates of the first Dirichlet eigenvalue by using diffusion processes*, Probab. Theory Related Fields **101** (1995), 363–369.

[10] F.-Y. Wang, *Gradient estimates of Dirichlet semigroups and applications to isoperimetric inequalities,* Ann. Probab. **32** (2004), 424–440.

[11] F.-Y. Wang, *Gradient estimates and the first Neumann eigenvalue on manifolds with boundary*, Stoch. Proc. Appl. **115** (2005), 1475–1486.

[12] F.-Y. Wang, *Estimates of the first Neumann eigenvalue and the log-Sobolev constant on non-convex manifolds,* Math. Nachr. **280** (2007), 1431–1439.

[13] F.-Y. Wang, *Harnack inequalities on manifolds with boundary and applications,* J. Math. Pures Appl. **94** (2010), 304–321.

[14] F.-Y. Wang, *Semigroup properties for the second fundamental form*, Documenta Math. **15** (2010), 543–559.

[15] F.-Y. Wang, *Analysis for Diffusion Processes on Riemannian Manifolds*, World Scientific, 2014.