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# Gradient Estimates on Dirichlet and Neumann Eigenfunctions

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## Abstract

By methods of stochastic analysis on Riemannian manifolds, we derive explicit constants  $c_1(D)$  and  $c_2(D)$  for a  $d$ -dimensional compact Riemannian manifold  $D$  with boundary such that

$$c_1(D)\sqrt{\lambda}\|\phi\|_\infty \leq \|\nabla\phi\|_\infty \leq c_2(D)\sqrt{\lambda}\|\phi\|_\infty$$

holds for any Dirichlet eigenfunction  $\phi$  of  $-\Delta$  with eigenvalue  $\lambda$ . In particular, when  $D$  is convex with non-negative Ricci curvature, the estimate holds for

$$c_1(D) = \frac{1}{de}, \quad c_2(D) = \sqrt{e} \left( \frac{\sqrt{2}}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{4\sqrt{2}} \right).$$

Corresponding two-sided gradient estimates for Neumann eigenfunctions are derived in the second part of the paper.

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## 1 Introduction

Let  $D$  be a  $d$ -dimensional compact Riemannian manifold with boundary  $\partial D$ . We write  $(\phi, \lambda) \in \text{Eig}(\Delta)$  if  $\phi$  is a Dirichlet eigenfunction of  $-\Delta$  in  $D$  with eigenvalue  $\lambda > 0$ . According to [7], there exist two constants  $c_1(D), c_2(D) > 0$  such that

$$(1.1) \quad c_1(D)\sqrt{\lambda}\|\phi\|_\infty \leq \|\nabla\phi\|_\infty \leq c_2(D)\sqrt{\lambda}\|\phi\|_\infty, \quad (\phi, \lambda) \in \text{Eig}(\Delta).$$

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An analogous statement for Neumann eigenfunctions has been derived in [5].

Concerning Dirichlet eigenfunctions, an explicit upper constant  $c_2(D)$  can be derived from the uniform gradient estimate of the Dirichlet semigroup in an earlier paper [10] of the third named author. More precisely, let  $K, \theta \geq 0$  be two constants such that

$$(1.2) \quad \text{Ric}_D \geq -K, \quad H_{\partial D} \geq -\theta,$$

where  $\text{Ric}_D$  is the Ricci curvature on  $D$  and  $H_{\partial D}$  the mean curvature of  $\partial D$ . Let

$$(1.3) \quad \alpha_0 = \frac{1}{2} \max \{ \theta, \sqrt{(d-1)K} \}.$$

Consider the semigroup  $P_t = e^{t\Delta}$  for the Dirichlet Laplacian  $\Delta$ . According to [10, Theorem 1.1] where  $c = 2\alpha_0$ , for any nontrivial  $f \in \mathcal{B}_b(D)$  and  $t > 0$ , the following estimate holds:

$$\frac{\|\nabla P_t f\|_\infty}{\|f\|_\infty} \leq 9.5\alpha_0 + \frac{2\sqrt{\alpha_0}(1+4^{2/3})^{1/4}(1+5 \times 2^{-1/3})}{(t\pi)^{1/4}} + \frac{\sqrt{1+2^{1/3}}(1+4^{2/3})}{2\sqrt{t\pi}} =: c(t).$$

Consequently, for any  $(\phi, \lambda) \in \text{Eig}(\Delta)$ ,

$$\|\nabla \phi\|_\infty \leq \|\phi\|_\infty \inf_{t>0} c(t) e^{\lambda t}.$$

In particular, when  $\text{Ric}_D \geq 0, H_{\partial D} \geq 0$ ,

$$(1.4) \quad \|\nabla \phi\|_\infty \leq \frac{\sqrt{e(1+2^{1/3})(1+4^{2/3})}}{\sqrt{2\pi}} \sqrt{\lambda} \|\phi\|_\infty, \quad (\phi, \lambda) \in \text{Eig}(\Delta).$$

In this paper, by using stochastic analysis of the Brownian motion on  $D$ , we develop two-sided gradient estimates; the upper bound given below in (1.8) improves the one in (1.4). Our result will also be valid for  $\alpha_0 \in \mathbb{R}$  satisfying

$$(1.5) \quad \frac{1}{2} \Delta \rho_{\partial D} \leq \alpha_0 \quad \text{outside the focal set,}$$

where  $\rho_{\partial D}$  is the distance to boundary. The case  $\alpha_0 < 0$  appears naturally in many situations, for instance when  $D$  is a closed ball with convex distance to the origin. Note that by [10, Lemma 2.3], if under (1.2) we define  $\alpha_0$  by (1.3) then condition (1.5) holds as a consequence.

For  $x \geq 0$ , in what follows in the limiting case  $x = 0$  we use the convention

$$\left( \frac{1}{1+x} \right)^{1/x} := \lim_{r \downarrow 0} \left( \frac{1}{1+r} \right)^{1/r} = \frac{1}{e}.$$

**Theorem 1.1.** *Let  $K, \theta \geq 0$  be two constants such that (1.2) holds and let  $\alpha_0$  be given by (1.3) or more generally satisfy (1.5). Then, for any nontrivial  $(\phi, \lambda) \in \text{Eig}(\Delta)$ ,*

$$(1.6) \quad \frac{\lambda}{\sqrt{de(\lambda+K)}} \leq \frac{\lambda}{\sqrt{d(\lambda+K)}} \left( \frac{\lambda}{\lambda+K} \right)^{\lambda/(2K)} \leq \frac{\|\nabla \phi\|_\infty}{\|\phi\|_\infty}$$

and

$$(1.7) \quad \frac{\|\nabla \phi\|_\infty}{\|\phi\|_\infty} \leq \begin{cases} \sqrt{e(\lambda+K)} & \text{if } \sqrt{\lambda+K} \geq 2A \\ \sqrt{e} \left( A + \frac{\lambda+K}{4A} \right) & \text{if } \sqrt{\lambda+K} \leq 2A, \end{cases}$$

where

$$A := 2\alpha_0^+ + \frac{\sqrt{2(\lambda + K)}}{\sqrt{\pi}} \exp\left(-\frac{\alpha_0^2}{2(\lambda + K)}\right).$$

In particular, when  $\text{Ric}_D \geq 0$ ,  $H_{\partial D} \geq 0$ ,

$$(1.8) \quad \frac{\sqrt{\lambda}}{\sqrt{de}} \leq \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leq \sqrt{\lambda} \left( \frac{\sqrt{2e}}{\sqrt{\pi}} + \frac{\sqrt{\pi e}}{4\sqrt{2}} \right), \quad (\phi, \lambda) \in \text{Eig}(\Delta).$$

*Proof.* This result follows from Theorem 2.1 and Theorem 2.2 below in the special case  $V = 0$ . In this case,  $\text{Ric}_D^V = \text{Ric}_D \geq -K$  is equivalent to (2.1) with  $n = d$ . More sophisticated upper bounds are given below in Theorem 2.2.  $\square$

By (1.8), if  $D$  is convex with non-negative Ricci curvature then (1.1) holds with

$$c_1(D) = \frac{1}{\sqrt{de}}, \quad c_2(D) = \frac{\sqrt{2e}}{\sqrt{\pi}} + \frac{\sqrt{\pi e}}{4\sqrt{2}}.$$

To give explicit values of  $c_1(D)$  and  $c_2(D)$  for positive  $K$  or  $\theta$ , let  $\lambda_1 > 0$  be the first Dirichlet eigenvalue of  $-\Delta$  on  $D$ . Then Theorem 1.1 implies that (1.1) holds for

$$c_1(D) = \frac{\sqrt{\lambda_1}}{\sqrt{de(\lambda_1 + K)}},$$

$$c_2(D) = \frac{\sqrt{e(\lambda_1 + K)}}{\sqrt{\lambda_1}} \mathbb{1}_{\{B > 2A\}} + \frac{\sqrt{e}}{\sqrt{\lambda_1}} \left( 2\alpha_0^+ + \sqrt{\frac{2(\lambda_1 + K)}{\pi}} + \frac{\lambda_1 + K}{4(2\alpha_0^+ + \sqrt{2(\lambda_1 + K)/\pi})} \right) \mathbb{1}_{\{B \leq 2A\}}$$

with

$$B = \sqrt{\lambda_1 + K} \quad \text{and} \quad A = 2\alpha_0^+ + \sqrt{\frac{2(\lambda_1 + K)}{\pi}}.$$

This is due to the fact that the expression for  $c_1(D)$  is an increasing function of  $\lambda$  and the expression for  $c_2(D)$  a decreasing function of  $\lambda$ . Since there exist explicit lower bound estimates on  $\lambda_1$  (see [9] and references within), this gives explicit lower bounds of  $c_1(D)$  and explicit upper bounds of  $c_2(D)$ .

The lower bound for  $\|\nabla\phi\|_\infty$  will be derived by using Itô's formula for  $|\nabla\phi|^2(X_t)$  where  $X_t$  is a Brownian motion (with drift) on  $D$ , see Subsection 2.1 for details. To derive the upper bound estimate, we will construct some martingales to reduce  $\|\nabla\phi\|_\infty$  to  $\|\nabla\phi\|_{\partial D, \infty} := \sup_{\partial D} |\nabla\phi|$ , and to estimate the latter in terms of  $\|\phi\|_\infty$ , see Subsection 2.2 for details.

Next, we consider the Neumann problem. Let  $\text{Eig}_N(\Delta)$  be the set of non-trivial eigenpairs  $(\phi, \lambda)$  for the Neumann eigenproblem, i.e.  $\phi$  is non-constant,  $\Delta\phi = -\lambda\phi$  with  $N\phi|_{\partial D} = 0$  for the unit inward normal vector field  $N$  of  $\partial D$ . Let  $\mathbb{I}_{\partial D}$  be the second fundamental form of  $\partial D$ ,

$$\mathbb{I}_{\partial D}(X, Y) = -\langle \nabla_X N, Y \rangle, \quad X, Y \in T_x \partial M, \quad x \in \partial M.$$

With a concrete choice of the function  $f$ , the next theorem implies (1.1) for  $(\phi, \lambda) \in \text{Eig}_N(\Delta)$  together with explicit constants  $c_1(D), c_2(D)$ .

**Theorem 1.2.** *Let  $K, \delta \in \mathbb{R}$  be constants such that*

$$(1.9) \quad \text{Ric}_D \geq -K, \quad \mathbb{I}_{\partial D} \geq -\delta.$$

*For  $f \in C_b^2(\bar{D})$  with  $\inf_D f = 1$  and  $N \log f|_{\partial D} \geq \delta$ , let*

$$c_\varepsilon(f) = \sup_D \left\{ \frac{4\varepsilon |\nabla \log f|^2}{1 - \varepsilon} + K - 2\Delta \log f \right\}, \quad \varepsilon \in (0, 1),$$

$$K(f) = \sup_D \{2|\nabla \log f|^2 + K - \Delta \log f\}.$$

Then for any non-trivial  $(\phi, \lambda) \in \text{Eig}_N(\Delta)$ , we have  $\lambda + c_\varepsilon(f) > 0$  and

$$\begin{aligned} \sup_{\varepsilon \in (0,1)} \frac{\varepsilon \lambda^2}{d\varepsilon(\lambda + c_\varepsilon(f))\|f\|_\infty^2} &\leq \sup_{\varepsilon \in (0,1)} \frac{\varepsilon \lambda^2}{d(\lambda + c_\varepsilon(f))\|f\|_\infty^2} \left(\frac{\lambda}{\lambda + c_\varepsilon(f)}\right)^{\lambda/c_\varepsilon(f)} \\ &\leq \frac{\|\nabla \phi\|_\infty^2}{\|\phi\|_\infty^2} \leq \frac{2\|f\|_\infty^2(\lambda + K(f))}{\pi} \left(1 + \frac{K(f)}{\lambda}\right)^{\lambda/K(f)} \\ &\leq 2e\|f\|_\infty^2 \frac{\lambda + K(f)}{\pi}. \end{aligned}$$

*Proof.* Under the conditions (1.2), Theorem 3.3 below applies with  $L = \Delta$ ,  $K_V = K$  and  $n = d$ . The desired estimates are immediate consequences.  $\square$

When  $\partial D$  is convex, i.e.  $\mathbb{I}_{\partial D} \geq 0$ , we may take  $f \equiv 1$  in Theorem 1.2 to derive the following result. According to Theorem 3.2 below, this result also holds for  $\partial D = \emptyset$  where  $\text{Eig}(\Delta)$  is the set of eigenpairs for the closed eigenproblem.

**Corollary 1.3.** *Let  $\partial D$  be convex or empty. If  $\text{Ric}_D^V \geq -K$  for some constant  $K$ , then for any non-trivial  $(\phi, \lambda) \in \text{Eig}_N(\Delta)$ , we have  $\lambda + K > 0$  and*

$$\frac{\lambda^2}{d\varepsilon(\lambda + K^+)} \leq \frac{\lambda^2}{d(\lambda + K)} \left(\frac{\lambda}{\lambda + K}\right)^{\lambda/K} \leq \frac{\|\nabla \phi\|_\infty^2}{\|\phi\|_\infty^2} \leq \frac{2(\lambda + K)}{\pi} \left(1 + \frac{K}{\lambda}\right)^{\lambda/K} \leq \frac{2e(\lambda + K^+)}{\pi}.$$

## 2 Proof of Theorem 1.1

In general, we will consider Dirichlet eigenfunctions for the symmetric operator  $L := \Delta + \nabla V$  on  $D$  where  $V \in C^2(D)$ . We denote by  $\text{Eig}(L)$  the set of pairs  $(\phi, \lambda)$  where  $\phi$  is a Dirichlet eigenfunction of  $-L$  on  $D$  with eigenvalue  $\lambda$ .

In the following two subsections, we consider the lower bound and upper bound estimates respectively.

### 2.1 Lower bound estimate

In this subsection we will estimate  $\|\nabla \phi\|_\infty$  from below using the following Bakry-Émery curvature-dimension condition:

$$(2.1) \quad \frac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f \rangle \geq -K|\nabla f|^2 + \frac{(Lf)^2}{n}, \quad f \in C^\infty(D),$$

where  $K \in \mathbb{R}$ ,  $n \geq d$  are two constants. When  $V = 0$ , this condition with  $n = d$  is equivalent to  $\text{Ric}_D \geq -K$ .

**Theorem 2.1** (Lower bound estimate). *Assume that (2.1) holds. Then*

$$(2.2) \quad \|\nabla \phi\|_\infty^2 \geq \|\phi\|_\infty^2 \sup_{t>0} \frac{\lambda^2(e^{Kt} - 1)}{nKe^{(\lambda+K)t}}, \quad (\phi, \lambda) \in \text{Eig}(L).$$

Consequently, for  $K^+ := \max\{0, K\}$  there holds

$$(2.3) \quad \|\nabla \phi\|_\infty^2 \geq \frac{\lambda^2\|\phi\|_\infty^2}{n(\lambda + K^+)} \left(\frac{\lambda}{\lambda + K^+}\right)^{\lambda/K^+} \geq \frac{\lambda^2\|\phi\|_\infty^2}{ne(\lambda + K^+)}, \quad (\phi, \lambda) \in \text{Eig}(L).$$

*Proof.* Let  $X_t$  be the diffusion process generated by  $\frac{1}{2}L$  in  $D$ , and let

$$\tau_D := \inf\{t \geq 0 : X_t \in \partial D\}.$$

By Itô's formula, we have

$$(2.4) \quad d|\nabla\phi|^2(X_t) = \frac{1}{2}L|\nabla\phi|^2(X_t) dt + dM_t, \quad t \leq \tau_D,$$

for some martingale  $M_t$ . By the curvature dimension condition (2.1) and  $L\phi = -\lambda\phi$ , we obtain

$$(2.5) \quad \frac{1}{2}L|\nabla\phi|^2 = \frac{1}{2}L|\nabla\phi|^2 - \langle \nabla L\phi, \nabla\phi \rangle - \lambda|\nabla\phi|^2 \geq -(K + \lambda)|\nabla\phi|^2 + \frac{\lambda^2}{n}\phi^2.$$

Therefore, (2.4) gives

$$d|\nabla\phi|^2(X_t) \geq \left( \frac{\lambda^2}{n}\phi^2 - (K + \lambda)|\nabla\phi|^2 \right)(X_t) dt + dM_t, \quad t \leq \tau_D.$$

Hence, for any  $t > 0$ ,

$$\begin{aligned} e^{(K+\lambda)t} \|\nabla\phi\|_\infty^2 &\geq \mathbb{E} \left[ |\nabla\phi|^2(X_{t \wedge \tau_D}) e^{(K+\lambda)(t \wedge \tau_D)} \right] \\ &\geq \frac{\lambda^2}{n} \mathbb{E} \left[ \int_0^{t \wedge \tau_D} e^{(K+\lambda)s} \phi(X_s)^2 ds \right] \\ &= \frac{\lambda^2}{n} \mathbb{E} \left[ \int_0^t 1_{\{s < \tau_D\}} e^{(K+\lambda)s} \phi(X_s)^2 ds \right]. \end{aligned}$$

Since  $\phi|_{\partial D} = 0$  and  $L\phi = -\lambda\phi$ , by Jensen's inequality we have

$$\mathbb{E} [1_{\{s < \tau_D\}} \phi(X_s)^2] \geq (\mathbb{E}[\phi(X_{s \wedge \tau_D})])^2 = e^{-\lambda s} \phi(x)^2,$$

where  $x = X_0 \in D$  is the starting point of  $X_t$ . Then, by taking  $x$  such that  $\phi(x)^2 = \|\phi\|_\infty^2$ , we arrive at

$$\begin{aligned} e^{(K+\lambda)t} \|\nabla\phi\|_\infty^2 &\geq \frac{\lambda^2}{n} \int_0^t e^{(K+\lambda)s} e^{-\lambda s} \phi(x)^2 ds \\ &= \frac{\lambda^2 \|\phi\|_\infty^2}{n} \int_0^t e^{Ks} ds = \frac{\lambda^2 (e^{Kt} - 1)}{nK} \|\phi\|_\infty^2. \end{aligned}$$

This completes the proof of (2.2).

Since (2.1) holds for  $K^+$  replacing  $K$ , we may and do assume that  $K \geq 0$ . By taking the optimal choice  $t = \frac{1}{K} \log(1 + \frac{K}{\lambda})$  (by convention  $t = \lambda^{-1}$  if  $K = 0$ ) in (2.2), we obtain

$$\|\nabla\phi\|_\infty^2 \geq \frac{\lambda^2 \|\phi\|_\infty^2}{\lambda + K} \left( \frac{\lambda}{\lambda + K} \right)^{\lambda/K} \geq \frac{\lambda^2 \|\phi\|_\infty^2}{ne(\lambda + K)}.$$

Hence (2.3) holds. □

## 2.2 Upper bound estimate

Let  $\text{Ric}_D^V = \text{Ric}_D - \text{Hess}_V$ . For  $K_0, \theta \geq 0$  such that  $\text{Ric}_D \geq -K_0$  and  $H_{\partial D} \geq -\theta$ , let

$$(2.6) \quad \alpha = \frac{1}{2} \left( \max \{ \theta, \sqrt{(d-1)K_0} \} + \|\nabla V\|_\infty \right)$$

We note that  $\frac{1}{2}L\rho_{\partial D} \leq \alpha$  by [10, Lemma 2.3].

**Theorem 2.2** (Upper bound estimate). *Let  $K_V, \theta \geq 0$  be constants such that*

$$\text{Ric}_D^V \geq -K_V, \quad H_{\partial D} \geq -\theta.$$

*Let  $\alpha \in \mathbb{R}$  be such that*

$$(2.7) \quad \frac{1}{2}L\rho_{\partial D} \leq \alpha.$$

1. *Assume  $\alpha \geq 0$ . Then, for any nontrivial  $(\phi, \lambda) \in \text{Eig}(L)$ ,*

$$(2.8) \quad \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leq \begin{cases} \sqrt{e(\lambda + K_V)} & \text{if } \sqrt{\lambda + K_V} \geq 2A \\ \sqrt{e} \left( A + \frac{\lambda + K_V}{4A} \right) & \text{if } \sqrt{\lambda + K_V} \leq 2A, \end{cases}$$

where

$$(2.9) \quad A := \alpha + \frac{\sqrt{2(\lambda + K_V)}}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{2(\lambda + K_V)}\right) + |\alpha| \wedge \frac{\sqrt{2}\alpha^2}{\sqrt{\pi(\lambda + K_V)}}.$$

*In particular, (2.8) holds with  $A$  replaced by*

$$(2.10) \quad A' := 2\alpha + \frac{\sqrt{2(\lambda + K_V)}}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{2(\lambda + K_V)}\right).$$

*We also have*

$$(2.11) \quad \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leq \sqrt{e} \left( \frac{2\alpha + \sqrt{2(\lambda + K_V)}}{\sqrt{\pi}} + \frac{\lambda + K_V}{4} \frac{\sqrt{\pi}}{2\alpha + \sqrt{2(\lambda + K_V)}} \right).$$

2. *Assume  $\alpha \leq 0$ . Then, for any nontrivial  $(\phi, \lambda) \in \text{Eig}(L)$ ,*

$$(2.12) \quad \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leq \begin{cases} \sqrt{e(\lambda + K_V)} & \text{if } \sqrt{\lambda + K_V} \geq 2A^* \\ \sqrt{e} \left( A^* + \frac{\lambda + K_V}{4A^*} \right) & \text{if } \sqrt{\lambda + K_V} \leq 2A^*, \end{cases}$$

where

$$(2.13) \quad A^* := \frac{\sqrt{2(\lambda + K_V)}}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{2(\lambda + K_V)}\right).$$

*In particular,*

$$(2.14) \quad \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leq \sqrt{\lambda + K_V} \left( \sqrt{\frac{2}{\pi}} + \frac{1}{4} \sqrt{\frac{\pi}{2}} \right) \sqrt{e}.$$

In addition, the following estimate holds:

$$(2.15) \quad \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty} \leq \begin{cases} \sqrt{e(\lambda + K_V)} & \text{if } \sqrt{\lambda + K_V} \geq 2\sqrt{e\hat{A}} \\ e\hat{A} + \frac{\lambda + K_V}{4\hat{A}} & \text{if } \sqrt{\lambda + K_V} < 2\sqrt{e\hat{A}}, \end{cases}$$

where

$$(2.16) \quad \hat{A} := \alpha + \frac{\sqrt{2\lambda}}{\sqrt{\pi}} e^{-\frac{\alpha^2}{2\lambda}} + |\alpha| \wedge \frac{\sqrt{2\alpha^2}}{\sqrt{\pi\lambda}}.$$

The strategy to prove Theorem 2.2 will be to first estimate  $\|\nabla\phi\|_\infty$  in terms of  $\|\phi\|_\infty$  and  $\|\nabla\phi\|_{\partial D, \infty}$  (see estimate (2.24) below) where  $\|f\|_{\partial D, \infty} := \|1_{\partial D}f\|_\infty$  for a function  $f$  on  $D$ . The end we construct appropriate martingales in terms of  $\phi$  and  $\nabla\phi$ .

We start by recalling the necessary facts about the diffusion process generated by  $\frac{1}{2}L$ , see for instance [1, 3]. For any  $x \in D$ , the diffusion  $X_t$  solves the SDE

$$(2.17) \quad dX_t = \frac{1}{2}\nabla V(X_t) dt + u_t \circ dB_t, \quad X_0 = x, \quad t \leq \tau_D,$$

where  $B_t$  is a  $d$ -dimensional Brownian motion,  $u_t$  is the horizontal lift of  $X_t$  onto the orthonormal frame bundle  $O(D)$  with initial value  $u_0 \in O_x(D)$ , and

$$\tau_D := \inf\{t \geq 0 : X_t \in \partial D\}$$

is the hitting time of  $X_t$  to the boundary  $\partial D$ . Setting  $Z := \nabla V$ , we have

$$(2.18) \quad du_t = \frac{1}{2}Z^*(u_t) dt + \sum_{i=1}^d H_i(u_t) \circ dB_t^i$$

where  $Z^*(u) := h_u(Z_{\pi(u)})$  and  $H_i(u) := h_u(ue_i)$  are defined by means of the horizontal lift  $h_u: T_{\pi(u)}D \rightarrow T_u O(D)$  at  $u \in O(D)$ . Note that formally  $h_{u_t}(u_t \circ dB_t) = \sum_i h_{u_t}(u_t e_i) \circ dB_t^i = \sum_i H_i(u_t) \circ dB_t^i$ .

For  $f \in C^\infty(D)$ , let  $a := df \in \Gamma(T^*D)$ . Setting  $m_t := u_t^{-1}a(X_t)$ , we see by Itô's formula that

$$(2.19) \quad dm_t \stackrel{\text{m}}{=} \frac{1}{2}u_t^{-1}(\square a + \nabla_Z a)(X_t) dt$$

where  $\square a = \text{tr } \nabla^2 a$  denotes the so-called connection (or rough) Laplacian on 1-forms and  $\stackrel{\text{m}}{=}$  equality modulo the differential of a local martingale.

Denote by  $Q_t: T_x D \rightarrow T_{X_t} D$  the solution, along the paths of  $X_t$ , to the covariant ordinary differential equation

$$DQ_t = -\frac{1}{2}(\text{Ric}_D^V)^\sharp Q_t dt, \quad Q_0 = \text{id}_{T_x D}, \quad t \leq \tau_D,$$

where  $D := u_t du_t^{-1}$  and where by definition

$$(\text{Ric}_D^V)^\sharp v = \text{Ric}_D^V(\cdot, v)^\sharp, \quad v \in T_x D.$$

Thus, condition  $\text{Ric}_D^V \geq -K_V$  implies

$$(2.20) \quad |Q_t v| \leq e^{\frac{K_V}{2}t} |v|, \quad t \leq \tau_D.$$



Finally, note that for any smooth function  $f$  on  $D$ , we have by the Weitzenböck formula:

$$\begin{aligned}
d(\Delta + Z)f &= d(-d^*df + (df)Z) \\
&= \Delta^{(1)}df + \nabla_Z df + \langle \nabla, Z, \nabla f \rangle \\
&= (\square + \nabla_Z)(df) - \text{Ric}_D^V(\cdot, \nabla f) \\
(2.21) \qquad &= (\square - \text{Ric}_D^V + \nabla_Z)(df)
\end{aligned}$$

where  $\Delta^{(1)}$  denotes the Hodge-deRham Laplacian on 1-forms.

Now let  $(\phi, \lambda) \in \text{Eig}(L)$ , i.e.  $L\phi = -\lambda\phi$ , where  $L = \Delta + Z$ . For  $v \in T_x D$ , consider the process

$$n_t(v) := (d\phi)(Q_t v).$$

Then

$$n_t(v) = \langle \nabla\phi(X_t), Q_t v \rangle = \langle u_t^{-1}(\nabla\phi)(X_t), u_t^{-1}Q_t v \rangle.$$

Using (2.19), we see by Itô's formula and formula (2.21) that

$$dn_t(v) \stackrel{m}{=} \frac{1}{2}(\square d\phi + \nabla_Z d\phi)(X_t) Q_t v dt + d\phi(X_t)(DQ_t v) dt = -\frac{\lambda}{2}n_t(v) dt.$$

It follows that

$$(2.22) \qquad e^{\lambda t/2} n_t(v) = e^{\lambda t/2} \langle \nabla\phi(X_t), Q_t v \rangle, \quad t \leq \tau_D,$$

is a martingale.

*Lemma 2.1.* Let  $(\phi, \lambda) \in \text{Eig}(L)$ . We keep the notation from above. Then, for any function  $h \in C^1([0, \infty); \mathbb{R})$ , the process

$$(2.23) \qquad N_t(v) := h_t e^{\lambda t/2} \langle \nabla\phi(X_t), Q_t v \rangle - e^{\lambda t/2} \phi(X_t) \int_0^t \langle \dot{h}_s Q_s v, u_s dB_s \rangle, \quad t \leq \tau_D,$$

is a martingale. In particular, for fixed  $t > 0$  and  $h \in C^1([0, t]; [0, 1])$  monotone such that  $h_0 = 1$  and  $h_t = 0$ , we have

$$\begin{aligned}
\|\nabla\phi\|_\infty &\leq \|\nabla\phi\|_{\partial D, \infty} \mathbb{P}\{t > \tau_D\} e^{(\lambda + K_V)t/2} \\
(2.24) \qquad &+ \|\phi\|_\infty e^{\lambda t/2} \mathbb{P}\{t \leq \tau_D\}^{1/2} \left( \int_0^t |\dot{h}_s|^2 e^{K_V s} ds \right)^{1/2}.
\end{aligned}$$

*Proof.* Indeed, from (2.22) we deduce that

$$h_t e^{\lambda t/2} \langle \nabla\phi(X_t), Q_t v \rangle - \int_0^t \dot{h}_s e^{\lambda s/2} \langle \nabla\phi(X_s), Q_s v \rangle ds, \quad t \leq \tau_D,$$

is a martingale as well. By the formula

$$e^{\lambda t/2} \phi(X_t) = \phi(X_0) + \int_0^t e^{\lambda s/2} \langle \nabla\phi(X_s), u_s dB_s \rangle$$

we see then that  $N_t(v)$  is a martingale. To check inequality (2.24), we deduce from the martingale property of  $\{N_{s \wedge \tau_D}(v)\}_{s \in [0, t]}$  that

$$\begin{aligned}
\|\nabla\phi\|_\infty &\leq \|\nabla\phi\|_{\partial D, \infty} \mathbb{E} \left[ \mathbf{1}_{\{t > \tau_D\}} e^{\lambda \tau_D/2} |h_{\tau_D}| |Q_{\tau_D}| \right] \\
&+ \|\phi\|_\infty e^{\lambda t/2} \mathbb{E} \left[ \mathbf{1}_{\{t \leq \tau_D\}} \sup_{|v| \leq 1} \left( \int_0^t \langle \dot{h}_s Q_s v, u_s dB_s \rangle \right)^2 \right]^{1/2}.
\end{aligned}$$

The claim follows by using (2.20). □

To estimate the boundary norm  $\|\nabla\phi\|_{\partial D,\infty}$ , we shall compare  $\phi(x)$  and

$$\psi(t, x) := \mathbb{P}(\tau_D^x > t), \quad t > 0,$$

for small  $\rho_{\partial D}(x) := \text{dist}(x, \partial D)$ . Let  $P_t^D$  be the Dirichlet semigroup generated by  $\frac{1}{2}L$ . Then

$$\psi(t, x) = P_t^D 1_D(x),$$

so that

$$(2.25) \quad \partial_t \psi(t, x) = \frac{1}{2}L\psi(t, \cdot)(x), \quad t > 0.$$

**Lemma 2.3.** *For any  $(\phi, \lambda) \in \text{Eig}(L)$ ,*

$$(2.26) \quad \|\nabla\phi\|_{\partial D,\infty} \leq \|\phi\|_\infty \inf_{t>0} e^{\lambda t/2} \|\nabla\psi(t, \cdot)\|_{\partial D,\infty}.$$

*Proof.* To prove (2.26), we fix  $x \in \partial D$ . For small  $\varepsilon > 0$ , let  $x^\varepsilon = \exp_x(\varepsilon N)$ , where  $N$  is the inward unit normal vector field of  $\partial D$ . Since  $\phi|_{\partial D} = 0$  and  $\psi(t, \cdot)|_{\partial D} = 0$ , we have

$$(2.27) \quad |\nabla\phi(x)| = |N\phi(x)| = \lim_{\varepsilon \rightarrow 0} \frac{|\phi(x^\varepsilon)|}{\varepsilon}, \quad |\nabla\psi(t, \cdot)(x)| = \lim_{\varepsilon \rightarrow 0} \frac{|\psi(t, x^\varepsilon)|}{\varepsilon}.$$

Let  $X_t^\varepsilon$  be the  $L$ -diffusion starting at  $x^\varepsilon$  and  $\tau_D^\varepsilon$  its first hitting time of  $\partial D$ . Note that

$$N_t := \phi(X_{t \wedge \tau_D^\varepsilon}^\varepsilon) e^{\lambda(t \wedge \tau_D^\varepsilon)/2}, \quad t \geq 0,$$

is a martingale. Thus, for each fixed  $t > 0$ , we can estimate as follows:

$$\begin{aligned} |\nabla\phi(x)| &= \lim_{\varepsilon \rightarrow 0} \frac{|\phi(x^\varepsilon)|}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\left| \mathbb{E}[\phi(X_t^\varepsilon) \mathbf{1}_{\{t < \tau_D^\varepsilon\}}] e^{\lambda(t \wedge \tau_D^\varepsilon)/2} \right|}{\varepsilon} \\ &\leq \|\phi\|_\infty e^{\lambda t/2} \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{E}[\mathbf{1}_{\{t < \tau_D^\varepsilon\}}]}{\varepsilon} \\ &\leq \|\phi\|_\infty e^{\lambda t/2} \lim_{\varepsilon \rightarrow 0} \frac{\psi(t, x^\varepsilon)}{\varepsilon} \\ &= \|\phi\|_\infty e^{\lambda t/2} |\nabla\psi(t, \cdot)|(x). \end{aligned}$$

Taking the infimum over  $t$  gives the claim. □

We now work out an explicit estimate for  $\|\nabla\psi(t, \cdot)\|_{\partial D,\infty}$ . Let  $\text{cut}(D)$  be the cut-locus of  $\partial D$ , which is a zero-volume closed subset of  $D$  such that  $\rho_{\partial D} := \text{dist}(\cdot, \partial D)$  is smooth in  $D \setminus \text{cut}(D)$ .

**Proposition 2.4.** *Let  $\alpha \in \mathbb{R}$  such that*

$$(2.28) \quad \frac{1}{2}L\rho_{\partial D} \leq \alpha.$$

*Then*

$$\|\nabla\psi(t, \cdot)\|_{\partial D,\infty} \leq \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds$$

$$(2.29) \quad \leq \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\alpha^2 t}{2}} + \min \left\{ |\alpha|, \frac{\alpha^2 \sqrt{2t}}{\sqrt{\pi}} \right\},$$

and

$$(2.30) \quad \|\nabla \psi(t, \cdot)\|_{\partial D, \infty} \leq \frac{\sqrt{2}}{\sqrt{\pi t}} + \alpha + \frac{\sqrt{t}}{\sqrt{2\pi}} \alpha^2$$

Notice that by [10, Lemma 2.3] the condition  $\frac{1}{2}L\rho_{\partial D} \leq \alpha$  holds for  $\alpha$  defined by (2.6).

*Proof.* Let  $x \in D$  and let  $X_t$  solve SDE (2.17). As shown in [6],  $(\rho_{\partial D}(X_t))_{t \leq \tau_D}$  is a semimartingale satisfying

$$(2.31) \quad \rho_{\partial D}(X_t) = \rho_{\partial D}(x) + b_t + \frac{1}{2} \int_0^t L\rho_{\partial D}(X_s) ds - l_t, \quad t \leq \tau_D,$$

where  $b_t$  is a real-valued Brownian motion starting at 0, and  $l_t$  a non-decreasing process which increases only when  $X_t^x \in \text{cut}(D)$ . Setting  $\varepsilon = \rho_{\partial D}(x)$ , we deduce from (2.31) together with  $\frac{1}{2}L\rho_{\partial D} \leq \alpha$ , that

$$(2.32) \quad \rho_{\partial D}(X_t(x)) \leq Y_t^\alpha(\varepsilon) := \varepsilon + b_t + \alpha t, \quad t \leq \tau_D.$$

Consequently, letting  $T^\alpha(\varepsilon)$  be the first hitting time of 0 by  $Y_t^\alpha(\varepsilon)$ , we obtain

$$(2.33) \quad \psi(t, x) \leq \mathbb{P}(t < T^\alpha(\varepsilon)).$$

On the other hand, since  $\psi(t, \cdot)$  vanishes on the boundary and is positive in  $D$ , we have for all  $y \in \partial D$

$$(2.34) \quad |\nabla \psi(t, y)| = \lim_{x \in D, x \rightarrow y} \frac{\psi(t, x)}{\rho_{\partial D}(x)}.$$

Hence, by (2.33), to prove the first inequality in (2.29) it is enough to establish that

$$(2.35) \quad \limsup_{\varepsilon \downarrow 0} \frac{\mathbb{P}(t < T^\alpha(\varepsilon))}{\varepsilon} \leq \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds.$$

It is well known that the (sub-probability) density  $f_{\alpha, \varepsilon}$  of  $T^\alpha(\varepsilon)$  is

$$(2.36) \quad f_{\alpha, \varepsilon}(s) = \frac{\varepsilon \exp(-(\varepsilon + \alpha s)^2 / (2s))}{\sqrt{2\pi s^3}},$$

which can be obtained by the reflection principle for  $\alpha = 0$  and the Girsanov transform for  $\alpha \neq 0$ . Thus

$$(2.37) \quad \begin{aligned} \mathbb{P}(t \geq T^\alpha(\varepsilon)) &= \varepsilon \int_0^t \frac{\exp(-(\varepsilon + \alpha s)^2 / (2s))}{\sqrt{2\pi s^3}} ds \\ &= \varepsilon \exp(-\alpha \varepsilon) \int_0^t \frac{e^{-\alpha^2 s / 2}}{\sqrt{2\pi s^3}} \exp\left(-\frac{\varepsilon^2}{2s}\right) ds \\ &= \exp(-\alpha \varepsilon) \int_0^{2t/\varepsilon^2} \frac{e^{-1/r}}{\sqrt{\pi r^3}} \exp\left(-\frac{\alpha^2 \varepsilon^2 r}{4}\right) dr, \end{aligned}$$

where we have made the change of variable  $r = 2s/\varepsilon^2$ . With the change of variable  $v = 1/r$  we easily check that

$$(2.38) \quad \int_0^\infty r^{-3/2} e^{-1/r} dr = \Gamma(1/2) = \sqrt{\pi},$$

and this allows to write

$$(2.39) \quad \mathbb{P}(t \geq T^\alpha(\varepsilon)) = \exp(-\alpha\varepsilon) \left( 1 - \int_{2t/\varepsilon^2}^\infty \frac{e^{-1/r}}{\sqrt{\pi r^3}} dr - \int_0^{2t/\varepsilon^2} \frac{e^{-1/r}}{\sqrt{\pi r^3}} \left( 1 - e^{-\alpha^2 \varepsilon^2 r/4} \right) dr \right).$$

As  $\varepsilon \rightarrow 0$ ,

$$\int_{2t/\varepsilon^2}^\infty \frac{e^{-1/r}}{\sqrt{\pi r^3}} dr = \int_{2t/\varepsilon^2}^\infty \frac{1}{\sqrt{\pi r^3}} dr + o(\varepsilon) = \frac{\varepsilon\sqrt{2}}{\sqrt{t}} + o(\varepsilon),$$

and with change of variable  $s = \frac{1}{2}\varepsilon^2 r$

$$\begin{aligned} \int_0^{2t/\varepsilon^2} \frac{e^{-1/r}}{\sqrt{\pi r^3}} \left( 1 - e^{-\alpha^2 \varepsilon^2 r/4} \right) dr &= \varepsilon \int_0^t \frac{e^{-\frac{\varepsilon^2}{2s}}}{\sqrt{2\pi s^3}} \left( 1 - e^{-\frac{\alpha^2 s}{2}} \right) ds \\ &= \varepsilon \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds + o(\varepsilon) \end{aligned}$$

by monotone convergence. Combining these with  $e^{-\alpha\varepsilon} = 1 - \alpha\varepsilon + o(\varepsilon)$ , we deduce from (2.39) that

$$(2.40) \quad \mathbb{P}(t \geq T^\alpha(\varepsilon)) = 1 - \varepsilon \left( \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds \right) + o(\varepsilon)$$

which yields (2.35).

Next, an integration by parts yields

$$(2.41) \quad \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds = \frac{\alpha^2}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{u}} e^{-\frac{\alpha^2 u}{2}} du - \frac{\sqrt{2}}{\sqrt{\pi t}} \left( 1 - e^{-\frac{\alpha^2 t}{2}} \right).$$

With the change of variable  $s = |\alpha| \sqrt{\frac{u}{t}}$  in the first term in the right we obtain

$$(2.42) \quad \frac{\alpha^2}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{u}} e^{-\frac{\alpha^2 u}{2}} du = |\alpha| \sqrt{\frac{2t}{\pi}} \int_0^{|\alpha|} e^{-\frac{s^2 t}{2}} ds.$$

We arrive at

$$(2.43) \quad f(\alpha) := \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds = \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\alpha^2 t}{2}} + \alpha + |\alpha| \sqrt{\frac{2t}{\pi}} \int_0^{|\alpha|} e^{-\frac{s^2 t}{2}} ds.$$

Bounding  $\sqrt{\frac{2t}{\pi}} \int_0^{|\alpha|} e^{-\frac{s^2 t}{2}} ds$  by  $\sqrt{\frac{2t}{\pi}} \int_0^\infty e^{-\frac{s^2 t}{2}} ds = 1$ , respectively bounding  $e^{-\frac{\alpha^2 t}{2}}$  by 1 in the integral yield (2.29).

The function

$$f(\alpha) = \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\alpha^2 t}{2}} + \alpha + |\alpha| \sqrt{\frac{2t}{\pi}} \int_0^{|\alpha|} e^{-\frac{s^2 t}{2}} ds$$

is smooth and an easy computation shows that

$$(2.44) \quad f(0) = \frac{\sqrt{2}}{\sqrt{\pi t}}, \quad f'(0) = 1, \quad f''(\alpha) = \frac{\sqrt{2t}}{\sqrt{\pi}} e^{-\frac{\alpha^2 t}{2}}$$

Using the fact that  $f(\alpha) - \alpha$  is even, we also get

$$(2.45) \quad f(\alpha) = \frac{\sqrt{2}}{\sqrt{\pi t}} + \alpha + \int_0^{|\alpha|} \frac{\sqrt{2t}}{\sqrt{\pi}} e^{-\frac{s^2 t}{2}} s ds \leq \frac{\sqrt{2}}{\sqrt{\pi t}} + \alpha + \frac{\sqrt{t}}{\sqrt{2\pi}} \alpha^2.$$

which yields (2.30).  $\square$

*Remark 2.2.* One could use estimate (2.24) (optimizing the right-hand side with respect to  $t$ ) together with Lemma 2.3 (again optimizing with respect to  $t$ ) to estimate  $\|\nabla\phi\|_\infty$  in terms of  $\|\phi\|_\infty$ . We prefer to combine the two steps.

**Lemma 2.5.** *Assume  $\text{Ric}_D^V \geq -K_V$  for some constant  $K_V \in \mathbb{R}$ . Let  $\alpha$  be determined by (2.28).*

(a) *If  $\alpha \geq 0$ , then for any  $(\phi, \lambda) \in \text{Eig}(L)$ ,*

$$\|\nabla\phi\|_\infty \leq \inf_{t>0} \max_{\varepsilon \in [0,1]} e^{\frac{(\lambda+K_V^+)t}{2}} \left\{ \varepsilon \left( \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\alpha^2 t}{2}} + \min \left( |\alpha|, \frac{\alpha^2 \sqrt{2t}}{\sqrt{\pi}} \right) \right) + \sqrt{\frac{1-\varepsilon}{t}} \right\} \|\phi\|_\infty,$$

as well as

$$\|\nabla\phi\|_\infty \leq \inf_{t>0} \max_{\varepsilon \in [0,1]} e^{(\lambda+K_V^+)t/2} \left\{ \varepsilon \left( \alpha + \sqrt{\frac{2}{\pi t}} + \frac{\sqrt{t}}{\sqrt{2\pi}} \alpha^2 \right) + \sqrt{\frac{1-\varepsilon}{t}} \right\} \|\phi\|_\infty$$

and

$$\|\nabla\phi\|_\infty \leq \inf_{t>0} \max_{\varepsilon \in [0,1]} e^{(\lambda+K_V^+)t/2} \left\{ \varepsilon \left( 2\alpha + \sqrt{\frac{2}{\pi t}} \right) + \sqrt{\frac{1-\varepsilon}{t}} \right\} \|\phi\|_\infty.$$

(b) *If  $\alpha \leq 0$ , then*

$$\|\nabla\phi\|_\infty \leq \inf_{t>0} \max_{\varepsilon \in [0,1]} e^{(\lambda+K_V^+)t/2} \left\{ \varepsilon \sqrt{\frac{2}{\pi t}} e^{-\frac{\alpha^2 t}{2}} + \sqrt{\frac{1-\varepsilon}{t}} \right\} \|\phi\|_\infty.$$

*In particular,*

$$\|\nabla\phi\|_\infty \leq \inf_{t>0} \max_{\varepsilon \in [0,1]} e^{(\lambda+K_V^+)t/2} \left\{ \varepsilon \sqrt{\frac{2}{\pi t}} + \sqrt{\frac{1-\varepsilon}{t}} \right\} \|\phi\|_\infty.$$

*Proof.* For fixed  $t > 0$  in (2.23), we take  $h \in C^1([0, t]; [0, 1])$  such that  $h_0 = 1$  and  $h_t = 0$ . Then, by the martingale property of  $\{N_{s \wedge \tau_D}(v)\}_{s \in [0, t]}$ , we obtain

$$(2.46) \quad \begin{aligned} & |\nabla_v \phi|(x) = |N_0(v)| = |\mathbb{E} N_{t \wedge \tau_D}(v)| \\ & = \left| \mathbb{E} \left[ \mathbf{1}_{\{t > \tau_D\}} e^{\lambda \tau_D / 2} h_{\tau_D} \langle \nabla \phi(X_{\tau_D}), Q_{\tau_D} v \rangle - \mathbf{1}_{\{t \leq \tau_D\}} e^{\lambda t / 2} \phi(X_t) \int_0^t \langle \dot{h}_s Q_s v, u_s dB_s \rangle \right] \right|. \end{aligned}$$

Note that using (2.20) along with Lemma 2.3 we may estimate

$$\left| \mathbb{E} \left[ \mathbf{1}_{\{t > \tau_D\}} e^{\lambda \tau_D / 2} h_{\tau_D} \langle \nabla \phi(X_{\tau_D}), Q_{\tau_D} v \rangle \right] \right|$$

$$\begin{aligned}
&\leq \mathbb{E} \left[ 1_{\{t > \tau_D\}} e^{\lambda \tau_D / 2} |h_{\tau_D}| \|\nabla \phi\|_{\partial D, \infty} e^{K_V \tau_D / 2} |v| \right] \\
&\leq \mathbb{E} \left[ 1_{\{t > \tau_D\}} e^{\lambda \tau_D / 2} |h_{\tau_D}| \|\phi\|_{\infty} \|\nabla \psi(t - \tau_D, \cdot)\|_{\partial D, \infty} e^{\lambda(t - \tau_D) / 2} e^{K_V \tau_D / 2} |v| \right] \\
&= \mathbb{E} \left[ 1_{\{t > \tau_D\}} |h_{\tau_D}| \|\phi\|_{\infty} \|\nabla \psi(t - \tau_D, \cdot)\|_{\partial D, \infty} e^{\lambda t / 2} e^{K_V \tau_D / 2} |v| \right] \\
&\leq e^{(\lambda + K_V^+) t / 2} \|\phi\|_{\infty} \mathbb{E} \left[ 1_{\{t > \tau_D\}} |h_{\tau_D}| \|\nabla \psi(t - \tau_D, \cdot)\|_{\partial D, \infty} |v| \right],
\end{aligned}$$

as well as

$$\mathbb{E} \left[ 1_{\{t \leq \tau_D\}} e^{\lambda t / 2} \phi(X_t) \int_0^t \langle \dot{h}_s Q_s v, u_s dB_s \rangle \right] \leq e^{\lambda t / 2} \|\phi\|_{\infty} \mathbb{P}\{t \leq \tau_D\}^{1/2} \left( \int_0^t |\dot{h}_s|^2 e^{K_V s} ds \right)^{1/2}.$$

Taking

$$h_s = \frac{t - s}{t}, \quad s \in [0, t],$$

we obtain thus from (2.46)

$$\begin{aligned}
|\nabla \phi(x)| &\leq \frac{e^{(\lambda + K_V^+) t / 2}}{t} \|\phi\|_{\infty} \mathbb{E} \left[ 1_{\{t > \tau_D\}} (t - \tau_D) \|\nabla \psi(t - \tau_D, \cdot)\|_{\partial D, \infty} \right] \\
&\quad + e^{\lambda t / 2} \|\phi\|_{\infty} \mathbb{P}\{t \leq \tau_D\}^{1/2} \frac{1}{t} \left( \frac{e^{K_V^+ t} - 1}{K_V^+} \right)^{1/2}.
\end{aligned}$$

Note that

$$\frac{e^{K_V^+ t} - 1}{K_V^+} \leq t e^{K_V^+ t}.$$

(i) By (2.29), assuming that  $\alpha \geq 0$ , we have on  $\{t > \tau_D\}$ :

$$\begin{aligned}
\frac{t - \tau_D}{t} \|\nabla \psi(t - \tau_D, \cdot)\|_{\partial D, \infty} &\leq \alpha \frac{t - \tau_D}{t} + \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sqrt{t - \tau_D}}{t} + \frac{t - \tau_D}{t} \int_0^{t - \tau_D} \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds \\
&\leq \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds \\
&\leq \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\alpha^2 t}{2}} + \min \left\{ \alpha, \frac{\alpha^2 \sqrt{2t}}{\sqrt{\pi}} \right\}.
\end{aligned}$$

Thus, letting  $\varepsilon = \mathbb{P}(t > \tau_D)$ , we obtain

$$|\nabla \phi(x)| \leq e^{(\lambda + K_V^+) t / 2} \|\phi\|_{\infty} \left[ \varepsilon \left( \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\alpha^2 t}{2}} + \min \left\{ \alpha, \frac{\alpha^2 \sqrt{2t}}{\sqrt{\pi}} \right\} \right) + \sqrt{\frac{1 - \varepsilon}{t}} \right].$$

(ii) Still under the assumption  $\alpha \geq 0$ , this time using estimate (2.30), we have on  $\{t > \tau_D\}$ :

$$\|\nabla \psi(t - \tau_D, \cdot)\|_{\partial D, \infty} \leq \frac{\sqrt{2}}{\sqrt{\pi(t - \tau_D)}} + \alpha + \frac{\sqrt{t - \tau_D}}{\sqrt{2\pi}} \alpha^2,$$

and thus letting  $\varepsilon = \mathbb{P}(t > \tau_D)$ , we get

$$|\nabla \phi(x)| \leq \frac{e^{(\lambda + K_V^+) t / 2}}{t} \|\phi\|_{\infty} \mathbb{E} \left[ 1_{\{t > \tau_D\}} \left( \sqrt{\frac{2}{\pi}} \sqrt{t - \tau_D} + \alpha(t - \tau_D) + \frac{(t - \tau_D)^{3/2}}{\sqrt{2\pi}} \alpha^2 \right) \right]$$

$$\begin{aligned}
& + e^{\lambda t/2} \|\phi\|_\infty \mathbb{P}\{t \leq \tau_D\}^{1/2} \frac{1}{t} \left( \frac{e^{K_V^+ t} - 1}{K_V^+} \right)^{1/2} \\
& \leq e^{(\lambda + K_V^+)t/2} \|\phi\|_\infty \left[ \varepsilon \left( \sqrt{\frac{2}{\pi t}} + \alpha + \frac{\sqrt{t}}{\sqrt{2\pi}} \alpha^2 \right) + \sqrt{\frac{1-\varepsilon}{t}} \right].
\end{aligned}$$

(iii) In the case  $\alpha \leq 0$ , we get from (2.29) in a similar way:

$$|\nabla\phi(x)| \leq e^{(\lambda + K_V^+)t/2} \|\phi\|_\infty \left\{ \varepsilon \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{\alpha^2 t}{2}} + \sqrt{\frac{1-\varepsilon}{t}} \right\}.$$

This concludes the proof of Lemma 2.5.  $\square$

*Proposition 2.3.* We keep the assumptions of Lemma 2.5.

(a) If  $\alpha \geq 0$ , then for any  $(\phi, \lambda) \in \text{Eig}(L)$ ,

$$\begin{aligned}
\|\nabla\phi\|_\infty & \leq \sqrt{e} \max_{\varepsilon \in [0,1]} \left\{ \varepsilon \left( \alpha + \frac{\sqrt{2(\lambda + K_V^+)}}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{2(\lambda + K_V^+)}\right) + \min\left(|\alpha|, \frac{\sqrt{2}\alpha^2}{\sqrt{\pi(\lambda + K_V^+)}}\right) \right) \right. \\
& \quad \left. + \sqrt{1-\varepsilon} \sqrt{(\lambda + K_V^+)} \right\} \|\phi\|_\infty,
\end{aligned}$$

as well as

$$\|\nabla\phi\|_\infty \leq \sqrt{e} \max_{\varepsilon \in [0,1]} \left\{ \varepsilon \left( \alpha + \frac{\sqrt{2(\lambda + K_V^+)}}{\sqrt{\pi}} + \frac{\alpha^2}{\sqrt{2\pi(\lambda + K_V^+)}} \right) + \sqrt{1-\varepsilon} \sqrt{(\lambda + K_V^+)} \right\} \|\phi\|_\infty$$

and

$$\|\nabla\phi\|_\infty \leq \sqrt{e} \max_{\varepsilon \in [0,1]} \left\{ \varepsilon \left( 2\alpha + \frac{\sqrt{2(\lambda + K_V^+)}}{\sqrt{\pi}} \right) + \sqrt{1-\varepsilon} \sqrt{(\lambda + K_V^+)} \right\} \|\phi\|_\infty$$

(b) If  $\alpha \leq 0$ , then

$$\|\nabla\phi\|_\infty \leq \sqrt{e} \max_{\varepsilon \in [0,1]} \left\{ \varepsilon \frac{\sqrt{2(\lambda + K_V^+)}}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{2(\lambda + K_V^+)}\right) + \sqrt{1-\varepsilon} \sqrt{(\lambda + K_V^+)} \right\} \|\phi\|_\infty.$$

*Proof.* Take  $t = 1/(\lambda + K_V^+)$  in Lemma 2.5.  $\square$

We are now ready to complete the proof of Theorem 2.2.

*Proof of Theorem 2.2.* The claims of Theorem 2.2 (with the exception of estimate (2.15)) follow directly from the inequalities in Proposition 2.3 together with the fact that for any  $A, B \geq 0$ ,

$$(2.47) \quad \max_{\varepsilon \in [0,1]} \{\varepsilon A + \sqrt{1-\varepsilon} B\} = B \mathbf{1}_{\{B > 2A\}} + \left( A + \frac{B^2}{4A} \right) \mathbf{1}_{\{B \leq 2A\}}. \quad \square$$

Finally, to check (2.15) we may go back to (2.24) from where we have

$$\|\nabla\phi\|_\infty \leq \varepsilon e^{(\lambda + K_V^+)t/2} \|\nabla\phi\|_{\partial D, \infty} + \sqrt{1-\varepsilon} e^{\lambda t/2} \|\phi\|_\infty \left( \int_0^t |\dot{h}_s|^2 e^{K_V s} ds \right)^{1/2}.$$

Taking

$$h_s = \frac{e^{-K_V t} - e^{-K_V s}}{e^{-K_V t} - 1}, \quad s \in [0, t],$$

we obtain

$$\|\nabla\phi\|_\infty \leq \inf_{t>0} \max_{\varepsilon \in [0,1]} \left\{ \varepsilon e^{(\lambda+K_V)t/2} \|\nabla\phi\|_{\partial D, \infty} + \|\phi\|_\infty e^{\lambda t/2} \sqrt{1-\varepsilon} \left( \frac{K_V}{1-e^{-K_V t}} \right)^{1/2} \right\}.$$

Noting that

$$\frac{K_V}{1-e^{-K_V t}} \leq \frac{K_V^+}{1-e^{-K_V^+ t}} \leq t^{-1} e^{K_V^+ t},$$

and taking  $t = (K_V^+ + \lambda)^{-1}$  we obtain

$$\|\nabla\phi\|_\infty \leq \sqrt{e} \max_{\varepsilon \in [0,1]} \left\{ \varepsilon \|\nabla\phi\|_{\partial D, \infty} + \sqrt{(1-\varepsilon)(\lambda + K_V^+)} \|\phi\|_\infty \right\}.$$

Applying Lemma 2.3 and Proposition 2.4 with  $t = 1/\lambda$ , we arrive at

$$\|\nabla\phi\|_\infty \leq \|\phi\|_\infty \max_{\varepsilon \in [0,1]} \left\{ e\varepsilon \left( \alpha + \frac{\sqrt{2\lambda}}{\sqrt{\pi}} e^{-\frac{\alpha^2}{2\lambda}} + |\alpha| \wedge \frac{\alpha^2 \sqrt{2}}{\sqrt{\pi\lambda}} \right) + \sqrt{e(1-\varepsilon)(\lambda + K_V^+)} \right\}.$$

The proof is then finished as above with observation (2.47).  $\square$

### 3 Proof of Theorem 1.2

As in Section 2, we consider  $L = \Delta + \nabla V$  and let  $\text{Eig}_N(L)$  be the set of corresponding non-trivial eigenpairs for the Neumann problem of  $L$ . We also allow  $\partial D = \emptyset$ , then we consider the eigenproblem without boundary. We first consider the convex case, then extend to the general situation. In this section,  $P_t$  denotes the (Neumann if  $\partial D \neq \emptyset$ ) semigroup generated by  $L/2$  on  $D$ . Let  $X_t$  be the corresponding (reflecting) diffusion process which solves the SDE

$$(3.1) \quad dX_t = u_t \circ dB_t + \frac{1}{2} \nabla V(X_t) dt + N(X_t) d\ell_t,$$

where  $B_t$  is a  $d$ -dimensional Euclidean Brownian motion,  $u_t$  the horizontal lift of  $X_t$  onto the orthonormal frame bundle, and  $\ell_t$  the local time of  $X_t$  on  $\partial D$ .

We will apply the following Bismut type formula for the Neumann semigroup  $P_t$ , see [15, Theorem 3.2.1], where the multiplicative functional process  $Q_s$  was introduced in [4].

**Theorem 3.1** ([15]). *Let  $\text{Ric}_D^V \geq -K_V$  and  $\mathbb{I}_{\partial D} \geq -\delta$  for some  $K_V \in C(\bar{D})$  and  $\delta \in C(\partial D)$ . Then there exists a  $\mathbb{R}^d \otimes \mathbb{R}^d$ -valued adapted continuous process  $Q_s$  with*

$$(3.2) \quad \|Q_t\| \leq \exp \left( \frac{1}{2} \int_0^t K_V(X_s) ds + \int_0^t \delta(X_s) d\ell_s \right), \quad s \geq 0,$$

such that for any  $t > 0$  and  $h \in C^1([0, t])$  with  $h(0) = 0$ ,  $h(t) = 1$ , there holds

$$(3.3) \quad \nabla P_t f = \mathbb{E} \left[ f(X_t) \int_0^t h'(s) Q_s dB_s \right], \quad f \in \mathcal{B}_b(D).$$



### 3.1 The case with convex or empty boundary

In this part we assume that  $\partial D$  is either convex or empty. When  $\partial D$  is empty,  $D$  is a Riemannian manifold without boundary and  $\text{Eig}_N(L)$  denotes the set of eigenpairs for the eigenproblem without boundary. In this case, if  $\text{Ric}^V \geq K_V$  for some constant  $K_V \in \mathbb{R}$ , then  $\lambda + K_V \geq 0$  for  $(\phi, \lambda) \in \text{Eig}_N(L)$ , see for instance [8].

**Theorem 3.2.** *Assume that  $\partial D$  is either convex or empty.*

(1) *If the curvature-dimension condition (2.1) holds, then for any  $(\phi, \lambda) \in \text{Eig}_N(L)$ ,*

$$\|\nabla\phi\|_\infty^2 \geq \frac{\lambda^2 \|\phi\|_\infty^2}{n(\lambda + K)} \left( \frac{\lambda}{\lambda + K} \right)^{\lambda/K} \geq \frac{\lambda^2 \|\phi\|_\infty^2}{ne(\lambda + K^+)}.$$

(2) *If  $\text{Ric}_D^V \geq -K_V$  for some constant  $K_V \in \mathbb{R}$ , then for any  $(\phi, \lambda) \in \text{Eig}_N(L)$ ,*

$$\frac{\|\nabla\phi\|_\infty^2}{\|\phi\|_\infty^2} \leq \frac{2(\lambda + K_V)}{\pi} \left( 1 + \frac{K_V}{\lambda} \right)^{\lambda/K_V} \leq \frac{2e(\lambda + K_V^+)}{\pi}.$$

*Proof.* (a) We start establishing the lower bound estimate. By Itô's formula, for any  $(\phi, \lambda) \in \text{Eig}_N(L)$  we have

$$(3.4) \quad d|\nabla\phi|^2(X_t) = \frac{1}{2}L|\nabla\phi|^2(X_t) dt + 2\mathbb{I}_{\partial D}(\nabla\phi, \nabla\phi)(X_t) d\ell_t + dM_t, \quad t \geq 0,$$

where  $\ell_t$  is the local time of  $X_t$  at  $\partial D$ , which is an increasing process. Since  $\mathbb{I}_{\partial D} \geq 0$ , and since (2.1) and  $L\phi = -\lambda\phi$  imply

$$\frac{1}{2}L|\nabla\phi|^2 \geq -(K + \lambda)|\nabla\phi|^2 + \frac{\lambda^2}{n}\phi^2,$$

we obtain

$$d|\nabla\phi|^2(X_t) \geq \left( \frac{\lambda^2}{n}\phi^2 - (\lambda + K)|\nabla\phi|^2 \right)(X_t) dt + dM_t, \quad t \geq 0.$$

Noting that for  $X_0 = x \in D$  we have

$$\mathbb{E}[\phi(X_s)^2] \geq (\mathbb{E}[\phi(X_s)])^2 = e^{-\lambda s} \phi(x)^2,$$

we arrive at

$$\begin{aligned} e^{(\lambda+K)t} \|\nabla\phi\|_\infty^2 &\geq e^{(\lambda+K)t} \mathbb{E}[|\nabla\phi|^2(X_t)] \geq \frac{\lambda^2}{n} \int_0^t e^{(\lambda+K)s} \mathbb{E}[\phi^2(X_s)] ds \\ &\geq \frac{\lambda^2}{n} \int_0^t e^{Ks} \phi(x)^2 ds = \frac{\lambda^2(e^{Kt} - 1)}{nK} \phi(x)^2. \end{aligned}$$

Multiplying by  $e^{-(\lambda+K)t}$ , choosing  $t = \frac{1}{K} \log(1 + \frac{K}{\lambda})$  (noting that  $\lambda + K \geq 0$ , in case  $\lambda + K = 0$  taking  $t \rightarrow \infty$ ), and taking the supremum over  $x \in D$ , we finish the proof of (1).

(b) Let  $\partial D$  be convex and  $\text{Ric}_D^V \geq -K_V$  for some constant  $K_V$ . Then Theorem 3.1 holds for  $\delta = 0$ , so that

$$\sigma_t := \left( \mathbb{E} \int_0^t |h'(s)|^2 \|Q_s\|^2 ds \right)^{1/2} \leq \left( \int_0^t |h'(s)|^2 e^{K_V s} ds \right)^{1/2}.$$

Taking

$$h(s) = \frac{\int_0^s e^{-K_V r} dr}{\int_0^t e^{-K_V r} dr}$$

we obtain

$$\sigma_t \leq \left( \frac{K_V}{1 - e^{-K_V t}} \right)^{1/2}.$$

Therefore,

$$\begin{aligned} \|\nabla P_t f\|_\infty &\leq \|f\|_\infty \mathbb{E} \left| \int_0^t h'(s) Q_s dB_s \right| \\ (3.5) \quad &\leq \|f\|_\infty \frac{2}{\sqrt{2\pi} \sigma_t} \int_0^\infty s \exp\left(-\frac{s^2}{2\sigma_t^2}\right) ds \\ &= \|f\|_\infty \frac{\sigma_t \sqrt{2}}{\sqrt{\pi}}, \quad t > 0, f \in \mathcal{B}_b(D). \end{aligned}$$

Applying this to  $(\phi, \lambda) \in \text{Eig}_N(L)$ , we obtain

$$e^{-\lambda t/2} |\nabla \phi| \leq \|\phi\|_\infty \frac{\sigma_t \sqrt{2}}{\sqrt{\pi}} \leq \|\phi\|_\infty \left( \frac{2K_V}{\pi(1 - e^{-2K_V t})} \right)^{1/2}, \quad t > 0.$$

Consequently,  $\lambda + K_V \geq 0$ . Taking  $t = \frac{1}{K_V} \log(1 + \frac{K_V}{\lambda})$  as above, we arrive at

$$\frac{\|\nabla \phi\|_\infty^2}{\|\phi\|_\infty^2} \leq \frac{2(\lambda + K_V)}{\pi} \left(1 + \frac{K_V}{\lambda}\right)^{\lambda/K_V}. \quad \square$$

### 3.2 The non-convex case

When  $\partial D$  is non-convex, a conformal change of metric may be performed to make  $\partial M$  convex under the new metric; this strategy has been used in [2, 12, 13, 14] for the study of functional inequalities on non-convex manifolds. According to [15, Theorem 1.2.5], for a strictly positive function  $f \in C^\infty(\bar{D})$  with  $\mathbb{I}_{\partial D} + N \log f|_{\partial D} \geq 0$ , the boundary  $\partial D$  is convex under the metric  $f^{-2}\langle \cdot, \cdot \rangle$ . For simplicity, we will assume that  $f \geq 1$ . Hence, we take as class of reference functions

$$\mathcal{D} := \left\{ f \in C^2(\bar{D}) : \inf f = 1, \mathbb{I}_{\partial D} + N \log f \geq 0 \right\}.$$

Assume (2.1) and  $\text{Ric}_D^V \geq -K_V$  for some constants  $n \geq d$  and  $K, K_V \in \mathbb{R}$ . For any  $f \in \mathcal{D}$  and  $\varepsilon \in (0, 1)$ , define

$$c_\varepsilon(f) := \sup_D \left\{ \frac{4\varepsilon |\nabla \log f|^2}{1 - \varepsilon} + \varepsilon K + (1 - \varepsilon)K_V - 2L \log f \right\}.$$

We let  $\lambda_1^N$  be the smallest non-trivial Neumann eigenvalue of  $-L$ . The following result implies  $\lambda_1 \geq -c_\varepsilon(f)$ .

**Theorem 3.3.** *Let  $f \in \mathcal{D}$ .*

(1) *If (2.1) and  $\text{Ric}_D^V \geq -K_V$  hold for some constants  $n \geq d$  and  $K, K_V \in \mathbb{R}$ . Then for any non-trivial  $(\phi, \lambda) \in \text{Eig}_N(L)$ , we have  $\lambda + c_\varepsilon(f) \geq 0$  and*

$$\frac{\|f\|_\infty^2 \|\nabla \phi\|_\infty^2}{\|\phi\|_\infty^2} \geq \sup_{\varepsilon \in (0,1)} \frac{\varepsilon \lambda^2}{n(\lambda + c_\varepsilon(f))} \left( \frac{\lambda}{\lambda + c_\varepsilon(f)} \right)^{\lambda/c_\varepsilon(f)} \geq \sup_{\varepsilon \in (0,1)} \frac{\varepsilon \lambda^2}{n\varepsilon(\lambda + c_\varepsilon(f)^+)}.$$

(2) Let  $\text{Ric}_D^V \geq -K_V$  for some  $K_V \in C(\bar{D})$ , and

$$K(f) = \sup_D \{2|\nabla \log f|^2 + K_V - L \log f\}.$$

Then for any non-trivial  $(\phi, \lambda) \in \text{Eig}_N(L)$ , we have  $\lambda + K(f) \geq 0$  and

$$\frac{\|\nabla \phi\|_\infty^2}{\|\phi\|_\infty^2 \|f\|_\infty^2} \leq \frac{2(\lambda + K(f))}{\pi} \left(1 + \frac{K(f)}{\lambda}\right)^{\lambda/K(f)} \leq \frac{2e(\lambda + K(f)^+)}{\pi}.$$

*Proof.* Let  $f \in \mathcal{D}$  and  $(\phi, \lambda) \in \text{Eig}_N(L)$ .

(1) On  $\partial D$  we have

$$\begin{aligned} N(f^2|\nabla \phi|^2) &= (Nf^2)|\nabla \phi|^2 + f^2N|\nabla \phi|^2 \\ &= f^2((N \log f^2)|\nabla \phi|^2 + 2\mathbb{I}_{\partial D}(\nabla \phi, \nabla \phi)) \\ (3.6) \quad &= 2f^2((N \log f)|\nabla \phi|^2 + \mathbb{I}_{\partial D}(\nabla \phi, \nabla \phi)) \geq 0. \end{aligned}$$

Next, by the Bochner-Weitzenböck formula, using that  $\text{Ric}_D^V \geq -K_V$  and  $L\phi = -\lambda\phi$ , we observe

$$\begin{aligned} \frac{1}{2}L|\nabla \phi|^2 &= \frac{1}{2}L|\nabla \phi|^2 - \langle \nabla L\phi, \nabla \phi \rangle - \lambda|\nabla \phi|^2 \\ &\geq \|\text{Hess}_\phi\|_{\text{HS}}^2 - (K_V + \lambda)|\nabla \phi|^2. \end{aligned}$$

Combining this with (2.5), for any  $\varepsilon \in (0, 1)$ , we obtain

$$\begin{aligned} &\frac{f^2}{2}L|\nabla \phi|^2 + \langle \nabla f^2, \nabla |\nabla \phi|^2 \rangle \\ &\geq -f^2(\varepsilon K + (1 - \varepsilon)K_V + \lambda)|\nabla \phi|^2 + \frac{\varepsilon\lambda^2}{n}f^2\phi^2 \\ &\quad + (1 - \varepsilon)f^2\|\text{Hess}_\phi\|_{\text{HS}}^2 - 2\|\text{Hess}_\phi\|_{\text{HS}} \times |\nabla f^2| \times |\nabla \phi| \\ &\geq -\left\{\frac{|\nabla \log f^2|^2}{1 - \varepsilon} + \varepsilon K + (1 - \varepsilon)K_V + \lambda\right\}f^2|\nabla \phi|^2 + \frac{\varepsilon\lambda^2}{n}f^2\phi^2. \end{aligned}$$

Combining this with (3.6) and applying Itô's formula, we obtain

$$\begin{aligned} d(f^2|\nabla \phi|^2)(X_t) &\stackrel{\text{m}}{=} \frac{1}{2}L(f^2|\nabla \phi|^2)(X_t) dt + N(f^2|\nabla \phi|^2)(X_t) d\ell_t \\ &\geq -\frac{1}{2}\left(f^2L|\nabla \phi|^2 + 2\langle \nabla f^2, \nabla |\nabla \phi|^2 \rangle + |\nabla \phi|^2Lf^2\right)(X_t) dt \\ &\geq \left\{\frac{\varepsilon\lambda^2}{n}f^2\phi^2 - \left(\frac{|\nabla \log f^2|^2}{1 - \varepsilon} + \varepsilon K + (1 - \varepsilon)K_V + \lambda - f^{-2}Lf^2\right)f^2|\nabla \phi|^2\right\}(X_t) dt \\ &\geq \left(\frac{\varepsilon\lambda^2}{n}\phi^2 - (\lambda + c_\varepsilon(f))f^2|\nabla \phi|^2\right)(X_t) dt. \end{aligned}$$

Hence, for  $X_0 = x \in D$ ,

$$\begin{aligned} \|f\|_\infty^2 \|\nabla \phi\|_\infty^2 e^{(\lambda + c_\varepsilon(f))t} &\geq \mathbb{E}\left[e^{c_\varepsilon(f)t}(f^2|\nabla \phi|^2)(X_t)\right] \\ &\geq \frac{\varepsilon\lambda^2}{n} \int_0^t e^{(\lambda + c_\varepsilon(f))s} \mathbb{E}[\phi(X_s)^2] ds \end{aligned}$$

$$\begin{aligned}
&\geq \frac{\varepsilon\lambda^2}{n} \int_0^t e^{c_\varepsilon(f)s} \phi(x)^2 ds \\
&= \frac{\varepsilon\lambda^2(e^{c_\varepsilon(f)t} - 1)}{nc_\varepsilon(f)} \phi(x)^2.
\end{aligned}$$

This implies  $\lambda + c_\varepsilon(f) \geq 0$  and

$$\begin{aligned}
\frac{\|f\|_\infty^2 \|\nabla\phi\|_\infty^2}{\|\phi\|_\infty^2} &\geq \sup_{t>0} \frac{\varepsilon\lambda^2 (e^{-\lambda t} - e^{-(\lambda+c_\varepsilon(f))t})}{nc_\varepsilon(f)} \\
&= \frac{\varepsilon\lambda^2}{n(\lambda + c_\varepsilon(f))} \left( \frac{\lambda}{\lambda + c_\varepsilon(f)} \right)^{\lambda/c_\varepsilon(f)} \geq \frac{\varepsilon\lambda^2}{ne(\lambda + c_\varepsilon(f)^+)}.
\end{aligned}$$

(2) The claim could be derived from [2, inequality (2.12)]. For the sake of completeness we include a sketch of the proof. For any  $p > 1$ , let

$$K_p(f) = \sup_D \{K_V + p|\nabla \log f|^2 - L \log f\}.$$

Note that  $p|\nabla \log f|^2 - L \log f = p^{-1}f^p Lf^{-p}$ . Since  $f \in \mathcal{D}$  implies  $\mathbb{I}_{\partial D} \geq -N \log f$ , we have

$$\begin{aligned}
\|Q_t\|^2 &\leq \exp\left(\int_0^t K_V(X_s) ds + 2 \int_0^t N \log f(X_s) d\ell_s\right) \\
&\leq \exp(K_p(f)t) \exp\left(-\frac{1}{p} \int_0^t (f^p Lf^{-p})(X_s) ds + 2 \int_0^t N \log f(X_s) d\ell_s\right).
\end{aligned}$$

As

$$\begin{aligned}
df^{-p}(X_t) &\stackrel{m}{=} \frac{1}{2} Lf^{-p}(X_t) dt + Nf^{-p}(X_t) d\ell_t \\
&= -f^{-p}(X_t) \left( -\frac{1}{2} f^p Lf^{-p}(X_t) dt + pN \log f(X_t) d\ell_t \right),
\end{aligned}$$

we obtain that

$$M_t := f^{-p}(X_t) \exp\left(-\frac{1}{2} \int_0^t f^p(X_s) Lf^{-p}(X_s) ds + p \int_0^t N \log f(X_s) d\ell_s\right)$$

is a (local) martingale. Proceeding as in the proof of [15, Corollary 3.2.8] or [2, Theorem 2.4], we get

$$\begin{aligned}
&\|f\|_\infty^{-p} \mathbb{E} \left[ \exp\left(-\frac{1}{2} \int_0^t f^p(X_s) Lf^{-p}(X_s) ds + p \int_0^t N \log f(X_s) d\ell_s\right) \right] \\
&\leq \mathbb{E} \left[ f^{-p}(X_t) \exp\left(-\frac{1}{2} \int_0^t f^p(X_s) Lf^{-p}(X_s) ds + p \int_0^t N \log f(X_s) d\ell_s\right) \right] \\
&= f^{-p}(x) \leq 1,
\end{aligned}$$

since  $f \geq 1$  by assumption. This shows that

$$\|Q_t\|^2 \leq e^{pK_p(f)t} \|f\|_\infty^p, \quad t \geq 0.$$

Combining this for  $p = 2$  with Theorem 3.1 and denoting  $K(f) = K_2(f)$ , we obtain

$$\sigma_t^2 := \mathbb{E} \int_0^t |h'(s)|^2 \|Q_s\|^2 ds \leq \|f\|_\infty^2 \int_0^t |h'(s)|^2 e^{K(f)s} ds.$$

Therefore, repeating step (b) in the proof of Theorem 3.2 with  $K(f)$  replacing  $K_V$ , we finish the proof of (2).  $\square$

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