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A conjecture which implies that there are infinitely many primes of the form $n! + 1$

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Abstract

Let $f(6) = 720$, and let $f(n+1) = f(n)!$ for every integer $n \geq 6$. For an integer $n \geq 6$, let $\Lambda_n$ denote the following statement: if a system $S \subseteq \{x_i = x_j : 1 \leq i < j \leq n\} \cup \{x_i \cdot x_j = x_{j+1} : 1 \leq i < j \leq n-1\}$ has at most finitely many solutions in integers $x_1, \ldots, x_n$ greater than 3, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq f(n)$. We conjecture that the statements $\Lambda_6, \ldots, \Lambda_9$ are true. We prove that the statement $\Lambda_9$ implies that there are infinitely many primes of the form $n! + 1$.

Key words and phrases: prime numbers of the form $n! + 1$.

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It is conjectured that $n! + 1$ is prime for infinitely many positive integers $n$, see [3]. In this note, we propose a conjecture which implies this.

Lemma 1. For every integers $x$ and $y$ greater than 1, $x! \cdot y = y!$ if and only if $x + 1 = y$.

For an integer $n \geq 6$, let $V_n$ denote the following system of equations:

\[
\begin{align*}
\forall i \in \{1, \ldots, n-1\} \setminus \{1, 3, 5\} & \quad x_i! = x_{i+1} \\
& \quad x_1! = x_6 \\
x_1 \cdot x_2 & = x_3 \\
x_3 \cdot x_4 & = x_5 \\
x_1 \cdot x_5 & = x_6 \\
\end{align*}
\]

The diagram in Figure 1 illustrates the construction of the system $V_n$.

![Diagram of system V_n](image)

Fig. 1  Construction of the system $V_n$
Let \( f(6) = 720 \), and let \( f(n + 1) = f(n)! \) for every integer \( n \geq 6 \).

**Lemma 2.** For every integer \( n \geq 6 \), the system \( \mathcal{V}_n \) has exactly one solution in integers greater than 3, namely \( (6, 4, 24, 5, 120, f(6), \ldots, f(n)) \).

**Proof.** By Lemma 1, \( x_2 + 2 = x_1 \). Hence, \( (x_2 + 2) \cdot x_2 = x_1 \cdot x_2 = x_3 = x_2! \). Therefore, \( x_2 = 4 \). The rest of the proof follows from the diagram in Figure 1. \(\square\)

Let
\[
T_n = \{ x_i! = x_j : 1 \leq i < j \leq n \} \cup \{ x_i \cdot x_j = x_{j+1} : 1 \leq i < j \leq n - 1 \}
\]
For an integer \( n \geq 6 \), let \( \Lambda_n \) denote the following statement: if a system \( S \subseteq T_n \) has at most finitely many solutions in integers \( x_1, \ldots, x_n \) greater than 3, then each such solution \( (x_1, \ldots, x_n) \) satisfies \( x_1, \ldots, x_n \leq f(n) \).

**Conjecture.** The statements \( \Lambda_6, \ldots, \Lambda_9 \) are true.

We present a heuristic reasoning that leads to the Conjecture. For every integer \( n \in \{6, 7, 8, 9\} \), we consider all subsystems of the system \( T_n \). We conjecture that the largest known solution is indeed the largest possible.

**Theorem 1.** For every statement \( \Lambda_n \), the bound \( f(n) \) cannot be decreased.

**Proof.** It follows from Lemma 2 because \( \mathcal{V}_n \subseteq T_n \). \(\square\)

**Theorem 2.** Every statement \( \Lambda_n \) holds true with an integer bound that depends on \( n \).

**Proof.** Indeed, for every integer \( n \geq 6 \), the system \( T_n \) has a finite number of subsystems. \(\square\)

**Lemma 3.** If a prime number \( x \) is greater than 3, then \( \frac{(x - 1)! + 1}{x} > 3 \).

**Lemma 4.** (Wilson’s theorem, [4] p. 89). For every integer \( x \geq 2 \), \( x \) is prime if and only if \( x \) divides \( (x - 1)! + 1 \).

Let \( \mathcal{A} \) denote the following system of equations:
\[
\begin{align*}
    x_1! &= x_2 \\
    x_2! &= x_3 \\
    x_3! &= x_4 \\
    x_4! &= x_5 \\
    x_5! &= x_6 \\
    x_6! &= x_7 \\
    x_7 \cdot x_5 &= x_8 \\
    x_8 \cdot x_7 &= x_9
\end{align*}
\]

Lemma 1 and the diagram in Figure 2 explain the construction of the system \( \mathcal{A} \).
Lemma 5. For every integer \( x_1 > 3 \), the system \( A \) is solvable in integers \( x_2, \ldots, x_9 \) greater than 3 if and only if \( x_1! + 1 \) is prime. In this case, the integers \( x_2, \ldots, x_9 \) are uniquely determined by the following equalities:

\[
\begin{align*}
  x_2 & = x_1! \\
  x_3 & = (x_1!)! \\
  x_4 & = ((x_1!)!)! \\
  x_5 & = x_1! + 1 \\
  x_6 & = (x_1!) + 1 \\
  x_7 & = \frac{(x_1!)! + 1}{x_1! + 1} \\
  x_8 & = (x_1!)! + 1 \\
  x_9 & = ((x_1!)! + 1)!
\end{align*}
\]

**Proof.** By Lemmas 1 and 3, for every integer \( x_1 > 3 \), the system \( A \) is solvable in integers \( x_2, \ldots, x_9 \) greater than 3 if and only if \( x_1! + 1 \) divides \( (x_1!)! + 1 \). Hence, the claim of Lemma 5 follows from Lemma 4. \( \square \)

**Theorem 3.** The statement \( \Lambda_9 \) implies that there are infinitely many primes of the form \( n! + 1 \).

**Proof.** Harvey Dubner proved that \( 872! + 1 \) is prime, see [2], [3], [5, p. 7], and [7]. Let \( x_1 = 872 \). By Lemma 5 there exists a unique tuple \( (x_2, \ldots, x_9) \) of integers greater than 3 such that the tuple \( (x_1, x_2, \ldots, x_9) \) solves the system \( A \). Hence,

\[
x_9 = ((x_1!)! + 1) > ((720!)! + 1) > ((720!)!)! = ((6!)!)! = (9)
\]

The statement \( \Lambda_9 \) and the inequality \( x_9 > f(9) \) imply that the system \( A \) has infinitely many solutions in integers \( x_1, \ldots, x_9 \) greater than 3. This conclusion and Lemma 5 imply that \( x_1! + 1 \) is prime for infinitely many integers \( x_1 > 3 \). \( \square \)

Let \( C \) denote the following system of equations:

\[
\begin{align*}
  x_1! & = x_4 \\
  x_2! & = x_5 \\
  x_3! & = x_6 \\
  x_2 \cdot x_3 & = x_4 \\
  x_3 \cdot x_5 & = x_6
\end{align*}
\]

Lemma 1 and the diagram in Figure 3 explain the construction of the system \( C \).
Lemma 6. For every integers $x_1, x_2$ greater than 3, the system $C$ is solvable in integers $x_3, x_4, x_5, x_6$ greater than 3 if and only if $x_1! = x_2(x_2 + 1)$. In this case, the integers $x_3, x_4, x_5, x_6$ are uniquely determined by the following equalities:

\[
\begin{align*}
    x_3 &= x_2 + 1 \\
    x_4 &= x_1! \\
    x_5 &= x_2! \\
    x_6 &= (x_2 + 1)!
\end{align*}
\]

Theorem 4. If the equation $x_1! = x_2(x_2 + 1)$ has at most finitely many solutions in positive integers, then the statement $\Lambda_6$ guarantees that each such solution $(x_1, x_2)$ belongs to the set $\{(2, 1), (3, 2)\}$.

Proof. Suppose that the antecedent holds. Then, the equation $x_1! = x_2(x_2 + 1)$ has at most finitely many solutions $(x_1, x_2) \in (\mathbb{N} \setminus \{0, 1, 2, 3\})^2$. By Lemma 6 the system $C$ is solvable in integers $x_3, x_4, x_5, x_6$ greater than 3. Since $C \subseteq T_6$, the statement $\Lambda_6$ implies that $x_1! = x_4 \leq f(6) = 720 = 6!$. Hence, $x_1 \in \{1, 2, 3, 4, 5, 6\}$. For every integer $x_1 \in \{1, 2, 3, 4, 5, 6\}$, $x_1!$ is a product of two consecutive integers if and only if $x_1 \in \{2, 3\}$. \hfill $\Box$

The question of solving the equation $y! = x(x + 1)$ was posed by P. Erdős, see [1]. F. Luca proved that the $abc$ conjecture implies that the equation $y! = x(x + 1)$ has only finitely many solutions in positive integers, see [6].

References


[3] C. K. Caldwell and Y. Gallot, On the primality of $n! \pm 1$ and $2 \times 3 \times 5 \times \cdots \times p \pm 1$, Math. Comp. 71 (2002), no. 237, 441–448.


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