A conjecture which implies that there are infinitely many primes of the form n!+1

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Abstract

Let $f(6) = 720$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 6$. For an integer $n \geqslant 6$, let Λ_n denote the following statement: if a system $\mathcal{S} \subseteq \left\{x_i! = x_j :\ 1 \leqslant i < j \leqslant n\right\} \cup \left\{x_i \cdot x_j = x_{j+1} :\ 1 \leqslant i < j \leqslant n-1\right\}$ has at most finitely many solutions in integers $x_1, \ldots, x_n$ greater than 3, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant f(n)$. We conjecture that the statements $\Lambda_6, \ldots, \Lambda_9$ are true. We prove that the statement Λ_9 implies that there are infinitely many primes of the form $n! + 1$.

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It is conjectured that $n! + 1$ is prime for infinitely many positive integers n, see [3]. In this note, we propose a conjecture which implies this.

Lemma 1. *For every integers x and y greater than* 1*, $x! \cdot y = y!$ if and only if $x + 1 = y$.*

For an integer $n \geqslant 6$, let $\mathcal{V}_n$ denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \ldots, n-1\} \setminus \{1,3,5\}\ x_i! = x_{i+1} \\ x_1! = x_6 \\ x_1 \cdot x_2 = x_3 \\ x_3 \cdot x_4 = x_5 \\ x_1 \cdot x_5 = x_6 \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{V}_n$.

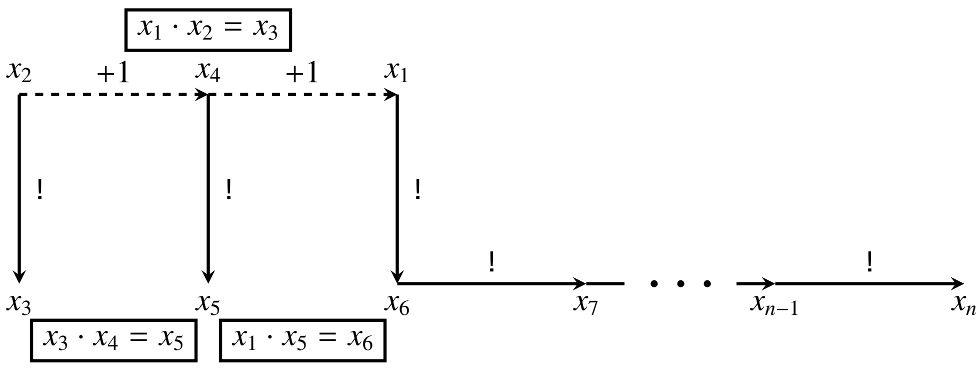

Fig. 1 Construction of the system $\mathcal{V}_n$

Let $f(6) = 720$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 6$.

Lemma 2. *For every integer $n \geqslant 6$, the system $\mathcal{V}_n$ has exactly one solution in integers greater than 3, namely $\big(6, 4, 24, 5, 120, f(6), \ldots, f(n)\big)$.*

Proof. By Lemma 1, $x_2 + 2 = x_1$. Hence, $(x_2 + 2) \cdot x_2 = x_1 \cdot x_2 = x_3 = x_2!$. Therefore, $x_2 = 4$. The rest of the proof follows from the diagram in Figure 1. □

Let

$$T_n = \Big\{x_i! = x_j : \ 1 \leqslant i < j \leqslant n\Big\} \cup \Big\{x_i \cdot x_j = x_{j+1} : \ 1 \leqslant i < j \leqslant n-1\Big\}$$

For an integer $n \geqslant 6$, let Λ_n denote the following statement: if a system $\mathcal{S} \subseteq T_n$ has at most finitely many solutions in integers $x_1, \ldots, x_n$ greater than 3, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant f(n)$.

Conjecture. *The statements $\Lambda_6, \ldots, \Lambda_9$ are true.*

We present a heuristic reasoning that leads to the Conjecture. For every integer $n \in \{6, 7, 8, 9\}$, we consider all subsystems of the system T_n. We conjecture that the largest known solution is indeed the largest possible.

Theorem 1. *For every statement Λ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 2 because $\mathcal{V}_n \subseteq T_n$. □

Theorem 2. *Every statement Λ_n holds true with an integer bound that depends on n.*

Proof. Indeed, for every integer $n \geqslant 6$, the system T_n has a finite number of subsystems. □

Lemma 3. *If a prime number x is grater than 3, then $\dfrac{(x-1)!+1}{x} > 3$.*

Lemma 4. *(Wilson's theorem, [4, p. 89]). For every integer $x \geqslant 2$, x is prime if and only if x divides $(x-1)!+1$.*

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_1! &= x_2 \\ x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 1 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

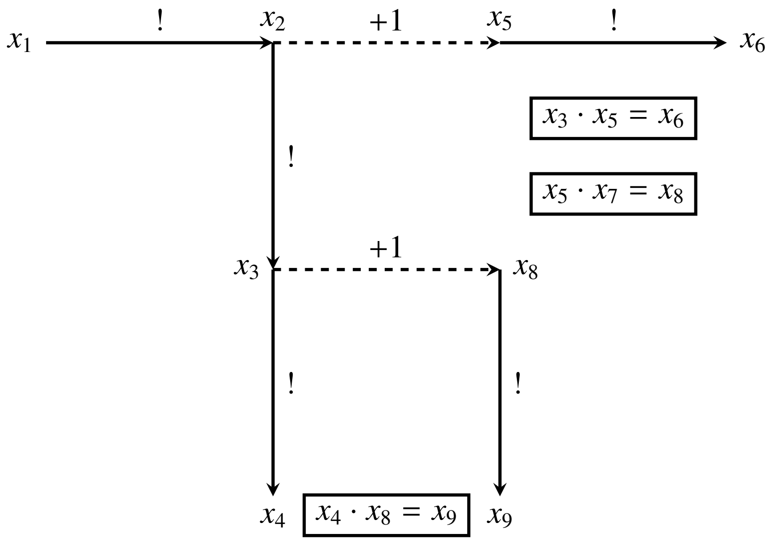

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 5. *For every integer $x_1 > 3$, the system $\mathcal{A}$ is solvable in integers $x_2, \ldots, x_9$ greater than 3 if and only if $x_1! + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_4 &= ((x_1!)!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \\ x_7 &= \frac{(x_1!)! + 1}{x_1! + 1} \\ x_8 &= (x_1!)! + 1 \\ x_9 &= ((x_1!)! + 1)! \end{aligned}$$

Proof. By Lemmas 1 and 3, for every integer $x_1 > 3$, the system $\mathcal{A}$ is solvable in integers $x_2, \ldots, x_9$ greater than 3 if and only if $x_1! + 1$ divides $(x_1!)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. □

Theorem 3. *The statement Λ_9 implies that there are infinitely many primes of the form $n! + 1$.*

Proof. Harvey Dubner proved that $872! + 1$ is prime, see [2], [3], [5, p. 7], and [7]. Let $x_1 = 872$. By Lemma 5, there exists a unique tuple $(x_2, \ldots, x_9)$ of integers greater than 3 such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Hence,

$$x_9 = ((x_1!)! + 1)! > ((720!)! + 1)! > ((720!)!)! = ((f(6)!)!)! = f(9)$$

The statement Λ_9 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions in integers $x_1, \ldots, x_9$ greater than 3. This conclusion and Lemma 5 imply that $x_1! + 1$ is prime for infinitely many integers $x_1 > 3$. □

Let C denote the following system of equations:

$$\begin{cases} x_1! = x_4 \\ x_2! = x_5 \\ x_3! = x_6 \\ x_2 \cdot x_3 = x_4 \\ x_3 \cdot x_5 = x_6 \end{cases}$$

Lemma 1 and the diagram in Figure 3 explain the construction of the system C.

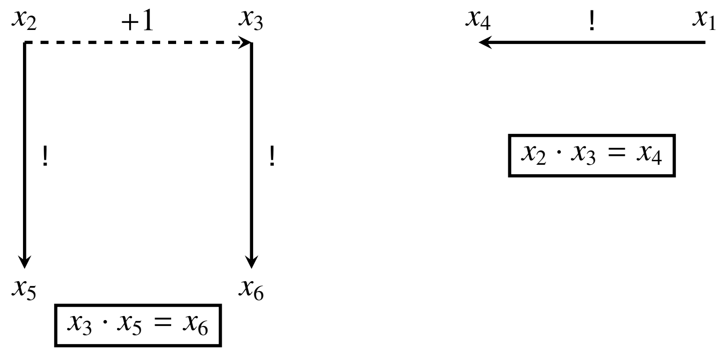

Fig. 3 Construction of the system C

Lemma 6. *For every integers x_1, x_2 greater than 3, the system C is solvable in integers x_3, x_4, x_5, x_6 greater than 3 if and only if $x_1! = x_2(x_2+1)$. In this case, the integers x_3, x_4, x_5, x_6 are uniquely determined by the following equalities:*

$$\begin{aligned} x_3 &= x_2 + 1 \\ x_4 &= x_1! \\ x_5 &= x_2! \\ x_6 &= (x_2+1)! \end{aligned}$$

Theorem 4. *If the equation $x_1! = x_2(x_2+1)$ has at most finitely many solutions in positive integers, then the statement Λ_6 guarantees that each such solution (x_1, x_2) belongs to the set $\{(2,1),(3,2)\}$.*

Proof. Suppose that the antecedent holds. Then, the equation $x_1! = x_2(x_2+1)$ has at most finitely many solutions $(x_1, x_2) \in (\mathbb{N} \setminus \{0,1,2,3\})^2$. By Lemma 6, the system C is solvable in integers x_3, x_4, x_5, x_6 greater than 3. Since $C \subseteq T_6$, the statement Λ_6 implies that $x_1! = x_4 \leqslant f(6) = 720 = 6!$. Hence, $x_1 \in \{1,2,3,4,5,6\}$. For every integer $x_1 \in \{1,2,3,4,5,6\}$, $x_1!$ is a product of two consecutive integers if and only if $x_1 \in \{2,3\}$. □

The question of solving the equation $y! = x(x+1)$ was posed by P. Erdös, see [1]. F. Luca proved that the *abc* conjecture implies that the equation $y! = x(x+1)$ has only finitely many solutions in positive integers, see [6].

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