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► **To cite this version:**

Apoloniusz Tyszka. A conjecture which implies that there are infinitely many primes of the form  $n!+1$ . 2017. hal-01625653v2

**HAL Id: hal-01625653**

**<https://hal.science/hal-01625653v2>**

Preprint submitted on 13 Nov 2017

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# A conjecture which implies that there are infinitely many primes of the form $n! + 1$

Apoloniusz Tyszka

## Abstract

Let  $f(6) = 720$ , and let  $f(n+1) = f(n)!$  for every integer  $n \geq 6$ . For an integer  $n \geq 6$ , let  $\Lambda_n$  denote the following statement: if a system  $\mathcal{S} \subseteq \{x_i! = x_j : 1 \leq i < j \leq n\} \cup \{x_i \cdot x_j = x_{j+1} : 1 \leq i < j \leq n-1\}$  has at most finitely many solutions in integers  $x_1, \dots, x_n$  greater than 3, then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq f(n)$ . We conjecture that the statements  $\Lambda_6, \dots, \Lambda_9$  are true. We prove that the statement  $\Lambda_9$  implies that there are infinitely many primes of the form  $n! + 1$ .

**Key words and phrases:** prime numbers of the form  $n! + 1$ .

**2010 Mathematics Subject Classification:** 11A41.

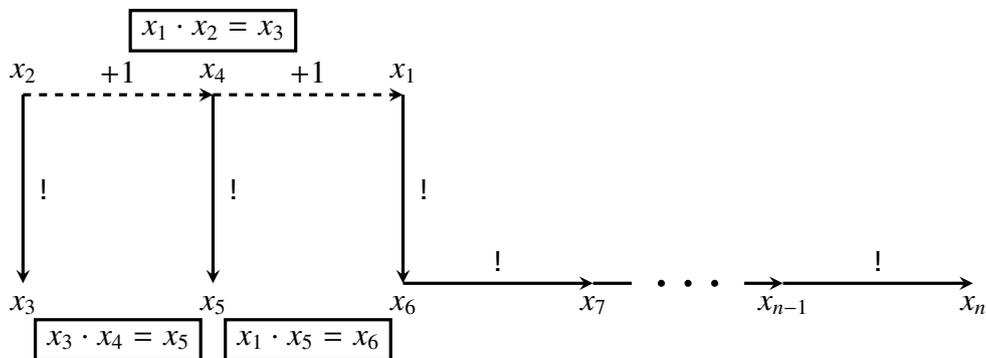
It is conjectured that  $n! + 1$  is prime for infinitely many positive integers  $n$ , see [3]. In this note, we propose a conjecture which implies this.

**Lemma 1.** For every integers  $x$  and  $y$  greater than 1,  $x! \cdot y = y!$  if and only if  $x + 1 = y$ .

For an integer  $n \geq 6$ , let  $\mathcal{V}_n$  denote the following system of equations:

$$\left\{ \begin{array}{l} \forall i \in \{1, \dots, n-1\} \setminus \{1, 3, 5\} \quad x_i! = x_{i+1} \\ x_1! = x_6 \\ x_1 \cdot x_2 = x_3 \\ x_3 \cdot x_4 = x_5 \\ x_1 \cdot x_5 = x_6 \end{array} \right.$$

The diagram in Figure 1 illustrates the construction of the system  $\mathcal{V}_n$ .



**Fig. 1** Construction of the system  $\mathcal{V}_n$

Let  $f(6) = 720$ , and let  $f(n + 1) = f(n)!$  for every integer  $n \geq 6$ .

**Lemma 2.** *For every integer  $n \geq 6$ , the system  $\mathcal{V}_n$  has exactly one solution in integers greater than 3, namely  $(6, 4, 24, 5, 120, f(6), \dots, f(n))$ .*

*Proof.* By Lemma 1,  $x_2 + 2 = x_1$ . Hence,  $(x_2 + 2) \cdot x_2 = x_1 \cdot x_2 = x_3 = x_2!$ . Therefore,  $x_2 = 4$ . The rest of the proof follows from the diagram in Figure 1.  $\square$

Let

$$T_n = \{x_i! = x_j : 1 \leq i < j \leq n\} \cup \{x_i \cdot x_j = x_{j+1} : 1 \leq i < j \leq n - 1\}$$

For an integer  $n \geq 6$ , let  $\Lambda_n$  denote the following statement: if a system  $\mathcal{S} \subseteq T_n$  has at most finitely many solutions in integers  $x_1, \dots, x_n$  greater than 3, then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq f(n)$ .

**Conjecture.** *The statements  $\Lambda_6, \dots, \Lambda_9$  are true.*

We present a heuristic reasoning that leads to the Conjecture. For every integer  $n \in \{6, 7, 8, 9\}$ , we consider all subsystems of the system  $T_n$ . We conjecture that the largest known solution is indeed the largest possible.

**Theorem 1.** *For every statement  $\Lambda_n$ , the bound  $f(n)$  cannot be decreased.*

*Proof.* It follows from Lemma 2 because  $\mathcal{V}_n \subseteq T_n$ .  $\square$

**Theorem 2.** *Every statement  $\Lambda_n$  holds true with an integer bound that depends on  $n$ .*

*Proof.* Indeed, for every integer  $n \geq 6$ , the system  $T_n$  has a finite number of subsystems.  $\square$

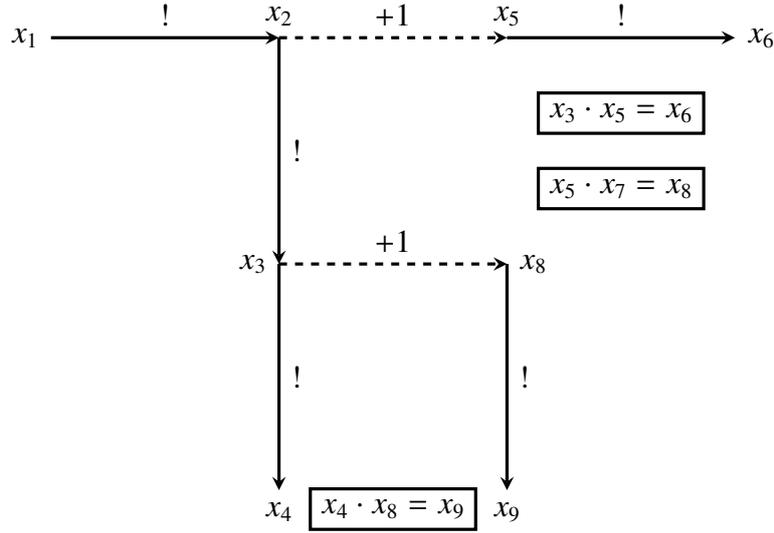
**Lemma 3.** *If a prime number  $x$  is greater than 3, then  $\frac{(x-1)! + 1}{x} > 3$ .*

**Lemma 4.** (Wilson's theorem, [4, p. 89]). *For every integer  $x \geq 2$ ,  $x$  is prime if and only if  $x$  divides  $(x-1)! + 1$ .*

Let  $\mathcal{A}$  denote the following system of equations:

$$\left\{ \begin{array}{l} x_1! = x_2 \\ x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 1 and the diagram in Figure 2 explain the construction of the system  $\mathcal{A}$ .



**Fig. 2** Construction of the system  $\mathcal{A}$

**Lemma 5.** For every integer  $x_1 > 3$ , the system  $\mathcal{A}$  is solvable in integers  $x_2, \dots, x_9$  greater than 3 if and only if  $x_1! + 1$  is prime. In this case, the integers  $x_2, \dots, x_9$  are uniquely determined by the following equalities:

$$\begin{aligned}
 x_2 &= x_1! \\
 x_3 &= (x_1!)! \\
 x_4 &= ((x_1!)!)! \\
 x_5 &= x_1! + 1 \\
 x_6 &= (x_1! + 1)! \\
 x_7 &= \frac{(x_1!)! + 1}{x_1! + 1} \\
 x_8 &= (x_1!)! + 1 \\
 x_9 &= ((x_1!)! + 1)!
 \end{aligned}$$

*Proof.* By Lemmas 1 and 3, for every integer  $x_1 > 3$ , the system  $\mathcal{A}$  is solvable in integers  $x_2, \dots, x_9$  greater than 3 if and only if  $x_1! + 1$  divides  $(x_1!)! + 1$ . Hence, the claim of Lemma 5 follows from Lemma 4.  $\square$

**Theorem 3.** The statement  $\Lambda_9$  implies that there are infinitely many primes of the form  $n! + 1$ .

*Proof.* Harvey Dubner proved that  $872! + 1$  is prime, see [2], [3], [5, p. 7], and [7]. Let  $x_1 = 872$ . By Lemma 5, there exists a unique tuple  $(x_2, \dots, x_9)$  of integers greater than 3 such that the tuple  $(x_1, x_2, \dots, x_9)$  solves the system  $\mathcal{A}$ . Hence,

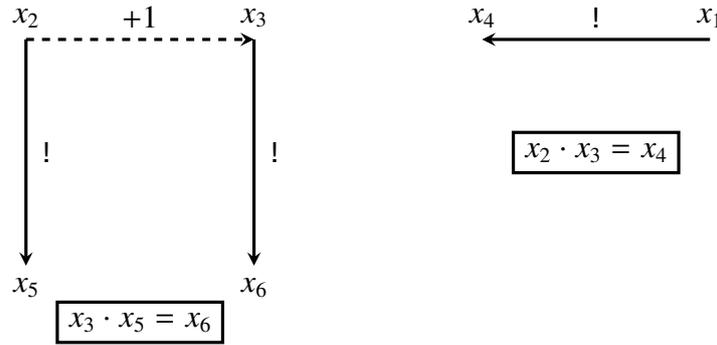
$$x_9 = ((x_1!)! + 1)! > ((720!)! + 1)! > ((720!)!)! = ((f(6)!)!)! = f(9)$$

The statement  $\Lambda_9$  and the inequality  $x_9 > f(9)$  imply that the system  $\mathcal{A}$  has infinitely many solutions in integers  $x_1, \dots, x_9$  greater than 3. This conclusion and Lemma 5 imply that  $x_1! + 1$  is prime for infinitely many integers  $x_1 > 3$ .  $\square$

Let  $C$  denote the following system of equations:

$$\left\{ \begin{array}{l}
 x_1! = x_4 \\
 x_2! = x_5 \\
 x_3! = x_6 \\
 x_2 \cdot x_3 = x_4 \\
 x_3 \cdot x_5 = x_6
 \end{array} \right.$$

Lemma 1 and the diagram in Figure 3 explain the construction of the system  $C$ .



**Fig. 3** Construction of the system  $C$

**Lemma 6.** For every integers  $x_1, x_2$  greater than 3, the system  $C$  is solvable in integers  $x_3, x_4, x_5, x_6$  greater than 3 if and only if  $x_1! = x_2(x_2 + 1)$ . In this case, the integers  $x_3, x_4, x_5, x_6$  are uniquely determined by the following equalities:

$$\begin{aligned} x_3 &= x_2 + 1 \\ x_4 &= x_1! \\ x_5 &= x_2! \\ x_6 &= (x_2 + 1)! \end{aligned}$$

**Theorem 4.** If the equation  $x_1! = x_2(x_2 + 1)$  has at most finitely many solutions in positive integers, then the statement  $\Lambda_6$  guarantees that each such solution  $(x_1, x_2)$  belongs to the set  $\{(2, 1), (3, 2)\}$ .

*Proof.* Suppose that the antecedent holds. Then, the equation  $x_1! = x_2(x_2 + 1)$  has at most finitely many solutions  $(x_1, x_2) \in (\mathbb{N} \setminus \{0, 1, 2, 3\})^2$ . By Lemma 6, the system  $C$  is solvable in integers  $x_3, x_4, x_5, x_6$  greater than 3. Since  $C \subseteq T_6$ , the statement  $\Lambda_6$  implies that  $x_1! = x_4 \leq f(6) = 720 = 6!$ . Hence,  $x_1 \in \{1, 2, 3, 4, 5, 6\}$ . For every integer  $x_1 \in \{1, 2, 3, 4, 5, 6\}$ ,  $x_1!$  is a product of two consecutive integers if and only if  $x_1 \in \{2, 3\}$ .  $\square$

The question of solving the equation  $y! = x(x + 1)$  was posed by P. Erdős, see [1]. F. Luca proved that the *abc* conjecture implies that the equation  $y! = x(x + 1)$  has only finitely many solutions in positive integers, see [6].

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