Pseudo-spectral methods for the spatial symplectic reduction of open systems of conservation laws

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Abstract

A reduction method is presented for systems of conservation laws with boundary energy flow. It is stated as a generalized pseudo-spectral method which performs exact differentiation by using simultaneously several approximation spaces generated by polynomials bases and suitable choices of port-variables. The symplecticity of this spatial reduction method is proved when used for the reduction of both closed and open systems of conservation laws, for any choice of collocation points (i.e. for any polynomial bases). The symplecticity of some more usual collocation schemes is discussed and finally their accuracy on approximation of the spectrum, on the example of the ideal transmission line, is discussed in comparison with the suggested reduction scheme.

1 Introduction

Hamiltonian operators are classically used to represent the dynamics of many closed systems of conservation laws. More recently port-Hamiltonian extensions have been introduced to model distributed parameter systems with boundary energy flow [47, 30]. Classical Hamiltonian examples such as electromagnetic fields obeying Maxwell equations or ideal fluid described by the Navier-Stokes equations may be considered using this port-Hamiltonian approach when systems with energy flows are considered.

This modelling approach has proven to be fruitful for the modelling, simulation and control of many hyperbolic systems such as transmission lines models [19], beam equations [27] or shallow water equations [22]. Quite surprisingly

it may also be applied to some parabolic examples such as transport phenomena in multi-scale adsorption columns [1], fuel cells [18] or Ionic Polymer-Metal Composites [34].

In the spatial reduction of distributed parameters systems, pseudo-spectral methods are often chosen because they lead to low order approximate model, with good spectral properties (in the linear case). When a polynomial basis is chosen for the approximation space, the derived pseudo-spectral method may be viewed as a collocation method where the collocation points are the zeros of the chosen polynomial . In this case, the reduced model is moreover stated in "natural" variables (the infinite dimensional state variables evaluated at the collocation points), making its physical meaning easy to catch [17]. Accurate spectral properties and low order models are key features for control engineers. These are the reasons why pseudo-spectral methods (and more specifically collocation methods) have become popular among them (see for instance [16, 3, 38]).

Obviously in these engineering applications only open systems are considered since they are both measured and actuated. Besides accuracy properties, either for long range simulation or for stabilizing control issues, it is of prime importance for the reduced model to remain in the same port-Hamiltonian form (i.e. with the same geometric structure and the same physical invariants). This is what we will call here spatial symplecticity of the reduction scheme.

In this paper, we suggest a polynomial pseudo-spectral method which preserves the geometric structure of port Hamiltonian models, the phenomenological laws and the conservation laws without introducing any undesired numerical dissipation. Doing so, we expect useful structural dynamical properties of the obtained reduced model for numerical simulation and control. Mixed finite elements methods [7, 19, 2, 23] may be viewed as a particular case of the methodology developed hereafter for the case of low order polynomial approximations. Besides this generalization, this paper provides a theoretical interpretation of implicit choices made in these earlier works.

The paper is organized as follows. In the section 2 we present some existing results on the Hamiltonian formulation of open distributed parameter systems. Definition, examples and representation results of Hamiltonian systems defined with respect to Dirac structures are recalled. Then the extension of Hamiltonian operators to Stokes-Dirac structure for the infinite dimensional case are briefly recalled. Finally, two hyperbolic 1D examples are presented: the ideal transmission line and the (nonlinear) shallow water model. In the section 3, we present a new geometric collocation scheme. First we define the different approximation subspaces according to the geometric nature of the approximated variables (differential forms of various degrees). Then, defining appropriate reduced boundary variables we define a reduced Dirac structure by performing exact differentiation. In the section 4 it is recalled how the closure equations defining the Hamiltonian may be projected onto the discretization basis and the resulting spatially discretized port Hamiltonian system is defined. This procedure is illustrated on the two examples of the ideal transmission line and the shallow water equations. While the previous sections have presented a polynomial spatial discretization scheme which, by construction, preserves the symplecticity of

Hamiltonian systems defined on Stokes-Dirac structures, in the section 5, we discuss the spatial symplecticity of another "classical" collocation schemes: it is shown that when chosen collocation points are zeroes of Gauss-Legendre polynomials, the discretization of closed Hamiltonian systems (in the sense that the boundary conditions are such that there is no energy flow through the boundaries) is symplectic. This will allow fair comparisons between the geometric collocation scheme proposed in this paper and another symplectic collocation scheme (although the latter scheme does not preserve the geometric structure for open systems). Comparisons concerning the spectrum approximation for an ideal transmission line are then proposed.

2 Extension of Hamiltonian operators and Dirac structures

The Dirac structure is a geometric structure introduced originally to gauge Poisson brackets for system with constraints [11, 10]. Dirac structures generalize as well Poisson brackets as presymplectic forms defined on some differential manifold $\mathcal{M}$ in terms of vector subbundles of the product bundle $T\mathcal{M} \times T^*\mathcal{M}$. Dirac structures are the graph of skew-symmetric tensors encompassing the tensor fields associated with the Poisson brackets and presymplectic forms. Dirac structures appear also for evolution equations expressed as Hamiltonian systems defined with respect to Hamiltonian operators [14] and have been used for the analysis of their integrability [15].

In the context of this paper we shall consider a class of Dirac structures which extend Poisson brackets and Hamiltonian systems in the sense that they are defined on larger subbundles than $T\mathcal{M} \times T^*\mathcal{M}$ for finite-dimensional Hamiltonian systems [46] or correspond to extensions of Hamiltonian operators for infinite-dimensional Hamiltonian systems [47, 25]. These latter extensions correspond to the definition of Hamiltonian systems for which the Hamiltonian function obeys a balance equation with a source term defining either the exchange of energy through the boundary of the system or some dissipative phenomenon in the spatial domain [47].

In this section we shall briefly recall the definitions of Dirac structures and Hamiltonian systems defined with respect to Dirac structures, and detail the particular case of the Hamiltonian formulation of a system of two conservation laws following [47].

2.1 Dirac structures on real vector spaces

Let $\mathscr{F}$ and $\mathscr{E}$ be two real vector spaces and assume that they are endowed with a *non degenerated bilinear form*[1] denoted by:

$$\begin{array}{rccc} \langle . | . \rangle : & \mathscr{F} \times \mathscr{E} & \rightarrow & \mathbb{R} \\ & (f,\, e) & \mapsto & \langle e |\, f \rangle \end{array} \tag{1}$$

The bilinear product leads to the definition of a *symmetric bilinear form* on the product space[2] $\mathscr{B} = \mathscr{F} \times \mathscr{E}$ as follows:

$$\begin{array}{rcl} \ll \cdot, \cdot \gg : \mathscr{B} \times \mathscr{B} & \rightarrow & \mathbb{R} \\ ((f_1,\, e_1)\,,\, (f,_2\, e_2)) & \mapsto & \ll (f_1, e_1), (f_2, e_2) \gg := \langle e_1 |\, f_2 \rangle + \langle e_2 |\, f_1 \rangle \end{array} \tag{2}$$

Definition 1. [10] [Dirac structure] A Dirac structure is a linear subspace $\mathscr{D} \subset \mathscr{B}$ such that $\mathscr{D} = \mathscr{D}^{\perp}$, with $\perp$ denoting the orthogonal complement with respect to the bilinear form $\ll, \gg$.

If a linear subspace $\mathscr{D} \subset \mathscr{B}$ satisfy only the isotropy condition $\mathscr{D} \subset \mathscr{D}^{\perp}$, what means that it is not maximal, one say that it is a *Tellegen structure* [20, chap. 5].

Dirac structures are a geometric perspective to skew-symmetric tensors, actually corresponding to their *graph*, which generalize the tensors associated with Poisson brackets or pre-symplectic forms as may be seen from the next example.

Example 2. Consider a finite-dimensional vector space V and its dual vector space V^* and define $\mathscr{F} = V$ and $\mathscr{E} = V^*$. Choose the canonical duality product as the non degenerated bilinear form (1) . Then it is easy to check that the graph of any skew-symmetric linear map $\omega : V \rightarrow V^*$ (such that $\langle \omega\,(v_1)\,|\, v_2 \rangle + \langle \omega\,(v_2)\,|\,(v_1) \rangle = 0,\ \forall\,(v_1,\, v_2) \in V \times V$) endowing the vector space V with a presympletic structure, defines a Dirac structure in $V \times V^*$. In an analoguous way the graph of of any skew-symmetric linear map $J : V^* \rightarrow V$ (such that $\langle w_1 |\, J\,(w_2) \rangle + \langle w_2 |\, J\,(w_1) \rangle = 0,\ \forall\,(w_1,\, w_2) \in V^* \times V^*$) endowing the vector space V with a Poisson structure, defines a Dirac structure in $V \times V^*$.

It should be noted that we have defined Dirac structure in vector spaces which is sufficient for this paper. A more general definition is given on differentiable manifold in [10, 12].

In the case of infinite-dimensional vector spaces different approaches have been developed. The first one considers Hilbert spaces and uses their inner product as non degenerated bilinear form (1) [39, 25, 20, chap.5]. The second

[1] These spaces and the bilinear product are actually more general than in the original definition [10, 15] where the two spaces are algebraic duals and the bilinear product is simply the duality product. An example of more general definition arises for instance when considering Hilbert spaces associated with operators and their duals [20, 25]. A special terminology is often adopted, stemming from network theory, namely the vector space $\mathscr{F}$ is called *space of flow variables* and the space $\mathscr{E}$ is called *space of effort variables*. The bilinear form (1) is also called *power product* as for physical systems it often has the dimension of power.

[2] This symmetric product has been called *+ pairing* in [10]. The product space $\mathscr{B}$ is often called *bond space*.

one, which we shall follows here, is based on the use of exterior forms as base vector spaces and the non degenerated bilinear form is based on their wedge product [47, 28, 26]; it will be presented in more details in the section 2.3.

Example 3. [39] Choose as vector spaces a Hilbert space: $\mathscr{F} = \mathscr{E} = \mathscr{H}$. And define the non degenerated bilinear form (1) to be the inner product of the Hilbert space. Then it may be checked that the the graph of any densily defined skew-symmetric operator A (satifying $\mathrm{Dom}\,(A) = \mathrm{Dom}\,(A^*)$, dense in $\mathscr{H}$ and satisfying $\langle A\, h_1 |\, h_2\rangle + \langle h_1 |\, A^*\, h_2\rangle = 0,\ \forall\,(h_1,\, h_2) \in \mathscr{H}$) is a Dirac structure .

Dirac structures admit more concrete definitions, based on some linear maps called representation of a Dirac structure [12, 20]. We shall use in the sequel only such representations for the finite-dimensional reduction of the system of conservation laws and therefore present in the sequel only the matrix representation of finite-dimensional Dirac structures. Assume now that the spaces $\mathscr{F}$ and $\mathscr{E}$ are finite-dimensional and for the sake of simplicity choose $\mathscr{F} = \mathscr{E} = \mathbb{R}^n$ with $n \in \mathbb{N}^*$ [3]. Define the non degenerated bilinear product (1) being the canonical Euclidean product in $\mathbb{R}^n$ composed with a signature[4] matrix σ:

$$\langle e |\, f\rangle = e^T \sigma\, f \quad \text{where } f \in \mathscr{F} = \mathbb{R}^n,\ e \in \mathscr{E} = \mathbb{R}^n \tag{3}$$

A Dirac structure in $\mathscr{F} \times \mathscr{E} = \mathbb{R}^n \times \mathbb{R}^n$ admits several matrix representations [10, 46, 12, 20, chap. 5] from which we shall present three.

Proposition 4. *[Image representation of a Dirac structure] A linear subspace $\mathscr{D} \subset \mathscr{F} \times \mathscr{E} = \mathbb{R}^n \times \mathbb{R}^n$ endowed with the symmetic product (2) associated with the bilinear product (3) is a Dirac structure if and only if there exist two $n \times n$ real matrices, denoted here E and F, and satisfying*

1. skew-symmetry: $E\sigma F^T + F\sigma E^T = 0$
2. $\mathrm{rank}[E : F] = n$

such that:

$$\mathscr{D} = \{(f, e) \in \mathscr{F} \times \mathscr{E} | f = E^T\lambda,\ e = F^T\lambda,\ \lambda \in \mathbb{R}^n\} \tag{4}$$

This description is called an *image representation* of the Dirac structure $\mathscr{D}$.

It is then immediate to deduce the dual representation called kernel representation [12].

Proposition 5. *[Kernel representation of a Dirac structure] A Dirac structure $\mathscr{D} \subset \mathscr{F} \times \mathscr{E} = \mathbb{R}^n \times \mathbb{R}^n$ endowed with the symmetric product (2) associated with the bilinear product (3) and admitting the image representation of the proposition 4 admits also the* kernel representation*:*

$$\mathscr{D} = \{(f, e) \in \mathscr{F} \times \mathscr{E} |\, (F\,\sigma)\ f + (E\,\sigma)\ e = 0\} \tag{5}$$

[3] A geometric definition of these representations may be found in [10] and [12]

[4] Signature matrices are diagonal matrices often used in network theory to represent the chosen sign convention in the *power product* (3)

Finally one may also represent a Dirac structure as the graph of some skew-symmetric tensor, the so-called input-output representation, as follows [12, 20].

Proposition 6. *[Input-output representation] Consider a Dirac structure* $\mathscr{D} \subset \mathscr{F} \times \mathscr{E} = \mathbb{R}^n \times \mathbb{R}^n$ *endowed with the symmetric product (2) associated with the bilinear product (3). There exist a decomposition of the space of flow variables:* $\mathscr{F} = \mathscr{F}_1 \oplus \mathscr{F}_2 \ni (f_1, f_2) = f$ *and the space of effort variables:* $\mathscr{E} = \mathscr{E}_1 \oplus \mathscr{E}_2 \ni (e_1, e_2) = e$ *and an* $n \times n$ *skew-symmetric matrix* J *such that the Dirac structure admits also the input-output representation:*

$$\mathscr{D} = \{(f, e) \in \mathscr{F} \times \mathscr{E} | \begin{pmatrix} f_1 \\ e_2 \end{pmatrix} = J \begin{pmatrix} e_1 \\ f_2 \end{pmatrix}\} \tag{6}$$

2.2 Hamiltonian systems defined with respect to finite-dimensional Dirac structures

In this section we shall recall briefly the definition of Hamiltonian systems defined with respect to some Dirac structure in the finite dimensional case and introduce the definition of port Hamiltonian systems in the particular case when the state space is a real vector space $\mathscr{F}$ for which, at any point $x \in \mathscr{F}$, the tangent space may be identified with $\mathscr{F}$ and the cotangent space identified with the dual $\mathscr{F}^*$ [5].

Definition 7. [10] [Implicit Hamiltonian system] Consider a real vector space, denoted by $\mathscr{F}$, of dimension n and its dual vector space $\mathscr{F}^*$. Consider a Dirac structure $\mathscr{D} \subset \mathscr{F} \times \mathscr{F}^*$. *An implicit Hamiltonian system* with respect to the Dirac structure $\mathscr{D}$ and generated by the Hamiltonian function $H \in C^\infty(\mathscr{F}, \mathbb{R})$, is defined by the implicit differential equation: $\left(\frac{dx}{dt}, \frac{\partial H}{\partial x}\right) \in \mathscr{D}$.

A consequence of the isotropy of the Dirac structure is the conservation of the Hamiltonian:

$$\frac{dH}{dt} = \left\langle \frac{\partial H}{\partial x} | \frac{dx}{dt} \right\rangle = 0 \tag{7}$$

which expresses the conservation of the energy for physical systems where the Hamiltonian is the total energy of the system.

Implicit Hamiltonian system encompass constrained Hamiltonian systems[6] as is illustrated on the following example but retain also all the symmetries of standard Hamiltonian systems [10, 5, 4, 44].

Example 8. Consider the simple LC circuit of the figure 1 composed of two capacitors and an inductor in parallel. Choose as state variables $x = (q_1, \phi, q_2) \in \mathbb{R}^3 = \mathscr{F}$ where q_i denotes the charge of the capacitors $i \in \{1, 2\}$ and ϕ the total magnetic flux in the inductor. The time variation $\frac{dx}{dt}(t)$ of the state variables at time t, may be identified with the following circuit variables: $(i_{C_1}, v_L, i_{C_2})^T \in$

[5] The reader is referred to [10] for the definition of implicit Hamiltonian systems on differentiable manifolds endowed with a Dirac bundle structure and also to [31, 12, 49].

[6] Such constrained Hamiltonian systems may be reduced to explicit pseudo-Hamiltonian systems on some submanifold as has been shown for instance in [45, 5].

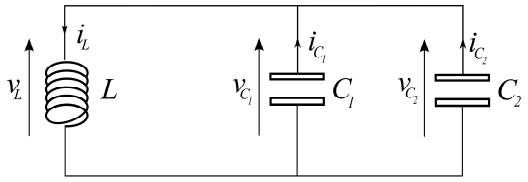

Figure 1: Closed LC circuit

$\mathbb{R}^3 = \mathscr{F}$. The total electro-magnetic energy may be expressed as a function: $H(x)$ and its gradient $\frac{\partial H}{\partial x}(x)$ may be identified as: $(v_{C_1}, i_L, v_{C_2})^T \in \mathbb{R}^3 = \mathscr{E}$ where v_{C_I} denote the voltages of the capacitor C_i and i_L the current in the inductor. Considering the bilinear form on $\mathscr{F} \times \mathscr{E}$ to be simply the euclidean product in $\mathbb{R}^3$ (i.e. the signature matrix σ is the identity), one may check that Kirchhoff's laws may be expressed[7] as:

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{F} \begin{pmatrix} i_{C_1} \\ v_L \\ i_{C_2} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}}_{E} \begin{pmatrix} v_{C_1} \\ i_L \\ v_{C_2} \end{pmatrix} = 0 \tag{8}$$

which define, according to proposition 4, a Dirac structure in $\mathscr{F} \times \mathscr{E}$. The dynamical system representing this LC circuit may thus be defined as an implicit Hamiltonian system according to the definition 7. The rank degeneracy of the matrix F corresponds to the mesh law applied to the circuit containing the two capacitors: $v_{C_1} - v_{C_2} = 0$ and may be interpreted as a constraint on the state variables: $\frac{\partial H}{\partial q_1}(x) - \frac{\partial H}{\partial q_2}(x) = 0$.

In this paper we shall consider an extension of these implicit Hamiltonian systems which is called port Hamiltonian system [46] and is defined with respect to Dirac structure in a product space encompassing not only the tangent and co-tangent spaces of the state space but also external variables representing the interaction of the system with its environment through its boundaries. In the sequel we shall define port Hamiltonian system on vector spaces but they may be defined on differentiable manifolds [46, 12] and more precise definitions have been given when the state space is a Lie group [31].

Definition 9. [46] [Port Hamiltonian system] Consider a real vector space, denoted by $\mathscr{F}^i = \mathscr{E}^i$, of dimension n and its dual vector space $\mathscr{F}^{i*}$ endowed with the canonical duality product denoted by $\langle . | . \rangle_i$. And define two other finite-dimensional vector spaces, called *port spaces*, denoted by $\mathscr{F}^e$ and $\mathscr{E}^e$ and endowed with a non generated bilinear form denoted by $\langle . | . \rangle_e$. Define the product vector spaces $\mathscr{F} = \mathscr{F}^i \times \mathscr{F}^e \ni \left(f^i, f^e\right)$ and $\mathscr{E} = \mathscr{E}^i \times \mathscr{E}^e \ni \left(e^i, e^e\right)$ endowed with the bilinear form defined by: $\left\langle \left(f^i, f^e\right) \mid \left(e^i, e^e\right) \right\rangle = \left\langle e^i | f^i \right\rangle_i + \left\langle e^e | f^e \right\rangle_e$. Consider a Dirac structure $\mathscr{D} \subset \mathscr{F} \times \mathscr{E}$. A *port Hamiltonian system* with respect to the

[7] These matrices may be constructed systematically and the reader is referred to [32, 43, 48]

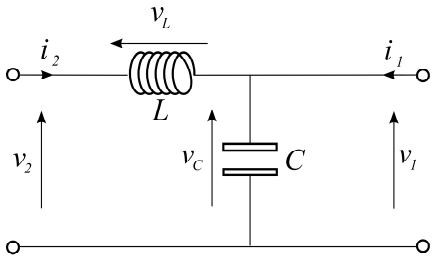

Figure 2: Open LC circuit

Dirac structure $\mathscr{D}$ and generated by the Hamiltonian function, $H \in C^\infty\left(\mathscr{F}^i, \mathbb{R}\right)$ is defined by the implicit differential equation: $\left(\left(\frac{dx}{dt}, f^e\right), \left(\frac{\partial H}{\partial x}, e^e\right)\right) \in \mathscr{D}$.

In general port Hamiltonian systems do not satisfy the Cauchy conditions as long as the system is not completed with some relations on the external port variables (f^e, e^e) . Nevertheless this formulation may be very usefull in order to define a *composition of Hamiltonian systems* and handle complex Hamiltonian systems composed of a set of interacting subsystems [31, 8]. It has also been used in control theory in order to generate Lyapunov function and so-called control Lyapunov functions to find stabilizing controllers [29, 37, 36]. The isotropy property of the Dirac structure translates now in a balance equation of the Hamiltonian (often the total energy of the system):

$$0 = \left\langle \frac{\partial H}{\partial x} | \frac{dx}{dt} \right\rangle + \langle e^e | f^e \rangle_e = \frac{dH}{dt} + \langle e^e | f^e \rangle_e \tag{9}$$

Example 10. Consider the simple LC circuit, represented in the figure 2, with two open ports, indexed by $k \in \{1, 2\}$, and the pairs of currents and voltages (i_k, v_k). Concerning the internal variables, the state variables may be chosen as: $x = (q, \phi) \in \mathbb{R}^2 = \mathscr{F}^i$ where q denote the charge of the capacitor and ϕ the total magnetic flux in the inductor. The time variation $\frac{dx}{dt}(t)$ of the state variables at time t, may be identified with the following circuit variables: $(i_C, v_L)^T \in \mathbb{R}^2 = \mathscr{F}^i$. The total electro-magnetic energy may be expressed as a function: $H(x)$ and its gradient $\frac{\partial H}{\partial x}(x)$ may be identified as: $(v_C, i_L,)^T \in \mathbb{R}^2 = \mathscr{E}^i$ where v_C denote the charge of the capacitor and i_L the current in the inductor. The two open ports define the external vector spaces, the space of external currents $\mathscr{F}^e = \mathbb{R}^2 \ni (i_1, i_2)$ and the space of external voltages $\mathscr{E}^e = \mathbb{R}^2 \ni (v_1, v_2)$ endowed with the euclidean product as bilinear form. Considering the bilinear form on $\mathscr{F} \times \mathscr{E}$ to be simply the euclidean product in $\mathbb{R}^4$ (i.e. the signature matrix σ is the identity), one may check that Kirchhoff's laws may be expressed

as:

$$\underbrace{\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{F} \begin{pmatrix} i_C \\ v_L \\ i_1 \\ i_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}}_{E} \begin{pmatrix} v_C \\ i_L \\ v_1 \\ v_2 \end{pmatrix} = 0 \quad (10)$$

and define a Dirac structure in $\mathscr{F} \times \mathscr{E}$ (check the conditions of proposition 4). The dynamical system of the *open* LC circuit may hence be defined as a port Hamiltonian system accordingly to the definition 9. It is easy to show that if one assigns some time function to the external variables i_1 and v_2, then the first and third lines of (10) lead to a well-posed set of differential equations with second member. The other two lines allows to compute the conjugated external variables v_1 and i_2 as observation variables allowing to write the energy balance equation (9). Another way of completing the port Hamiltonian system would be to assign again some time function to i_1 but define some dissipative relation on the port 2, such as $v_2 = R\, i_2$ where R is the resistance of some charge added at the port 2.

2.3 Stokes-Dirac structures extending Hamiltonian operators

In this section we shall briefly recall the extension of Hamiltonian systems, derived from a system of two conservation laws, defined with respect to Dirac structures extending the Hamiltonian operator associated with the Hamiltonian system [47]. In this paper we consider the case of a 1-dimensional spatial domain $Z = [0,\, L]$ being a finite interval on the real line but the definition has actually been introduced for spatial domains of any dimension [47]. We shall, however, keep the notation of exterior differential forms [41, 9] (also called k-forms) in order to make explicit the discretization procedure suggested below.

2.3.1 Stokes-Dirac structure for Hamiltonian systems of two conservation laws

Let us first recall some definitions and notations used in the sequel. We shall define the conserved quantities as 1-forms on the interval $Z = [0,\, L]$, whose space will be denoted $\Omega^1(Z)$. Ones a coordinate, denoted by z and corresponding to some measure, is chosen on the interval Z, a 1-form $\alpha \in \Omega^1(Z)$, is written with an abuse of notation: $\alpha = \alpha\,(z)\, dz$ where $\alpha\,(.)$ denotes a smooth function. Hence the state space of a system of two conservation laws is the product space $\Omega^1(Z) \times \Omega^1(Z)$. The space of 0-forms, that is smooth functions on the interval Z, is denoted by $\Omega^0(Z)$.

We shall denote the exterior product of k-forms by $\wedge$ and the exterior derivation by d.[8] Furthermore we shall use the Hodge star associated with the measure

[8] Actually in the case of a 1-dimensional domain these operations become quite trivial. The

dz of the real interval Z and denote it by $\star$. In the coordinates z, the Hodge star product of the 1-form $\alpha(z)\,dz$ is simply the 0-form: $\alpha(x)$.

Between 0-forms $\Omega^0(Z) \ni \beta$ and 1-forms $\Omega^1(Z) \ni \alpha$, one may define a *bilinear form*:

$$< \beta | \alpha >:= \int_Z \beta \wedge \alpha \quad (\in \mathbb{R}) \tag{11}$$

which is simply expressed in coordinates by: $< \beta | \alpha >:= \int_Z \beta(z)\ \alpha(z)\, dz$. The bilinear form (11) is *non-degenerate* in the sense that if $< \beta | \alpha >= 0$ for all α (respectively for all β), then $\beta = 0$ (respectively $\alpha = 0$).

The pairing defined above may also be used in order to define the variational derivative of functional on 1-forms in terms of 0-forms according to the general definition suggested in [47]. Consider an energy density 1-form $\mathcal{H} : \Omega^1(Z) \times Z \rightarrow \Omega^1(Z)$ and denote by $H := \int_Z \mathcal{H} \in \mathbb{R}$ the associated functional. Then for any 1-form $\omega \in \Omega^1(Z)$ and any variation $\Delta\omega \in \Omega^1(Z)$ with compact support strictly included in Z and any $\varepsilon \in \mathbb{R}$, it may be proven that[47] :

$$H(\omega + \varepsilon\Delta\omega) = \int_Z \mathcal{H}(\omega + \varepsilon\Delta\omega) = \int_Z \mathcal{H}(\omega) + \varepsilon \int_Z \left[\frac{\delta H}{\delta \omega} \wedge \Delta\omega\right] + \mathrm{O}\left(\varepsilon^2\right)$$

for a uniquely defined 0-form which will be denoted $\frac{\delta H}{\delta \omega} \in \Omega^0(Z)$ and which is called the *variational derivative* of H with respect to $\alpha \in \Omega^1(Z)$.

Finally we shall also consider real functions defined on the boundary of the spatial domain $\partial Z = \{0,\, L\}$ as 0-forms defined on ∂Z, endowed with the non-degenerated bilinear form:

$$\langle \gamma_1,\, \gamma_2 \rangle_\partial = \gamma_1(L)\,\gamma_2(L) - \gamma_1(0)\,\gamma_2(0) \quad \gamma_i \in \Omega^0(\partial Z),\ i = 1, 2$$

We shall now consider systems of two conservation laws in canonical interaction and then represent them using Dirac structures in the open case (i.e. with boundary energy flows).

Definition 11. Consider the two conserved quantities as being two 1-forms: $q \in \Omega^1(Z)$ and $p \in \Omega^1(Z)$. Consider also the system of conservation laws, with flux variables β_q and β_p for each conserved quantity, defined by the *Hamiltonian density function* $\mathcal{H} : \Omega^1(Z) \times \Omega^1(Z) \times Z \rightarrow \Omega^1(Z)$ resulting in the total Hamiltonian $H := \int_Z \mathcal{H}(q,\, p) \in \mathbb{R}$. The *system of two canonically interacting conservation laws* is then defined by:

$$\frac{\partial}{\partial t}\begin{pmatrix} q \\ p \end{pmatrix} + \mathrm{d}\begin{pmatrix} \beta_q \\ \beta_p \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} \beta_q \\ \beta_p \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta q} \\ \frac{\delta H}{\delta p} \end{pmatrix} \tag{12}$$

where $\varepsilon \in \{-1, +1\}$ depends on the fluxes sign convention on the physical domain.

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wedge product of 0-forms, i.e. functions, is simply their product and the wedge product of a 0-form with a 1-form is again simply the product of the 1-form by the 0-form. The only non-trivial derivation acts on 0-forms and is written in the coordinates z: $\mathrm{d}\beta(z) = \frac{\partial \beta}{\partial z}(z)\ dz$.

This system of two conservation laws may be also written as follows:

$$\frac{\partial}{\partial t}\begin{pmatrix} q \\ p \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta q} \\ \frac{\delta H}{\delta p} \end{pmatrix} \tag{13}$$

that is as an infinite-dimensional Hamiltonian system defined with respect to the *matrix differential operator*:

$$\mathcal{J} = \varepsilon \begin{pmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{pmatrix} \tag{14}$$

and generated by the Hamiltonian function H [35].[9]

In order to generate a Hamiltonian systems, the matrix differential operator $\mathcal{J}$ defined in (14) should satisfy the properties of a *Hamiltonian operator*, that is it should be skew-symmetric and satisfy the Jacobi identities. A short calculus shows that the skew-symmetry holds only for functions *with domain strictly included in the spatial domain* Z. This assumption is satisfied for Dirichlet or Neumann boundary conditions but one might be interested in more general (dynamic) boundary conditions where some energy is exchanged through the boundary of the spatial domain.

Therefore the matrix differential operator $\mathcal{J}$ is extended to a Dirac structure, called Stokes-Dirac structure [47, 28, 26] as follows.

Proposition 12. *[47] Consider the product spaces of k-forms:*

$$\mathcal{F} = \Omega^1(Z) \times \Omega^1(Z) \times \Omega^0(\partial Z) \ni (f_p, f_q, f_b) \tag{15}$$

$$\mathcal{E} = \Omega^0(Z) \times \Omega^0(Z) \times \Omega^0(\partial Z) \ni (e_p, e_q, e_b) \tag{16}$$

Consider the linear subspace $\mathscr{D}$ *of the bond space* $\mathcal{B} = \mathcal{F} \times \mathcal{E}$*:*

$$\begin{array}{rcl} \mathscr{D} & = & \{(f_p, f_q, f_b, e_p, e_q, e_b) \in \mathcal{F} \times \mathcal{E} | \\ & & \left[\begin{array}{c} f_p \\ f_q \end{array}\right] = \varepsilon \left[\begin{array}{cc} 0 & d \\ d & 0 \end{array}\right] \left[\begin{array}{c} e_p \\ e_q \end{array}\right], \\ & & \left[\begin{array}{c} f_b \\ e_b \end{array}\right] = \left[\begin{array}{cc} \varepsilon & 0 \\ 0 & -1 \end{array}\right] \left[\begin{array}{c} e_{p|\partial Z} \\ e_{q|\partial Z} \end{array}\right] \Big\} \end{array} \tag{17}$$

where $\varepsilon \in \{-1, +1\}$ *and* $|_{\partial Z}$ *denotes restriction to the boundary* ∂Z*. Then* D *is a Dirac structure with respect to the non degenerated bilinear form between* $\mathcal{F}$ and $\mathcal{E}$*:*

$$\langle (e_p, e_q, e_b) \,|\, (f_p, f_q, f_b) \rangle = \int_Z [e_p \wedge f_p + e_q \wedge f_q] + \langle e_b,\, f_b \rangle_\partial \tag{18}$$

[9] In the coordinates z, the Hamiltonian system (14) may be written using functions as:

$$\frac{\partial}{\partial t}\begin{pmatrix} q(z) \\ p(z) \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta q}(z) \\ \frac{\delta H}{\delta p}(z) \end{pmatrix}$$

In this case the functional spaces may be defined as Hilbert spaces: some more general cases have been studied in [25].

This theorem is proved by using the properties of the exterior derivation; actually it corresponds, for general dimensions of the spatial domain, to Stokes' theorem [47, 28, 26]. For the sake of clarity we shall just explicit the condition of isotropy: $\mathscr{D} \subset \mathscr{D}^{\perp}$. Computing the symmetric bilinear product and using integration by parts gives:

$$\int_0^L e_1^p(z)de_2^q(z) + e_1^q(z)de_2^p(z) + e_2^p(z)de_1^q(z) + e_2^q(z)de_1^p(z) \\ - e_{\partial 1}^0 f_{\partial 2}^0 + e_{\partial 1}^L f_{\partial 2}^L - f_{\partial 1}^0 e_{\partial 2}^0 + f_{\partial 1}^L e_{\partial 2}^L = \\ - \int_0^L d(e_1^p(z)e_2^q(z)) + d(e_1^q(z)e_2^p(z)) - e_{\partial 1}^0 f_{\partial 2}^0 + e_{\partial 1}^L f_{\partial 2}^L - f_{\partial 1}^0 e_{\partial 2}^0 + f_{\partial 1}^L e_{\partial 2}^L = 0 \tag{19}$$

For the proof of the condition of co-isotropy : $\mathscr{D}^{\perp} \subset \mathscr{D}$, the reader is referred to [47] and in the 1-dimensional case to [26].

Remark 13. We haved derived the Dirac structure extending the canonical Hamiltonian operator[10] defined in (14). It should be noted that one may construct, in a similar way, the Stokes-Dirac structures extending the Hamiltonian operator associated with higher order Hamiltonian operators [25] or for spatial dimension higher than 1. In this latter case, the restriction of a function to the two boundary points which is used in the definition of the port boundary variables, becomes the trace operator.

As a consequence of proposition 12 one may define a Hamiltonian system with respect to this Stokes-Dirac structure as follows.

Definition 14. The boundary *port-Hamiltonian system of two conservation laws* with state space $\Omega^1(Z) \times \Omega^1(Z) \ni (q,\, p)$ and boundary port variables spaces $\Omega^0(\partial Z) \times \Omega^0(\partial Z) \ni (f_b,\, e_b)$, is the Hamiltonian system defined with respect to the Stokes-Dirac structure D given in proposition 12 and generated by the Hamiltonian functional $H\,(q,\, p)$, as follows:

$$\left(\left(\left(-\frac{\partial p}{\partial t}, -\frac{\partial q}{\partial t}\right), f_b\right), \left(\left(\frac{\delta H}{\delta p}, \frac{\delta H}{\delta q}\right), e_b\right)\right) \in D$$

The choice of boundary conditions has obviously to be added to the definition of a boundary port-Hamiltonian system in order to define a Cauchy problem. In fact a boundary port Hamiltonian system defines a class of well-posed systems. For any solution, the isotropy condition of the Dirac structure implies the balance equation on the Hamiltonian:

$$\frac{dH}{dt} = \langle e_b,\, f_b \rangle_{\partial} \tag{20}$$

Remark 15. One may also define port variables with support in the spatial domain by considering higher dimensional Hamiltonian operators and the associated Dirac structure [47, 27].

[10] One may find a very stimulating discussion about the notion of canonical Hamiltonian operator in [33].

2.3.2 Examples of boundary port Hamiltonian systems

In this section we present two examples of boundary port Hamiltonian systems. They are, firstly, the (linear) ideal transmission line and, secondly, the (nonlinear) shallow water equations with non-separated Hamiltonian.

Example 16 (Ideal transmission line)**.** Consider an ideal lossless transmission line defined on the interval $Z = [0,\, L]$. The state variables are the charge density 1-form $q = q(t,z)dz \in \Omega^1([0,L])$, and the flux density 1-form $p = p(t,z)dz \in \Omega^1([0,L])$ where $t \geq 0$ denotes the time variable. The total energy stored at time t in the transmission line is given as

$$\begin{aligned} H(q,\,p) &= \int_0^L \frac{1}{2}\left(\frac{1}{C(z)} \star q \wedge q + \frac{1}{L(z)} \star p \wedge p\right) dz \\ &= \int_0^L \frac{1}{2}\left(\frac{q^2(t,z)}{C(z)} + \frac{p^2(t,z)}{L(z)}\right) dz \end{aligned} \tag{21}$$

where $C(z)$, $L(z)$ are respectively the distributed lineic capacitance and inductance of the line. Its variational derivatives with respect to the state variables are:

$$\begin{aligned} \tfrac{\delta H}{\delta q} &= \tfrac{1}{C(z)} \star q &&= V(t,z) \quad \text{(voltage)} \\ \tfrac{\delta H}{\delta p} &= \tfrac{1}{L(z)} \star p &&= I(t,z) \quad \text{(current)} \end{aligned} \tag{22}$$

The dynamics of the transmission line equation may be expressed as the Hamiltonian system:

$$\frac{\partial}{\partial t}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta q} \\ \frac{\delta H}{\delta p} \end{pmatrix} \tag{23}$$

augmented, according to (2.3), with the boundary variables

$$\begin{aligned} f_b^0(t) &= V(t,0), \quad & f_b^1(t) &= V(t,L) \\ e_b^0(t) &= -I(t,0), \quad & e_b^1(t) &= -I(t,L) \end{aligned} \tag{24}$$

which are simply the voltage and the currents at both boundary points of the spatial domain. The resulting energy-balance is

$$\frac{dH}{dt} = \langle e_b,\, f_b \rangle_\partial = -\left(I(t,L)V(t,L) - I(t,0)V(t,0)\right) \tag{25}$$

Example 17 (The shallow water equation)**.** We consider the case of a 1D shallow water flow of length, L, defined on the spatial domain, $Z = [0, L]$, with a non uniform reach such as the one represented on figure 3. Such flows are usually modelled using the shallow water equations, also known as Saint-Venant equations [21, 13]. For simplicity we consider frictionless and horizontal flows. Developments for the general case with frictions and slope may be found in [22]. Quite natural energy state variables are the lineic mass and momentum

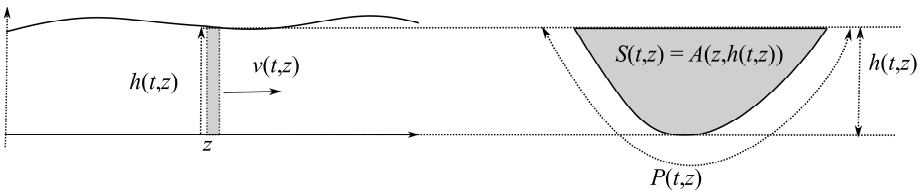

Figure 3: Schematic longitudinal view (left) and wetted cross section (right) of a shallow water flow in a non-uniform canal or river reach. $z \in [0, L]$ is the longitudinal spatial coordinate, $h(t, z)$ the water level, $v(t, z)$ the water horizontal velocity, $P(t, z)$ the wetted perimeter and $S(t, z)$ the wetted cross section area. The cross section of the reach is defined using the function $A(z, h)$ which relates the water level, $h(t, z)$, and the wetted cross section area, $S(t, z)$.

densities, respectively $q(t, z) = \rho S(t, z)dz \in \Omega^1([0, L])$ and $p(t, z) = \rho v(t, z)dz \in \Omega^1([0, L])$, where ρ is the water mass density. Indeed, the total energy stored in the reach may be quite easily computed as the sum of kinetic and potential energies:

$$\begin{aligned} H(q,p) &= H_{pot}(q) + H_{cin}(q,p) \\ &= \int_0^L \rho \left(g \left(hA(z,h) - \int_0^h A(z,\xi)d\xi \right) + \frac{S(t,z)v^2(t,z)}{2} \right) dz \end{aligned} \tag{26}$$

where g denotes the gravity acceleration. The variational derivatives of this total energy with respect to the states variables are the two potentials (functions)

$$\begin{aligned} \delta_q \mathcal{H} &= \frac{v^2}{2} + gh \\ \delta_p \mathcal{H} &= Sv \end{aligned} \tag{27}$$

which are respectively the hydrodynamic pressure, $p_{dyn}(t, z)$, and the water flow, $Q(t, z)$. Considering the boundary port-Hamiltonian system of two conservation laws from definition 14, the canonical dynamics reads:

$$\left\{ \begin{array}{l} \frac{\partial S}{\partial t} = -\frac{\partial}{\partial z}(Sv) \\ \frac{\partial v}{\partial t} = -\frac{\partial}{\partial z}\left(\frac{v^2}{2} + gh \right) \end{array} \right. \tag{28}$$

and the boundary port variables are

$$\left[\begin{array}{c} e_\partial \\ f_\partial \end{array} \right] = \left[\begin{array}{c} -\delta_q \mathcal{H}|_{\partial[0,L]} \\ \delta_p \mathcal{H}|_{\partial[0,L]} \end{array} \right] = \left[\begin{array}{c} -p_{dyn}|_{\partial[0,L]} \\ Q|_{\partial[0,L]} \end{array} \right] \tag{29}$$

Equations (28) are exactly the classical shallow water equations. The boundary port-variables are the hydrodynamic pressures and the water flows at both ends of the reach. The energy balance reads here

$$\frac{dH}{dt} = \int_{\partial[0,L]} e_\partial \wedge f_\partial = p_{dyn}(0,t)Q(0,t) - p_{dyn}(L,t)Q(L,t) \tag{30}$$

3 A geometric discretization scheme using polynomial bases

In this section we shall suggest a pseudo-spectral discretization method which is adapted to the geometric nature of the variables (0- or 1-forms) and furthermore discretizes the Stokes-Dirac structure into a finite-dimensional Dirac structure. We will use, for the spatial discretization, polynomial approximation bases (with Lagrange interpolation) in such a way that the reduced variables will be approximations of the distributed ones at chosen "collocation" points. Usually these points are chosen as zeros of orthogonal polynomials in order to reduce the oscillations of the solution. Firstly we show how the choice of the bases for the effort and flow variables allows the exact discretization of the exterior derivative and the restriction of the effort variables to the boundary points. Secondly we analyze the product between the approximation spaces for the effort and flow variables and show that it is degenerated. Hence it may not be used to defined a reduced Dirac structure. Thirdly we use the kernel of this product in order to project the effort variables and define in such a way the desired reduced Dirac structure.

3.1 Polynomial approximation and discretization of the Stokes-Dirac structure

Following the work of Bossavit [6, 7], we shall account for the geometric nature of the effort and flow variables in order to define the spaces of approximations. In [19] the mixed-finite element method has been adapted in order to reduce the Stokes-Dirac structure to finite-dimensional Dirac structure. In this paper we shall consider polynomial approximation bases.

According to their definition in the proposition 12, the effort variables are approximated in a basis of polynomial 0-forms and the flux variables are approximated in a basis of polynomial 1-forms one. Furthermore we wish to discretize exactly the exterior derivation which applies to the effort variables in the definition of the Dirac structure in the proposition 12. Hence the polynomials approximating the 0-forms should be of degree greater by 1 than the degree of the polynomials approximating the 1-forms. Hence we suggest to define the

approximations as follows :

$$e^q(z) = \sum_{i=0}^{N} e_i^q \varphi_i(z)\,, \quad e_i^q \in \mathbb{R} \tag{31}$$

$$e^p(z) = \sum_{i=0}^{N} e_i^p \varphi_i(z)\,, \quad e_i^p \in \mathbb{R} \tag{32}$$

$$f^q(z) = \sum_{k=0}^{N-1} f_k^q \psi_k(z)dz\,, \quad f_i^q \in \mathbb{R} \tag{33}$$

$$f^p(z) = \sum_{k=0}^{N-1} f_k^p \psi_k(z)dz\,, \quad f_i^p \in \mathbb{R} \tag{34}$$

where $\varphi_i(z)$ and $\psi_k(z)$ are interpolating Lagrange polynomials, respectively of degree N and $N-1$ defined as

$$\varphi_i(z) = \prod_{j=0, j\neq i}^{N} \frac{z-\zeta_j}{\zeta_i-\zeta_j} \;;\; \psi_k(z) = \prod_{l=0, l\neq k}^{N-1} \frac{z-z_l}{z_k-z_l} \tag{35}$$

satisfying $\varphi_i(\zeta_j) = \delta_{ij}$ and $\psi_k(z_l) = \delta_{kl}$, $\zeta_j \in]0, L[$, $j = 0, ..., N$ being the interpolating points associated to the φ_j base, while $z_l \in]0, L[$ are those of the ψ_k base, $l \in \{0, ..., N-1\}$.

Let us further denote by $\Omega_r^0(Z)$ the space of 0-forms generated by the functions $\varphi_i(z)$ and by $\Omega_r^1(Z)$ the space of 1-forms generated by the 1-forms $\psi_k(z)dz$. Let us define the vector spaces of the coordinates of the approximating forms as the space of flow variables: $\mathscr{F}_r = \mathbb{R}^{2N+2} \ni (\mathbf{f}^p, \mathbf{f}^q, f_\partial^0, f_\partial^L)$ and the space of effort variables: $\mathscr{E}_r = \mathbb{R}^{2N+4} \ni (\mathbf{e}^p, \mathbf{e}^q, e_\partial^0, e_\partial^L)$ [11].
Inserting relations (31) - (34) into the definition of the canonical Hamiltonian operator (14), and evaluating the approximations at the collocation points z_l, one compute the restriction of the exterior derivation and the Hamiltonian operator to the approximation spaces. This leads to the following matrix relations on the coefficients of the approximations:

$$\begin{aligned} f_k^q &= \sum_{i=0}^{N} D_{k,i} e_i^p \\ f_k^p &= \sum_{i=0}^{N} D_{k,i} e_i^q \end{aligned} \tag{36}$$

where D is a $N \times (N+1)$ matrix obtained by evaluating the derivation of the polynomial 0-forms at the collocation points used for the interpolation polyno-

[11] We use $(\mathbf{f}^p, \mathbf{f}^q)$ and $(\mathbf{e}^p, \mathbf{e}^q)$ to denote the coordinate vectors of respectively the flows and efforts approximation forms. Due to the interpolation property of the Lagrange polynomials (35), these vector coordinates are respectively $(f_k^p, f_k^q) = (f^p(z_k), f^q(z_k))$ and $(e_i^p, e_i^q) = (e^p(z_i), e^q(z_i))$.

mials of the 1-forms:

$$D_{k,i} = \frac{d\varphi_i}{dz}(z_k)$$

The boundary port variables are defined accordingly to (17) as the polynomial interpolation of the effort variables e^p, e^q (0-forms) at the two-boundary points $z = 0$ and $z = L$:

$$\begin{aligned} e^0_\partial &= e^q(0) = \sum_{i=0}^{N} e^q_i \varphi_i(0) \\ e^L_\partial &= e^q(L) = \sum_{i=0}^{N} e^q_i \varphi_i(L) \\ f^0_\partial &= e^p(0) = \sum_{i=0}^{N} e^p_i \varphi_i(0) \\ f^L_\partial &= e^p(L) = \sum_{i=0}^{N} e^p_i \varphi_i(L) \end{aligned} \tag{37}$$

Equations (36) and (37) giving the projections of the Stokes-Dirac structure (17) in the chosen approximation spaces may be summarized in the matrix form:

$$\begin{pmatrix} \mathbf{f}^q \\ \mathbf{f}^p \\ f^0_\partial \\ f^L_\partial \\ e^0_\partial \\ e^L_\partial \end{pmatrix} = \begin{pmatrix} 0 & D \\ D & 0 \\ 0 & \varphi(0)^T \\ 0 & \varphi(L)^T \\ \varphi(0)^T & 0 \\ \varphi(L)^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}^q \\ \\ \mathbf{e}^p \end{pmatrix} \tag{38}$$

which expresses the boundary effort variables $\left(e^0_\partial, e^L_\partial\right) \in \mathbb{R}^2$, and the flow variables $\left(\mathbf{f}^p, \mathbf{f}^q, f^0_\partial, f^L_\partial\right) \in \mathscr{F}_r = \mathbb{R}^{2N+2}$ in terms of the effort variables $(\mathbf{e}^q, \mathbf{e}^p) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$. The vectors $\varphi(0) \in \mathbb{R}^{N+1}$ and $\varphi(L) \in \mathbb{R}^{N+1}$ denote the vectors of the polynomials $\varphi_i(z)$ evaluated respectively at the boundary points $z = 0$ and $z = L$.

3.2 Restricted bilinear product and Stokes' theorem

Consider now the bilinear product (18) and evaluate the associated symmetrized bilinear product, according to (2), using the polynomial approximations of the effort and flow variables (31), (32), (33) and (34). This leads to the following symmetric bilinear form on the product space of reduced effort and flow variables $\mathscr{F}_r \times \mathscr{E}_r$:

$$\begin{pmatrix} \mathbf{e}^p_1 \\ \mathbf{e}^q_1 \\ e^0_1 \\ e^L_1 \\ \mathbf{f}^p_1 \\ \mathbf{f}^q_1 \\ f^0_1 \\ f^L_1 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 0 & M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ M^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}^p_2 \\ \mathbf{e}^q_2 \\ e^0_2 \\ e^L_2 \\ \mathbf{f}^p_2 \\ \mathbf{f}^q_2 \\ f^0_2 \\ f^L_2 \end{pmatrix} \tag{39}$$

where M is the $(N+1) \times N$ matrix whose elements are

$$M_{i,k} = \int_0^L \varphi_i(z)\psi_k(z)dz \tag{40}$$

As a consequence of the choice of the approximation spaces with different dimension for the reduced effort and flow variables, the matrix M^T has a non trivial kernel $Ker(M^T) \subset \mathscr{E}_r$. Hence the symmetric pairing (39) is degenerated and one cannot use it for the definition of a Dirac structure according to the definition 1.

However the relations (38) still define a vector subspace of the *bond space* $\mathscr{F}_r \times \mathscr{E}_r$ where relations corresponding to Stokes' theorem are satisfied. This may be expressed in terms of the isotropy of this vector subspace with respect to the degenerated product (39).[12]

Proposition 18. *The subspace*

$$\mathscr{D}_r = \left\{(\mathbf{f}^p, \mathbf{f}^q, f^0_\partial, f^L_\partial, \mathbf{e}^p, \mathbf{e}^q, e^0_\partial, e^L_\partial) \in \mathscr{F}_r \times \mathscr{E}_r \text{ / satisfying (38)}\right\}$$

satisfies the isotropy condition: $\mathscr{D}_r \subset \mathscr{D}_r^\perp$ with respect to the symmetric power product (39).

Proof. We shall consider the expression (19) of the isotropy condition of the Stokes-Dirac structure and substitute therein the expressions of discrete (reduced) effort and flux variables in (31) (32), (33) and (34) and the reduced bilinear product defined in (40). One obtains:

$$\begin{aligned} 0 = &\int_0^L e_1^p(z)de_2^q(z) + e_1^q(z)de_2^p(z) + e_2^p(z)de_1^q(z) + e_2^q(z)de_1^p(z) \\ &- e^0_{\partial 1}f^0_{\partial 2} + e^L_{\partial 1}f^L_{\partial 2} - f^0_{\partial 1}e^0_{\partial 2} + f^L_{\partial 1}e^L_{\partial 2} \\ = &\,\mathbf{e}_1^{pT}M\mathbf{f}_2^p + \mathbf{e}_1^{qT}M\mathbf{f}_2^q + \mathbf{f}_1^{pT}M^T\mathbf{e}_2^p + \mathbf{f}_1^{qT}M^T\mathbf{e}_2^q \\ &- e^0_{\partial 1}f^0_{\partial 2} + e^L_{\partial 1}f^L_{\partial 2} - f^0_{\partial 1}e^0_{\partial 2} + f^L_{\partial 1}e^L_{\partial 2} \end{aligned} \tag{41}$$

which proves the isotropy of $\mathscr{D}_r$. □

It is useful to note that the isotropy condition (41) results in a relation between the matrices appearing in the discretization of the Stokes-Dirac structure (see relations (36) and (37)) and in the discretization of bilinear product (39). Indeed, using these discretized relations, the bilinear product in (41) may be expressed in terms of bilinear product in the effort variables ($\mathbf{e}_1^q$, $\mathbf{e}_1^p$) and ($\mathbf{e}_2^q$, $\mathbf{e}_2^p$)(which are the coordinates of the space of 0-forms):

$$\mathbf{e}_1^q(MD + D^TM^T - T^0 + T^L)\mathbf{e}_2^p + \mathbf{e}_1^p(MD + D^TM^T - T^0 + T^L)\mathbf{e}_2^q = 0 \tag{42}$$

[12] In [20, chap.5] a similar isotropic subspace is called *Tellegen structure*, however it is there defined with respect to a non-degenerated symmetric bilinear product.

where T^0 is the $(N+1)\times(N+1)$ matrix with elements $T^0_{ij}=\varphi_i(0)\varphi_j(0)$, and T^L the $(N+1)\times(N+1)$ matrix with elements $T^L_{ij}=\varphi_i(L)\varphi_j(L)$. Since (42) holds for any $(\mathbf{e}_1^q,\mathbf{e}_2^p)\in\mathbb{R}^{2N+2}$ and for any $(\mathbf{e}_1^p,\mathbf{e}_2^q)\in\mathbb{R}^{2N+2}$, we deduce:

$$MD+D^TM^T-T^0+T^L=0 \tag{43}$$

This last relation may be regarded as a *discrete Stokes theorem* resulting from the translation in the chosen finite approximation spaces of Stokes' theorem for the integration of differential forms. It relates thus logically the *discrete derivation operator* D, the *discrete bilinear product* M and the *discrete trace operator* T^L-T^0.

Remark 19. Consider the case when $T^L-T^0=0$ in (43), which occurs for systems where the effort variables are 0 at the boundaries as well as their approximations, yielding to $\varphi_i(0)=0$, $i=1,..,N+1$. This corresponds to the case where there is no exchange of energy through the boundaries. One gets then $MD+D^TM^T=0$. In this case the matrix MD is skew-symmetric and defines a Poisson tensor. The condition (43) may thus be viewed as the extension of a skew-symmetry property characterizing Dirac structures.

3.3 Dirac structure on a reduced coordinate space

The fact that the discretized relations (38) satisfy the isotropy condition but do not define a Dirac structure is related to the fact that the dimension of the spaces of effort and flow variables are different and that the power product (39) is degenerated and admits the kernel:

$$\ker(\ll\cdot,\cdot\gg)=0_{\mathscr{F}_r}\times 0_{\mathscr{F}_r}\times\ker(M^T)\times\ker(M^T)$$

Hence in the sequel we shall define a space of effort variables which is the quotient of the Tellegen structure by the kernel of M^T. Therefore we shall define the following effort variables $\tilde{\mathbf{e}}^q$, $\tilde{\mathbf{e}}^p\in\mathbb{R}^N$ defined as

$$\begin{aligned}\tilde{\mathbf{e}}^q&=M^T\mathbf{e}^q\\ \tilde{\mathbf{e}}^p&=M^T\mathbf{e}^p\end{aligned} \tag{44}$$

which are indeed, since the matrix M has rank N, coordinate vectors for the quotient space $\mathscr{E}_r\diagup\ker(M^T)$. The degenerated bilinear product (39) reduces then simply, modulo (44), to the *non-degenerated* symmetric bilinear product (2) induced by the Euclidean product in $\mathbb{R}^{2N+2}$.

Let us now observe that the discretized relations (38) and the definition (44) may be written in terms of the image representation (see the proposition 4):

$$\begin{pmatrix}\mathbf{f}^q\\ \mathbf{f}^p\\ f^0_\partial\\ f^L_\partial\end{pmatrix}=\underbrace{\begin{pmatrix}0&D\\ D&0\\ 0&\varphi(0)^T\\ 0&\varphi(L)^T\end{pmatrix}}_{E^T}\begin{pmatrix}\mathbf{e}^q\\ \mathbf{e}^p\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}\tilde{\mathbf{e}}^q\\ \tilde{\mathbf{e}}^p\\ e^0_\partial\\ e^L_\partial\end{pmatrix}=\underbrace{\begin{pmatrix}M^T&0\\ 0&M^T\\ \varphi(0)^T&0\\ \varphi(L)^T&0\end{pmatrix}}_{F^T}\begin{pmatrix}\mathbf{e}^q\\ \mathbf{e}^p\end{pmatrix} \tag{45}$$

which defines a Dirac structure as stated in the following proposition.

Proposition 20. *Define the flow variables,* $\tilde{\mathbf{f}}^T := \left(\mathbf{f}^q, \mathbf{f}^p, f_\partial^0, f_\partial^L\right)$ *and the effort variables,* $\tilde{\mathbf{e}}^T := \left(\tilde{\mathbf{e}}^q, \tilde{\mathbf{e}}^p, e_\partial^0, e_\partial^L\right)$*, in the bond space,* $\tilde{\mathscr{F}} \times \tilde{\mathscr{E}} = \mathbb{R}^{2N+2} \times \mathbb{R}^{2N+2}$*, endowed with the bilinear Euclidean product. Define the structure matrices,*

$$F = \begin{pmatrix} M & 0 & \varphi(0) & \varphi(L) \\ 0 & M & 0 & 0 \end{pmatrix} \quad \textit{and} \quad E = \begin{pmatrix} 0 & D^T & 0 & 0 \\ D^T & 0 & \varphi(0) & \varphi(L) \end{pmatrix} \tag{46}$$

Then the vector subspace of $\tilde{\mathscr{F}} \times \tilde{\mathscr{E}}$ *defined by:*

$$\mathscr{D}_r = \left\{ \tilde{\mathbf{f}} \in \tilde{\mathscr{F}},\ \tilde{\mathbf{e}} \in \tilde{\mathscr{E}} \,\middle|\, \tilde{\mathbf{f}} = E^T\lambda\,,\ \tilde{\mathbf{e}} = F^T\lambda\,,\ \lambda \in \mathbb{R}^{2N+2} \right\}$$

is a Dirac structure.

Proof. We shall check the two conditions on the structure matrices given in the proposition 4.

(i) skew-symmetry: $E\sigma F^T + F\sigma E^T = 0$

Let us compute:

$$\begin{aligned} E\sigma F^T + F\sigma E^T = &\begin{pmatrix} 0 & D^T & 0 & 0 \\ D^T & 0 & \varphi(0) & \varphi(L) \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} M^T & 0 \\ 0 & M^T \\ \varphi(0)^T & 0 \\ \varphi(L)^T & 0 \end{pmatrix} \\ &+ \begin{pmatrix} M & 0 & \varphi(0) & \varphi(L) \\ 0 & M & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & D \\ D & 0 \\ 0 & \varphi(0)^T \\ 0 & \varphi(L)^T \end{pmatrix} \end{aligned} \tag{47}$$

where 1 denotes an identity matrix of appropriate dimension. After some elementary calculations (47) gives:

$$E\sigma F^T + F\sigma E^T = \begin{pmatrix} 0 & D^T M^T + MD - T^0 + T^L \\ D^T M^T + MD - T^0 + T^L & 0 \end{pmatrix} \tag{48}$$

which, according to (43) , implies: $E\sigma F^T + F\sigma E^T = 0$

(ii) rank condition: $[E : F]$ is full rank $(2N + 2)$

Actually we shall show that the matrix E defined in (46) has rank $(2N+2)$. Using the structure of the matrix E, it is sufficient to show that

$$\{D_1^T, D_2^T, ..., D_N^T, \varphi(0)\}$$

is an independent set where $D_l^T = \frac{d\varphi}{dz}(z_l)$ are the N column vectors of the transpose discrete derivation matrix D^T (the $(N + 1) \times N$ matrix defined in (36)). The z_l, $l = 1, 2, ..., N$ are the N interpolating points chosen for the polynomial basis $\{\psi_k\}$.

By contradiction, let us assume that the set under consideration is dependent. In this case one can write $\varphi(0)$ as a linear combination of the $\frac{d\varphi}{dz}(z_l)$. The polynomials $\varphi'(z)$ are of order $(N-1)$ and consequently are uniquely determined by their values $\frac{d\varphi}{dz}(z_l)$ at the interpolation points z_l, $i = 1, 2, ..., N$. Hence these N interpolating conditions would be sufficient to characterize uniquely the Nth. order polynomials

$$\varphi(z) = \varphi(0) + \int_0^z \varphi'(\zeta) d\zeta$$

This clearly contradicts the classical uniqueness result on polynomial interpolation.
Consequently $(\varphi'(z_1), \varphi'(z_2), ..., \varphi(0))$ is an independent (and maximal) set in $\mathbb{R}^{n+1}$. Thus,

$$\begin{pmatrix} D^T & \varphi(0) & 0 & 0 \\ 0 & 0 & D^T & \varphi(0) \end{pmatrix} \tag{49}$$

is an independent and maximal set in $\mathbb{R}^{2n+2}$ and thus the concatenated matrix $[E : F]$ is full rank $2n + 2$. □

According to the proposition 6, there exist also a input-output representation of the Dirac structure $\mathscr{D}_r$ defined in the proposition 20. This input-output represntation is definied in the following proposition.

Proposition 21. *These input-output representation of the Dirac structure $\mathscr{D}_r$ defined in the proposition 20 is defined by:*

$$\begin{pmatrix} \mathbf{f}^q \\ f_\partial^L \\ \mathbf{f}^p \\ -e_\partial^0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \begin{pmatrix} D \\ \varphi(L) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-1} \\ \begin{pmatrix} D \\ -\varphi(0) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(L) \end{pmatrix}^{-1} & 0 \end{pmatrix}}_{J} \begin{pmatrix} \tilde{\mathbf{e}}^q \\ e_\partial^L \\ \tilde{\mathbf{e}}^p \\ f_\partial^0 \end{pmatrix} \tag{50}$$

The matrix J results from simple matrix operations. We can check directly that it is skew-symmetric. Computing the anti-diagonal element's sum

$$\begin{aligned} \begin{pmatrix} D \\ \varphi(L) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-1} + \left(\begin{pmatrix} D \\ -\varphi(0) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(L) \end{pmatrix}^{-1} \right)^T &= \\ \left(M\varphi(L)^T\right)^{-1} \left(MD + D^T M^T - T^0 + T^L\right) \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-1} &= 0 \end{aligned} \tag{51}$$

using the *discrete Stokes' theorem* (43).

Remark 22. Actually the input output representation (50) is associated to a particular choice of inputs and outputs. It may be generalized by choosing as input, u, and output, y, variables any linear combinations of the boundary port variables $e_\partial := e_q \,|_{\partial Z}$ and $f_\partial := e_p \,|_{\partial Z}$:

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} W \\ \tilde{W} \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \tag{52}$$

The definition of the matrices W and $\tilde{W}$ is not further discussed here but may be related to the definition of boundary control systems associated with the port Hamiltonian systems [25].

4 Reduced Hamiltonian system

In the previous section we have presented the reduction of the Stokes-Dirac structure on a polynomial approximation space as a finite-dimensional Dirac

structure. In this section, we will derive the corresponding approximation of the boundary Port Hamiltonian system by restricting the Hamiltonian functional to the approximation spaces in the cases of our two running examples: the lossless transmission line and the shallow water equations.

4.1 The lossless transmission line

Recall that the Hamiltonian functional is defined by:

$$H = H^q + H^p = \int_0^L \frac{\star q(z,t)}{2C(z)} q(z,t) + \frac{\star p(z,t)}{2L(z)} p(z,t) \tag{53}$$

and consider its time variation:

$$\frac{dH}{dt} = \int_0^L \frac{\star q(z,t)}{C(z)} \dot{q}(z,t) + \frac{\star p(z,t)}{L(z)} \dot{p}(z,t) \tag{54}$$

which may be identified with the general expression in terms of the variational derivative of the Hamiltonian:

$$\frac{dH}{dt} = \int_Z \delta_q H \wedge \dot{q} + \delta_p H \wedge \dot{p} = \int_Z e^q \wedge f^q + e^p \wedge f^p \tag{55}$$

The Hamiltonian $H(p,q)$ depends on the 1-forms p and q. They are approximated using the ψ polynomial base as in section 3

$$\begin{aligned} q(z,t) &= \sum_{k=0}^{N-1} q_k(t)\psi_k(z)dz \\ p(z,t) &= \sum_{k=0}^{N-1} p_k(t)\psi_k(z)dz \end{aligned} \tag{56}$$

Inserting (56) into (54) the time variation of the Hamiltonian, restricted to the approximation space is:

$$\frac{d\bar{H}}{dt} = \mathbf{q}^T(t) C \dot{\mathbf{q}}(t) + \mathbf{p}^T(t) L \dot{\mathbf{p}}(t) \tag{57}$$

where $\mathbf{q}$ and $\mathbf{p}$ are respectively the vectors with coordinates q_k and p_k for $k \in \{1, \ldots, N-1\}$ and

$$C_{ij} = \int_0^L \frac{\psi_i(z)\psi_j(z)}{C(z)} dz$$

and

$$L_{ij} = \int_0^L \frac{\psi_i(z)\psi_j(z)}{L(z)} dz$$

Defining the Hamiltonian on the coefficients of the approximations by $\bar{H}(\mathbf{q}, \mathbf{p})$, its time variation is:

$$\frac{d\bar{H}}{dt} = \sum_{i=0}^{N-1} \frac{\partial \bar{H}}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial \bar{H}}{\partial p_i} \frac{dp_i}{dt} \tag{58}$$

Identifying (58) with (57) we get the expression of the gradient of $\bar{H}$:

$$\begin{aligned} e_i^q(t) &= \frac{\partial \bar{H}}{\partial q_i} = \sum_{j=0}^{N} C_{ij} q_j(t) \\ e_i^p(t) &= \frac{\partial \bar{H}}{\partial p_i} = \sum_{j=0}^{N} L_{ij} p_j(t) \end{aligned} \tag{59}$$

Note that, by construction, the matrices (C_{ij}) and (L_{ij}) are symmetric. The reduced Hamiltonian system is then defined with respect to the Dirac structure defined in the proposition 20 and generated by the Hamiltonian $\bar{H}(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\sum_{j=0}^{N} q_i C_{ij} q_j + \frac{1}{2}\sum_{j=0}^{N} p_i L p_j$.

4.2 Shallow water equations

In this example, the Hamiltonian of the system depends on the the lineic mass density $q(z,t) = \rho S(z,t)dz$ and the lineic momentum density $p(z,t) = \rho v(z,t)dz$. Again we consider that the state variables will be approximated using the ψ polynomial approximation base, according to (56), where here $q_k(t) = \rho S(z_k,t)$ and $p_k(t) = \rho v(z_k,t)$ are respectively proportional to the wetted cross section area and water mean velocity at the collocation point z_k. Let us recall that the time variation of the Hamiltonian functional is given by:

$$\begin{aligned} \frac{dH}{dt} &= \int_Z \delta_q H \wedge \dot{q} + \delta_p H \wedge \dot{p} \\ &= \int_Z (\frac{v^2}{2} + gh)\dot{q} + (Sv)\dot{p} \end{aligned} \tag{60}$$

Inserting the previous polynomial approximation in this last expression gives:

$$\frac{d\bar{H}}{dt} = F^T \dot{\mathbf{q}}(t) + G^T \dot{\mathbf{p}}(t) \tag{61}$$

where

$$F_k = \int_0^L (\frac{v^2(z,t)}{2} + gh(z,t))\psi_k(z)\, dz \tag{62}$$

and

$$G_k = \int_0^L (S(t,z)v(t,z))\psi_k(z)\, dz \tag{63}$$

Note that the reach geometry usually defines a relation

$$S(z) = A(z, h(z)) \tag{64}$$

between the wetted cross section area, A, and the water level, h, at any given location, z. Hence, equations (62) and (63) in fact defines efforts vector:

$$\begin{aligned} \mathbf{e}_q &:= F(\mathbf{q}, \mathbf{p}) \\ \mathbf{e}_p &:= G(\mathbf{q}, \mathbf{p}) \end{aligned} \tag{65}$$

In this case, the efforts are nonlinear functions of the state variables $\mathbf{q}$ and $\mathbf{p}$ since the Hamiltonian is not quadratic. Moreover, they are implicitly defined trough the relation (64) which specifies the flow (reach) geometry. For instance a canal reach with a rectangular cross section geometry gives a relation $S = Bh$ where B is the canal width. In this case (64) is trivially invertible and equations (62) and (63) define explicitly the effort variables.

5 Symplectic collocation methods

We would like to compare the reduction method we proposed in this paper with an existing collocation scheme from a geometrical point of view. More precisely we would like to investigate both symplecticity and spectral properties of the considered reduction schemes. The reduction method we developed here is symplectic for closed and open systems (systems with boundary energy flows), by construction, since it preserves both the Dirac structure and the Hamiltonian. Usually, *classical collocation schemes* are not. This will indeed be numerically checked on the example of an ideal (lossless) transmission line in section 6. However a classical collocation scheme is indeed symplectic for closed Hamiltonian systems (only) when the chosen collocation points are zeros of either Legendre polynomials, as it will be shown hereafter. It will thus be possible to perform numerical comparisons between our method and this classical collocation scheme.

To perform this comparison, we choose as the polynomial approximating basis for the collocation scheme, the same representation as previously for the approximation of 1-forms (state variables), namely the ψ_i, i=1,...,N basis. Indeed, the reduced system is finally expressed in terms of coordinates of the state variables in this basis.

5.1 Symplecticity of spatial discretization schemes and classical collocation schemes

For the sake of simplicity, we will consider throughout this section the Dirichlet conditions $q(-1) = 0$ and $p(+1) = 0$. For instance, in the transmission line example, this corresponds to zero voltage at $z = -1$ and zero current at $z = 1$. There are obviously other possible boundary conditions resulting in a closed system (with no energy flows through the boundaries). The following proposition states the formal skew-symmetry property of the canonical differential operator (14), for these boundary conditions.

Proposition 23. *Consider the canonical differential operator (14)*

$$\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$$

in a space domain $[-1, +1]$*, and acting on some state variables* $q, p \in C^1(-1, +1)$*.*

This canonical operator is formally skew-symmetric with respect to the integration when considering the Dirichlet conditions $q(-1) = 0$ and $p(+1) = 0$.

Proof. Actually,

$$\begin{aligned}\int_{-1}^{+1} \begin{pmatrix} q(z) & p(z) \end{pmatrix} \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \begin{pmatrix} q(z) \\ p(z) \end{pmatrix} &= \int_{-1}^{+1} q(z)dp(z) + p(z)dq(z) \\ &= \int_{-1}^{+1} d(q(z)p(z)) = [q(z)p(z)]_{-1}^{+1} = 0\end{aligned} \tag{66}$$

□

The symplecticity we will be interested in may be defined as the stability of the skew-symmetry, for this canonical differential operator, in the reduction schemes. The considered collocation schemes will include the boundary conditions in the approximation basis definition. Hence, with the Dirichlet conditions $q(-1) = 0$, $p(+1) = 0$ and the previously defined polynomial approximation basis $\{\psi_i\}_{i\in\{1,\dots,N-1\}}$, the approximations of the 1-forms $q(z)$ and $p(z)$ will be defined as

$$\begin{aligned} q(z) &= \sum_{i=1}^{N} \alpha_i \psi_i(z)(1-z) \\ p(z) &= \sum_{i=1}^{N} \beta_i \psi_i(z)(1+z) \end{aligned} \tag{67}$$

Including homogeneous boundary conditions in the definition of the approximation basis is quite usual in collocation methods and gives the best results in terms of accuracy and minimization of the boundary effects [17]. We will make use of these discretization bases for the numerical comparisons of what will be termed as a *classical* collocation scheme with the suggested discretization scheme based on Stokes-Dirac structures.

5.2 Symplectic collocation using Gauss-Legendre points

The property of symplecticity for Gauss-Legendre points is known from [24] and [42] who both discovered in 1988, independently, that the 4th order Runge-Kutta method is symplectic when its parameters are chosen as Gauss-Legendre points. The resulting so-called Gauss-Legendre-Runge-Kutta method is extensively used for time integration of Hamiltonian systems. This method was later extended to mulitsymplectic integration (space and time integration) [40].
In this section, we will show that the collocation method is also symplectic when the collocation points are chosen as zeros of the Legendre polynomial of the corresponding order. We will make use of the well-know property of Gauss-Legendre quadrature stating that one can integrate exactly a polynomial $P(z)$

of order $2N-1$ with N collocation points chosen as the zeros of the Legendre polynomial of order N, that is:

$$\int_{-1}^{1} P(z)dz = \sum_{i=1}^{n} w_i P(z_i) \tag{68}$$

where z_i are zeros of the Nth order Legendre polynomial and were w_i are appropriate weights.
One should notice that the collocation method developed in this section method is adapted only for closed systems (i.e. with boundary conditions such that the energy flow through the boundary is zero). In this case, we may expect that this symplectic Legendre collocation is equivalent to the mixed collocation scheme developed in the previous section since in the latter we perform exact integration of bilinear forms defined on the polynomial approximation space as it will be the case for the Legendre collocation.

Theorem 24. *The Gauss-Legendre collocation points defined as the zeros the N-order Legendre polynomial preserves the skew-symmetry of the canonical differential operator defined in proposition 23 in the corresponding N-dimensional approximation space.*

Proof. We have to show that the canonical operator from proposition 23 is projected in the considered approximation space to a non degenerated skew-symmetric finite dimensional bilinear operator. We will thus first define this reduced bilinear operator. We will make use of the zeros of the N-th order Legendre polynomial, denoted hereafter z_k, to define the approximation basis with N Lagrange interpolation polynomials of order $N-1$:

$$\psi_i(z) = \prod_{k=0, k\neq i}^{N-1} \frac{z - z_k}{z_i - z_k} \tag{69}$$

which satisfy $\psi_i(z_j) = \delta_i^j$. Hence the form q and p will be approximated as

$$\begin{aligned} q(z) &= \sum_{i=1}^{N} \alpha_i \psi_i(z)(1+z) \\ p(z) &= \sum_{i=1}^{N} \beta_i \psi_i(z)(1-z) \end{aligned} \tag{70}$$

with polynomials of order N in such a way that the Dirichlet boundary conditions $q(-1) = 0$ et $p(1) = 0$ are satisfied in the whole approximation space. For the sequel, we will make use of the notation:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix} \tag{71}$$

Substituting (70) in the canonical bilinear differential operator

$$\begin{pmatrix} q(z) & p(z) \end{pmatrix} \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \begin{pmatrix} q(z) \\ p(z) \end{pmatrix} \tag{72}$$

gives the following expression for the coordinates of this reduced bilinear operator evaluated at the Legendre collocation points z_k:

$$\Phi(p,q)|_{z_k} := \begin{pmatrix} \alpha^T & \beta^T \end{pmatrix} \begin{pmatrix} 0 & M^1 \\ M^2 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{73}$$

where $\alpha, \beta \in \mathbb{R}^N$, are respectively the coordinates of any q and p polynomials in the approximation basis and where

$$\begin{aligned} M^1_{ij} &= \psi_i(z)(1+z)\frac{d}{dz}(\psi_j(z)(1-z))\Big|_{z_k} \\ M^2_{ij} &= \psi_i(z)(1-z)\frac{d}{dz}(\psi_j(z)(1+z))\Big|_{z_k} \end{aligned} \tag{74}$$

Note that $\psi_i(z)(1+z)\frac{d}{dz}(\psi_j(z)(1-z))$ and $\psi_i(z)(1-z)\frac{d}{dz}(\psi_j(z)(1+z))$ are polynomials of order $2N-1$ in z. Hence they may be exactly integrated using the quadrature formulae and

$$\begin{aligned} \int_{-1}^{1} \psi_i(z)(1-z)\frac{d}{dz}\big(\psi_j(z)(1+z)\big) &= \sum_{k=1}^{n} w_k \psi_i(z)(1-z)\frac{d}{dz}\big(\psi_j(z)(1+z)\big)\Big|_{z_k} \\ \int_{-1}^{1} \psi_i(z)(1+z)\frac{d}{dz}\big(\psi_j(z)(1-z)\big) &= \sum_{k=1}^{n} w_k \psi_i(z)(1+z)\frac{d}{dz}\big(\psi_j(z)(1-z)\big)\Big|_{z_k} \end{aligned} \tag{75}$$

where w_k are the Legendre weights. Let us denote Ω the diagonal matrix with the diagonal elements w_k. Then we have proven

$$\int_{-1}^{1} \Phi(p,q) = \begin{pmatrix} \alpha^T & \beta^T \end{pmatrix} \begin{pmatrix} 0 & \Omega M^1 \\ M^2\Omega & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{76}$$

On the other hand

$$\int_{-1}^{1} \Phi(p,q) = \begin{pmatrix} \alpha^T & \beta^T \end{pmatrix} \begin{pmatrix} 0 & B^1 \\ B^2 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{77}$$

with

$$\begin{aligned} B^1 &= \Psi(z)(1+z)\frac{d}{dz}(\Psi(z)(1-z)) \\ B^2 &= \Psi(z)(1-z)\frac{d}{dz}(\Psi(z)(1+z)) \end{aligned} \tag{78}$$

Integration by part gives

$$\begin{aligned} B^1_{ij} &= \int_{-1}^{1} \psi_i(z)(1+z)\frac{d}{dz}(\psi_j(z)(1-z))dz \\ &= \Big[\psi_i(z)(1+z)\psi_j(z)(1-z)\Big]_{-1}^{1} - \int_{-1}^{1} \frac{d}{dz}(\psi_i(z)(1+z))\psi_j(z)(1-z)dz \\ &= -B^2_{ji} \end{aligned} \tag{79}$$

which proves $B^1 = -B^{2^T}$. Finally, comparing (76) and (77) gives $\Omega M^1 = -(M^2\Omega)^T$ which implies $M^1 = -M^{2^T}$. The matrix defining the canonical bilinear operator projected to the approximation space and evaluated at the N collocation points is thus skew-symmetric. □

Remark 25. The Chebyshev collocation points are also widely used in numerical analysis in view of their many advantages: minimization of Runge phenomenons, minimization of the Lagrange interpolation error, easiness of computations, among others [17]. It is somewhat surprising to observe that for Chebyshev collocation points also, the classical collocation method seems still to be symplectic. Unfortunately, to the best of our knowledge, the following assertion is only a conjecture. *The Gauss-Chebyshev collocation points defined as*

$$z_k = \cos\left(\frac{(2k-1)\pi}{2N}\right) \ , \ k = 1, ..., N$$

reduce the skew-symmetry of the canonical differential operator defined in proposition 23 to skew-symmetry of the reduced finite dimensional bilinear operator in the corresponding N-dimensional approximation space.

6 Numerical example

For the sake of comparison, we will consider a lossless transmission line with constant parameters (inductance $L = 2$, capacitance $C = 3$). In this particular example, the resulting PDE is linear. Thus, we can formally compute the model dynamical spectrum from the underlying eigenvalue problem and compare it with the spectrum of the finite dimensional model obtained using the discretization scheme developed in this paper. Symmetric Dirichlet boundary conditions $q(0) = 0$ and $p(L) = 0$ have been chosen to complete the ideal transmission line model written in the form :

$$\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \begin{pmatrix} \frac{q(z)}{C} \\ \frac{p(z)}{L} \end{pmatrix} = \lambda \begin{pmatrix} q \\ p \end{pmatrix} \tag{80}$$

The theoretical eigenvalues of this ideal transmission line example are given in table 1. Comparisons will be made with the spectrum of reduced models obtained with 8 interior collocation points (hence with reduced models of order 16). To achieve the symmetric Dirichlet boundary conditions, the chosen polynomial bases have been augmented (in the "classical collocation" case) by considering the developments

$$q(z) = \sum_{i=1}^{8} q_i \psi_i(z) z, \qquad p(z) = \sum_{i=1}^{8} p_i \psi_i(z)(z - L) \tag{81}$$

which automatically satisfy $q(0) = 0$ and $p(L) = 0$. In this case (closed system), resulting eigenvalues are given in table 2 hereafter where it is verified that

theoretical eigenvalues
0.00000000000000±0.32063745754047i
0.00000000000000±0.96191237262140i
0.00000000000000±1.60318728770233i
0.00000000000000±2.24446220278326i
0.00000000000000±2.88573711786419i
0.00000000000000±3.52701203294513i
0.00000000000000±4.16828694802606i
0.00000000000000±4.80956186310699i

Table 1: Spectrum of the considered ideal transmission line model (first eight couples of conjugated eigenvalues)

Legendre collocation	Chebyshev collocation
0.00000000000000±0.32063745754047i	0.00000000000000±0.32063745744583i
0.00000000000000±0.96191238151097i	0.00000000000000±0.96191420177938i
0.00000000000000±1.60321563740732i	0.00000000000000±1.60338416589471i
0.00000000000000±2.24771134219675i	0.00000000000000±2.25154046253161i
0.00000000000000±2.94830702957368i	0.00000000000000±3.00613251338085i
0.00000000000000±3.98374558826631i	0.00000000000000±4.38041246765385i
0.00000000000000±6.38606332778675i	0.00000000000000±8.49771153485658i
0.00000000000000±18.76179259703422i	0.00000000000000±31.96188912930034i

Table 2: Spectrum approximation for the ideal transmission line example using the classical scheme with 8 interior collocation points which are zeros of corresponding Legendre (left) and Chebyshev (right) polynomials.

classical collocation methods are symplectic either with Legendre (theorem 24) or Chebyshev (conjecture in remark 25) collocation points.

It may be noticed that other choices of collocation points result in non symplectic schemes as illustrated in table 3 hereafter where 8 interior uniformly distributed collocation points have been chosen. This results in dissipative and even unstable modes for the reduced model. On the contrary, the geometric collocation scheme proposed in this paper remains symplectic whatever the choice of collocation points is. In table 3, this geometric collocation scheme has been used with the same uniformly distributed 8 interior collocation points. The eigenvalues of the model resulting from our "geometric" collocation method are obtained by the diagonalization of the matrix $J \times Q$ from the input-output representation (50) when substituting the efforts from (59) and again considering

Geometric collocation	Classical collocation
0.00000000000000±0.32063745754047i	-0.00000000000001±0.32063749232832i
0.00000000000000±0.96191238151097i	0.00000000000003±0.96142344694331i
0.00000000000000±1.60321563740732i	-0.00000000000001±1.55336611678737i
0.00000000000000±2.24771134219675i	-0.49234496522402±2.25266731541568i
0.00000000000000±2.94830702957369i	0.49234496522400±2.25266731541571i
0.00000000000000±3.98374558826630i	-0.91327478638258±2.15975871233621i
0.00000000000000±6.38606332778657i	0.91327478638259±2.15975871233621i
0.00000000000000±18.76179259703309i	-0.00000000000000±4.73727006377608i

Table 3: Spectrum approximation for the ideal transmission line example. Right column: using a classical scheme with 8 interior uniformly distributed collocation points. Left column: using the *geometric* or *mixed* collocation method with the same 8 interior uniformly distributed collocation points.

Dirichlet conditions $q(0) = 0$ and $p(L) = 0$. One obtains

$$\begin{pmatrix} f^q \\ f^L_\partial \\ f^p \\ e^0_\partial \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \begin{pmatrix} D \\ \varphi(L) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(0) \end{pmatrix}^{-1} \\ \begin{pmatrix} D \\ -\varphi(0) \end{pmatrix} \begin{pmatrix} M^T \\ \varphi(L) \end{pmatrix}^{-1} & 0 \end{pmatrix} \begin{pmatrix} C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{J \times Q} \begin{pmatrix} q \\ e^L_\partial \\ p \\ f^0_\partial \end{pmatrix} \tag{82}$$

Among the possible classical collocation methods described above, the Gauss-Legendre collocation scheme is the more accurate. We will therefore compare this scheme with our geometric scheme using the same collocation points (classical orthogonal collocation is indeed a reference method for high precision spectrum approximation). The results are given in table 4 hereafter.

Geometric collocation	Gauss-Legendre collocation
0.00000000000000±0.32063745754047i	0.00000000000000±0.32063745754047i
0.00000000000000±0.96191238151097i	0.00000000000000±0.96191238151097i
0.00000000000000±1.60321563740732i	0.00000000000000±1.60321563740732i
0.00000000000000±2.24771134219675i	0.00000000000000±2.24771134219675i
0.00000000000000±2.94830702957369i	0.00000000000000±2.94830702957368i
0.00000000000000±3.98374558826630i	0.00000000000000±3.98374558826631i
0.00000000000000±6.38606332778657i	0.00000000000000±6.38606332778675i
0.00000000000000±18.76179259703309i	0.00000000000000±18.76179259703422i

Table 4: Spectrum approximation for the ideal transmission line example. Right column: using a classical scheme with 8 interior Legendre collocation points. Left column: using the *geometric* or *mixed* collocation method with the same 8 interior Legendre collocation points.

It is remarkable that Gauss-Legendre collocation and geometric collocation

give essentially the same approximated eigenvalues. This has to be considered simultaneously with the fact that both methods perform exact integration of the bilinear power product in the same approximation space of polynomials. However, the spectrum accuracy in the geometric collocation do not depend on the choice of collocation points (except for floating point numerical errors issues when solving the corresponding finite dimensional eigenvalue problem). In this sense, the geometric scheme gives the best possible accuracy for the spectrum approximation and leave open the choice of collocation points (which could then be optimized according to other criteria).

Finally we would like to point out that the geometric collocation method developed here is designed for open systems (with boundary energy flows). For these systems this geometric scheme remains symplectic in the sense that it preserves both the Hamiltonian and the interconnection Dirac structure. It may be seen from table 5 hereafter that this is not the case for classical collocation schemes, even for the Gauss-Legendre collocation method.

Time varying boundary conditions are required to represent open systems. They are usually expressed directly in classical collocation schemes using the following constraints:

$$\begin{aligned} q(0,t) &= \sum_{i=1}^{N} \psi_i(0) q_i = u_1(t) \\ p(L,t) &= \sum_{i=1}^{N} \psi_i(L) p_i = u_2(t) \end{aligned} \tag{83}$$

where $u_1(t)$ and $u_2(t)$ are the transmission line "inputs". Therefore, the eigenvalue problem for such open systems may be written:

$$\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \begin{pmatrix} q(z) \\ p(z) \end{pmatrix} = \lambda \begin{pmatrix} q(z) \\ p(z) \end{pmatrix} \tag{84}$$

with the Dirichlet boundary conditions:

$$\begin{aligned} q(0) &= \sum_{i=1}^{N} \psi_i(0) q_i = 0 \\ p(L) &= \sum_{i=1}^{N} \psi_i(L) p_i = 0 \end{aligned} \tag{85}$$

The obtained eigenvalues are reported in table 5 hereafter for 8 interior Legendre collocation points and compared with the results from the geometric method. Besides the loss of accuracy, it may be noticed that dissipative and even unstable modes appear in classical collocation case.

7 CONCLUSIONS AND FUTURE WORKS

In this paper we have suggested an adaptation of the so-called collocation method in order to preserve the geometric structure of a class of Hamiltonian

Geometric collocation	Gauss-Legendre collocation
0.00000000000000±0.32063745754047i	0.00000000000000±0.28966097209118i
0.00000000000000±0.96191238151097i	0.00000000000000±0.95732943443983i
0.00000000000000±1.60321563740732i	0.00000000000000±1.62271610834325i
0.00000000000000±2.24771134219675i	0.00000000000000±2.34315842499615i
0.00000000000000±2.94830702957369i	0.00000000000000±2.96231787828937i
0.00000000000000±3.98374558826630i	1.08122738779532±3.73134482798125i
0.00000000000000±6.38606332778657i	-1.08122738779532±3.73134482798126i
0.00000000000000±18.76179259703309i	-3.23331090942184±0.00000000000000

Table 5: Spectrum approximation for the *open* ideal transmission line example with Dirichlet boundary conditions (85). Right column: using a classical scheme with 8 interior Legendre collocation points. Left column: using the *geometric* or *mixed* collocation method with the same 8 interior Legendre collocation points.

systems representing open physical systems, i.e. with energy flow through the boundary of their spatial domain. These Hamiltonian systems are endowed with a geometric structure, called Dirac structure, which in the case of Hamiltonian systems of conservation laws takes a canonical form called Stokes-Dirac structure. The spatial discretization presented in this paper preserves this structure after the reduction by projection on polynomial bases which are differently chosen according to the degree of the differential forms that they approximate. Doing this both the exterior derivative and the boundary operator may be discretized exactly. However one obtains in a first instance a Tellegen structure defined with respect to a degenerate pairing as the dimensions of the spaces of flow and effort variables are not equal. Then a Dirac structure has been obtained as the quotient of the Tellegen structure with respect to the kernel of the degenerate pairing.

This method could be called "mixed colocation method" and indeed generalizes previously suggested discretization methods using mixed finite-elements. Completing this discretized Dirac structure with the approximated closure relations associated with the variational derivative of the Hamiltonian, one obtains a reduced model which has the structure of a finite-dimensional port Hamiltonian system. This port Hamiltonian system form allows the use of passivity-based control laws with the accurate spectral properties that pseudo-spectral methods provide.

Another point of interest is the analysis of the symplecticity of spatial discretization schemes based on collocation methods. It has been shown that classical collocation methods may exhibit symplecticity with particular choices of collocation points (zeroes of Legendre or Chebyshev polynomials) for closed Hamiltonian systems (with boundary conditions implying no energy transfer through their spatial boundaries such as Dirichlet or von Neumann homogeneous boundary conditions). It has to be notice however that the proposed collocation scheme (which is symplectic by construction) remains symplectic whatever the choice of the collocation points is (which is not the case of classi-

cal collocation schemes). Moreover the geometric collocation scheme preserves Stokes-Dirac structures and thus perform geometric reduction for open systems (which again is not the case for classical collocation schemes).

In view of the many advantages that this method does offer, we are currently extending these results to two dimensional case still within the port-Hamiltonian formalism. Furthermore this structure preserving spatial discretization method will be also used for the synthesis of stabilizing boundary control and open the way to relating the controllers obtained using the infinite-dimensional models and their finite dimensional approximations.

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