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► To cite this version:

Vitaly L Krupenin. On the description of strongly nonlinear vibroconducting and vibrogenerating media. Journal of Machinery Manufacture and Reliability, 2016, 45 (4), pp.297-306. 10.3103/S1052618816040087. hal-01623049

HAL Id: hal-01623049

<https://hal.science/hal-01623049>

Submitted on 25 Oct 2017

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On the Description of Strongly Nonlinear Vibroconducting and Vibrogenerating Media

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Abstract—A quite general class of problems is considered, the analysis of which makes it possible to describe the mechanism of for the generation and propagation of broadband vibration in machine structures. Mathematical models based on the theory of strongly nonlinear continuous media with a complex structure are presented. Different types of models of a “linear carrier medium plus attached equipment containing impact pairs” class are presented. It is assumed that the impact is Newtonian. The necessary defining relations are given. An example of vibroconductor analysis with internal breaks is given.

1. It is known [1, 2] that vibration is transmitted from point to point through vibroconductors (waveguides), which are the medium in which elastic waves, due to the presence of limiting surfaces, propagate in directions depending on the design features of the vibroconducting object and the character of interaction between structural elements. Recently developed methods of acoustic dynamics of machines, including the methods for analyzing the vibration propagation features and combating vibration are based on the assumption of linearity of vibroconductors or their weak nonlinearities. However, appearance of at least one break in a vibroconductor [3, 4], resulting in impacts of its design elements, causes a significant change in the dynamic qualities of the system as a whole, and the vibroconducting medium is at the same time a vibrogenerating one. The same thing happens when elements directly connected to the vibroconductor collide. If similar parasitic impact pairs are rare and the vibroconductor is linear from the impact points to the observation points, then the generated broadband vibration after passage through actually present mechanical filters can be considered as certain high-frequency, low-intensity noise. If there is a larger number of impact pairs in any way attached to the vibroconductor and directly related to the vibration transfer process, the elastic transmitting medium transforms from linear to strongly nonlinear. In this case, the use of traditional calculation methods is problematical.

Linear (or close to linear) specific models of continuous media with a complex structure are considered in [1]. One feature of the models is the presence of two original parts of the medium, namely: carrier and attached parts. Accordingly, the dynamic equations of such media consist of two groups. The first group of equations describes the carrier part, and the second group describes the attached part. Just like any multipolar mechanics model, here the understanding of point is substantially revised, the state of which can be defined by an arbitrary number of kinematic parameters. This approach, along with others, was fundamentally development in [5], etc.

In [3, 6–8], strongly nonlinear models of a similar medium were considered, in which distributed impact elements were used to take into account multiple collisions in attached equipment components. Such models make it possible to describe the process of formation and propagation of vibratory fields in complex composite structures. In addition, the models make it possible to obtain a number of design formulas and significant defining relations. The need to access them is dictated by the fact that in machine structures, just multiple systematic collisions of subsystem elements are often responsible for the form of the generated global vibrofields and the vibroactivity of structures as a whole.

Returning to the two groups of equations of motion, we postulate the existence of an elastic (elastoviscous) carrier medium, i.e. the model contains the equations of motion of the carrier parts (the examples are as follows: the classical Lamé equation, the equation of longitudinal or transverse oscillations of rods, etc.), to which boundary conditions are added.

Fig. 1.

Assuming that it is as if the collisions are spread within a certain spatial domain, the equations of motion of attached equipment are added to the equations of motion of carrier parts. The coherence mechanism of both parts determines the global structure of the generated vibration field.

2. The representative case postulates the existence of some elastically dissipative carrier medium described by the displacement vector $u(x, t)$, ($u, x \in \mathbf{R}^3, x \in \Xi \subseteq \mathbf{R}^3, t \in \mathbf{R}$), being subject to the classical Lamé equation with a dissipative term (see [1, 7]):

$$\rho u_{tt} = (\lambda + \mu) \text{grad} \text{div} u + \mu \Delta u + D[u] + F, \quad (1)$$

where ρ is the density of the medium; λ, μ are the Lamé parameters describing the elastic properties of the medium; Ξ is the domain of the change in coordinates; Δ is the Laplacian; $D[\dots]$ is the operator responsible to dissipative loss. Let the external (volume and surface) forces $F = F_1 + F_0$, where F_1 is the specified vector; F_0 is the effect of attached systems containing the impact pairs.

Note that the carrier medium model can itself have a complex structure, so Eq. (1) can be replaced by a more common one. Equation (1) should satisfy to the boundary conditions.

Let the attached equipment be elastically amortized:

$$F_0(x, t) = -c_1(x)[u(x, t) - y_1^{(I)}(x, t)] - c_2(x)[u(x, t) - y_1^{(II)}(x, t)], \quad (2)$$

where it is assumed that the interactive (impact) pair consisting of two contacting linear stationary sclerononomous subsystems $\mathbf{A}^{(I)}(x)$ and $\mathbf{A}^{(II)}(x)$ is suspended in the each point of medium, each of these subsystems is described with the finite set of functions $y_j^{(I, II)}(x, t), j = 1, 2, \dots, N$.

For convenience, an identical number of degrees of freedom for both subsystems were assumed: the extra degrees of freedom can be fictitious. The points of suspension were identified in (2) as $y_1^{(I, II)}(x, t)$, and the point of contact is denoted as $y_n^{(I, II)}(x, t), n \leq N$. Let us assume that subsystems $\mathbf{A}^{(I, II)}(x)$ are defined with families of distributed dynamic compliance operators $L_{qk}^{(I, II)}(x; p) = O(p^{-2}), p \rightarrow \infty$; indices q and k change from 1 to N ; $p = \partial/\partial t, x, z \in \Xi \subseteq \mathbf{R}^3$.

Here and below, we assume that interaction in every element is direct, central, and one-dimensional: let $y^0 = y_n^{(I)} - y_n^{(II)}$ be the relative convergence of the points of contact and let the coordinate y^0 change along some axis. The interaction force $\Phi_1(y^0, y_t^0) = \lambda \Phi(y^0) + \Phi_2(y^0, y_t^0)$, $\lambda \gg 1$ is a large parameter [9]. Here, the first term in the right-hand side determines the elastic component of the interaction force, and the second term determines the dissipative component. The function Φ_1 determines the hypothesis of noninstantaneous (non-Newtonian) impact; it can directly depend on x , since the character of interaction can change from point to point. Let us assume that Φ is a threshold function [6, 10]:

$$\Phi(y^0) \in \{\Phi\}_\Delta \equiv \{\Phi(y^0); \Phi(y^0) = \psi(y^0 - \Delta)\eta(y^0 - \Delta); \Delta \geq 0; \psi(y^0) \equiv 0; y^0 \geq 0\},$$

Fig. 2.

where $\psi(u)$ is continuously differentiable on the entire numerical axis, monotonically increases, and is a convex function for $u \geq 0$ (Fig. 1); $\eta(u)$ is a unit function. In the case when Newtonian interaction is rejected, the dissipative component Φ_2 is also described by some threshold function [6, 10]. The methods for analyzing such systems are discussed in [6–10].

For the points of suspension and interaction we have the operator equations

$$y_1^{(I,II)} = L_{11}^{(I,II)}(p)c_{1,2}(x)u \pm L_{n1}^{(I,II)}(p)\Phi_1(y^0, y_t^0) + f_1^{(I,II)}, \quad (3)$$

$$y_1^{(I,II)} = L_{nn}^{(I,II)}(p)c_{1,2}(x)u \pm L_{nn}^{(I,II)}(p)\Phi_1(y^0, y_t^0) + f_n^{(I,II)}, \quad (4)$$

where for system $\mathbf{A}^{(I)}$ in (3) and (4) we select the plus sign, and for system $\mathbf{A}^{(II)}$, we select the minus sign. Some functions $f_{1,n}^{(I,II)}$ describing an additional external effect can be introduced into these equations.

The communication mechanisms of the carrier and attached parts determine the structure of the global vibrofield. This approach does not take into account information on the features of specific parts of the system, nor effects whose appearance is possible only with allowance for the discreteness of the model.

3. The proposed model is complicated for analysis and needs some simplifications. Let us give examples of more realistic models of this kind.

2D Structure of Carrier Part: Discrete Model

Let us consider a string lattice [9, 11] composed of two mutually perpendicular families of linear elastic strings fastened at the ends and having lengths of L_1 and L_2 (Fig. 2). Each string is numbered using the indexes $k = 0, 1, N_1$ and $q = 0, 1, \dots, N_2$. At the nodes are point perfectly solid bodies with masses m_{kq} .

It is assumed that the lattice cells are identical and rectangular, but the lengths and widths of their sides are not equal among themselves and the lattice itself is anisotropic. The strings are inertia-free. Their fastenings at the nodes are considered perfectly hard, and the tensions are so large that possible changes during linear oscillations can be neglected.

Let the length of each horizontal side of cells be ΔL_1 , that of the vertical side be ΔL_2 ; let the tension of horizontal strings be T_1 , and that of vertical strings be T_2 . Thus, the dynamics of the lattice structure can describe by the displacement function of the lattice nodes $u_{kq}(t)$, $k = 0, 1, \dots, N_1$, $q = 0, 1, \dots, N_2$. Each of the functions $u_{kq}(t)$ changes along some axis perpendicular to the plane of static equilibrium of the lattice.

Let us consider that all actuating nonconservative forces are small. Let us denote them as $\varepsilon g_{kq}(t, u_{kq}, \dot{u}_{kq})$, where ε is the small parameter. Since each particle lies on two strings, we obtain N the equations of motion $N = (N_1 - 1)(N_2 - 1)$

$$m_{kq}\ddot{u}_{kq} + c_1(2u_{kq} - u_{(k-1,q)} - u_{(k+1,q)}) + c_2(2u_{kq} - u_{(k,q-1)} - u_{(k,q+1)}) = \varepsilon g_{kq}(t, u_{kq}, \dot{u}_{kq}) + F_{kq}, \quad (5)$$

where $c_{1,2} = T_{1,2}/\Delta L_{1,2}$ are the elasticity coefficients. The fastening boundary conditions can be written as [9] $u_{kq} = 0$, at $k = 0, N_1$; $q = 0, N_2$. Equations (5) represent the first group of equations; in the discrete model, they describe the carrier part of medium with a complex structure. Let the so-called impact oscillator be suspended in each nodal body of the lattice [10] (Fig. 2). Then let w_{kq} be the coordinate of the body forming the oscillator; M_{kq} is its mass; Ω_{kq} is a natural frequency; Δ_{kq} are the values of the adjusting gaps (of interferences); $k = 0, 1, \dots, N_1$, $q = 0, 1, \dots, N_2$. Thus, the equations of motion of the attached equipment will be

$$M_{kq}\ddot{w}_{kq} + M_{kq}\Omega_{kq}^2(w_{kq} - u_{kq}) + \Phi_{1kq}(w_{kq}, \dot{w}_{kq}) = \varepsilon f_{kq}(t, w_{kq}, \dot{w}_{kq}), \quad (6)$$

where $f_{kq}(t, w_{kq}, \dot{w}_{kq})$ are the additional forces acting on the oscillator.

It is seen that the forces in Eq. (5) are $F_{kq} = M_{kq}\Omega_{kq}^2(w_{kq} - u_{kq})$.

The interaction forces are determined by the threshold functions (see point 2 and [9]) $\Phi_{1kq}(w_{kq}, \dot{w}_{kq}) = \lambda \Phi_{kq}(w_{kq}) + \Phi_{2kq}(w_{kq}, \dot{w}_{kq})$, $\lambda \gg 1$. We believe that for $\lambda \rightarrow \infty$, the considered hypothesis of interaction transforms into Newton's hypothesis: the impact is instantaneous; the impact force is set with a singular generalized function. This needs be postulated, since it is possible to construct its correct proof under the assumption of the elasticity of interaction [10]. It is necessary to determine the displacements $\{u_{kq}, w_{kq}\}$ for the specified forces $\{q_{kq}, f_{kq}\}$.

Continual Model

In system of equations (5), (6), we turn to long waves, believing that at all k and q : $m_{kq} = m$. Let the number of strings in both families be sufficiently large and quantities $\Delta L_{1,2}$ be small. At the same time, let the surface density $\gamma = m/(\Delta L_1 \Delta L_2)$ and linear tensions $\theta_x = T_1/\Delta L_1$, $\theta_y = T_2/\Delta L_2$ have orders $O(1)$. System of equations (1) can be continualized, considering it as a finite-difference analog of continuous systems with distributed parameters. Let us consider a sufficiently smooth function $u(x, y, t)$, coinciding (in this case) in the lattice nodes with the original functions $u_{kq}(t)$.

We replace the expressions $(2u_{kq} - u_{(k-1,q)} - u_{(k+1,q)})$ and $(2u_{kq} - u_{(k,q-1)} - u_{(k,q+1)})$ with the analogs

$$2u_{kq} - u_{(k-1,q)} - u_{(k+1,q)} = 2u(x, y, t) - u(x - \Delta L_1, y, t) - u(x + \Delta L_1, y, t),$$

$$2u_{kq} - u_{(k,q-1)} - u_{(k,q+1)} = 2u(x, y, t) - u(x, y - \Delta L_2, t) - u(x, y + \Delta L_2, t).$$

Then, Eq. (5) can be represented as

$$\gamma \Delta L_1 \Delta L_2 u_{tt} = \Delta L_1 \Delta L_2 (\theta_x u_{xx} + \theta_y u_{yy}) + (\Delta L_1)^3 (\Delta L_2) (\theta_x / 12) u_{xxxx} + (\Delta L_1) (\Delta L_2)^3 (\theta_y / 12) u_{yyyy} + \dots$$

Here, instead of differences determining (1), the series in powers of ΔL_1 and ΔL_2 corresponding to them were written out. In this case, within the accuracy of small quantities of order $O[(\Delta L_{1,2})^2]$ in the original approximation, we turn to the anisotropic membrane equation $\gamma u_{tt} - \theta_x u_{xx} - \theta_y u_{yy} + \dots = \varepsilon g(t, u, u_t, \dots) + O[(\Delta L_{1,2})^2]$.

If we consider the most popular isotropic membrane model when the stresses are $\theta_x = \theta_y \equiv \theta$ for $\varepsilon = 0$, then we arrive at the well-known and thoroughly studied equation of a classical membrane tightened along the contour of deflection, which is $u = u(t, x, y)$: $u_{tt} = c^2(u_{xx} + u_{yy})$, $c^2 = \theta \gamma^{-1}$.

The boundary conditions for the lattice lead to the membrane being fastened on all its edges. If the lengths of its sides are l_1 and l_2 , than the limiting conditions are written as $u(0, y, t) = u(l_1, y, t) = u(x, 0, t) = u(x, l_2, t) = 0$.

Fig. 3.

Thus, taking into account the representations for force F appearing in (1), we obtain the following equation for the carrier part of the medium:

$$u_{tt} = c^2(u_{xx} + u_{yy}) + M(x, y)\Omega^2(x, y)(w - u) + \varepsilon g(t, u, \dot{u}). \quad (7)$$

Here the forces and other physical factors are replaced by their area distributions. Equations (6) are transformed as follows:

$$M(x, y)w_{tt}(t, x, y) + M(x, y)\Omega^2(x, y)[w(t, x, y) - u(t, x, y)] + \Phi_1(w, w_t) = \varepsilon f(t, w, w_t). \quad (8)$$

Equations (7), (8) give the simplest model of a two-dimensional strongly nonlinear medium with a complex structure. Generalization can be performed with allowance for more complex models of carrier media (plates, beam and rod lattices, etc.) and owing to more general models of attached equipment (point 2). In [12] at considering the discrete model exposed to the random broadband excitation, the lattice with the attached elastic chains of point bodies, provided with single-sided and double-sided catchers was studied. In the case of Newtonian interaction, the analysis of this model was partially carried out in [11].

Equipped Strongly Nonlinear Rod

A one-dimensional analog of model (1)–(4) was studied in detail in [6, 8]. Instead of Eq. (1) for a rod of length l , we have

$$\rho u_{tt} - Eu_{xx} + c_1(u - y_n^{(I)}) + c_1(u - y_n^{(II)}) - D[u] + F = 0 \quad x \in [0, l], \quad (9)$$

where the boundary conditions are as follows: $u(0, t) = 0$; $Eu_x(l, t) = P \cos \omega t [u(l, t) = \mu \cos \omega t]$. The meaning of the notation is obvious. Relations (2)–(4) retain their form.

Figure 3 shows the continuum model of a strongly nonlinear vibroconductor with elastically amortized equipment, as well as the carrier part of the system and symbolically attached equipment (interacting in the subsystem buffers $A^{(I)}(x)$ and $A^{(II)}(x)$).

Rod with Internal Breaks

Figure 4 shows a discrete model of a rod with internal breaks. For a large concentration of elastically attached systems, we can use the long-wave approximation. We postulate the existence of a one-dimensional medium with the determining equations

$$\rho u_{tt} - Eu_{xx} - \Phi_1(y^0, y_t^0) = 0, \quad (10)$$

$$m(y_{tt}^0 + u_{tt}) + \Phi_1(y^0, y_t^0) = 0, \quad (11)$$

Fig. 4.

where ρ and E are the linear mass and Young's modulus of the rod; it is as though the interacting equipment is spread along the length of the rod; m is the linear mass of distributed impact elements; $y^0(x, t)$ is the distribution of their relative displacement. The boundary conditions are given after Eq. (9).

Note that use of this model is restricted, since it is possible to be outside the scope of continuous solutions; in certain frequency ranges, the model should be refined.

Remarks

The vibrofields in the considered objects are determined by two main mechanisms. The attached equipment acts on the carrier part like impact vibration dampers [1]. An exception is narrow frequency ranges near the eigenfrequencies of linear oscillations, where this action is similar to linear dampers. The carrier part filters the passing vibration. Just these mechanisms, manifesting themselves in different ways at different points of carrier structures, can generate specific dynamic effects.

Ending the brief review of linear carrier medium models plus interactive attached equipment, we note that when writing the necessary equations of motion, one can use different assumptions on the structure of interacting elements. Two subsystems can contact, or attached equipment can interact with restraints, or there can be double-sided elements, etc.

4. Let us assume that the interaction is elastic $\Phi_1(y^0, y_i^0) = \lambda\Phi(y^0)$ (point 2). If we accept the hypothesis that the interaction is instantaneous ($\lambda \rightarrow \infty$), then the impact force is expressed by singular general functions. To analyze processes and systems in which it is important to take into account the finite duration of interaction, the method of singularization was proposed in [9, 10]; in accordance with which, taking into account the transition to the long-wave approximation, we write for a single interaction

$$\lambda\Phi[y^0(x, t)] \approx J_\lambda(x)\delta[t - \varphi_\lambda(x) - \tau_\lambda(x)], \quad (12)$$

where $J_\lambda(x)$ is the interaction momentum density; $\varphi_\lambda(x)$ is the distribution of instants of onset of interaction; $\tau_\lambda(x) = 1/2t_\lambda(x)$; $t_\lambda(x)$ is the distribution of times of interaction; $\delta(x)$ is the Dirac δ -function; and $x \in \Xi_1$ is the interaction region.

Let us denote by $\Delta(x)$ the distribution of coordinates of the onset (and, accordingly, end) of interaction:

$$y^0[x, \varphi_\lambda(x)] = y^0[x, \varphi_\lambda(x) + t_\lambda(x)] = \Delta(x). \quad (13)$$

The momentum density and total interaction momentum are, respectively,

$$J_\lambda(x) = \int_0^{t_\lambda(x)} \lambda\Phi[y^0(x, t)]dt, \quad I_\lambda = \int_{\Xi_1} J_\lambda(x)dx. \quad (14)$$

The functions $J_\lambda(x)$ and $\varphi_\lambda(x)$ are determined during concretization of the problem and, finally, determine the form of the desired solutions. When considering T -periodic processes, we express force (12) through a T -periodic δ -function (the Dirac comb) or dual Dirac comb in the case of a symmetric double-side catcher

$$\delta^T(t) = T^{-1} \sum_{\sigma=-\infty}^{\infty} \exp(i\sigma\omega t) = \sum_{K=-\infty}^{\infty} \delta(t - KT), \quad \omega = 2\pi/T^{-1},$$

$$\delta^{T/2}(t) = 1/2T^{-1} \sum_{\sigma=-\infty}^{\infty} \exp[i(2\sigma+1)\omega t] = \sum_{K=-\infty}^{\infty} [\delta(t - KT) - \delta(t - T/2 - KT)], \quad \omega = 2\pi/T^{-1}.$$

5. The set of desired displacement fields of structural elements of medium has the form $\mathbf{U} = \{u(x, t); y_0(x, t); y_1(x, t); \dots; y_N(x, t)\}$, at that the first element describes the carrier part of medium and the rest depict the attached parts. In the considered models, the nonlinear forces are determined only by the presence of subsystems interacting among themselves or with fastened catchers. For the considered type of

models, one can indicate the structure of the set of elements $\mathbf{U}$. The displacement field $u(x, t)$ is summed as the result of the action of external (mass and surface) forces—let us denote it $u_1(x, t)$ —and the result of the action of interaction forces in the attached equipment, i.e.,

$$u(x, t) = u_1(x, t) + \int_{\Xi_1} \int_{-\infty}^t V(x, z, t-s) \Phi_1[y^0(x, s), y_t^0(x, s)] ds dz, \quad (15)$$

where $V(x, z, t-s)$ is the corresponding Green's function determined by the structures of linear systems: the carrier and attached parts of the medium. For interacting elements of the medium (attached parts), we find representations of the type

$$\begin{aligned} y_q(x, t) = & y_{q1}(x, t) - \int_{-\infty}^t H_q(t-s) \Phi_1[y^0(x, s), y_t^0(x, s)] \\ & + \int_{\Xi_1} \int_{-\infty}^t Y_q(x, z, t-s) \Phi_1[y^0(x, s), y_t^0(x, s)] ds dz, \end{aligned} \quad (16)$$

where $y_q(x, t)$ is the displacement of the structural element of one of the interacting subsystems; $H_q(t-s)$ is the Green's function at the interaction point; the second term in the right-hand side of Eq. (16) is displacement occurring as a direct result of interaction at this point; the third term is displacement resulting from interaction, but appearing as a result of its motion of caused by other structural elements of the medium; $Y_q(x, z, t-s)$ is the Green's function. Now it is easy to write the relations for the relative coordinate and other elements of the medium.

When analyzing periodic vibroimpact processes in Eqs. (15), (16), it is advisable to switch to periodic Green's functions (PFG). Examples of analyzing such models in the case of acceptance of analogs of Newton's hypothesis are given in [3, 5–8]. For complication of models of the carrier parts of a medium, e.g., taking into account the microstructure, accordingly, the representations of displacements of the structural elements will change (they will be complex).

In [1], a model of was considered medium in which the attached equipment was simulated with a continual set of isotropic oscillators. Similar constructions were also used in part for the models presented in this work, but owing to the strong nonlinearity of the attached part, to describe the frequency properties of the system, it is sufficient to give this part a finite number of degrees of freedom.

6. As an example, we consider the problem of a vibroconductor with internal breaks (Fig. 4). Let the linear mass of impact elements be $m(x) \equiv 1$. We write Eqs. (10), (11) as

$$\rho u_{tt} - E u_{xx} + \lambda \Phi(y^0) = 0, \quad (17)$$

$$y_{tt}^0 + u_{tt} + \lambda \Phi(y^0) = 0, \quad (18)$$

where the elastic interaction force belongs to a class of symmetric threshold functions, which is constructed using class $\{\Phi\}_\lambda$, described in point 2 as $\Phi(y^0) = \Phi_1(y^0) - \Phi_1(-y^0)$; $\Phi_1(y^0) \in \{\Phi\}_\Delta$. We assume that the loading at the contact zones is linear:

$$\begin{aligned} \lambda \Phi[y^0(x, t)] = & \lambda \{ [y^0(x, t) - \Delta(x)] \eta [y^0(x, t) - \Delta(x)] \\ & - [y^0(x, t) + \Delta(x)] \eta [y^0(x, t) + \Delta(x)] \}. \end{aligned}$$

Let $\Delta(x) = \text{const}$. We determine $\varepsilon = O(\lambda^{-1/2})$. The boundary conditions are as follows:

$$u(0, t) = 0, \quad E u_x(l, t) = \varepsilon P_1 \cos \omega t, \quad P_1 = 0, \quad (19)$$

which corresponds to the case of fastening of the left end of the rod and the application to its right end of T -periodic force action; $T = 2\pi\omega^{-1}$.

In accordance with (15), (16) and [10], we write the integral equations of symmetric T -periodic oscillations:

$$u(x, t) = u_1(x, t) + \int_0^l \int_0^{T/2} \chi(x, z, t-s) \lambda \Phi[y^0(z, s)] dz ds, \quad (20)$$

$$y^0(x, t) = -u(x, t) - \int_0^{T/2} \chi_1(t-s) \lambda \Phi[y^0(x, s)] ds, \quad (21)$$

where χ and χ_1 are PFGs. The field of displacement of a structural element neglecting collisions is

$$u_1(x, t) = [P_1 \sin \omega_0 x \cos \omega t] [E \omega_0 \cos \omega_0 l]^{-1}, \quad (22)$$

and, in accordance with [7], in this case $\omega_0 = \omega a^{-1}$, where $a = \sqrt{E \rho^{-1}}$; the PFG $\chi_1(t)$ is given by the series [10]

$$\chi_1(t) = T^{-1} \sum_{k=-\infty}^{\infty} -(2k+1)^{-2} \omega^{-2} \exp[i(2k+1)\omega t].$$

For $0 < t \leq T/2$, this series can be written in finite form: $\chi_1(t) = [1/2t - 1/4\pi\omega^{-1}]$ [10]. Outside the interval $]0, T/2]$, this representation should continue on the periodicity based on symmetry conditions. The result is a periodic saw-tooth function (“triangular sine”).

The PFG $\chi(x, z, t)$ corresponds to a linear wave operator and the type of boundary conditions being considered. Using [10], we find

$$\begin{aligned} \chi(x, z, t) &= (lT\rho)^{-1} \sum_{n=0}^{\infty} (\Omega_n \cos 1/4\Omega_n T)^{-1} \sin(h_n x) \sin(h_n z) \cos[\Omega_n(t - 1/4T)], \\ h_n &= [(2l)^{-1}(2n+1)\pi], \quad \Omega_n = 1/2a(2n+1)\pi l^{-1}, \quad a = \sqrt{E/\rho}, \quad 0 < t \leq \frac{1}{2}T. \end{aligned} \quad (23)$$

It turns out [6, 8] that for PFG (23), another representation takes place:

$$\chi(x, z, t) = 4(Ta^2\rho)^{-1} \sum_{k=1}^{\infty} G_{2k+1}(x, z) \cos[(2k+1)\omega t]. \quad (24)$$

Here, the functions

$$\begin{aligned} G_k(x, z) &= [-k\omega_0 \cos k\omega_0]^{-1} \cos k\omega_0(z-l) \sin k\omega_0 x, \quad 0 \leq x \leq l, \\ G_k(x, z) &= [-k\omega_0 \cos k\omega_0]^{-1} \cos k\omega_0(x-l) \sin k\omega_0 z, \quad z \leq x \leq l \end{aligned} \quad (25)$$

are determined by the Green’s functions of the Sturm–Liouville problem, in this case, problems with the left fastened end and right nonfastened end [13].

According to (12), for symmetric double-impact modes of motion [7, 8], we write the force density of impact interaction as

$$\Phi(y^0) = J_\lambda(x) \delta^{T/2}(t) [t - \phi_\lambda(x) - \tau_\lambda(x)]. \quad (26)$$

Taking into account the given relations, introducing (20) into (21), we get a double-functional representation [4, 7, 8] of the form

$$y^0(x, t) = -u_1(x, t) - J_\lambda(x) \chi_1[t - \phi_\lambda(x) - \tau_\lambda(x)] - \int_0^t J_\lambda(z) \chi[(x, z, t - \phi_\lambda(x) - \tau_\lambda(x))] dz. \quad (27)$$

The double-functional representations of form (27) determined, under our assumptions, the sought displacements. For the accepted interaction hypotheses, we can assume for large momenta [9] that $\tau_\lambda(x) = \tau_\lambda = \pi/2\sqrt{\lambda} = \text{const}(x)$. We use (13) and (22). We have

$$\begin{aligned} J(x) &= J_{\lambda 0} - \chi_1^{-1} \int_0^t J(z) \chi[x, z, \phi_\lambda(x) - \phi_\lambda(z)] dz \\ &\quad - \varepsilon P_1 [E \omega_0 \chi_1 \cos \omega_0 l]^{-1} \sin \omega_0 x \cos[\omega_0 \phi_\lambda(x)]. \end{aligned} \quad (28)$$

Here, $\omega_0 = \omega a^{-1}$, $\chi_1 = \chi_1(\tau_\lambda)$, $J_{\lambda 0} = -\Delta/\chi_1(\tau_\lambda)$ is the momentum density for the perfectly solid carrier part of medium; $\chi_1(\tau_\lambda) = 1/2\tau_\lambda - 1/4\pi\omega^{-1}$. The second condition determining the parameters of motion gives (14) or other equivalent relations. We have from formula (20)

$$u(x, t) = u_1(x, t) + \int_0^l J_\lambda(z) \chi[x, z, t - \varphi_\lambda(z) - \tau_\lambda] dz. \quad (29)$$

7. Let us consider passage of a vibration near the first resonance frequency. It is shown in [6–8] that in this case, in such systems, in-phase forms of nonlinear oscillations are achieved. Let the linear density of equipment be significantly less than the linear mass of the carrier rod $\rho \gg 1$.

In the considered Hamiltonian case and in this frequency range, the phase values are as follows: $\omega_0 \varphi_\lambda = 0, \pi$. Under the action of real dissipation, only a larger value of the impact pulse can correspond to stable operation in the originating system ($\rho \rightarrow \infty$), so for $\omega_0 < \pi/2l$ we select $\omega_0 \varphi_\lambda = 0$; meanwhile, for $\pi/2l < \omega_0 < \pi/l$, we select $\omega_0 \varphi_\lambda = \pi$.

It should be especially emphasized that when analyzing the situation in above-resonance ranges $\omega_0 > \pi/l$, it is necessary to introduce anti-phase forms of nonlinear oscillations of different kinds [8]. For vibroconductors with internal breaks, the description of such forms requires special considerations going beyond the scope of this article, because it is necessary to refine the model.

Limiting ourselves to the most important case of synchronization of hammers near the first resonance range, we note that if $\varphi = \text{const}$, then Eq. (29) is the Fredholm equation. Believing ρ^{-1} to be small, outside the small neighborhood of $\omega_0 > 0.5\pi/l$ using the successive approximations method, we find the first approximation for the impulse density $J_\lambda(x)$, assuming that in the zero-order (generic) approximation $J_\lambda(x) \approx J_{\lambda 0} = -\Delta/\gamma_1(\tau_\lambda)$. After calculations taking into account (23)–(28), we have

$$J_\lambda(x) = J_{\lambda 0} \left\{ 1 - \frac{8}{\pi \rho} \sum_{k=0}^{\infty} \frac{\cos[(2k+1)\omega_0(x-l)]}{(2k+1)^2 \{1 - \cos[(2k+1)\omega_0 l]\}} \right\} + 4\varepsilon \frac{P_1 a}{\pi E} \frac{|\sin \omega_0 x|}{|\cos \omega_0 l|},$$

$$\omega_0 \neq \pi/2l, \quad \omega_0 < \pi/l.$$

From formula (15), with an accuracy to small quantities of highest order, we find

$$u(x, t) = 4\varepsilon \frac{P_1 a \sin \omega_0 x}{\pi E \cos \omega_0 l} \cos \omega t + \frac{8\Delta}{\pi^2 \rho} \sum_{k=0}^{\infty} \left\{ \frac{\cos[(2k+1)\omega_0(x-l)]}{(2k+1)^2 \cos[(2k+1)\omega_0 l]} - 1 \right\} \\ \times \cos[(2k+1)\omega(t - \varphi_\lambda)] + \dots$$

From here, we can find that for $\omega_0 < \pi/2l$, $\varphi_\lambda = 0$, the vibroconductor is transparent to the fundamental tone and the structure is actively oscillating. If $\omega_0 > \pi/2l$, $\varphi_\lambda = \pi$, then the vibroconductor is locked and the fundamental tone of vibrations practically does not pass [6, 8]. In both cases, the representative set of highest harmonic components is generated in the structure.

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Translated by M. Kromin