Differential Geometry Revisited by Biquaternion Clifford Algebra

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DIFFERENTIAL GEOMETRY REVISITED BY BIQUATERNION CLIFFORD ALGEBRA

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Abstract. In the last century, differential geometry has been expressed within various calculi: vectors, tensors, spinors, exterior differential forms and recently Clifford algebras. Clifford algebras yield an excellent representation of the rotation group and of the Lorentz group which are the cornerstones of the theory of moving frames. Though Clifford algebras are all related to quaternions via the Clifford theorem, a biquaternion formulation of differential geometry does not seem to have been formulated so far. The paper develops, in $3D$ Euclidean space, a biquaternion calculus, having an associative exterior product, and applies it to differential geometry. The formalism being new, the approach is intended to be pedagogical. Since the methods of Clifford algebras are similar in other dimensions, it is hoped that the paper might open new perspectives for a 4D hyperbolic differential geometry. All the calculi presented here can easily be implemented algebraically on Mathematica and numerically on Matlab. Examples, matrix representations, and a Mathematica work-sheet are provided.

Keywords: Clifford algebras, quaternions, biquaternions, differential geometry, rotation group $SO(3)$, hyperquaternion algebra

1 Introduction

Much of differential geometry is still formulated today within the $3D$ vector calculus which was developed at the end of the nineteenth century. In recent years, new mathematical tools have appeared, based on Clifford algebras [1–10] which give an excellent representation of groups, such as the rotation group $SO(3)$ or the Lorentz group, which are the cornerstones of the theory of moving frames. Since the methods of Clifford algebras can easily be transposed to other dimensions, the question naturally arises of whether it is possible to rewrite differential geometry within a Clifford algebra in order to open new perspectives for $4D$ modeling. Such an extension might proceed as follows. A $4D$ tetraquaternion calculus has already been presented in [7, 8]. A moving surface $OM = f(t, u, v)$

can be viewed as a hypersurface (with normal n) in a $4D$ pseudo-euclidean space. The invariants are then obtained by diagonalizing the second fundamental form via a rotation around n combined with a Lorentz boost along n, generalizing the methods presented here. Though Clifford algebras can be presented in various ways, the originality of the paper lies in the use biquaternions. We shall first introduce quaternions and Clifford algebras together with a demonstration of Clifford's theorem relating Clifford algebras to quaternions. Then, we shall develop the biquaternion calculus (with its associative exterior product) and show how classical differential geometry can be reformulated within this new algebraic framework.

2 Clifford algebras: historical perspective

2.1 Hamilton's quaternions and biquaternions

In 1843, W. R. Hamilton (1805-1865) discovered quaternions [11–17] which are a set of four real numbers:

$$\begin{aligned} a &= a_0 + a_1 i + a_2 j + a_3 k && (1)\\ &= (a_0, a_1, a_2, a_3) && (2)\\ &= (a_0, \overrightarrow{a}) && (3) \end{aligned}$$

where i, j, k multiply according to the rules

$$\begin{aligned} i^2 = j^2 = k^2 = ijk &= -1 && (4)\\ ij = -ji &= k && (5)\\ jk = -kj &= i && (6)\\ ki = -ik &= j. && (7) \end{aligned}$$

The conjugate of a quaternion is given by

$$a_c = a_0 - a_1 i - a_2 j - a_3 k. \tag{8}$$

Hamilton was to give a $3D$ interpretation of quaternions; he named a_0 the scalar part and $\overrightarrow{a}$ the vector part. The product of two quaternions a and b is defined by

$$\begin{aligned} ab = &(a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3)\\ &+(a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)i\\ &+(a_0b_2 + a_2b_0 + a_3b_1 - a_1b_3)j\\ &+(a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1)k \end{aligned} \tag{9}$$

and in a more condensed form

$$ab = (a_0b_0 - \overrightarrow{a} \cdot \overrightarrow{b}, a_0\overrightarrow{b} + b_0\overrightarrow{a} + \overrightarrow{a} \times \overrightarrow{b}) \tag{10}$$

where $\overrightarrow{a} \cdot \overrightarrow{b}$ and $\overrightarrow{a} \times \overrightarrow{b}$ are respectively the usual scalar and vector products. Quaternions (denoted by $\mathbb{H}$) constitute a non commutative field without zero divisors (i.e. $ab = 0$ implies a or $b = 0$). At the end of the nineteenth century, the classical vector calculus was obtained by taking $a_0 = b_0 = 0$ and by separating the dot and vector products. Hamilton also introduced complex quaternions he called biquaternions which we shall use in the next parts.

2.2 Clifford algebras and theorem

About the same time Hamilton discovered the quaternions, H. G. Grassmann (1809-1877) had the fundamental idea of a calculus composed of n generators $e_1, e_2, ...e_n$ multiplying according to the rule $e_i e_j = -e_j e_i (i \neq j)$ [18–21]. In 1878, W. K. Clifford (1845-1878) was to give a precise algebraic formulation thereof and proved the Clifford theorem relating Clifford algebras to quaternions. Though Clifford did not claim any particular originality, his name was to become attached to these algebras[22, 23].

Definition 1. *Clifford's algebra C_n is defined as an algebra (over $\mathbb{R}$) composed of n generators $e_1, e_2, ..., e_n$ multiplying according to the rule $e_i e_j = -e_j e_i$ $(i \neq j)$ and such that $e_i^2 = \pm 1$. The algebra C_n contains 2^n elements constituted by the n generators, the various products $e_i e_j, e_i e_j e_k, ...$ and the unit element 1.*

Examples of Clifford algebras (over $\mathbb{R}$) are

1. complex numbers $\mathbb{C}$ $(e_1 = i, e_1^2 = -1)$.
2. quaternions $\mathbb{H}$ $(e_1 = i, e_2 = j, e_i^2 = -1)$.
3. biquaternions $\mathbb{H} \otimes \mathbb{C}$ $(e_1 = Ii, e_2 = Ij, e_3 = Ik, I^2 = -1, e_i^2 = 1$, I commuting with $i, j, k)$. Matrix representations of biquaternions are given in the appendix.
4. tetraquaternions $\mathbb{H} \otimes \mathbb{H}$ $(e_0 = j, e_1 = kI, e_2 = kJ, e_3 = kK, e_0^2 = -1, e_1^2 = e_2^2 = e_3^2 = 1$, where the small i, j, k commute with the capital $I, J, K)$ [7, 8].

All Clifford algebras are related to quaternions via the following theorem.

Theorem 1. *If $n = 2m$ (m : integer), the Clifford algebra C_{2m} is the tensor product of m quaternion algebras. If $n = 2m - 1$, the Clifford algebra C_{2m-1} is the tensor product of $m - 1$ quaternion algebras and the algebra $(1, \omega)$ where ω is the product of the $2m$ generators $(\omega = e_1 e_2 ... e_{2m})$ of the algebra C_{2m}.*

Proof. The above examples of Clifford algebras prove the Clifford theorem up to $n = 4$. For any n, Clifford's theorem can be proved by recurrence as follows [24, p. 378]. The theorem being true for $n = (2, 4)$, suppose that the theorem is true for $C_{2(n-1)}$, to $C_{2(n-1)}$ one adds the quantities

$$f = e_1 e_2 ... e_{2(n-1)} e_{2n-1}, g = e_1 e_2 ... e_{2(n-1)} e_{2n} \tag{11}$$

which anticommute among themselves and commute with the elements of $C_{2(n-1)}$; hence, they constitute a quaternionic system which commutes with $C_{2(n-1)}$. From the various products between f, g and the elements of $C_{2(n-1)}$ one obtains a basis of C_{2n} which proves the theorem. □

Hence, Clifford algebras can be formulated as hyperquaternion algebras the latter being defined as either a tensor product of quaternion algebras or a subalgebra thereof.

3 Biquaternion Clifford algebra

3.1 Definition

The algebra (over $\mathbb{R}$) has three anticommuting generators $e_1 = Ii, e_2 = Ij, e_3 = Ik$ with $e_1^2 = e_2^2 = e_3^2 = 1$ ($I^2 = -1$, I commuting with i, j, k). A complete basis of the algebra is given in the following table

1	$i = e_3e_2$	$j = e_1e_3$	$k = e_2e_1$
$I = e_1e_2e_3$	$Ii = e_1$	$Ij = e_2$	$Ik = e_3$

(12)

A general element of the algebra can be written

$$A = p + Iq \tag{13}$$

where $p = p_0 + p_1i + p_2j + p_3k$ and $q = q_0 + q_1i + q_2j + q_3k$ are quaternions. The Clifford algebra contains scalars p_0, vectors $I(0, q_1, q_2, q_3)$, bivectors $(0, p_1, p_2, p_3)$ and trivectors (pseudo-scalars) Iq_0 where all coefficients (p_i, q_i) are real numbers; we shall call these multivector spaces respectively V_0, V_1, V_2 and V_3. The product of two biquaternions $A = p + Iq$ and $B = p' + Iq'$ is defined by

$$AB = (pp' - qq') + I(pq' + qp') \tag{14}$$

where the products in parentheses are quaternion products. The conjugate of A is defined as

$$A_c = (p_c + Iq_c) \tag{15}$$

with p_c and q_c being the quaternion conjugates with $(AB)_c = B_cA_c$. The dual of A noted A^* is defined by

$$A^* = IA \tag{16}$$

and the commutator of two Clifford numbers by

$$[A, B] = \frac{1}{2}(AB - BA). \tag{17}$$

3.2 Interior and exterior products

Products between vectors and multivectors In this section we shall adopt the general approach used in [4] though our algebra differs as well as several formulas. The product of two general elements of the algebra being given, one can define interior and exterior products of two vectors a $(= a_1iI + a_2jI + a_3kI)$ and b via the obvious identity

$$ab = \frac{1}{2}(ab + ba) - [-\frac{1}{2}(ab - ba)] \tag{18}$$

$$= a.b - a \wedge b \tag{19}$$

with $a.b$ being the interior product

$$a.b = \frac{1}{2}(ab + ba) \tag{20}$$
$$= a_1b_1 + a_2b_2 + a_3b_3 \in V_0 \tag{21}$$

and $a \wedge b$ the exterior product

$$a \wedge b = -\frac{1}{2}(ab - ba) \tag{22}$$
$$= (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k \in V_2 \tag{23}$$

which has the same components as the pseudo-vector $\overrightarrow{a} \times \overrightarrow{b}$. Next we define the interior products $a.A_p$ and $A_p.a$ (with $2 \leq p \leq 3$ and $A_p = v_1 \wedge v_2 \wedge ... \wedge v_p$, $v_i \in V_1$)

$$a.A_p = \Sigma_{k=1}^{p}(-1)^k(a.v_k)v_1 \wedge ... \wedge v_{k-1} \wedge v_{k+1} \wedge ... \wedge v_p \tag{24}$$

together with

$$A_p.a \equiv (-1)^{p-1}a.A_p. \tag{25}$$

Explicitly, we have

$$a.(v_1 \wedge v_2) = -(a.v_1)v_2 + (a.v_2)v_1 \tag{26}$$

$$a.(v_1 \wedge v_2 \wedge v_3) = -(a.v_1)(v_2 \wedge v_3) + (a.v_2)(v_1 \wedge v_3) - (a.v_3)(v_1 \wedge v_2). \tag{27}$$

The interior product $a.A_p$ allows the definition of the multivector $a \wedge A_p$ and $A_p \wedge a$ via the relations

$$aA_p = a.A_p - a \wedge A_p \tag{28}$$

$$A_pa = A_p.a - A_p \wedge a \tag{29}$$

with

$$A_p \wedge a = (-1)^p a \wedge A_p. \tag{30}$$

Multiplying both sides of Eq. (29) with $(-1)^p$ and applying Eqs. (25, 30), we obtain

$$(-1)^p A_pa = -a.A_p - a \wedge A_p \tag{31}$$

Combining Eqs. (28, 31), we obtain the formulas valid in all cases $(1 \leq p \leq 3)$

$$a.A_p = \frac{1}{2}\left[aA_p - (-1)^p A_pa\right] \in V_{p-1} \tag{32}$$

$$a \wedge A_p = -\frac{1}{2}\left[aA_p + (-1)^p A_pa\right] \in V_{p+1} \tag{33}$$

A_3 being a pseudo-scalar, commuting with any Clifford number, we have in particular $a \wedge A_3 = 0$.

Table 1. Interior and exterior products with their corresponding expressions in the classical vector calculus (with $B = b \wedge c, B_1 = a \wedge b, B_2 = c \wedge d, T_1 = a \wedge b \wedge c, T_2 = f \wedge g \wedge h$ and $a, b, c, d, e, f, g, h \in V_1, T \in V_3$)

Multivector calculus	Classical vector calculus
$a.b = \frac{1}{2}(ab + ba) \in V_0$	$\vec{a}\,.\,\vec{b}$
$a \wedge b = -\frac{1}{2}(ab - ba) \in V_2$	$\vec{a} \times \vec{b}$
$a.B = \frac{1}{2}(aB - Ba) \in V_1$	$\vec{a} \times \left(\vec{b} \times \vec{c}\right)$
$a \wedge B = -\frac{1}{2}(aB + Ba) \in V_3$	$\vec{a}\,.\left(\vec{b} \times \vec{c}\right)$
$B_1.B_2 = -\frac{1}{2}(B_1B_2 + B_2B_1) \in V_0$	$\left(\vec{a} \times \vec{b}\right).\left(\vec{c} \times \vec{d}\right)$
$[B_1, B_2] = \frac{1}{2}(B_1B_2 - B_2B_1) \in V_2$	$\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right)$
$T_1.T_2 = -\frac{1}{2}(T_1T_2 + T_2T_1) \in V_0$	$\left[\vec{a}\,.\left(\vec{b} \times \vec{c}\right)\right]\left[\vec{f}\,.\left(\vec{g} \times \vec{h}\right)\right]$
$B.T = -\frac{1}{2}(BT + TB) \in V_1$	$\left(\vec{a} \times \vec{b}\right)\left[\vec{f}\,.\left(\vec{g} \times \vec{h}\right)\right]$
$a.T = \frac{1}{2}(aT + Ta) \in V_2$	$\vec{a}\left[\vec{f}\,.\left(\vec{g} \times \vec{h}\right)\right]$

Products between multivectors Other interior and exterior products between two multivectors A_p and B_q are defined for $p \leq q$ [4]

$$A_p \cdot B_q \equiv (v_1 \wedge v_2 \wedge \cdots \wedge v_{p-1}) \cdot (v_p \cdot B_q) \tag{34}$$

$$A_p \wedge B_q \equiv v_1 \wedge (v_2 \wedge \cdots \wedge v_p) \wedge B_q) \tag{35}$$

with

$$A_p \cdot B_q = (-1)^{p(q+1)} B_q \cdot A_p \tag{36}$$

which defines $B_q \cdot A_p$ for $q \geq p$. The various products are given in Table 1.

Associativity A major property of the exterior product is its associativity which is expressed as (with $v_i \in V_1$) [4].

$$(v_1 \wedge v_2) \wedge v_3 = v_1 \wedge (v_2 \wedge v_3) \tag{37}$$

Proof.

$$(v_1 \wedge v_2) \wedge v_3 = v_3 \wedge (v_1 \wedge v_2) \tag{38}$$

$$= \frac{1}{2}\left[-v_3 (v_1 \wedge v_2) - (v_1 \wedge v_2) v_3\right] \tag{39}$$

$$= \frac{1}{4}\left[v_3 (v_1v_2 - v_2v_1) + (v_1v_2 - v_2v_1) v_3\right] \tag{40}$$

$$v_1 \wedge (v_2 \wedge v_3) = \frac{1}{2}\left[-v_1 (v_2 \wedge v_3) - (v_2 \wedge v_3) v_1\right] \tag{41}$$

$$= \frac{1}{4}\left[v_1 (v_2v_3 - v_3v_2) + (v_2v_3 - v_3v_2) v_1\right]. \tag{42}$$

Since $v_3v_1v_2 - v_2v_1v_3 = -v_1v_3v_2 + v_2v_3v_1$ because of

$$(v_3v_1 + v_1v_3)\, v_2 = v_2\,(v_3v_1 + v_1v_3)\,, \tag{43}$$

Eq. (37) is established. □

3.3 General formulas

Among general formulas, one has with $(a, b, c, d\,) \in V_1$, $(B,\, B_i\,) \in V_2$ and F, G, H being any elements

$$(a \wedge b)\,.B = a.\,(b.B) = -b.\,(a.B) \tag{44}$$

$$(a \wedge b)\,.\,(c \wedge d) = (a.c)\,(b.d) - (a.d)\,(b.c) \tag{45}$$

$$[F, [G, H]] - [G, [F, H]] = [[F, G]\,, H] \tag{46}$$

$$B_2.\,(B_1.a) - B_1.\,(B_2.a) = [B_2, B_1]\,.a \tag{47}$$

$$a.A_p = \left(a \wedge A_p^*\right)^* \tag{48}$$

$$a \wedge A_p = \left(a.A_p^*\right)^* \tag{49}$$

$$B_1 \wedge B_2^* = (B_1.B_2)^* \tag{50}$$

$$B_1.B_2 = B_1^*.B_2^* \tag{51}$$

Proof. Eq. (44) results from the definition (34). Eq. (45) follows from

$$(a \wedge b)\,.\,(c \wedge d) = a.\,[b.\,(c \wedge d)] \tag{52}$$

with

$$b.\,(c \wedge d) = (b.d)\,c - (b.c)\,d \tag{53}$$

hence,

$$a.\,[b.\,(c \wedge d)] = (a.c)\,(b.d) - (b.c)\,(a.d)\,. \tag{54}$$

Eq. (46) is simply the Jacobi identity which entails Eq. (47). Eq. (48) is established as follows (with $n = 3$)

$$a \wedge A_p^* = a \wedge A_{n-p} \tag{55}$$

$$= -\frac{1}{2}\left[aA_{n-p} + (-1)^{n-p}\, A_{n-p}a\right] \tag{56}$$

$$= -\frac{1}{2}\left[aA_{n-p} - (-1)^{p}\, A_{n-p}a\right] \tag{57}$$

$$= -\frac{I}{2}\left[aA_{p} - (-1)^{p}\, A_{p}a\right] = -\,(a.A_p)^* \tag{58}$$

hence, we obain since $(A^*)^* = -A$ the relation

$$\left(a \wedge A_p^*\right)^* = (a.A_p)\,. \tag{59}$$

Eq. (49) follows from

$$a.A_p^* = a.A_{n-p} \tag{60}$$
$$= \frac{1}{2}\left[aA_{n-p} - (-1)^{n-p} A_{n-p}a\right] \tag{61}$$
$$= \frac{1}{2}\left[aA_{n-p} + (-1)^{p} A_{n-p}a\right] \tag{62}$$
$$= \frac{I}{2}\left[aA_{p} + (-1)^{p} A_{p}a\right] \tag{63}$$

thus we get

$$\left(a.A_p^*\right)^* = \frac{-1}{2}\left[aA_p + (-1)^p A_pa\right] = a \wedge A_p. \tag{64}$$

Eq. (50) results from

$$B_1 \wedge B_2^* = B_2^* \wedge B_1 \tag{65}$$
$$= \frac{-I}{2}\left(B_2B_1 + B_1B_2\right) = \left(B_1.B_2\right)^* \tag{66}$$

and Eq. (51) from

$$\left(B_1.B_2\right) = \frac{-1}{2}\left(B_1B_2 + B_2B_1\right) \tag{67}$$
$$= \frac{1}{2}\left(IB_1IB_2 + IB_2IB_1\right) = \left(B_1^*.B_2^*\right).\square \tag{68}$$

4 Multivector geometry

4.1 Analytic geometry

The equation of a straight line parallel to the vector u and going through the point a is expressed by

$$(x-a) \wedge u = 0 \tag{69}$$

yielding the solution

$$x - a = \lambda u \tag{70}$$
$$x = \lambda u + a \qquad (\lambda \in \mathbb{R}). \tag{71}$$

Similarly, the equation of a plane going through the point a parallel to the plane $B = u \wedge v$ is expressed by

$$(x-a) \wedge (u \wedge v) = 0 \tag{72}$$

with the solution

$$x - a = \lambda u + \mu v \tag{73}$$
$$x = \lambda u + \mu v + a \qquad (\lambda, \mu \in \mathbb{R}) \tag{74}$$

4.2 Orthogonal projections

Orthogonal projection of a vector on a vector The orthogonal projection of a vector $u = u_{\parallel} + u_{\perp}$ on a vector a with $u_{\perp} \cdot a = 0$, $u_{\parallel} \wedge a = 0$ is obtained as follows. Since

$$ua = u \cdot a - u \wedge a \tag{75}$$

one has

$$u_{\parallel} a = u_{\parallel} \cdot a = u \cdot a \tag{76}$$
$$u_{\perp} a = -u_{\perp} \wedge a = -u \wedge a \tag{77}$$

therefore

$$u_{\parallel} = (u \cdot a) a^{-1} \tag{78}$$
$$u_{\perp} = -(u \wedge a) a^{-1}. \tag{79}$$

Orthogonal projection of a vector on a plane Similarly, to obtain the orthogonal projection of a vector $u = u_{\parallel} + u_{\perp}$ on a plane $B = a \wedge b$ (with $u_{\perp} \cdot B = 0$, $u_{\parallel} \wedge B = 0$) one writes

$$uB = u \cdot B - u \wedge B \tag{80}$$

hence, the solution is (with $B^{-1} = B_c / BB_c$)

$$u_{\parallel} = (u \cdot B) B^{-1} \tag{81}$$
$$u_{\perp} = -(u \wedge B) B^{-1}. \tag{82}$$

Orthogonal projection of a plane on a plane As another example, let us give the orthogonal projection of a plane $B_1 = B_{1\parallel} + B_{1\perp}$ on the plane $B_2 = a \wedge b$ with $B_{1\perp} \cdot B_2 = 0$, and $\left[B_{1\parallel}, B_2\right] = 0$. Using the relation

$$B_1 B_2 = -B_1 \cdot B_2 + [B_1, B_2], \tag{83}$$

we obtain

$$B_{1\parallel} = -(B_1 \cdot B_2) B_2^{-1} \tag{84}$$
$$B_{1\perp} = \{[B_1, B_2]\} B_2^{-1}. \tag{85}$$

5 Differential operators and integrals

5.1 Differential operators

In Cartesian coordinates, the nabla operator $\nabla = Ii\frac{\partial}{\partial x_1} + Ij\frac{\partial}{\partial x_2} + Ik\frac{\partial}{\partial x_3}$ acting on a scalar f, a vector a $(= a_1 Ii + a_2 Ij + a_3 Ik)$, a bivector B $(= B_1 i + B_2 j + B_3 k)$ and a trivector T $(= \tau I)$ yields respectively

$$\nabla f = Ii\frac{\partial}{\partial x_1} + Ij\frac{\partial}{\partial x_2} + Ik\frac{\partial}{\partial x_3} = grad f \in V_1 \tag{86}$$

$$\nabla .a=\frac{\partial a_1}{\partial x_1}+\frac{\partial a_2}{\partial x_2}+\frac{\partial a_3}{\partial x_3}=diva\in V_0 \tag{87}$$

$$\nabla\wedge a=\left(\frac{\partial a_3}{\partial x_2}-\frac{\partial a_2}{\partial x_3}\right)i+\left(\frac{\partial a_1}{\partial x_3}-\frac{\partial a_3}{\partial x_2}\right)j+\left(\frac{\partial a_2}{\partial x_1}-\frac{\partial a_1}{\partial x_2}\right)k \tag{88}$$

$$=rota\in V_2 \tag{89}$$

$$\nabla .B=\left(\nabla\wedge B^*\right)^*=\left(rotB^*\right)^*\in V_1 \tag{90}$$

$$\nabla\wedge B=\left(\nabla .B^*\right)^*=\left(divB^*\right)^*\in V_3 \tag{91}$$

$$\nabla .T=\nabla T=-\left(\nabla T^*\right)^*=-\left(gradT^*\right)^*\in V_2. \tag{92}$$

Hence, the various operators can be expressed with the usual ones $(grad, div, rot)$ and the duality. Among a few properties of the nabla operator, one has

$$\nabla^2=\triangle,\nabla\wedge\left(\nabla\wedge f\right)=0,\nabla\wedge\left(\nabla\wedge a\right)=0 \tag{93}$$

$$\triangle a=\nabla\left(\nabla a\right)=\nabla\left(\nabla .a-\nabla\wedge a\right)=\nabla\left(\nabla .a\right)-\nabla .\left(\nabla\wedge a\right), \tag{94}$$

where the last equation results from

$$\nabla\left(\nabla\wedge a\right)=\nabla .\left(\nabla\wedge a\right)-\nabla\wedge\left(\nabla\wedge a\right)=\nabla .\left(\nabla\wedge a\right). \tag{95}$$

5.2 Integrals and theorems

The length, surface and volume integrals are respectively for a curve $x(u)$, surface $x(u,v)$ and a volume $x(u,v,w)$

$$L=\int ds=\int\sqrt{\left(dx\right)^2}=\int\sqrt{\left(\frac{dx}{du}\right)^2}du \tag{96}$$

$$S=\int\int\sqrt{-\left(\frac{\partial x}{\partial u}\wedge\frac{\partial x}{\partial v}\right)^2}dudv \tag{97}$$

$$V=\int\int\int\sqrt{-\left(\frac{\partial x}{\partial u}\wedge\frac{\partial x}{\partial v}\wedge\frac{\partial x}{\partial w}\right)^2}dudvdw. \tag{98}$$

The formulas exhibit immediately the transformation properties under a change of coordinates.

Stokes' theorem is expressed for a vector a (with $dl=dx, dS=dl_1\wedge dl_2$)

$$\oint a.dl=\int\left(\nabla\wedge a\right).dS; \tag{99}$$

the same formula can be used for a bivector B by taking $a=B^*$

$$\oint B^*.dl=\int\left(\nabla\wedge B^*\right).dS=-\int\left(\nabla .B\right)^*.dS \tag{100}$$

$$=\int rotB^*.dS. \tag{101}$$

Ostrogradsky's theorem for a bivector B yields (with $d\tau = dl_1 \wedge dl_2 \wedge dl_3$)

$$\oint B.dS = \int (\nabla \wedge B) .d\tau = \int (divB^*)^* .d\tau \tag{102}$$

$$= \int (divB^*) \, dV. \tag{103}$$

For a vector a, one obtains with $B = -a^*$

$$-\oint a^*.dS = \int (diva) \, dV \tag{104}$$

which transforms, since $a^*.dS = -a.dS^*$, into

$$\oint a.dS^* = \int (diva) \, dV. \tag{105}$$

6 Orthogonal groups $O(3)$ and $SO(3)$

Definition 2. *The symmetric of x with respect to a plane is obtained by drawing the perpendicular to the plane and by extending this perpendicular by an equal length.*

Let x be a vector, x' its symmetric to a plane and a a unit vector perpendicular to the plane. From the geometry, $x' - x$ is perpendicular to the plane and thus parallel to a; similarly, $x' + x$ is parallel to the plane and thus perpendicular to a. Consequently, one has

$$x' = x + \lambda a, a \cdot \left(\frac{x' + x}{2} \right) = 0; \tag{106}$$

hence, one obtains (with $a \cdot a = a^2 = 1$)

$$a \cdot \left(x + \frac{\lambda a}{2} \right) = 0 \tag{107}$$

yielding $\lambda = -\frac{2(a \cdot x)}{a \cdot a}$ and

$$x' = x - \frac{2(a \cdot x)a}{a \cdot a} = x - \frac{(ax + xa)a}{a \cdot a} \tag{108}$$

$$= -axa. \tag{109}$$

Definition 3. *The orthogonal group $O(3)$ is the group of linear operators which leave invariant the quadratic form $x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$*

Theorem 2. *Every rotation of $O(3)$ is the product of an even number ≤ 3 of symmetries, any reflection is the product of an odd number ≤ 3 of symmetries.*

The special orthogonal group $SO(3)$ is constituted by rotations i.e. of proper transformations $f(x)$ of determinant equal to 1 (i.e. $\alpha = f(e_1) \wedge f(e_2) \wedge f(e_3) = I$). A reflection is an improper transformation of determinant equal to -1 (i.e. $\alpha = -I$). Combining two orthogonal symmetries, we obtain

$$x' = (ba)\, x\, (ab) = rxr_c \tag{110}$$

with $r = ba$, $r_c = a_c b_c = ab$, $rr_c = 1$. One can express r as

$$r = \left(\cos\frac{\theta}{2} + u \sin\frac{\theta}{2} \right) = e^{\frac{1}{2}u\theta} \tag{111}$$

with $u = u_1 i + u_2 j + u_3 k$ ($u^2 = -1$). Eq. (110) represents a conical rotation of the vector x by an angle θ around the unit vector $u^* = Iu$. One verifies that the rotation conserves the norm $x'^2 = x^2$. The same equation holds for any element A of the algebra $A' = rAr_c$ since the product of two vectors x, y transforms as

$$x'y' = (rxr_c)\,(ryr_c) = r(xy)r_c \tag{112}$$

and similarly for the product of three vectors as well as a linear combination of such products. The above formulas allow to easily express the classical moving frames such as the Frenet and Darboux frames within the Grassmannian scheme.

7 Curves

7.1 Generalities

Consider a $3D$ curve $x(t)$ $(= x_1(t)e_1 + x_2(t)e_2 + x_3(t)e_3)$ where e_i is the canonical orthonormal basis ($e_1 = Ii$, $e_2 = Ij$, $e_3 = Ik$). Taking the length of the curve s as parameter we have $x = f(s)$ with $ds = \sqrt{(dx)^2}$. The tangent unit vector at a point $M(x)$ is

$$T = \frac{dx}{ds}, T^2 = \left(\frac{dx}{ds}\right)^2 = 1. \tag{113}$$

The equation of the tangent at a point $M(x)$ is given by

$$(X - x) \wedge \frac{dx}{ds} = 0 \tag{114}$$

where X is a generic point of the tangent. The equations of the plane perpendicular to the curve and of the osculating plane at the point M read respectively

$$(X - x)\,.\frac{dx}{ds} = 0 \tag{115}$$

$$(X - x) \wedge \frac{dx}{ds} \wedge \frac{d^2x}{ds^2} = 0 \tag{116}$$

where X is a generic point of the plane [25].

7.2 Frenet frame

The Frenet frame (v_i) attached to the curve $x(t)$ is given by

$$v_i = r e_i r_c \tag{117}$$

where $r = e^{\frac{1}{2}u\theta}$ expresses the rotation of angle θ around the unit vector u^* $\left(r r_c = 1, u^2 = -1\right)$ and e_i is the canonical orthonormal basis. After differentiation, one obtains (using the relation $dr_c r = -r_c dr$ resulting from the differentiation of $r r_c = 1$)

$$dv_i = r\left(r_c dr e_i + e_i dr_c r\right) r_c \tag{118}$$
$$= r\left(d\iota_F . e_i\right) r_c \tag{119}$$
$$= r\left(De_i\right) r_c \tag{120}$$

with

$$d\iota_F = 2r_c dr = 2e^{-\frac{1}{2}u\theta} e^{\frac{1}{2}u\theta}\left(\frac{d\theta}{2}u + \frac{\theta}{2}du\right) \tag{121}$$
$$= (d\theta u + \theta du) \tag{122}$$
$$= (da)\ i + (db)\ j + (dc)\ k) \in V_2 \tag{123}$$

and

$$De_i = d\iota_F . e_i = \frac{1}{2}\left(d\iota_F e_i - e_i d\iota_F\right). \tag{124}$$

We shall call $d\iota_F = 2r_c dr$ the affine connection bivector. Explicitly, one has

$$De_1 = (dc)\, e_2 - (db)\, e_3 \tag{125}$$
$$De_2 = -\,(dc)\, e_1 + (da)\, e_3 \tag{126}$$
$$De_3 = (db)\, e_1 - (da)\, e_2. \tag{127}$$

The Frenet frame is defined by the affine connection bivector

$$d\iota_F = 2r_c dr = (\tau ds)\, i + (\rho ds)\, k \tag{128}$$

where $\rho = 1/R$ is the curvature and $\tau = 1/T$ the torsion. This gives the Frenet equations

$$De_1 = (\rho ds)\, e_2 \tag{129}$$
$$De_2 = -\,(\rho ds)\, e_1 + (\tau ds)\, e_3 \tag{130}$$
$$De_3 = (-\tau ds)\, e_2. \tag{131}$$

7.3 Curvature and torsion

To obtain the curvature and torsion we define α and β

$$\alpha = \frac{dx}{ds} \wedge \frac{d^2x}{ds^2} = \rho v_1 \wedge v_2 \tag{132}$$

$$\beta = \frac{dx}{ds} \wedge \frac{d^2x}{ds^2} \wedge \frac{d^3x}{ds^3} = I\rho^2\tau \tag{133}$$

and using the Lagrange equation

$$(v_1 \wedge v_2)^2 = (v_1.v_2)^2 - (v_1)^2 (v_2)^2 = -1 \tag{134}$$

we obtain the invariants

$$\rho = \sqrt{-\alpha^2} = \sqrt{-\left(\frac{dx}{ds} \wedge \frac{d^2x}{ds^2}\right)^2} \tag{135}$$

$$\tau = \frac{I\beta}{\alpha^2} = \frac{I\left(\frac{dx}{ds} \wedge \frac{d^2x}{ds^2} \wedge \frac{d^3x}{ds^3}\right)}{\left(\frac{dx}{ds} \wedge \frac{d^2x}{ds^2}\right)^2}. \tag{136}$$

Under a change of parameter t, one has using $\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$

$$\frac{dx}{dt} \wedge \frac{d^2x}{dt^2} = \left(\frac{ds}{dt}\right)^3 \alpha \tag{137}$$

and thus one obtains the curvature

$$\rho = \sqrt{-\left(\frac{dx}{dt}\right)^{-6}\left(\frac{dx}{dt} \wedge \frac{d^2x}{dt^2}\right)^2}. \tag{138}$$

For the torsion, proceeding similarly, we get under a change of parameter

$$\frac{dx}{dt} \wedge \frac{d^2x}{dt^2} \wedge \frac{d^3x}{dt^3} = \left(\frac{ds}{dt}\right)^6 \beta \tag{139}$$

and thus

$$\tau = \frac{I\left(\frac{dx}{dt} \wedge \frac{d^2x}{dt^2} \wedge \frac{d^3x}{dt^3}\right)}{\left(\frac{dx}{dt} \wedge \frac{d^2x}{dt^2}\right)^2} \in V_0. \tag{140}$$

7.4 Example

As example, consider the curve $x(t) = (2\cos t)Ii + (2\sin t)Ij + (t)Ik$. The line element is $ds = \sqrt{dx^2} = \sqrt{5}dt$; writing $x' = \frac{dx}{dt}$, etc., we have

$$x' \wedge x'' = (2\sin t)i - (2\cos t)j + 4k, \tag{141}$$

$$(x' \wedge x'')^2 = -20 \tag{142}$$

$$x' \wedge x'' \wedge x''' = 4I. \tag{143}$$

The curvature and torsion are respectively

$$\rho = \sqrt{-(x')^{-6}(x' \wedge x'')^2} = \frac{2}{5} \tag{144}$$

$$\tau = \frac{I(x' \wedge x'' \wedge x''')}{(x' \wedge x'')^2} = \frac{1}{5}. \tag{145}$$

The equation of the osculating plane is (with $X = (X_1)\, Ii + (X_2)\, Ij + (X_3)\, Ik$ being a generic point of the plane)

$$(X - x) \wedge \frac{dx}{ds} \wedge \frac{d^2x}{ds^2} = \frac{I}{\sqrt{5}} \left(-2t + 2X_3 - X_2 \cos t + X_1 \sin t\right) = 0. \tag{146}$$

The Frenet basis v_i is

$$v_1 = \frac{1}{\sqrt{5}} \left[(-2 \sin t)\, Ii + (2 \cos t)\, Ij + Ik\right] \tag{147}$$

$$v_2 = (-\cos t)\, Ii - (\sin t)\, Ij \tag{148}$$

$$v_3 = (v_1 \wedge v_2)^* = \frac{1}{\sqrt{5}} \left[(\sin t)\, Ii - (\cos t)\, Ij + 2Ik\right]. \tag{149}$$

The basis v_i is obtained via the following rotations. First, the frame is brought into its initial position (at $t = 0$) via the rotation $f_0 = f_1 f_2$ with $\tan\theta = \frac{1}{2}$ and

$$f_1 = e^{k\frac{\pi}{2}} = \frac{1}{\sqrt{2}} (1 + k) \tag{150}$$

$$f_2 = e^{-j\frac{\theta}{2}} = \left(\sqrt{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}}} - j\sqrt{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{5}}}\right) \tag{151}$$

yielding

$$f_0 = \sqrt{\left(\frac{1}{4} + \frac{1}{2\sqrt{5}}\right)} + i\sqrt{\frac{1}{4} - \frac{1}{2\sqrt{5}}} - j\sqrt{\frac{1}{4} - \frac{1}{2\sqrt{5}}} + k\sqrt{\frac{1}{4} + \frac{1}{2\sqrt{5}}}. \tag{152}$$

Next, follows the rotation due to the affine connection bivector

$$f_3 = \cos\frac{t}{2} + \left(\frac{i}{\sqrt{5}} + \frac{2k}{\sqrt{5}}\right) \sin\frac{t}{2}. \tag{153}$$

The end result is $r = f_1 f_2 f_3$ and explicitly

$$r = \left(A \cos\frac{t}{2} - C \sin\frac{t}{2}\right) + i\left(B \cos\frac{t}{2} + D \sin\frac{t}{2}\right) \tag{154}$$

$$+j\left(-B \cos\frac{t}{2} + D \sin\frac{t}{2}\right) + k\left(A \cos\frac{t}{2} + C \sin\frac{t}{2}\right) \tag{155}$$

with

$$A = \frac{1}{10}\sqrt{5\left(5 + 2\sqrt{5}\right)}, B = \frac{1}{10}\sqrt{5\left(5 - 2\sqrt{5}\right)} \tag{156}$$

$$C = \frac{1}{10}\left(\sqrt{5 - 2\sqrt{5}} + 2\sqrt{5 + 2\sqrt{5}}\right), \tag{157}$$

$$D = \frac{1}{10}\left(-2\sqrt{5 - 2\sqrt{5}} + \sqrt{5 + 2\sqrt{5}}\right). \tag{158}$$

Finally, one verifies that $v_i = r e_i r_c$.

8 Surfaces

8.1 Generalities

Consider in a $3D$ Euclidean space a surface $x = f(u, v)$. The tangent plane is given by $f_u \wedge f_v$ ($f_u = \frac{\partial f}{\partial u}, f_v = \frac{\partial f}{\partial v}$) and the unit normal by

$$h = \frac{(f_u \wedge f_v)^*}{\sqrt{-(f_u \wedge f_v)^2}} \tag{159}$$

with (f_u, f_v, h) being a direct trieder $(f_u \wedge f_v \wedge h = \lambda I, \lambda > 0)$. Take an orthonormal moving frame of basis vectors v_i and a vector a (of components A with respect to the moving frame) attached to the surface. This frame is obtained by rotating the canonical frame e_i; hence, one has

$$v_i = re_ir_c, a = rAr_c. \tag{160}$$

Differentiating these relations, one obtains $dv_i = r(De_i)r_c, da = r(DA)r_c$ with

$$De_i = d\iota.e_i, DA = dA + d\iota.A \tag{161}$$

where DA is the covariant differential (dA being a differentiation with respect to the components only) and $d\iota = 2r_cdr$.

8.2 Darboux frame

The Darboux frame (v_{iD}) is obtained for a curve on the surface from the Frenet frame by a rotation of an algebraic angle $\alpha = \angle(v_{3F}, h)$ around v_{1F}. Hence

$$v_{iD} = (r_D)e_i(r_D)_c \tag{162}$$

with $r_D = r_1r$ and

$$r_1 = \cos\frac{\alpha}{2} - (v_{1F})^* \sin\frac{\alpha}{2} = rpr_c \tag{163}$$

where $p = \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}$; hence $r_D = r_1r = rp$. Consequently, the rotation bivector $d\iota_D = 2(r_D)_c dr_D$ can be expressed as follows

$$d\iota_D = 2(rp)_c d(rp) = 2p_cr_c(drp + rdp) \tag{164}$$

$$= p_c(d\iota_F)p + 2p_cdp \tag{165}$$

$$= (\tau_rds)i - (\rho_nds)j + (\rho_gds)k \tag{166}$$

where $\rho_g = \rho\cos\alpha$ is the geodesic curvature, $\rho_g = -\rho\sin\alpha$ the normal curvature and $\tau_r = \frac{d\alpha}{ds} + \tau$ the relative torsion. The Darboux equations thus become

$$dv_{iD} = r_D(d\iota_D.e_i)(r_D)_c \tag{167}$$

and explicitly

$$dv_{1D} = (\rho_g v_{2D} + \rho_n v_{3D})\, ds \tag{168}$$
$$dv_{2D} = (-\rho_g v_{1D} + \tau_r v_{3D})\, ds \tag{169}$$
$$dv_{3D} = (-\rho_n v_{1D} - \tau_r v_{2D})\, ds. \tag{170}$$

If the Darboux frame is rotated in the tangent plane by an angle $\theta(s)$ around the unit normal h one obtains, applying the same reasoning as above, for the rotation $r = r_D f$ with $f = \cos\frac{\theta}{2} + k\sin\frac{\theta}{2}$. The new vectors v_i of the frame become $v_i = re_i r_c$ and the affine connection bivector transforms into

$$d\iota = 2r_c dr \tag{171}$$
$$= 2f_c\,(r_D)_c\,[(dr_D)\,f + r_D df] \tag{172}$$
$$= f_c\,(d\iota_D)\,f + 2f_c df. \tag{173}$$

Applying this formula, one obtains

$$d\iota = ds\,(\tau_r\cos\theta - \rho_n\sin\theta)\,i + ds\,(-\rho_n\cos\theta - \tau_r\sin\theta)\,j \tag{174}$$
$$+ (\rho_g ds + d\theta)\,k. \tag{175}$$

8.3 Integrability conditions

The $3D$ Euclidean space being without torsion and curvature, this entails that dM and dv_i are integrable.

Integrability of dM Consider a point on the surface, one has

$$dM = r\,(DM)\,r_c = f_u du + f_v dv \tag{176}$$

where DM are the components expressed in the moving frame (with the moving vectors $v_i = r\,(e_i)\,r_c$). One has

$$DM = \omega_1 e_1 + \omega_2 e_2 \tag{177}$$
$$= (A_1 du + B_1 dv)\,e_1 + (A_2 du + B_2 dv)\,e_2 \tag{178}$$

with

$$A_1 = (r_c f_u r)\,.e_1 = f_u.v_1, B_1 = (r_c f_v r)\,.e_1 = f_v.v_1 \tag{179}$$
$$A_2 = (r_c f_u r)\,.e_2 = f_u.v_2, B_2 = (r_c f_v r)\,.e_2 = f_v.v_2. \tag{180}$$

The affine connection bivector $d\iota = 2r_c dr$ can be written [15, II, p. 410] as $d\iota = \iota_1 du + \iota_2 dv$ (with $\iota_1 = a_1 i + b_1 j + c_1 k$, $\iota_2 = a_2 i + b_2 j + c_2 k$). The integrability condition of dM is expressed by the condition

$$\Delta\,(DM) - D\,(\Delta M) = 0 \tag{181}$$

with

$$\varDelta\left(DM\right)=\delta\left(DM\right)+\delta\iota.DM \tag{182}$$

$$D\left(\varDelta M\right)=d\left(\varDelta M\right)+d\iota.\varDelta M. \tag{183}$$

This leads to the relation

$$\delta\left(DM\right)-d\left(\varDelta M\right)=d\iota.\varDelta M-\delta\iota.DM. \tag{184}$$

Explicitly, it reads

$$\frac{\partial A_1}{\partial v}-\frac{\partial B_1}{\partial u}=-B_2c_1+A_2c_2 \tag{185}$$

$$\frac{\partial A_2}{\partial v}-\frac{\partial B_2}{\partial u}=B_1c_1-A_1c_2 \tag{186}$$

$$A_1b_2+B_2a_1=A_2a_2+B_1b_1. \tag{187}$$

The linear Eqs. (185,186) determine c_1, c_2 ; the result is

$$c_1=\frac{-1}{A_1B_2-A_2B_1}\left[A_1\left(\frac{\partial A_1}{\partial v}-\frac{\partial B_1}{\partial u}\right)+A_2\left(\frac{\partial A_2}{\partial v}-\frac{\partial B_2}{\partial u}\right)\right] \tag{188}$$

$$c_2=\frac{-1}{A_1B_2-A_2B_1}\left[B_2\left(\frac{\partial A_2}{\partial v}-\frac{\partial B_2}{\partial u}\right)+B_1\left(\frac{\partial A_1}{\partial v}-\frac{\partial B_1}{\partial u}\right)\right] \tag{189}$$

Integrability of dv_i The integrability conditions of dv_i are obtained similarly, with $dv_i=rDe_ir_c$. The condition $\delta dv_i-d\delta v_i=0$ leads to the relation

$$\varDelta\left(De_i\right)-D\left(\varDelta e_i\right)=0 \tag{190}$$

with $De_i=d\iota.e_i$, $\varDelta e_i=\delta\iota.e_i$ and

$$\varDelta\left(De_i\right)=\delta\left(d\iota.e_i\right)+\delta\iota.\left(d\iota.e_i\right) \tag{191}$$

$$D\left(\varDelta e_i\right)=d\left(\delta\iota.e_i\right)+d\iota.\left(\delta\iota.e_i\right). \tag{192}$$

Applying Eq. (47), one has

$$\delta\iota.\left(d\iota.e_i\right)-d\iota.\left(\delta\iota.e_i\right)=-\left[d\iota,\delta\iota\right].e_i \tag{193}$$

hence, the integrability condition of dv_i can be expressed as

$$\left(\delta d\iota-d\delta\iota\right).e_i=\left[d\iota,\delta\iota\right].e_i \tag{194}$$

and thus $(\delta d\iota-d\delta\iota)=[d\iota,\delta\iota]$ or

$$\frac{\partial\iota_1}{\partial v}-\frac{\partial\iota_2}{\partial u}=\left[\iota_1,\iota_2\right]. \tag{195}$$

Explicitly, these equations read [15, II, p. 412]

$$\frac{\partial a_1}{\partial v}-\frac{\partial a_2}{\partial u}=b_1c_2-b_2c_1 \tag{196}$$

$$\frac{\partial b_1}{\partial v}-\frac{\partial b_2}{\partial u}=c_1a_2-c_2a_1 \tag{197}$$

$$\frac{\partial c_1}{\partial v}-\frac{\partial c_2}{\partial u}=a_1b_2-a_2b_1. \tag{198}$$

8.4 Curvature lines and curvature: first method

Consider a moving frame $v_i = re_i r_c$ with $De_i = d\iota.e_i$ ($d\iota = 2r_c dr$) and $DM = \omega_1 e_1 + \omega_2 e_2$ expressed in the moving frame. The fundamental form Π can be expressed as

$$\Pi = -De_3.DM = -(d\iota.e_3).DM \tag{199}$$
$$= DM.(e_3.d\iota) = (DM \wedge e_3).d\iota. \tag{200}$$

The affine connection bivector $d\iota = dai + dbj + dck$ can be developed on $\omega_1 (= A_1 du + B_1 dv), \omega_2 \ (= A_2 du + B_2 dv\ )$ as follows [26, p. 209]

$$da = L_{21}\omega_1 + L_{22}\omega_2 = a_1 du + a_2 dv \tag{201}$$
$$db = -L_{11}\omega_1 - L_{12}\omega_2 = b_1 du + b_2 dv. \tag{202}$$

Identifying the coefficients of du, dv, and solving the linear system, one obtains

$$L_{11} = \frac{b_2 A_2 - b_1 B_2}{A_1 B_2 - A_2 B_1}, L_{22} = \frac{a_2 A_1 - a_1 B_1}{A_1 B_2 - A_2 B_1} \tag{203}$$

$$L_{12} = \frac{b_1 B_1 - b_2 A_1}{A_1 B_2 - A_2 B_1}, L_{21} = \frac{a_1 B_2 - a_2 A_2}{A_1 B_2 - A_2 B_1}. \tag{204}$$

Due to the integrability condition Eq. (187), we have $L_{12} = L_{21}$ and thus the fondamental form Π becomes

$$\Pi = L_{11}\omega_1^2 + L_{22}\omega_2^2 + 2L_{12}\omega_1\omega_2. \tag{205}$$

If we rotate the frame by an angle Φ around v_3 we have $v_i' = f v_i f_c$ with

$$f = \cos(\Phi/2) - v_3^* \sin(\Phi/2) \tag{206}$$
$$= r\left[\cos(\Phi/2) - e_3^* \sin(\Phi/2)\right] r_c \tag{207}$$
$$= r\left[\cos(\Phi/2) + k\sin(\Phi/2)\right] r_c. \tag{208}$$

The total rotation is $R = fr = rp$ with $p = \cos(\Phi/2) + k\sin(\Phi/2)$ and the affine connection bivector transforms into

$$d\iota' = 2R_c dR = 2p_c r_c (drp) = p_c (d\iota) p \tag{209}$$
$$= da' i + db' j + dc' k \tag{210}$$

with

$$da' = \cos(\Phi)da + \sin(\Phi)db = L_{12}'\omega_1' + L_{22}'\omega_2' \tag{211}$$
$$db' = -\sin(\Phi)da + \cos(\Phi)db = -L_{11}'\omega_1' - L_{12}'\omega_2' \tag{212}$$
$$dc' = dc. \tag{213}$$

The vector $DM' = \omega_1' e_1 + \omega_2' e_2$ transforms in the same way i.e.,

$$\omega_1' = \cos(\Phi)\omega_1 + \sin(\Phi)\omega_2 \tag{214}$$

$$\omega_2' = -\sin(\Phi)\omega_1 + \cos(\Phi)\omega_2. \tag{215}$$

The coefficients L'_{ij} are obtained by expressing (da, db) of Eqs. (211, 212) in terms of $(L_{ij,}\omega_i)$ via the Eqs. (201, 202) and then by writing the ω_i in terms of $\omega_i^{'}$; one finds

$$L'_{11} = L_{11}\cos^2(\Phi) + L_{22}\sin^2(\Phi) + L_{12}\sin(2\Phi) \tag{216}$$

$$L'_{22} = L_{11}\sin^2(\Phi) + L_{22}\cos^2(\Phi) - L_{12}\sin(2\Phi) \tag{217}$$

$$L'_{12} = L_{12}\cos(2\Phi) + \frac{1}{2}\sin(2\Phi)(L_{22} - L_{11}). \tag{218}$$

The curvature lines are the lines for which the fundamental form Π becomes diagonal, i.e., when $L'_{12} = 0$, or

$$\tan 2\Phi = \frac{2L_{12}}{(L_{11} - L_{22})}. \tag{219}$$

Along these lines, the curvature is defined by $De_3 = -K\,(DM)$ with

$$De_3 = db' e_1 - da' e_2 \tag{220}$$

$$da' = L'_{22}\omega'_2, db = -L'_{11}\omega'_1 \tag{221}$$

and $DM = \omega'_1 e_1$ or $DM = \omega'_2 e_{2;}$ hence, the curvatures are given by $K_1 = L'_{11,} K_2 = L'_{22}$ i.e., as a function of the angle Φ. To obtain the standard formulas, we write

$$\cos 2\Phi = \frac{L_{11} - L_{22}}{\left[(L_{11} - L_{22})^2 + 4L_{12}^2\right]^{1/2}}, \sin 2\Phi = \frac{2L_{12}}{\left[(L_{11} - L_{22})^2 + 4L_{12}^2\right]^{1/2}} \tag{222}$$

and use $\cos^2\Phi = \frac{1}{2}(1 + \cos 2\Phi), \sin^2\Phi = \frac{1}{2}(1 - \cos 2\Phi)$; after rearrangement, we get

$$K_1 = L'_{11} = \frac{1}{2}\left(L_{11} + L_{22} - \sqrt{(L_{11} - L_{22})^2 + 4L_{12}^2}\right) \tag{223}$$

$$K_2 = L'_{12} = \frac{1}{2}\left(L_{11} + L_{22} + \sqrt{(L_{11} - L_{22})^2 + 4L_{12}^2}\right). \tag{224}$$

Hence, the Gaussian curvature K is given by

$$K = K_1 K_2 = L_{11}L_{22} - L_{12}^2 \tag{225}$$

$$= \frac{a_1 b_2 - a_2 b_1}{A_1 B_2 - A_2 B_1} = \frac{\left(\frac{\partial c_1}{\partial v} - \frac{\partial c_2}{\partial u}\right)}{A_1 B_2 - A_2 B_1} \tag{226}$$

where we have made use of the integrability condition Eq. (198). Replacing c_1, c_2 by their expressions of Eqs. (188, 189), we finally get for the Gaussian curvature

$$K = \frac{1}{(A_1 B_2 - A_2 B_1)} \left\{ \begin{array}{c} \frac{\partial}{\partial u}\left[\frac{B_2\left(\frac{\partial A_2}{\partial v} - \frac{\partial B_2}{\partial u}\right) + B_1\left(\frac{\partial A_1}{\partial v} - \frac{\partial B_1}{\partial u}\right)}{(A_1 B_2 - A_2 B_1)}\right] \\ -\frac{\partial}{\partial v}\left[\frac{A_1\left(\frac{\partial A_1}{\partial v} - \frac{\partial B_1}{\partial u}\right) + A_2\left(\frac{\partial A_2}{\partial v} - \frac{\partial B_2}{\partial u}\right)}{(A_1 B_2 - A_2 B_1)}\right] \end{array} \right\}. \tag{227}$$

The Gaussian curvature thus depends only on the metric $ds = \sqrt{(dM)^2}$, as stated by Gauss' theorem. The mean curvature H is given by

$$H = \frac{1}{2}(K_1 + K_2) = \frac{1}{2}(L_{11} + L_{22}) \tag{228}$$

$$= \frac{-b_1B_2 + b_2A_2 + a_2A_1 - a_1B_1}{2(A_1B_2 - A_2B_1)}. \tag{229}$$

8.5 Gaussian and mean curvature: second method

The Gaussian and mean curvatures can also be derived as follows [25]. The curvature K is defined by the relation

$$dv_3 = -KdM. \tag{230}$$

Developing that equation we have

$$\frac{\partial v_3}{\partial u}du + \frac{\partial v_3}{\partial v}dv = -K(x_u du + x_v dv) \tag{231}$$

or

$$\left(Kx_u + \frac{\partial v_3}{\partial u}\right)du + \left(Kx_v + \frac{\partial v_3}{\partial v}\right)dv = 0. \tag{232}$$

The two vectors in parentheses are parallel and thus

$$\left(Kx_u + \frac{\partial v_3}{\partial u}\right) \wedge \left(Kx_v + \frac{\partial v_3}{\partial v}\right) = 0 \tag{233}$$

which gives the equation

$$K^2(x_u \wedge x_v) + K\left(\frac{\partial v_3}{\partial u} \wedge x_v + x_u \wedge \frac{\partial v_3}{\partial v}\right) + \frac{\partial v_3}{\partial u} \wedge \frac{\partial v_3}{\partial v} = 0. \tag{234}$$

Multiplying with the exterior product on the left by $n = (x_u \wedge x_v)^*$ and using Eq. (50)

$$(x_u \wedge x_v)^* \wedge (x_u \wedge x_v) = -\left[(x_u \wedge x_v)^2\right]^* = I(n^2) \tag{235}$$

we obtain the formula [25]

$$K^2 I(n^2) + K\left[n \wedge \frac{\partial v_3}{\partial u} \wedge x_v + n \wedge x_u \wedge \frac{\partial v_3}{\partial v}\right] + n \wedge \frac{\partial v_3}{\partial u} \wedge \frac{\partial v_3}{\partial v} = 0. \tag{236}$$

Calling K_1, K_2 the two roots of the equation, one has for the Gaussian curvature

$$K = K_1K_2 = \frac{-I}{n^2}\left(n \wedge \frac{\partial v_3}{\partial u} \wedge \frac{\partial v_3}{\partial v}\right). \tag{237}$$

The mean curvature is given by

$$H = \frac{1}{2}(K_1 + K_2) = \frac{I}{2n^2}\left(n \wedge x_u \wedge \frac{\partial v_3}{\partial v} + n \wedge \frac{\partial v_3}{\partial u} \wedge x_v\right). \tag{238}$$

8.6 Curves on surfaces: asymptotic, curvature and geodesic lines

Consider on a surface, at a point OM, an orthonormal frame v_1, v_2, v_3 (with v_3 being the unit normal). One has (with $\omega_1 = \cos\Phi ds,\ \omega_2 = \sin\Phi ds$) [15, II, p. 413]

$$d(OM) = (v_1 \cos\Phi + v_2 \sin\Phi)\, ds \tag{239}$$
$$= \omega_1 v_1 + \omega_2 v_2 \tag{240}$$

$$d^2(OM) = \frac{d^2(OM)}{ds^2} ds = (v_2 dc - v_3 db) \cos\Phi \tag{241}$$
$$+ (-v_1 dc + v_3 da) \sin\Phi + (v_2 \cos\Phi - v_1 \sin\Phi)\, d\Phi \tag{242}$$
$$= -v_1 \sin\Phi (dc + d\Phi) + v_2 \cos\Phi\, (dc + d\Phi) \tag{243}$$
$$+ v_3 (\sin\Phi da - \cos\Phi db)\,. \tag{244}$$

Asymptotic lines The normal curvature is defined by

$$\rho_n(s) = \frac{d^2(OM)}{ds^2} . v_3 \tag{245}$$
$$= \left(\sin\Phi \frac{da}{ds} - \cos\Phi \frac{db}{ds} \right). \tag{246}$$

Developing da, db on ω_1, ω_2 via Eqs. (201, 202), one obtains after rearrangement

$$\rho_n(s) = L_{22} \sin^2\Phi + L_{11} \cos^2\Phi + 2 \sin\Phi \cos\Phi L_{12}. \tag{247}$$

The asymptotic line is defined by $\rho_n(s) = 0$, leading to

$$\tan\Phi = \frac{-L_{12} \pm \sqrt{L_{12}^2 - L_{11}L_{22}}}{L_{22}} \tag{248}$$

under the assumption that $L_{12}^2 - L_{11}L_{22} \geq 0$.

Curvature lines The relative torsion $\tau_r(s)$ is expressed by

$$I\tau_r(s) = \frac{d(OM)}{ds} \wedge v_3 \wedge \frac{dv_3}{ds} \tag{249}$$
$$= (v_1 \cos\Phi + v_2 \sin\Phi) \wedge v_3 \wedge \left(\frac{db}{ds} v_1 - \frac{da}{ds} v_2 \right) \tag{250}$$
$$= \left(\cos\Phi \frac{da}{ds} + \frac{db}{ds} \sin\Phi \right) v_1 \wedge v_2 \wedge v_3. \tag{251}$$

Developing da, db on ω_1, ω_2 as above, we get

$$\tau_r(s) = \begin{bmatrix} \cos\Phi\, (L_{12} \cos\Phi + L_{22} \sin\Phi) \\ - \sin\Phi\, (L_{11} \cos\Phi + L_{12} \sin\Phi) \end{bmatrix} \tag{252}$$
$$= \left[L_{12} \cos 2\Phi + \frac{\sin 2\Phi}{2} (L_{22} - L_{11}) \right]. \tag{253}$$

The curvature lines are defined by $\tau_r(s) = 0$, hence

$$\tan 2\Phi = \frac{2L_{12}}{L_{11} - L_{22}} \tag{254}$$

as we have already obtained previously in Eq. (219).

Geodesics The geodesic curvature $\rho_g(s)$ is given by

$$I\rho_g(s) = \frac{d(OM)}{ds} \wedge \frac{d^2(OM)}{ds^2} \wedge v_3 \tag{255}$$

$$= \frac{1}{ds}\left(v_1 \cos\Phi + v_2 \sin\Phi\right) \tag{256}$$

$$\wedge \begin{bmatrix} (d\Phi + dc)\left(-v_1 \sin\Phi + v_2 \cos\Phi\right) \\ +v_3\left(\sin\Phi da - \cos\Phi db\right) \end{bmatrix} \wedge v_3 \tag{257}$$

$$= \left(\frac{dc}{ds} + \frac{d\Phi}{ds}\right) I. \tag{258}$$

The geodesic lines correspond to $\rho_g(s) = 0$ and thus to $dc + d\Phi = 0$ or equivalently

$$c_1 du + c_2 dv + d\left(Arc\tan\frac{\omega_2}{\omega_1}\right) = 0. \tag{259}$$

Hence, the equation of the geodesic is given by [15, II, p. 414]

$$c_1 du + c_2 dv + d\left(Arc\tan\frac{A_2 du + B_2 dv}{A_1 du + B_1 dv}\right) = 0 \tag{260}$$

where c_1, c_2 are expressed by Eqs. (188, 189).

8.7 Example

Consider as surface the sphere of radius r, $x(t) = (r\sin\theta\cos\varphi)Ii + (r\sin\theta\sin\varphi)Ij + (r\cos\theta)Ik$. One has $dx = \omega_1 v_1 + \omega_2 v_2$ with $\omega_1 = rd\theta, \omega_1 = r\sin\theta d\varphi$ $(A_1 = r, B_1 = 0, A_2 = 0, B_2 = r\sin\theta)$ and

$$v_1 = (\cos\theta\cos\varphi)\, Ii + (\cos\theta\sin\varphi)\, Ij - (\sin\theta)\, Ik \tag{261}$$

$$v_2 = -(\sin\varphi)\, Ii + (\cos\varphi)\, Ij \tag{262}$$

$$v_3 = (v_1 \wedge v_2)^* = (\sin\theta\cos\varphi)\, Ii + (\sin\theta\sin\varphi)\, Ij + (\cos\theta)\, Ik. \tag{263}$$

This frame $(v_i = re_i r_c)$ is obtained from the canonical basis e_i via a rotation r_1 of φ around e_3, followed by a rotation r_2 of θ around the axis $r_1 e_2 r_{1c}$, hence,

$$r = r_1 r_2 = e^{k\frac{\varphi}{2}} e^{j\frac{\theta}{2}} \tag{264}$$

$$= \cos\frac{\theta}{2}\cos\frac{\varphi}{2} - \left(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\right) i \tag{265}$$

$$+ \left(\sin\frac{\theta}{2}\cos\frac{\varphi}{2}\right) j + \left(\cos\frac{\theta}{2}\sin\frac{\varphi}{2}\right) k. \tag{266}$$

The affine connection bivector $d\iota = 2r_c dr$ is

$$\begin{aligned} d\iota &= -\left(d\varphi \sin\theta\right) i + \left(d\theta\right) j + \left(d\varphi \cos\theta\right) k && (267)\\ &= \left(da\right) i + \left(db\right) j + \left(dc\right) k, && (268)\end{aligned}$$

leading to (with $da = a_1 d\theta + a_2 d\varphi, etc.$)

$$\begin{aligned} &a_1 = 0, a_2 = \sin\theta, b_1 = 1, b_2 = 0 && (269)\\ &c_1 = 0, c_2 = \cos\theta. && (270)\end{aligned}$$

The Gaussian and mean curvature are respectively

$$\begin{aligned} K &= \frac{a_1 b_2 - a_2 b_1}{A_1 B_2 - A_2 B_1} = \frac{1}{r^2} && (271)\\ H &= \frac{-b_1 B_2 + b_2 A_2 + a_2 A_1 - a_1 B_1}{2\left(A_1 B_2 - A_2 B_1\right)} = -\frac{1}{r}. && (272)\end{aligned}$$

9 Conclusion

The paper has presented a biquaternion calculus, having an associative exterior product, and shown how differential geometry can be expressed within this new algebraic framework. The method presented here can be extended to other spaces such as a pseudo-Euclidean 4D space. It is hoped that this paper will further interest in these new algebraic tools and provide new perspectives for geometric modeling.

Acknowledgments. This work was conducted in the framework of LabEx Celya and PRIMES (Physics Radiobiology Medical Imaging and Simulation).

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A Representation of biquaternions by 4 × 4 real matrices

$$e_1 = Ii = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, e_2 = Ij = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e_3 = Ik = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, e_3e_2 = i = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$e_1e_3 = j = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, e_2e_3 = k = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$e_1e_2e_3 = I = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, 1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

B Representation of biquaternions by 2 × 2 complex Pauli matrices

(i': ordinary complex imaginary)

$$e_1 = Ii = \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, e_2 = Ij = \sigma_2 = \begin{bmatrix} 0 & -i' \\ i' & 0 \end{bmatrix}$$

$$e_3 = Ik = \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e_3e_2 = i = \begin{bmatrix} 0 & -i' \\ -i' & 0 \end{bmatrix}$$

$$e_1e_3 = j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, e_2e_3 = k = \begin{bmatrix} -i' & 0 \\ 0 & i' \end{bmatrix}$$

$$e_1e_2e_3 = I = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

C Work-sheet: biquaternions (Mathematica)

<<Quaternions\`

(*product of two biquaternions $a = a_1 + Ia_2, b = b_1 + Ib_2, a_i, b_i \in \mathbb{H}$; a double star ** means a quaternion product*)

$$CP[a\_, b\_] := \{(a[[1]] ** b[[1]]) - (a[[2]] ** b[[2]]),$$
$$(a[[2]] ** b[[1]]) + (a[[1]] ** b[[2]])\}$$

(*conjugate K*)

$$K[a\_] := \{Quaternion[a[[1,1]], -a[[1,2]], -a[[1,3]], -a[[1,4]]],$$
$$Quaternion[a[[2,1]], -a[[2,2]], -a[[2,3]], -a[[2,4]]]\}$$

(∗sum and difference∗)

$$csum[a\_, b\_] := \{a[[1]] + b[[1]], a[[2]] + b[[2]]\}$$
$$cdif[a\_, b\_] := \{a[[1]] - b[[1]], a[[2]] - b[[2]]\}$$

(∗multiplication by a scalar∗)

$$fclif[f\_, a\_] := \{f * a[[1]], f * a[[2]]\}$$

(∗products $\frac{1}{2}(ab+ba)$, $\frac{1}{2}(ab-ba)$∗)

$$int[a\_, b\_] := \{fclif[1/2, csum[CP[a,b], CP[b,a]]]\}$$
$$ext[a\_, b\_] := \{fclif[1/2, cdif[CP[a,b], CP[b,a]]]\}$$

(∗products $-\frac{1}{2}(ab+ba)$, $-\frac{1}{2}(ab-ba)$∗)

$$mint[a\_, b\_] := \{fclif[-1/2, csum[CP[a,b], CP[b,a]]]\}$$
$$mext[a\_, b\_] := \{fclif[-1/2, cdif[CP[a,b], CP[b,a]]]\}$$

(∗example: product of two biquaternions A and B, $w = AB$∗)

$$A = \{Quaternion[1,3,0,4], Quaternion[2,1,5,1]\}$$
$$B = \{Quaternion[1,7,8,1], Quaternion[2,1,0,1]\}$$
$$w = Simplify[CP[A,B]]$$

(∗result∗)

$$\{Quaternion[-26,-31,23,30], Quaternion[-51,19,28,-15]\}$$