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Variational approximation of size-mass energies for $k$-dimensional currents

A. Chambolle∗  L. Ferrari†  B. Merlet‡

In this paper we produce a Γ-convergence result for a class of energies $F^k_{\varepsilon,a}$ modeled on the Ambrosio-Tortorelli functional. For the choice $k=1$ we show that $F^1_{\varepsilon,a}$ Γ-converges to a branched transportation energy whose cost per unit length is a function $f_{a-1}$ depending on a parameter $a>0$ and on the codimension $n-1$. The limit cost $f_{a}(m)$ is bounded from below by $1+m$ so that the limit functional controls the mass and the length of the limit object. In the limit $a\downarrow 0$ we recover the Steiner energy.

We then generalize the approach to any dimension and codimension. The limit objects are now $k$-currents with prescribed boundary, the limit functional controls both their masses and sizes. In the limit $a\downarrow 0$, we recover the Plateau energy defined on $k$-currents, $k<n$. The energies $F^k_{\varepsilon,a}$ then can be used for the numerical treatment of the $k$-Plateau problem.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a convex, bounded open set. We consider vector measures $\sigma \in \mathcal{M}(\Omega, \mathbb{R}^n)$ of the form

$$\sigma = m \nu \mathcal{H}^1|_\Sigma,$$

where $\Sigma$ is a 1-dimensional rectifiable set oriented by a Borel measurable tangent map $\nu: \Sigma \to S^{n-1}$ and $m: \Sigma \to \mathbb{R}_+$ is a Borel measurable function representing the multiplicity. We write $\sigma = (m, \nu, \Sigma)$ for such measures. Given a cost function $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ we introduce the functional

$$F(\sigma) := \begin{cases} \int_{\Sigma} f(m) \, d\mathcal{H}^1 & \text{if } \sigma = (m, \nu, \Sigma), \\ +\infty & \text{otherwise in } \mathcal{M}(\Omega, \mathbb{R}^d). \end{cases} \quad (1.1)$$

Next, given $\mathcal{S} = \{x_1, \cdots, x_n\} \subset \Omega$ a finite set of points and $c_1, \cdots, c_n \in \mathbb{R}$ such that $\sum_{j=1}^n c_j = 0$, we consider the optimization problem $F(\sigma)$ for $\sigma \in \mathcal{M}(\Omega, \mathbb{R}^n)$ satisfying

$$\nabla \cdot \sigma = \sum_{j=1}^n c_j \delta_{x_j} \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \quad (1.2)$$

The setting is similar to the one from Beckman [22] and Xia [25]. We model transport nets connecting a given set of sources $\{x_j \in \mathcal{S} : c_j > 0\}$ to a given set of wells $\{x_j \in \mathcal{S} : c_j < 0\}$ via vector valued measures. For numerical reasons, we wish to approximate the measure $\sigma = (m, \nu, \Sigma)$ by a diffuse object (a smooth vector field). For this, we introduce below a family of corresponding “diffuse” functionals $F_{\varepsilon,a}$ that converge towards (1.1) in the sense of Γ-convergence [9, 10, 14]. This general idea has proved
to be effective in a variety of contexts such as fracture theory, optimal partitions problems and image segmentation [8, 15, 16, 21]. More recently this tool has been used to approximate energies depending on one dimensional sets, for instance in [20] the authors take advantage of a functional similar to the one from Modica and Mortola defined on vector valued measures to approach the branched transportation problem [5]. With similar techniques approximations of the Steiner minimal tree problem ([2], [17] and [21]) have been proposed in [8, 7].

In the present paper we first extend to any ambient dimension \( n \geq 2 \) the phase-field approximation for a branched transportation energy introduced in [5] for \( n = 2 \). In particular the approximate functionals \( F_{\epsilon,a} \) are modeled on the one from Ambrosio and Tortorelli [4]. We also extend the construction to any dimension and co-dimension. Indeed, for \( 1 \leq k \leq n-1 \) integer, we consider \( k \)-rectifiable currents \( \sigma = (\theta, \epsilon, \Sigma) \) where \( \Sigma \) is a countably \( k \)-rectifiable set with approximate tangent \( k \)-plane defined by a simple unit multi-vector \( \xi(x) = \xi_1(x) \wedge \cdots \wedge \xi_k(x) \) and \( m : \Sigma \to \mathbb{R}_+ \) is a Borel measurable function (the multiplicity). The functional (1.1) extends to \( k \)-currents \( \sigma \) as follows,

\[
F(\sigma) := \begin{cases} 
\int_{\Sigma} f(m(x)) \, d\mathcal{H}^k & \text{if } \sigma = (m, \xi, \Sigma), \\
+\infty & \text{otherwise.}
\end{cases}
\]

Let us define the approximate functionals and describe our main results in the case \( k = 1 \). For our phase field approximations we relax the condition on the vector measure \( \sigma \) replacing it by a vector field \( \sigma_\epsilon \in L^2(\Omega, \mathbb{R}^n) \). We then need to mollify condition (1.2). Let \( \rho : \mathbb{R}^n \to \mathbb{R}_+ \) be a classical radial mollifier such that \( \text{supp} \rho \subset B_1(0) \) and \( \int_{B_1(0)} \rho = 1 \). For \( \epsilon > 0 \), we set \( \rho_\epsilon = \epsilon^{-n} \rho(\cdot/\epsilon) \). We substitute for (1.2) the condition

\[
\nabla \cdot \sigma_\epsilon = \left( \sum_{j=1}^{n} c_j \delta_{x_j} \right) * \rho_\epsilon = \sum_{j=1}^{n} c_j \rho_\epsilon(\cdot - x_j) \quad \text{in } \mathcal{D}'(\mathbb{R}^n). 
\]

\[
(1.3)
\]

Remark 1. Notice that in (1.2) (1.3) the equality holds in \( \mathcal{D}'(\mathbb{R}^n) \) and not only in \( \mathcal{D}'(\Omega) \) so that there is no flux trough \( \partial \Omega \).

We also consider the functions \( u \in W^{1,p}(\Omega, [\eta, 1]) \) such that \( u \equiv 1 \) on \( \partial \Omega \) where \( \eta = \eta(\epsilon) \) satisfies

\[
\eta \equiv a \epsilon^n 
\]

for some \( a \in \mathbb{R}_+ \). We denote by \( X_\epsilon(\Omega) \) the set of pairs \((\sigma, u)\) satisfying the above hypotheses. This set is naturally embedded in \( \mathcal{M}(\Omega, \mathbb{R}^n) \times L^2(\Omega) \). For \((\sigma, u) \in \mathcal{M}(\Omega, \mathbb{R}^n) \times L^2(\Omega) \) we set

\[
F_{\epsilon,a}(\sigma, u; \Omega) := \begin{cases} 
\int_{\Omega} \left[ \epsilon^{p-1} |\nabla u|^p + \frac{(1-u)^2}{\epsilon^{n-1}} + \frac{|u|^2}{\epsilon} \right] \, dx & \text{if } (\sigma, u) \in X_\epsilon(\Omega), \\
+\infty & \text{in the other cases.}
\end{cases}
\]

\[
(1.4)
\]

Let \( X \) be the subset of \( \mathcal{M}(\Omega, \mathbb{R}^n) \times L^2(\Omega) \) consisting of those couples \((\sigma, u)\) such that \( u \equiv 1 \) and \( \sigma = (m, \nu, \Sigma) \) satisfies the constraint (1.2). Given any sequence \( \epsilon \equiv (\epsilon_i)_{i \in \mathbb{N}} \) of positive numbers such that \( \epsilon_i \downarrow 0 \), we show that the above family of functionals \( \Gamma \)-converges to

\[
F_a(\sigma, u; \Omega) := \begin{cases} 
\int_{\Sigma} f_a(m(x)) \, d\mathcal{H}^1(x) & \text{if } (\sigma, u) \in X \text{ and } \sigma = m \nu \mathcal{H}^1 \mathcal{L} \Sigma, \\
+\infty & \text{otherwise.}
\end{cases}
\]

\[
(1.5)
\]

The function \( f_a : \mathbb{R}_+ \to \mathbb{R}_+ \) (introduced and studied in the appendix) is the minimum value of some optimization problem depending on \( a \) and on the codimension \( n-1 \) (we note \( f_a^d \), with \( d = n-k \) in the general case \( 1 \leq k \leq n-1 \)). In particular we prove that \( f_a \) is lower semicontinuous, subadditive, increasing, \( f_a(0) = 0 \) and that there exists some \( c > 0 \) such that

\[
\frac{1}{c} \leq \frac{f_a(m)}{\sqrt{1 + a m^2}} \leq c \quad \text{for } m > 0.
\]
The Γ-convergence holds for the topology of the weak-∗ convergence for the sequence of measures \((σ_ε)\) and for the strong \(L^2\) convergence for the phase field \((u_ε)\). For a sequence \((σ_ε, u_ε)\) we write \((σ_ε, u_ε) \to (σ, u)\) if \(σ_ε \rightharpoonup σ\) and \(\|u_ε - u\|_{L^2} \to 0\). In the sequel we first establish that the sequence of functionals \((F_{ε,a})\) is coercive with respect this topology.

**Theorem 1.1.** Assume that \(a > 0\). For any sequence \((σ_ε, u_ε) \subset M(\overline{Ω}, \mathbb{R}^n) \times L^2(Ω)\) with \(ε \downarrow 0\), such that

\[
F_{ε,a}(σ_ε, u_ε; Ω) \leq F_0 < +∞,
\]

then there exists \(σ \in M(\overline{Ω}, \mathbb{R}^n)\) such that, up to a subsequence, \((σ_ε, u_ε) \to (σ, 1) \in X\).

Then we prove the Γ-liminf inequality

**Theorem 1.2.** Assume that \(a ≥ 0\). For any sequence \((σ_ε, u_ε) \in M(\overline{Ω}, \mathbb{R}^n) \times L^2(Ω)\) that converges to \((σ, u) \in M(\overline{Ω}, \mathbb{R}^n) \times L^2(Ω)\) as \(ε \downarrow 0\) it holds

\[
\liminf_{ε \downarrow 0} F_{ε,a}(σ_ε, u_ε; Ω) \geq F_a(σ, u; Ω).
\]

We also establish the corresponding Γ-limsup inequality

**Theorem 1.3.** Assume that \(a ≥ 0\). For any \((σ, u) \in M(\overline{Ω}, \mathbb{R}^n) \times L^2(Ω)\) there exists a sequence \((σ_ε, u_ε) \subset M(\overline{Ω}, \mathbb{R}^n) \times L^2(Ω)\) such that

\[
(σ_ε, u_ε) \rightharpoonup (σ, u) \quad \text{in} \quad M(\overline{Ω}, \mathbb{R}^n) \times L^2(Ω)
\]

and

\[
\limsup_{ε \downarrow 0} F_{ε,a}(σ_ε, u_ε; Ω) \leq F_a(σ, u; Ω).
\]

As already stated, we only considered the case \(k = 1\) in this introduction. Section 4 is devoted to the extension of Theorems 1.1, 1.2 and 1.3 in the case where the 1-currents (vector measures) are replaced with \(k\)-currents.

Notice that the coercivity of the family of functionals only holds in the case \(a > 0\). However, as \(a \downarrow 0\) we have the important phenomena:

\[
f_a \rightharpoonup c1_{(0, +∞)} \quad \text{pointwise},
\]

for some \(c > 0\). As a consequence (1.5) is an approximation of \(cH^1(Σ)\) for \(a > 0\) small and the minimization of (1.4) in \(X_ε(Ω)\) provides an approximation of the Steiner problem associated to the set of points \(S\), for a suitable choice of the weights in (1.2). In the case \(k > 1\), we obtain a variational approximation of the \(k\)-Plateau problem.

**Structure of the paper:** In Section 2 we introduce some notation and recall some useful facts about vector measures and currents, we also anticipate the optimization problem defining the cost function \(f^d_a\) and state some results which are proved in Appendix A. In Section 3 we establish Theorems 1.1, 1.2 and 1.3. In Section 4 we extend these results to the case \(1 \leq k \leq n - 1\). In Section 5 we discuss the limit \(a \downarrow 0\).

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2. Preliminaries and notation

The canonical orthonormal basis of $\mathbb{R}^n$ is denoted by the vectors $e_1, \ldots, e_n$. $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$ and given an integer value $k$ we denote with $\omega_k$ the measure of the unit ball in $\mathbb{R}^k$, i.e. $\mathcal{L}^k(B_k(0))$. For a point $x \in \mathbb{R}^n$ we note $x = (x_1; x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. For any Borel-measurable set $A \subset \mathbb{R}^n$ we denote with $1_A(x)$ the characteristic function of the set $A$

$$1_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Given a vector space $Y$ and its dual $Y'$ for $\omega \in Y$ and $\sigma \in Y'$ we write $\langle \omega, \sigma \rangle$ for the dual pairing.

2.1. Measures and vector measures

We denote with $\mathcal{M}(\Omega)$ the vector space of Radon measures in $\Omega$ and with $\mathcal{M}(\Omega, \mathbb{R}^n) = \mathcal{M}(\Omega)^n$ the vector space of vector valued measures. For a measure $\mu \in \mathcal{M}(\Omega)$ we denote with $|\mu|$ its total variation, in the vector case $\mu \in \mathcal{M}(\Omega, \mathbb{R}^n)$ we write $\mu = \nu |\mu|$ where $\nu$ is a $|\mu|$-measurable map into $\mathbb{S}^{n-1}$. We say that a measure is supported on a Borel set $E$ if $|\mu|(\Omega \setminus E) = 0$. For an integer $k < n$ we denote with $\mathcal{H}^k$ the $k$-dimensional Hausdorff measure as in [F]. Given a set $E \subset \Omega$, such that, $\mathcal{H}^k(E)$ is finite for some $k$ the restriction $\mathcal{H}^k \mathcal{L} E$ defines a Radon measure in the space $\mathcal{M}(\Omega)$. A set $E \subset \Omega$ is said to be countably $k$-rectifiable if up to a $\mathcal{H}^k$ negligible set $N$, $E \setminus N$ is contained in a countable union of $C^1$ $k$-dimensional manifolds.

2.2. Currents

We denote with $\mathcal{D}^k(\Omega)$ the vector space of compactly supported smooth $k$-differential forms. For a $k$-differential form $\omega$ its comass is defined as

$$\|\omega\| = \sup \{ \langle \omega, \xi \rangle : \xi \text{ is a unit, simple } k\text{-vector} \}$$

Let $\mathcal{D}_k(\Omega)$ be the dual to $\mathcal{D}^k(\Omega)$ i.e. the space of $k$-currents with its weak-$*$ topology. We denote with $\partial$ the boundary operator that operates by duality as follows

$$\langle \partial \sigma, \omega \rangle = \langle \sigma, d \omega \rangle \quad \text{for all } (k-1)\text{-differential forms } \omega.$$ 

The mass of a $k$-current $M(\sigma)$ is the supremum of $\langle \sigma, \omega \rangle$ among all $k$-differential forms with comass bounded by 1. For any $k$-current $\sigma$ such that both $\sigma$ and $\partial \sigma$ are of finite mass we say that $\sigma$ is a normal $k$-current and we write $\sigma \in N_k(\Omega)$. On the space $\mathcal{D}_k(\Omega)$ we can define the flat norm by

$$F(\sigma) = \inf \left\{ M(R) + M(S) : \sigma = R + \partial S \text{ where } S \in \mathcal{D}_{k+1}^\flat(\Omega) \text{ and } R \in \mathcal{D}_k(\Omega) \right\},$$

which metrizes the weak-$*$ topology on currents on compact subsets of $N_k(\Omega)$. By the Radon-Nikodym theorem we can identify a $k$-current $\sigma$ with finite mass with the vector valued measure $\nu|\mu_\sigma$ where $\mu_\sigma$ is a finite positive valued measure and $\nu$ is a $\mu_\sigma$-measurable map in the set of unitary $k$-vectors for the mass norm. In particular the action of $\sigma$ on $\omega$ can be written as

$$\langle \sigma, \omega \rangle = \int_{\Omega} \langle \omega, \nu \rangle \, d\mu_\sigma.$$ 

For a finite mass $k$-current the mass of $\sigma$ coincides with the total variation of the measure $\mu_\sigma$. A $k$-current $\sigma$ is said to be $k$-rectifiable if we can associate to it a triplet $(\theta, \nu, \Sigma)$ such that

$$\langle \sigma, \omega \rangle = \int_{\Sigma} \theta(\omega, \nu) \, d\mathcal{H}^k$$

where $\Sigma$ is a countably $k$-rectifiable subset of $\Omega$, $\nu$ at $\mathcal{H}^k$ a.e. point is a unit simple $k$-vector that spans the tangent plane to $\Sigma$ and $\theta$ is an $L^1(\Omega, \mathcal{H}^k \mathcal{L} \Sigma)$ function that can be assumed positive. We will denote with $R_k(\Omega)$ the space of these $k$-rectifiable currents. Among these we name out the subset $P_k(\Omega)$ of $k$-rectifiable currents for which $\Sigma$ is a finite union of polyhedra, these will be called polyhedral chains. Finally the flat chains $F_k(\Omega)$ consist of the closure of $P_k(\Omega)$ in the weak-$*$ topology. By the scheme of Federer [F] 4.1.24 it holds

$$P_k(\Omega) \subset N_k(\Omega) \subset F_k(\Omega).$$
Remark 2 (1-Currents and Vector Measures). Since the vector spaces $\Lambda^1 \mathbb{R}^n$, $\Lambda^1 \mathbb{R}^n$ identify with $\mathbb{R}^n$, any vector measure $\sigma \in \mathcal{M}(\Omega, \mathbb{R}^n)$ with finite mass identifies with a 1-current with finite mass and viceversa. The divergence operator acting on measures is defined by duality as the boundary operator for currents. In the following $\sigma \in \mathcal{M}(\Omega, \mathbb{R}^n)$ is called a rectifiable vector measure if it is 1-rectifiable as 1-current. In the same fashion we define polyhedral 1-measures.

2.3. Functionals defined on flat chains

For $f : \mathbb{R} \mapsto \mathbb{R}^+$ an even function we define a functional

$$P_k(\Omega) \mapsto \mathbb{R}_+,$$

$$P = \sum_j (m_j, \nu_j, \Sigma_j) \mapsto \mathcal{F}(P) = \sum_j f(m_j)\mathcal{H}^k(\Sigma_j),$$

on the space of polyhedral currents. Under the assumption that $f$ is lower semi-continuous and subadditive, $\mathcal{F}$ can be extended to a lower semi-continuous functional by relaxation

$$F_k(\Omega) \mapsto \mathbb{R}_+,$$

$$P \mapsto \mathcal{F}(P) = \inf \left\{ \liminf_{P_j \to P} \mathcal{F}(P_j) : (P_j) \subset P_k(\Omega) \text{ and } P_j \to P \right\}.$$

as shown in [23 Section 6]. Furthermore, in [12] the authors show that if $f(t)/t \to \infty$ as $t \to 0$, then $\mathcal{F}(\sigma) < \infty$ if and only if $\sigma$ is rectifiable and for any such $\sigma$ the functional takes the explicit form

$$\mathcal{F}(\sigma) = \int_{\Sigma} f(m(x)) \, d\mathcal{H}^k(x) \quad \text{if } \sigma = (m, \nu, \Sigma). \quad (2.1)$$

To conclude this subsection let us recall a sufficient condition for a flat chain to be rectifiable, proved by White in [24 Corollary 6.1].

Theorem 2.1. Let $\sigma \in F_k(\Omega)$. If $M(\sigma) + M(\partial \sigma) < \infty$ and if there exists a set $\Sigma \subset \Omega$ with finite $k$-dimensional Hausdorff measure such that $\sigma = \alpha \ll \Sigma$ then $\sigma \in R_k(\Omega)$ i.e., $\sigma$ writes as $(m, \nu, \Sigma)$.

In the context of vector measures the theorem writes as

Theorem 2.2. Let $\sigma \in \mathcal{M}(\Omega, \mathbb{R}^n)$. If $|\sigma|(\Omega) + |\nabla \cdot s|(\Omega) < \infty$, $\nabla \cdot \sigma$ is at most a countable sum of Dirac masses and there exists $\Sigma$ with $\mathcal{H}^k(\Sigma) < \infty$ and $\sigma = \alpha \ll \Sigma$ then $\sigma$ is a rectifiable vector measure in the sense expressed in Subsection 2.2.

2.4. Reduced problem results in dimension $n-k$

This subsection is devoted at introducing some notation and results corresponding to the case $k = 0$. In the sequel, these results are used to describe the energetical behaviour of the $(n-k)$-dimensional slices of the configuration $(\sigma_\varepsilon, u_\varepsilon)$. We postpone the proofs to Appendix A. We set $d = n - k$, $p > d$ and consider $\varepsilon$ to be a sequence such that $\varepsilon \downarrow 0$. Let $B_\varepsilon(0) \subset \mathbb{R}^d$ be the ball of radius $r$ centered in the origin, we consider the functional

$$E_{\varepsilon, a}(\vartheta, u; B_\varepsilon) := \int_{B_\varepsilon} \left[ \varepsilon^{p-d} |\nabla u|^p + \frac{(1 - u)^2}{\varepsilon^d} + \frac{u|\vartheta|^2}{\varepsilon} \right] \, dx$$

where $u \in W^{1,p}(B_\varepsilon)$ is constrained to satisfy the lower bound $u \geq a \varepsilon^d + 1 =: \eta$ and $\vartheta \in L^2(B_\varepsilon)$ is such that $\text{supp}(\vartheta) \subset B_\varepsilon$ with $0 < \bar{r} < r$, $\|\vartheta\|_1 = m$. This leads to define the set

$$Y_{\varepsilon, a}(m, r, \bar{r}) = \left\{ (\vartheta, u) \in L^2(B_\varepsilon) \times W^{1,p}(B_\varepsilon, [\eta, 1]) : \|\vartheta\|_1 = m \text{ and } \text{supp}(\vartheta) \subset B_\varepsilon \right\},$$

and the optimization problem

$$f_{\varepsilon, a}(m, r, \bar{r}) = \inf_{Y_{\varepsilon, a}(m, r, \bar{r})} E_{\varepsilon, a}(\vartheta, u; B_\varepsilon). \quad (2.2)$$
Let \( f^d_a : [0, +\infty) \rightarrow R_+ \) be defined as
\[
f^d_a(m) = \begin{cases} 
\min_{\tilde{r} > 0} \left\{ \frac{a m^2}{\omega_d \tilde{r}^d} + \omega_d \tilde{r}^d + (d - 1) \omega_d q^d_\infty(0, \tilde{r}) \right\}, & \text{for } m > 0, \\
0, & \text{for } m = 0,
\end{cases}
\]
(2.3)
with
\[
q^d_\infty(\xi, \tilde{r}) := \inf \left\{ \int_0^{+\infty} t^{d-1} \left[ |v'|^p + (1 - v)^2 \right] \, dt : v(\tilde{r}) = \xi \text{ and } \lim_{t \to +\infty} v(t) = 1 \right\},
\]
(2.4)
for \( \tilde{r} > 0, \xi \geq 0 \). We have the following results

**Proposition 2.1.** For any \( r > \tilde{r} > 0 \), it holds
\[
\liminf_{\varepsilon \downarrow 0} f^d_{\varepsilon,a}(m, r, \tilde{r}) \geq f^d_a(m).
\]
(2.5)
There exists a uniform constant \( \kappa := \kappa(d, p) \) such that
\[
f^d_a(m) \geq \kappa \quad \text{for every } m > 0.
\]
(2.6)

**Proposition 2.2.** For fixed \( m > 0 \) let \( r_* \) be the minimizing radius in the definition of \( f^d_a(m) \) (2.3). For any \( \delta > 0 \) and \( \varepsilon \) small enough there exist a function \( \vartheta_\varepsilon = c I_{B_{r_*}} \) with \( c > 0 \) such that \( \int_{B_{r_*}} \vartheta_\varepsilon = m \) and a nondecreasing radial function \( u_\varepsilon : B_r \rightarrow [\eta, 1] \) such that \( u_\varepsilon(0) = \eta, u_\varepsilon = 1 \) on \( \partial B_r \) and
\[
E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon; B_r) \leq f^d_a(m) + \delta.
\]
(2.7)

**Proposition 3.2.** The function \( f^d_a \) is continuous in \((0, +\infty), \) increasing, sub-additive and \( f^d_a(0) = 0. \)

### 3. The 1-dimensional problem

#### 3.1. Compactness

We prove the compactness Theorem \([1, 1]\) for the family of functionals \((F_{\varepsilon,a})_\varepsilon.\) Let us consider a family of functions \((\sigma_\varepsilon, u_\varepsilon)_{\varepsilon \downarrow 0},\) such that \((\sigma_\varepsilon, u_\varepsilon) \in X_\varepsilon(\Omega)\) and
\[
F_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; \Omega) \leq F_0.
\]
As a first step we prove:

**Lemma 3.1.** Assume \( a > 0. \) There exists \( C \geq 0, \) only depending on \( \Omega, F_0 \) and \( a \) such that
\[
\int_\Omega |\sigma_\varepsilon| \leq C, \quad \forall \varepsilon.
\]
(3.1)
As a consequence there exist a positive Radon measure \( \mu \in (R^n, R_+) \) supported in \( \overline{\Omega} \) and a vectorial Radon measure \( \sigma \in \mathcal{M}(\overline{\Omega}, R^n) \) with \( \nabla \cdot \sigma = \sum a_j \delta_{x_j} \) and \( |\sigma| \leq \mu \) such that up to a subsequence
\[
u_\varepsilon \rightharpoonup 1 \text{ in } L^2(\Omega), \quad |\sigma_\varepsilon| \rightharpoonup \mu \text{ in } \mathcal{M}(R^n), \quad \sigma_\varepsilon \rightharpoonup \sigma \text{ in } \mathcal{M}(R^n, R^n).
\]
Proof. We divide the proof into three steps.

**Step 1.** We start by proving the uniform bound (3.1). Let \( \lambda \in (0, 1] \) and let
\[
\Omega_\lambda := \{ x \in \Omega : u_\varepsilon > \lambda \}.
\]
Being \( \sigma_\varepsilon \) square integrable we identify the measure \( \sigma_\varepsilon \) with its density with respect to \( \mathcal{L}^n. \) Therefore splitting the total variation of \( \sigma_\varepsilon, \) we write
\[
|\sigma_\varepsilon|(\Omega) = \int_\Omega |\sigma_\varepsilon| \, dx = \int_{\Omega}\lambda |\sigma_\varepsilon| \, dx + \int_{\Omega\setminus\Omega_\lambda} |\sigma_\varepsilon| \, dx.
\]
We estimate each term separately. By Cauchy-Schwarz inequality we have
\[
\int_{\Omega} |\sigma_\varepsilon| \leq \left( \int_{\Omega} \frac{u_\varepsilon |\sigma_\varepsilon|^2}{\varepsilon} \right)^{1/2} \left( \int_{\Omega} \frac{\varepsilon}{u_\varepsilon} \right)^{1/2}.
\]
Since \( \lambda < u_\varepsilon < 1 \) on \( \Omega_\lambda \) and \( \int_{\Omega_\lambda} (u_\varepsilon |\sigma_\varepsilon|^2)/\varepsilon \) \( dx \) being bounded by \( F_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon) \) from the previous we get
\[
\int_{\Omega_\lambda} |\sigma_\varepsilon| \leq \left( \int_{\Omega_\lambda} \frac{u_\varepsilon |\sigma_\varepsilon|^2}{\varepsilon} \right)^{1/2} \sqrt{\frac{\Omega_\varepsilon}{\lambda}} \leq \sqrt{\frac{\Omega_\varepsilon}{\lambda}} F_0.
\]
Next, in \( \Omega \setminus \Omega_\lambda \), by Young inequality, we have
\[
2 \int_{\Omega \setminus \Omega_\lambda} |\sigma_\varepsilon| \leq \int_{\Omega \setminus \Omega_\lambda} \frac{u_\varepsilon |\sigma_\varepsilon|^2}{\varepsilon} + \int_{\Omega \setminus \Omega_\lambda} \frac{\varepsilon}{u_\varepsilon}.
\]
Using \( u_\varepsilon \geq \eta(\varepsilon), \eta/\varepsilon^n = a \) and \((1 - \lambda)^2 \leq (1 - u_\varepsilon)^2 \) in \( \Omega \setminus \Omega_\lambda \), we obtain
\[
\int_{\Omega \setminus \Omega_\lambda} |\sigma_\varepsilon| \leq \frac{1}{2} \int_{\Omega} \frac{u_\varepsilon |\sigma_\varepsilon|^2}{\varepsilon} + \frac{\varepsilon^n}{2\eta (1 - \lambda)^2} \int_{\Omega} \frac{(1 - u_\varepsilon)^2}{\varepsilon^{n-1}} \leq \frac{F_0}{2} + \frac{F_0}{2 a (1 - \lambda)^2}.
\]
Hence
\[
|\sigma_\varepsilon|(\Omega) \leq \frac{F_0}{2} + \frac{F_0}{2 a (1 - \lambda)^2} + \sqrt{\frac{\Omega_\varepsilon}{\lambda} F_0}.
\]
As \( \varepsilon > 0 \), this yields \( |\sigma_\varepsilon|(\Omega) \leq C_* F_0 \).

**Step 2.** We easily see from \( \int_{\Omega} (1 - u_\varepsilon) \leq F_0 \varepsilon^{n-1} \) that \( u_\varepsilon \to 1 \) in \( L^2(\Omega) \) as \( \varepsilon \downarrow 0 \).

**Step 3.** The existence of the Radon measures \( \mu \) and \( \sigma \) such that, up to extraction, \( |\sigma_\varepsilon| \rightharpoonup \mu \) and \( \sigma_\varepsilon \rightharpoonup \sigma \) follows from \( \text{(3.1)} \). The properties on the support of \( \mu \), on the divergence of \( \sigma \) and the fact that \( |\sigma| \leq \mu \) follow from the respective properties of \( \sigma_\varepsilon \).

We have just showed that the limit \( \sigma \) of a family \( (\sigma_\varepsilon, u_\varepsilon) \), equibounded in energy is bounded in mass. In what follows, we assume \( a \geq 0 \) and that \( \sigma_\varepsilon \) is bounded in mass. We show that the limiting \( \sigma \) is rectifiable.

**Proposition 3.1.** Assume \( a \geq 0 \) and that the conclusions of Lemma \( \text{(3.3)} \) hold true. There exists a Borel subset \( \Sigma \) with finite length and a Borel measurable function \( \nu : \Sigma \to S^{n-1} \) such that \( \sigma = \nu|\sigma|_{\big| \Sigma} \). Moreover, we have the following estimate,
\[
\mathcal{H}^1(\Sigma) \leq C_* F_0,
\]
where the constant \( C_* \geq 0 \) only depends on \( d \) and \( p \).

This proposition together with Lemma \( \text{(3.4)} \) and Theorem \( \text{(2.2)} \) leads to

**Proposition 3.2.** \( \sigma \) is a 1-rectifiable vector measure and in particular \( \Sigma \) is a countably \( \mathcal{H}^1 \)-rectifiable set.

The latter ensures that the limit couple \( (\sigma, 1) \) belongs to \( X \) and concludes the proof of Theorem \( \text{(1.1)} \). We now establish Proposition \( \text{(3.1)} \).

**Sketch of the proof:** We first define \( \Sigma \). Then we show in Lemma \( \text{(3.3)} \) that for \( x \in \Sigma \), we have \( \liminf_{\varepsilon \downarrow 0} F_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; B(x, r_j)) \geq \kappa r_j \) for a sequence of radii \( r_j \downarrow 0 \) and \( \kappa > 0 \). The proof of the lemma is based on slicing and on the results of Appendix \( \text{A} \). The proposition then follows from an application of the Besicovitch covering theorem.

First we introduce the Borel set
\[
\tilde{\Sigma} := \left\{ x \in \Omega : \forall r > 0, |\sigma|(B_r(x)) > 0 \text{ and } \exists \nu = \nu(x) \in S^{n-1} \text{ such that } \nu = \lim_{r \downarrow 0} \frac{\sigma(B_r(x))}{|\sigma|(B_r(x))} \right\}.
\]
We observe that by Besicovitch derivation theorem,
\[ \sigma = \nu|\sigma|\downarrow \Sigma. \]
Next we fix \( \theta \in (0, 1/4^n) \) and define
\[ \Gamma := \left\{ x \in \Sigma : \exists \tau_0 > 0 \text{ such that } \frac{|\sigma|(B_{\tau/4}(x))}{|\sigma|(B_{\tau}(x))} \leq \theta \text{ for every } \tau \in (0, \tau_0) \right\}. \]
We show that this set is \(|\sigma|\)-negligible.

**Lemma 3.2.** We have \(|\sigma|(\Gamma) = 0.\)

**Proof.** Let \( x \in \Gamma. \) Applying the inequality \(|\sigma|(B_{\tau/4}(x)) \leq \theta|\sigma|(B_{\tau}(x))\) with \( \tau = r_k = 4^{-k}r_0, \ k \geq 0, \) we get \(|\sigma|(B_{r_k}) \leq \theta^k|\sigma|(B_{r_0}). \) Hence there exists \( C \geq 0 \) such that
\[ |\sigma|(B_{\tau}(x)) \leq Cr^{(\ln 1/\theta)/(\ln 4)}. \]
Noting, \( \lambda = (\ln \frac{1}{\theta})/(\ln 4) \), we have by assumption \( \lambda > n. \) Therefore, for every \( \xi > 0 \) there exists \( r_\xi = r_\xi(x) \in (0, 1) \) such that
\[ |\sigma|(B_{\xi}(x)) \leq \xi|\sigma|(B_{r_\xi}(x)). \]
Now, for \( R > 0, \) we cover \( \Gamma \cap B_R \) with balls of the form \( B_{r_\xi(x)}(x). \) Using Besicovitch covering theorem, we have
\[ \Gamma \cap B_R \subseteq \bigcup_{j=1}^{N(n)} B_j \]
where \( N(n) \) only depends on \( n \) and each \( B_j \) is a (finite or countable) disjoint union of balls of the form \( B_{r_\xi(x)}(x_k). \) Then we get
\[ |\sigma|(\Gamma \cap B_R) \leq \sum_{j=1}^{N(n)} |\sigma|(B_j) \leq N(n)|\sigma|(B_R) \leq N(n)|\sigma(R+1)|. \]
Sending \( \xi \) to 0 and then \( R \) to \( \infty, \) we obtain \(|\sigma|(\Gamma) = 0. \)

Set \( \Sigma := \tilde{\Sigma} \setminus \Gamma, \) from Lemma 3.2, we have \( \sigma = \nu|\sigma|\downarrow \Sigma. \) Recall that \( \mathcal{F} = \{x_1, \cdots, x_{n_\nu}\}. \)

**Lemma 3.3.** For every \( x \in \Sigma \setminus \mathcal{F}, \) there exists a sequence \( (r_j) = (r_j(x)) \subseteq (0, 1) \) with \( r_j \downarrow 0 \) such that
\[ \liminf_{r_j \downarrow 0} \mathcal{F}_{r_\xi, \nu}(\sigma(x), u_\omega; B(x, r_j)) \geq \frac{\sqrt{2}}{\kappa} r_j, \]
where \( \kappa \) is the constant of Proposition 2.1.

**Proof.** Let \( x \in \Sigma \setminus \mathcal{F}. \) Without loss of generality, we assume \( x = 0 \) and \( \nu(x) = e_1. \) Let \( \xi > 0 \) be a small parameter to be fixed later. From the definition of \( \Sigma, \) there exists a sequence \( (r_j) = (r_j(x)) \subseteq (0, d(x, \mathcal{F})) \) such that for every \( j \geq 0, \)
\[ |\sigma|(B_{r_j}) \cdot e_1 \geq (1 - \xi)|\sigma|(B_{r_j}) \quad \text{and} \quad |\sigma|(B_{r_j/4}) \geq \theta|\sigma|(B_{r_j}). \quad (3.2) \]
Let us fix \( j \geq 0 \) and set, to simplify the notation, \( r = r_j \) and \( r_* = r/\sqrt{2}. \) Recall the notation \( x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^n-1 \) and define the cylinder
\[ C_{r_*} := \{ x : |x_1| \leq r_* \quad \text{and} \quad |x'| \leq r_* \} \]
so that \( C_{r_*} \subset B_r \) and \( B_{r_j/4} \subset C_{r_*/2}, \) as shown in figure 1. Let \( \chi \in C_\infty^0(\mathbb{R}^{n-1}, [0, 1]) \) be a radial cut-off function such that \( \chi(x') = 1 \) if \( |x'| \leq \frac{1}{2} \) and \( \chi(x') = 0 \) for \( |x'| \geq \frac{3}{4}. \) Then, we note \( \chi_{r_*}(x') = \chi(x'/r_*). \) and for \( s \in [-r, r], \) we set
\[ g_\xi(s) := e_1 \cdot \int_{B_{r_*}} \sigma(x, x') \chi_{r_*}(x') \, dx'. \]
Since \( \sigma_\xi \) is divergence free, \( e_1 \cdot \sigma_\xi(\cdot, s) \) has a meaning on the hyperplane \( \{x_1 = s\} \) in the sense of trace,
moreover, \( g_\varepsilon \) is continuous. Now, let us fix \( \hat{r} \in [(1 - \xi)r_\ast, r_\ast] \) such that \( \mu([\hat{r}, \hat{r}]) = 0 \) (which holds true for a.e. \( \hat{r} \in [(1 - \xi)r_\ast, r_\ast] \)) and let us define the mean value,
\[
\bar{g}_\varepsilon := \frac{1}{2\hat{r}} \int_{-\hat{r}}^{\hat{r}} g_\varepsilon(s) \, ds.
\]

From \( \sigma_\varepsilon \xrightarrow{\varepsilon \to 0} \sigma, \ |\sigma_\varepsilon| \xrightarrow{\varepsilon \to 0} \mu \), we have
\[
\lim_{\varepsilon \downarrow 0} \bar{g}_\varepsilon = \left( \frac{1}{2\hat{r}} \int_{(-\hat{r},\hat{r})} \chi_{r, \ast}(x') \, d\sigma(s,x') \right) 
\cdot e_1 =: \overline{m}.
\]

From (3.2), we see that \( \overline{m} > 0 \) for \( \xi \) small enough. Indeed, we have
\[
(1 - \xi)|\sigma|([B_r \setminus B_{r/4}) \leq +2\hat{r}\overline{m} + \sigma(B_r) \cdot e_1 = \int_{B_r} \left( 1 - \chi_{r, \ast}(x')1_{[-\hat{r},\hat{r})} \right) \, d\sigma(s,x') \cdot e_1 \leq 2\hat{r}\overline{m} + \int_{B_r} \left( 1 - \chi_{r, \ast}(x')1_{[-\hat{r},\hat{r})} \right) \, d\sigma(s,x') \leq 2\hat{r}\overline{m} + |\sigma|(B_r) - \int_{B_r} \chi_{r, \ast}(x')1_{[-\hat{r},\hat{r})} \, d|\sigma|(s,x') \cdot e_1.
\]

Since by construction \( \chi_{r, \ast}(x')1_{[-\hat{r},\hat{r})} \geq 1_{B_{r/4}} \), using the second inequality of (3.2), we have
\[
\overline{m} \geq \frac{1}{2\hat{r}}(\theta - \xi)|\sigma|(B_r) > 0,
\]
for \( \xi \) small enough. Similarly, denoting \( \Pi : \mathbb{R}^n \to \mathbb{R}^{n-1}, \ (t,x') \mapsto x' \) the orthogonal projection onto the last \((n-1)\) coordinates, we deduce again from (3.2) that
\[
|\Pi \sigma|(C_{r_\ast}) \leq \frac{\sqrt{2\overline{m}}}{\theta - \xi} \hat{r}.
\] (3.3)

Now, for \( \varepsilon \) small enough, we have \( \nabla \cdot \sigma_\varepsilon = 0 \) in \( C_{r_\ast} \). Using this, we have for almost every \( s, t \in [-\hat{r}, \hat{r}] \), with \( s < t \),
\[
g_\varepsilon(t) - g_\varepsilon(s) = \int_s^t \left[ \int_{B_{r/4}} \sigma_\varepsilon(x', h) \cdot \nabla' \chi_{r, \ast}(x') \, dx' \right] \, dh.
\]

Integrating in \( s \) over \((-\hat{r}, \hat{r})\), we get for almost every \( t \in [-r, r] \),
\[
g_\varepsilon(t) - g_\varepsilon = \frac{1}{2\hat{r}} \int_{(-\hat{r}, \hat{r}) \times B_{r/4}} \phi_t(x', h) \cdot \sigma_\varepsilon(x', h) \, dx' \, dh
\]
with
\[
\phi_t(h, x') = \begin{cases} (h + \hat{r}) \nabla' \chi_{r, \ast}(x') & \text{if } h < t, \\ (h - \hat{r}) \nabla' \chi_{r, \ast}(x') & \text{if } h > t. \end{cases}
\]
We deduce the following convergence

\[ g_\varepsilon(t) - \overline{m} \xrightarrow{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{[-\tilde{r}, \tilde{r}] \times B_\varepsilon} \phi_t(h, x') \cdot d\sigma(h, x'). \]

in the \( L^1(-\tilde{r}, \tilde{r}) \) topology. Using (3.3), we see that the above right hand side is bounded by \( c \frac{\sqrt{\varepsilon}}{\theta - \xi} \overline{m} \).

Taking into account (3.3) and the continuity of \( g_\varepsilon \), we conclude that

\[ \liminf_{\varepsilon \downarrow 0} g_\varepsilon(t) \geq \left(1 - c \frac{\sqrt{\varepsilon}}{\theta - \xi}\right) \overline{m} \quad \text{for} \quad t \in [-\tilde{r}, \tilde{r}]. \]

Next, by decomposing the integral we have

\[ F_{\varepsilon, a}(\sigma_\varepsilon, u_\varepsilon; B_r) \geq \int_{-\tilde{r}}^{\tilde{r}} \int_{B_{\tilde{r}?'}} \left[ \varepsilon^{p-n+1} |\nabla u_\varepsilon|^p + \frac{(1 - u_\varepsilon)^2}{\varepsilon^{n-1}} + \frac{u_\varepsilon |\sigma_\varepsilon|^2}{\varepsilon} \right] dx' dt \]

\[ \geq \int_{-\tilde{r}}^{\tilde{r}} \int_{B_{\tilde{r}?'}} \left[ \varepsilon^{p-n+1} |\nabla u_\varepsilon|^p + \frac{(1 - u_\varepsilon)^2}{\varepsilon^{n-1}} + \frac{u_\varepsilon |\chi_{r_\varepsilon}(x')\sigma_\varepsilon|^2}{\varepsilon} \right] dx' dt. \]

Let us set

\[ \vartheta_\varepsilon^f(x') := |\chi_{r_\varepsilon}(x')\sigma_\varepsilon(t, x')|. \]

By construction \( \vartheta_\varepsilon^f \) has the properties:

- \( \vartheta_\varepsilon^f \in L^1(B_{\tilde{r}?'}, \underline{m}) \),
- \( \liminf_{\varepsilon \downarrow 0} \int_{B_{\tilde{r}?'}} \vartheta_\varepsilon^f(x') dx' \geq \liminf_{\varepsilon \downarrow 0} g_\varepsilon(t) \geq \left(1 - c \frac{\sqrt{\varepsilon}}{\theta - \xi}\right) \overline{m} = \overline{m} > 0 \),
- \( \text{supp}(\vartheta_\varepsilon^f) \subset B_{\tilde{r}'} \) with \( \tilde{r} := \frac{3}{4} r_\varepsilon < r_\varepsilon \).

By definition of the minimization problem introduced in Subsection 2.4, we have

\[ F_{\varepsilon, a}(\sigma_\varepsilon, u_\varepsilon; B_r) \geq \int_{-\tilde{r}}^{\tilde{r}} \left[ \inf_{(\vartheta, u) \in Y_{r_\varepsilon}(\tilde{m}, r)} E_{\varepsilon, a}(\vartheta, u; B_r) \right] dt = \int_{-\tilde{r}}^{\tilde{r}} f_\varepsilon^f(\tilde{m}) dt. \]

Taking the infimum limit, by Fatou's lemma and equation (2.6) of Proposition 2.1 we get

\[ \liminf_{\varepsilon \downarrow 0} F_{\varepsilon, a}(\sigma_\varepsilon, u_\varepsilon; B_r) \geq \int_{-\tilde{r}}^{\tilde{r}} \liminf_{\varepsilon \downarrow 0} f_\varepsilon^f(\tilde{m}) dt \geq 2 \tilde{r} \kappa. \]

The latter holds for almost every \( \tilde{r} \in [(1 - \xi) r_\varepsilon, r_\varepsilon] \) and eventually, since the \( r_\varepsilon = r/\sqrt{2} \), we conclude

\[ \liminf_{\varepsilon \downarrow 0} F_{\varepsilon, a}(\sigma_\varepsilon, u_\varepsilon; B_r) \geq \sqrt{2} \kappa r. \]

The proof of Proposition 3.1 is then obtained via the Besicovitch covering theorem [13].

### 3.2. \( \Gamma \)-liminf inequality

In this subsection we prove the \( \Gamma \) – lim inf inequality stated in Theorem 1.2.

**Proof of Theorem 1.2**. With no loss of generality we assume that \( \liminf_{\varepsilon \downarrow 0} F_{\varepsilon, a}(\sigma_\varepsilon, u_\varepsilon) < +\infty \) otherwise the inequality is trivial. For a Borel set \( A \subset \Omega \), we define

\[ H(A) := \liminf_{\varepsilon \downarrow 0} F_{\varepsilon, a}(\sigma_\varepsilon, u_\varepsilon; A), \]

so that \( H \) is a subadditive set function. By assumption, the limit measure \( \sigma \) is 1-rectifiable; we write \( \sigma = m \nu H^1, \Sigma \). Furthermore we can assume \( \sigma \) to be compactly supported in \( \Omega \). Consider a convex open set \( \Omega_0 \) such that \( \text{supp}(\nabla \cdot \sigma) \subset \mathcal{K} \subset \Omega_0 \subset \subset \Omega \) and let \( h := [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n \) be a smooth homotopy of the identity map on \( \mathbb{R}^n \) onto a contraction of \( \overline{\Omega} \) into \( \overline{\Omega}_0 \) such that \( h(t, \cdot) \) restricted to \( \Omega_0 \) is the
identity map, for any \( t \in [0, 1] \). Let \( \sigma_t = b(t, \cdot) \sigma \), indeed \( \lim_{t \downarrow 0} F(\sigma_t, 1) \geq F(\sigma, 1) \) as \( \sigma_t \to \sigma \). Further \( \nabla \cdot \sigma_t = \nabla \cdot \sigma \) since \( b(t, \cdot) \) is the identity on \( \mathcal{S} \). Now we claim that
\[
\liminf_{r \downarrow 0} \frac{H \left( B(x, r) \right)}{2r} \geq f_a(m(x)) \quad \text{for } \mathcal{H}^1\text{-almost every } x \in \Sigma.
\]
(3.4)
Let us fix \( \lambda \geq 1 \) and let us note \( f_{a, \lambda}(t) := \min(f_a(t), \lambda) \). We then introduce the Radon measure
\[
H'_\lambda(A) := \int_{\Sigma \cap A} f_{a, \lambda}(m) \, d\mathcal{H}^1.
\]
Now, let \( \delta \in (0, 1) \). Assuming that (3.4) holds true, there exists \( \Sigma' \subset \Sigma \) with \( \mathcal{H}^1(\Sigma \setminus \Sigma_0) = 0 \) such that for every \( x \in \Sigma_0 \), there exists \( r_0(x) > 0 \) with
\[
(1 + \delta)H \left( B(x, r) \right) \geq 2rf_{a, \lambda}(m(x)) \quad \text{for every } r \in (0, r_0(x)).
\]
By the Besicovitch differentiation Theorem, there exists \( \Sigma_1 \subset \Sigma \) with \( \mathcal{H}^1(\Sigma \setminus \Sigma_1) = 0 \) such that for every \( x \in \Sigma_1 \), there exists \( r_1(x) > 0 \) with
\[
(1 + \delta)2rf_{a, \lambda}(m(x)) \geq H'_\lambda \left( B(x, r) \right) \quad \text{for every } r \in (0, r_1(x)).
\]
We consider the family \( B \) of closed balls \( B(x, r) \) with \( x \in \Sigma_0 \cap \Sigma_1 \) and \( 0 < r < \min(r_0(x), r_1(x)) \) and we apply the Vitali-Besicovitch covering theorem [1 Theorem 2.19.] to the family \( B \) and to the Radon measure \( H'_\lambda \). We obtain a disjoint family of closed balls \( B' \subset B \) such that
\[
H'_\lambda(\Omega) = H'_\lambda(\Sigma) = \sum_{B(x, r) \in B'} H'_\lambda \left( B(x, r) \right) \leq (1 + \delta)^2 \sum_{B(x, r) \in B'} H \left( B(x, r) \right) \leq (1 + \delta)^2 H(\Omega).
\]
Sending \( \lambda \) to infinity and then \( \delta \) to 0, we get the lower bound \( H(\Omega) \geq \int_\Sigma f_a(m) \, d\mathcal{H}^1 \) which proves the theorem.

Let us now establish the claim (3.4). Since \( \sigma \) is a rectifiable measure, we have for \( \mathcal{H}^1\text{-almost every } x \in \Sigma,
\[
\frac{1}{2r} \int_0^r \varphi(x + ry) \, d|\sigma|(y) \xrightarrow{r \downarrow 0} m(x) \int_\mathbb{R} \varphi(t \nu(x)) \, dt \quad \text{for every } \varphi \in C_c(\mathbb{R}^n),
\]
and
\[
\frac{1}{2r} \int_{B(x, r) \cap \Sigma} |\nu(y) - \nu(x)| \, d|\sigma|(y) \xrightarrow{r \downarrow 0} 0.
\]
Let \( x \in \Sigma \setminus \mathcal{S} \) be such a point. Without loss of generality, we assume \( x = 0, \nu(0) = e_1 \) and \( \mathbf{m} := m(0) > 0 \). Let \( \delta \in (0, 1) \). Our goal is to establish a precise lower bound for \( \mathcal{F}_{x, \nu}(\sigma, u_x; C) \) where \( C \) is a cylinder of the form
\[
C_\nu^\delta := \{ x \in \mathbb{R}^n : |x_1| < \delta r, |x'| < r \}.
\]
For this we proceed as in the proof of Lemma 33 here, the rectifiability of \( \sigma \) simplifies the argument. Let \( \chi^\delta \in C_c^\infty(\mathbb{R}^{n-1}, [0, 1]) \) be a radial cut-off function with \( \chi^\delta(x') = 1 \) if \( |x'| \leq \delta/2 \), \( \chi^\delta(x') = 0 \) if \( |x'| \geq \delta \). For \( \varepsilon > 0 \) and \( r \in (0, \delta d(0, \partial \Omega)) \), we define for \( s \in (-r, r) \),
\[
g_{\varepsilon}^\delta (s) := e_1 \cdot \int_{\mathbb{R}^{n-1}} \sigma_\varepsilon(s, x') \chi^\delta(x'/r) \, dx'.
\]
We also introduce the mean value
\[
\overline{g}_{\varepsilon}^\delta := \frac{1}{2r} \int_{-r}^{r} g_{\varepsilon}^\delta(s) \, ds.
\]
From (3.5), we have for \( r > 0 \) small enough,
\[
\overline{g}_{\varepsilon}^\delta := \frac{1}{2r} \int_{-r}^{r} e_1 \cdot \int_{\mathbb{R}^{n-1}} \sigma_\varepsilon(s, x') \chi^\delta(x'/r) \, dx \, ds \geq (1 - \delta)\mathbf{m}.
\]
For such $r > 0$, we deduce from $\sigma_\varepsilon \rightharpoonup \sigma$ that for $\varepsilon > 0$ small enough
\[ g_\varepsilon^{\delta,r} := \frac{1}{2r} \int_{-r}^{r} g_\varepsilon^{\delta,r}(s) \, ds \geq (1 - 2\delta)\overline{m}. \] (3.7)

We study the variation of $g_\varepsilon^{\delta,r}(s)$. Using $\nabla \cdot \sigma_\varepsilon = 0$ in $C^p$, we compute as in the proof of Lemma 3.3.
\[ g_\varepsilon^{\delta,r}(t) - g_\varepsilon^{\delta,r} = \frac{1}{2r} \int_{(r-r,r) \times B_{2r}} \phi_t(x', h) \cdot \sigma_\varepsilon(x', h) \, dx' \, dh \]
with
\[ \phi_t(h, x') = \begin{cases} (h + \varepsilon) \nabla' \chi^\delta(x'/r) & \text{if } h < t, \\ (h - \varepsilon) \nabla' \chi^\delta(x'/r) & \text{if } h > t. \end{cases} \]

Using again the convergence $\sigma_\varepsilon \rightharpoonup \sigma$, we deduce
\[ g_\varepsilon^{\delta,r}(t) - g_\varepsilon^{\delta,r} \xrightarrow{\varepsilon \downarrow 0} \frac{1}{2r} \int_{(r-r,r) \times B_{2r}} \phi_t(x', h) \cdot \sigma(x', h) \, dx' \, dh, \]
in $L^1(-r,r)$. Now, since $c_1 \cdot \nabla' \chi^\delta \equiv 0$, we deduce from (3.6) that the right hand side goes to 0 as $r \downarrow 0$. Hence, for $r > 0$ small enough,
\[ \left| \frac{1}{2r} \int_{(r-r,r) \times B_{2r}} \phi_t(x', h) \cdot \sigma(x', h) \, dx' \, dh \right| \leq \delta \overline{m}. \]
Using (3.7), we conclude that for $r > 0$ small enough and then for $\varepsilon > 0$ small enough, we have
\[ g_\varepsilon^{\delta,r}(t) \geq (1 - 3\delta)\overline{m}, \quad \text{for a.e. } t \in (-r,r). \]

By definition of the codimension-0 problem, we conclude that
\[ \mathcal{F}_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; C^\delta_r) \geq 2r f_{a,\varepsilon}^{-1}((1 - 3\delta)\overline{m}). \]
Sending $\varepsilon \downarrow 0$, we obtain
\[ H(C^\delta_r) \geq 2r f_{a,\varepsilon}^{-1}((1 - 3\delta)\overline{m}). \]
We notice that $H(B_{\sqrt{1 + \delta^2}r}) \geq H(C^\delta_r)$. Dividing by $2\sqrt{1 + \delta^2}r$ and taking the liminf as $r \downarrow 0$, we get
\[ \liminf_{r \downarrow 0} \frac{H(B_{\sqrt{1 + \delta^2}r})}{2\sqrt{1 + \delta^2}r} \geq \frac{f_a((1 - 3\delta)\overline{m})}{\sqrt{1 + \delta^2}}. \]
Sending $\delta$ to 0, we get (3.4) by lower semi-continuity of $f_a$. \hfill \Box

3.3. $\Gamma$-limsup inequality

Proof of Theorem 1.3.

Let us suppose $\mathcal{F}(\sigma, u; \Omega) < +\infty$, so that in particular $u \equiv 1$. From Xia [26], we can assume $\sigma$ to be supported on a finite union of compact segments and to have constant multiplicity on each of them, namely polyhedral vector measures are dense in energy. We first construct a recovery sequence for a measure $\sigma$ concentrated on a segment with constant multiplicity. Then we show how to deal with the case of a polyhedral vector measures.

Step 1. ($\sigma$ concentrated on a segment.) Assume that $\sigma$ is supported on the segment $I = [0, L] \times \{0\}$ and writes as $m \cdot e_1 \mathcal{H}^1_{\perp I}$. Consider $m$ constant so that $\nabla \cdot \sigma = m(\delta_{(0,0)} - \delta_{(L,0)})$ and
\[ \mathcal{F}(\sigma, 1; \Omega) = f_a(m) \mathcal{H}^1(I) = L f_a(m). \]

For $\delta > 0$ fixed, we consider the profiles
\[ \varphi_{\varepsilon}(t) := \begin{cases} \eta, & \text{for } 0 \leq t \leq r \varepsilon, \\ v_h \left( \frac{t}{\varepsilon} \right), & \text{for } r \varepsilon \leq t \leq r, \\ 1 & \text{for } r \leq t, \end{cases} \]
and
\[ \vartheta_{\varepsilon} = \frac{m \chi_{B_{r \varepsilon}}(x')}{\omega_{n-1} (r \varepsilon)^{n-1}}. \]
with $r_*$ and $v_*\xi$, defined in Proposition 2.2, with $d = n - 1$. Assume $r_* \geq 1$ and let $d(x,I)$ be the distance function from the segment $I$ and introduce the sets

$$I_{r_*,\epsilon} := \{ x \in \Omega : d(x,I) \leq r_\epsilon \}, \quad \text{and} \quad I_r := \{ x \in \Omega : d(x,I) \leq r \}.$$ 

Set $u_\epsilon(x) = \overline{u}_\epsilon(d(x,I))$ and $\sigma_\epsilon^1 = mH^1(I \times r_\epsilon \rho_\epsilon$, where $\rho_\epsilon$ is the mollifier of equation 1.3. We first construct the vector measures

$$\sigma_\epsilon^1 = \sigma_\epsilon^1 e_1 \quad \text{and} \quad \sigma_\epsilon^2(x_1,x') = \vartheta_\epsilon(|x'|) e_1.$$ 

Alternatively, $\sigma_\epsilon^2 = \sigma \ast \tilde{\rho}_\epsilon$ for the choice $\tilde{\rho}_\epsilon(x_1,x') = \chi_{B_{r_\epsilon}}(x')/\omega_{n-1}(\epsilon r_\epsilon)^{n-1}$. Let us highlight some properties of $\sigma_\epsilon^1$ and $\sigma_\epsilon^2$. Both vector measures are radial in $x'$, with an abuse of notation we denote $\overline{\sigma}_\epsilon^1(x_1,s) = \overline{\sigma}_\epsilon^1(x_1,|x'|)$. Since, both $\sigma_\epsilon^1$ and $\sigma_\epsilon^2$ are obtained through convolution it holds $\text{supp}(\sigma_\epsilon^1) \cup \text{supp}(\sigma_\epsilon^2) \subset I_{r_*,\epsilon}$ and they are oriented by the vector $e_1$ therefore $|\sigma_\epsilon^1| = \overline{\sigma}_\epsilon^1$ and $|\sigma_\epsilon^2| = \vartheta_\epsilon$. Furthermore for any $x_1$, it holds

$$\int_{\{x_1 \times B_{r_\epsilon}^\circ\}} [\overline{\sigma}_\epsilon^1(x_1,x') - \vartheta_\epsilon(x')] \, dx' = 0 \quad (3.8)$$

We construct $\sigma_\epsilon$ by interpolating between $\sigma_\epsilon^1$ and $\sigma_\epsilon^2$. To this aim consider a cutoff function $\zeta_\epsilon : \mathbb{R} \to \mathbb{R}_+$ satisfying

$$\zeta_\epsilon(t) = 1 \quad \text{for } t \leq r_\epsilon \epsilon \text{ or } t \geq L - r_\epsilon \epsilon,$$

$$\zeta_\epsilon(t) = 0 \quad \text{for } 2r_\epsilon \epsilon \leq t \leq L - 2r_\epsilon \epsilon,$$

and set

$$\begin{cases} 
\sigma_\epsilon^3 \cdot e_1 = 0, \\
\sigma_\epsilon^3 \cdot e_i(x_1,x') = \zeta_\epsilon^i(x_1) - \frac{x_i}{|x'|^{n-1}} \int_0^{|x'|} s^{n-2} [\overline{\sigma}_\epsilon^1(x_1,s) - \vartheta_\epsilon(s)] \, ds, \quad \text{for } i = 2, \ldots, n.
\end{cases}$$

The integral corresponds to the difference of the fluxes of $\overline{\sigma}_\epsilon^1$ and $\overline{\sigma}_\epsilon^2$ through the $(n-1)$-dimensional disk $\{x_1\} \times B'$. For $\sigma_\epsilon^3$ we have the following

$$\nabla \cdot \sigma_\epsilon^3 = -\zeta_\epsilon^i(x_1) \sum_{i=2}^n \left[ \frac{1}{|x'|^{n-1}} - \frac{(n-1)x_i^2}{|x'|^{n+1}} \right] \int_0^{|x'|} s^{n-2} [\overline{\sigma}_\epsilon^1(x_1,s) - \vartheta_\epsilon(s)] \, ds$$

$$+ \frac{x_i^2}{|x'|^2} \left[ \overline{\sigma}_\epsilon^1(x_1,|x'|) - \vartheta_\epsilon(|x'|) \right] = -\zeta_\epsilon^i(x_1) \left[ \overline{\sigma}_\epsilon^1(x_1,|x'|) - \vartheta_\epsilon(|x'|) \right] \quad (3.9)$$

Let

$$\sigma_\epsilon = \zeta_\epsilon \sigma_\epsilon^1 + (1 - \zeta_\epsilon) \sigma_\epsilon^2 + \sigma_\epsilon^3.$$ 

In force of equation 3.9 and from construction of $\sigma_\epsilon^1$, $\sigma_\epsilon^2$ and $\zeta_\epsilon$ we have

$$\nabla \cdot \sigma_\epsilon = \nabla \cdot (\zeta_\epsilon \sigma_\epsilon^1) + \nabla \cdot ((1 - \zeta_\epsilon) \sigma_\epsilon^2) + \nabla \cdot \sigma_\epsilon^3$$

$$= \zeta_\epsilon \nabla \cdot \sigma_\epsilon^1 + (1 - \zeta_\epsilon) \nabla \cdot \sigma_\epsilon^2 + \nabla \cdot \sigma_\epsilon^3$$

$$= \zeta_\epsilon \nabla \cdot \sigma_\epsilon^1 = \nabla \cdot (\sigma \ast \rho_\epsilon).$$

In addition for any $(x_1,x')$ such that $|x'| \geq r_\epsilon \epsilon$ from (3.8) we derive

$$\sigma_\epsilon^3 \cdot e_i(x_1,x') = -\zeta_\epsilon^i(x_1) \int_0^{|x'|} \frac{x_i}{|x'|^{n-2}} s^{n-1} [\overline{\sigma}_\epsilon^1(x_1,s) - \vartheta_\epsilon(s)] \, ds = 0$$

which justifies $\text{supp}(\sigma_\epsilon) \subset I_{r_*,\epsilon}$. Let us now prove

$$\limsup_{\epsilon \to 0} \mathcal{F}_{\epsilon,a}(\sigma_\epsilon, u_\epsilon; \Omega) \leq L f_a(m) + C\delta.$$ 

We split $\Omega$ as the union of $\Omega \setminus I_r$, $G_{r,\epsilon} := I_r \setminus [2 \epsilon, L - 2 \epsilon] \times \mathbb{R}^{n-1}$ and $D_\epsilon$ and $D'_\epsilon$, as show in figure 2.
where \( D_\varepsilon = \{ x_1 \leq 2 r_* \varepsilon \} \cap I_{r_* \varepsilon} \) and \( D'_\varepsilon = \{ x_1 \geq L - 2 r_* \varepsilon \} \cap I_{r_* \varepsilon} \). On \( \Omega \setminus I_r \) we notice that \( \sigma_\varepsilon = 0 \) and \( u_\varepsilon = 1 \) therefore

\[
F_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; \Omega \setminus I_r) = 0.
\]

Observe that \( |D_\varepsilon| = |D'_\varepsilon| = C \varepsilon^n \), then we have the upper bound

\[
\int_{D_\varepsilon} |\sigma_\varepsilon|^2 \, dx \leq 2 \frac{m^2 \varepsilon^2}{\varepsilon^{n-2}} \left( \int_{B_1} \rho^2 \, dx + C \right).
\]

Taking into consideration this estimate we obtain

\[
F_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; D_\varepsilon) = F_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; D'_\varepsilon) \leq \frac{(1 - \eta)^2}{\varepsilon^{n-1}} \mathcal{X}^n(D_\varepsilon) + 2 \frac{m^2 \varepsilon^2}{\varepsilon^{n-2}} \eta.
\]  
(3.10)

Finally on \( C_{r, \varepsilon} \) both \( \sigma_\varepsilon \) and \( u_\varepsilon \) are independent of \( x_1 \) and are radial in \( x' \) then by Fubini’s theorem and Proposition 22 we get

\[
F_{\varepsilon,a}(\sigma_\varepsilon, u_\varepsilon; C_{r, \varepsilon}) = \int_{L-2 \varepsilon r_*}^{L-2 \varepsilon r_*} \int_{B_1} E_{\varepsilon,a}(\vartheta_{\varepsilon}, u_\varepsilon) \leq L (f_a(m) + C \delta).
\]

Adding all together gives the desired estimate. It remains to discuss the case \( r_* < 1 \). From the point of view of the construction of \( \sigma_\varepsilon \) we need to replace the functions \( \tilde{\zeta}_\varepsilon \) with

\[
\tilde{\zeta}_\varepsilon(t) = 1 \quad \text{for } t \leq \varepsilon \text{ or } t \geq L - \varepsilon,
\]

\[
\tilde{\zeta}_\varepsilon(t) = 0 \quad \text{for } 2 \varepsilon \leq t \leq L - 2 \varepsilon,
\]

and

\[
|\tilde{\zeta}_\varepsilon| \leq \frac{1}{\varepsilon}.
\]

This choice ensures that \( \sigma_\varepsilon \) has all the properties previously obtained with \( r_* \varepsilon \) replaced by \( \varepsilon \) accordingly. Define

\[
w_\varepsilon(t) := \begin{cases} 
\eta, & \text{for } t \leq \sqrt{3}\varepsilon \\
\frac{1 - \eta}{\sqrt{3}} (t - \sqrt{3}) + \eta, & \text{for } \sqrt{3}\varepsilon \leq t \leq r.
\end{cases}
\]

and set

\[
u_\varepsilon = \min\{\overline{\nu}_\varepsilon(d(x, I)), w_\varepsilon(|x|), w_\varepsilon(|x - (L; 0)|)\}.
\]

with these choices for \( u_\varepsilon \) and \( \sigma_\varepsilon \) the estimates follow analogously with small differences in the constants.

\textbf{Step 2. (Case of a generic \( \sigma \) in polyhedral form.)} Indeed, in force of the results quoted in Subsection 2.3 it is sufficient to show equation (1.3) for a polyhedral vector measure. Following the same notation introduced therein let

\[
\sigma = \sum_{j=1}^{N} m_j \mathcal{H}^1 \cap \Sigma_j \nu_j.
\]

With no loss of generality we can assume that the segments \( \Sigma_j \) intersect at most at their extremities. We consider measures \( \sigma \) satisfying constraint (1.2) so that if a point \( P \) belongs to \( \Sigma_{j_1}, \ldots, \Sigma_{j_p} \) it must satisfy of Kirchhoff law,

\[
\sum_{j_{i_p}} z_j m_j = \begin{cases} 
\epsilon_i, & \text{if } P \in \mathcal{J}.
0, & \text{otherwise.}
\end{cases}
\]

(3.11)
where \( z_j \) is +1 if \( P \) is the ending point of the segment \( \Sigma_j \) with respect to its orientation, and −1 if it is the starting point. Let \( \sigma_j^\varepsilon \) and \( u_j^\varepsilon \) be the sequences constructed above for each segment \( I_k \) and define
\[
\sigma^\varepsilon = \sum_{j=1}^N \sigma_j^\varepsilon \quad \text{and} \quad u^\varepsilon = \min_j \{ u_j^\varepsilon \}.
\]

Let \( P_j \) and \( Q_j \) be respectively the initial and final point of the segment \( \Sigma_j \) and recall that, by the construction made above, for each \( j \)
\[
\nabla \cdot \sigma_j^\varepsilon = m_j (\delta_{P_j} - \delta_{Q_j}) \ast \rho^\varepsilon
\]
then by linearity of the divergence operator, it holds
\[
\nabla \cdot \sigma^\varepsilon = \sum_{j=1}^N \nabla \cdot \sigma_j^\varepsilon = \sum_{j=1}^N m_j (\delta_{P_j} - \delta_{Q_j}) \ast \rho^\varepsilon
\]
and the latter satisfies constraint (1.3) in force of equation (3.11). To conclude let us prove that
\[
\limsup_{\varepsilon \downarrow 0} F_{\varepsilon,a}(\sigma^\varepsilon, u^\varepsilon; \Omega) \leq \sum_{j=1}^N f_a(m_j) H^1(\Sigma_j).
\]
(3.12)
Indeed the following inequality holds true
\[
F_{\varepsilon,a}(\sigma^\varepsilon, u^\varepsilon; \Omega) \leq \sum_{j=1}^N F_{\varepsilon,a}(\sigma_j^\varepsilon, u_j^\varepsilon; \Omega).
\]
Suppose
\[
supp(\sigma_j^{i_1}) \cap supp(\sigma_j^{i_2}) \cap \cdots \cap supp(\sigma_j^{i_p}) \neq \emptyset
\]
for some \( j_1, \ldots, j_p \) and \( \varepsilon \). Let \( r_j^{i_1}, \ldots, r_j^{i_p} \) be the radii introduced above for each of these measures, let \( r^* = \max\{ r_j^{i_1}, \ldots, r_j^{i_p}, 1 \} \), set \( m = \max\{ m_{j_1}, \ldots, m_{j_p} \} \) and consider \( D_{j_1}, \ldots, D_{j_p} \) as defined previously. Since
\[
\sum_{k=1}^{j_p} |\sigma_k^\varepsilon|^2 \leq C \sum_{k=1}^{j_p} |\sigma_k^\varepsilon|^2
\]
and \( u^\varepsilon \leq u_j^\varepsilon \) for any \( j \), we have the following inequality
\[
F_{\varepsilon,a}(\sigma^\varepsilon, u^\varepsilon; \Omega) \leq C \sum_{k=1}^{j_p} F_{\varepsilon,a}(\sigma_k^\varepsilon, u_k^\varepsilon; D_k)
\]
And by inequality (3.10) follows
\[
F_{\varepsilon,a}(\sigma^\varepsilon, u^\varepsilon; \Omega) \leq C \left( \frac{(1-\eta)^2}{\varepsilon^{n-1}} \sum_{k=1}^{j_p} L^n(D_k) + 2 m^2 r^2 \eta \varepsilon^{n-2} \right).
\]
Which vanishes as $\varepsilon \downarrow 0$. Let us remark that the intersection $\text{supp}(\sigma^j_2) \cap \text{supp}(\sigma^j_2) \cap \ldots \cap \text{supp} \sigma^j_p$ is non empty for any $\varepsilon$ only if the segments $\Sigma_{j_1}, \ldots, \Sigma_{j_p}$ have a common point. Since we are considering a polyhedral vector measure composed by $N$ segments the worst case scenario is that we have $2N$ intersections in which at most $N$ segments intersects. We conclude

$$F_{c,a}(\sigma, u; \Omega) \leq \sum_{j=1}^N F_{c,a}(\sigma^j_2, u^j_2; \Omega) + C(N) \left( \frac{(1-\eta)^2}{\varepsilon^{n-1}} \sum_{k=j_1}^p \mathcal{L}^n(D_k) + 2\frac{\eta}{\varepsilon^{n-2}} \right)$$

which, passing to the limit, yields inequality (3.12). \hfill \Box

### 4. The $k$-dimensional problem

#### 4.1. Setting

Let $\sigma_0 \in P_k(\Omega)$ a polyhedral $k$-current with finite mass and let $\mathcal{F} := \text{supp}(\partial \sigma_0)$ be compactly contained in $\Omega$. We want to minimize a functional of the type (2.1) where the set of candidates ranges among all currents $D_k(\Omega)$ such that

$$\partial \sigma = \partial \sigma_0 \quad \text{in} \ D^k(\mathbb{R}^n).$$

Let us introduce a parameter $\eta = \eta(\varepsilon)$ which satisfies

$$\eta(\varepsilon) = a\varepsilon^{n-k+1} \quad \text{for} \ a \in \mathbb{R}_+$$

and let $X_{c}(\Omega)$ be the set of couples $(\sigma, u)$ such that $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ and has trace 1 on $\partial \Omega$ and $\sigma$ is of finite mass with density absolutely continuous with respect to $\mathcal{L}^n$. In this case we identify the current $\sigma$ with its $L^1(\Omega, \Lambda_k(\mathbb{R}^n))$ density. Furthermore as in equation (1.3) given a convolution kernel $\rho_{\varepsilon}$ we impose the constraint

$$\partial \sigma_{\varepsilon} = (\partial \sigma_0) * \rho_{\varepsilon} \quad \text{in} \ D^k(\mathbb{R}^n).$$

For $(\sigma, u) \in D_k(\Omega) \times L^2(\Omega)$ let

$$F_{c,a}(\sigma, u; \Omega) := \int_{\Omega} \left[ \varepsilon^{p-n+k} |\nabla u_{\varepsilon}|^p + \frac{(1-u_{\varepsilon})^2}{\varepsilon^{n-k}} + \frac{u_{\varepsilon}|\sigma_{\varepsilon}|^2}{\varepsilon} \right] \, dx,
\quad \text{if} \ (\sigma, u) \in X_{c}(\Omega),$$

$$+\infty, \quad \text{otherwise.}$$

Let us denote with $X$ the set of couples $(\sigma, u)$ such that $\sigma$ is a $k$-rectifiable current satisfying (4.1) and $u \equiv 1$. In this section we show that for any sequence $\varepsilon \downarrow 0$ the $\Gamma$-limit of the family $(F_{c,a})_{\varepsilon \in \mathbb{R}_+}$ is the functional

$$F_{c,a}(\sigma, u; \Omega) = \begin{cases} \int_{\text{supp} \sigma} f_{u}^{n-k}(m(x)) \, d\mathcal{H}^k(x), & \text{if} \ (\sigma, u) \in X \\
+\infty, & \text{otherwise in } M(\Omega, \mathbb{R}^n) \times L^2(\Omega) \end{cases}$$

Where the function $f_u^{n-k} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the function obtained in Appendix A for the choice $d = n - k$ and is endowed with the same properties stated for $f$ in Section 3. In particular under the assumption $p > n - k$ we first prove a compactness theorem.

**Theorem 4.1.** Assume that $a > 0$. For any sequence $\varepsilon \downarrow 0$, $(\sigma_{\varepsilon}, u_{\varepsilon}) \in D_k(\Omega) \times L^2(\Omega)$ such that

$$F_{c,a}(\sigma_{\varepsilon}, u_{\varepsilon}; \Omega) \leq F_0 < +\infty$$

then $u_{\varepsilon} \rightarrow 1$ and there exists a rectifiable $k$-current $\sigma \in D_k(\Omega)$ such that, up to a subsequence, $\sigma_{\varepsilon} \rightharpoonup^* \sigma$ and $(\sigma, 1) \in X$.

Then we show the $\Gamma$-convergence result, namely

**Theorem 4.2.** Assume that $a \geq 0$.

1. For any $(\sigma, u) \in D_k(\Omega) \times L^2(\Omega)$ and any sequence $(\sigma_{\varepsilon}, u_{\varepsilon}) \in D_k(\Omega) \times L^2(\Omega)$ such that $(\sigma_{\varepsilon}, u_{\varepsilon}) \rightarrow (\sigma, u)$ it holds

$$\liminf_{\varepsilon \downarrow 0} F_{c,a}(\sigma_{\varepsilon}, u_{\varepsilon}; \Omega) \geq F_{c,a}(\sigma, u; \Omega).$$
2. For any couple \((\sigma, u) \in \mathcal{D}_k(\overline{\Omega}) \times L^2(\Omega)\) there exists a sequence \((\sigma_\varepsilon, u_\varepsilon) \in \mathcal{D}_k(\overline{\Omega}) \times L^2(\Omega)\) such that 
\[(\sigma_\varepsilon, u_\varepsilon) \to (\sigma, u)\] and 
\[\limsup_{\varepsilon \to 0} \mathcal{F}^k_{\varepsilon, a}(\sigma_\varepsilon, u_\varepsilon; \Omega) \leq \mathcal{F}^k_a(\sigma, u; \Omega).\]

\[4.2. \text{Compactness and } k\text{-rectifiability}\]

**Proof of Proposition 4.1.** By the same procedure of Lemma 3.1 we obtain

\[|\sigma_\varepsilon|(\Omega) \leq \frac{F_0}{2} + \frac{F_0}{2a(1 - \lambda)^2} + \sqrt{\frac{[\Omega]_F F_0}{\lambda}}\] (4.1)

and

\[\int_\Omega (1 - u_\varepsilon)^2 \leq \varepsilon^{n-k} F_0.\]

Therefore by the weak compactness of \(\mathcal{D}_k(\Omega)\) we obtain the existence of a limit \(k\)-current \(\sigma\) a limit measure \(\mu\) and a subsequence \(\varepsilon\) such that \(\sigma_\varepsilon \rightharpoonup \sigma, \ |\sigma_\varepsilon| \rightharpoonup \mu\). As in the 1-dimensional case it is still necessary to prove the rectifiability of the limit current. This is obtained by showing that the support of \(\sigma\) is of finite size.

**Step 1. (Preliminaries and good representative for \(v \in \Lambda_k(\mathbb{R}^n)\).)** Let us introduce the set

\[\mathcal{I} := \{I = (i_1, \ldots, i_k) : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}, \quad e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}\]

So that \(\Lambda_k(\mathbb{R}^n)\) is the Euclidean space with basis \(\{e_I\}_{I \in \mathcal{I}}\). Let \(v \in \Lambda_k(\mathbb{R}^n)\) and consider the problem

\[a_0 = \max\{a \in \mathbb{R} : v = af_1 \wedge \cdots \wedge f_k + t : (f_1, \ldots, f_n) \text{ orthonormal basis}, t \in (f_1 \wedge \cdots \wedge f_k)^\perp\}.\]

Notice that \(a_0 \geq 1/\sqrt{|\mathcal{I}|}\). Assume that the optimum for the preceding problem is obtained with 
\((f_1, \ldots, f_n) = (e_1, \ldots, e_n)\). We note

\[v = a_0 e_{I_0} + \sum_{I \in \mathcal{I}_1} a_I e_I + \sum_{I \in \mathcal{J}} a_I e_I\]

with

\[I_0 = e_1 \wedge \cdots \wedge e_k, \quad \mathcal{I}_1 := \{I = (i_1, \ldots, i_k) : I \in \mathcal{I} : 1 \leq i_1 < \cdots < i_{k-1} \leq k < i_k \leq n\}, \quad \mathcal{J} := \mathcal{I} \setminus (\mathcal{I}_1 \cap I_0).\]

We claim that \(a_I = 0\) for \(I \in \mathcal{I}_1\). Indeed, let \(I_1 = (e_1, \ldots, e_{l-1}, e_{l+1}, \ldots, e_k, e_h) \in \mathcal{I}_1\) and for \(\phi \in \mathbb{R}\), let \(e^\phi\) be orthonormal base defined as

\[e_i = e_i^\phi \quad \text{for } i \neq \{l, h\}, \quad e_l = \cos(\phi)e_i^\phi - \sin(\phi)e_h^\phi, \quad e_h = \sin(\phi)e_i^\phi + \cos(\phi)e_h^\phi.\]

In this basis

\[v = (a_0 \cos(\phi) + a_{I_1}(-1)^{k-l}\sin(\phi))e_h^\phi + t^\phi, \quad \text{with } w^\phi \in (e^\phi)^\perp.\]

By optimality of \((e_1, \ldots, e_n)\) we deduce \(a_{I_1} = 0\) which proves the claim. Hence we write

\[v = a_0 e_{I_0} + t, \quad \text{with } t \in \text{span}\{e_I : I \in \mathcal{J}\}.\] (4.2)

Now we let \(\theta \in (0, 1/4^n)\) and \(\Sigma\) be the set of points for which there exists a sequence \(r_j \downarrow 0\) such that

\[\sigma(B_{r_j}(x)) \to w(x) \in \text{SA}_k(\mathbb{R}^n) \quad \text{and} \quad \frac{|\sigma|(B_{r_j/4}(x))}{|\sigma|(B_{r_j}(x))} \geq \theta.\]

In particular \(w\) is a \(|\sigma|\)-measurable map and we have \(\sigma = w|\sigma|\ll \Sigma\).

**Step 2. (Flux of \(\sigma_\varepsilon\) through a small \((n-k)\)-disk.)** Consider a point \(x \in \Sigma \setminus \mathcal{J}\), with no loss of generality we assume \(x = 0\). Let \(v = w(0)\), up to a change of basis, by equation (4.2) we write

\[v = a_0 e_{I_0} + t, \quad \text{with } t \in \text{span}\{e_I : I \in \mathcal{J}\}.\]
Let \( j \) sufficiently small, such that \( B_{r_j} \cap \mathcal{S} = \emptyset \) and
\[
\sigma(B_{r_j}) \cdot v \geq (1 - \xi) |\sigma| (B_{r_j}). \tag{4.3}
\]
Set, to simplify notation, \( r_j = r \) and \( r_s = r/\sqrt{2} \). For \( x \in \mathbb{R}^n \) we write \((x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}\) for the usual decomposition and denote \( B_{r_j}', B_{r_s}'' \) the \( k \)-dimensional and the \((n-k)\)-dimensional ball respectively. Let \( \chi \in C^\infty (B_{r_s}'') \) be a radial cut-off function with \( \chi (x') = 1 \) for \(|x'| \leq 1/2\) and \( \chi (x'') = 0 \) for \(|x''| \geq 3/4\). Set \( \chi_{r_s} (x'') = \chi (x''/r_s) \), then since \( \sigma_e \) is a \( L^1 \) function for \( \varepsilon > 0 \) we can define
\[
g_\varepsilon (x') : = \int_{B_{r_s}'} \chi_{r_s} (x'') \langle \sigma_e, e_{I_0} \rangle \, dx'' = \int_{B_{r_s}''} \chi_{r_s} (x'') \sigma^{I_0}_e \, dx'' \tag{4.4}
\]
for any \( x' \in B_{r_j}' \). Let us compute \( \partial g_\varepsilon (x') \) for \( l \in \{ 1, \ldots, k \} \). Since \( \partial \sigma_e = 0 \) in \( B_r \), it holds \( \langle \sigma_e, d\omega \rangle = 0 \) for any smooth \((k-1)\)-differential form \( \omega \in D^{k-1} (B_r) \). Choosing \( \omega \) of the form
\[
\omega = \beta (x) \, dx_1 \wedge \ldots \wedge dx_{l-1} \wedge dx_{l+1} \wedge \cdots \wedge dx_k
\]
we obtain
\[
d\omega = (-1)^{l-1} \partial_l \beta (x) \, dx_1 \wedge \cdots \wedge dx_k + (-1)^{k-1} \sum_{h=k+1}^d \partial_h \beta (x) \, dx_1 \wedge \ldots \wedge dx_{l-1} \wedge dx_{l+1} \wedge \cdots \wedge dx_k \wedge dx_h.
\]
Denote \( \sigma^l_\varepsilon = \langle \sigma, e^l \rangle \), then imposing \( \langle \sigma_e, d\omega \rangle = 0 \) for every \( \beta \in C^\infty (B_r) \) in \( \ref{4.5} \) yields
\[
(-1)^{k-l} \partial_l \sigma^0_\varepsilon + \sum_{h \in \{ k+1, \ldots, d \}} \sum_{I = (1, \ldots, l-1, l+1, \ldots, k, h)} \partial_h \sigma^I_\varepsilon = 0.
\]
Hence,
\[
\partial_l g_\varepsilon (x') = \frac{(-1)^{k-l}}{r_s} \sum_{h \in \{ k+1, \ldots, d \}} \sum_{I = (1, \ldots, l-1, l+1, \ldots, k, h)} \partial_h \chi_{r_s} (x'') \sigma^I_\varepsilon \, dx''. \tag{4.6}
\]
Let us introduce the notation
\[
\sigma^I_\varepsilon := \sum_{l \in I_\varepsilon} \sigma^l_\varepsilon e_l,
\]
denoting with \( \nabla' \) the gradient with respect to \( x' \), equation \( \ref{4.6} \) rewrites as
\[
\nabla' g_\varepsilon (x') = \frac{1}{r_s} \int_{B_{r_s}''} Y \left( \frac{x}{r_s} \right) \sigma^I_\varepsilon \, dx''. \tag{4.7}
\]
Where \( Y \) is smooth and compactly supported in \( B_{r_s}'' \) and with values into the linear maps : \( \text{span} \{ e_{I_l} : I \in I_\varepsilon \} \to \mathbb{R}^k \). Let us prove that, for some \( \tilde{r} \), the functions \( g_\varepsilon \) converge in \( \text{BV} - \varepsilon \) to some \( g \). First for a.e. choice of \( \tilde{r} \in [(1 - \xi) r_s, r_s] \) it must hold \( \mu ( \partial B'_{r_s} \times B''_{r_s} ) = 0 \) so that
\[
g_\varepsilon (x') = \int_{B_{r_s}'} \chi_{r_s} (x'') \langle \sigma_e, e_{I_0} \rangle \, dx'' \xrightarrow{\varepsilon \downarrow 0} \int_{B_{r_s}''} \chi_{r_s} (x'') \, d\sigma =: g (x'). \tag{4.8}
\]
Secondly we define the mean value
\[
\overline{g} := \frac{1}{|B_{r_s}'|} \int_{B_{r_s}'} g (x') \, dx' = \frac{1}{|B_{r_s}'|} \int_{B_{r_s}'} \left[ \int_{B_{r_s}''} \chi_{r_s} (x'') \, d\sigma \right]^0 \, dx'.
\]
and taking advantage of \( \ref{4.3} \) and the definition of \( \Sigma \), we see that
\[
\overline{g} \geq \left( \frac{\theta}{\sqrt{2}} - \xi \right) \frac{|\sigma| (B_r)}{|B_{r_s}'|} > 0.
\]
On the other hand, denoting \( \Pi : \mathbb{R}^n \to \mathbb{R}^{n-k} \), \( x \mapsto x'' \), from (4.2), we have
\[
|\Pi \sigma|(B'_x \times B''_x) \leq \sqrt{3\xi} \left( \frac{\theta}{\sqrt{|D|}} - \xi \right) |B'_x| \mathcal{H}.
\]

Now from (4.7) - (4.8) and the latter we obtain
\[
\langle D'g, \phi \rangle = \frac{1}{r_\varepsilon} \int_{B'_x \times B''_x} \phi(x') Y \left( \frac{x''}{r_\varepsilon} \right) \, d\sigma \quad \text{and} \quad |D'g|(B'_x) \leq \frac{C |B'_x| \sqrt{\xi |\mathcal{H}|}}{r_\varepsilon}.
\]

Finally from Poincaré-Wirtinger inequality and the convergence \( g_\varepsilon \to g \) in \( L^1(B'_x) \) is easy to show that for any sufficiently small \( \varepsilon \) the sets
\[
A_\varepsilon = \left\{ x \in B' : g_\varepsilon(x) \geq \frac{\mathcal{H}}{8} \right\}
\]

are such that \( |A_\varepsilon| \geq |B'_x|/2 \).

**Step 3. (Conclusion.)** Set \( \partial \varepsilon(x', x'') = |\chi_{\varepsilon}(x'')\sigma_{\varepsilon}^0| \) and observe that for fixed \( x' \) by construction
\[
\int_{B_{r_\varepsilon}} \partial \varepsilon(x', x'') \, dx'' = g_\varepsilon(x').
\]

Therefore for any \( x' \in A_\varepsilon \) it holds \( \int_{B_{r_\varepsilon}} \partial \varepsilon(x', x'') \, dx'' \geq \mathcal{H}/8 \). Furthermore \( \text{supp}(\partial \varepsilon(x')) \subset B'_x \) with \( \tilde{r} := \frac{3}{2} r_\varepsilon < r_\varepsilon \). Now, by Fubini
\[
\mathcal{F}_{k,a}(\sigma_{\varepsilon}, u_\varepsilon; B_\varepsilon) \geq \int_{A_\varepsilon} \int_{B_{\tilde{r}}} \left[ \varepsilon^{p-n+k} |\nabla u_\varepsilon|^p + \frac{(1-u_\varepsilon)^2}{\varepsilon^{n-k}} + \frac{u_\varepsilon |\sigma_{\varepsilon}|^2}{\varepsilon} \right] \, dx'' \, dx'
\]
\[
\geq \int_{A_\varepsilon} \int_{B_{\tilde{r}}} \left[ \varepsilon^{p-n+k} |\nabla u_\varepsilon|^p + \frac{(1-u_\varepsilon)^2}{\varepsilon^{n-k}} + \frac{u_\varepsilon |\partial \varepsilon(x', x'')|^2}{\varepsilon} \right] \, dx'' \, dx'
\]

With the notation introduced in Subsection 2.4 and by definition of \( A_\varepsilon \)
\[
\mathcal{F}_{k,a}(\sigma_{\varepsilon}, u_\varepsilon; B_\varepsilon) \geq \int_{A_\varepsilon} \inf_{(\theta, u) \in \mathcal{Y}_{\varepsilon,a}(m, r)} \mathcal{E}_{k,a}(\theta, u) \, dx' = \int_{A_\varepsilon} f^t_{\varepsilon} (\mathcal{H}/8) \, dx' = f^t_{\varepsilon} (\mathcal{H}/8) |A_\varepsilon|.
\]

Taking the infimum limit, by Proposition 2.1 in particular equation (2.6) we get
\[
\lim_{\varepsilon \downarrow 0} \mathcal{F}_{k,a}(\sigma_{\varepsilon}, u_\varepsilon; B_\varepsilon) \geq \lim_{\varepsilon \downarrow 0} f^t_{\varepsilon} (\mathcal{H}/8) |A_\varepsilon| \geq \kappa \frac{|B'_x|}{2}.
\]

Recall that the latter stands for a.e. \( \tilde{r} \in [1 - \xi] r_\varepsilon, r_\varepsilon \) and \( r_\varepsilon = r/\sqrt{2} \) thus we may rewrite
\[
\lim_{\varepsilon \downarrow 0} \mathcal{F}_{k,a}(\sigma_{\varepsilon}, u_\varepsilon; B_\varepsilon) \geq \kappa \frac{\omega_k r^k}{24 k/2}.
\]

As in Lemma 3.3 we conclude applying Besicovitch theorem to obtain \( H^k(\Sigma) < +\infty \). Finally, thanks to the latter and equation (1.1), Theorem 2.1 applies and \( \sigma \) is a \( k \)-rectifiable current. \( \square \)

### 4.3. \( \Gamma \)-liminf inequality

**Proof of item 1) of Theorem 4.2.** With no loss of generality we assume that \( \liminf_{\varepsilon \downarrow 0} \mathcal{F}_{k,a}(\sigma_{\varepsilon}, u_\varepsilon) < +\infty \) otherwise the inequality is trivial. For a Borel set \( A \subset \Omega \), we define
\[
H^k(A) := \liminf_{\varepsilon \downarrow 0} \mathcal{F}_{k,a}(\sigma_{\varepsilon}, u_\varepsilon; A),
\]

so that \( H^k \) is a subadditive set function. By assumption, the limit current \( \sigma \) is \( k \)-rectifiable; we write \( \sigma = m \nu H^k \ll \Sigma \). We claim that
\[
\liminf_{r \downarrow 0} \frac{H^k(B(x, r))}{\omega_k r^k} \geq f^a_{n-k}(m(x)) \quad \text{for } H^k\text{-almost every } x \in \Sigma.
\]

(4.10)
Assuming the latter the proof is achieved as in Theorem \[1.2\] To establish the claim (4.10) we restrict our attention to a single point and we assume \(x = 0, m = m(0)\) and \(\nu(0) = e_1 \wedge \cdots \wedge e_k\) then for any \(\xi > 0\) there exists \(r_0 = r(\xi)\) such that

\[
(\sigma, e_1 \wedge \cdots \wedge e_k)(B_r) \geq (1 - \xi)|\sigma|(B_r) \quad \text{and} \quad (1 - \xi)m \leq \frac{|\sigma|(B_r)}{\omega_k r^k} \leq (1 + \xi)m, \quad \text{for } r \leq r_0. \quad (4.11)
\]

Let \(\delta\) be an infinitesimal quantity and set, for \(r < r_0\), \(\hat{r} = \sqrt{1 - \delta^2} r\) and \(\tilde{r} = \delta r\) and define the cylinder

\[C_{\delta,r}(e_1, \cdots, e_n) = \{ (x'; x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} : |x'| \leq \hat{r} \text{ and } |x''| \leq \tilde{r} \}.\]

Let \(\chi(x'')\) be the radial cutoff introduced in the previous proposition and set \(\chi_{\tilde{r}}(x'') = \chi(x''/\tilde{r})\), \(\sigma^0 = (\sigma_{\tilde{r}}, e_1 \wedge \cdots \wedge e_k)\) and for any \(x' \in B_{\tilde{r}}\) set

\[g_{\tilde{r}}(x') := \int_{B_{\tilde{r}}'} \chi_{\tilde{r}}(x'') d\sigma^0 = \int_{B_{\tilde{r}}'} \chi_{\tilde{r}}(x'') d\sigma^0,
\]
as in equation \[4.4\]. Up to a smaller choice for \(r_0\) we can assume \(B_r \cap \mathcal{I} = \emptyset\) therefore \(\partial_{\mathcal{I}} B_r = 0\), and from equations \[4.4\] - \[4.7\] it holds

\[
\nabla^* g_{\tilde{r}}(x') = \frac{1}{\tilde{r}} \int_{B_{\tilde{r}}'} Y \left( \frac{x}{\tilde{r}} \right) d\sigma_{\tilde{r}}.
\]

For a.e. choice of \(\delta\) it holds \(|\sigma(\partial B_{\tilde{r}}' \times B_{\tilde{r}}'') = 0\) therefore, for any such choice, \(\gamma_{\tilde{r}}\) converges in \(BV(B_{\tilde{r}})\) to

\[g(x) := \int_{B_{\tilde{r}}'} \chi_{\tilde{r}}(x'') d\sigma^0 \quad \text{and} \quad (D'g, \phi) = \frac{1}{\tilde{r}} \int_{B_{\tilde{r}}' \times B_{\tilde{r}}'} \phi(x') Y \left( \frac{x''}{\tilde{r}} \right) d\sigma_{\tilde{r}}.
\]

Now we use \[4.11\] to improve the estimates on \(g\) and \(|D'g|\). Indeed, for \(\delta\) sufficiently small, \(\tilde{r} < \hat{r}/2\) therefore \(B_{\tilde{r}} \subset B_{\hat{r}}' \times B_{\hat{r}}''\) and

\[
\lim_{\varepsilon \to 0} \mathbb{I}_\varepsilon \geq (1 - \xi) \frac{1}{|B_{\hat{r}}'|} \int_{B_{\hat{r}}' \times B_{\hat{r}}''} \chi_{\hat{r}}(x') d|\sigma| \geq (1 - \xi)^2 m.
\]

and denoting \(\Pi : \mathbb{R}^n \to \mathbb{R}^{n-k}, x \mapsto x''\) we have

\[|\Pi| |\partial B_r| \leq (1 + \xi) \sqrt{k} m \quad \text{and} \quad |D'g|((B_{\hat{r}}' \times B_{\hat{r}}'')) \leq C |B_{\hat{r}}'| \sqrt{\xi} m \overline{\tilde{r}}.
\]

Choose \(r\) sufficiently small then by Poincaré - Wirtinger inequality there exists a set \(A\) of almost full measure in \(B_{\tilde{r}}\) such that \(g_{\tilde{r}}(a') \geq (1 - \xi)^2 m\), and following the proof of the previous lemma (Step 3) up to equation \[4.9\] we get

\[
\liminf_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,a}^k(\sigma_{\varepsilon}, u_{\varepsilon}; B_r) \geq \liminf_{\varepsilon \to 0} f_{\varepsilon,a}^{n-k} ((1 - \xi)^2 m, r, \tilde{r}) |A|.
\]

Since \(\xi\) and \(\delta\) are arbitrary and \(|A|\) can be chosen arbitrary close to \(|B_{\tilde{r}}|\) applying Proposition \[2.1\] with \(d = n - k\) to the latter we conclude

\[
\liminf_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,a}^k(\sigma_{\varepsilon}, u_{\varepsilon}; B_r) \geq f_a^{n-k} (m) \omega_k r^k.
\]

\[
\square
\]

4.4. \(\Gamma\)-limsup inequality

For the lim-sup inequality, we start by approximating \(\sigma\) with a polyhedral current: given \(\delta > 0\), there exists a \(k\) polyhedral current \(\tilde{\sigma}\) satisfying \(\partial \tilde{\sigma} = \partial \sigma_0\) and with \(F(\tilde{\sigma} - \sigma) < \delta\) and \(\mathcal{F}_a(\tilde{\sigma}) < \mathcal{F}_a(\sigma) + \varepsilon\). This result of independent interest is established in \[11\]. A similar result has been proved recently by Colombo et al. in \[12\] Prop. 2.6 (see also \[23\] Section 6)). The authors build an approximation of a \(k\)-rectifiable current in flat norm and in energy but their construction creates new boundaries and can not ensure the condition \(\partial \sigma = \partial \sigma_0\).
Proof of item 2) of Theorem 4.2

By [11] Theorem 1.1 and Remark 1.6 we can assume that $\sigma$ is a polyhedral current. We show how to produce the approximating $(\sigma_k, u_k)$ for $\sigma$ supported on a single $k$–dimensional simplex $Q$. We assume with no loss of generality that $Q \subset \mathbb{R}^k$, and that $\sigma$ writes as

$$m \mathcal{H}^k \llcorner Q \land (e_1 \land \cdots \land e_k).$$

For $\delta > 0$ fixed, we consider the optimal profiles

$$\bar{v}_{\varepsilon}(t) := \begin{cases} \eta, & \text{for } 0 \leq t \leq r_\varepsilon, \\ \nu_k \left( \frac{t}{\varepsilon} \right), & \text{for } r_\varepsilon \leq t \leq r, \\ 1, & \text{for } t \leq r, \end{cases} \quad \text{and} \quad \vartheta_{\varepsilon} = \frac{m \chi_{B_{r_\varepsilon}^{\varepsilon}}(\cdot)}{\omega_{n-1} (\varepsilon r_\varepsilon)^{n-k}},$$

with $r_\varepsilon$ and $v_3$, defined in Proposition 2.2, for the choice $d = n - k$. We denote $\partial Q$ the relative boundary of $Q$ and given a set $S$ we write $d(x, S)$ for the distance function from $S$. Recall that we use the notation $S_i$ for the $t$-enlargement of the set $S$ and $S'$ to denote its projection into $\mathbb{R}^k$. We first assume, as did for the case $k = 1$, $r_\varepsilon \geq 1$, and introduce $\zeta_{\varepsilon}$ a 0-form depending on the first $k$ variables $x'$, satisfying

$$\zeta_{\varepsilon}(x') = \begin{cases} 1, & \text{for } x' \in (\partial Q)'_{t_\varepsilon} := \{ x \in \Omega : d(x', \partial Q) \leq r_\varepsilon \} , \\
0, & \text{for } x' \in \Omega \setminus (\partial Q)'_{2r_\varepsilon}, \end{cases}$$

$$| \partial \zeta_{\varepsilon} | \leq \frac{1}{r_\varepsilon}. $$

Then we proceed by steps, first set $\overline{\pi}_{\varepsilon} := (|\sigma| * \rho_\varepsilon)$

$$\sigma_1^1 = \pi_{\varepsilon} e_1 \land \cdots \land e_k \quad \text{and} \quad \sigma_2^1(x', x'') = \vartheta_{\varepsilon}(|x''|) \land (e_1 \land \cdots \land e_k).$$

and observe that $\text{supp}(\sigma_1^1) \cup \text{supp}(\sigma_2^1) \subset Q_{r_\varepsilon}$, both $\sigma_1^1$ and $\sigma_2^1$ are radial in $x''$ and with a small abuse of notation we denote $\overline{\sigma}_{\varepsilon}^1(x', s) = \sigma_1^1(x', |x''|)$, finally for any $x'$

$$\int_{\{x'\} \times B_{r_\varepsilon}^{\varepsilon}} \left[ \overline{\sigma}_{\varepsilon}^1(x', |x''|) - \vartheta_{\varepsilon}(|x''|) \right] \omega_{n-1} \, dx'' = 0. $$

Now we take advantage of $\zeta_{\varepsilon}$ in order to interpolate between $\sigma_1^1$ and $\sigma_2^1$, note that such interpolation may affect the boundary of the new current therefore we first introduce $\sigma_3^1$ which corrects this defect. In particular set

$$\sigma_3^1(x', x'') = - \sum_{i=k+1}^{n} \frac{x_i}{|x''|^{n-k}} \int_{0}^{1} \left[ \overline{\sigma}_{\varepsilon}^1(x', s) \vartheta_{\varepsilon}(s) \right] \omega_{n-1} \, ds \land e_i, $$

and

$$\sigma_{\varepsilon} = \sigma_1^1 \llcorner \zeta_{\varepsilon} + \sigma_2^1 \llcorner (1 - \zeta_{\varepsilon}) + \sigma_3^1. $$

With this choice by a calculation similar to equation (3.9) it holds

$$\partial \sigma_{\varepsilon} = -\partial \sigma + \rho_\varepsilon \llcorner \zeta_{\varepsilon} - \sigma_1^1 \llcorner \omega_{n-1} \omega_{n-1} \llcorner (1 - \zeta_{\varepsilon}) + \sigma_2^1 \llcorner \omega_{n-1} \omega_{n-1} \llcorner \zeta_{\varepsilon} + \partial \sigma_3^1 = (\partial \sigma) + \rho_\varepsilon. $$

On the other hand the phase-field is simply defined as $u_{\varepsilon}(x) = \pi_{\varepsilon}(d(x, Q))$. In the case $r_\varepsilon < 1$ we need to modify the construction. For $\sigma_{\varepsilon}$ it is sufficient to replace every occurrence of $\zeta_{\varepsilon}$ with $\tilde{\zeta}_{\varepsilon}$, which satisfies

$$\tilde{\zeta}_{\varepsilon}(x') = \begin{cases} 1, & \text{for } x' \in (\partial Q)'_{t_\varepsilon} := \{ x \in \Omega : d(x', \partial Q) \leq \varepsilon \} , \\
0, & \text{for } x' \in \Omega \setminus (\partial Q)'_{2r_\varepsilon} \end{cases}$$

$$| \partial \zeta_{\varepsilon} | \leq \frac{1}{\varepsilon}. $$

Now let

$$w_{\varepsilon}(t) := \begin{cases} \eta, & \text{for } t \leq \sqrt{3} \varepsilon, \\
\frac{1 - \eta}{r - \sqrt{3}} (t - \sqrt{3}) + \eta, & \text{for } \sqrt{3} \varepsilon \leq t \leq r, \end{cases}$$

and set

$$u_{\varepsilon} = \min \{ \pi_{\varepsilon}(d(x, Q)), w_{\varepsilon}(d(x, \partial Q)) \}. $$
Remark 3. Given a polyhedral current $\sigma$ such that $\partial\sigma = \partial\sigma_0$ we perform our construction on each simplex and define $\sigma_\varepsilon$ as the sum of these elements. The linearity of the boundary operator grants that $\partial\sigma_\varepsilon = \partial\sigma_0 * \rho_\varepsilon$. The phase field is chosen as the pointwise minimum of the local phase fields. Finally the estimation for the $\Gamma$-limsup inequality is achieved in the same manner as Theorem 1.3.

5. Discussion about the results

By Lemma A.4 for any fixed $d = n - k$ the cost function $f^d_a$ pointwise converges as $a \downarrow 0$ to the function

$$f(m) = \begin{cases} \kappa, & \text{for } m > 0, \\ 0, & \text{if } m = 0, \end{cases}$$

where $\kappa$ is the constant value obtained in Proposition 2.1 and depends on $d$. This condition is sufficient to prove that the family of functionals $F^k_a$, parametrized in $a$, $\Gamma$-converges to the functional

$$F^k(\sigma; \Omega) := \begin{cases} \kappa H^k(\Sigma \cap \Omega), & \text{for } \sigma = m \nu H^k(\Sigma), \\ +\infty, & \text{otherwise.} \end{cases}$$

As a matter of fact for any sequence $\sigma_a \rightharpoonup \sigma$ in $D_k(\Omega)$ it holds

$$\liminf_{a \downarrow 0} F^k_a(\sigma; \Omega) \geq F^k(\sigma; \Omega)$$

since $f^d_a(m) \geq \kappa$. On the other hand setting $\sigma_a := \sigma$ we construct a recovery sequence for any $\sigma$ and obtain the $\Gamma$-limsup inequality

$$\limsup_{a \downarrow 0} F^k_a(\sigma_a; \Omega) = \limsup_{a \downarrow 0} F^k_a(\sigma; \Omega) = F^k(\sigma; \Omega).$$

This allows to interpret our result as an approximation of the Plateau problem in any dimension and co-dimension.

A. Reduced problem in dimension $n - k$

A.1. Auxiliary problem

In this appendix we show the results previously enunciated in Subsection 2.3 with the notation introduced therein let us define the auxiliary set

$$\mathcal{Y}_{\varepsilon,a}(m,r) = \{ (\vartheta,u) \in L^2(B_r) \times W^{1,p}(B_r,[\eta,1]) : \| \vartheta \|_1 = m \text{ and } u|_{\partial B_r} \equiv 1 \},$$

and the associated minimization problem

$$\mathcal{F}_{\varepsilon,a}(m,r) = \inf_{\mathcal{Y}_{\varepsilon,a}(m,r)} E_{\varepsilon,a}(\vartheta,u;B_r). \quad (A.1)$$

First we show that both $f^d_{\varepsilon,a}(m,r,\bar{r})$ and $\mathcal{F}_{\varepsilon,a}(m,r)$ are bounded by the same constant as $\varepsilon \downarrow 0$ and that the value of the second term is achieved by a radially symmetric couple of $\mathcal{Y}_{\varepsilon,a}(m,r)$. These two facts are then used to show that for each $m$ the limit values of $\mathcal{F}_{\varepsilon,a}(m,r)$ and $f^d_{\varepsilon,a}(m,r,\bar{r})$ as $\varepsilon \downarrow 0$ are equal and independent of the choices $(r,\bar{r})$ to the extent that $0 < \bar{r} < r$. Let us start by showing the first two properties.

**Lemma A.1.** For each $\varepsilon$, $m > 0$ and $r > 0$

a) there exists a constant $C = C(m) \leq C_0 \sqrt{1 + m^2}$ such that

$$f^d_{\varepsilon,a}(m,r,\bar{r}) < C \quad \text{and} \quad \mathcal{F}_{\varepsilon,a}(m,r) < C.$$
b) Both the problem defined in equation (2.2) and equation (A.1) admit a minimizer. Moreover among the minimizers of $E_{\epsilon,a}$ in $\text{Y}_{\epsilon,a}(m,r)$ it is possible to choose a radially symmetric couple $(\vartheta_{\epsilon}, u_{\epsilon})$ such that $u_{\epsilon}$ is radially non-decreasing and $\vartheta_{\epsilon}$ is radially non-increasing.

Proof. a) To show the bound it is sufficient to define

$$u_{\epsilon}(x) := \begin{cases} \eta & \text{if } |x| < r_{1}\epsilon, \\ \eta + \frac{1 - \eta}{(r_{2} - r_{1})\epsilon}(|x| - r_{1}\epsilon) & \text{if } r_{1}\epsilon \leq |x| < r_{2}\epsilon, \\ \frac{m}{|B_{r_{1}\epsilon}|} & \text{if } |x| < r_{1}\epsilon, \\ 0 & \text{if } r_{2}\epsilon \leq |x| < r, \end{cases}$$

$$\vartheta_{\epsilon}(x) := \begin{cases} \frac{m}{|B_{r_{1}\epsilon}|} & \text{if } |x| < r_{1}\epsilon, \\ 0 & \text{if } r_{1}\epsilon \leq |x| < r. \end{cases}$$

Evaluating the energy we get, for any choice of $r_{1} < r_{2} < r$,

$$E_{\epsilon,a}(u_{\epsilon}, \vartheta_{\epsilon}) \leq \frac{a m^{2}}{\omega_{d} r_{1}^{d}} + \omega_{d} \left( r_{1}^{d} + \frac{1}{(r_{2} - r_{1})^{d}} \left( \frac{r_{2}^{d} - r_{1}^{d}}{d} - \frac{r_{2}^{d+1} - r_{1}^{d+1}}{d+1} \right) \right).$$

As soon as $r_{1}\epsilon < \tilde{r}$, we have $(\vartheta_{\epsilon}, u_{\epsilon}) \in Y_{\epsilon,a}(m,r,\tilde{r}) \cap \overline{Y}_{\epsilon,a}(m,r)$. Choosing $r_{1} = (\sqrt{a}m)^{1/d}$ and $r_{2} = (1 + \sqrt{a}m)^{1/d}$, we get

$$\max\{f_{\epsilon,a}^{d}(m,r,\tilde{r}), \mathcal{J}_{\epsilon,a}^{d}(m,r)\} \leq C_{0} \sqrt{1 + m^{2}}.$$

b) To show the existence of minimizers for both minimization problems we use the direct method of the Calculus of Variation. The lower semicontinuity of the integral with integrand $|u|^{2}$ is ensured by Ioffe’s theorem [1, theorem 5.8]. Now given any minimizing couple $(\tilde{\vartheta}_{\epsilon}, \tilde{u}_{\epsilon}) \in Y_{\epsilon,a}(m,r)$, let $\tilde{\vartheta}_{\epsilon}$ be the decreasing Steiner rearrangement of $\tilde{\vartheta}_{\epsilon}$ and $\tilde{u}_{\epsilon}$ the increasing rearrangement of $\tilde{u}_{\epsilon}$. Indeed, since $\tilde{u}_{\epsilon}$ has range in $[\eta, 1]$, we still have $u_{\epsilon} |_{\partial B_{r}} \equiv 1$. Polya’s Szego and Hardy-Littlewood’s inequalities ensure

$$E_{\epsilon,a}(\tilde{\vartheta}_{\epsilon}, \tilde{u}_{\epsilon}) \leq E_{\epsilon,a}(\tilde{\vartheta}_{\epsilon}, \tilde{u}_{\epsilon}).$$

Let us prove the asymptotic equivalence of the values $f_{\epsilon,a}^{d}(m,r,\tilde{r})$ and $\mathcal{J}_{\epsilon,a}^{d}(m,r)$ as $\epsilon \downarrow 0$.

Lemma A.2 (Equivalence of the two problems). For any $\tilde{r} < r$ and $m > 0$ it holds

$$|f_{\epsilon,a}^{d}(m,r,\tilde{r}) - \mathcal{J}_{\epsilon,a}^{d}(m,r)| \xrightarrow{\epsilon \downarrow 0} 0.$$

Proof. Step 1: $|f_{\epsilon,a}^{d}(m,r,\tilde{r}) - \mathcal{J}_{\epsilon,a}^{d}(m,r)| + O(1)$

Consider for each $\epsilon$ the radially symmetric and monotone couple $(\vartheta_{\epsilon}, u_{\epsilon}) \in Y_{\epsilon,a}(m,r)$ as introduced in the previous lemma. Take $\xi \in (\eta, 1)$ and let us set

$$r_{\xi} := \sup\{t \in (0, r) : u_{\epsilon}(t) \leq \xi\} \quad \text{with } r_{\xi} = 0 \text{ if the set is empty. (A.2)}$$

By Cauchy-Schwarz inequality it holds

$$C \geq \int_{B_{r_{\xi}}} \frac{u_{\epsilon}|\vartheta_{\epsilon}|^{2}}{\epsilon} \, dx \geq \xi \left( \frac{\int_{B_{r_{\xi}}} |\vartheta_{\epsilon}| \, dx}{\omega_{d} r_{\xi}^{d}} \right)^{2}.$$ 

Let us define $\Delta_{\xi} := \int_{B_{r_{\xi}}} |\vartheta_{\epsilon}|$, the latter ensures that $\Delta_{\xi} \in o(\epsilon a^{\gamma/2})$. Let us now set $\hat{\vartheta}_{\epsilon} = \left( \frac{m \vartheta_{\epsilon}}{\mathcal{J}_{\epsilon,a}^{d}(m,r)} \right) 1_{B_{r_{\xi}}}$ which is not null for $\epsilon$ small. We have $(\hat{\vartheta}_{\epsilon}, u_{\epsilon}) \in Y_{\epsilon,a}(m,r,\tilde{r})$ if and only if $r_{\xi} \leq \tilde{r}$. Indeed, this holds as

$$C \geq \int_{B_{r_{\xi}}} \frac{(1 - u_{\epsilon})^{2}}{\epsilon d} \, dx \geq \omega_{d} (1 - \xi)^{2} \left( \frac{r_{\xi}}{\epsilon} \right)^{d},$$

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which ensures that \( r_\varepsilon = O(\varepsilon) \). Finally let us evaluate the energy

\[
E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon) = \int_{B_{\varepsilon}} \left[ \varepsilon^{p-d} |\nabla u_\varepsilon|^p + \frac{(1 - u_\varepsilon)^2}{\varepsilon^d} + \frac{u_\varepsilon |\vartheta_\varepsilon|^2}{\varepsilon} \right] \, dx
\]

\[
= \int_{B_{\varepsilon}} \left[ \varepsilon^{p-d} |\nabla u_\varepsilon|^p + \frac{(1 - u_\varepsilon)^2}{\varepsilon^d} \right] \, dx + \int_{B_{\varepsilon}^{\varepsilon}} \frac{u_\varepsilon m^2 |\vartheta_\varepsilon|^2}{\varepsilon^d} \, dx
\]

\[
\leq \frac{m^2}{\varepsilon} \omega_{d-1} \int_{B_{\varepsilon}^{\varepsilon}} E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon) = [1 + O(1)] E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon).
\]

Passing to the infimum we get

\[
f_{\varepsilon,a}^d(m, r, \tilde{r}) \leq \tilde{f}_{\varepsilon,a}^d(m, r) + O(1).
\]

**Step 2:** \( \int_{B_{\varepsilon}^{\varepsilon}} E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon) \leq f_{\varepsilon,a}^d(m, r, \tilde{r}) + O(1) \)

Consider a minimizing couple \((\vartheta_\varepsilon, u_\varepsilon)\) such that

\[
f_{\varepsilon,a}^d(m, r, \tilde{r}) = E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon).
\]

Let \( \chi \) be a smooth cutoff function such that \( \chi(x) = 1 \) if \(|x| \leq \tilde{r}\) and \( \chi(x) = 0 \) if \(|x| > \frac{\tilde{r} + \tilde{r}}{2}\) and set \( v_\varepsilon = \chi u_\varepsilon + (1 - \chi) \). By construction \((\vartheta_\varepsilon, v_\varepsilon) \in \mathcal{Y}_{\varepsilon}(m, r)\), furthermore, since \( u_\varepsilon \in [0, 1]\), it holds that \( u_\varepsilon \leq v_\varepsilon\) and \((1 - u_\varepsilon)^2 \geq (1 - v_\varepsilon)^2\). Moreover as \( v_\varepsilon \equiv u_\varepsilon \) on \( B_{\varepsilon}\) we have \( f_{B_{\varepsilon}} u_\varepsilon |\vartheta_\varepsilon|^2 \, dx = \int_{B_{\varepsilon}} v_\varepsilon |\vartheta_\varepsilon|^2 \, dx \).

Eventually, we estimate the gradient component of the energy as follows

\[
\int_{B_{\varepsilon}} \varepsilon^{p-d} |\nabla v_\varepsilon|^p \, dx = \int_{B_{\varepsilon}} \varepsilon^{p-d} |\chi \nabla u_\varepsilon + (u_\varepsilon - 1)\nabla \chi|^p \, dx
\]

\[
\leq \int_{B_{\varepsilon}} \varepsilon^{p-d} (|\nabla u_\varepsilon| + |\nabla \chi|)^p \, dx
\]

\[
\leq \int_{B_{\varepsilon}} \varepsilon^{p-d} |\nabla u_\varepsilon|^p \, dx + C(r, \varepsilon) \left( E_{\varepsilon,a}^{1-p}(\vartheta_\varepsilon, v_\varepsilon) \varepsilon^{\frac{p-d}{p-1}} + \varepsilon^{p-d} \right)
\]

where we have used the inequality \(|a + b|^p \leq |a|^p + C_p(|a|^{p-1}|b| + |b|^p)\) and Holder inequality. We get

\[
\tilde{f}_{\varepsilon,a}^d(m, r) \leq E_{\varepsilon,a}(\vartheta_\varepsilon, v_\varepsilon) \leq E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon) + O(\varepsilon^{\frac{p-d}{p-1}}) = f_{\varepsilon,a}^d(m, r) + O(1)
\]

**Step 3:** Combining inequalities \((A.3)\) and \((A.4)\) we obtain \( f_{\varepsilon,a}^d(m, r, \tilde{r}) - \tilde{f}_{\varepsilon,a}^d(m, r) = o(1)\).

**A.2. Study of the transition energy**

Given two values \( r_1 < r_2 \) let us introduce the functional

\[
G^d(v; (r_1, r_2)) := \int_{r_1}^{r_2} \varepsilon^{d-1} \left[ |v'|^p + (1 - v)^2 \right]
\]

and for any triplet \((\xi, r_1, r_2) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+\) we set

\[
q^d(\xi, (r_1, r_2)) := \inf \left\{ G^d(v; (r_1, r_2)) : v \in W^{1,p}(r_1, r_2), v(r_1) = \xi \text{ and } v(r_2) = 1 \right\}.
\]

This value represents the cost of the transition from \( \xi \) to 1 in the ring \( B_{r_2} \setminus B_{r_1} \). We will say that a function \( v \) is admissible for the triplet \((\xi, r_1, r_2)\) if it is a competitor in the above minimization problem.

Let us investigate the properties of the function introduced.

**Lemma A.3.** For any fixed triplet \((\xi, r_1, r_2) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+\) the infimum in equation \((A.5)\) is a minimum. Moreover there is a unique function achieving the minimum which is nondecreasing with range in the interval \([\xi, 1]\). Finally the function \( g \) satisfies the following properties

1. \( r_2 \mapsto q^d(\xi, r_1, r_2) \) is nonincreasing,
2. \( r_1 \mapsto q^d(\xi, r_1, r_2) \) is nondecreasing.

3. \( \xi \mapsto q^d(\xi, r_1, r_2) \) is nonincreasing, and \( g(1, r_1, r_2) = 0. \)

Recalling the definition (2.4) of \( q^d_{\infty} \), we have \( q^d_{\infty}(\xi, r) = q^d(\xi, r_1, \infty) \), and \( q^d_{\infty}(0, 0) > 0 \). Furthermore for any \( r > 0 \) the map \( \xi \mapsto q^d_{\infty}(\xi, r) \) is convex and continuous on \((0, +\infty)\).

**Proof.** Let \((\xi, r_1, r_2) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+\), the infimum is actually a minimum by means of the direct method of the calculus of variations. Such minimum is absolutely continuous on the interval \((0, 1)\) and \( g(1, r_1, r_2) \) is nonincreasing. Let \( \phi : (r_1, r_2) \mapsto (\xi, r_1, r_2) \) be a minimizer of \((A.5)\) set

\[
\tau = \min\{\max(v, \xi), 1\}
\]

then \( G^d(\tau; (r_1, r_2)) \leq G^d(v; (r_1, r_2)) \) if \( v \neq \tau \). As a consequence for every minimizer of \((A.5)\) we have \( \xi \leq v \leq 1 \). Similarly setting

\[
\tau(s) = \max\{v(t) : r_1 \leq t \leq s\}
\]

we have \( G^d(\tau; (r_1, r_2)) \leq G^d(v; (r_1, r_2)) \) if \( v \neq \tau \). Hence \( v \) is nondecreasing. Let us now study the

monotonicity of \( g \). To do so let \( v \) be the minimizer for \((\xi, r_1, r_2)\):

1. Let \( r_2 > r_1 \) and let us extend \( v \) by 1 on the interval \((r_2, r_2)\). We have

\[
q^d(\xi, r_1, r_2) = G^d(v; (r_1, r_2)) = G^d(v; (r_1, r_2)) \geq q^d(\xi, r_1, r_2).
\]

Hence \( r_2 \mapsto g \) is nonincreasing.

2. Let \( 0 < r_1 < r_2 \) and set \( \Delta = r_1^d - r_2^d > 0 \) and \( \tau_2 = (r_2^d - \Delta)^{\frac{1}{d}} < r_2 \). Define the diffeomorphism

\[
\phi : (r_1, r_2) \mapsto (\tau_1, \tau_2),
\]

\[
s \mapsto [s^d - \Delta]^{1/d}.
\]

Let \( v \) be the minimizer of \((A.5)\) and \( \tau(s) = v \circ \phi(s) \). Let us remark that \( \phi'(s) = s^{d-1}/\phi(s)^{d-1} \), thus it holds

\[
q^d(\xi, r_1, r_2) = \int_{r_1}^{r_2} t^{d-1} \left[ |v'|^p + (1 - v)^2 \right] \, dt = \int_{r_1}^{r_2} \phi(s)^{d-1} \left[ \frac{|\tau'|^p}{|\phi(s)|^p} + (1 - \tau)^2 \right] \phi'(s) \, ds
\]

\[
= \int_{r_1}^{r_2} s^{d-1} \left[ \left( 1 + \frac{\Delta}{s^d - \Delta} \right) |\tau'|^p + (1 - \tau)^2 \right] \, ds \geq q^d(\xi, r_1, r_2) \geq q^d(\xi, \tau_1, r_2).
\]

Therefore \( r_1 \mapsto q^d \) is nondecreasing.
3. Let \(0 \leq \xi < \xi \leq 1\) and \(v\) the absolutely continuous, nondecreasing minimizer of problem \(q^d(\xi, r_1, r_2)\). Then there exists \(\tau \in (r_1, r_2)\) for which \(v(\tau) = \xi\). Hence

\[
q^d(\xi, r_1, r_2) \geq g(\xi, \tau, r_2) \geq g(\xi, r_1, r_2).
\]

Hence, \(\xi \mapsto q^d\) is nonincreasing. Finally, for \(\xi = 1\) consider the constant function \(v \equiv 1\) to get \(g(1, r_1, r_2) = 0\).

Indeed, in view of the monotonicity, for every \(r_1\) and \(r_2\) we have

\[
g(0, r_1, r_2) \geq g(0, 0, +\infty) = q^d_\infty(0, 0).
\]

Let us show \(q^d_\infty(0, 0) > 0\). As a matter of facts, taken the minimizer \(v\) for the problem \((2.4)\), there exists \(r \in (0, +\infty)\) such that \(v(r) = 1/2\) and we have

\[
q^d_\infty(0, 0) \geq \int_0^r t^{d-1} \left[ |v'|^p + (1 - v)^2 \right] dt = \int_0^r t^{d-1} |v'|^p dt + \frac{r^d}{4d}.
\]

A direct evaluation gives

\[
\min \left\{ \int_0^r t^{d-1} |v'|^p dt : v(r) = 0 \text{ and } v(r) = 1/2 \right\} = \frac{c}{r}
\]

and we obtain the estimate

\[
q^d_\infty(0, 0) \geq \frac{c}{r} + \frac{r^d}{4d} > 0.
\]

Lastly, let us show that for any \(r\) the function \(q^d_\infty(\cdot, r)\) is convex. Consider two values \(\xi_1, \xi_2 \in (0, 1)\) and the associated minimizers \(v_1, v_2\) for the respective energy \(q^d_\infty(\cdot, r)\). Indeed, for any \(\lambda \in (0, 1)\) the function \(\lambda v_1 + (1 - \lambda) v_2\) is a competitor for the minimization problem \(q^d_\infty(\lambda \xi_1 + (1 - \lambda) \xi_2, r)\), therefore it holds

\[
q^d_\infty(\lambda \xi_1 + (1 - \lambda) \xi_2, r) \leq \int_0^r t^{d-1} \left[ |\lambda v_1 - (1 - \lambda) v_2|^p + (1 - \lambda) v_1 + (1 - \lambda) v_2 \right] dt
\]

\[
\leq \lambda q^d_\infty(\xi_1, r) + (1 - \lambda) q^d_\infty(\xi_2, r).
\]

Thus \(q^d_\infty(\cdot, r)\) is continuous in the open interval \((0, 1)\). To show the continuity in 0 let \(\xi\) be small and \(v = \arg\min q^d_\infty(\xi, r)\). Set

\[
h(t) := \begin{cases} 
\frac{1}{1 - \sqrt{\xi}} (t - \xi), & t < \sqrt{\xi}, \\
t, & t \geq \sqrt{\xi},
\end{cases}
\]

and observe that \(h \circ v\) is a competitor for the problem \(q^d_\infty(0, r)\). Then

\[
q^d_\infty(0, r) \leq \int_r^\infty t^{d-1} \left[ |(h \circ v)'|^p + (1 - h \circ v)^2 \right] dt
\]

\[
\leq \frac{1}{(1 - \sqrt{\xi})^p} q^d_\infty(\xi, r) + \int_r^\infty t^{d-1} \left[ (1 - h \circ v)^2 - (1 - v)^2 \right] dt
\]

Let us estimate the second addend in the latter. By the definition of \(f\) we have

\[
\int_r^\infty t^{d-1} \left[ (1 - h \circ v)^2 - (1 - v)^2 \right] dt = \int_{v < \sqrt{\xi}} t^{d-1} \left[ (1 - h \circ v - v)^2 (v - h \circ v)^2 \right] dt
\]

\[
\leq 4\xi \int_{v < \sqrt{\xi}} t^{d-1} dt
\]

\[
\leq \frac{4\xi}{(1 - \sqrt{\xi})^2} q^d_\infty(\xi, r).
\]

Since \(q^d_\infty(\cdot, r)\) is monotone we have

\[
|q^d_\infty(0, r) - q^d_\infty(\xi, r)| \leq \max \left\{ \frac{1 - (1 - \sqrt{\xi})^p}{(1 - \sqrt{\xi})^p}, \frac{4\xi}{(1 - \sqrt{\xi})^2} \right\} \kappa,
\]

which shows that \(q^d_\infty(\cdot, r)\) is continuous in 0.
We deal with each addend separately. First observe that by Cauchy-Schwarz inequality, it holds
\[ b_{\varepsilon} \geq \frac{m^2}{\int_{B_r} \frac{1}{u_{\varepsilon}} \, dx} \geq \int_{B_{r\varepsilon}} \frac{(1 - u_{\varepsilon})^2}{\varepsilon} \, dx + \frac{m^2}{\varepsilon \left( \int_{B_{r\varepsilon} \setminus B_{r\varepsilon}} \frac{1}{u_{\varepsilon}} \, dx + \int_{B_{r\varepsilon}} \frac{1}{u_{\varepsilon}} \, dx \right)} \]

taking into account \( \eta \leq u_{\varepsilon} \leq \xi \) in \( B_{r\varepsilon} \), \( \xi \leq u_{\varepsilon} \leq 1 \) in \( B_r \setminus B_{r\varepsilon} \) and \( \eta = a e^{d+1} \) we obtain
\[ b_{\varepsilon} \geq \omega_d (1 - \xi)^2 \left( \frac{r_{\varepsilon}}{\varepsilon} \right)^d + \frac{m^2}{\omega_d \left( \frac{r_{\varepsilon}}{\varepsilon} \right)^d + \omega_d \varepsilon r_{\varepsilon}^d}. \]

Since \( b_{\varepsilon} \leq \tilde{f}_{\varepsilon,a}(m, r) \leq C(m) \) we deduce that \( r_{\varepsilon}/\varepsilon \) belongs to a fixed compact subset \( K = K(m, \xi) \) of \( (0, +\infty) \). Up to extracting a subsequence, which we do not relabel, we can suppose \( r_{\varepsilon}/\varepsilon \) to converge to some \( \tilde{r} > 0 \). Let us now consider the term \( a_{\varepsilon} \). Let \( v_{\tilde{r}} \) be the radial profile of \( u_{\varepsilon} \)
\[ a_{\varepsilon} = \int_{B_{r\varepsilon}} \left[ \varepsilon^{d-1} |\nabla u_{\varepsilon}|^p + \frac{(1 - u_{\varepsilon})^2}{\varepsilon^d} \right] \, dx = (d - 1) \omega_d \int_{r_{\varepsilon}/\varepsilon} r_{\varepsilon}/\varepsilon \, \left[ |v_{\varepsilon}|^p + (1 - v_{\varepsilon})^2 \right] \, dt. \]

With the notation introduced in Subsection A.2 and Lemma A.3 therein we deduce
\[ \liminf_{\varepsilon \downarrow 0} a_{\varepsilon} \geq (d - 1) \omega_d \liminf_{\varepsilon \downarrow 0} q_{\varepsilon}^d (\xi; (r_{\varepsilon}/\varepsilon, r/\varepsilon)) \geq (d - 1) \omega_d q_{\infty}^d (\xi, \tilde{r}), \]
where \( q_{\infty}^d \) has been defined in (2.4). Combining inequality (A.7) and the latter we get
\[ \liminf_{\varepsilon \downarrow 0} \tilde{f}_{\varepsilon,a}(m, r) \geq (d - 1) \omega_d q_{\infty}^d (\xi, \tilde{r}) + (1 - \xi)^2 \omega_d \tilde{r}^d + \frac{a m^2}{\omega_d \tilde{r}^d}. \]

Sending \( \xi \) to 0 we have, by continuity (Lemma A.3) \( q_{\infty}^d (\xi, \tilde{r}) \rightarrow q_{\infty}^d (0, \tilde{r}) \). Then taking the infimum in \( \tilde{r} \), we obtain
\[ \liminf_{\varepsilon \downarrow 0} \tilde{f}_{\varepsilon,a}(m, r) \geq \min_{\tilde{r}} \left\{ \left( d - 1 \right) \omega_d q_{\infty}^d (0, \tilde{r}) + \omega_d \tilde{r}^d + \frac{a m^2}{\omega_d \tilde{r}^d} \right\}. \]

Again by Lemma A.3 the function \( q_{\infty}^d (0, \tilde{r}) \) is nondecreasing in \( \tilde{r} \), and \( q_{\infty}^d (0, 0) > 0 \) therefore setting
\[ \kappa := (d - 1) \omega_d q_{\infty}^d (0, 0) \leq f_a^d (m) \]
we conclude the proof of Proposition 2.1.
A.4. Proof of Proposition 2.2

Let \( \delta > 0 \), by Lemma A.3 for \( \varepsilon \) sufficiently small

\[ q^d(\eta; (r_*, r/\varepsilon)) \leq q^d_\infty(0, r_*) + \delta. \]

Let

\[ v_\delta(t) = \arg\min \left\{ q^d \left( v; \left( r_*, \frac{r}{\varepsilon} \right) \right) \right\} \text{ for } 0 \leq t \leq r_*, \]

and set

\[ u_\varepsilon(t) := \begin{cases} \eta & \text{for } 0 \leq t \leq r_*, \\ v_\delta \left( \frac{t}{\varepsilon} \right) & \text{for } r_* \varepsilon \leq t \leq r. \end{cases} \]

Set \( \vartheta_\varepsilon(s) \) to be constant equal to \( \frac{m}{\omega_d(\varepsilon r_*)^d} \) on the ball \( B_{r_*} \) and zero outside. Indeed, the couple \((\vartheta_\varepsilon, u_\varepsilon(|x|))\) belongs to \( Y_{\varepsilon,a}(m, r) \). That is because \( u_\varepsilon \) is greater than \( \eta \) and attains value 1 at the border of \( B_r \) and

\[ \int_{B_r} \vartheta_\varepsilon(x) \, dx = \frac{m}{\omega_d(\varepsilon r_*)^d} \omega_d(\varepsilon r_*)^d = m. \]

Let us show that the couple \((\vartheta_\varepsilon, u_\varepsilon)\) defined satisfy inequality (2.7). Taking advantage of the radial symmetry of the functions we get

\[
E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon) = \int_{r_*}^r \rho^{d-1} \left[ \frac{(1 - u_\varepsilon)}{\varepsilon^d} \right] \, dt + \int_{r_*}^r \frac{(1 - \eta)^2}{\varepsilon^d} \omega_d(\varepsilon r_*)^d + \frac{m^2}{\varepsilon} \omega_d(\varepsilon r_*)^d.
\]

By simplifying the expression and considering the change of variable \( s = \frac{t}{\varepsilon} \) in the latter it holds

\[
E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon) = (d - 1) \omega_d \int_{r_*}^r s^{d-1} \left[ (1 - u_\varepsilon) \right] \, ds + (1 - \eta)^2 \omega_d r_\delta^d + \frac{m^2}{\varepsilon} \omega_d r_\delta^d
\]

\[
\leq (d - 1) \omega_d q^d(\eta; (r_*, r/\varepsilon)) + (1 - \eta)^2 \omega_d r_\delta^d + \frac{m^2}{\varepsilon} \omega_d r_\delta^d.
\]

Then, by Lemma A.3 for \( \varepsilon \) sufficiently small we have

\[
E_{\varepsilon,a}(\vartheta_\varepsilon, u_\varepsilon) \leq \frac{a m^2}{\omega_d r_\delta^d} + \omega_d r_\delta^d + (d - 1) \omega_d q^d_\infty(0, r_*) + (d - 1) \omega d r_\delta^d \leq f^d_a(m) + C \delta,
\]

which ends the proof of Proposition 2.2

A.5. Proof of Proposition 2.3

Propositions 2.1, 2.2 and lemma A.2 ensure that

\[ f^d_a(m) = \lim_{\varepsilon \downarrow 0} T^d_{\varepsilon,a}(m, r) = \lim_{\varepsilon \downarrow 0} f^d_{\varepsilon,a}(m, r, \tilde{r}) \]

independently of the choices for \( r \) and \( \tilde{r} < r \). For the sake of clarity we introduce

\[ T(m, r) := \left\{ \frac{a m^2}{\omega_d r^d} + \omega_d r^d + (d - 1) \omega_d q^d_\infty(0, r) \right\} \]

and recall that \( f^d_a(m) = \min_r \, T(m, r) \) for \( m > 0 \) and \( f^d_a(0) = 0 \), see (2.3).

Proof.

Let us prove the continuity of \( f^d_a \) on \((0, +\infty)\). For \( m_1, m_2 \in (0, +\infty) \) and for \( i = 1, 2 \) let \( r_i \) be such that \( f^d_a(m_i) = T(m_i, r_i) \). On one hand comparing with \( r = 1 \) it holds

\[ \frac{m_i^2}{\omega_{d-1} r_i^d} \leq f^d_a(m_i) \leq T(m_i, 1) \quad (A.8) \]

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We have already shown that
\[ \omega_{d-1} r^d \leq f^d_a(m_1) \leq T(m_1, 1). \]
Consequently \( \omega_{d-1} r^d \) belongs to the compact set \([m_i / T(m_i, 1), T(m_i, 1)]\). Now remark that
\[ f^d_a(m_1) \leq T(m_1, r_2) = f^d_a(m_2) + T(m_1, r_2) - T(m_2, r_2) \]
thus
\[ |f^d_a(m_1) - f^d_a(m_2)| \leq |T(m_1, r_2) - T(m_2, r_2)| \leq \frac{|m_1^2 - m_2^2|}{\omega_{d-1} \min\{r_1^2, r_2^2\}} \]
and taking into account inequality (A.8) we have
\[ |f^d_a(m_1) - f^d_a(m_2)| \leq (m_1 + m_2) \max\left\{ \frac{T(m_1, 1)}{m_1^2}, \frac{T(m_2, 1)}{m_2^2} \right\} |m_1 - m_2|. \]

Observing that \( T(\cdot, 1) \) is continuous we conclude that \( f^d_a \) is continuous on \((0, +\infty)\). Next, we see that \( f^d_a \) is non-decreasing. Let \( 0 < m_1 < m_2 \) and \( r > 0 \). Let \((\vartheta, u) \in Y_{\epsilon,a}(m_2, r)\) such that \( E_{\epsilon,a}(\vartheta, u; B_r) = f^d_a(m_2, r) \). Set \( \overline{\vartheta} = m_1 \vartheta / m_2 \) and remark that the couple \((\overline{\vartheta}, u)\) belongs to \( Y_{\epsilon,a}(m_1, r)\). Therefore we have the following set of inequalities
\[ f^d_a(m_1, r) \leq E_{\epsilon,a}(\overline{\vartheta}, u; B_r) = E_{\epsilon,a}\left( \frac{m_1 \vartheta}{m_2}, u; B_r \right) < E_{\epsilon,a}(\vartheta, u; B_r) = f^d_a(m_2, r). \]

Passing to the limit as \( \epsilon \downarrow 0 \) we obtain
\[ f^d_a(m_1) \leq f^d_a(m_2). \]

Let us now prove the sub-additivity. For a radius \( r \) consider the competitors \((\vartheta_j, u_j) \in Y_{\epsilon,a}(m_j, r)\) for \( j = 1, 2 \). Consider the ball \( B_{2r+1} \) centered in the origin and two points \( x_1, x_2 \) such that the balls \( B_r(x_1), B_r(x_2) \) are disjoint and contained in \( B_{2r+1}\). Set
\[ \overline{\vartheta}(x) := \begin{cases} \vartheta_1(x - x_1), & x \in B_r(x_1), \\
\vartheta_2(x - x_2), & x \in B_r(x_2), \\
0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \overline{u}(x) := \begin{cases} u_1(x - x_1), & x \in B_r(x_1), \\
u_2(x - x_2), & x \in B_r(x_2), \\
1, & \text{otherwise}, \end{cases} \]
and observe that the couple \((\overline{\vartheta}, \overline{u})\) belongs to \( Y(m_1 + m_2, 2r+1)\). Being the balls \( B_r(x_j) \) disjoint we have
\begin{align*}
\overline{f}^d_{\epsilon,a}(m_1 + m_2, r_1 + r_2) & \leq E_{\epsilon,a}(\vartheta_1(x - x_1), u_1(x - x_1); B_r(x_1)) + E_{\epsilon,a}(\vartheta_2(x - x_2), u_2(x - x_2); B_r(x_2)) \\
& = \overline{f}^d_{\epsilon,a}(m_1, r) + f^d_a(m_2, r).
\end{align*}
Passing to the limit as \( \epsilon \downarrow 0 \), and recalling that it is independent of the choice of the radius, we get
\[ f^d_a(m_1 + m_2) \leq f^d_a(m_1) + f^d_a(m_2). \]

We conclude the appendix by showing that

**Lemma A.4.** For any sequence \( a_i \downarrow 0 \) it holds
\[ f^d_{a_i} \longrightarrow \kappa \mathbf{1}_{(0, \infty)} \tag{pointwise}. \]

**Proof.** We have already shown that \( f^d_a(m) \geq \kappa \) for \( m > 0 \). For \( m > 0 \) choose \( \hat{\kappa} = (\sqrt[4]{a m})^{1/d} \), then by definition it holds
\[ \kappa \leq f^d_a(m) \leq (d - 1) \omega_d \hat{q}^d_{\infty}(0, (\sqrt[4]{a m})^{1/d}) + \omega_d \sqrt[4]{a m} + \frac{\sqrt[4]{a m}}{\omega_d}. \]
Finally simply recall that \((d - 1) \omega_d \hat{q}^d_{\infty}(0, 0) = \kappa \) and that \( q^d_{\infty}(0, \cdot) \) is continuous. \( \square \)
References


