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# Dynamic boundary control synthesis of coupled PDE-ODEs for communication networks under fluid flow modeling

Nicolás Espitia, Antoine Girard, Nicolas Marchand and Christophe Prieur

**Abstract**—This paper deals with dynamic boundary control synthesis of communication networks which are modeled under fluid-flow modeling and compartmental representation. The boundary control synthesis of the resulting linearized coupled hyperbolic PDE-ODEs is carried by means of Lyapunov techniques and LMIs formulation. Two specific control functions with constraints are studied. Input-to state stability of the linearized system of an optimal equilibrium is guaranteed while minimizing the asymptotic gain due to the control actions.

## I. INTRODUCTION

Systems of conservations laws, namely systems described in macroscopic setting by hyperbolic partial differential equations (PDES) have gained a lot of attention. This is because hyperbolic PDES have been found useful for modeling various physical networks, and also find utility in engineering systems for modeling time delays. Concerning physical networks of different nature, we can point out: e.g. hydraulic, road traffic, gas networks ([2], [4], [11]). Stabilization and control of such systems have been also widely studied. Several results on that with more examples of physical systems can be found in a recent book [1].

Inspired by traffic flow on road networks (e.g. [10]), several studies deal with flow of information on communication networks (e.g. [5]). In addition to macroscopic models, compartmental systems are also known as suitable framework to describe conservation laws in networks. See for instance [12], or [3] where congestion control of compartmental networks is studied, still under fluid-flow modeling. They exploit properties of positive systems to establish stability of the network and prevent congestion by means of nonlinear controls.

Motivated by communication networks under fluid-flow modeling, we have recently introduced a model combining macroscopic models using hyperbolic PDEs, with ordinary differential equations (ODEs). Indeed, in [8] we have proposed a model of network of compartments for which we establish that: nodes are compartments for the dynamics of servers, composed mainly by buffers (modeled by ODEs), and

the links connecting them are compartments representing delays of the flow traveling through transmission lines (modeled by hyperbolic PDEs). The coupling between both is defined at the boundaries of the hyperbolic PDEs. It results in what we called there, coupled PDE-ODE with *dynamic boundary condition*. We have established also that the network operates in *free-flow*, referring to the fact that in literature of conservation laws by macroscopic setting, there exist critical densities that split the operation of the network in two zones: free-flow zone and congested one (according to the so-called *fundamental diagram of flow-density*) (see e.g [14], [6]).

Now, in this paper as a continuation and part of the perspectives of the work [8], we deal with control and input-to-state stabilization of the network around an optimal operation point, namely, an optimal free-flow steady state already characterized. The main contribution of this paper pertains to the control synthesis that is carried out by Lyapunov analysis which leads to sufficient condition for input-to-state stability under LMIs formulation. Moreover, ISS with respect to exogenous input demands that affect the behavior of the network is stated. It is worth recalling that one of the main contribution in [8] was precisely the fact that one can guarantee ISS without any control action but the throughput of the networks remains quite far from the desired equilibrium point and perhaps could lead the network to get congested. Here, by means of suitable control actions, we intend to reduce the impact by allowing the network to operate as close as possible to the optimal equilibrium. Formally, it translates in a minimization of the *asymptotic gain* in the ISS framework. In addition, in this paper it turns out that constraints on the control variables must be respected, thus limitation in amplitude of the gains has to be considered when solving the LMIs involved in the synthesis. Optimization issues are also considered for the minimization of the asymptotic gain when performing the control synthesis.

This paper is organized as follows. In Section II we bring back the model introduced in [8]. In Section III we present the main results regarding the synthesis, control constraints and optimization issues. Section IV provides a numerical example to illustrate the results. Finally, conclusions are given in Section V. Due to space limitation, proofs of some results are omitted.

## II. FLUID-FLOW MODELING OF COMMUNICATION NETWORKS

In this section, we briefly bring back the model of communication networks under compartmental fluid-flow dynamics that we have recently introduced in [8], using coupled

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hyperbolic PDE and ODEs. A general network is illustrated in Figure 1 where each node, as illustrated in Figure 2, represents a server in which a buffer stores information to be processed. These nodes are modeled by ODEs, while the other compartments considered in this network are the transmission lines when delays in time may exist since traffic flow takes a while in traveling from one buffer to another. Transmission lines are then modeled by linear hyperbolic equations. The

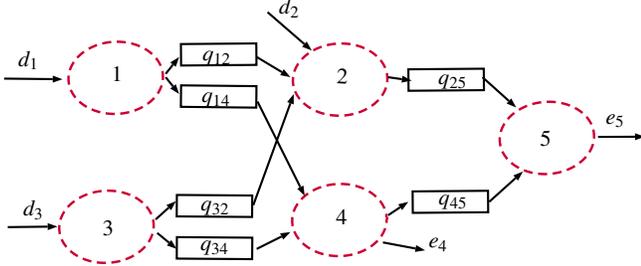


Fig. 1. Example of a compartmental network.

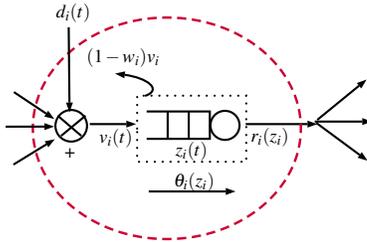


Fig. 2. Compartment: buffer.

model which has been studied in [8] is as follows <sup>1</sup>:

$$\begin{cases} \partial_t q_{ij}(t,x) + \lambda_{ij} \partial_x q_{ij}(t,x) = 0, & i \in \mathcal{I}_n, j \in \mathcal{D}_i \\ \dot{z}_i(t) = w_i(t) d_i(t) + \sum_{k \in \mathcal{U}_i} w_i(t) q_{ki}(t,1) - r_i(z_i(t)) \end{cases} \quad (1)$$

$x \in [0, 1], t \in \mathbb{R}^+$ , where  $z_i$  represents the quantity of information to be processed before it is sent to the transmission line,  $q_{ij}$  represents the flow of information traveling through the line. In this model  $\lambda_{ij}$  are the average velocities of packets in each line.

The dynamic boundary condition is given by

$$q_{ij}(t,0) = u_{ij}(t) r_i(z_i(t)), \quad r_i \geq 0 \quad (2)$$

with  $0 \leq u_{ji}(t) \leq 1, j \in \mathcal{D}_i, i \in \mathcal{R}$  (index set of routing compartments).

The output function is given by,

$$e_i(t) = u_i(t) r_i(z_i(t)) \quad (3)$$

with  $\sum_{j \in \mathcal{D}_i} u_{ij}(t) + u_i(t) = 1$  ( $u_i(t) \equiv 0$  if  $i \notin \mathcal{I}_{out}$ ). Finally, the initial conditions are given by

$$\begin{cases} q_{ij}(0,x) = q_{ij}^0(x), & x \in [0,1] \\ z_i(0) = z_i^0. \end{cases} \quad (4)$$

<sup>1</sup>  $\mathcal{I}_n$  is the set of the number of compartments, numbered from 1 to  $n$ .  $\mathcal{D}_i \subset \mathcal{I}_n$  is the index set of downstream compartments connected directly to compartment  $i$ .  $\mathcal{U}_i \subset \mathcal{I}_n$  is the index set of upstream compartments connected directly to compartment  $i$ .

It is worth recalling that the traffic flow may be routed to different compartments of the network until reaching the destination and can be modulated by suitable actuators. On one hand, there are control actions  $u_{ij}(\cdot)$  as in (2)-(3) that route the flow of information through different paths of the network. We call them *routing splitting controls*. The routing takes place at the output of the server. On the other hand, we have control actions  $w_i(\cdot)$  that reject packets (traffic flow) before they enter to the buffers. For instance, in (1),  $d_i(t)$  is the actual external input flow demand, then  $w_i d_i$  is a fraction of such a demand and  $w_i(t) q_{ki}(t,1)$  is a fraction of the incoming flow entering to the buffer  $i, 0 \leq w_i(t) \leq 1$ . In addition, let us recall that in (3),  $r_i$  is the processing rate function that is given by ([3])

$$r_i(z_i) = \frac{\varepsilon_i z_i}{1 + z_i} \quad (5)$$

with  $\varepsilon_i > 0$  as the maximal processing capacity of each server.

In this work, we want the network to operate at some equilibrium point, to be precise, at some *free-flow steady-state*. Moreover, we want to study the stability of the equilibrium when control actions are taken into account by means of a closed-loop setting. The analysis will be carried out using again a linearized system.

Then, by assuming a given constant input flow demand  $d_i^*$ , the system (1)-(4) admits an optimal equilibrium point  $\{q_{ki}^*, z_i^*, u_{ij}^*, u_i^*, w_i^*, e_i^*\}$  with  $q_{ij}^*$  time and space invariant [8]. The characterization of the optimal *free-flow steady state* has been done by using two criteria: maximizing the total output flow and minimizing the total mean travel time (see [8] for further details about the characterization). Then, the linearized system around an optimal free-flow steady state is as follows:

$$\begin{cases} \partial_t y(t,x) + \Lambda \partial_x y(t,x) = 0 \\ \dot{Z}(t) = AZ(t) + G_y y(t,1) + B_w W(t) + D \tilde{d}(t) \end{cases} \quad (6)$$

with dynamic boundary condition

$$y(t,0) = G_z Z(t) + B_u U(t) \quad (7)$$

and initial condition

$$\begin{cases} y(0,x) = y^0(x), & x \in [0,1] \\ Z(0) = Z^0. \end{cases} \quad (8)$$

where  $y: \mathbb{R}^+ \times [0,1] \rightarrow \mathbb{R}^m$  with  $m$  given by

$$m := \sum_{i \in \mathcal{I}_n} \text{card}(\mathcal{D}_i) \quad (9)$$

$\Lambda$  is a diagonal positive definite matrix in  $\mathbb{R}^{m \times m}$  such that  $\Lambda = \text{diag}(\lambda_{ij}), i \in \mathcal{I}_n, j \in \mathcal{D}_i$ .  $W: \mathbb{R}^+ \rightarrow \mathbb{R}^n, Z: \mathbb{R}^+ \rightarrow \mathbb{R}^n$  and  $U: \mathbb{R}^+ \rightarrow \mathbb{R}^l$  where  $l$  is given by:

$$l := \sum_{i \in \mathcal{R}} (\text{card}(\mathcal{D}_i) - 1) + \sum_{i \in \mathcal{R} \cap \mathcal{I}_{out}} \text{card}(\mathcal{D}_i)$$

The rest of matrices are of appropriate dimension and their detailed characterization are given in [8]. It remains to say that  $\tilde{d}(t)$  is the input flow that can be viewed in the following as an input disturbance. We assume that  $\tilde{d}$  is in  $\mathcal{C}_{pw}(\mathbb{R}^+; \mathbb{R}^n)$ . Finally,  $y(0,x) = y^0(x) \in L^2([0,1]; \mathbb{R}^m)$  and  $Z(0) = Z^0 \in \mathbb{R}^n$ .

### III. DYNAMIC BOUNDARY CONTROL OF THE COMMUNICATION NETWORK

Since we deal with input flows demands that are viewed as perturbations, it has been proved in [8], that the system (6)-(8) is input-to-state stable in open-loop with respect to those inputs. We recall the notion of ISS in  $L^2$ .

**Definition 1:** [Input-to-state stability ISS] The system (6)-(8) is input-to-state stable (ISS) with respect to  $\tilde{d} \in \mathcal{C}_{pw}(\mathbb{R}^+; \mathbb{R}^n)$ , if there exist  $\nu > 0$ ,  $C_1 > 0$  and  $C_2 > 0$  such that, for every  $Z^0 \in \mathbb{R}^n$ ,  $y^0 \in L^2([0, 1]; \mathbb{R}^m)$ , the solution satisfies, for all  $t \in \mathbb{R}^+$ ,

$$\begin{aligned} (\|Z(t)\|^2 + \|y(t, \cdot)\|_{L^2([0, 1]; \mathbb{R}^m)}^2) \leq \\ C_1 e^{-2\nu t} (\|Z^0\|^2 + \|y^0\|_{L^2([0, 1]; \mathbb{R}^m)}^2) + C_2 \sup_{0 \leq s \leq t} \|\tilde{d}(s)\|^2 \end{aligned} \quad (10)$$

$C_2$  is called the asymptotic gain.

In this work we are interested in designing control actions in order to minimize the asymptotic gain obtained in open loop. It can be carried out by means of a closed-loop setting using the following control functions

$$W(t) = \begin{bmatrix} K_z & K_y \end{bmatrix} \begin{bmatrix} Z(t) \\ y(t, 1) \end{bmatrix}; \quad U(t) = \begin{bmatrix} L_z & L_y \end{bmatrix} \begin{bmatrix} Z(t) \\ y(t, 1) \end{bmatrix}$$

with  $K_z \in \mathbb{R}^{n \times n}$ ,  $K_y \in \mathbb{R}^{n \times m}$ ,  $L_z \in \mathbb{R}^{l \times n}$  and  $L_y \in \mathbb{R}^{l \times m}$  named as control gains in the sequel. Therefore, the linearized coupled PDE-ODE system (6)-(8) becomes:

$$\begin{cases} \partial_t y(t, x) + \Lambda \partial_x y(t, x) = 0 \\ \dot{Z}(t) = (A + B_w K_z) Z(t) + (G_y + B_w K_y) y(t, 1) + D \tilde{d}(t) \end{cases} \quad (11)$$

with dynamic boundary condition

$$y(t, 0) = (G_z + B_u L_z) Z(t) + B_u L_y y(t, 1) \quad (12)$$

and initial condition

$$\begin{cases} y(0, x) = y^0(x), & x \in [0, 1] \\ Z(0) = Z^0. \end{cases} \quad (13)$$

#### A. ISS control synthesis without constraints

In this subsection we first study the control synthesis without any constraint on the control gains. Next, we will add some optimization objective which results in the minimization of the asymptotic gain.

**Theorem 1:** [Control synthesis] Let  $\lambda = \min\{\lambda_{ij}\}_{i \in \mathcal{S}_n, j \in \mathcal{S}_i}$ . Assume that there exist  $\mu, \gamma > 0$ , a symmetric positive definite matrix  $P \in \mathbb{R}^{n \times n}$  a diagonal positive matrix  $Q \in \mathbb{R}^{m \times m}$ , as well as control gains  $K_z \in \mathbb{R}^{n \times n}$ ,  $K_y \in \mathbb{R}^{n \times m}$ ,  $L_z \in \mathbb{R}^{l \times n}$  and  $L_y \in \mathbb{R}^{l \times m}$  such that the following matrix inequality, holds :

$$M_c = \begin{pmatrix} M_1 & M_2 & M_3 \\ \star & M_4 & 0 \\ \star & \star & M_5 \end{pmatrix} \leq 0 \quad (14)$$

with

- $M_1 = \mathcal{A}^T P + P \mathcal{A} + 2\mu \lambda P + \mathcal{G}_z^T Q \Lambda \mathcal{G}_z$ ;
- $M_2 = P \mathcal{G}_y + \mathcal{G}_z^T Q \Lambda B_u L_y$ ;

- $M_3 = PD$ ;
- $M_4 = -e^{-2\mu} Q \Lambda + L_y^T B_u^T Q \Lambda B_u L_y$ ;
- $M_5 = -\gamma I$ .

where

$$\begin{cases} \mathcal{A} := A + B_w K_z \\ \mathcal{G}_z := G_z + B_u L_z \\ \mathcal{G}_y := G_y + B_w K_y \end{cases} \quad (15)$$

Then, the closed-loop system (11)-(13) is input-to-state stable (ISS) with respect to inputs  $\tilde{d} \in \mathcal{C}_{pw}(\mathbb{R}^+; \mathbb{R}^n)$ , and the asymptotic gain  $C_2$  satisfies

$$C_2 \leq \frac{\gamma}{2\nu} e^{2\mu}. \quad (16)$$

with  $\nu = \mu \lambda$ .

Particularly, with  $K_z = 0$ ,  $K_y = 0$ ,  $L_z = 0$  and  $L_y = 0$ , we obtain the main result in [8].

*Proof:* Due to space constraints, we provide a sketch of the proof of Theorem 1. Let us consider the following Lyapunov function  $V$  defined for all  $y \in L^2([0, 1]; \mathbb{R}^m)$  and  $Z \in \mathbb{R}^n$  as follows:

$$V(y, Z) = Z^T P Z + \int_0^1 y^T Q y e^{-2\mu x} dx \quad (17)$$

Computing the time derivative of  $V$  along the solutions of (11), integrating by parts, using the boundary condition (12) and the fact that  $Q$  is a diagonal positive definite matrix (thus  $\underline{\lambda} Q \leq \Lambda Q$  holds), yields to:

$$\begin{aligned} \dot{V} \leq & -2\nu Z^T P Z - 2\nu \int_0^1 y^T Q y e^{-2\mu x} dx \\ & + Z^T (A^T P + P A + K_z^T B_w^T P + P B_w K_z + 2\mu \lambda P + G_z^T Q \Lambda G_z \\ & + G_z^T Q \Lambda B_u L_z + L_z^T B_u^T Q \Lambda G_z + L_z^T B_u^T Q \Lambda B_u L_z) Z \\ & + y^T(\cdot, 1) (-e^{-2\mu} Q \Lambda + L_y^T B_u^T Q \Lambda B_u L_y) y(\cdot, 1) \\ & + y^T(\cdot, 1) (G_y^T P + K_y^T B_w^T P + L_y^T B_u^T Q \Lambda G_z + L_y^T B_u^T Q \Lambda B_u L_z) Z \\ & + Z^T (P G_y + P B_w K_y + G_z^T Q \Lambda B_u L_y + L_z^T B_u^T Q \Lambda B_u L_y) y(\cdot, 1) \\ & + \tilde{d}^T D^T P Z + Z^T P D \tilde{d} \end{aligned} \quad (18)$$

Adding  $\gamma \|\tilde{d}\|^2 - \gamma \|\tilde{d}\|^2$  to (18), for some  $\gamma > 0$ , we have

$$\dot{V} \leq -2\nu V + \begin{bmatrix} Z \\ y(\cdot, 1) \\ \tilde{d} \end{bmatrix}^T M_c \begin{bmatrix} Z \\ y(\cdot, 1) \\ \tilde{d} \end{bmatrix} + \gamma \|\tilde{d}\|^2$$

with  $\nu = \mu \lambda$  and where  $M_c$  is defined as (14). Therefore, as long as  $M_c \leq 0$ , we have

$$\dot{V} \leq -2\nu V + \gamma \|\tilde{d}\|^2 \quad (19)$$

Using the Comparison principle, one can end up with

$$\begin{aligned} \|Z(t)\|^2 + \|y(t, \cdot)\|_{L^2([0, 1]; \mathbb{R}^m)}^2 \\ \leq C_1 e^{-2\nu t} (\|Z^0\|^2 + \|y^0\|_{L^2([0, 1]; \mathbb{R}^m)}^2) + \frac{\gamma}{2\nu} e^{2\mu} \|\tilde{d}\|_\infty^2 \end{aligned} \quad (20)$$

with  $C_1 = \max\{\lambda_{\max}(P), \lambda_{\max}(Q)\} e^{2\mu}$ . Therefore, the system (6) satisfies the ISS property with respect to the disturbance input  $\tilde{d}$ . The asymptotic gain satisfies:

$$C_2 \leq \frac{\gamma}{2\nu} e^{2\mu}. \quad (21)$$

It concludes the proof of Theorem 1.  $\blacksquare$

**Remark 1:** Finding the control gains for Theorem 1, may lead to obtain, in the worst case, the same asymptotic gain as if no control actions are taken (i.e. open loop). Let us recall that the control objective in this paper is to minimize the asymptotic gain. One way to minimize it, is minimizing an estimate of an upper bound of it. Therefore we could just take as objective function  $\frac{\gamma}{2\nu}e^{2\mu}$ , subject to  $M_c \leq 0$ . We do not handle directly this optimization problem but an extension of it while including some constraints on the control gains. We are going to present it in the sequel.

### B. ISS control synthesis with constraints

Due to the nature of the communication network (1)-(4) where the two control functions are involved, it is important to point out that one must take into account the constraints on those control variables, i.e.  $u_{ij}(t) \in [0, 1]$  and  $w_i(t) \in [0, 1]$ ,  $j \in \mathcal{D}_i$ ,  $i \in \mathcal{I}_n$ . It implies in turn, constraints on the control functions for the linearized system (6)-(8). That is,  $U_{ij}(t) \in [-u_{ij}^*, 1 - u_{ij}^*]$  and  $W_i(t) \in [-w_i^*, 1 - w_i^*]$ . Therefore, putting suitable constraints on the norm of the gains  $K_z$ ,  $K_y$ ,  $L_z$  and  $L_y$ , would result on the desired control constraints. It is important however to point out that we have to impose  $L_y = 0$  in order to well handle some technical issues in our analysis as we will see in Proposition 1.

Let us first define the set of admissible initial conditions to be considered in the sequel.

**Definition 2:** Let  $\mathcal{E}^0$  be the set of all admissible initial condition and input disturbances for the closed loop system (11)-(13) as follows:

$$\begin{aligned} \mathcal{E}^0 = \{ & Z^0 \in \mathbb{R}^n; y^0 \in L^2([0, 1]; \mathbb{R}^m); \tilde{d} \in \mathcal{C}_{pw}(\mathbb{R}^+; \mathbb{R}^n) : \\ & (e^{2\mu} \lambda_{\max}(P) \|Z^0\|^2 + e^{2\mu} \lambda_{\max}(Q) \|y^0\|_{L^2([0, 1], \mathbb{R}^m)}^2 \\ & + \frac{\gamma}{2\nu} e^{2\mu} \sup_{s \in [0, \infty)} \|\tilde{d}(s)\|^2 \leq \mathcal{M}^2 \} \end{aligned} \quad (22)$$

for a given  $\mathcal{M} > 0$ .

The problem now is to find the control gains  $K_z$ ,  $K_y$ ,  $L_z$  with limitation on amplitude such that the control functions for the closed-loop system (11)-(13) satisfy

$$\begin{cases} \|W_i(t)\| \leq \min\{|-w_i^*|, |1 - w_i^*|\} = \delta_i^w \\ \|U_{ij}(t)\| \leq \min\{|-u_{ij}^*|, |1 - u_{ij}^*|\} = \delta_{ij}^u \quad i \in \mathcal{I}_n, j \in \mathcal{D}_i \end{cases} \quad (23)$$

and that hypothesis of Theorem 1 hold.

**Proposition 1:** Let  $Z^0, y^0, \tilde{d}$  be in  $\mathcal{E}^0$ . Let  $\delta_i^w, \delta_{ij}^u$  be given and  $p \in [0, 1]$ . If  $\beta_z = \mathcal{M}$ ,  $\beta_y = \sqrt{\sum_{k=1}^m \|G_z^T(k)\|^2} \mathcal{M}$  and  $\|K_{zi}\| \leq \frac{p\delta_i^w}{\beta_z}$ ,  $\|K_{yi}\| \leq \frac{(1-p)\delta_{ij}^u}{\beta_y}$ ,  $\|L_{zi}\| \leq \frac{\delta_{ij}^u}{\beta_z}$ , then, for all  $t \geq 0$ ,  $\|Z(t)\| \leq \beta_z$ ,  $\|y(t, 1)\| \leq \beta_y$ , and (23) holds.

By just extending what was stated in Remark 1, the optimization problem can be formulated as a constrained optimization one by putting the conditions on the control gains provided in Proposition 1, that is,

### Problem 1:

$$\begin{aligned} & \text{minimize} \quad \frac{\gamma}{2\nu} e^{2\mu} \\ & \text{subject to} \quad M_c \leq 0; \\ & \|K_{zi}\| \leq \frac{p\delta_i^w}{\beta_z}; \|K_{yi}\| \leq \frac{(1-p)\delta_{ij}^u}{\beta_y}; \|L_{zi}\| \leq \frac{\delta_{ij}^u}{\beta_z} \end{aligned} \quad (24)$$

The constraints of this problem however must be well transformed in order to handle them numerically. This concerns the main result of this subsection. Before we state the main result, let us consider the following proposition which is a variation of the result in [9][Proposition 2.6] or in [13, Section 3.2] and is going to be useful during the proof of Theorem 2.

**Proposition 2:** Let  $\tilde{\delta} > 0$  be given. If there exist matrices  $S$  and  $T$  of appropriate dimension and a positive scalar  $\tilde{\eta}$  such that

$$\begin{pmatrix} \tilde{\eta}He(S) - \tilde{\eta}^2 I & T \\ * & \tilde{\delta}^2 I \end{pmatrix} \geq 0 \quad (25)$$

then  $\|TS^{-1}\| \leq \tilde{\delta}$ . ( $He(S) := S^T + S$ .)

**Theorem 2:** Assume that there exist  $\mu, \gamma > 0$ , a symmetric positive definite matrix  $X$ , a diagonal positive matrix  $Q_3 \in \mathbb{R}^{m \times m}$ , as well as matrices  $Y_{K_z} \in \mathbb{R}^{n \times n}$ ,  $Y_{K_y} \in \mathbb{R}^{n \times m}$  and  $Y_{L_z} \in \mathbb{R}^{l \times n}$  such that: (26) is satisfied, and

$$\begin{aligned} & \begin{pmatrix} \eta He(X) - \eta^2 I & Y_{K_z} \\ * & \left(\frac{p\delta_i^w}{\beta_z}\right)^2 I \end{pmatrix} \geq 0; \begin{pmatrix} \eta He(X) - \eta^2 I & Y_{L_z} \\ * & \left(\frac{\delta_{ij}^u}{\beta_z}\right)^2 I \end{pmatrix} \geq 0 \\ & \begin{pmatrix} \eta He(Q_3) - \eta^2 I & Y_{K_y} \\ * & \left(\frac{(1-p)\delta_{ij}^u}{\beta_y}\right)^2 I \end{pmatrix} \geq 0; \end{aligned} \quad (27)$$

for some  $\eta > 0$  and  $\delta_i^w, \delta_{ij}^u, \beta_z, \beta_y$  given in Proposition 1.

Then, by setting  $K_z = Y_{K_z} X^{-1}$ ,  $K_y = Y_{K_y} Q_3^{-1}$  and  $L_z = Y_{L_z} X^{-1}$  it holds  $\|Y_{K_z} X^{-1}\| \leq \frac{p\delta_i^w}{\beta_z}$ ,  $\|Y_{K_y} Q_3^{-1}\| \leq \frac{(1-p)\delta_{ij}^u}{\beta_y}$  and  $\|Y_{L_z} X^{-1}\| \leq \frac{\delta_{ij}^u}{\beta_z}$ , the constraints on (24) are satisfied and Theorem 1 applies.

*Proof:* The inequality  $M_c \leq 0$  in (24) is a BMI that can be transformed into a proper LMI (provided  $\mu$  fixed). To do so, first note that, even with  $L_y = 0$  as we have imposed for our analysis,  $M_c$  given by (14) can be rewritten as follows:

$$\begin{aligned} M_c = & \begin{pmatrix} \mathcal{A}^T P + P\mathcal{A} + 2\mu\lambda P & P\mathcal{G}_y & PD \\ * & -e^{-2\mu} Q\Lambda & 0 \\ * & * & -\gamma I \end{pmatrix} \\ & + \begin{pmatrix} \mathcal{G}_z^T \\ 0 \\ 0 \end{pmatrix} Q\Lambda \begin{pmatrix} \mathcal{G}_z & 0 & 0 \end{pmatrix} \leq 0 \end{aligned} \quad (28)$$

Applying the Schur Complement on (28) and performing the change of variable  $Q_2 = Q\Lambda$  (being  $Q_2$  still diagonal), we get that  $\mathcal{A}^T P + P\mathcal{A} + 2\mu\lambda P \leq 0$  and

$$\begin{pmatrix} \mathcal{A}^T P + P\mathcal{A} + 2\mu\lambda P & P\mathcal{G}_y & PD & \mathcal{G}_z^T \\ * & -e^{-2\mu} Q_2 & 0 & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -Q_2^{-1} \end{pmatrix} \leq 0 \quad (29)$$

$$\tilde{M}_c = \begin{pmatrix} XA^T + AX + 2\mu\lambda X + Y_{K_z}^T B_w^T + B_w Y_{K_z} & G_y Q_3 + B_w Y_{K_y} & D & XG_z^T + Y_{L_z}^T B_u^T \\ \star & -e^{-2\mu} Q_3 & 0 & 0 \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -Q_3 \end{pmatrix} \leq 0 \quad (26)$$

Multiplying on both sides of the previous matrix by a diagonal matrix  $\text{diag}(I, Q_2^{-1}, I, I)$ , and performing the change of variable  $Q_3 = Q_2^{-1}$  (being  $Q_3$  still diagonal), we obtain the following equivalent matrix inequality

$$\begin{pmatrix} \mathcal{A}^T P + P\mathcal{A} + 2\mu\lambda P & P G_y Q_3 & PD & G_z^T \\ \star & -e^{-2\mu} Q_3 & 0 & 0 \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -Q_3 \end{pmatrix} \leq 0 \quad (30)$$

Now, multiplying on both sides of the previous matrix by a diagonal matrix  $\text{diag}(P^{-1}, I, I, I)$ , using (15) and performing the following change of variables  $X = P^{-1}$ ,  $Y_{K_z} = K_z X$ ,  $Y_{L_z} = L_z X$  and  $Y_{K_y} = K_y Q_3$ , we obtain  $\tilde{M}_c$  given by (26). Hence, it can be noticed that (26) is now a LMI (provided  $\mu$  fixed). Next, applying Proposition 2 we have that if  $\begin{pmatrix} \eta \text{He}(X) - \eta^2 I & Y_{K_z} \\ \star & (\frac{p\delta_{ij}^w}{\beta_z})^2 I \end{pmatrix} \geq 0$ ,  $\begin{pmatrix} \eta \text{He}(X) - \eta^2 I & Y_{L_z} \\ \star & (\frac{\delta_{ij}^u}{\beta_z})^2 I \end{pmatrix} \geq 0$  and  $\begin{pmatrix} \eta \text{He}(Q_3) - \eta^2 I & Y_{K_y} \\ \star & (\frac{(1-p)\delta_{ij}^w}{\beta_y})^2 I \end{pmatrix} \geq 0$ , for some  $\eta > 0$ , then  $\|Y_{K_z} X^{-1}\| \leq \frac{p\delta_{ij}^w}{\beta_z}$ ,  $\|Y_{L_z} X^{-1}\| \leq \frac{\delta_{ij}^u}{\beta_z}$  and  $\|Y_{K_y} Q_3^{-1}\| \leq \frac{(1-p)\delta_{ij}^w}{\beta_y}$  respectively. Note that the constraints on (24) hold for the original variables. An immediate consequence is that Theorem 1 holds. With this, we conclude the proof. ■

**Remark 2:** When considering the following problem (Problem 2) where the objective function in (31) is the same as in Problem 1, one can obtain a sub-optimal solution of the Problem 1.

**Problem 2:**

$$\text{minimize } \frac{\gamma}{2\nu} e^{2\mu}$$

$$\text{subject to } \tilde{M}_c \leq 0; \quad (\tilde{M}_c \text{ as in (26)})$$

$$\begin{pmatrix} \eta \text{He}(X) - \eta^2 I & Y_{K_z} \\ \star & (\frac{p\delta_{ij}^w}{\beta_z})^2 I \end{pmatrix} \geq 0; \quad \begin{pmatrix} \eta \text{He}(X) - \eta^2 I & Y_{L_z} \\ \star & (\frac{\delta_{ij}^u}{\beta_z})^2 I \end{pmatrix} \geq 0 \\ \begin{pmatrix} \eta \text{He}(Q_3) - \eta^2 I & Y_{K_y} \\ \star & (\frac{(1-p)\delta_{ij}^w}{\beta_y})^2 I \end{pmatrix} \geq 0; \quad (31)$$

**Remark 3:** Note that the objective function in Problem 2 is nonlinear, involving variables  $\gamma$  and  $\mu$  ( $\nu = \lambda\mu$ ). However, in order to numerically handle this optimization problem, we combine the line search algorithm on  $\mu$ . In that case, we deal with an optimization problem within the semi-definite programming framework where the line search on  $\mu$  leads to successive LMIs.

#### IV. NUMERICAL SIMULATIONS

Let us consider a network under compartmental setting as represented in Figure 3 which is made up of 4 buffers along

with 5 transmission lines borrowed from [8]. The index sets involved in the example are:  $\mathcal{S}_n = \{1, 2, 3, 4\}$ ,  $\mathcal{S}_{in} = \{1\}$ ,  $\mathcal{S}_{out} = \{4\}$ ,  $\mathcal{U}_1 = \emptyset$ ,  $\mathcal{U}_2 = \{1\}$ ;  $\mathcal{U}_3 = \{1, 2\}$ ,  $\mathcal{U}_4 = \{2, 3\}$ ,  $\mathcal{D}_1 = \{2, 3\}$ ,  $\mathcal{D}_2 = \{3, 4\}$ ,  $\mathcal{D}_3 = \{4\}$ ,  $\mathcal{D}_4 = \emptyset$ . Let us consider the following network parameters: i) for the processing capacities;  $\varepsilon_1 = 100$ ,  $\varepsilon_2 = 50$ ,  $\varepsilon_3 = 80$  and  $\varepsilon_4 = 80$ , ii) the transport velocities;  $\lambda_{12} = 1$ ,  $\lambda_{13} = 1.2$ ,  $\lambda_{23} = 1.7$ ,  $\lambda_{24} = 0.5$  and  $\lambda_{34} = 2$  and iii) the critical traffic densities for free-flow condition;  $\sigma_{12} = 50$ ,  $\sigma_{13} = 50$ ,  $\sigma_{23} = 50$ ,  $\sigma_{24} = 30$  and  $\sigma_{34} = 100$ . For a given constant input flow demand  $d_1^* = 100$ , an optimal free-flow equilibrium point can be found. With,  $w_1^* = 0.95$ ,  $w_2^* = 0.95$ ,  $w_3^* = 0.95$ ,  $w_4^* = 0.95$  and  $u_{12}^* = 0.36$ ,  $u_{23}^* = 0.54$ . The steady-states are  $z_1^* = 19$ ,  $z_2^* = 1.98$ ,  $z_3^* = 13.12$ ,  $z_4^* = 5.6$  and  $q_{12}^* = 35$ ,  $q_{13}^* = 60$ ,  $q_{23}^* = 18.25$ ,  $q_{24}^* = 15$  and  $q_{34}^* = 74.3$ . We obtain that the maximal flow of the network  $e_4^* = 84.8$ .

When solving Problem 2, at each iteration, hypothesis of Theorem 2 are satisfied. Once the optimization problem is solved, Theorem 1 holds with optimal matrices  $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.55 & -4.88 \times 10^{-6} & -7.05 \times 10^{-6} \\ 0 & -4.88 \times 10^{-6} & 1.13 & -1.83 \times 10^{-6} \\ 0 & -7.05 \times 10^{-6} & -1.83 \times 10^{-6} & 1.11 \end{pmatrix}$ ,  $Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1.08 & 0 \\ 0 & 0 & 0 & 1.07 \\ 0 & 0 & 0 & 1.1 \end{pmatrix}$  along with  $\mu = 0.38$  and  $\nu = 0.19$ .  $\gamma = 0.58$ . For space limitations, we do not provide the values of control gains  $K_z$ ,  $K_y$  and  $L_z$ . The asymptotic gain is bounded by  $C_2 \leq \frac{\gamma}{2\nu} e^{2\mu} = 3.3$ . We recall that in open loop (i.e. when  $L_z = K_y = K_z = 0$ ), such a bound was given by 40.48 (see [8]). Therefore, we reduced considerably the asymptotic gain, thus the impact of the input flow demands in the behavior of the network while converging to desired equilibrium point in free-flow. On a frame of 40s, we close the loop at  $t = \frac{1}{\lambda} = 2$ . Figure 4 shows the total output of the network in open loop (black line) and in closed loop (red dashed line). It can be observed that thanks to the control actions, we can guarantee a faster convergence to the equilibrium and it remains closer than the open loop case. For both cases, the input flow demand has the profile depicted in Figure 5. In addition, Figure 6 shows the control functions devoted to reject traffic flow. It can be noticed that they respect the constraints.

#### V. CONCLUSION

In this paper, we have studied a dynamic boundary control synthesis of communication networks by means of Lyapunov techniques and LMIs formulation. The control functions meet the constraints and input-to state stability of the linearized system of an optimal equilibrium is guaranteed while minimizing the asymptotic gain. It could be fruitful to study event-based control schemes inspired by what it has been done for linear hyperbolic systems in [7]. We believe that event-based of the continuous-time controllers we have proposed in this

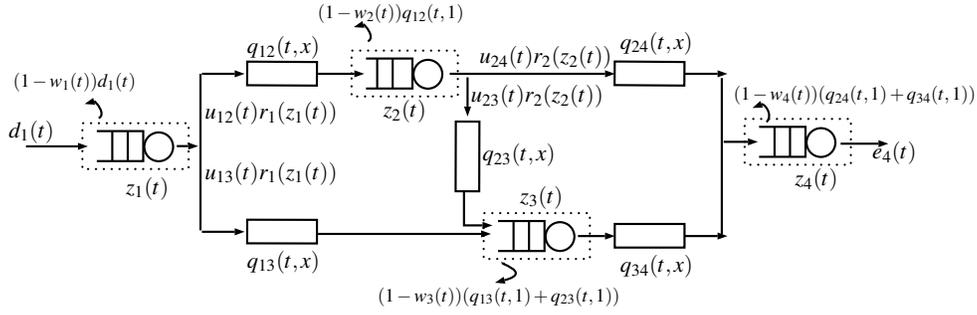


Fig. 3. Network of compartments made up of 4 buffers and 5 transmission lines.

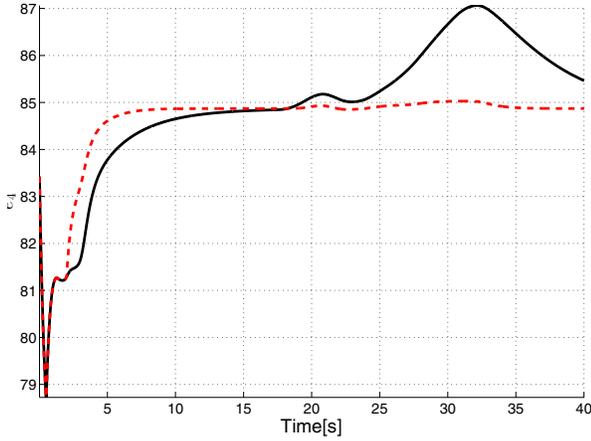


Fig. 4. Total output flow of the network: in open loop (black line) and in closed loop (red dashed line).

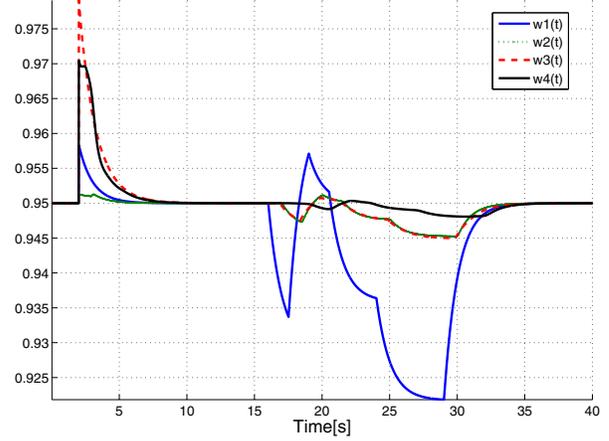


Fig. 6. Time evolution of control functions  $w_i$ .

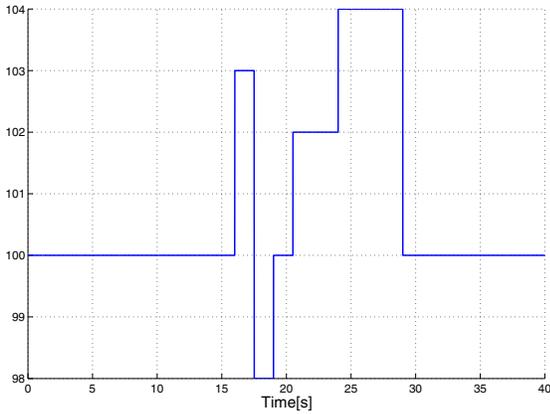


Fig. 5. Input flow demand  $d_1(t)$ .

paper, will represent a realistic approach of the action on the network. Finally, it could be interesting to generalize the work using nonlinear small-gain theorems.

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