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Abstract. In this article we show how to generalize to the Darbouxian, Liouvillian and Riccati case the extactic curve introduced by J. Pereira. With this approach, we get new algorithms for computing , if it exists, a rational, Darbouxian, Liouvillian or Riccati first integral with bounded degree of a polynomial planar vector field. We give probabilistic and deterministic algorithms. The arithmetic complexity of our probabilistic algorithm is in $\tilde{\mathcal{O}}(N^{\omega+1})$, where N is the bound on the degree of a representation of the first integral and $\omega \in [2;3]$ is the exponent of linear algebra. This result improves previous algorithms.

Introduction

In this article, we design an algorithm that given a planar polynomial vector field with degree d

$$(S)\ :\quad \left\{\begin{array}{lcl}\dot{x} & = & A(x,y),\\ \dot{y} & = & B(x,y),\end{array}\right. \qquad A,B\in\mathbb{K}[x,y], \qquad \deg(A),\deg B\leq d$$

and some bound $N\in\mathbb{N}$, computes first integrals of (S) of "size" (for some appropriate definition) lower than N.
The field $\mathbb{K}$ is an effective field of characteristic zero, i.e, one can perform arithmetic operations and test equality of two elements (typically, $\mathbb{K}=\mathbb{Q}$ or $\mathbb{Q}(\alpha)$, where α is an algebraic number).

First integrals are *non-constant functions* $\mathcal{F}$ that are constant along the solutions $\big(x(t),y(t)\big)$ of (S). This property can be rewritten as being solution of a partial differential equation

$$\text{(Eq)}\qquad A(x,y)\partial_x\mathcal{F}(x,y)+B(x,y)\partial_y\mathcal{F}(x,y)=0,$$

which can be also written $D_0(\mathcal{F})=0$ with D_0 the derivation

$$D_0=A(x,y)\partial_x+B(x,y)\partial_y.$$

Let us remark that multiplying A,B by some arbitrary non zero polynomial does not change the solutions of this equation. Thus in the rest of the article, we will always consider $A\wedge B=1$, thus excluding the case $A=0$ or $B=0$ as a trivial one. Indeed, in this case $\mathcal{F}(x,y)=x$ or y is then a first integral.

We need to precise in which class of functions we are searching $\mathcal{F}$. The most simple class are rational first integrals, for which we can easily define the notion of size by the degree of its numerator and denominator.

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However, larger class of functions can be used. The first method for computing first integrals in a symbolic way can be credited to G. Darboux in 1878, see [Dar78]. Darboux's method allows to compute what we call nowadays "Darbouxian" first integral. This kind of functions generalizes rational functions because Darbouxian functions are written with rational functions and logarithms. There exists even more general functions than Darbouxian functions, for example we can consider elementary or Liouvillian functions. There exists theoretical results about these kinds of first integrals, see [PS83, Sin92], and also algorithms for computing these first integrals, see [Man93, MM97, ADDdM05]. Roughly speaking, these algorithms compute what we call nowadays Darboux polynomials and then combine them in order to construct a first integral. The computation of Darboux polynomials is a difficult problem. Indeed, computing a bound on the degree of the irreducible Darboux polynomials of a derivation is still an open problem. This problem is Poincaré's problem. Thus in practice, algorithms ask the users a bound on the "size" of the first integral they want to compute. The computation of irreducible Darboux with degree smaller than N can be done in polynomial time, see [Chè11]. Unfortunately, it is a theoretical result, the exponent in the complexity is bigger than 10. Moreover, the algorithm proposed in [ADDdM05] to compute Liouvillian first integrals has an exponential time complexity in terms of the degree d of the derivation. Indeed, the recombination step used to construct the first integral from Darboux polynomials is of combinatorial nature.

In this paper we give an algorithm which computes a symbolic first integral: rational, Darbouxian, Liouvillian and Riccati, with "size" bounded by N, with $\tilde{\mathcal{O}}(N^{\omega+1}+d^2N^2)$ arithmetic operations in $\mathbb{K}$. We recall that $\omega \in [2;3]$ is the exponent of linear algebra over $\mathbb{K}$ and the soft-O notation $\tilde{\mathcal{O}}()$ indicates that polylogarithmic factors are neglected. Furthermore, in the following we suppose that the bound N tends to infinity and d is fixed. We have mentioned the term d^2N^2 in order to give the dependance relatively to the degree d. In particular, it shows that our algorithm is polynomial in d.
Our algorithm is thus more efficient than the existing ones.
Our strategy generalizes to the Darbouxian, Liouvillian and Riccati cases the algorithm proposed in [BCCW16] for computing rational first integrals. Our method avoids the computation of Darboux polynomials and then do not need a recombination step.

Now, we recall the definition of Darbouxian, Liouvillian first integrals and introduce a new definition: Riccati first integrals.

Definition 1. A *rational first integral* of (S) is a first integral $\mathcal{F} \in \overline{\mathbb{K}}(x,y)$.
A *Darbouxian first integral* of (S) is a first integral $\mathcal{F}$ of (S) of the form

$$\mathcal{F}(x,y) = \int G(x,y)dx + F(x,y)dy$$

where $F, G \in \overline{\mathbb{K}}(x,y)$ and $G(x,y)dx + F(x,y)dy$ is closed, or equivalently

$$\mathcal{F}(x,y) = \frac{P(x,y)}{Q(x,y)} + \sum_i \ln H_i(x,y)$$

where $P, Q, H_i \in \overline{\mathbb{K}}[x, y]$.
A *Liouvillian first integral* of (S) is a first integral $\mathcal{F}$ of (S) of the form

$$\mathcal{F}(x,y) = \int R(x,y)B(x,y)dx - R(x,y)A(x,y)dy$$

where $R(x,y) = \exp \int G(x,y)dx + F(x,y)dy$ (called *the integrating factor*), F, G belong to $\overline{\mathbb{K}}(x,y)$, $G(x,y)dx + F(x,y)dy$ and $R(x,y)B(x,y)dx - R(x,y)A(x,y)dy$ are closed.
A *Riccati first integral* of (S) is a first integral of the form $\mathcal{F}_1/\mathcal{F}_2$ where $\mathcal{F}_1, \mathcal{F}_2$ are two independent solutions over $\overline{\mathbb{K}(x)}$ of a second order differential equation

$$(EqR) \quad \partial_y^2 \mathcal{F}_i + G(x,y)\partial_y \mathcal{F}_i + F(x,y)\mathcal{F}_i = 0$$

with $F, G \in \overline{\mathbb{K}}(x,y)$.

The classical result of the equivalence of the two representations of a Darbouxian first integral is proved in [Pic02, Chr99, DDdM02b], [Rup86, Satz 2], and in [Sch00, Lemma 2 p. 205]. Singer [Sin92] proves that a vector field admitting a first integral built by successive integrations, exponentiations and algebraic extensions of $\mathbb{K}(x,y)$ (so a Liouvillian function), also admits a Liouvillian first integral of the form given in Definition 1. Similarly, we will prove in Proposition 14 that a vector field admitting a first integral built by successive integrations, exponentiations, algebraic and Riccati extensions of $\mathbb{K}(x,y)$ (see Definition 13 in Section 1), also admits a Riccati first integral of the form given by Definition 1.

A vector field admitting a first integral built by successive integrations and algebraic extensions of $\mathbb{K}(x,y)$ does not always admits a Darbouxian first integral of the form given in Definition 1. This is due to the possible appearance of algebraic extensions in the 1-form $G(x,y)dx + F(x,y)dy$. However, as we will prove in Proposition 14, such vector field then admits what we call a k-Darbouxian first integral

Definition 2. A *k-Darbouxian first integral* of (S) is a first integral $\mathcal{F}$ of (S) of the form

$$\mathcal{F}(x,y) = \int G(x,y)dx + F(x,y)dy$$

where $k \in \mathbb{N}^*$, $F^k, G^k \in \overline{\mathbb{K}}(x,y)$ and $G(x,y)dx + F(x,y)dy$ closed.

Putting $k = 1$ recovers the classical Darbouxian first integrals.
Using that $\mathcal{F}$ is a first integral and so $D_0(\mathcal{F}) = 0$, we have moreover

$$\frac{G(x,y)}{B(x,y)} = -\frac{F(x,y)}{A(x,y)},$$

and this defines a hyperexponential function $R(x,y)$. Now writing $\mathcal{F}(x,y) = \int R(x,y)B(x,y)dx - R(x,y)A(x,y)dy$, we recognize the form of a Liouvillian first integral. So k-Darbouxian functions define an intermediary class between Darbouxian and Liouvillian functions.

We will not consider elementary first integrals. In [PS83], Prelle and Singer have proved that the study of elementary first integral can be reduced to the study of Liouvillian first integral with an algebraic integrating factor. This meets our Definition of k-Darbouxian first integral. However, elementary first integrals require the additional condition that the 1-form can be integrated in elementary terms, and such integration problems will not be considered in this article.

Rational, k-Darbouxian and Liouvillian first integrals are particular cases of a Riccati first integrals, by simply taking $\mathcal{F}_1$ as the first integral and $\mathcal{F}_2 = 1$. Indeed, a rational, k-Darbouxian or Liouvillian first integral always satisfies a second order differential equation in y. In equation (EqR), it is always possible to multiply $\mathcal{F}_1, \mathcal{F}_2$ by a same hyperexponential function, leaving unchanged the quotient. This allows to force the Wronskian to 1, which allows us to put $G = 0$.

This suggests to represent rational, Darbouxian and Liouvillian first integrals by a differential equation in y of which they are solution.

- A rational first integral is a first integral solution of

$$\text{(Rat)} \qquad \mathcal{F} - F(x,y) = 0 \qquad F \in \overline{\mathbb{K}}(x,y) \setminus \overline{\mathbb{K}}.$$

- A k-Darbouxian first integral is a first integral solution of

$$\text{(D)} \qquad \partial_y \mathcal{F} - F(x,y) = 0, \qquad F^k \in \overline{\mathbb{K}}(x,y) \setminus \{0\}.$$

- A Liouvillian first integral is a first integral solution of

$$\text{(L)} \qquad \partial_y^2 \mathcal{F} - F(x,y)\partial_y \mathcal{F} = 0 \qquad F \in \overline{\mathbb{K}}(x,y).$$

- A Riccati first integral is a first integral quotient of two independent solutions over $\overline{\mathbb{K}(x)}$ of

$$\text{(Ric)} \qquad \partial_y^2 \mathcal{F} - F(x,y)\mathcal{F} = 0, \qquad F \in \overline{\mathbb{K}}(x,y).$$

These four equations will be the four canonical equations representing respectively each type of first integral. Once one of the above equation is found, it is possible to recover the first integral by single variable integration and linear differential equation solving.

Each case is included in the next one, leading to a ranking on the classes of first integrals

$$\text{Rational} < k\text{-Darbouxian} < \text{Liouvillian} < \text{Riccati}$$

Each type of equation can be represented by a single rational fraction, thus also giving us a notion of "size".

Definition 3. The degree of a rational, Darbouxian, Liouvillian, Riccati first integral is respectively the maximum of the degree of numerator and denominator of F (or F^k in the k-Darbouxian case) in the four above equations.

Our algorithm will give an output with coefficients in $\mathbb{K}$. We will see that we can always suppose F with coefficients in $\mathbb{K}$. This does not change the degree of F, see Corollaries 30, 35, 40. The main theorem of the article is the following:

Theorem 4. *Let d be the maximum of $\deg(A)$ and $\deg(B)$. The problem of finding symbolic (rational, k-Darbouxian, Liouvillian, Riccati) first integrals with degree smaller than N can be solved in a probabilistic way with $\tilde{\mathcal{O}}(N^{\omega+1}+d^2N^2)$ arithmetic operations in $\mathbb{K}$, plus the factorization of a univariate polynomial with degree at most N.*
More precisely, there exists an algorithm with inputs A, B, $k \in \mathbb{N}^*$, a bound N, and parametrized by initial conditions $z \in \mathbb{K}^3$ such that the possible outputs are:

- *a differential equation of one of the forms* (Rat), (D), (L), (Ric) *leading to a symbolic first integral,*
- *"None" meaning that there exists no symbolic first integral with degree smaller than N,*
- *"I don't know".*

Furthermore, if z avoids the roots of a non-zero polynomial with degree $\mathcal{O}(N^4)$ then the algorithm returns an equation leading to a symbolic first integral or "None". Moreover, if (S) admits a symbolic first integral with degree smaller than N then the output is a differential equation with minimal degree.

The parameter k is necessary for k-Darbouxian first integrals. An equation admitting a $k \geq 2$-Darbouxian first integral also admits a Liouvillian first integral. So for the (default) input $k = 1$, we detect Darbouxian first integrals, but the possible $k \geq 2$-Darbouxian first integral could stay unnoticed or seen as a Liouvillian first integral. The reduction of such Liouvillian first integral to a k-Darbouxian first integral comes down to an integration problem, i.e. testing if the integrating factor is an algebraic function, which will not be considered in this article.

As we use the dense representation of polynomials and $\deg(F) \leq N$, the size of the output of our algorithm is in $\mathcal{O}(N^2)$. Thus our algorithm has a sub-quadratic complexity if we use linear algebra algorithms with $\omega < 3$.

As the last possible output can only appear when z is a root of a non-zero polynomial with degree $\mathcal{O}(N^4)$ and as we are considering fields in characteristic zero, we can say that for almost all z the algorithm detects symbolic first integrals.

Repeating the probabilistic algorithm in order to avoid bad values for z provides a deterministic algorithm with a polynomial time complexity:

Corollary 5. *The probabilistic algorithm can be turned into a deterministic one. The deterministic algorithm uses at most $\tilde{\mathcal{O}}(d^2N^{\omega+9}+d^4N^{10})$ arithmetic operations in $\mathbb{K}$, plus the factorization of a univariate polynomial with degree at most N.*

In practice the complexity of the deterministic algorithm is better, see Section 7.

The algorithm can even sometimes return equations of degree higher than N. This situation can appear for example when we are looking for a Darbouxian first integral with degree smaller than N and there exists a rational first integral of degree $2N$. We give such examples in the Section 7.
In these kinds of situations the degree of the first integral is bigger than N. This means that the algorithm gives a minimal solution in terms of the degree of the first integral and not in terms of its class. However, if the algorithm returns "None" then the algorithm ensures that the equation (S) does not admit a first integral in a lower class of degree $\leq N$.

Strategy description and theoretical contributions. In this paper we generalize the approach given in [BCCW16] for computing rational first integral. The main idea was to compute a solution $y(x_0, y_0; x)$ as a power series in x with coefficients in $\mathbb{K}(x_0, y_0)$ of

$$(E) \ : \quad \partial_x y(x_0, y_0; x) = \frac{B(x, y(x_0, y_0; x))}{A(x, y(x_0, y_0; x))}, \text{ and } y(x_0, y_0; x_0) = y_0,$$

and then find a polynomial $\mathcal{F}$ vanishing at this power series solution. If the power series order is large enough, it is sufficient for recovering $\mathcal{F}$. The determination of $\mathcal{F}$ indeed comes down to a linear algebra problem. Furthermore, the polynomial $\mathcal{F}$ as the following form: $P(x, y)Q(x_0, y_0) - Q(x, y)P(x_0, y_0)$ where P/Q is a rational first integral. Thus the computation of $\mathcal{F}$ gives a rational first integral.
In [BCCW16], in order to avoid computations in $\mathbb{K}(x_0, y_0)$, two solutions with random intial conditions $x_0^\star, y_0^\star$ are used and give a probabilistic and then a deterministic algorithm.

In this article, the new ingredient is the following: we consider derivatives of the flow $y(x_0, y_0; x)$ relatively to y_0. We set

$$\bar{y}(x) = \partial_{y_0} y(x_0, y_0; x), \quad \bar{\bar{y}}(x) = \partial_{y_0}^2 y(x_0, y_0; x), \quad \bar{\bar{\bar{y}}}(x) = \partial_{y_0}^3 y(x_0, y_0; x).$$

With a direct computation we remark that the functions $y(x), \bar{y}(x), \bar{\bar{y}}(x), \bar{\bar{\bar{y}}}(x)$ are solutions of some differential systems (S'):

$$(S'_3) \begin{cases} (S'_2) \begin{cases} (S'_1) \begin{cases} (S'_0) \quad \partial_x y = \dfrac{B}{A} \\[2ex] \partial_x \bar{y} = \bar{y} \partial_y \left(\dfrac{B}{A}\right) \end{cases} \\[4ex] \partial_x \bar{\bar{y}} = \bar{\bar{y}} \partial_y \left(\dfrac{B}{A}\right) + \bar{y}^2 \partial_y^2 \left(\dfrac{B}{A}\right) \end{cases} \\[4ex] \partial_x \bar{\bar{\bar{y}}} = \bar{\bar{\bar{y}}} \partial_y \left(\dfrac{B}{A}\right) + 3\bar{\bar{y}}\bar{y} \partial_y^2 \left(\dfrac{B}{A}\right) + \bar{y}^3 \partial_y^3 \left(\dfrac{B}{A}\right) \end{cases}$$

The system (S') gives a method to compute the flow $y(x_0, y_0; x)$ and finitely many of its derivatives as series in x: we only have to solve (S') using the Newton method for initial condition $x = x_0, y = y_0, \bar{y} = 1, \bar{\bar{y}} = 0, \bar{\bar{\bar{y}}} = 0$.
As in [BCCW16], in order to get efficient algorithms in practice we will consider random intial conditions $x_0^\star, y_0^\star$. This leads to probabilistic algorithms and then to deterministic algorithms.

In the following we will sometimes omit the dependence relatively to x_0, y_0 in the notations. We will write $y(x)$ instead of $y(x_0, y_0, x)$. Furthermore, we will also denote sometimes $\bar{\bar{y}}$ (respectively $\bar{\bar{\bar{y}}}$) by $\bar{y}^{(2)}$ (respectively $\bar{y}^{(3)}$) in order to write statements and algorithms in a short and uniform way in terms of $\bar{y}^{(r)}$, where $r \in [[0; 3]]$.

Now, we introduce new variables $\bar{y}$, $\bar{\bar{y}}$ and $\bar{\bar{\bar{y}}}$ and we define a polynomial derivation D_r associated to the system (S'_r).

Definition 6. The system (S_1') is associated to the derivation D_1 in $\mathbb{K}[x,y,\bar{y}]$:

$$D_1 = A^2\partial_x + AB\partial_y + \bar{y}A^2\partial_y\left(\frac{B}{A}\right)\partial_{\bar{y}}.$$

The system (S_2') is associated to the derivation D_2 in $\mathbb{K}[x,y,\bar{y},\bar{\bar{y}}]$:

$$D_2 = A^3\partial_x + A^2B\partial_y + \bar{y}A^3\partial_y\left(\frac{B}{A}\right)\partial_{\bar{y}} + A^3\Big(\bar{\bar{y}}\partial_y\left(\frac{B}{A}\right) + \bar{y}^2\partial_y^2\left(\frac{B}{A}\right)\Big)\partial_{\bar{\bar{y}}}.$$

The system (S_3') is associated to the derivation D_3 in $\mathbb{K}[x,y,\bar{y},\bar{\bar{y}},\bar{\bar{\bar{y}}}]$:

$$\begin{aligned} D_3 \quad = \quad & A^4\partial_x + A^3B\partial_y + \bar{y}A^4\partial_y\left(\frac{B}{A}\right)\partial_{\bar{y}} + A^4\Big(\bar{\bar{y}}\partial_y\left(\frac{B}{A}\right) + \bar{y}^2\partial_y^2\left(\frac{B}{A}\right)\Big)\partial_{\bar{\bar{y}}} \\ & + A^4\Big(\bar{\bar{\bar{y}}}\partial_y\left(\frac{B}{A}\right) + 3\bar{\bar{y}}\bar{y}\partial_y^2\left(\frac{B}{A}\right) + \bar{y}^3\partial_y^3\left(\frac{B}{A}\right)\Big)\partial_{\bar{\bar{\bar{y}}}}. \end{aligned}$$

Let us now consider $\mathcal{F}$ a Darbouxian first integral of D_0 such that $\partial_y\mathcal{F} = F$. We have

$$\mathcal{F}\big(x, y(x)\big) = \mathcal{F}(x_0, y_0)$$

and thus the derivative relatively to y_0 of this equation gives:

$$\partial_y\mathcal{F}\big(x, y(x)\big)\bar{y}(x) = \partial_{y_0}\mathcal{F}(x_0, y_0)$$

with $\bar{y}_0 = 1$ since $y(x_0, y_0; x_0) = y_0$. Therefore if $\mathcal{F}$ is a Darbouxian first integral of D_0 with $\partial_y\mathcal{F} = F$ we get

$$F\big(x, y(x)\big)\bar{y}(x) = F(x_0, y_0),$$

where $F \in \mathbb{K}(x, y)$.

Now the computation of F comes down to solving this equation knowing $y(x), \bar{y}(x)$ as series. This situation is similar to the one studied for rational first integral. Let us remark moreover that the rational function $F(x,y)\bar{y} \in \mathbb{K}(x,y,\bar{y})$ is constant on $\big(x, y(x), \bar{y}(x)\big)$, where the initial condition is $y(x_0) = y_0$ and $\bar{y}(x_0) = 1$. In Section 1, we prove that $F(x,y)\bar{y}$ is a rational first integral for (S_1') and then the even more general result:

Proposition 7. *The system (S) admits a rational first integral associated to* (Rat) *if and only if $F(x,y)$ is a first integral of (S_0').*
The system (S) admits a Darbouxian first integral associated to (D) *if and only if $\bar{y}F(x,y)$ is a first integral of (S_1').*
The system (S) admits a Liouvillian first integral associated to (L) *if and only if $\bar{y}F(x,y) + \bar{\bar{y}}/\bar{y}$ is a first integral of (S_2').*
The system (S) admits a Riccati first integral associated to (Ric) *if and only if $4\bar{y}^2F(x,y) - 2\bar{\bar{\bar{y}}}/\bar{y} + 3\bar{\bar{y}}^2/\bar{y}^2$ is a first integral of (S_3').*

Then the computation of symbolic first integrals is reduced to the computation of a rational first integral with a given structure of a differential system (S_r') . The existence and the computation of these rationals first integrals can then be done thanks to generalized extactic curves. More precisely, this can be done with linear algebra only. For example, we get this kind of result, see Section 3:

Theorem 8 (Liouvillian extactic curve Theorem)**.**
Let $\tilde{\mathcal{E}}_{D_2}^N$ be the matrix

$$\tilde{\mathcal{E}}_{2,D_2}^N = \Big(D_2^k \big(x^i y^j \bar{y}^\alpha (\bar{\bar{y}})^\beta \big) \Big)$$

where $0 \leq k \leq 3(N+1)(N+2)/2$, $0 \leq i+j \leq N$, $(\alpha, \beta) \in \{(1,0); (2,0); (0,1)\}$. *We have* $\det \tilde{\mathcal{E}}_{D_2}^N \in \mathbb{K}[x, y, \bar{y}, \bar{\bar{y}}]$ *and we set* $\tilde{E}_{D_2}^N(x,y) = \det \tilde{\mathcal{E}}_{D_2}^N(x, y, 1, 0)$.

1. *If* $\tilde{E}_{D_2}^N(x,y) = 0$ *then the derivation* D_0 *has a Liouvillian first integral with degree smaller than* N *or a Darbouxian first integral with degree smaller than* $2N + 3d - 1$ *or a rational first integral with degree smaller than* $4N + 8d - 3$.
2. *If* D_0 *has a rational or a Darbouxian or a Liouvillian first integral with degree smaller than* N *then* $\tilde{E}_{D_2}^N(x,y) = 0$.

We call this kind of theorem an "extactic curve theorem". Indeed, the matrix $\tilde{\mathcal{E}}_{D_2}^N$ corresponds to the study of a high order of contact between a solution of the differential system (S) and a Liouvillian function. This generalizes the situation introduced by Pereira in [Per01] for rational first integrals.

The main step in our algorithms will be the computation of a non-trivial element in the kernel of a matrix $\tilde{\mathcal{E}}_{D_r}^N$. From such an element we will show that we can construct easily the rational function F appearing in equations (Rat), (D), (L), (Ric). Therefore, the computation of symbolic first integrals with bounded degree is reduced to a linear algebra problem.

Related results. The computation of symbolic first integrals can be credited to G. Darboux in 1878, see [Dar78]. In this paper Darboux introduced what we call nowadays *Darboux polynomials*. A polynomial $f \in \mathbb{K}[x, y]$ is a Darboux polynomial for D_0 means that f divides $D_0(f)$. Thus f is an invariant algebraic curve. Darboux has shown how to find a Darbouxian first integral thanks to a recombination of Darboux polynomials.
This approach has been generalized in order to compute elementary first integrals by Prelle and Singer in [PS83]. This method has been implemented and studied in [Man93, MM97].
In [Sin92], Singer has given a theoretical characterization of Liouvillian first integrals. This characterization is the main ingredient of the algorithm proposed by Duarte et al. in [ADDdM05, DDdM02a, DDdM02b].
The interested reader can also consult the following surveys [Sch93, Gor01, DLA06, Zha17] for more results about Darboux polynomials and first integrals.

Roughly speaking, all the previous algorithms proceed as follows: first compute Darboux polynomials with bounded degree and second recombine them in order to find a first integral.
These two steps correpond to two practical difficulties. The computation of Darboux polynomials with bounded degree can be performed in polynomial time, see [Chè11]. This method is based on the so-called extactic curve inroduced by Pereira in [Per01] and uses a number of binary operations that is polynomial in the bound N, the degree d and the logarithm of the height of A and B. Unfortunately, the arithmetic complexity of this computation is in $\mathcal{O}(d^{\omega+1} N^{4\omega+4})$, see [Chè11].

The recombination part can be solved with linear algebra if we are looking for Darbouxian first integrals. However, if we are looking for a Liouvillian first integral then the recombination step used in [ADDdM05, DDdM02a, DDdM02b] uses at least 2^d arithmetics operations. Indeed, this algorithm tries to solve a family of equation. Each equation of this family is constucted from a polynomial $\prod_i f_i^{e_i}$, where f_i is a Darboux polynomial and e_i is an unknown integer. A condition on the degree of the output leads to a condition on the degree of $\prod_i f_i^{e_i}$. With this approach if D_0 has d Darboux polynomials and the bound on the degree of $\prod_i f_i^{e_i}$ is bigger than d then we have to study at least 2^d situations.
In [ADDdM05, DDdM02a, DDdM02b], the authors compute a Darbouxian integrating factor $\mathcal{R} = e^{P/Q} \prod_i f_i^{c_i}$ in order to find a Liouvillian first integral. With our approach the integrating factor $\mathcal{R}$ is related to the equation (L) in the follwoing way:

$$\frac{\partial_y \mathcal{R}}{\mathcal{R}} = F - \frac{\partial_y A}{A}.$$

Thus our bound on the degree of F corresponds to a bound on the degree of the polynomials P, Q, f_i.

In [FG10], Ferragut and Giacomini have proposed a method to compute rational first integrals with bounded degree. This approach does not follow the previous strategy. The idea is to computed a bivariate polynomial annihilating $y(x_0, y_0; x)$ written as a power series solution of a first order equation. From this polynomial we can then deduce a rational first integral if it exists. Unfortunately, the precision needed on the power series to get a correct output was not explicitly given.
In [BCCW16], the authors have improved the Ferragut-Giacomini's method. They have given an explicit bound on the precision needed on the power series to get a rational first integral when it exists. Furthermore, the main step of this algorithm is reduced to linear algebra only. The complexity of the probabilistic algorithm is then in $\tilde{\mathcal{O}}(N^{2\omega})$. However, as remaked by G. Villard this complexity can be lowered to $\tilde{\mathcal{O}}(N^{\omega+1})$ with an application of Hermite-Padé approximation. This approach was just study as an heuristic in [BCCW16].

The algorithm proposed in this article is based on a generalization of the extactic curve and follows the idea used in [FG10] and [BCCW16]. We give then a uniform strategy with a uniform complexity to compute rational, Darbouxian, Liouvillian, and Riccati first integrals. Furthermore, we explain in Section 1 why our approach cannot be generalized to another class of functions.

Structure of the paper. In the first section of this article we prove Proposition 7, i.e. we show how the computation of a symbolic first integral can be reduced to the computation of a rational first integral of a differential system (S'_r). In the second section, we define and study extactic hypersurfaces. We give a precise statement for the following idea: if an hypersurface has a sufficiently big order of contact with a generic solution of a differential system then this order of contact is infinite. This result will be useful in our algorithm in order to compute a solution with a sufficient precision in order to construct a first integral. As a byproduct we show that the computation of a rational first integral of a derivation in $\mathbb{K}[x_1, \dots, x_n]$ can be reduced to a linear algebra problem. In Section 3, we define the Darbouxian, Liouvillian and the Riccati extactic curve. We prove that these curves allow us to characterize the existence of symbolic first integrals with bounded degree. In Section 4 we study the evaluation of the extactic curves. In particular we characterize non-generic solutions. In Section 5, we give and prove our algorithms based on the previous results. In Section 6 we study the complexity of our algorithms. At last, in Section 7 we give some examples.

Notations. $\mathbb{K}[x,y]_{\leq N}$: vector space of polynomials in x, y with coefficients in $\mathbb{K}$ of total degree less than N.
$div = \partial_x A + \partial_y B$

1. First integrals and differential invariants

1.1. **Representation of first integrals.** Let us first prove that equations (Rat), (D), (L), (Ric) used to represent first integrals allow to recover them. The next proposition explains then why (Rat), (D), (L), (Ric) are admissible outputs when we are looking for rational or k-Darbouxian or Liouvillian or Riccati first integrals.

Proposition 9.
A rational first integral is uniquely defined by equation (Rat).
A k-Darbouxian first integral is defined up to addition of a constant by equation (D).
A Liouvillian first integral is defined up to affine transformation by equation (L).
A Riccati first integral is defined up to homography by equation (Ric).

Proof. According to equation (Rat), a rational first integral is simply F, and so defined uniquely by equation (Rat).
A k-Darbouxian first integral $\mathcal{F}$ satisfies an equation (D). Indeed, with our definition we have $\partial_y \mathcal{F} = F$. We also know that it should satisfy the equation of first integrals. This gives

$$\partial_y \mathcal{F} = F(x,y), \qquad A(x,y)\partial_x \mathcal{F}(x,y) + B(x,y)\partial_y \mathcal{F}(x,y) = 0.$$

Thus we know the derivative of $\mathcal{F}$ with respect to x and y, and so (D) defines $\mathcal{F}$ up to an addition of a constant.
A Liouvillian first integral $\mathcal{F}$ satisfies an equation (L). Let us note

$$R(x,y) = \partial_y \mathcal{F}(x,y).$$

Equation (L) becomes

$$\frac{\partial_y R}{R} = F(x,y).$$

We now use the first integral equation (Eq), dividing it by A and differentiate in y, giving

$$\partial_x \partial_y \mathcal{F} + \partial_y \left(\frac{B}{A} \partial_y \mathcal{F} \right) = 0$$

$$\partial_x R + \partial_y \left(\frac{B}{A} R \right) = 0$$

$$\partial_x R + \partial_y \left(\frac{B}{A} \right) R + \frac{B}{A} F R = 0$$

$$\frac{\partial_x R}{R} = -\partial_y \left(\frac{B}{A} \right) - \frac{B}{A} F.$$

Therefore we know the logarithmic derivatives of R with respect to x and y, and thus we obtain R up to a multiplication by a constant. Then noting that

$$\partial_y \mathcal{F}(x,y) = R(x,y), \quad \partial_x \mathcal{F}(x,y) = -\frac{B}{A}(x,y)R(x,y)$$

we obtain $\mathcal{F}$ from R up to addition of a constant. Thus equation (L) defines $\mathcal{F}$ up to an affine transformation.

A Riccati first integral is a quotient of two solutions $\mathcal{F}_1, \mathcal{F}_2$ of equation (Ric) independent over $\overline{\mathbb{K}(x)}$. Knowing that the quotient is a first integral, we have moreover

$$D_0\left(\frac{\mathcal{F}_1}{\mathcal{F}_2}\right) = \frac{1}{\mathcal{F}_2^2}\left(D_0(\mathcal{F}_1)\mathcal{F}_2 - \mathcal{F}_1 D_0(\mathcal{F}_2)\right) = 0$$

and thus

$$\frac{D_0(\mathcal{F}_1)}{\mathcal{F}_1} = \frac{D_0(\mathcal{F}_2)}{\mathcal{F}_2}$$

Let us note $\Omega = D_0(\mathcal{F}_i)/\mathcal{F}_i$ (for $i=1$ or 2 as they are equal), the functions $\mathcal{F}_1, \mathcal{F}_2$ are solutions of the PDE system

$$\partial_y^2 \mathcal{F}_i - F(x,y)\mathcal{F}_i = 0, \quad D_0(\mathcal{F}_i) - \Omega(x,y)\mathcal{F}_i = 0. \tag{1.1}$$

Let us now consider a solution of this system. Due to the first equation, we can write it

$$C_1(x)\mathcal{F}_1(x,y) + C_2(x)\mathcal{F}_2(x,y).$$

Now injecting this in the second equation gives

$$C_1\big(D_0(\mathcal{F}_1) - \Omega\mathcal{F}_1\big) + C_2\big(D_0(\mathcal{F}_2) - \Omega\mathcal{F}_2\big) + A\partial_x C_1 \mathcal{F}_1 + A\partial_x C_2 \mathcal{F}_2 = 0.$$

Thus

$$\partial_x C_1(x)\mathcal{F}_1(x,y) + \partial_x C_2(x)\mathcal{F}_2(x,y) = 0.$$

As the functions $\mathcal{F}_1, \mathcal{F}_2$ are independent over the functions in x, we have $\partial_x C_1(x) = 0$, $\partial_x C_2(x) = 0$, and so C_1, C_2 are constants. Thus the dimension over $\mathbb{K}$ of the vector space of solutions of (1.1) is exactly 2. So the system (1.1) defines $\mathcal{F}_1, \mathcal{F}_2$ up to a change of basis. This change of basis acts on the quotient $\mathcal{F}_1/\mathcal{F}_2$ as a homography. □

The canonic equations of the output of our algorithm thus define the first integral up to a composition by a simple single variable function.

Below, we give a necessary and sufficient criterion for ensuring that an equation (Rat), (D), (L), (Ric) leads to a first integral.

Proposition 10.

- *Equation* (Rat) *leads to a rational first integral if and only if*
$$D_0(F) = 0,\ F \in \overline{\mathbb{K}}(x,y) \setminus \overline{\mathbb{K}}.$$
- *Equation* (D) *leads to a k-Darbouxian first integral if and only if*
$$D_0(F) = -AF\partial_y(B/A),\ F^k \in \overline{\mathbb{K}}(x,y) \setminus \{0\}.$$
- *Equation* (L) *leads to a Liouvillian first integral if and only if*
$$D_0(F) = -A\partial_y(B/A)F - A\partial_y^2(B/A),\ F \in \overline{\mathbb{K}}(x,y).$$

- *Equation* (Ric) *leads to a Riccati first integral if and only if*

$$D_0(F)=-2A\partial_y(B/A)F+\frac{1}{2}A\partial_y^3(B/A),\ F\in\overline{\mathbb{K}}(x,y).$$

Proof. In the rational case, this is simply the definition of first integrals.
In the k-Darbouxian case, we have

$$\partial_y\mathcal{F}=F(x,y),\quad D_0(\mathcal{F})=0.$$

So this is equivalent to

$$\partial_y\mathcal{F}=F(x,y),\quad \partial_x\mathcal{F}=-\frac{B}{A}F.$$

So a necessary and sufficient condition for a function $\mathcal{F}$ to exist (at least locally) is the closed form condition,

$$\begin{aligned}\partial_xF=-\partial_y\left(\frac{B}{A}F\right)\quad &\iff\quad \partial_xF=-\partial_y\Big(\frac{B}{A}\Big)F-\frac{B}{A}\partial_yF\\ &\iff\quad A\partial_xF+B\partial_yF=-A\partial_y\Big(\frac{B}{A}\Big)F\\ &\iff\quad D_0(F)=-A\partial_y\Big(\frac{B}{A}\Big)F\end{aligned}$$

which gives the condition of the proposition.

In the Liouvillian case, the first integral $\mathcal{F}$ has to solve the PDE system

$$(\star)\quad \partial_y^2\mathcal{F}-F\partial_y\mathcal{F}=0,\qquad (\star\star)\quad D_0(\mathcal{F})=0.$$

The derivative relatively to x of $(\star)$ gives:

$$(A)\quad \partial_x\partial_y^2\mathcal{F}-\partial_xF\partial_y\mathcal{F}-F\partial_x\partial_y\mathcal{F}=0.$$

The derivative relatively to y of $(\star\star)$ divided by A an then simplified thanks to $(\star)$ gives:

$$\partial_y\partial_x\mathcal{F}+\partial_y\Big(\frac{B}{A}\Big)\partial_y\mathcal{F}+\frac{B}{A}F\partial_y\mathcal{F}=0.$$

The derivative relatively to y of the previous equality gives:

$$(B)\quad \partial_y^2\partial_x\mathcal{F}+\partial_y^2\Big(\frac{B}{A}\Big)\partial_y\mathcal{F}+\partial_y\Big(\frac{B}{A}\Big)\partial_y^2\mathcal{F}+\partial_y\Big(\frac{B}{A}F\Big)\partial_y\mathcal{F}+\frac{B}{A}F\partial_y^2\mathcal{F}=0.$$

The difference $(A)-(B)$ simplified thanks to $(\star)$ gives:

$$-\partial_xF\partial_y\mathcal{F}-F\partial_x\partial_y\mathcal{F}-\partial_y^2\Big(\frac{B}{A}\Big)\partial_y\mathcal{F}-2\partial_y\Big(\frac{B}{A}\Big)F\partial_y\mathcal{F}-\frac{B}{A}\partial_yF\partial_y\mathcal{F}-\frac{B}{A}F\partial_y^2\mathcal{F}=0.$$

The equation $(\star\star)$ implies:

$$\begin{aligned}0\quad &=\quad -\partial_xF\partial_y\mathcal{F}-F\partial_y\Big(-\frac{B}{A}\partial_y\mathcal{F}\Big)-\partial_y^2\Big(\frac{B}{A}\Big)\partial_y\mathcal{F}-2\partial_y\Big(\frac{B}{A}\Big)F\partial_y\mathcal{F}-\frac{B}{A}\partial_yF\partial_y\mathcal{F}\\ &\qquad -\frac{B}{A}F\partial_y^2\mathcal{F}\\ &=\quad -\left(\partial_xF+\frac{B}{A}\partial_yF+\partial_y(B/A)F+\partial_y^2(B/A)\right)\partial_y\mathcal{F}.\end{aligned}$$

If $\partial_y \mathcal{F} = 0$ then $\mathcal{F}$ only depend on x. This is impossible as this would imply $A = 0$ or $\mathcal{F}$ constant. So the only possibility left is

$$\partial_x F + \frac{B}{A}\partial_y F + \partial_y(B/A)F + \partial_y^2(B/A) = 0$$

This is the condition of the proposition.

Conversely, we suppose that $D_0(F) = -A\partial_y(B/A)F - A\partial_y^2(B/A)$. We are going to prove that in this situation D_0 has a Liouvillian first integral with integrating factor $\mathfrak{R} = e^{\mathfrak{F}}$, where $\mathfrak{F}$ is the integral of a rational closed 1-form. We set

$$\Omega_1 = F - \frac{\partial_y A}{A}, \quad \Omega_2 = \frac{-div - B\Omega_1}{A}.$$

We are going to show that $\partial_x(\Omega_1) = \partial_y(\Omega_2)$. Then this implies that there exists a Darbouxian function $\mathfrak{F}$ such that $\Omega_1 = \partial_y \mathfrak{F}$ and $\Omega_2 = \partial_x \mathfrak{F}$. By construction we have $A\Omega_2 + B\Omega_1 = -div$, and so $B\mathfrak{R}dx - A\mathfrak{R}dy$ is closed, where $\mathfrak{R} = e^{\mathfrak{F}}$. Therefore $\mathfrak{R}$ is the integrating factor of a Liouvillian first integral, and we get the desired result. Thus, now we are going to prove that $\partial_x(\Omega_1) = \partial_y(\Omega_2)$.

$$\begin{aligned}
\partial_x(\Omega_1) - \partial_y(\Omega_2) &= \partial_x F - \partial_x\Big(\frac{\partial_y A}{A}\Big) + \partial_y\Big(\frac{div}{A}\Big) + \partial_y\Big(\frac{B}{A}F\Big) - \partial_y\Big(\frac{B}{A}\frac{\partial_y A}{A}\Big) \\
&= \partial_x F + \frac{\partial_y^2 B}{A} - \frac{\partial_y B \partial_y A}{A^2} + \partial_y\Big(\frac{B}{A}\Big)F + \frac{B}{A}\partial_y F - \partial_y\Big(\frac{B}{A}\Big)\frac{\partial_y A}{A} \\
&\quad -\frac{B}{A}\partial_y\Big(\frac{\partial_y A}{A}\Big) \\
&= \partial_x F + \Big(\frac{B}{A}\Big)\partial_y F + \partial_y\Big(\frac{B}{A}\Big)F \\
&\quad +\frac{\partial_y^2 B}{A} - \frac{\partial_y B \partial_y A}{A^2} - \partial_y\Big(\frac{B}{A}\Big)\frac{\partial_y A}{A} - \frac{B}{A}\partial_y\Big(\frac{\partial_y A}{A}\Big).
\end{aligned}$$

Now we remark that

$$\partial_y^2\Big(\frac{B}{A}\Big) = \frac{\partial_y^2 B}{A} - \frac{\partial_y B \partial_y A}{A^2} - \partial_y\Big(\frac{B}{A}\Big)\frac{\partial_y A}{A} - \frac{B}{A}\partial_y\Big(\frac{\partial_y A}{A}\Big).$$

Thus

$$\partial_x(\Omega_1) - \partial_y(\Omega_2) = \partial_x F + \Big(\frac{B}{A}\Big)\partial_y F + \partial_y\Big(\frac{B}{A}\Big)F + \partial_y^2\Big(\frac{B}{A}\Big).$$

By hypothesis the right hand side of this equation is equal to zero. This gives the desired result.

In the Riccati case, a first integral is a quotient of two functions $\mathcal{F}_1, \mathcal{F}_2$, common solutions of a PDE system of the form

$$(\lozenge) \quad \partial_y^2 \mathcal{F}_i - F\mathcal{F}_i = 0, \qquad (\lozenge\lozenge) \quad D_0(\mathcal{F}_i) - \Omega\mathcal{F}_i = 0.$$

The derivative relatively to x of $(\lozenge)$ and then simplified by $(\lozenge\lozenge)$ gives:

$$(C) \quad \partial_x\partial_y^2\mathcal{F}_i - \partial_x F.\mathcal{F}_i - F\Big(\frac{\Omega}{A}\mathcal{F}_i - \frac{B}{A}\partial_y\mathcal{F}_i\Big).$$

The derivative relatively to y of $(\lozenge\lozenge)$ divided by A and then simplified thanks to $(\lozenge)$ gives:

$$\partial_y\partial_x\mathcal{F}_i + \partial_y\Big(\frac{B}{A}\Big)\partial_y\mathcal{F}_i + \frac{B}{A}F\mathcal{F}_i - \partial_y\Big(\frac{\Omega}{A}\Big)\mathcal{F}_i - \frac{\Omega}{A}\partial_y\mathcal{F}_i = 0.$$

The derivative relatively to y of the previous equality and simplified thanks to $(\lozenge)$ gives:

$$(E) \qquad 0=\partial_y^2\partial_x\mathcal{F}_i+\partial_y^2\Big(\frac{B}{A}\Big)\partial_y\mathcal{F}_i+\partial_y\Big(\frac{B}{A}\Big)F\mathcal{F}_i+\partial_y\Big(\frac{B}{A}F\Big)\mathcal{F}_i+\frac{B}{A}F\partial_y\mathcal{F}_i-\partial_y^2\Big(\frac{\Omega}{A}\Big)\mathcal{F}_i$$
$$-\partial_y\Big(\frac{\Omega}{A}\Big)\partial_y\mathcal{F}_i-\partial_y\Big(\frac{\Omega}{A}\Big)\partial_y\mathcal{F}_i-\frac{\Omega}{A}F\mathcal{F}_i.$$

The difference $(C)-(E)$ gives:

$$\left(-\partial_xF-\partial_y\Big(\frac{B}{A}\Big)F-\partial_y\Big(\frac{B}{A}F\Big)+\partial_y^2\Big(\frac{\Omega}{A}\Big)\right)\mathcal{F}_i+\left(2\partial_y\Big(\frac{\Omega}{A}\Big)-\partial_y^2\Big(\frac{B}{A}\Big)\right)\partial_y\mathcal{F}_i=0.$$

The functions $\mathcal{F}_1,\mathcal{F}_2$ are independent.Thus the Wronskian of $\mathcal{F}_1,\mathcal{F}_2$ in y is not 0, which implies that the above linear form in $\mathcal{F}_i,\partial_y\mathcal{F}_i$ should be 0. We obtain the equalities

$$2\partial_y\left(\frac{\Omega}{A}\right)-\partial_y^2\left(\frac{B}{A}\right)=0,$$
$$-\partial_xF-\frac{B}{A}\partial_yF-2\partial_y\Big(\frac{B}{A}\Big)F+\partial_y^2\Big(\frac{\Omega}{A}\Big)=0.$$

Therefore, we get:

$$\partial_xF+\frac{B}{A}\partial_yF=-2\partial_y\Big(\frac{B}{A}\Big)F+\frac{1}{2}\partial_y^3\Big(\frac{B}{A}\Big).$$

This is the condition given by the proposition.

Conversely, let us prove that if this condition is satisfied, then (Ric) leads to a Riccati first integral. Let us choose

$$\Omega=\frac{1}{2}A\partial_y\left(\frac{B}{A}\right)$$

and prove that the system

$$(1.2) \qquad \partial_y^2\mathcal{F}-F\mathcal{F}=0, \quad D(\mathcal{F})-\Omega\mathcal{F}=0$$

has two independent solutions. Differentiating in x the first equation gives

$$\partial_y^2\partial_x\mathcal{F}-\partial_xF\mathcal{F}-F\partial_x\mathcal{F}=0.$$

Let us note $\mathcal{G}=\partial_x\mathcal{F}$, and so this equation rewrites

$$(\sharp) \quad \partial_y^2\mathcal{G}-F\mathcal{G}=\partial_xF\mathcal{F}.$$

So this is a linear differential equation with a non homogeneous term $\partial_xF(x,y)\mathcal{F}$. Let us now consider $\mathcal{F}_1,\mathcal{F}_2$ a basis of solutions of $\partial_y^2\mathcal{F}-F(x,y)\mathcal{F}=0$, i.e. two solutions independent over *the constant field of functions in* x.

We now want to solve equation $(\sharp)$ with $\mathcal{F}=\mathcal{F}_i,\ i=1,2$

$$(\sharp\sharp) \quad \partial_y^2\mathcal{G}-F\mathcal{G}=\partial_xF\mathcal{F}_i(x,y)$$

We already know a basis of solutions of the homogeneous part, and we guess as particular solution

$$\mathcal{G}=-\frac{B}{A}\partial_y\mathcal{F}_i+\frac{1}{2}\mathcal{F}_i\partial_y\left(\frac{B}{A}\right).$$

Indeed, thanks to the relation $\partial_y^2\mathcal{F}-F\mathcal{F}=0$, we get:

$$\partial_y\mathcal{G}=-\frac{1}{2}\partial_y\Big(\frac{B}{A}\Big)\partial_y\mathcal{F}_i-\frac{BF}{A}\mathcal{F}_i+\frac{1}{2}\mathcal{F}_i\partial_y^2\Big(\frac{B}{A}\Big).$$

This gives:

$$\begin{aligned}
\partial_y^2 \mathcal{G} &= -\frac{1}{2}\partial_y\Big(\frac{B}{A}\Big)F\mathcal{F}_i - \partial_y\Big(\frac{BF}{A}\Big)\mathcal{F}_i - \frac{BF}{A}\partial_y\mathcal{F}_i + \frac{1}{2}\mathcal{F}_i\partial_y^3\Big(\frac{B}{A}\Big)\\
&= F\mathcal{G} + \left(-F\partial_y\Big(\frac{B}{A}\Big) - \partial_y\Big(\frac{BF}{A}\Big) + \frac{1}{2}\partial_y^3\Big(\frac{B}{A}\Big)\right)\mathcal{F}_i\\
&= F\mathcal{G} + \left(-2\partial_y\Big(\frac{B}{A}\Big)F - \frac{B}{A}\partial_y F + \frac{1}{2}\partial_y^3\Big(\frac{B}{A}\Big)\right)\mathcal{F}_i
\end{aligned}$$

As by hypothesis, we have

$$-2\partial_y\Big(\frac{B}{A}\Big)F - \frac{B}{A}\partial_y F + \frac{1}{2}\partial_y^3\Big(\frac{B}{A}\Big) = \partial_x F,$$

this proves that $\mathcal{G}$ is a particular solution.

So the solutions of ($\sharp\sharp$) are respectively for $i = 1, 2$ of the form

$$\mathcal{G} = C_1\mathcal{F}_1 + C_2\mathcal{F}_2 - \frac{B}{A}\partial_y\mathcal{F}_1 + \frac{1}{2}\mathcal{F}_1\partial_y\left(\frac{B}{A}\right)$$

$$\mathcal{G} = C_3\mathcal{F}_1 + C_4\mathcal{F}_2 - \frac{B}{A}\partial_y\mathcal{F}_2 + \frac{1}{2}\mathcal{F}_2\partial_y\left(\frac{B}{A}\right)$$

where the C_i *depend on* x *only*. So we deduce that there exists C_1, C_2, C_3, C_4 functions of x only such that

$$\partial_x\mathcal{F}_1 = C_1\mathcal{F}_1 + C_2\mathcal{F}_2 - \frac{B}{A}\partial_y\mathcal{F}_1 + \frac{1}{2}\mathcal{F}_1\partial_y\left(\frac{B}{A}\right),$$

$$\partial_x\mathcal{F}_2 = C_3\mathcal{F}_1 + C_4\mathcal{F}_2 - \frac{B}{A}\partial_y\mathcal{F}_2 + \frac{1}{2}\mathcal{F}_2\partial_y\left(\frac{B}{A}\right).$$

We now search solutions of equations (1.2) of the form

$$E_1\mathcal{F}_1 + E_2\mathcal{F}_2$$

with E_1, E_2 functions of x only. Injecting it in equations (1.2), we obtain 0 for the first, and for the second

$$(E_1C_1 + E_2C_3 + \partial_x E_1)\mathcal{F}_1 + (E_1C_2 + E_2C_4 + \partial_x E_2)\mathcal{F}_2 = 0$$

As $\mathcal{F}_1, \mathcal{F}_2$ are independent over functions in x, this is equivalent to the system

$$\partial_x E_1 = -E_1C_1 - E_2C_3, \quad \partial_x E_2 = -E_1C_2 - E_2C_4.$$

This is a 2×2 linear system, and so admits two independent solutions. Then these two solutions E_1, E_2 give two independent solutions of equations (1.2) of the form $E_1\mathcal{F}_1 + E_2\mathcal{F}_2$. Their quotient is then a first integral. □

During the previous proof we have shown the following:

Corollary 11. *If D_0 has a Riccati first integral $\mathcal{F}_1/\mathcal{F}_2$ then we can suppose that*

$$D_0(\mathcal{F}_i) = \frac{1}{2}A\partial_y\Big(\frac{B}{A}\Big)\mathcal{F}_i.$$

1.2. **A Casale Theorem.** We have defined 4 type of first integrals. We are going to prove that there are no other types of first integrals with algebraic-differential properties. Recall that the flow is defined by

$$\partial_x y(x_0,y_0;x) = \frac{B(x,y(x_0,y_0;x))}{A(x,y(x_0,y_0;y))}, \text{ and } y(x_0,y_0;x_0) = y_0$$

and we are interested in $\partial_{y_0}^i y(x_0,y_0;x)$, $i = 0\dots3$. These functions belong to $\mathbb{K}(x_0,y_0)[[x-x_0]]$, and we note them $y(x_0,y_0;x), \bar{y}(x_0,y_0;x), \bar{\bar{y}}(x_0,y_0;x), \bar{\bar{\bar{y}}}(x_0,y_0;x)$ respectively. Sometimes we will not write the dependance on x_0,y_0. The $y,\bar{y},\bar{\bar{y}},\bar{\bar{\bar{y}}}$ are now seen as functions in x, solutions of some differential system (S'):

$$(S'_3)\left\{\begin{array}{l}(S'_2)\left\{\begin{array}{l}(S'_1)\left\{\begin{array}{l}(S'_0)\quad \partial_x y = \dfrac{B}{A}\\[2ex] \partial_x \bar{y} = \bar{y}\partial_y\left(\dfrac{B}{A}\right)\end{array}\right.\\[4ex] \partial_x\bar{\bar{y}} = \bar{\bar{y}}\partial_y\left(\dfrac{B}{A}\right) + \bar{y}^2\partial_y^2\left(\dfrac{B}{A}\right)\end{array}\right.\\[4ex] \partial_x\bar{\bar{\bar{y}}} = \bar{\bar{\bar{y}}}\partial_y\left(\dfrac{B}{A}\right) + 3\bar{\bar{y}}\bar{y}\partial_y^2\left(\dfrac{B}{A}\right) + \bar{y}^3\partial_y^3\left(\dfrac{B}{A}\right)\end{array}\right.$$

and by construction, their initial conditions are

$$y(x_0,y_0;x_0) = y_0, \bar{y}(x_0,y_0;x_0) = 1, \bar{\bar{y}}(x_0,y_0;x_0) = 0, \bar{\bar{\bar{y}}}(x_0,y_0;x) = 0.$$

The system (S') gives a method to compute the flow $y(x_0,y_0;x)$ and finitely many of its derivatives as series in x: we only have to solve (S') using the Newton method for initial condition $x = x_0, y = y_0, \bar{y} = 1, \bar{\bar{y}} = 0, \bar{\bar{\bar{y}}} = 0$.

If $\mathcal{F}$ is a first integral of D_0 then we have $\mathcal{F}\big(x,y(x_0,y_0;x)\big) = \mathcal{F}(x_0,y_0)$. As mentioned in the introduction, the derivation relatively to y_0 of this relation gives with our notations:

$$\partial_y\mathcal{F}\big(x,y(x)\big)\bar{y}(x) = \partial_{y_0}\mathcal{F}(x_0,y_0).$$

Therefore if $\mathcal{F}$ is a Darbouxian first integral, we have $\partial_y\mathcal{F} = F \in \overline{\mathbb{K}}(x,y)$ and then

$$F\big(x,y(x)\big)\bar{y}(x) = F(x_0,y_0).$$

Thus the rational function $F(x,y)\bar{y} \in \mathbb{K}(x,y,\bar{y})$ is constant on $\big(x,y(x),\bar{y}(x)\big)$, where the initial condition is $y(x_0) = y_0$ and $\bar{y}(x_0) = 1$. Below we prove that $F(x,y)\bar{y}$ is moreover a rational first integral for (S'_1).

In the same way we have

$$\partial_y^2\mathcal{F}\big(x,y(x_0,y_0;x)\big)\Big(\partial_{y_0}y(x_0,y_0;x)\Big)^2 + \partial_y\mathcal{F}\big(x,y(x_0,y_0;x)\big)\partial_{y_0}^2 y(x_0,y_0;x) = \partial_{y_0}^2\mathcal{F}(x_0,y_0).$$

If $\mathcal{F}$ is Liouvillian then $\partial_y^2\mathcal{F}/\partial_y\mathcal{F} = F$. We get with our notations:

$$F\big(x,y(x)\big)\bar{y}(x) + \frac{\bar{\bar{y}}(x)}{\bar{y}(x)} = F(x_0,y_0).$$

We are also going to prove that $F(x,y)\bar{y} + \bar{\bar{y}}/\bar{y}$ is a rational first integral of (S'_2).

At last, for a Riccati first integral similar computations give a rational expression in $x, y, \bar{y}, \dots, \bar{\bar{\bar{y}}}$ which happens to be a rational first integral for (S'_3).

The reason why we stop at $r = 3$ is the following.

Theorem 12 (Casale)**.** *If there exists a non-constant rational invariant of the form*

$$J\big(x, y(x_0, y_0; x), \bar{y}(x_0, y_0; x), \dots\big) = J\big(x_0, y(x_0, y_0; x_0), \bar{y}(x_0, y_0; x_0), \dots\big)$$

then there exists an invariant of one the forms

- $h\big(x_0, y(x_0, y_0; x_0)\big) = h\big(x, y(x_0, y_0; x)\big)$
- $h\big(x_0, y(x_0, y_0; x_0)\big) = h\big(x, y(x_0, y_0; x)\big)\bar{y}(x_0, y_0; x)^k$ *with* $k \in \mathbb{N}^*$
- $h\big(x_0, y(x_0, y_0; x_0)\big) = h\big(x, y(x_0, y_0; x)\big)\bar{y}(x_0, y_0; x) + \bar{\bar{y}}(x_0, y_0; x)/\bar{y}(x_0, y_0; x)$
- $h\big(x_0, y(x_0, y_0; x_0)\big) = h\big(x, y(x_0, y_0; x)\big)\bar{y}^2(x_0, y_0; x) - 3\bar{\bar{y}}^2(x_0, y_0; x)/\bar{y}^2(x_0, y_0; x) + 2\bar{\bar{\bar{y}}}(x_0, y_0; x)/\bar{y}(x_0, y_0; x)$

with h rational.

This result is Proposition 1.18 and Theorem 1.19 of Casale in [Cas06] applied to the map $y_0 \mapsto \varphi(x_0, y_0; x)$, and restricted to the case of rational invariants instead of meromorphic. Now Casale's invariants can be seen as first integrals of the systems (S'_r) and satisfy the equations $D_r(J) = 0$. We will associate for each class of first integral a Casale's invariant for the flow. This gives the following proposition stated in the introduction:

Proposition 7. *The system (S) admits a rational first integral associated to (Rat) if and only if $F(x, y)$ is a first integral of (S'_0), where $F \in \overline{\mathbb{K}}(x, y) \setminus \mathbb{K}$.*
The system (S) admits a k-Darbouxian first integral associated to (D) if and only if $\bar{y}F(x, y)$ is a first integral of (S'_1), where $F^k \in \overline{\mathbb{K}}(x, y) \setminus \{0\}$.
The system (S) admits a Liouvillian first integral associated to (L) if and only if $\bar{y}F(x, y) + \bar{\bar{y}}/\bar{y}$ is a first integral of (S'_2), where $F \in \overline{\mathbb{K}}(x, y)$.
The system (S) admits a Riccati first integral associated to (Ric) if and only if $4\bar{y}^2F(x, y) - 2\bar{\bar{\bar{y}}}/\bar{y} + 3\bar{\bar{y}}^2/\bar{y}^2$ is a first integral of (S'_3), where $F \in \overline{\mathbb{K}}(x, y)$.

Proof. By definition a rational first integral of (S'_0) is a rational function $F(x, y)$ such that $F(x, y(x_0, y_0; x)) = F(x_0, y_0)$. We deduce then

$$\begin{aligned} & \partial_x F\big(x, y(x_0, y_0; x)\big) + \partial_y F\big(x, y(x_0, y_0; x)\big)\frac{B}{A}\big(x, y(x_0, y_0; x)\big) = 0 \\ \Longleftrightarrow \quad & \partial_x F(x_0, y_0) + \partial_y F\big(x_0, y_0\big)\frac{B}{A}(x_0, y_0) = 0 \\ \Longleftrightarrow \quad & D_0(F) = 0. \end{aligned}$$

This gives the desired conclusion in the rational case.

In the k-Darbouxian case, we have $\bar{y}F(x,y)$ first integral of (S_1') if and only if $D_1\big(\bar{y}F\big)=0$. This gives:

$$\begin{aligned} D_1\big(\bar{y}F\big)=0 \quad &\Longleftrightarrow \quad \bar{y}\left(\partial_x F+\frac{B}{A}\partial_y F+\partial_y\Big(\frac{B}{A}\Big)F\right)=0\\ &\Longleftrightarrow \quad D_0(F)=-A\partial_y\Big(\frac{B}{A}\Big)F \end{aligned}$$

Then, Proposition 10 gives the desired conclusion.

In the Liouvillian case, we have $\bar{y}F(x,y)+\bar{\bar{y}}/\bar{y}$ first integral of (S_2') if and only if $D_2(\bar{y}F+\bar{\bar{y}}/\bar{y})=0$. This gives:

$$\begin{aligned} D_2(\bar{y}F+\bar{\bar{y}}/\bar{y})=0 \quad &\Longleftrightarrow \quad \bar{y}\left(\partial_x F+\frac{B}{A}\partial_y F+\partial_y\Big(\frac{B}{A}\Big)F+\partial_y^2\Big(\frac{B}{A}\Big)\right)=0\\ &\Longleftrightarrow \quad D_0(F)=-A\partial_y\Big(\frac{B}{A}\Big)F-A\partial_y^2\Big(\frac{B}{A}\Big) \end{aligned}$$

As before we get the desired conclusion thanks to Proposition 10.

In the Riccati case, we have $4\bar{y}^2F(x,y)-2\bar{\bar{\bar{y}}}/\bar{y}+3\bar{\bar{y}}^2/\bar{y}^2$ first integral of (S_3') if and only if $D_3(4\bar{y}^2F(x,y)-2\bar{\bar{\bar{y}}}/\bar{y}+3\bar{\bar{y}}^2/\bar{y}^2=0$. This gives:

$$\begin{aligned} &D_3(4\bar{y}^2F(x,y)-2\bar{\bar{\bar{y}}}/\bar{y}+3\bar{\bar{y}}^2/\bar{y}^2)=0\\ \Longleftrightarrow \quad &\bar{y}^2\left(\partial_x F+\frac{B}{A}\partial_y F+2A\partial_y\Big(\frac{B}{A}\Big)F-\frac{1}{2}A\partial_y^3\Big(\frac{B}{A}\Big)\right)=0\\ \Longleftrightarrow \quad &D_0(F)=-2A\partial_y\Big(\frac{B}{A}\Big)F+\frac{1}{2}A\partial_y^3\Big(\frac{B}{A}\Big) \end{aligned}$$

We conclude using Proposition 10. □

Definition 13. Let $(K;\partial_x,\partial_y)$ be a differential field. An algebraic extension $L\supset K$ is a differential field such that $L=K(f)$ with f algebraic over K. An exponential extension $L\supset K$ is a differential field such that $L=K(\exp f)$ with $f\in K$.
A primitive extension $L\supset K$ is a differential field such that $L=K(f)$ with

$$df=\partial_x f dx+\partial_y f dy$$

a 1-form with coefficients in K.
A Riccati extension $L\supset K$ is a differential field such that $L=K(f)$ with df a 1-form with coefficients in $K[f]_{\leq 2}$.

Proposition 14. *The system (S) admits a first integral in a field built by successive algebraic and primitive extensions over $\mathbb{K}(x,y)$ if and only if it admits a k-Darbouxian first integral.*
The system (S) admits a first integral in a field built by successive algebraic, exponential, primitive extensions, over $\mathbb{K}(x,y)$ if and only if it admits a Liouvillian first integral.
The system (S) admits a first integral in a field built by successive algebraic extensions, exponential, primitive and Riccati extensions over $\mathbb{K}(x,y)$ if and only if it admits a Riccati first integral.

Proof. If (S) admits a first integral built by successive algebraic and primitive extensions over $\mathbb{K}(x,y)$, then by Theorem 4.2, Theorem 1.19 and Proposition 1.18 of Casale [Cas06] there exists $k\in\mathbb{N}^*, F\in\overline{\mathbb{K}}(x,y)$ such that $\bar{y}^kF(x,y)$ is a first integral of (S_1'). By Proposition 7, then (S) admits a k-Darbouxian first integral. The converse is immediate as a k-Darbouxian first integral is the integral of an algebraic 1-form.

The Liouvillian case is Singer's result, and so only Riccati case is left. Again thanks to Theorem 4.2, Theorem 1.19 and Proposition 1.18 of Casale [Cas06], if (S) admits a first integral in a field built by successive algebraic exponential, primitive and Riccati extensions over $\mathbb{K}(x,y)$, then

$$4\bar{y}^2F(x,y)-2\bar{\bar{\bar{y}}}/\bar{y}+3\bar{\bar{y}}^2/\bar{y}^2$$

is a first integral of (S_3'). Using again Proposition 7, (S) admits a Riccati first integral.

For the converse, we have a first integral which is the quotient of two solutions $\mathcal{F}_1,\mathcal{F}_2$ of a linear second order differential equation in y. We also know thanks to Corollary 11 that $\mathcal{F}_1,\mathcal{F}_2$ satisfy the equation

$$D_0(\mathcal{F})=\frac{1}{2}A\partial_y\left(\frac{B}{A}\right)\mathcal{F},$$

and thus writing $\partial_y\mathcal{F}$ in function of $\partial_x\mathcal{F}$, we obtain another linear second order differential equation in x of the following kind: $\partial_x^2\mathcal{F}=R\partial_x\mathcal{F}+S\mathcal{F}$, where R,S belong to $\mathbb{K}(x,y)$.

Therefore, $f_1=\partial_x\mathcal{F}_i/\mathcal{F}_i$ is a solution of the following Riccati associated equation:

$$\partial_x f_1=Rf_1+S-f_1^2.$$

Furthermore, $f_2=\partial_y\mathcal{F}_i/\mathcal{F}_i$ is a solution of the Riccati equation: $\partial_y f_2=F-f_2^2$.

Now we also have

$$Af_1+Bf_2=\frac{1}{2}A\partial_y\left(\frac{B}{A}\right)$$

and thus

$$\begin{aligned}
\partial_y f_1 &= -\partial_y\left(\frac{B}{A}\right)f_2-\frac{B}{A}\partial_y f_2+\partial_y\left(\frac{1}{2}A\partial_y\left(\frac{B}{A}\right)\right)\\
&= -\partial_y\left(\frac{B}{A}\right)f_2-\frac{B}{A}(F-f_2^2)+\partial_y\left(\frac{1}{2}A\partial_y\left(\frac{B}{A}\right)\right)\\
&= -\partial_y\left(\frac{B}{A}\right)\left(\frac{1}{2}\frac{A}{B}\partial_y\left(\frac{B}{A}\right)-\frac{A}{B}f_1\right)-\frac{B}{A}\Big[F-\Big(\frac{1}{2}\frac{A}{B}\partial_y\left(\frac{B}{A}\right)-\frac{A}{B}f_1\Big)^2\Big]\\
&\quad+\partial_y\left[\frac{1}{2}A\partial_y\left(\frac{B}{A}\right)\right]\in\mathbb{K}(x,y)[f_1]_{\le 2}
\end{aligned}$$

Symmetrically, we also obtain that $\partial_x f_2\in\mathbb{K}(x,y)[f_2]_{\le 2}$. Then we can construct a (four successive) Riccati extension L_1 of $\mathbb{K}(x,y)$ containing

$$\partial_y\mathcal{F}_1/\mathcal{F}_1,\partial_y\mathcal{F}_2/\mathcal{F}_2,\partial_x\mathcal{F}_1/\mathcal{F}_1,\partial_x\mathcal{F}_2/\mathcal{F}_2.$$

Now taking a primitive extension and then an exponential extension for each $\mathcal{F}_i$, we obtain a field L_2 containing $\mathcal{F}_1,\mathcal{F}_2$, and thus the first integral $\mathcal{F}_1/\mathcal{F}_2$. $\square$

2. Extactic hypersurfaces

As already remarked in Proposition 7, the existence of a Darbouxian, or Liouvillian or Riccati first integral is equivalent to the existence of a rational first integral with a special structure for a derivation associated to the problem. Furthermore, in this situation we have new variables $\bar{y},\bar{\bar{y}},\bar{\bar{\bar{y}}}$. In the following we will need to study rational first integral for derivations with several variables $x, y, \bar{y}, \bar{\bar{y}}, \bar{\bar{\bar{y}}}$. To reduce the amount of notations, in this subsection we study rational first integrals for derivation with variables $x, y_1, \dots, y_n$. Our main tool will be the extactic curve. This curve has been discovered independently by Lagutinski and Pereira, see [Per01]. It allows to characterize the situation where a derivation has a rational first integral with bounded degree. Here, we define and prove the main property of this object for a derivation in $\mathbb{K}(x, y_1, \dots, y_n)$ and we will get extactic hypersurfaces.

We consider a derivation

$$D = f_0\partial_x + \sum_{j=1}^{n} f_j\partial_{y_j}, \text{ where } f_j \in \mathbb{K}[x, y_1, \dots, y_n]$$

with $f_0 \neq 0$ and we consider the associated differential system:

$$(S_n) \quad \partial_x y_j(x) = \frac{f_j\big(x, y_1(x), \dots, y_n(x)\big)}{f_0\big(x, y_1(x), \dots, y_n(x)\big)}, \text{ for } j = 1, \dots, n.$$

We want to characterize the existence of a rational first integral with degree smaller than N for this kind of differential system. The idea is to study the order of contact between a solution of (S_n) and a polynomial.

Definition 15. A parametrized curve $\big(x, y(x)\big)$ where $y(x) = \big(y_1(x) \dots, y_n(x)\big)$ and an implicit hypersurface $g(x, y_1, \dots, y_n) = 0$ have a contact of order ν at $(x_0, y_{1,0}, \dots, y_{n,0}) = \big(x_0, y_1(x_0), \dots, y_n(x_0)\big)$ when ν is the biggest integer such that:

$$g\big(x, y(x - x_0)\big) = 0 \mod (x - x_0)^{\nu - 1}.$$

When we consider a planar vector field, the idea to discover a rational first integral is to compute algebraic curves with a "high" order of contact with a generic solutions of (S_0'). Actually, here "high" means infinite. However we will see that if the order of contact is large enough (a bound will be given latter) then this order of contact will be infinite. We are going to use the same approach in the multivariate case.

In control theory this kind of idea is also classical. Risler, in [Ris96], and Gabrielov [Gab95] have shown that if the order of contact is big enough then this order of contact is infinite. More precisely, in [Gab95] the author shows that if $y(x)$ is a solution of a differential system with degree d and g is a polynomial with degree k such that $g(y(x)) \neq 0$ then the order of contact between $g = 0$ and $y(x)$ is smaller than $2^{2n+1}\sum_{j=1}^{n+1}[k + (j-1)(d-1)]^{2n+2}$.

In order to compute the order of contact between g and a solution

$$\big(x, y(x)\big) = \big(x, y_1(x), \dots, y_n(x)\big)$$

we have to compute the Taylor expansion of $g(x, y(x))$ at x_0. As

$$f_0\partial_x\Big(g\big(x, y(x)\big)\Big) = D(g)\big(x, y(x)\big),$$

and $f_0 \neq 0$ we deduce easily that

$$\partial_x^i g\big(x, y(x)\big) = 0, \text{ for } i = 1, \dots, l \iff D^i(g)(x_0, y_{1,0}, \dots, y_{n,0}) = 0, \text{ for } i = 1, \dots, l.$$

where $D^0(g) = g$ and $D^i(g) = D\big(D^{i-1}(g)\big)$.

The study of the order of contact at a generic point $(x_0, y_{1,0}, \dots, y_{n,0})$ leads us to consider the following map:

Definition 16. Let D be a derivation on $\mathbb{K}[x, y_1 \dots, y_n]$, V be a finite dimensional linear subspace of $\mathbb{K}[x, y_1 \dots, y_n]$ and x_0, $\underline{y}_0 = (y_{1,0}, \dots, y_{n,0})$ be new variables. We set $\mathbb{L} = \mathbb{K}(x_0, \underline{y}_0)$. We consider the linear $\mathbb{L}$-morphism:

$$\begin{array}{rcl} \mathcal{E}_D^V : \mathbb{L} \otimes_{\mathbb{K}} V & \longrightarrow & \mathbb{L}^l \\ g(x, y_1, \dots, y_n) & \longmapsto & \big(g(x_0, \underline{y}_0), D(g)(x_0, \underline{y}_0), D^2(g)(x_0, \underline{y}_0), \dots, D^{l-1}(g)(x_0, \underline{y}_0)\big) \end{array}$$

where $l = \dim_{\mathbb{K}} V$, $D^k(g) = D\big(D^{k-1}(g)\big)$ and D is by abuse of notation the extension of the derivation D to $\mathbb{L}[x, y_1, \dots, y_n]$, i.e.

$$D\Big(\sum_\alpha c_\alpha(x_0, \underline{y}_0) x^{\alpha_1} y^{\alpha_2}\Big) = \sum_\alpha c_\alpha(x_0, \underline{y}_0) D(x^{\alpha_1} y^{\alpha_2}).$$

The determinant of this linear map is denoted by $E_D^V(x_0, \underline{y}_0)$. Moreover, we note $E_D^N(x_0, \underline{y}_0) := E_D^{\mathbb{K}[x, y_1, \dots, y_n]_{\leq N}}(x_0, \underline{y}_0)$ and call this hypersurface the Nth extactic hypersurface.

If $\{g_1, \dots, g_l\}$ is a basis of V then the associated extactic hypersurface is given by

$$E_D^V(x_0, \underline{y}_0) = \begin{vmatrix} g_1(x_0, \underline{y}_0) & g_2(x_0, \underline{y}_0) & \dots & g_k(x_0, \underline{y}_0) \\ D(g_1)(x_0, \underline{y}_0) & D(g_2)(x_0, \underline{y}_0) & \dots & D(g_k)(x_0, \underline{y}_0) \\ \vdots & \vdots & \vdots & \vdots \\ D^{k-1}(g_1)(x_0, \underline{y}_0) & D^{k-1}(g_2)(x_0, \underline{y}_0) & \dots & D^{k-1}(g_k)(x_0, \underline{y}_0) \end{vmatrix}.$$

The extactic hypersurface is related to invariant algebraic hypersurfaces. We call these kinds of hypersurface Darboux polynomials.

Definition 17. A non constant polynomial $M \in \overline{\mathbb{K}}[x, y_1, \dots, y_n]$ is a *Darboux polynomial for D* if M divides $D(M)$ in $\overline{\mathbb{K}}[x, y_1, \dots, y_n]$. We call the polynomial $\Lambda = D(M)/M$ the *cofactor associated with the Darboux polynomial M*.

Proposition 18. *If $g \in V$ is a Darboux polynomial of a derivation D then $g(x_0, \underline{y}_0)$ is a factor of $E_D^V(x_0, \underline{y}_0)$.*

Proof. As g is a Darboux polynomial we have $D(g) = \Lambda.g$ where $\Lambda \in \mathbb{K}[x, y_1, \dots, y_n]$. Thus there exist polynomials Λ_j such that $D^j(g) = \Lambda_j.g$. Thus, we have

$$\mathcal{E}_D^V(g) = g(x_0, \underline{y}_0).\big(1, \Lambda(x_0, \underline{y}_0), \dots, \Lambda_l(x_0, \underline{y}_0)\big)$$

and $g(x_0, \underline{y}_0)$ is a factor of a column of matrix representation of $\mathcal{E}_D^V$. It follows: g is factor of $E_D^V(x_0, \underline{y}_0)$. $\square$

We remark that by definition the factors of $E_D^V(\underline{x}_0)$ which are not Darboux polynomials correspond to algebraic hypersurfaces with order of contact with a solution $y(x)$ bigger than $l = \dim_{\mathbb{K}} V$.

We also remark that the determinant $E_D^V(x_0, \underline{y}_0)$ corresponds to a Wronskian and we recall the following classical lemma, see [Bro05, Lemma 3.3.5]:

Lemma 19. *Let (F, D) be a differential field. Then $g_1, \dots, g_k \in F$ are linearly dependent over* $\ker D$ *if and only if $W(g_1, \dots, g_k) = 0$, where*

$$W(g_1,\dots,g_k) = \begin{vmatrix} g_1 & g_2 & \dots & g_k \\ D(g_1) & D(g_2) & \dots & D(g_k) \\ \vdots & \vdots & \vdots & \vdots \\ D^{k-1}(g_1) & D^{k-1}(g_2) & \dots & D^{k-1}(g_k) \end{vmatrix}$$

is the Wronskian of $g_1, \dots, g_n$ relatively to D.

This leads to the following proposition:

Proposition 20. *We have the following equivalence:*
There exists a $\mathbb{K}$ vector space V such that $E_D^V(\underline{x}) = 0$ if and only if D has a rational first integral.

Proof. We denote by $g_1, \dots, g_l$ a basis of V. This basis gives a basis of the $\mathbb{L}$ vector space $\mathbb{L} \otimes_{\mathbb{K}} V$.
Then $E_D^V(x_0, \underline{y}_0) = W(g_1(x_0, \underline{y}_0), \dots, g_l(x_0, \underline{y}_0))$.
By Lemma 19 applied with $F = \mathbb{L}$ and $\tilde{D} = f_0(x_0, \underline{y}_0)\partial_{x_0} + \sum_i f_i(x_0, \underline{y}_0)\partial_{y_{i,0}}$, we deduce:

$$E_D^V(x_0, \underline{y}_0) = 0 \iff g_1(x_0, \underline{y}_0), \dots, g_l(x_0, \underline{y}_0) \text{ are linearly dependent over } \ker \tilde{D}.$$

As $g_1(x_0, \underline{y}_0), \dots, g_l(x_0, \underline{y}_0)$ are linearly independent over $\mathbb{K}$, this means that $\ker \tilde{D} \neq \mathbb{K}$. Thus $\tilde{D}$ and then D has a rational first integral.

Conversely, if D has a rational first integral G_1/G_2 with degree N then we set $V = \mathbb{K}[x, y_1, \dots, y_n]_{\leq N}$ and we can consider a basis of V where $G_1 = g_1$ and $G_2 = g_2$. We have $g_1(x_0, \underline{y}_0)/g_2(x_0, \underline{y}_0) \in \ker \tilde{D}$ and

$$g_1(x_0, \underline{y}_0) - \frac{g_1(x_0, \underline{y}_0)}{g_2(x_0, \underline{y}_0)} g_2(x_0, \underline{y}_0) = 0.$$

Thus we have a non-trivial relation over $\ker \tilde{D}$ then

$$W(g_1(x_0, \underline{y}_0), \dots, g_l(x_0, \underline{y}_0)) = E_D^N(x_0, \underline{y}_0) = 0.$$

□

Remark 21. The previous proof shows that if D has a rational first integral with degree N then $E_D^N(x_0, \underline{y}_0) = 0$.

This kind of result is not new, see [Per01]. We have given here a proof in order to emphasize the relation between the extactic curve and the Wronskian. The following example shows however that it is possible to have $E_D^N(x_0, \underline{y}_0) = 0$ and no rational first integral with degree N.

Example 22. Consider the following derivation

$$D = x\partial_x + (3x - 2y_1)\partial_{y_1} - 3x^3\partial_{y_2}.$$

This derivation has two polynomial first integrals with degree 3:

$$P_1(x, y_1, y_2) = x^2 y_1 + y_2, \quad P_2(x, y_1, y_2) = x^3 + y_2.$$

For this derivation we have $E_D^2(x_0,\underline{y}_0)=0$, but D has no rational first integral with degree 2. Indeed, a direct computation with a computer algebra system shows that the only Darboux polynomials for this derivation with degree smaller than 2 are: x, y_1-x and their products.
Now, we explain why the second extactic curve is equal to zero. As P_1 and P_2 are first integrals we have: $\Delta P_i(x_0,\underline{y}_0,x,y(x))=0$, where $\Delta P_i(x_0,\underline{y}_0,x,y_1,y_2)=P_i(x,y_1,y_2)-P_i(x_0,y_{1,0},y_{2,0})$. Thus

$$\begin{aligned} P(x_0,\underline{y}_0;x,y_1,y_2) &= x.\Delta P_1(x_0,\underline{y}_0,x,y_1,y_2)-y_1.\Delta P_2(x_0,\underline{y}_0,x,y_1,y_2)\\ &= xy_2-y_1y_2-x(x_0^2y_{1,0}+y_{2,0})+y_1(x_0^3+y_{2,0}) \end{aligned}$$

has degree 2 in $\mathbb{K}(x_0,\underline{y}_0)[x,y_1,y_2]$ and satisfies

$$P(x_0,\underline{y}_0;x,y(x))=0.$$

Thus $P\in\ker\mathcal{E}_D^{\mathbb{K}[x,y_1,y_2]_{\le 2}}$ and $E_D^2(x_0,\underline{y}_0)=0$.

Now, we show that the computation of $\ker\mathcal{E}_D^N$ gives rational first integrals. More precisely, we are going to exhibit a structure for the element in $\ker\mathcal{E}_D^V$.

Proposition 23. *Let $\{b_i\}$ be a basis of $\mathbb{K}[x,y_1,\dots,y_n]$. Consider $g_1(x_0,\underline{y}_0,x,\underline{y}),\dots,g_r(x_0,\underline{y}_0,x,\underline{y})$ a basis of $\ker\mathcal{E}_D^V$ in reduced row echelon form. Then we can write each g_i in the following form:*

$$g_i(x_0,\underline{y}_0,x,\underline{y})=\sum_{j\in J_i}c_j(x_0,\underline{y}_0)b_j,$$

where J_i is a finite set, $c_j(x_0,\underline{y}_0)\in\ker\tilde{D}$ and $\tilde{D}=f_0(x_0,\underline{y}_0)\partial_{x_0}+\sum_i f_i(x_0,\underline{y}_0)\partial_{y_{i,0}}$. Furthermore, there exists j_0 such that $c_{j_0}(x_0,\underline{y}_0)\notin\mathbb{K}$.

This result says that the computation of a reduced row echelon basis of $\ker\mathcal{E}_D^V$ gives a non-trivial rational first integral: $c_{j_0}(x,\underline{y})$.

Proof. Consider $\{g_1,\dots,g_r\}$ a basis of $\ker\mathcal{E}_D^V$ in reduced row echelon form and we set $g_i(x_0,\underline{y}_0,x,\underline{y})=\sum_{j\in J_i}p_j(x_0,\underline{y}_0)b_j$.

As $g_i\in\ker\mathcal{E}_D^V$ we deduce that $W(b_j(x_0,\underline{y}_0);j\in J_i)=0$. By Lemma 19 we have $b_j(x_0,\underline{y}_0)$ are linearly related over $\ker\tilde{D}$, where $j\in J_i$.
Then there exists $c_j(x_0,\underline{y}_0)\in\ker\tilde{D}$ such that $\sum_{j\in J_i}c_j(x_0,\underline{y}_0)b_j(x_0,\underline{y}_0)=0$. As $\{b_j(x_0,\underline{y}_0)\,|\,j\in J_i\}$ is a family of linearly independent elements over $\mathbb{K}$, we deduce that there exists j_0 such that $c_{j_0}(x_0,\underline{y}_0)\notin\mathbb{K}$. Furthermore, we have:

$$\begin{aligned} 0 &= \tilde{D}\big(\sum_{j\in J_i}c_j(x_0,\underline{y}_0)b_j(x_0,\underline{y}_0)\big)\\ &= \sum_{j\in J_i}\tilde{D}(c_j)b_j(x_0,\underline{y}_0)+\sum_{j\in J_i}c_j(x_0,\underline{y}_0)\tilde{D}\big(b_j(x_0,\underline{y}_0)\big)\\ &= \sum_{j\in J_i}c_j(x_0,\underline{y}_0)\tilde{D}\big(b_j(x_0,\underline{y}_0)\big). \end{aligned}$$

In the same way, we get $\sum_{j\in J_i}c_j(x_0,\underline{y}_0)\tilde{D}^j\big(b_j(x_0,\underline{y}_0)\big)=0$.
It follows: $\sum_{j\in J_i}c_j(x_0,\underline{y}_0)b_i\in\ker\mathcal{E}_D^V$. This polynomial has the same support than the polynomial g_i and the basis $\{g_1,\dots,g_r\}$ is in reduced row echelon form, thus we get the desired result. $\square$

Now, we are going to give an explicit statement for "if the contact between an hypersurface and an orbit is big enough then the orbit is included in the hypersurface."

Theorem 24. *Let $y(x)$ be a solution of (S_n) satisfying $y(x_0) = \underline{y}_0$.*
If $P \in \mathbb{L} \otimes_{\mathbb{K}} V$ and $P(x_0, \underline{y}_0; x, y(x)) = 0 \mod (x - x_0)^l$ then $P(x_0, \underline{y}_0; x, y(x)) = 0$.

Proof. Consider $P(x_0, \underline{x}_0, x, \underline{y})$ such that $P(x_0, \underline{y}_0, x, y(x)) = 0 \mod (x - x_0)^l$. The Taylor expansion of P shows that $P(x_0, \underline{x}_0, x, \underline{y}) \in \ker \mathcal{E}_D^V$.
We can write $P(x_0, \underline{y}_0, x, \underline{y})$ in the following form:

$$P(x_0, \underline{y}_0, x, \underline{y}) = \sum_i \lambda_i(x_0, \underline{y}_0) g_i(x_0, \underline{y}_0, x, \underline{y}),$$

where $g_i(x_0, \underline{y}_0, x, \underline{y})$ satisfy Proposition 23.

We have:

$$\begin{aligned} g_i(x, y(x), x, \underline{y}) &= \sum_{j \in J_i} c_j(x, y(x)) b_j \\ &= \sum_{j \in J_i} c_j(x_0, \underline{y}_0) b_j, \text{ because } c_j(x_0, \underline{y}_0) \in \ker \tilde{D}, \\ &= g_i(x_0, \underline{y}_0, x, \underline{y}) \end{aligned}$$

Furthermore, $g_i(x_0, \underline{y}_0, x_0, \underline{y}_0) = 0$ because $g_i(x_0, \underline{y}_0, x, \underline{y}) \in \ker \mathcal{E}_D^V$.
Thus $g_i(x, y(x), x, y(x)) = 0$.
As $g_i(x, y(x), x, \underline{y}) = g_i(x_0, \underline{y}_0, x, \underline{y})$ we get

$$0 = g_i(x, y(x), x, y(x)) = g_i(x_0, \underline{y}_0, x, y(x)).$$

Then $P(x_0, \underline{y}_0, x, y(x)) = \sum_i \lambda_i(x_0, \underline{y}_0) g_i(x_0, \underline{y}_0, x, y(x)) = 0$. □

This result means that for a generic point if the order of contact with a polynomial of degree N is bigger than $\dim_{\mathbb{K}} \mathbb{K}[x, y_1, \dots, y_n]_{\leq N}$ then this order of contact is infinite.

3. Extactic curves

3.1. **Rational extactic curve.**

We here recover a classical result for the extactic curve in two variables. Let us note

$$\tilde{\mathcal{E}}_{D_0}^N = \mathcal{E}_{D_0}^{\mathbb{K}[x,y]_{\leq N}}, \quad \tilde{E}_{D_0}^N = E_{D_0}^{\mathbb{K}[x,y]_{\leq N}},$$

and call a rational function $F \in \bar{\mathbb{K}}(x, y)$ *indecomposable* when it cannot be written $f \circ g$, with $f \in \overline{\mathbb{K}}(T)$, $g \in \overline{\mathbb{K}}[x, y]$ and $\deg(f) \geq 2$.

Theorem 25 (Bivariate rational extactic curve theorem)**.**
The derivation D_0 has an indecomposable rational first integral with degree N if and only if $\tilde{E}_{D_0}^N = 0$ and $\tilde{E}_{D_0}^{N-1} \neq 0$. Moreover, this indecomposable first integral can always be assumed to have coefficients in $\mathbb{K}$.

This theorem says that the minimal degree of a rational first integral corresponds to the minimal index where the extactic curve vanishes. This theorem is not new, see e.g. [Per01, CLP07]. We have recalled it in order to have a self contained paper. Furthermore, we are going to generalize this result for the study of Darbouxian, Liouvillian and Riccati first integrals.

Proof. If D_0 has a rational first integral P/Q with degree N then by Remark 21 we have $\tilde{E}_{D_0}^{N}=0$.
Now, suppose that $\tilde{E}_{D_0}^{N-1}=0$ then Theorem 24 implies that all solutions of the differential system vanishes a polynomial with degree smaller than $N-1$. As we have supposed P/Q indecomposable then a corollary of the Bertini-Krull theorem see e.g. [BC11], implies that we can assume P and Q irreducible with degree N. Thus we get the desired contradiction.

If $\tilde{E}_{D_0}^{N}=0$ then by Proposition 20, D_0 has a rational first integral P/Q. We can assume that this first integral is indecomposable and thus have a minimal degree. As before, we can assume that P and Q are irreducible. Furthermore, Remark 21 implies that the degree of this first integral cannot be strictly smaller than N. Thus suppose that this degree is strictly bigger than N. Therefore, the curve $P=0$ corresponds to an irreducible orbit of the differential system, but Theorem 24 implies that all orbits are included in algebraic curves with degree smaller than N. This gives the desired contradiction. To conclude, the indecomposable first integral of degree N has a priori coefficients in $\bar{\mathbb{K}}$, and thus $\tilde{\mathcal{E}}_{D_0}^{N}$ has a non trivial kernel. However, as the coefficients of $\tilde{\mathcal{E}}_{D_0}^{N}$ are in $\mathbb{K}(x_0,y_0)$, this implies that $\tilde{\mathcal{E}}_{D_0}^{N}$ also as a non trivial element in its kernel with coefficients in $\mathbb{K}(x_0,y_0)$. And thus that the system admits a first integral of degree N with coefficients in $\mathbb{K}$ (and so thus also indecomposable). □

3.2. **Darbouxian extactic curve.**
In this subsection we are going to apply the result of Section 2 to the derivation D_1. Then in the following $\big(y(x),\bar{y}(x)\big)$ is a solution of (S_1') satisfying the initial condition $y(x_0)=y_0, \bar{y}(x_0)=\bar{y}_0$, where x_0, y_0 and $\bar{y}_0$ are variables.

Now, we are going to generalize Theorem 25 to the Darbouxian case.

Definition 26. We set $\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)=\mathcal{E}_{D_1}^{V_1}(x_0,y_0,1)$, where

$$V_1:=\mathbb{K}[x,y]_{\leq N}\oplus\mathbb{K}[x,y]_{\leq N}\bar{y}^k, \quad l_1=\dim(V_1).$$

The N-th k-Darbouxian extactic curve is defined by

$$\tilde{E}_{D_1}^{N,k}(x_0,y_0)=E_{D_1}^{V_1}(x_0,y_0,1).$$

Let us begin by two Lemmas about this Darbouxian extactic curve.

Lemma 27. *We have the following equivalence:*

$$\bar{y}^kP-Q\in\ker\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$$
$$\Updownarrow$$
$$\bar{y}^kP-\bar{y}_0^kQ\in\ker\mathcal{E}_{D_1}^{V_1}(x_0,y_0,\bar{y}_0).$$

This Lemma essentially says that evaluating $\bar{y}_0=1$ in the definition of the extactic curve does not loose much information.

Proof. We denote by $\psi(x)$ the solution of (S_1') such that $\psi(x_0)=y_0$, $\bar\psi(x_0)=1$. We consider the transformation

$$T(y,\bar y)=(y,\bar y_0\bar y).$$

We set

$$\big(\psi_T(x),\bar\psi_T(x)\big):=T(\psi(x),\bar\psi(x)\big)=\big(\psi(x),\bar y_0\bar\psi(x)\big).$$

Now, we are going to show that $\big(\psi_T(x),\bar\psi_T(x)\big)$ is a solution of (S_1') with initial conditions $\psi_T(x_0)=y_0$, $\bar\psi_T(x_0)=\bar y_0$. Indeed,

$$\partial_x\bar\psi_T(x)=\bar y_0\partial_x\bar\psi(x)=\bar y_0\bar\psi(x)\partial_y\Big(\frac{B}{A}\Big)\big(x,\psi(x)\big)=\bar\psi_T(x)\partial_y\Big(\frac{B}{A}\Big)\big(x,\psi_T(x)\big).$$

Now suppose that $\bar y^kP-Q\in\ker\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$ then

$$\bar\psi^k(x)P\big(x,\psi(x)\big)-Q\big(x,\psi(x)\big)=0 \mod (x-x_0)^{l_1}.$$

Thus

$$\bar y_0^k\bar\psi^k(x)P\big(x,\psi(x)\big)-\bar y_0^kQ\big(x,\psi(x)\big)=0 \mod (x-x_0)^{l_1}.$$

This gives

$$\bar\psi_T^k(x)P\big(x,\psi_T(x)\big)-\bar y_0^kQ\big(x,\psi_T(x)\big)=0 \mod (x-x_0)^{l_1}.$$

Therefore $\bar y^kP-\bar y_0^kQ\in\ker\mathcal{E}_{D_1}^{V_1}$.
The converse is straightforward. $\square$

Lemma 28. *Consider a non trivial solution $\bar y^kP-Q\in\ker\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$, then:*

- *If $P=0$ then $Q\in\ker\tilde{\mathcal{E}}_{D_0}^{N}$.*
- *If $Q=0$ then $P\in\ker\tilde{\mathcal{E}}_{D_0}^{N}$.*
- *If $PQ\neq 0$ and $Q\notin\ker\tilde{\mathcal{E}}_{D_0}^{N}$ then:*

$$\Big(D_0((P/Q)^{1/k})+A(P/Q)^{1/k}\partial_y(B/A)\Big)\big(x,y(x)\big)=0.$$

The two first cases are pathological ones, i.e. we compute the Darbouxian extactic curve but it appears that a rational first integral exists.

Proof. As $\bar y^kP-Q\in\ker\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$ we get using Lemma 27

$$\bar y_0^kQ(x,y)-\bar y^kP(x,y)\in\ker\mathcal{E}_{D_1}^{V_1}(x_0,y_0,\bar y_0).$$

Thus

$$\bar y_0^kQ(x,y(x))-\bar y(x)^kP(x,y(x))=0 \mod (x-x_0)^{l_1}.$$

By Theorem 24, we deduce that $\bar y_0^kQ(x,y(x))-\bar y(x)^kP(x,y(x))=0$.
If $P=0$ then we get $Q(x,y(x))=0 \mod (x-x_0)^{l_1}$, then $Q\in\ker\tilde{\mathcal{E}}_{D_0}^{N}$.
If $Q=0$ then we get $\bar y(x)^kP(x,y(x))=0 \mod (x-x_0)^{l_1}$. We have $\bar y(x)\neq 0$ as $\bar y(x_0)=\bar y_0$ and thus $P\in\ker\tilde{\mathcal{E}}_{D_0}^{N}$.

Now we suppose that $PQ\neq 0$, and $Q\notin\ker\tilde{\mathcal{E}}_{D_0}^{N}$. We have then $Q\big(x,y(x)\big)\neq 0$ and:

$$\bar y(x)^k\frac{P}{Q}\big(x,y(x)\big)=\bar y_0^k.$$

and thus

$$\bar y(x)\left(\frac{P}{Q}\right)^{1/k}\big(x,y(x)\big)=\xi\bar y_0,\quad \text{where } \xi \text{ is a } k-\text{root of unity.}$$

The derivation relatively to x of this relation and the fact that $\bar{y}(x)$ is a solution of (S_1') gives:

$$\bar{y}(x)\partial_y\Big(\frac{B}{A}\Big)(x,y(x))\left(\frac{P}{Q}\right)^{1/k}(x,y(x))+$$
$$\bar{y}(x)\left(\partial_x\Big(\Big(\frac{P}{Q}\Big)^{1/k}\Big)(x,y(x))+\partial_y\Big(\Big(\frac{P}{Q}\Big)^{1/k}\Big)(x,y(x))\frac{B}{A}(x,y(x))\right)=0$$

As $\bar{y}(x)\neq 0$ we get:

$$\left(A\left(\frac{P}{Q}\right)^{1/k}\partial_y\Big(\frac{B}{A}\Big)+D_0\Big(\Big(\frac{P}{Q}\Big)^{1/k}\Big)\right)(x,y(x))=0.$$

This gives the desired result. □

We now prove the main result about the Darbouxian extactic curve.

Theorem 29 (Darbouxian extactic curve theorem)**.**

1. *If $\tilde{E}_{D_1}^{N,k}(x_0,y_0)=0$ then the derivation D_0 has a k-Darbouxian first integral with degree smaller than N or a rational first integral with degree smaller than $2N+2d-1$. Moreover the defining equation of the first integral, (Rat) or (D), has coefficients in $\mathbb{K}$.*
2. *If D_0 has a rational or a k-Darbouxian first integral with degree smaller than N then $\tilde{E}_{D_1}^{N,k}(x_0,y_0)=0$.*

Proof of Theorem 29. First, consider a non trivial solution $\bar{y}P-Q\in\ker\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$. If $P=0$ then by Lemma 28 we get $Q\in\ker\tilde{\mathcal{E}}_{D_0}^{N}$. Theorem 25 implies that the derivation D_0 has a rational first integral with degree smaller than N with coefficients in $\mathbb{K}$.
If $Q=0$, we deduce in the same way that D_0 has a rational first integral with degree smaller than N with coefficients in $\mathbb{K}$.

Now we suppose that $PQ\neq 0$.
If $Q\in\ker\mathcal{E}_{D_0}^{N}$, then by Theorem 25, D_0 has a rational first integral with degree smaller than N with coefficients in $\mathbb{K}$.
Now, we suppose that Q do not belong to $\ker\mathcal{E}_{D_0}^{N}$. By Lemma 28, we have

$$\Big(D_0((P/Q)^{1/k})+A(P/Q)^{1/k}\partial_y(B/A)\Big)(x,y(x))=0.$$

We introduce the polynomial

$$G:=APQ(P/Q)^{-1/k}\big(D_0((P/Q)^{1/k})+A(P/Q)^{1/k}\partial_y(B/A)\big)$$

we have $G\big(x,y(x)\big)=0$. Therefore, if $G=0$ then Proposition 10 gives the existence of a k-Darbouxian first integral. Moreover, the equation of type (D) giving the existence of a Darbouxian first integral has coefficient in $\mathbb{K}$ since $P,Q\in\mathbb{K}(x_0,y_0)[x,y]$.
Now, we consider the situation where $G\neq 0$ then G is a non-zero polynomial with degree smaller than $2N+2d-1$ such that $G\in\ker\tilde{\mathcal{E}}_{D_0}^{2N+2d-1}$. By Theorem 25, D_0 has a rational first integral with degree smaller than $2N+2d-1$ with coefficients in $\mathbb{K}$. This concludes the first part of the proof.

Now, we suppose that D_0 has a rational or a k-Darbouxian first integral with degree smaller than N.
If D_0 has a k-Darbouxian first integral $\mathcal{F}$ with degree smaller than N then we have

$\partial_y \mathcal{F} = (P/Q)^{1/k}$, with $P, Q \in \bar{\mathbb{K}}[x,y]$, and $\deg P, \deg Q \leq N$. Proposition 7 implies that $\bar{y}^k P/Q$ is a rational first integral of (S_1'). This gives

$$\bar{y}(x)^k \frac{P\big(x, y(x)\big)}{Q\big(x, y(x)\big)} = c,$$

where $c \in \bar{\mathbb{K}}(x_0, y_0)$, putting $\bar{y}_0 = 1$. Thus $\bar{y}^k P(x,y) - cQ(x,y) \in \bar{\mathbb{K}} \otimes_{\mathbb{K}} \ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0, y_0)$. Now as $A, B \in \mathbb{K}[x,y]$, the coefficients of $\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0, y_0)$ are in $\mathbb{K}(x_0, y_0)$, and so we deduce the existence of $\tilde{P}, \tilde{Q} \in \mathbb{K}(x_0, y_0)[x,y]$ with degree N such that $\bar{y}^k \tilde{P} - \tilde{Q}$ belong to $\ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0, y_0)$. Thus $\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0, y_0) = 0$ and we get the desired conclusion. If D_0 has a rational first integral P/Q with degree smaller than N, then we can suppose $P/Q \in \mathbb{K}(x,y)$ thanks to Theorem 25 and

$$P(x,y)Q(x_0, y_0) - Q(x,y)P(x_0, y_0) \in \ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0, y_0).$$

Thus $\tilde{E}_{D_1}^{N,k}(x_0, y_0) = 0$. □

As a corollary, we obtain the following results about the coefficient field extensions of the possible first integrals.

Corollary 30. *Suppose that D_0 has a k-Darbouxian first integral and no rational first integral. We have:*
There exists $F \in \mathbb{K}(x,y)$ with $\deg F \leq N$ such that equation (D) gives a k-Darbouxian first integral if and only if there exists $\tilde{F} \in \overline{\mathbb{K}}(x,y)$ with $\deg \tilde{F} \leq N$ such that equation (D) with $\tilde{F}$ gives a k-Darbouxian first integral.

Proof. Suppose that there exists $\tilde{F} \in \overline{\mathbb{K}}(x,y)$ with $\deg \tilde{F} = N$ such that equation (D) gives a k-Darbouxian first integral. Then by second part of Theorem 29, we have $\tilde{E}_{D_1}^{N,k}(x_0, y_0) = 0$. Applying now the first part, we obtain either a rational first integral (forbidden by our assumption) or a k-Darbouxian first integral given by an equation (D) with a rational function with degree smaller than N and with coefficients in $\mathbb{K}$. □

3.3. Liouvillian Extactic curve.

In this subsection we are going to apply the result of Section 2 to the derivation D_2. Then in the following $\big(y(x), \bar{y}(x), \bar{\bar{y}}(x)\big)$ is a solution of (S_2') satisfying the initial condition $y(x_0) = y_0$, $\bar{y}(x_0) = \bar{y}_0$, $\bar{\bar{y}}(x_0) = \bar{\bar{y}}_0$ where x_0, y_0, $\bar{y}_0$ and $\bar{\bar{y}}_0$ are variables.

Now, we are going to generalize Theorem 25 to the Liouvillian case.

Definition 31. We set $\tilde{\mathcal{E}}_{D_2}^{N}(x_0, y_0) = \mathcal{E}_{D_2}^{V_2}(x_0, y_0, 1, 0)$ where

$$V_2 := \mathbb{K}[x,y]_{\leq N}\bar{y}^2 \oplus \mathbb{K}[x,y]_{\leq N}\bar{\bar{y}} \oplus \mathbb{K}[x,y]_{\leq N}\bar{y}, \quad l_2 = \dim(V_2).$$

The N-th Liouvillian extactic curve is

$$\tilde{E}_{D_2}^{N}(x_0, y_0) = E_{D_2}^{V_2}(x_0, y_0, 1, 0).$$

We begin by proving the following two Lemmas.

Lemma 32. *We have the following equivalence:*

$$P(x,y)\bar{y}^2 + Q(x,y)\bar{\bar{y}} + R(x,y)\bar{y} \in \ker \tilde{\mathcal{E}}_{D_2}^{N}(x_0, y_0)$$

$$\Updownarrow$$

$$P(x,y)\bar{y}^2 + Q(x,y)\bar{\bar{y}} + \Big(\bar{y}_0 R(x,y) - \frac{\bar{\bar{y}}_0}{\bar{y}_0} Q(x,y)\Big)\bar{y} \in \ker \mathcal{E}_{D_2}^{V_2}(x_0, y_0, \bar{y}_0, \bar{\bar{y}}_0)$$

Proof. We denote by $\psi(x)$ the solution of (S_2') such that $\psi(x_0)=y_0$, $\bar{\psi}(x_0)=1$, $\bar{\bar{\psi}}(x_0)=0$.
We consider the transformation

$$T(y,\bar{y},\bar{\bar{y}})=(y,\bar{y}_0\bar{y},\bar{y}_0^2\bar{\bar{y}}+\bar{\bar{y}}_0\bar{y}).$$

We set

$$\big(\psi_T(x),\bar{\psi}_T(x),\bar{\bar{\psi}}_T(x)\big):=T\big(\psi(x),\bar{\psi}(x),\bar{\bar{\psi}}(x)\big)=\big(\psi(x),\bar{y}_0\bar{\psi}(x),\bar{y_0}^2\bar{\bar{\psi}}(x)+\bar{\bar{y}}_0\bar{\psi}(x)\big).$$

Now, we are going to show that $\big(\psi_T(x),\bar{\psi}_T(x),\bar{\bar{\psi}}_T(x)\big)$ is a solution of (S_2') with initial conditions $\psi_T(x_0)=y_0$, $\bar{\psi}_T(x_0)=\bar{y}_0$, $\bar{\bar{\psi}}_T(x_0)=\bar{\bar{y}}_0$. Indeed we have:

$$\partial_x\bar{\psi}_T(x)=\bar{y}_0\partial_x\bar{\psi}(x)=\bar{y}_0\bar{\psi}(x)\partial_y\Big(\frac{B}{A}\Big)\big(x,\psi(x)\big)=\bar{\psi}_T(x)\partial_y\Big(\frac{B}{A}\Big)\big(x,\psi_T(x)\big).$$

Furthermore,

$$\begin{aligned}\partial_x\bar{\bar{\psi}}_T(x) &= \bar{y}_0^2\partial_x\bar{\bar{\psi}}(x)+\bar{\bar{y}}_0\partial_x\bar{\psi}(x)\\ &= \bar{y}_0^2\bar{\bar{\psi}}(x)\partial_y\Big(\frac{B}{A}\Big)\big(x,\psi(x)\big)+\bar{y}_0^2\big(\bar{\psi}(x)\big)^2\partial_y^2\Big(\frac{B}{A}\Big)\big(x,\psi(x)\big)+\bar{\bar{y}}_0\bar{\psi}(x)\partial_y\Big(\frac{B}{A}\Big)\big(x,\psi(x)\big)\\ &= \big(\bar{y}_0^2\bar{\bar{\psi}}(x)+\bar{\bar{y}}_0\bar{\psi}(x)\big)\partial_y\Big(\frac{B}{A}\Big)\big(x,\psi(x)\big)+\big(y_0\bar{\psi}(x)\big)^2\partial_y^2\Big(\frac{B}{A}\Big)\big(x,\psi(x)\big)\\ &= \bar{\bar{\psi}}_T(x)\partial_y\Big(\frac{B}{A}\Big)\big(x,\psi_T(x)\big)+\big(\bar{\psi}_T(x)\big)^2\partial_y^2\Big(\frac{B}{A}\Big)\big(x,\psi_T(x)\big)\end{aligned}$$

Now suppose that $P(x,y)\bar{y}^2+Q(x,y)\bar{\bar{y}}+R(x,y)\bar{y}\in\ker\tilde{\mathcal{E}}^N_{D_2}(x_0,y_0)$ then

$$P(x,\psi(x))\bar{\psi}^2(x)+Q(x,\psi(x))\bar{\bar{\psi}}(x)+R(x,\psi(x))\bar{\psi}(x)=0 \mod (x-x_0)^{l_2}$$

Thus

$$\begin{aligned}&\bar{y}_0^2P(x,\psi(x))\bar{\psi}^2(x)+\bar{y}_0^2Q(x,\psi(x))\bar{\bar{\psi}}(x)+\bar{\bar{y}}_0Q(x,\psi(x))\bar{\psi}(x)\\ &\qquad -\bar{\bar{y}}_0Q(x,\psi(x))\bar{\psi}(x)+\bar{y}_0^2R(x,\psi(x))\bar{\psi}(x) = 0 \mod (x-x_0)^{l_2}\end{aligned}$$

This gives

$$\begin{aligned}&P(x,\psi_T(x))\bar{\psi}_T^2(x)+Q(x,\psi_T(x))\bar{\bar{\psi}}_T(x)\\ +&\Big(-\frac{\bar{\bar{y}}_0}{\bar{y}_0}Q(x,\psi_T(x))+\bar{y}_0R(x,\psi_T(x)\Big)\bar{\psi}_T(x) = 0 \mod (x-x_0)^{l_2}\end{aligned}$$

Therefore

$$P(x,y)\bar{y}^2+Q(x,y)\bar{\bar{y}}+\Big(\bar{y}_0R(x,y)-\frac{\bar{\bar{y}}_0}{\bar{y}_0}Q(x,y)\Big)\bar{y}\in\ker\mathcal{E}^{V_2}_{D_2}(x_0,y_0,\bar{y}_0,\bar{\bar{y}}_0).$$

The converse is straightforward. □

Lemma 33. *Consider a non trivial solution* $P(x,y)\bar{y}^2+Q(x,y)\bar{\bar{y}}+R(x,y)\bar{y}$ *in* $\ker\tilde{\mathcal{E}}^N_{D_2}(x_0,y_0)$*, then:*

- *If* $Q=0$ *then* $P\bar{y}+R\in\ker\tilde{\mathcal{E}}^N_{D_1}(x_0,y_0)$.
- *If* $Q\neq 0$ *and* $Q\notin\ker\mathcal{E}^N_{D_0}$ *then:*

$$\bar{y}(x)\Big(D_0(P/Q)+A(P/Q)\partial_y(B/A)+A\partial_y^2(B/A)\Big)\big(x,y(x)\big)+\bar{y}_0D_0(R/Q)\big(x,y(x)\big)=0$$

Proof. As $P(x,y)\bar{y}^2+Q(x,y)\bar{\bar{y}}+R(x,y)\bar{y}\in\ker\tilde{\mathcal{E}}_{D_2}^N(x_0,y_0)$, we get using Lemma 32

$$P(x,y)\bar{y}^2+Q(x,y)\bar{\bar{y}}+\Big(\bar{y}_0R(x,y)-\frac{\bar{\bar{y}}_0}{\bar{y}_0}Q(x,y)\Big)\bar{y}\in\ker\mathcal{E}_{D_2}^{V_2}(x_0,y_0,\bar{y}_0,\bar{\bar{y}}_0)$$

and thus

$$P(x,y(x))\bar{y}(x)^2+Q(x,y(x))\bar{\bar{y}}(x)+\Big(\bar{y}_0R(x,y(x))-\frac{\bar{\bar{y}}_0}{\bar{y}_0}Q(x,y(x))\Big)\bar{y}(x)=0 \mod (x-x_0)^{l_2}$$

Then Theorem 24 applied with the four variables $x,y,\bar{y},\bar{\bar{y}}$ gives

$$P(x,y(x))\bar{y}(x)^2+Q(x,y(x))\bar{\bar{y}}(x)+\Big(\bar{y}_0R(x,y(x))-\frac{\bar{\bar{y}}_0}{\bar{y}_0}Q(x,y(x))\Big)\bar{y}(x)=0.$$

If $Q(x,y)=0$ as a polynomial then we have

$$P(x,y(x))\bar{y}(x)+\bar{y}_0R(x,y(x))=0.$$

Thus $P\bar{y}+\bar{y}_0R\in\ker\mathcal{E}_{D_1}V_1(x_0,y_0,\bar{y}_0)$ and by Lemma 27 we get $P\bar{y}+R\in\ker\tilde{\mathcal{E}}_{D_1}^N(x_0,y_0)$.

If $Q(x,y)\neq 0$ as a polynomial and $Q\notin\ker\mathcal{E}_{D_0}^N$ then $Q\big(x,y(x)\big)\neq 0$. We set $F=P/Q$ and $G=R/Q$ then we have:

$$F\big(x,y(x)\big)\bar{y}(x)+\frac{\bar{\bar{y}}(x)}{\bar{y}(x)}-\frac{\bar{\bar{y}}_0}{\bar{y}_0}+\bar{y}_0G\big(x,y(x)\big)=0.$$

The derivation relatively to x of the previous expression and the relation given by the differential system (S_2') gives:

$$\begin{aligned}
0 &= \partial_xF\big(x,y(x)\big)\bar{y}(x)+\partial_yF\big(x,y(x)\big)\frac{B}{A}\big(x,y(x)\big)\bar{y}(x)+F\big(x,y(x)\big)\bar{y}(x)\partial_y\Big(\frac{B}{A}\Big)\big(x,y(x)\big)\\
&\quad+\frac{1}{\bar{y}^2(x)}\Big(\bar{\bar{y}}(x)\bar{y}(x)\partial_y\Big(\frac{B}{A}\Big)\big(x,y(x)\big)+(\bar{y}(x))^3\partial_y^2\Big(\frac{B}{A}\Big)\big(x,y(x)\big)-\bar{\bar{y}}(x)\bar{y}(x)\partial_y\Big(\frac{B}{A}\Big)\big(x,y(x)\big)\Big)\\
&\quad+\bar{y}_0\partial_xG\big(x,y(x)\big)+\bar{y}_0\partial_yG\big(x,y(x)\big)\frac{B}{A}\big(x,y(x)\big)\\
&= \partial_xF\big(x,y(x)\big)\bar{y}(x)+\partial_yF\big(x,y(x)\big)\frac{B}{A}\big(x,y(x)\big)\bar{y}(x)+F\big(x,y(x)\big)\bar{y}(x)\partial_y\Big(\frac{B}{A}\Big)\big(x,y(x)\big)\\
&\quad+\bar{y}(x)\partial_y^2\Big(\frac{B}{A}\Big)\big(x,y(x)\big)+\bar{y}_0\partial_xG\big(x,y(x)\big)+\bar{y}_0\partial_yG\big(x,y(x)\big)\frac{B}{A}\big(x,y(x)\big)
\end{aligned}$$

This gives:

$$\begin{aligned}
0 &= \bar{y}(x)\Big(\partial_xF\big(x,y(x)\big)+\partial_yF\big(x,y(x)\big)\frac{B}{A}\big(x,y(x)\big)+F\big(x,y(x)\big)\partial_y\Big(\frac{B}{A}\Big)\big(x,y(x)\big)\\
&\quad+\partial_y^2\Big(\frac{B}{A}\Big)\big(x,y(x)\big)\Big)+\bar{y}_0\partial_xG\big(x,y(x)\big)+\bar{y}_0\partial_yG\big(x,y(x)\big)\frac{B}{A}\big(x,y(x)\big)
\end{aligned}$$

Thus

$$\begin{aligned}
0 &= \bar{y}(x)\Big(D_0(F)+(AF)\partial_y(B/A)+A\partial_y^2(B/A)\Big)\big(x,y(x)\big)\\
&\quad+\bar{y}_0D_0(G)\big(x,y(x)\big).
\end{aligned}$$

This gives the desired conclusion. □

Now we can state the generalization of Theorem 25 for the Liouvillian case.

Theorem 34 (Liouvillian extactic curve Theorem)**.**

(1) *If* $\tilde{E}^N_{D_2}(x_0,y_0)=0$ *then the derivation* D_0 *has a Liouvillian first integral with degree smaller than* N *or a Darbouxian first integral with degree smaller than* $2N+3d-1$ *or a rational first integral with degree smaller than* $4N+8d-3$*. Moreover the defining equation of the first integral, equation (Rat), (D) or (L), has coefficients in* $\mathbb{K}$*.*
(2) *If* D_0 *has a rational or a Darbouxian or a Liouvillian first integral with degree smaller than* N *then* $\tilde{E}^N_{D_2}(x_0,y_0)=0$*.*

Proof of Theorem 34. First suppose that $\tilde{E}^N_{D_2}(x_0,y_0)=0$, giving a non trivial solution

$$P(x,y)\bar{y}^2+Q(x,y)\bar{\bar{y}}+R(x,y)\bar{y}\in\ker\tilde{\mathcal{E}}^N_{D_2}(x_0,y_0).$$

If $Q=0$ then by Lemma 33, then $P\bar{y}+R\in\ker\tilde{\mathcal{E}}^N_{D_1}(x_0,y_0)$ and Theorem 29 gives the existence of a Darbouxian first integral with degree smaller than N and defining equation (D) with coefficients in $\mathbb{K}$ or a first integral with degree smaller than $2N+2d-1$ with coefficients in $\mathbb{K}$.

If $Q\neq 0$ and $Q\in\ker\mathcal{E}^N_{D_0}$ then by Theorem 25 there exists a rational first integral with degree smaller than N with coefficients in $\mathbb{K}$.

If $Q\neq 0$ and $Q\notin\ker\mathcal{E}^N_{D_0}$ then by Lemma 33 we have two situations:
In the first situation we have:

$$D_0(P/Q)+A(P/Q)\partial_y(B/A)+A\partial_y^2(B/A)=0.$$

In this case, Proposition 10 gives the existence of a Liouvillian first integral. As $\deg P,\deg Q\leq N$, we deduce the existence of a Liouvillian first integral with degree smaller than N. Moreover, the equation of type (L) giving the existence of a Liouvillian first integral has coefficients in $\mathbb{K}$ since $P,Q\in\mathbb{K}(x_0,y_0)[x,y]$.
In the second situation we have $D_0(P/Q)+A(P/Q)\partial_y(B/A)+A\partial_y^2(B/A)\neq 0$ and

$$\bar{y}(x)\Big(D_0(P/Q)+A(P/Q)\partial_y(B/A)+A\partial_y^2(B/A)\Big)\big(x,y(x)\big)+\bar{y}_0D_0(R/Q)\big(x,y(x)\big)=0.$$

In this case, we set $P_1=A^2Q^2\big(D_0(P/Q)+A(P/Q)\partial_y(B/A)+A\partial_y^2(B/A)\big)$, $Q_1=A^2Q^2D_0(R/Q)$ and we obtain $\bar{y}P_1+\bar{y}_0Q_1\in\ker\mathcal{E}^{V_1}_{D_1}(x_0,y_0,\bar{y}_0)$. Thus thanks to Lemma 27 we get $\bar{y}P_1+Q_1\in\ker\tilde{\mathcal{E}}^{2N+3d-1,1}_{D_1}(x_0,y_0)$. Theorem 29 gives the existence of a Darbouxian first integral with degree smaller than $2N+3d-1$ or the existence of a rational first integral with degree smaller than $4N+8d-3$ with coefficients in $\mathbb{K}$.

Now, we study the second part of the theorem.
If D_0 has a Liouvillian first integral with degree smaller than N then we have $\partial_y^2\mathcal{F}/\partial_y\mathcal{F}=P/Q$, with $P,Q\in\bar{\mathbb{K}}[x,y]$ and $\deg(P),\deg(Q)\leq N$. Proposition 7 implies that $\bar{y}P/Q+\bar{\bar{y}}/\bar{y}$ is a first integral of (S_2'). This gives

$$\bar{y}(x)P/Q\big(x,y(x)\big)+\bar{\bar{y}}(x)/\bar{y}(x)=c,$$

where $c\in\bar{\mathbb{K}}(x_0,y_0)$ putting $\bar{y}_0=1,\bar{\bar{y}}_0=0$. Thus

$$-cQ\big(x,y(x)\big)\bar{y}(x)+P\big(x,y(x)\big)\bar{y}^2(x)+Q\big(x,y(x)\big)\bar{\bar{y}}(x)=0,$$

and then
$$-cQ\bar{y}+P\bar{y}^2+Q\bar{\bar{y}}\in\bar{\mathbb{K}}\otimes_{\mathbb{K}}\ker\tilde{\mathcal{E}}^N_{D_2}(x_0,y_0).$$
As the coefficients of $\tilde{\mathcal{E}}^N_{D_2}(x_0,y_0)$ are in $\mathbb{K}(x_0,y_0)$, this implies that there exists an element with coefficients in $\mathbb{K}(x_0,y_0)$ in $\ker\tilde{\mathcal{E}}^N_{D_2}(x_0,y_0)$ and thus $\tilde{E}^N_{D_2}(x_0,y_0)=0$. The situation where D_0 has a rational or a Darbouxian first integral with degree smaller than N can be done in the same way. □

As before, we deduce that the computation of $F\in\mathbb{K}(x,y)$ is not restrictive.

Corollary 35. *Suppose that D_0 has a Liouvillian first integral and no Darbouxian nor rational first integral. There exists $F\in\mathbb{K}(x,y)$ with $\deg F\le N$ such that equation (L) gives a Liouvillian first integral if and only if there exists $\tilde{F}\in\overline{\mathbb{K}}(x,y)$ with $\deg\tilde{F}\le N$ such that equation (L) with $\tilde{F}$ gives a Liouvillian first integral.*

Proof. We just apply to a Liouvillian first integral with a defining equation (L) with coefficients in $\bar{\mathbb{K}}$ the second part of Theorem 34, proving that $\tilde{E}^N_{D_2}(x_0,y_0)=0$. Then we apply the first part proving there exists a Liouvillian first integral with degree smaller than N or a Darbouxian first integral with degree smaller than $2N+3d-1$ or a rational first integral with degree smaller than $4N+8d-3$ with coefficients in $\mathbb{K}$. As the last two are forbidden by assumption, D_0 admits a Liouvillian first integral given by an equation (L) with a rational function with degree smaller than N and with coefficients in $\mathbb{K}$. □

3.4. Riccati extactic curve.

In this subsection we are going to apply the result of Section 2 to the derivation D_3. Then in the following $\big(y(x),\bar{y}(x),\bar{\bar{y}}(x),\bar{\bar{\bar{y}}}(x)\big)$ is a solution of (S_3') satisfying the initial condition $y(x_0)=y_0$, $\bar{y}(x_0)=\bar{y}_0$, $\bar{\bar{y}}(x_0)=\bar{\bar{y}}_0$, $\bar{\bar{\bar{y}}}(x_0)=\bar{\bar{\bar{y}}}_0$ where x_0, y_0, $\bar{y}_0$, $\bar{\bar{y}}_0$ and $\bar{\bar{\bar{y}}}_0$ are variables.

Now, we are going to generalize Theorem 25 to the Riccati case.

Definition 36. We set $\tilde{\mathcal{E}}^N_{D_3}(x_0,y_0)=\mathcal{E}^{V_3}_{D_3}(x_0,y_0,1,0,0)$, where
$$V_3:=\mathbb{K}[x,y]_{\le N}\bar{y}^4\oplus\mathbb{K}[x,y]_{\le N}(3\bar{\bar{y}}^2-2\bar{\bar{\bar{y}}}\bar{y})\oplus\mathbb{K}[x,y]_{\le N}\bar{y}^2,\quad l_3=\dim(V_3).$$
The N-th Riccati extactic curve is defined by
$$\tilde{E}^N_{D_3}(x_0,y_0)=E^{V_3}_{D_3}(x_0,y_0,1,0,0).$$

As before, we begin by proving two Lemmas.

Lemma 37. *We have the following equivalence:*
$$4P(x,y)\bar{y}^4+Q(x,y)(3\bar{\bar{y}}^2-2\bar{\bar{\bar{y}}}\bar{y})+R(x,y)\bar{y}^2\in\ker\tilde{\mathcal{E}}^N_{D_3}(x_0,y_0)$$
$$\Updownarrow$$
$$4P(x,y)\bar{y}^4+Q(x,y)(3\bar{\bar{y}}^2-2\bar{\bar{\bar{y}}}\bar{y})+\left(R(x,y)\bar{y}_0^2-\left(3\frac{\bar{\bar{y}}_0^2}{\bar{y}_0^2}-2\frac{\bar{\bar{\bar{y}}}_0}{\bar{y}_0}\right)Q(x,y)\right)\bar{y}^2\in\ker\mathcal{E}^{V_3}_{D_3}(x_0,y_0,\bar{y}_0,\bar{\bar{y}}_0,\bar{\bar{\bar{y}}}_0)$$

Proof. We denote by $\psi(x)$ the solution of (S_3') such that $\psi(x_0)=y_0$, $\bar{\psi}(x_0)=1$, $\bar{\bar{\psi}}(x_0)=0$, $\bar{\bar{\bar{\psi}}}(x_0)=0$.
We consider the transformation
$$T(y,\bar{y},\bar{\bar{y}},\bar{\bar{\bar{y}}})=(y,\bar{y}_0\bar{y},\bar{y}_0^2\bar{\bar{y}}+\bar{\bar{y}}_0\bar{y},\bar{y}_0^3\bar{\bar{\bar{y}}}+3\bar{y}_0\bar{\bar{y}}_0\bar{\bar{y}}+\bar{\bar{\bar{y}}}_0\bar{y}).$$

We set

$$\begin{aligned}&\big(\psi_T(x), \bar{\psi}_T(x), \bar{\bar{\psi}}_T(x), \bar{\bar{\bar{\psi}}}_T(x)\big) := T\big(\psi(x), \bar{\psi}(x), \bar{\bar{\psi}}(x), \bar{\bar{\bar{\psi}}}(x)\big)\\ &= \big(\psi(x), \bar{y}_0\bar{\psi}(x), \bar{y}_0^2\bar{\bar{\psi}}(x) + \bar{\bar{y}}_0\bar{\psi}(x), \bar{y}_0^3\bar{\bar{\bar{\psi}}}(x) + 3\bar{y}_0\bar{\bar{y}}_0\bar{\bar{\psi}}(x) + \bar{\bar{\bar{y}}}_0\bar{\psi}(x)\big).\end{aligned}$$

Now, we are going to show that $\big(\psi_T(x), \bar{\psi}_T(x), \bar{\bar{\psi}}_T(x), \bar{\bar{\bar{\psi}}}_T(x)\big)$ is a solution of (S_3') with initial conditions

$$\psi_T(x_0) = y_0, \bar{\psi}_T(x_0) = \bar{y}_0, \bar{\bar{\psi}}_T(x_0) = \bar{\bar{y}}_0, \bar{\bar{\bar{\psi}}}_T(x_0) = \bar{\bar{\bar{y}}}_0.$$

We have already proved in Lemma 32 that:

$$\partial_x\bar{\psi}_T(x) = \bar{\psi}_T(x)\partial_y\Big(\frac{B}{A}\Big)\big(x, \psi_T(x)\big).$$

and

$$\partial_x\bar{\bar{\psi}}_T(x) = \bar{\bar{\psi}}_T(x)\partial_y\Big(\frac{B}{A}\Big)\big(x, \psi_T(x)\big) + \big(\bar{\psi}_T(x)\big)^2\partial_y^2\Big(\frac{B}{A}\Big)\big(x, \psi_T(x)\big).$$

Finally

$$\begin{aligned}\partial_x\bar{\bar{\bar{\psi}}}_T(x) =&\bar{y}_0^3\partial_x\bar{\bar{\bar{\psi}}}(x) + 3\bar{\bar{y}}_0\bar{y}_0\partial_x\bar{\bar{\psi}}(x) + \bar{\bar{\bar{y}}}_0\partial_x\bar{\psi}(x)\\ =&\bar{y}_0^3\bar{\bar{\bar{\psi}}}(x)\partial_y\Big(\frac{B}{A}\Big)\big(x, \psi(x)\big) + 3\bar{y}_0^3\bar{\bar{\psi}}(x)\bar{\psi}(x)\partial_y^2\Big(\frac{B}{A}\Big)\big(x, \psi(x)\big) + \bar{y}_0^3\bar{\psi}(x)^3\partial_y^3\Big(\frac{B}{A}\Big)\big(x, \psi(x)\big)\\ &+ 3\bar{\bar{y}}_0\bar{y}_0\bar{\bar{\psi}}(x)\partial_y\Big(\frac{B}{A}\Big)\big(x, \psi(x)\big) + 3\bar{\bar{y}}_0\bar{y}_0\big(\bar{\psi}(x)\big)^2\partial_y^2\Big(\frac{B}{A}\Big)\big(x, \psi(x)\big)\\ &+ \bar{\bar{\bar{y}}}_0\bar{\psi}(x)\partial_y\Big(\frac{B}{A}\Big)\big(x, \psi(x)\big)\\ =&\Big(\bar{y}_0^3\bar{\bar{\bar{\psi}}}(x) + 3\bar{\bar{y}}_0\bar{y}_0\bar{\bar{\psi}}(x) + \bar{\bar{\bar{y}}}_0\bar{\psi}(x)\Big)\,\partial_y\Big(\frac{B}{A}\Big)\big(x, \psi(x)\big)\\ &+ \Big(3\bar{y}_0^3\bar{\bar{\psi}}(x)\bar{\psi}(x) + 3\bar{\bar{y}}_0\bar{y}_0\big(\bar{\psi}(x)\big)^2\Big)\,\partial_y^2\Big(\frac{B}{A}\Big)\big(x, \psi(x)\big)\\ &+ \bar{y}_0^3\bar{\psi}(x)^3\partial_y^3\Big(\frac{B}{A}\Big)\big(x, \psi(x)\big)\\ =&\bar{\bar{\bar{\psi}}}_T(x)\partial_y\Big(\frac{B}{A}\Big)\big(x, \psi(x)\big) + 3\bar{\bar{\psi}}_T(x)\bar{\psi}_T(x)\partial_y^2\Big(\frac{B}{A}\Big)\big(x, \psi(x)\big) + \bar{\psi}_T(x)^3\partial_y^3\Big(\frac{B}{A}\Big)\big(x, \psi(x)\big).\end{aligned}$$

Now suppose that $4P(x,y)\bar{y}^4 + Q(x,y)(3\bar{\bar{y}}^2 - 2\bar{\bar{\bar{y}}}\bar{y}) + R(x,y)\bar{y}^2 \in \ker\tilde{\mathcal{E}}_{D_3}^N(x_0,y_0)$ then

$$4P(x,\psi(x))\bar{\psi}^4(x) + Q(x,\psi(x))(3\bar{\bar{\psi}}^2(x) - 2\bar{\bar{\bar{\psi}}}(x)\bar{\psi}(x)) + R(x,\psi(x))\bar{\psi}^2(x) = 0 \mod (x-x_0)^{l_3}$$

Thus

$$\begin{aligned}&4P(x,\psi(x))\bar{y}_0^4\bar{\psi}^4(x)\\ &+Q(x,\psi(x))\Big(3\bar{y}_0^4\bar{\bar{\psi}}^2(x) - 2\big(\bar{y}_0^3\bar{\bar{\bar{\psi}}}(x) + 3\bar{y}_0\bar{\bar{y}}_0\bar{\bar{\psi}}(x) + \bar{\bar{\bar{y}}}_0\bar{\psi}(x)\big)\bar{y}_0\bar{\psi}(x) + 6\bar{y}_0^2\bar{\bar{y}}_0\bar{\bar{\psi}}(x)\bar{\psi}(x) + 2\bar{\bar{\bar{y}}}_0\bar{y}_0\bar{\psi}^2(x)\Big)\\ &+R(x,\psi(x))\bar{y}_0^4\bar{\psi}^2(x) = 0 \mod (x-x_0)^{l_3}\end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned}&4P(x,\psi(x))\bar{\psi}_T^4(x)\\ &+Q(x,\psi(x))\Big(3\bar{y}_0^4\bar{\bar{\psi}}^2(x) - 2\bar{\bar{\bar{\psi}}}_T(x)\bar{\psi}_T(x) + 6\bar{y}_0^2\bar{\bar{y}}_0\bar{\bar{\psi}}(x)\bar{\psi}(x) + 2\bar{\bar{\bar{y}}}_0\bar{y}_0\bar{\psi}^2(x)\Big)\\ &+R(x,\psi(x))\bar{y}_0^2\bar{\psi}_T^2(x) = 0 \mod (x-x_0)^{l_3}\end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned}
&4P(x,\psi(x))\bar{\psi}_T^4(x)\\
&+Q(x,\psi(x))\Big(3(\bar{y}_0^2\bar{\bar{\psi}}(x)+\bar{\bar{y}}_0\bar{\psi}(x))^2-6\bar{y}_0^2\bar{\bar{y}}_0\bar{\bar{\psi}}(x)\bar{\psi}(x)-3\bar{\bar{y}}_0^2\bar{\psi}^2(x)\\
&-2\bar{\bar{\bar{\psi}}}_T(x)\bar{\psi}_T(x)+6\bar{y}_0^2\bar{\bar{y}}_0\bar{\bar{\psi}}(x)\bar{\psi}(x)+2\bar{\bar{\bar{y}}}_0\bar{y}_0\bar{\psi}^2(x)\Big)\\
&+R(x,\psi(x))\bar{y}_0^2\bar{\psi}_T^2(x)=0 \mod (x-x_0)^{l_3}
\end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned}
&4P(x,\psi(x))\bar{\psi}_T^4(x)\\
&+Q(x,\psi(x))\Big(3\bar{\bar{\psi}}_T^2(x)-3\bar{\bar{y}}_0^2\bar{\psi}^2(x)-2\bar{\bar{\bar{\psi}}}_T(x)\bar{\psi}_T(x)+2\bar{\bar{\bar{y}}}_0\bar{y}_0\bar{\psi}^2(x)\Big)\\
&+R(x,\psi(x))\bar{y}_0^2\bar{\psi}_T^2(x)=0 \mod (x-x_0)^{l_3}
\end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned}
&4P(x,\psi_T(x))\bar{\psi}_T^4(x)\\
&+Q(x,\psi_T(x))(3\bar{\bar{\psi}}_T(x)^2-2\bar{\bar{\bar{\psi}}}_T(x)\bar{\psi}_T(x))\\
&\left(\left(-3\tfrac{\bar{\bar{y}}_0^2}{\bar{y}_0^2}+2\tfrac{\bar{\bar{\bar{y}}}_0}{\bar{y}_0}\right)Q(x,\psi_T(x))+R(x,\psi_T(x))\bar{y}_0^2\right)\bar{\psi}_T^2(x)=0 \mod (x-x_0)^{l_3}
\end{aligned}$$

Therefore

$$4P(x,y)\bar{y}^4+Q(x,y)(3\bar{\bar{y}}^2-2\bar{\bar{\bar{y}}}\bar{y})+\left(R(x,y)\bar{y}_0^2-\left(3\frac{\bar{\bar{y}}_0^2}{\bar{y}_0^2}-2\frac{\bar{\bar{\bar{y}}}_0}{\bar{y}_0}\right)Q(x,y)\right)\bar{y}^2$$

belongs to $\ker\mathcal{E}_{D_3}^{V_3}(x_0,y_0,\bar{y}_0,\bar{\bar{y}}_0,\bar{\bar{\bar{y}}}_0)$. The converse is straightforward. □

Lemma 38. *Consider a non trivial solution*

$$4P(x,y)\bar{y}^4+Q(x,y)(3\bar{\bar{y}}^2-2\bar{\bar{\bar{y}}}\bar{y})+R(x,y)\bar{y}^2\in\ker\tilde{\mathcal{E}}_{D_3}^N(x_0,y_0),$$

then:

- *If* $Q=0$ *then* $4P\bar{y}^2+R\in\ker\tilde{\mathcal{E}}_{D_1}^{N,2}(x_0,y_0)$.
- *If* $Q\neq 0$ *and* $Q\notin\ker\mathcal{E}_{D_0}^N$ *then:*

$$\bar{y}(x)^2\Big(4D_0(P/Q)+8A(P/Q)\partial_y(B/A)-2A\partial_y^3(B/A)\Big)\big(x,y(x)\big)+\bar{y}_0^2D_0(R/Q)\big(x,y(x)\big)=0.$$

Proof. As $4P(x,y)\bar{y}^4+Q(x,y)(3\bar{\bar{y}}^2-2\bar{\bar{\bar{y}}}\bar{y})+R(x,y)\bar{y}^2\in\ker\tilde{\mathcal{E}}_{D_3}^N(x_0,y_0)$, we have using Lemma 37

$$4P\big(x,y(x)\big)\bar{y}^4(x)+Q\big(x,y(x)\big)(3\bar{\bar{y}}^2(x)-2\bar{\bar{\bar{y}}}(x)\bar{y}(x))$$
$$+\left(R\big(x,y(x)\big)\bar{y}_0^2-\left(3\frac{\bar{\bar{y}}_0^2}{\bar{y}_0^2}-2\frac{\bar{\bar{\bar{y}}}_0}{\bar{y}_0}\right)Q\big(x,y(x)\big)\right)\bar{y}^2(x)=0 \mod (x-x_0)^{l_3}$$

Then Theorem 24 applied with the five variables $x,y,\bar{y},\bar{\bar{y}},\bar{\bar{\bar{y}}}$ gives

$$4P\big(x,y(x)\big)\bar{y}^4(x)+Q\big(x,y(x)\big)(3\bar{\bar{y}}^2(x)-2\bar{\bar{\bar{y}}}(x)\bar{y}(x))$$
$$+\left(R\big(x,y(x)\big)\bar{y}_0^2-\left(3\frac{\bar{\bar{y}}_0^2}{\bar{y}_0^2}-2\frac{\bar{\bar{\bar{y}}}_0}{\bar{y}_0}\right)Q\big(x,y(x)\big)\right)\bar{y}^2(x)=0.$$

If $Q(x,y)=0$ as a polynomial then we have $4P\big(x,y(x)\big)\bar{y}^4(x)+\bar{y}_0^2R\big(x,y(x)\big)\bar{y}^2(x)=0$. As $\bar{y}(x)\neq 0$, we get $4P\big(x,y(x)\big)\bar{y}^2(x)+\bar{y}_0^2R\big(x,y(x)\big)=0$. And thus $4P\bar{y}^2+R$ belongs to $\ker\tilde{\mathcal{E}}_{D_1}^{N,2}(x_0,y_0)$ by Lemma 27.

If $Q(x,y) \neq 0$ as a polynomial and $Q \notin \ker \mathcal{E}_{D_0}^N$ then $Q\big(x,y(x)\big) \neq 0$. We set $F = P/Q$ and $G = R/Q$ then we have:

$$(\star) \quad 4F\big(x,y(x)\big)\bar{y}^2(x) + \left(3\frac{\bar{\bar{y}}^2(x)}{\bar{y}^2(x)} - 2\frac{\bar{\bar{\bar{y}}}(x)}{\bar{y}(x)}\right) + \bar{y}_0^2 G\big(x,y(x)\big) - \left(3\frac{\bar{\bar{y}}_0^2}{\bar{y}_0^2} - 2\frac{\bar{\bar{\bar{y}}}_0}{\bar{y}_0}\right) = 0.$$

The derivation relatively to x of $3\frac{\bar{\bar{y}}^2(x)}{\bar{y}^2(x)} - 2\frac{\bar{\bar{\bar{y}}}(x)}{\bar{y}(x)}$ and the relation given by the differential system (S_3') gives:

$$\begin{aligned}
\partial_x\left(3\frac{\bar{\bar{y}}^2(x)}{\bar{y}^2(x)} - 2\frac{\bar{\bar{\bar{y}}}(x)}{\bar{y}(x)}\right) &= \Bigg[3\left(2\bar{\bar{y}}\Big[\bar{y}\partial_y\Big(\frac{B}{A}\Big) + \bar{y}^2\partial_y^2\Big(\frac{B}{A}\Big)\Big]\frac{1}{\bar{y}^2}\right)\\
&\quad -6\bar{\bar{y}}^2\bar{y}\partial_y\Big(\frac{B}{A}\Big)\frac{1}{\bar{y}^3}\\
&\quad -2\left(\bar{\bar{y}}\partial_y\Big(\frac{B}{A}\Big) + 3\bar{\bar{y}}\bar{y}\partial_y^2\Big(\frac{B}{A}\Big) + \bar{y}^3\partial_y^3\Big(\frac{B}{A}\Big)\right)\frac{1}{\bar{y}}\\
&\quad +2\bar{\bar{\bar{y}}}\bar{y}\partial_y\Big(\frac{B}{A}\Big)\frac{1}{\bar{y}^2}\Bigg]\big(x,y(x),\bar{y}(x),\bar{\bar{y}}(x),\bar{\bar{\bar{y}}}(x)\big)\\
&= -2\bar{y}^2(x)\partial_y^3\Big(\frac{B}{A}\Big)\big(x,y(x)\big).
\end{aligned}$$

Then the derivation relatively to x of $(\star)$ gives:

$$\begin{aligned}
0 &= 4A^{-1}(x,y(x))D_0(F)(x,y(x))\bar{y}^2(x) + 8F(x,y(x))\bar{y}^2(x)\partial_y\Big(\frac{B}{A}\Big)\\
&\quad -2\bar{y}^2(x)\partial_y^3\Big(\frac{B}{A}\Big)\big(x,y(x)\big) + \bar{y}_0^2A^{-1}(x,y(x))D_0(G)(x,y(x))
\end{aligned}$$

Thus

$$\begin{aligned}
0 &= \bar{y}(x)^2\Big(4D_0(F) + 8AF\partial_y(B/A) - 2A\partial_y^3(B/A)\Big)\big(x,y(x)\big)\\
&\quad +\bar{y}_0^2D_0(G)\big(x,y(x)\big).
\end{aligned}$$

This gives the desired conclusion. □

Now we can state the generalization of Theorem 34 for the Riccati case.

Theorem 39 (Riccati extactic curve Theorem)**.**

(1) *If $\tilde{E}_{D_3}^N(x_0,y_0) = 0$ then the derivation D_0 has a Riccati first integral with degree smaller than N or a 2-Darbouxian first integral with degree smaller than $2N+4d-1$ or a rational first integral with degree smaller than $4N+10d-3$. Moreover the defining equation of the first integral, (Rat), (D) or (Ric), has coefficients in $\mathbb{K}$.*

(2) *If D_0 has a rational or a 2-Darbouxian or a Riccati first integral with degree smaller than N then $\tilde{E}_{D_3}^N(x_0,y_0) = 0$.*

Proof of Theorem 39. First consider a non trivial solution

$$4P(x,y)\bar{y}^4 + Q(x,y)(3\bar{\bar{y}}^2 - 2\bar{\bar{\bar{y}}}\bar{y}) + R(x,y)\bar{y}^2 \in \ker \tilde{\mathcal{E}}_{D_3}^N(x_0,y_0).$$

If $Q = 0$ then by Lemma 38, then $4P\bar{y}^2 + R \in \ker \tilde{\mathcal{E}}_{D_1}^{N,2}(x_0,y_0)$ and Theorem 29 gives the existence of a 2-Darbouxian first integral with degree smaller than N and defining equation (D) with coefficients in $\mathbb{K}$ or a first integral with degree smaller than $2N+2d-1$ and with coefficients in $\mathbb{K}$.

If $Q \neq 0$ and $Q \in \ker \tilde{\mathcal{E}}_{D_0}^N$ then by Theorem 25 there exists a rational first integral with degree smaller than N with coefficients in $\mathbb{K}$.

If $Q \neq 0$ and $Q \notin \ker \mathcal{E}_{D_0}^N$ then by Lemma 38 we have two situations:
In the first situation we have:

$$4D_0(P/Q) + 8A(P/Q)\partial_y(B/A) - 2A\partial_y^3(B/A) = 0.$$

In this case, Proposition 10 gives the existence of a Riccati first integral. As $\deg P, \deg Q \leq N$, we deduce the existence of a Riccati first integral with degree smaller than N. Moreover, the equation of type (Ric) giving the existence of a Riccati first integral has coefficients in $\mathbb{K}$ since $P, Q \in \mathbb{K}(x_0,y_0)[x,y]$.
In the second situation we have $4D_0(P/Q) + 8A(P/Q)\partial_y(B/A) - 2A\partial_y^3(B/A) \neq 0$ and

$$\bar{y}(x)^2\Big(4D_0(P/Q)+8A(P/Q)\partial_y(B/A)-2A\partial_y^3(B/A)\big)\big(x,y(x)\Big)+\bar{y}_0^2 D_0(R/Q)\big(x,y(x)\big) = 0.$$

In this case, we set

$$P_1 = A^3Q^2\big(4D_0(P/Q) + 8A(P/Q)\partial_y(B/A) - 2A\partial_y^3(B/A)\big),$$

$$Q_1 = A^3Q^2 D_0(R/Q)$$

and we obtain $\bar{y}^2 P_1 + \bar{y}_0^2 Q_1 \in \ker \tilde{\mathcal{E}}_{D_1}^{2N+4d-1,2}(x_0,y_0,\bar{y}_0)$. Therefore by Lemma 27, $\bar{y}^2P_1 + Q_1 \in \ker \mathcal{E}_{D_1}^{2N+4D-1,2}(x_0,y_0)$, thus Theorem 29 gives the existence of a 2-Darbouxian first integral with degree smaller than $2N+4d-1$ or the existence of a rational first integral with degree smaller than $4N+10d-3$. Moreover the equation giving this first integral has coefficients in $\mathbb{K}$ since $P_1, Q_1 \in \mathbb{K}(x_0, y_0[x,y]$.

Now, we study the second part of the theorem.
If D_0 has a Riccati first integral with degree smaller than N then we have the following relation $\partial_y^2\mathcal{F}/\mathcal{F} = P/Q$, with $P, Q \in \bar{\mathbb{K}}[x,y]$ and $\deg(P), \deg(Q) \leq N$. Proposition 7 implies that $4\bar{y}^2P/Q + 3\bar{\bar{y}}^2/\bar{y}^2 - 2\bar{\bar{\bar{y}}}/\bar{y}$ is a first integral of (S_3'). This gives $4\bar{y}^2(x)P/Q\big(x,y(x)\big) + 3\bar{\bar{y}}^2(x)/\bar{y}^2(x) - 2\bar{\bar{\bar{y}}}(x)/\bar{y}(x) = c$, where $c \in \bar{\mathbb{K}}(x_0,y_0)$ putting $\bar{y}(x_0) = 1, \bar{\bar{y}}(x_0) = 0, \bar{\bar{\bar{y}}}(x_0) = 0$. Thus

$$4P\big(x,y(x)\big)\bar{y}^4(x) + Q\big(x,y(x)\big)(3\bar{\bar{y}}^2(x) - 2\bar{\bar{\bar{y}}}(x)\bar{y}(x)) - cQ\big(x,y(x)\big)\bar{y}^2(x) = 0,$$

and then

$$4P\bar{y}^4 + Q(3\bar{\bar{y}}^2 - 2\bar{\bar{\bar{y}}}\bar{y}) - cQ\bar{y}^2 \in \bar{\mathbb{K}} \otimes_{\mathbb{K}} \ker \tilde{\mathcal{E}}_{D_3}^N(x_0,y_0).$$

Thus there exists a non trivial element with coefficients in $\mathbb{K}$ in $\ker \tilde{\mathcal{E}}_{D_3}^N(x_0,y_0)$, and thus $\tilde{E}_{D_3}^N(x_0,y_0) = 0$. The situation where D_0 has a rational or a 2-Darbouxian first integral with degree smaller than N can be done in the same way. □

As before, we remark that the computation of $F \in \mathbb{K}(x,y)$ is not restrictive.

Corollary 40. *Suppose that D_0 has a Riccati first integral and no 2-Darbouxian nor rational first integral. We have:*
There exists $F \in \mathbb{K}(x,y)$ with $\deg F \leq N$ such that equation (Ric) gives a Riccati first integral if and only if there exists $\tilde{F} \in \overline{\mathbb{K}}(x,y)$ with $\deg \tilde{F} \leq N$ such that equation (Ric) gives a Riccati first integral

Proof. Given a Riccati first integral with coefficients in $\bar{\mathbb{K}}$, we apply the second part of Theorem 39, giving $\tilde{E}_{D_3}^{N}(x_0,y_0)=0$. We then apply the first part, and knowing that 2-Darbouxian and rational first integrals are forbidden, the only possibility left is a the existence of a Riccati first integral of degree $\leq N$ with an equation of type (Ric) with coefficients in $\mathbb{K}$. $\square$

4. Evaluations of extactic curves

In the algorithms, we will not compute the extactic curves as polynomials in x_0, y_0. We will only compute the extactic curves evaluated at a random point. If the extactic curve is a non-zero polynomial, then almost surely its evaluation at a random point will not be zero. However, theoretically this can happen, and thus we want to bound the algebraic set on which such kind of bad situations can happen.

Definition 41. We denote by $\Sigma_{D_r,N,k}$ (where k is omited when $r\neq 1$) the following algebraic variety:

$$\Sigma_{D_r,N,k}=\mathcal{V}\big(p\times p \text{ minors of } \tilde{\mathcal{E}}_{D_r}^{N,k}(x_0,y_0),\ \text{where } p=\operatorname{rank}\tilde{\mathcal{E}}_{D_r}^{N,k}\big).$$

With this definition we can now state a specialized version of Theorem 24. In the following in order to have a uniform statement we set $\bar{y}^{(2)}=\bar{\bar{y}}$ and $\bar{y}^{(3)}=\bar{\bar{\bar{y}}}$.

Lemma 42. *Let $P\in\ker\tilde{\mathcal{E}}_{D_r}^{N,k}(x_0^\star,y_0^\star)$ and $(x_0^\star,y_0^\star)\notin\Sigma_{D_r,N,k}$ then*

$$P\big(x,y_\star(x),\dots,\bar{y}_\star^{(r)}(x)\big)=0,$$

where $y_\star(x),\dots,\bar{y}_\star^{(r)}(x)$ is a solution of (S_r') with initial condition $y_\star(x_0^\star)=y_0^\star$, $\bar{y}_\star(x_0^\star)=1$, $\bar{y}_\star^{(2)}(x_0^\star)=\bar{y}_\star^{(3)}(x_0^\star)=0$.

Proof. If $(x_0^\star,y_0^\star)\notin\Sigma_{D_r,N,k}$ then

$$\dim_{\mathbb{K}}\ker\tilde{\mathcal{E}}_{D_r}^{N,k}(x_0^\star,y_0^\star,1,0,\dots,0)=\dim_{\mathbb{K}(x_0,y_0)}\ker\tilde{\mathcal{E}}_{D_r}^{N,k}(x_0,y_0).$$

By Lemma 27, Lemma 32 and Lemma 37, we have

$$\dim_{\mathbb{K}(x_0,y_0)}\ker\tilde{\mathcal{E}}_{D_r}^{N,k}(x_0,y_0)=\dim_{\mathbb{L}_r}\ker\mathcal{E}_{D_r}^{V_r},$$

where $\mathbb{L}_r=\mathbb{K}(x_0,y_0,\bar{y}_0,\dots,\bar{y}_0^{(r)})$. Thus in this situation, if $P\in\ker\tilde{\mathcal{E}}_{D_r}^{N,k}(x_0^\star,y_0^\star)$ then there exists $\mathcal{P}(x_0,y_0,\dots,\bar{y}_0^{(r)},x,y,\dots,\bar{y}^{(r)})$ in $\ker\mathcal{E}_{D_r}^{V_r}$ such that

$$\mathcal{P}(x_0^\star,y_0^\star,1,0,0;x,y,\bar{y},\dots,\bar{y}^{(r)})=P(x,y,\dots,\bar{y}^{(r)}).$$

By Theorem 24, we have

$$\mathcal{P}(x_0,y_0,\dots,\bar{y}_0^{(r)},x,y(x),\dots,\bar{y}^{(r)}(x))=0$$

then

$$P\big(x,y_\star(x),\dots,\bar{y}_\star^{(r)}(x)\big)=0.$$

$\square$

In the following, we will need some explicit bounds on the degree of the minors of $\tilde{\mathcal{E}}_{D_i}^{N,k}$.

Lemma 43. *The degree of a minor of* $\mathcal{E}_{D_0}^N$ *is smaller than*

$$\mathcal{B}_0(d,N) := \frac{N(N+1)(N+2)}{2} + (d-1)\frac{(N+1)^2(N+2)^2-(N+1)(N+2)}{8}.$$

The degree of a minor of $\tilde{\mathcal{E}}_{D_1}^{N,k}$ *is smaller than*

$$\begin{aligned}\mathcal{B}_1(d,N) &:= Nl_1 + \frac{(2d-1)(l_1-1)l_1}{2}\\ &= N(N+1)(N+2) + (2d-1)\frac{(N+1)^2(N+2)^2-(N+1)(N+2)}{2}.\end{aligned}$$

The degree of a minor of $\tilde{\mathcal{E}}_{D_2}^N$ *is smaller than*

$$\begin{aligned}\mathcal{B}_2(d,N) &:= Nl_2 + \frac{(3d-1)(l_2-1)l_2}{2}\\ &= \frac{3N(N+1)(N+2)}{2} + \frac{3d-1}{2}\Big[\Big(\frac{3}{2}(N+1)(N+2)\Big)^2 - \frac{3}{2}(N+1)(N+2)\Big].\end{aligned}$$

The degree of a minor of $\tilde{\mathcal{E}}_{D_3}^N$ *is smaller than*

$$\begin{aligned}\mathcal{B}_3(d,N) &:= Nl_2 + \frac{(4d-1)(l_3-1)l_3}{2}\\ &= \frac{3N(N+1)(N+2)}{2} + \frac{4d-1}{2}\Big[\Big(\frac{3}{2}(N+1)(N+2)\Big)^2 - \frac{3}{2}(N+1)(N+2)\Big].\end{aligned}$$

Proof. The degree of a minor of $\tilde{\mathcal{E}}_{D_0}^N$ is smaller than the degree of the extactic curve. The annouced bound is given in [Chè11]. We apply here the same strategy for $\tilde{\mathcal{E}}_{D_1}^{N,k}$: Let $\{v_i\}$ be basis of V_1. The degree in x_0, y_0 of a minor $\mathcal{M}$ of $\tilde{\mathcal{E}}_{D_1}^N$ satisfies

$$\deg \mathcal{M} \leq \sum_{k=0}^{l_1-1} \deg D_1^k(v_i).$$

As $\deg D_1^k(v_i) \leq k(2d-1)+N$, we get

$$\deg \mathcal{M} \leq \sum_{k=0}^{l_1-1} k(2d-1)+N \leq Nl_1 + (2d-1)\sum_{k=0}^{l_1-1} k \leq Nl_1 + \frac{(2d-1)(l_1-1)l_1}{2}.$$

The bounds for $\tilde{\mathcal{E}}_{D_2}^N, \tilde{\mathcal{E}}_{D_3}^N$ are obtained in the same way. □

Corollary 44. *The algebraic variety* $\Sigma_{D_r,N,k}$ *is included in an algebraic hypersurface with degree smaller than* $\mathcal{B}_r(d,N)$.

The following set will be also useful to characterize some special situations.

Definition 45. We denote by $\mathfrak{S}_N$ the following set:
If D has no rational first integrals of degree $\leq N$ then

$$\mathfrak{S}_N = \left\{ (x_0,y_0,x_1,y_1) \in \mathbb{K}^4 \;\middle|\; \begin{array}{l} (x_0,y_0) \text{ or } (x_1,y_1) \text{ vanishes an irreducible} \\ \text{Darboux polynomial of degree } \leq N \end{array} \right\}$$

If D has an indecomposable rational first integral P/Q of degree $p \leq N$ then

$$\mathfrak{S}_N = \left\{ (x_0,y_0,x_1,y_1) \in \mathbb{K}^4 \;\middle|\; \begin{array}{l} P(x_0,y_0)Q(x_1,y_1) = P(x_1,y_1)Q(x_0,y_0) \text{ or} \\ (x_0,y_0) \text{ or } (x_1,y_1) \text{ vanishes an irreducible} \\ \text{Darboux polynomial of degree } < p \end{array} \right\}$$

This set corresponds to situations we try to avoid. More precisely, when D_0 has no rational first integral we do not want to get a solution $\big(x, y(x)\big)$ corresponding to a Darboux polynomial. When D_0 has a rational first integral we do not want to get an orbit with degree strictly smaller than the degree of generic orbit. At last, when considering two initial conditions with D_0 having a rational first integral we do not want them to be on the same level of this first integral. Now, we give a bound on this set:

Lemma 46. *The algebraic variety $\mathfrak{S}_N$ is included in an algebraic hypersurface with degree smaller than $\big(d(d+1)+12\big)N$.*

Proof. If D_0 has no rational first integral then by the Darboux-Jouanolou theorem D_0 has at most $d(d+1)/2$ irreducible Darboux polynomials. Therefore, if $(x_0, y_0, x_1, y_1) \in \mathfrak{S}_N$ then (x_0, y_0) or (x_1, y_1) vanishes the product of $d(d+1)/2$ bivariate polynomials with degree smaller than N. This gives a bound on the degree which is lower than the bound of the Lemma.
If D_0 has a rational first integral with degree $p \leq N$ then all irreducible Darboux polynomials divide a linear combination $\lambda P - \mu Q$ where P/Q is an indecomposable rational first integral with degree p. By the Darboux-Jouanolou theorem we know that all but finitely many irreducible Darboux polynomials are of the form $\lambda P - \mu Q$ and have degree p. The set $\sigma(P, Q)$ of $(\lambda : \mu) \in \mathbb{P}^1(\overline{\mathbb{K}})$ such that $\lambda P - \mu Q$ is reducible or has a degree strictly smaller than p is the set of remarkable values. Sometimes this set is called the spectrum of P/Q. It is proved in [Chè15] that $|\sigma(P, Q)| \leq d(d+1)/2 + 5$. So if $(x_0, y_0, x_1, y_1) \in \mathfrak{S}_N$ then it vanishes the polynomial

$$P(x_0, y_0)Q(x_1, y_1) - P(x_1, y_1)Q(x_0, y_0)$$

of degree $2N$ or (x_0, y_0) or (x_1, y_1) vanishes a polynomial $\lambda P - \mu Q$ where $(\lambda : \mu)$ belongs to $\sigma(P, Q)$. Multiplying these polynomials together gives the bound on the degree of the Lemma

$$2N + N\left(5 + \frac{d(d+1)}{2}\right) + N\left(5 + \frac{d(d+1)}{2}\right) = (d(d+1) + 12)N.$$

□

5. The first integral algorithms

In the following sections we are going to describe our algorithms. As mentionned before we are going to compute rational first integrals for the derivations D_0, D_1, D_2, D_3. These rational first integrals are computed thanks to the extactic curves. We need then to compute the flow and a non trivial solution for $\tilde{\mathcal{E}}_{D_i}^N$.

Compute flow series
`Input:` $A(x, y), B(x, y) \in \mathbb{K}[x, y]$, $x_0^\star, y_0^\star$, $N \in \mathbb{N}$, $r \in [[0; 3]]$.
`Output:` $r+1$ series $y(x), \dots, \bar{y}^{(r)}(x)$ solutions of $(S'_r) \mod (x - x_0^\star)^\sigma$ where $\sigma = (r+1)\frac{(N+1)(N+2)}{2}$, with initial condition $y(x_0^\star) = y_0^\star$, $\bar{y}(x_0^\star) = 1$, $\bar{y}^{(2)}(x_0^\star) = \bar{y}^{(3)}(x_0^\star) = 0$.
This subroutine is performed with the algorithm given in [BCO+07].

For the following subroutine, we need a weighted degree in order to specified the output. We use the following weighted degree:

$\text{w-deg}\big(P(x,y,\bar{y},\bar{y}^{(2)},\bar{y}^{(3)}\big) = \deg P(x,y,\bar{y}^{N+1},(\bar{y}^{(2)})^{2N+2},(\bar{y}^{(3)})^{3N+3})$.

Compute solution extactic kernel

`Input:` $A(x,y), B(x,y) \in \mathbb{K}[x,y]$, $y(x),\dots,\bar{y}^{(r)}(x) \in \mathbb{K}[[x]]/(x-x_0^\star)^\sigma$, N, $r \in [[0;3]]$, $k \in \mathbb{N}^*$ (k is omited if $r \neq 1$).

`Output:` A non trivial solution, if it exists, with minimal weighted degree of $\tilde{\mathcal{E}}_{D_r}^{N,k}$, or "None".

(1) Compute a Hermite-Padé approximation of
$$\big(y(x),\dots,y^N(x),\bar{y}(x),\dots,\bar{y}(x)y^N(x),\dots,\bar{y}^{(r)}(x),\dots,\bar{y}^{(r)}(x)y^N(x)\big)$$
with minimal weighted degree, see Section 6.

(2) Construct from this approximation a polynomial $J \in \mathbb{K}[x,y,\bar{y},\bar{y}^{(2)},\bar{y}^{(3)}]$, such that $J\big(x,y(x),\dots,y^{(r)}(x)\big) = 0 \mod (x-x_0^\star)^\sigma$.

(3) If $J \in V_r$ and $\deg_{x,y} J \leq N$ then Return J, Else Return "None".

5.1. **Rational first integrals.**
An algorithm which computes a rational first integral with degree smaller than N has been described in [BCCW16]. In this article, rational first integrals are computed using the same approach. We recall here the description of such algorithm.

The following algorithm search for Darboux polynomials factors of a polynomial P which vanish at some point (x_0, y_0).

Build Rational first integral

`Input:` $P, A, B \in \mathbb{K}[x,y]$, $(x_0^\star, y_0^\star) \in \mathbb{K}^2$.

`Output:` An irreducible Darboux polynomial $\mathcal{P}(x,y) \in \mathbb{K}[x,y]$, such that $\mathcal{P}(x_0^\star,y_0^\star) = 0$, or 1.

(1) Compute the factorization $\gcd\big(P, D_0(P)\big) = \prod_{j=1}^{l} L_j(x,y)$, where L_i are irreducible in $\mathbb{K}[x,y]$ and set $i := 1$.

(2) While $L_i(x_0^\star, y_0^\star) \neq 0$ do $i := i+1$.

(3) If $i \leq l$ then Return L_i Else Return 1.

The algorithm for computing the rational first integrals is then the following.

Compute Rational first integral

`Input:` $A, B \in \mathbb{K}[x,y]$, $(x_0^\star, y_0^\star), (x_0^\star, y_1^\star) \in \mathbb{K}^2$, $N \in \mathbb{N}$

`Output:` An equation $(Eq_0) : \mathcal{F} - F = 0$ where $F(x,y) \in \mathbb{K}(x,y) \setminus \mathbb{K}$, or "None" or "I don't know".

(1) For $y_i^\star$ in $\{y_0^\star, y_1^\star\}$ do
   (a) If $A(x_0^\star, y_i^\star) = 0$ then Return "I don't know".
   (b) Compute flow series$(A, B, x_0^\star, y_i^\star, N, 0)$=: $y(x)$.
   (c) Compute solution extactic kernel$(A, B, y(x), N, 0)$=:$\mathcal{S}$.
   If $\mathcal{S}$="None", then Return "None", else $\mathcal{S} =: P$.
   (d) Build Rational first integral$(A, B, P, x_0^\star, y_i^\star)$=: $\mathcal{P}_i$.

(2) If $\mathcal{P}_0/\mathcal{P}_1 \notin \mathbb{K}$ and $D_0(\mathcal{P}_0/\mathcal{P}_1) = 0$ then Return$((Eq_0) : \mathcal{F} - \mathcal{P}_0/\mathcal{P}_1 = 0)$, else Return "I don't know".

5.2. **Darbouxian first integrals.**
This section describes how to use the results given in the previous section in order to get an efficient probabilistic algorithm for searching Darbouxian first integrals.

Build Darbouxian first integral
`Input:` $A(x,y), B(x,y),\ P(x,y), Q(x,y) \in \mathbb{K}[x,y]$ with $(P,Q) \neq (0,0)$, (x_0,y_0) in $\mathbb{K}^2$, $k \in \mathbb{N}^*$ (by default $k=1$).
`Output:` An equation $(Eq_1) : \partial_y \mathcal{F} - F^{1/k} = 0$, where $F(x,y) \in \mathbb{K}(x,y) \setminus \{0\}$, or a polynomial $\mathcal{P}(x,y)$.

(1) If $P = 0$ then Return$\big($Build Rational first integral$(Q, A, B, x_0, y_0)\big)$.
(2) If $Q = 0$ then Return$\big($Build Rational first integral$(P, A, B, x_0, y_0)\big)$.
(3) $R_1 := APQ(P/Q)^{-1/k}\Big(D_0((P/Q)^{1/k}) + A(P/Q)^{1/k}\partial_y(B/A)\Big)$.
(4) If $R_1 = 0$ then Return $(Eq_1) : \partial_y \mathcal{F} - (P/Q)^{1/k} = 0$
Else Return$\big($Build Rational first integral $(R_1, A, B, x_0, y_0)\big)$.

With the previous algorithm we can now describe how we compute Darbouxian first integral.

Compute Darbouxian first integral
`Input:` $A, B \in \mathbb{K}[x,y]$, $(x_0^\star, y_0^\star), (x_0^\star, y_1^\star) \in \mathbb{K}^2$, $N \in \mathbb{N}$, $k \in \mathbb{N}^*$ (by default $k=1$)
`Output:` An equation $(Eq_1) : \partial_y \mathcal{F} - F^{1/k} = 0$, where $F(x,y) \in \mathbb{K}(x,y) \setminus \{0\}$, or an equation $(Eq_0) : \mathcal{F} - F = 0$ where $F(x,y) \in \mathbb{K}(x,y) \setminus \mathbb{K}$, or "None" or "I don't know".

(1) For $y_i^\star$ in $\{y_0^\star, y_1^\star\}$ do
   (a) If $A(x_0^\star, y_i^\star) = 0$ then Return "I don't know".
   (b) Compute flow series$(A, B, x_0^\star, y_i^\star, N, 1)$=: $y(x), \bar{y}(x)$.
   (c) Compute solution extactic kernel$(A, B, y(x), \bar{y}(x), N, 1, k)$=:$\mathcal{S}$.
   If $\mathcal{S}$="None", then Return "None", else $\mathcal{S} =: \bar{y}^k P - Q$.
   (d) Build Darbouxian first integral$(A, B, P, Q, x_0^\star, y_i^\star, k)$=: $\mathcal{P}_i$.
   (e) If $\mathcal{P}_i \notin \mathbb{K}[x,y]$ then Return$(\mathcal{P}_i)$
(2) If $\mathcal{P}_0/\mathcal{P}_1 \notin \mathbb{K}$ and $D_0(\mathcal{P}_0/\mathcal{P}_1) = 0$ then Return$((Eq_0) : \mathcal{F} - \mathcal{P}_0/\mathcal{P}_1 = 0)$,
else Return "I don't know".

Proposition 47. *The algorithm* Compute Darbouxian first integral *satisfies the following properties:*

- *If it returns "None" then there are no k-Darbouxian nor rational first integral with degree smaller than N.*
- *If it returns an equation (Eq_0) or (Eq_1) then this equation leads to a first integral.*
- *If it returns "I don't know", then $(x_0^\star, y_0^\star, x_0^\star, y_1^\star)$ belongs to*

$$(\mathbb{K}^2 \times (A^{-1}(0) \cup \Sigma_{D_0,N} \cup \Sigma_{D_1,N,k})) \cup ((A^{-1}(0) \cup \Sigma_{D_0,N} \cup \Sigma_{D_1,N,k}) \times \mathbb{K}^2) \cup \mathfrak{S}_{2N+2d-1}.$$

Proof. If the algorithm returns "None", this means that $\tilde{E}_{D_1}^{N,k}(x_0^\star, y_0^\star) \neq 0$. Theorem 29 implies that D_0 has no rational nor k-Darbouxian first integral with degree smaller than N.

If the algorithm returns (Eq_1) this means that we have $R_1 = 0$ in Build Darbouxian first integral. Proposition 10 gives then the desired result.

If the algorithm returns (Eq_0) then we necessarily have a rational first integral since Step 2 checks if $D_0(\mathcal{P}_0/\mathcal{P}_1) = 0$ and $\mathcal{P}_0/\mathcal{P}_1 \notin \mathbb{K}$.

Now we prove the last point of the proposition. We suppose that $(x_0^\star, y_0^\star, x_0^\star, y_1^\star)$ do not belong to

$$(\mathbb{K}^2 \times (A^{-1}(0) \cup \Sigma_{D_0,N} \cup \Sigma_{D_1,N,k})) \cup ((A^{-1}(0) \cup \Sigma_{D_0,N} \cup \Sigma_{D_1,N,k}) \times \mathbb{K}^2) \cup \mathfrak{S}_{2N+2d-1}.$$

First, if $\dim_{\mathbb{K}} \ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0^\star, y_i^\star) = 0$ then in Step 1c of Compute Darbouxian first integral we have $\mathcal{S} =$"None". Thus the algorithm returns "None".

Second, we suppose that $\dim_{\mathbb{K}} \ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0^\star, y_i^\star) \neq 0$.
In Step 1c of Compute Darbouxian first integral we have $\mathcal{S} = \bar{y}^k P - Q$.
• If $P = 0$, then the computed solution has the form $(Q, 0)$ with $Q \neq 0$ and $Q \in \ker \tilde{\mathcal{E}}_{D_0}^{N}(x_0^\star, y_i^\star)$. Furthermore $(x_0^\star, y_i^\star) \notin \Sigma_{D_0,N}$, thus by Lemma 42 we get $Q\big(x, y(x)\big) = 0$. Then a factor of Q is a Darboux polynomial which vanishes at $(x_0^\star, y_i^\star)$ with $(x_0^\star, y_0^\star, x_0^\star, y_1^\star) \notin \mathfrak{S}_N$. As $(x_0^\star, y_i^\star) \notin \Sigma_{D_0,N}$, we also deduce that there exists a rational first integral with degree p smaller than N. Thus in Step 1d of Compute Darbouxian first integral the polynomial $\mathcal{P}_i$ is a Darboux polynomial with degree p since in Compute solution extactic solution we compute a solution with minimal degree and $(x_0^\star, y_0^\star, x_0^\star, y_1^\star) \notin \mathfrak{S}_N$.
We claim that if we have obtained a solution with $P = 0$ for $(x_0^\star, y_0^\star)$ then we necessary have the same situation for $(x_0^\star, y_1^\star)$. Indeed, we have already remarked that in this situation D_0 admits a rational first integral f/g with degree p smaller than N. Thus as $(x_0^\star, y_0^\star, x_0^\star, y_1^\star) \notin \mathfrak{S}_N$ there exists an irreducible Darboux polynomial $\mathcal{Q}$ with degree p which gives a non-trivial solution $(\mathcal{Q}, 0)$ in $\ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0^\star, y_1^\star)$. As in Compute solution extactic kernel we compute a non-trivial solution with minimal weighted degree this proves our claim.
Therefore, in this situation we have $\mathcal{P}_0 = \lambda_0 f - \mu_0 g$, and $\mathcal{P}_1 = \lambda_1 f - \mu_1 g$, where $(\lambda_i : \mu_i) \notin \sigma(f, g)$. Now as $(x_0^\star, y_0^\star, x_0^\star, y_1^\star) \notin \mathfrak{S}_N$, we have $(\lambda_0 : \mu_0) \neq (\lambda_1 : \mu_1)$. Then $\mathcal{P}_0/\mathcal{P}_1$ is not constant and thus gives a first integral.

• If $P \neq 0$ then $Q \neq 0$. Indeed, if $Q = 0$ then as $(x_0^\star, y_i^\star) \notin \Sigma_{D_1,N,k}$ we have, thanks to Lemma 42, $\bar{y}(x)^k P\big(x, y(x)\big) = 0$. Since $\bar{y}(x) \neq 0$ we deduce that $P\big(x, y(x)\big) = 0$. Therefore a factor $\mathcal{P}$ of P is a Darboux polynomial which vanishes at $(x_0^\star, y_i^\star)$. It would give a solution $(\mathcal{P}, 0) \in \ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0^\star, y_i^\star)$. This is absurd since Compute solution extactic kernel returns a solution with minimal weighted degree. It follows $Q \neq 0$.
Furthermore, we have $Q \notin \ker \tilde{\mathcal{E}}_{D_0}^{N}(x_0^\star, y_i^\star)$. Indeed, since $(x_0^\star, y_i^\star) \notin \Sigma_{D_0,N}$, the contrary would imply $Q\big(x, y(x)\big) = 0$, thus $(Q, 0) \in \ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0^\star, y_1^\star)$. This is absurd since $\bar{y}^k P - Q$ is a solution with minimal weighted degree.
In Build Darbouxian first integral, we thus compute R_1.
If $R_1 = 0$ then by Proposition 10 we get a Darbouxian first integral.
Now, we suppose that $R_1 \neq 0$.
As $(x_0^\star, y_i^\star) \notin \Sigma_{D_1,N,k}$ we have $\bar{y}(x)^k P\big(x, y(x)\big) - Q\big(x, y(x)\big) = 0$ thanks to Lemma 42.

Then, with the same strategy used in Lemma 28, we get $R_1\big(x, y(x)\big) = 0$. Therefore a factor $\mathcal{P}_i$ of R_1 gives an irreducible Darboux polynomial with degree smaller than $2N+2d-1$ which vanishes at $(x_0^\star, y_i^\star)$. As $(x_0^\star, y_0^\star, x_0^\star, y_1^\star) \notin \mathfrak{S}_{2N+2d-1}$, this implies that D_0 admits a rational first integral of degree $\leq 2N+2d-1$ and $\mathcal{P}_0, \mathcal{P}_1$ define two different levels of this first integral. Then $\mathcal{P}_0/\mathcal{P}_1$ is not constant and thus gives a rational first integral. Therefore, the test at Step 2 is satisfied, and the algorithm returns a rational first integral. Thus when $R_1 \neq 0$ the algorithm will never return "I don't know". This concludes the proof. □

Proposition 48. *We set*

$$\mathcal{D}(d, N) = 2d + 2\mathcal{B}_0(d, N) + 2\mathcal{B}_1(d, N) + 2\Big(\frac{d(d+1)}{2} + 6\Big)(2N + 2d - 1).$$

There exists a polynomial H with degree smaller than $\mathcal{D}(d, N)$ such that:
If $H(x_0^\star, y_0^\star, y_1^\star) \neq 0$ then Compute Darbouxian first integral *returns "None" or an equation leading to a first integral.*

Proof. From Corollary 44 we deduce the existence of a polynomial $\tilde{H}$ such that:

$$\Sigma_{D_0,N} \cup \Sigma_{D_1,N,k} \cup \mathcal{V}(A) \subset \mathcal{V}(\tilde{H})$$

where $\deg(\tilde{H}) \leq d + \mathcal{B}_0(d, N) + \mathcal{B}_1(d, N)$. We also have from Lemma 46 the existence of a polynomial $\tilde{\tilde{H}}$ such that:

$$\mathfrak{S}_{2N+2d-1} \subset \mathcal{V}(\tilde{\tilde{H}})$$

Thus the polynomial

$$H(x_0^\star, y_0^\star, y_0^\star) = \tilde{H}(x_0^\star, y_0^\star)\tilde{H}(x_0^\star, y_1^\star)\tilde{\tilde{H}}(x_0^\star, y_0^\star, x_0^\star, y_1^\star)$$

vanishes on the set given in Proposition 47 in the "I don't know" part. So if $H(x_0^\star, y_0^\star, y_1^\star) \neq 0$ then Compute Darbouxian first integral returns "None" or an equation leading to a frst integral, and the degree of H satisfies the degree bound. □

Corollary 49. *Let Ω a finite subset of $\mathbb{K}$ of cardinal $|\Omega|$ greater than $\mathcal{D}(d, N)$ and assume that in* Compute Darbouxian first integral *$x_0^\star$, $y_0^\star$, $y_1^\star$ are chosen independently and uniformly at random in Ω. Then,* Compute Darbouxian first integral *returns "None" or an equation leading to a first integral with probability at least*

$$1 - \frac{\mathcal{D}(d, N)}{|\Omega|}.$$

Proof. This follows from Proposition 48, and Zippel-Schwartz's lemma, see [gGG99]. □

Remark 50. In practice the "practical" probability will be much better, see Section 7.

Proposition 51. *If D_0 admits a rational or Darbouxian first integral with degree smaller than N then* Compute Darbouxian first integral *returns an equation with minimal degree.*

Proof. This follows directly from the fact that Compute solution extactic kernel returns a solution with minimal weighted degree. □

5.3. **Liouvillian first integrals.** This section describes how to use the results given in the previous section in order to get an efficient probabilistic algorithm for searching Liouvillian first integrals.

Build Liouvillian first integral

`Input:` $A(x,y), B(x,y),\ P(x,y), Q(x,y), R(x,y) \in \mathbb{K}[x,y]$ such that $(P,Q,R) \neq 0$, $(x_0, y_0) \in \mathbb{K}^2$.
`Output:` An equation $(Eq_2) : \partial_y^2 \mathcal{F} - F(x,y)\partial_y \mathcal{F} = 0$, or $(Eq_1) : \partial_y \mathcal{F} - F = 0$, where $F(x,y) \in \mathbb{K}(x,y) \setminus \{0\}$, or a polynomial $\mathcal{P}(x,y) \in \mathbb{K}[x,y]$.

(1) If $Q = 0$ then Return(Build Darbouxian first integral(A, B, P, R, x_0, y_0))

(2) Compute $P_1 := A^3Q^2\big(D_0(P/Q) + A(P/Q)\partial_y(B/A) + A\partial_y^2(B/A)\big)$, $Q_1 := A^3Q^2D_0(R/Q)$.
(3) If $P_1 = 0$ then Return $(Eq_2) : \partial_y^2\mathcal{F} - (P/Q)\partial_y\mathcal{F} = 0$
Else Return$\big($Build Darbouxian first integral$(A, B, P_1, Q_1, x_0, y_0)\big)$

Compute Liouvillian first integral

`Input:` $A, B \in \mathbb{K}[x,y]$, $(x_0^\star, y_0^\star)$, $(x_0^\star, y_1^\star) \in \mathbb{K}^2$, $N \in \mathbb{N}$
`Output:` An equation $(Eq_2) : \partial_y^2\mathcal{F} - F\partial_y\mathcal{F} = 0$, or $(Eq_1) : \partial_y\mathcal{F} - F = 0$, or an equation $(Eq_0) : \mathcal{F} - F = 0$ where $F(x,y) \in \mathbb{K}(x,y)$, or "None" or "I don't know".

(1) For $y_i^\star$ in $\{y_0^\star, y_1^\star\}$ do
  (a) If $A(x_0^\star, y_i^\star) = 0$ then Return "I don't know".
  (b) Compute flow series$(A, B, x_0^\star, y_i^\star, N, 2)$=: $y(x), \bar{y}(x), \bar{y}^{(2)}(x)$.
  (c) Compute solution extactic kernel$(A, B, y(x), \bar{y}(x), \bar{y}^{(2)}(x), N, 2)$=:$\mathcal{S}$.
  If $\mathcal{S}$="None", then Return("None"),
  else $\mathcal{S} =: P(x,y)\bar{y}^2 + Q(x,y)\bar{y}^{(2)} + R(x,y)\bar{y}$.
  (d) Build Liouvillian first integral$(A, B, P, Q, R, x_0^\star, y_i^\star)$=: $\mathcal{P}_i$.
  (e) If $\mathcal{P}_i \notin \mathbb{K}[x,y]$ then Return$(\mathcal{P}_i)$
(2) If $\mathcal{P}_0/\mathcal{P}_1 \notin \mathbb{K}$ and $D_0(\mathcal{P}_0/\mathcal{P}_1) = 0$ then Return$((Eq_0) : \mathcal{F} - \mathcal{P}_0/\mathcal{P}_1 = 0)$,
else Return "I don't know".

Proposition 52. *The algorithm* Compute Liouvillian first integral *satisfies the following properties:*

- *If it returns "None" then there are no Liouvillian nor Darbouxian nor rational first integral with degree smaller than N.*
- *If it returns an equation (Eq_0) or (Eq_1) or (Eq_2) then this equation leads to a non-trivial first integral.*
- *If it returns "I don't know", then $(x_0^\star, y_0^\star, x_0^\star, y_1^\star)$ belongs to*

$$\big(\mathbb{K}^2\times(\mathcal{V}(A)\cup\Sigma_{D_0,N}\cup\Sigma_{D_1,N}\cup\Sigma_{D_2,N})\big)\cup\big((\mathcal{V}(A)\cup\Sigma_{D_0,N}\cup\Sigma_{D_1,N}\cup\Sigma_{D_2,N})\times\mathbb{K}^2\big)\cup\mathfrak{S}_{4N+8d-3}.$$

Proof. If the algorithm returns "None", this means that we have $\tilde{E}_{D_2}^N(x_0^\star, y_0^\star) \neq 0$ or $\tilde{E}_{D_2}^N(x_0^\star, y_1^\star) \neq 0$. Theorem 34 implies that D_0 has no rational nor Darbouxian nor Liouvillian first integral with degree smaller than N.

If the algorithm returns (Eq_2) this means that we have $P_1 = 0$ in Build Liouvillian first integral. Proposition 10 gives then the desired result.

If the algorithm returns (Eq_1) this result is correct thanks to Proposition 47. Indeed, the algorithm returns (Eq_1) when Build Liouvillian first integral uses Build Darbouxian first integral.

If the algorithm returns (Eq_0) then the output is correct as shown in Proposition 47.

Now,we prove the last point of the proposition and we suppose $(x_0^\star, y_0^\star, x_0^\star, y_1^\star)$ do not belong to

$$(\mathbb{K}^2\times(\mathcal{V}(A)\cup\Sigma_{D_0,N}\cup\Sigma_{D_1,N}\cup\Sigma_{D_2,N}))\cup((\mathcal{V}(A)\cup\Sigma_{D_0,N}\cup\Sigma_{D_1,N}\cup\Sigma_{D_2,N})\times\mathbb{K}^2)\cup\mathfrak{S}_{4N+8d-3}.$$

First, if $\dim_{\mathbb{K}} \ker \tilde{\mathcal{E}}_{D_2}^N(x_0^\star, y_i^\star) = 0$ then in Step 1c of Compute Liouvillian first integral we have $\mathcal{S} =$"None". Thus the algorithm returns "None".

Second, we suppose that $\dim_{\mathbb{K}} \ker \tilde{\mathcal{E}}_{D_2}^N(x_0^\star, y_i^\star) \neq 0$.
In Step 1c of Compute Liouvillian first integral we have $\mathcal{S} = P\bar{y}^2 + Q\bar{\bar{y}} + R\bar{y}$.
• If $P = Q = 0$ then as in Proposition 47 we deduce that in this situation the algorithm returns a rational first integral with minimal degree.
• If $Q = 0$ and $P \neq 0$ then we deduce that $P\bar{y} + R \in \tilde{\mathcal{E}}_{D_1}^N(x_0^\star, y_i^\star)$. Then Proposition 47 allows us to conclude in this situation.
• If $Q \neq 0$ then $Q \notin \ker \tilde{\mathcal{E}}_{D_0}^N(x_0^\star, y_i^\star)$. Indeed, as $(x_0^\star, y_i^\star) \notin \Sigma_{D_0,N}$, this would imply the existence of a rational first integral with degree smaller than N. Therefore, this would give a non-trivial element $(0, 0, \mathcal{R}) \in \ker \tilde{\mathcal{E}}_{D_2}^N(x_0^\star, y_i^\star)$. This is impossible since the computed solution has a minimal weighted degree.
Then in Build Liouvillian first integral, we compute P_1.
If $P_1 = 0$ then by Proposition 10 we get a Liouvillian first integral.
Now, we suppose $P_1 \neq 0$.
As $(x_0^\star, y_i^\star) \notin \Sigma_{D_2,N}$, we have thanks to Lemma 42

$$P\big(x, y(x)\big)\bar{y}^2(x) + Q\big(x, y(x)\big)\bar{\bar{y}}(x) + R\big(x, y(x)\big)\bar{y}(x) = 0.$$

Thus with the strategy used in Lemma 33 we get $P_1\big(x, y(x)\big)\bar{y}(x) + Q_1\big(x, y(x)\big) = 0$, where $\deg(P_1), \deg(Q_1) \leq 2N + 3d - 1$. Then by Lemma 28, the algorithm Build Darbouxian first integral gives either a Darbouxian first integral with degree smaller than $2N + 3d - 1$, or a rational first integral with degree smaller than $4N + 8d - 3$ or a Darboux polynomial with degree smaller than $4N + 8d - 3$. In this last case, we continue to Step 2 and we have a Darboux polynomial $\mathcal{P}_i$ with degree smaller than $4N + 8d - 3$ vanishing at $(x_0^\star, y_i^\star)$. As $(x_0^\star, y_0^\star, x_0^\star, y_1^\star) \notin \mathfrak{S}_{4N+8d-3}$, this implies that D_0 admits a rational first integral of degree $\leq 4N + 8d - 3$ and $\mathcal{P}_0, \mathcal{P}_1$ define two different levels of this first integral. Then $\mathcal{P}_0/\mathcal{P}_1$ is not constant and thus gives a rational first integral. So the checking at Step 2 is satisfied, and the algorithm returns a rational first integral. □

Proposition 53. *We set*

$$\mathcal{L}(d, N) = 2d + 2\mathcal{B}_0(d, N) + 2\mathcal{B}_1(d, N) + 2\mathcal{B}_2(d, N) + 2\Big(\frac{d(d+1)}{2} + 6\Big)(4N + 8d - 3).$$

There exists a polynomial H_L with degree smaller than $\mathcal{L}(d,N)$ such that:
If $H_L(x_0^\star, y_0^\star, y_1^\star) \neq 0$ then Compute Liouvillian first integral *returns "None" or an equation leading to a first integral.*

Proof. The proof is done exactly in the same way as the proof of Proposition 53. □

Corollary 54. *Let Ω a finite subset of $\mathbb{K}$ of cardinal $|\Omega|$ greater than $\mathcal{L}(d,N)$ and assume that in* Compute Liouvillian first integral *$x_0^\star$, $y_0^\star$, $y_1^\star$ are chosen independently and uniformly at random in Ω. Then,* Compute Liouvillian first integral *returns "None " or an equation leading to a first integral with probability at least*

$$1 - \frac{\mathcal{L}(d,N)}{|\Omega|}.$$

Proposition 55. *If D_0 admits a rational or Darbouxian or Liouvillian first integral with degree smaller than N then* Compute Liouvillian first integral *returns an equation with minimal degree.*

Proof. As in the Darbouxian case this is a direct consequence of the minimality of the weighted degree of a solution in Compute solution extactic kernel. □

5.4. **Riccati first integrals.** This section describes how to use the results given in the previous section in order to get an efficient probabilistic algorithm for searching Riccati first integrals.

Build Riccati first integral
`Input:` $A(x,y), B(x,y),\ P(x,y), Q(x,y), R(x,y) \in \mathbb{K}[x,y]$ such that $(P,Q,R) \neq 0$, $(x_0, y_0) \in \mathbb{K}^2$.
`Output:`An equation $(Eq_3) : \partial_y^2 \mathcal{F} - F(x,y)\mathcal{F} = 0$, or $(Eq_1) : \partial_y \mathcal{F} - \sqrt{F(x,y)} = 0$, where $F(x,y) \in \mathbb{K}(x,y)$, or a polynomial $\mathcal{P}(x,y) \in \mathbb{K}[x,y]$.

(1) If $Q = 0$ then Return(Build Darbouxian first integral$(A, B, P, R, x_0, y_0, 2)$)

(2) Compute $P_1 := A^4Q^2\big(4D_0(P/Q) + 8A(P/Q)\partial_y(B/A) - 2A\partial_y^3(B/A)\big)$,
$Q_1 := A^4Q^2D_0(R/Q)$.
(3) If $P_1 = 0$ then Return $(Eq_3) : \partial_y^2\mathcal{F} - (P/Q)\mathcal{F} = 0$
Else Return$\big($Build Darbouxian first integral$(A, B, P_1, Q_1, x_0, y_0, 2)\big)$

Compute Riccati first integral
`Input:` $A, B \in \mathbb{K}[x,y]$, $(x_0^\star, y_0^\star)$, $(x_0^\star, y_1^\star) \in \mathbb{K}^2$, $N \in \mathbb{N}$
`Output:` An equation $(Eq_3) : \partial_y^2\mathcal{F} - F\mathcal{F}$, or $(Eq_1) : \partial_y\mathcal{F} - \sqrt{F} = 0$, or an equation $(Eq_0) : \mathcal{F} - F = 0$ where $F(x,y) \in \mathbb{K}(x,y)$, or "None" or "I don't know".

(1) For $y_i^\star$ in $\{y_0^\star, y_1^\star\}$ do
  (a) If $A(x_0^\star, y_i^\star) = 0$ then Return "I don't know".
  (b) Compute flow series$(A, B, x_0^\star, y_i^\star, N, 3)$=: $y(x), \bar{y}(x), \bar{y}^{(2)}(x), \bar{y}^{(3)}(x)$.
  (c) Compute solution extactic kernel$(A, B, y(x), \bar{y}(x), \bar{y}^{(2)}(x), \bar{y}^{(3)}(x), N, 3)$=:$\mathcal{S}$.
  If $\mathcal{S}$="None", then Return("None"),
  else $\mathcal{S} =: 4P(x,y)\bar{y}^4 + Q(x,y)(3(\bar{y}^{(2)})^2 - 2\bar{y}^{(3)}\bar{y}) + R(x,y)\bar{y}^2$.
  (d) Build Riccati first integral$(A, B, P, Q, R, x_0^\star, y_i^\star)$=: $\mathcal{P}_i$.
  (e) If $\mathcal{P}_i \notin \mathbb{K}[x,y]$ then Return$(\mathcal{P}_i)$

(2) If $\mathcal{P}_0/\mathcal{P}_1 \notin \mathbb{K}$ and $D_0(\mathcal{P}_0/\mathcal{P}_1) = 0$ then Return($(Eq_0) : \mathcal{F} - \mathcal{P}_0/\mathcal{P}_1 = 0$), else Return "I don't know".

Proposition 56. *The algorithm* Compute Riccati first integral *satisfies the following properties:*

- *If it returns "None" then there are no Riccati nor 2-Darbouxian nor rational first integral with degree smaller than N.*
- *If it returns an equation (Eq_0) or (Eq_1) or (Eq_3) then this equation leads to a non-trivial first integral.*
- *If it returns "I don't know", then $(x_0^\star, y_0^\star, x_0^\star, y_1^\star)$ belongs to*

$$\big(\mathbb{K}^2\times(\mathcal{V}(A)\cup\Sigma_{D_0,N}\cup\Sigma_{D_1,N,2}\cup\Sigma_{D_3,N})\big)\cup\big((\mathcal{V}(A)\cup\Sigma_{D_0,N}\cup\Sigma_{D_1,N,2}\cup\Sigma_{D_3,N})\times\mathbb{K}^2\big)\cup\mathfrak{S}_{4N+10d-3}.$$

Proof. If the algorithm returns "None", this means that we have $\tilde{E}^N_{D_3}(x_0^\star, y_0^\star) \neq 0$ or $\tilde{E}^N_{D_3}(x_0^\star, y_1^\star) \neq 0$. Theorem 39 implies that D_0 has no rational nor 2-Darbouxian nor Riccati first integral with degree smaller than N.

If the algorithm returns (Eq_3) this means that we have $P_1 = 0$ in Build Riccati first integral. Proposition 10 gives then the desired result.

If the algorithm returns (Eq_1) this result is correct thanks to Proposition 47. Indeed, the algorithm returns (Eq_1) when Build Riccati first integral uses Build Darbouxian first integral.

If the algorithm returns (Eq_0) then the output is correct as shown in Proposition 47.

Now,we prove the last point of the proposition and we suppose $(x_0^\star, y_0^\star, x_0^\star, y_1^\star)$ do not belong to

$$(\mathbb{K}^2\times(\mathcal{V}(A)\cup\Sigma_{D_0,N}\cup\Sigma_{D_1,N,2}\cup\Sigma_{D_3,N}))\cup((\mathcal{V}(A)\cup\Sigma_{D_0,N}\cup\Sigma_{D_1,N,2}\cup\Sigma_{D_3,N})\times\mathbb{K}^2)\cup\mathfrak{S}_{4N+10d-3}.$$

First, if $\dim_{\mathbb{K}} \ker \tilde{\mathcal{E}}^N_{D_3}(x_0^\star, y_i^\star) = 0$ then in Step 1c of Compute Riccati first integral we have $\mathcal{S} =$"None". Thus the algorithm returns "None".

Second, we suppose that $\dim_{\mathbb{K}} \ker \tilde{\mathcal{E}}^N_{D_3}(x_0^\star, y_i^\star) \neq 0$.
In Step 1c of Compute Riccati first integral we have

$$\mathcal{S} = 4P(x,y)\bar{y}^4 + Q(x,y)(3\bar{\bar{y}}^2 - 2\bar{\bar{\bar{y}}}\bar{y}) + R(x,y)\bar{y}^2.$$

• If $P = Q = 0$ then as in Proposition 47 we deduce that in this situation the algorithm returns a rational first integral with minimal degree.
• If $Q = 0$ and $P \neq 0$ then we deduce that $4P\bar{y}^2 + R \in \tilde{\mathcal{E}}^{N,2}_{D_1}(x_0^\star, y_i^\star)$. Then Proposition 47 allows us to conclude in this situation.
• If $Q \neq 0$ then $Q \notin \ker \tilde{\mathcal{E}}^N_{D_0}(x_0^\star, y_i^\star)$. Indeed, as $(x_0^\star, y_i^\star) \notin \Sigma_{D_0,N}$, this would imply the existence of a rational first integral with degree smaller than N. Therefore, this would give a non-trivial element $(0, 0, \mathcal{R}) \in \ker \tilde{\mathcal{E}}^N_{D_3}(x_0^\star, y_i^\star)$. This is impossible since the computed solution has a minimal weighted degree.
Then in Build Riccati first integral, we compute P_1.
If $P_1 = 0$ then by Proposition 10 we get a Riccati first integral.

Now, we suppose $P_1 \neq 0$.
As $(x_0^\star, y_i^\star) \notin \Sigma_{D_3,N}$, we have

$$4P\big(x, y(x)\big)\bar{y}(x)^4 + Q\big(x, y(x)\big)(3\bar{\bar{y}}(x)^2 - 2\bar{\bar{\bar{y}}}(x)\bar{y}(x)) + R\big(x, y(x)\big)\bar{y}(x)^2 = 0$$

thanks to Lemma 42. Thus with the strategy used in Lemma 38 we get

$$P_1\big(x, y(x)\big)\bar{y}(x)^2 + Q_1\big(x, y(x)\big) = 0,$$

where $\deg(P_1), \deg(Q_1) \leq 2N + 4d - 1$. Then by Lemma 28, the algorithm Build Darbouxian first integral gives either a 2-Darbouxian first integral with degree smaller than $2N+4d-1$, or a rational first integral with degree smaller than $4N+10d-3$ or a Darboux polynomial with degree smaller than $4N + 10d - 3$. In this last case, we continue to Step 2 and we have a Darboux polynomial $\mathcal{P}_i$ with degree smaller than $4N + 10d - 3$ vanishing at $(x_0^\star, y_i^\star)$. As $(x_0^\star, y_0^\star, x_0^\star, y_1^\star) \notin \mathfrak{S}_{4N+10d-3}$, this implies that D_0 admits a rational first integral of degree $\leq 4N + 10d - 3$ and $\mathcal{P}_0, \mathcal{P}_1$ define two different levels of this first integral. Then $\mathcal{P}_0/\mathcal{P}_1$ is not constant and thus gives a rational first integral. Therefore the test at Step 2 is satisfied, and the algorithm returns a rational first integral. □

As before we deduce the following results:

Proposition 57. *We set*

$$\mathcal{R}(d, N) = 2d + 2\mathcal{B}_0(d, N) + 2\mathcal{B}_1(d, N) + 2\mathcal{B}_3(d, N) + \big(d(d+1) + 12\big)(4N + 10d - 3)$$

There exists a polynomial H_R with degree smaller than $\mathcal{R}(d, N)$ such that:
If $H_R(x_0^\star, y_0^\star, y_1^\star) \neq 0$ then Compute Riccati first integral returns "None" or an equation leading to a first integral.

Proof. The proof is done exactly in the same way as the proof of Proposition 53. □

Corollary 58. *Let Ω a finite subset of $\mathbb{K}$ of cardinal $|\Omega|$ greater than $\mathcal{R}(d, N)$ and assume that in Compute Riccati first integral $x_0^\star$, $y_0^\star$, $y_1^\star$ are chosen independently and uniformly at random in Ω. Then, Compute Riccati first integral returns "None" or an equation leading to a first integral with probability at least*

$$1 - \frac{\mathcal{R}(d, N)}{|\Omega|}.$$

Proposition 59. *If D_0 admits a rational or 2-Darbouxian or Riccati first integral with degree smaller than N then Compute Riccati first integral returns an equation with minimal degree.*

Proof. As in the Darbouxian case this is a direct consequence of the minimality of the weighted degree of a solution in Compute solution extactic kernel. □

5.5. **Deterministic algorithms.** In this section we show how to get a deterministic algorithm from our probabilistic ones. We give explicitly the deterministic algorithm for the Riccati case below. The Darbouxian and Liouvillian can be obtained in the same way.

Deterministic computation Riccati first integral
`Input:` $A, B \in \mathbb{K}[x, y]$, such that $A(x, y) \neq 0$, $N \in \mathbb{N}$
`Output:` An equation $(Eq_3) : \partial_y^2 \mathcal{F} - F\mathcal{F} = 0$, or $(Eq_1) : \partial_y \mathcal{F} - \sqrt{F} = 0$, where $F(x, y) \in \mathbb{K}(x, y)$, or an equation $(Eq_0) : \mathcal{F} - F = 0$ where $F(x, y) \in \mathbb{K}(x, y) \setminus \mathbb{K}$,

or "None".

(1) Set $i := 0$, $x_0^\star := -1$.
(2) While $i \leq \mathcal{R}(d, N) + 1$ do
    (a) $x_0^\star := x_0^\star + 1$, $\Omega := \emptyset$.
    (b) While $A(x_0^\star, y) = 0$ do $x_0^\star := x_0^\star + 1$.
    (c) While $|\Omega| \leq \mathcal{R}(d, N) + 1$ do
        (i) Choose two random elements $y_0^\star, y_1^\star \in \mathbb{K} \setminus \Omega$ such that $y_0^\star \neq y_1^\star$ and $A(x_0^\star, y_j^\star) \neq 0$.
        (ii) $\mathcal{E} :=$Compute Riccati first integral$(A, B, (x_0^\star, y_0^\star), (x_0^\star, y_1^\star), N)$.
        (iii) If $\mathcal{E} =$"None", then Return "None".
        (iv) If $\mathcal{E} =$"I don't know" then $\Omega := \Omega \cup \{y_0^\star, y_1^\star\}$, Else Return $\mathcal{E}$.
    (d) $i := i + 1$.
(3) Return "None".

Proposition 60. *The algorithm* Deterministic computation Riccati first integral *is correct.*

Proof. The deterministic algorithm repeats the probabilistic algorithm. If the probabilistic returns an equation or "None" then this output is correct thanks to Proposition 56.
We want to get $x_0^\star, y_0^\star, y_1^\star$ such that $H(x_0^\star, y_0^\star, y_1^\star) \neq 0$. As we use the probabilistic algorithm with at most $\mathcal{R}(d, N) + 1$ different values for $x_0^\star$ and $\mathcal{R}(d, N) + 1$ different values for $(y_0^\star, y_1^\star)$ we necessarily avoid situations where $H(x_0^\star, y_0^\star, y_1^\star)$ is equal to zero. Then Proposition 57 implies that the probabilistic algorithm returns an output different from "I don't know" and we get the desired output. $\square$

6. Complexity results

In this section we study the arithmetic complexity of our algorithms. We focus on the dependency on the degree bound N and we recall that we assume that $N \geq d$, where $d = \max(\deg(A), \deg(B))$ denotes the degree of the polynomial vector field. This hypothesis is natural because if a derivation has a polynomial first integral of degree N, then necessarily $d \leq N - 1$. More precisely, we suppose that d is fixed and N tends to infinity.

All the complexity estimates are given in terms of arithmetic operations in $\mathbb{K}$. We use the notation $f \in \tilde{\mathcal{O}}(g)$, roughly speaking this means that we neglect the logarithmic factors in the expression of the complexity. For a precise definition, see [gGG99, Definition 25.8].
We suppose that the Fast Fourier Transform can be used so that two univariate polynomials with coefficients in $\mathbb{K}$ and degree bounded by r can be multiplied in $\tilde{\mathcal{O}}(r)$, see [gGG99, Corollary 8.19].
We further assume that two matrices of size n with entries in $\mathbb{K}$ can be multiplied using $\mathcal{O}(n^\omega)$, where $2 \leq \omega \leq 3$ is the matrix multiplication exponent, see [gGG99, Ch. 12].

The algorithm Compute flow series is a direct application of the algorithm given in [BCO+07]. In our situation , the number of arithmetic operations needed to perform this subroutine is in $\tilde{\mathcal{O}}(L\sigma + \sigma)$. Here L is the number of arithmetic operations needed to evaluate the rational functions defining the system (S'_r). Thus, we have $L \in \mathcal{O}(d^2)$. Furthermore, σ is the precision on the power series, then $\sigma \in \mathcal{O}(N^2)$. It thus follows that the computation modulo $(x - x_0^\star)^\sigma$ of $y(x)$, $y^2(x)$,...,$y^N(x)$, $\bar{y}(x)$, $y(x)\bar{y}(x)$, ...,$y^N(x)\bar{y}^{(3)}(x)$ can done with at most $\mathcal{O}(d^2N^2)$ arithmetic operations.

In Compute solution extactic kernel we need to find a nontrivial solution for $\mathcal{E}^N_{r,D_r}$. This can be done with an Hermite-Padé approximation. We recall this setting: We have m polynomials $f_i(x) \in \mathbb{K}[x]$, a precision σ, a shift $s = (s_1, \dots, s_m)$ and we want to compute m polynomials $p_i(x) \in \mathbb{K}[x]$ such that

$$\sum_{i=1}^{m} p_i.f_i = 0 \mod x^\sigma.$$

The set of all solutions $(p_1, \dots, p_m)$ is a $\mathbb{K}[x]$-module. A s-minimal approximate basis is a basis of this module and furthermore an element of this basis has minimal s-degree among all solutions of the problem. We recall that the s-degree of $(p_1, \dots, p_m)$ is $\max_i \deg(p_i + s_i)$.
We can compute such a basis with $\tilde{\mathcal{O}}\big(m^{\omega-1}(\sigma + \xi)\big)$ arithmetic operations in $\mathbb{K}$, where $\xi = \sum_i (s_i - \min(s))$, see [BL94], [ZL12, Theorem 5.3] and [JNSV17].
In our situation we have $r \in [[0;3]]$, $m = (r+1)(N+1)$, $\sigma = (r+1)\frac{(N+1)(N+2)}{2}$.
When $r = 1$ we set:

$$(f_1, \dots, f_m) = \big(1, y(x), y^2(x), \dots, y^N(x), \bar{y}(x), \bar{y}(x)y(x), \bar{y}(x)y^2(x), \dots, \bar{y}(x)y^N(x)\big),$$
$$s = (0, 1, 2, \dots, N, N+1, \dots, 2N+1).$$

When $r = 2$ we set

$$\begin{aligned}(f_1, \dots, f_m) \quad = \quad & \big(\bar{y}(x), \bar{y}(x)y(x), \bar{y}(x)y^2(x), \dots, \bar{y}(x)y^N(x),\\ & \bar{y}^2(x), \bar{y}^2(x)y(x), \bar{y}^2(x)y^2(x), \dots, \bar{y}^2(x)y^N(x),\\ & \bar{y}^{(2)}(x), \bar{y}^{(2)}(x)y(x), \bar{y}^{(2)}(x)y^2(x), \dots, \bar{y}^{(2)}(x)y^N(x)\big),\end{aligned}$$
$$s = (0, 1, 2, \dots, N, N+1, \dots, 2N+1, 2N+2, \dots, 3N+2).$$

When $r = 3$ we set

$$\begin{aligned}(f_1, \dots, f_m) \quad = \quad & \big(\bar{y}^4(x), \bar{y}^4(x)y(x), \bar{y}^4(x)y^2(x), \dots, \bar{y}^4(x)y^N(x),\\ & \Psi(x), \Psi(x)y(x), \Psi(x)y^2(x), \dots, \Psi(x)y^N(x),\\ & \bar{y}^2(x), \bar{y}^2(x)y(x), \bar{y}^2(x)y^2(x), \dots, \bar{y}^2(x)y^N(x)\big)\end{aligned}$$

where $\Psi(x) = 3\bar{\bar{y}}^2(x) - 2\bar{\bar{\bar{y}}}(x)\bar{y}(x)$, and

$$s = (0, 1, 2, \dots, N, N+1, \dots, 2N+1, 2N+2, \dots, 3N+2).$$

We remark that from a solution $(p_1, \dots, p_m)$ we get:

- when $r = 1$, a polynomial
$$Q(x,y) + P(x,y)\bar{y} = \sum_{i=0}^{N} p_i(x)y^i + \sum_{i=0}^{N} p_{N+1+i}(x)y^i\bar{y},$$
- when $r = 2$, a polynomial
$$R(x,y)\bar{y} + P(x,y)\bar{y}^2 + Q(x,y)\bar{y}^{(2)} = \sum_{i=0}^{N} p_i(x)y^i\bar{y} + \sum_{i=0}^{N} p_{N+1+i}(x)y^i\bar{y}^2 + \sum_{i=0}^{N} p_{2N+2+i}(x)y^i\bar{y}^{(2)},$$

- when $r = 3$, a polynomial

$$P(x,y)\bar{y}^4+Q(x,y)\Psi+R(x,y)\bar{y}^2 = \sum_{i=0}^{N} p_i(x)y^i\bar{y}^4+\sum_{i=0}^{N} p_{N+1+i}(x)y^i\Psi+\sum_{i=0}^{N} p_{2N+2+i}(x)y^i\bar{y}^2,$$

where $\Psi = 3\bar{\bar{y}}^2 - 2\bar{\bar{\bar{y}}}\bar{y}$.

Therefore a solution with a minimal s-degree corresponds to a polynomial solution of $\mathcal{E}^N_{r,D_r}$ with minimal weighted degree. Thus the subroutine Compute solution extactic kernel can be done with at most $\tilde{\mathcal{O}}(N^{\omega-1}N^2) = \tilde{\mathcal{O}}(N^{\omega+1})$ arithmetic operations in $\mathbb{K}$.

The algorithm Build Darboux computes a gcd of bivariate polynomials with degree in $\mathcal{O}(N)$. This subroutine can be done with at most $\tilde{\mathcal{O}}(N^2)$ arithmetic operations in $\mathbb{K}$, see [gGG99]. Furthermore, we need to factorize a polynomial with degree at most N, this can be done in a probabilistic (respectively deterministic) way with $\tilde{\mathcal{O}}(N^3)$ (respectively $\tilde{\mathcal{O}}(N^{\omega+1})$) arithmetic operations plus the factorization of an univariate polynomial in $\mathbb{K}[T]$ with degree N, see [BLS+04, Lec06].

At last, in our algorithms we test if $D_0(\mathcal{P}_0/\mathcal{P}_1) = 0$. This step corresponds to the multiplication of bivariate polynomials with degree in $\mathcal{O}(N)$. Therefore this step can be done with at most $\tilde{\mathcal{O}}(N^2)$ arithmetic operations in $\mathbb{K}$.

In conclusion our probabilistic algorithms use at most $\tilde{\mathcal{O}}(N^{\omega+1} + d^2N^2)$ arithmetic operations in $\mathbb{K}$ plus the factorization of a univariate polynomial with degree at most N. This is the complexity given in Theorem 4.

As $\mathcal{R}(d,N) \in \mathcal{O}(dN^4)$, the deterministic algorithm uses at most $\tilde{\mathcal{O}}(d^2N^{\omega+9} + d^4N^{10})$ arithmetic operations in $\mathbb{K}$ plus the factorization of a univariate polynomial in $\mathbb{K}[T]$ with degree at most N.

7. Examples

The algorithms developed in the previous sections have been implemented in Maple. This implementation is available with some examples at:
`http://combot.perso.math.cnrs.fr/software.html`,
`https://www.math.univ-toulouse.fr/~cheze/Programme.html`.
The computations for the following examples have been done on a Macbook pro 2013, intel core i7 2.8 Ghz.
For practical reasons, the implemented version of our algorithms do not use the Hermite-Padé algorithm to find a solution of the extactic kernel. We just solve a linear system. Furthermore, the solutions $y(x)$,..., $\bar{\bar{\bar{y}}}(x)$ are computed from $y(x)$ and then integrated. For example, we compute $\bar{y}(x)$ with the formula:

$$\bar{y}(x) = exp\Big(\int \frac{B}{A}(x,y(x))dx\Big).$$

7.1. **The Darbouxian case.** Let us consider the system

$$\dot{x} = x^2 + 2xy + y^2 - 4x + 4y - 2, \;\; \dot{y} = x^2 + 2xy + y^2 + 4x - 4y - 2.$$

The algorithm Compute Darbouxian first integral returns in $0.2s$, when $N = 3$:

$$\frac{\partial \mathcal{F}}{\partial y} + \frac{14(x^2 + 2xy + y^2 - 4x + 4y - 2)}{11(x-y)(x^2 + 2xy + y^2 - 2)},$$

which after integration leads to the Darbouxian first integral

$$\mathcal{F}(x,y)=\sqrt{2}\ln\left(x+y-\sqrt{2}\right)-\sqrt{2}\ln\left(x+y+\sqrt{2}\right)+\ln(x-y).$$

Now noting

$$z=\frac{x+y+\sqrt{2}}{x+y-\sqrt{2}},\quad w=x-y,$$

we have for the first integral level $\mathcal{F}(x,y)=c$

$$w=e^c z^{\sqrt{2}}.$$

This curve is not algebraic for almost all c, and thus the system does not admit a rational first integral.

The initial points used in the execution of the algorithm above were $(1,8),(1,3)$. To get "I don't know", we need for example to use two bad points, i.e. two points vanishing a Darboux polynomial. From each point, we will obtain a Darboux polynomial, and thus the algorithm will try the quotient as a first integral, which will not work as the vector field has no rational first integral. We can choose for example $(1,1),(1,\sqrt{2}-1)$. Such bad pairs of initial points were never encountered when using (small) random initial points. In particular, the probabilistic algorithm is the only algorithm necessary to use in practice, and we never have to rerun it with several initial points.

If we use the algorithm Compute Darbouxian first integral with $N=2$ then the output is "None". This is correct and means that there exists no Darbouxian first integral with degree smaller than 2.

Now let us modify a little the previous example

$$\dot{x}=2\lambda^2x-2\lambda^2y+\lambda^2-x^2-2xy-y^2,\;\;\dot{y}=2\lambda^2y-2\lambda^2x+\lambda^2-x^2-2xy-y^2.$$

The algorithm Compute Darbouxian first integral returns with $\lambda=100$ and $N=3$

$$\frac{\partial\mathcal{F}}{\partial y}-\frac{312(x^2+2xy+y^2-20000x+20000y-10000)}{469(x-y)(y+100+x)(y-100+x)}$$

in $0.2s$ which after integration leads to the Darbouxian first integral

$$\mathcal{F}(x,y)=100\ln(x+y-100)-100\ln(x+y+100)+\ln(x-y).$$

Now the exponential of $\mathcal{F}$ gives a rational first integral

$$\left(\frac{x+y-100}{x+y+100}\right)^{100}(x-y),$$

which is of degree 101.

We remark that if we want to compute a rational first integral we can use Compute Darbouxian first integral. In this case the bound N is a bound on the degree of the product of the irreducible Darboux polynomials used to write the rational first integral (3 in the previous example) and not a bound on the degree of the first integral (101 in the previous example). The difference between these bounds is important when the rational first integral has one or several factors with large multiplicities.

7.2. **Comparison with the Avelar-Duarte-da Mota's algorithm.** Now let us compare our algorithm with the algorithm proposed by J. Avellar, L.G.S. Duarte, L.A.C.P. da Mota, denoted in the following by: ADM algorithm, see [ADdM12]. First, we consider a vector field of the form $(-\partial_y G, \partial_x G)$ with

$$G = x + \sum_{i=1}^{m} i \ln(x + y - i),$$

after multiplying by a common denominator.
We find the Darbouxian first integral in the following times.

m	1	2	3	4	5	6
Compute Darbouxian first integral	0.015	0.031	0.297	2.090	4.883	17.51
ADM algorithm	0.094	0.047	0.078	0.109	0.109	0.187

Unexpectedly, the ADM algorithm fails with $m \geq 7$. We see that the ADM algorithm computation times are much better than ours. This is because the Darboux polynomials are all of degree 1, and the ADM algorithm computes them first. After there is an exponential combinatoric step, but here it is negligible at those low m.

Let us now compare with a growing degree Darboux polynomial case

$$G = x + \ln(x + y^m - 1).$$

We find the Darbouxian first integral in the following times.

m	1	2	3	4	5	6
Compute Darbouxian first integral	0.015	0.015	0.125	0.843	1.809	6.412
ADM algorithm	0.094	0.078	1.123	$> 10^3$	$> 10^3$	$> 10^3$

The timings of our algorithm have the same order of magnitude, but the ADM algorithm becomes almost unusable. This is because the computation of Darboux polynomials is very expansive even for low degrees. In other words, as soon as the Darboux polynomials become a little to complicate, the ADM algorithm is not usable. Our algorithm never computes Darboux polynomials, and thus avoids this problem.

7.3. **The Liouvillian case.** Consider the system

$$\dot{x} = 2x^2 - 2y^2 - 1, \ \dot{y} = 2x^2 - 2y^2 - 3.$$

The algorithm Compute Liouvillian first integral returns in $0.3s$ when $N = 3$

$$\frac{\partial^2 \mathcal{F}}{\partial y^2} - \frac{2(x+y)(2x^2 - 4xy + 2y^2 - 1)}{2x^2 - 2y^2 - 1} \frac{\partial \mathcal{F}}{\partial y}.$$

After integration, this gives the first integral

$$\mathcal{F}(x, y) = \sqrt{\pi} \operatorname{erf}(x - y) + (x + y) e^{-(x-y)^2}.$$

Now noting $z = x - y, w = x + y$, we have on the level $\mathcal{F}(x, y) = c$

$$w = (c - \sqrt{\pi} \operatorname{erf}(z)) e^{z^2}$$

This function is never algebraic for $c \in \mathbb{C}$ as erf is not even elementary. Thus the system does not admit a rational first integral. The function $\mathcal{F}$ is holomorphic and thus all solution curves (outside the straight line at infinity) are levels of the form $\mathcal{F}(x, y) = c \in \mathbb{C}$. As none of these curves are algebraic, the system does not admit any Darboux polynomial. As the poles of a Darbouxian first integral are

Darboux polynomials, a Darbouxian first integral should be a polynomial, which is again not possible as there are no rational first integrals. Thus the system admits a Liouvillian first integral but no first integrals of lower class.

Example 185 of Kamke is an Abel equation with a Liouvillian first integral

$$\dot{x} = -x^7,\ \dot{y} = y^2(5x^3 + 2x^2y + 2y).$$

Compute Liouvillian first integral gives in 55.9s with $N = 7$:

$$\frac{\partial^2 \mathcal{F}}{\partial y^2} + \frac{x^6 + 7x^3y + 6x^2y^2 + 6y^2}{2y(x^6 + 2x^3y + x^2y^2 + y^2)} \frac{\partial \mathcal{F}}{\partial y} = 0.$$

This system admits a lower class first integral, a 4-Darbouxian first integral of degree 32, which can be recovered by integration of this equation, giving

$$\tilde{\mathcal{F}}(x,y) = \int \frac{y^{3/2}\sqrt{x}(5x^3 + 2x^2y + 2y)}{(x^6 + 2x^3y + x^2y^2 + y^2)^{5/4}} dx + \frac{x^{15/2}}{(x^6 + 2x^3y + x^2y^2 + y^2)^{5/4}\sqrt{y}} dy.$$

7.4. **The Riccati case.** Example 43 of Kamke is an Abel equation, and with $a = 3, b = 17$ admits a Riccati first integral

$$\dot{x} = 1,\ \dot{y} = -(9x^2 + 36x + 17)y^3 - 3xy^2.$$

Compute Riccati first integral gives in $390s$ with $N = 9$:

$$\frac{\partial^2 \mathcal{F}}{\partial y^2} - \frac{3P}{4(9x^2y + 36xy + 17y - 6)^2 y^3} \mathcal{F} = 0,$$

with

$$\begin{aligned} P \quad = \quad & 81x^4y^3 + 648x^3y^3 - 18x^3y^2 + 1602x^2y^3 - 180x^2y^2 \\ & + 1224xy^3 + 3x^2y - 466xy^2 + 289y^3 + 24xy - 204y^2 + 36y - 2. \end{aligned}$$

This equation together with the equation of the first integral defines a PDE system with a two dimensional space of solutions. The first integral in Kamke's book is written using Bessel functions, thus the solutions of this PDE system can be expressed in terms of Bessel functions here, but this is not an easy task.

In general, Abel equations are of the form

$$\frac{\partial y}{\partial x} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x).$$

These can be seen as a generalization of the Riccati equation. However, in contrary to the Riccati equation, they are not all solvable in algebraic-differential terms. Still many integrable families are known, Abel integrability is typically searched by looking into a known table list up to some transformations. Our algorithm can detect any integrable cases, even belonging to an unknown new integrable family.

7.5. **The generic case.** In a generic situation a vector field has no symbolic first integral. Let us now consider a random quadratic vector field

$$\dot{x} = 2x^2 + xy - 2y^2 - 1,\ \ \dot{y} = 2x^2 - 2y^2 + y - 3.$$

We do not find any Liouvillian nor Riccati first integrals (and thus neither Darbouxian, 2-Darbouxian or rational first integral) up to degree 9, with the following timings.

N	1	2	3	4	5	6	7	8	9
Liouvillian	0.016	0.109	0.640	2.433	8.143	24.71	69.31	175.8	453.9
Riccati	0.031	0.125	0.671	2.543	8.439	25.93	72.04	177.6	563.3

7.6. **Rational first integral with degree bigger than N.** Let us consider the following example

$$\dot{x} = \lambda x^3 - \lambda x y^2 - 2\mu y^2 - \lambda x, \;\; \dot{y} = \lambda x^2 y - \lambda y^3 - 2\mu x y - \lambda y$$

with $\lambda, \mu \in \mathbb{Z}$. This vector field always admits the first integral

$$I(x,y) = \lambda \ln\left(\frac{x}{y} - \frac{\sqrt{x^2-y^2}}{y}\right) + \mu \ln\left(\frac{x^2-y^2+1}{x^2-y^2-1} - 2\frac{\sqrt{x^2-y^2}}{x^2-y^2-1}\right)$$

which is a 2-Darbouxian first integral, which is of degree 8. Indeed, we have: $\partial_y I - F = 0$, where $F^2 = P/Q$ and

$$\begin{array}{rcl} P & = & \lambda^2 x^6 - 2\,\lambda^2 x^4 y^2 + \lambda^2 x^2 y^4 - 4\,\lambda\,\mu\,x^3 y^2 + 4\,\lambda\,\mu\,x y^4 - 2\,\lambda^2 x^4 + 2\,\lambda^2 x^2 y^2 \\ & & +4\,\mu^2 y^4 + 4\,\lambda\,\mu\,x y^2 + \lambda^2 x^2 \\ Q & = & x^6 y^2 - 3\,x^4 y^4 + 3\,x^2 y^6 - y^8 - 2\,x^4 y^2 + 4\,x^2 y^4 - 2\,y^6 + x^2 y^2 - y^4. \end{array}$$

As $\lambda/\mu \in \mathbb{Q}$, we can however build from this a rational first integral (with degree depending on λ/μ). And this is also a particular case of a Liouvillian first integral, which is then of degree 8. This kind of example is build by searching radical extension of $\bar{\mathbb{K}}(x,y)$ with groups of unit of rank ≥ 2. The first integral is then a linear combination of logs of these units.

For this example, we here display the timings in seconds of the algorithms Rational, Darbouxian, Liouvillian and Ricatti first integrals with initial points $(2,5),(2,3)$. The degree columns are the minimum N for which the output is not trivial.

(λ, μ)	Rat deg	time	D deg	time	L deg	time	Ric deg	time
(1,0)	1	0.016	1	0.032	1	0.047	1	0.064
(0,1)	2	0.032	1	0.015	1	0.062	2	0.109
(1,1)	3	0.063	2	0.109	2	0.344	3	1.935
(2,1)	4	0.109	3	0.312	2	0.187	3	1.903
(1,2)	5	0.296	4	0.889	4	4.867	5	21.45
(3,1)	5	0.140	4	0.951	3	1.685	5	26.61
(1,3)	7	1.997	5	3.510	5	19.38	6	220.6
(4,1)	6	0.858	5	3.541	4	6.381	5	21.72
(3,2)	7	2.418	5	3.603	5	21.03	6	61.71
(2,3)	8	3.947	6	9.376	5	18.16	6	54.60
(1,4)	9	8.751	6	9.781	6	55.21	8	391.9
(5,1)	7	2.184	5	3.728	5	19.30	7	148.6
(1,5)	11	33.59	7	24.27	7	148.4	8	205^*
(6,1)	8	4.758	6	10.76	6	63.59	7	156.49
(5,2)	9	9.001	6	8.955	6	51.08	7	134.6
(4,3)	10	16.70	7	28.14	6	62.90	7	174.1
(3,4)	11	31.23	7	26.58	6	60.08	8	264^*
(2,5)	12	60.01	8	65.58	7	166.6	8	267^*
(1,6)	13	99.39	8	63.82	8	389^*	8	232^*

In many cases (all except those with $\star$), Darbouxian, Liouvillian and Ricatti algorithms have returned the rational first integral even if it is of degree larger than N. Remark that degree cannot be higher than 8 for Liouvillian or Ricatti first integrals, because the 2-Darbouxian first integral is always present and its degree is not growing.

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References

[ADDdM05] J. Avellar, L.G.S. Duarte, S.E.S. Duarte, and L.A.C.P. da Mota. Integrating first-order differential equations with liouvillian solutions via quadratures: a semi-algorithmic method. *Journal of Computational and Applied Mathematics*, 182(2):327 – 332, 2005. 2, 8, 9

[ADdM12] J. Avellar, L.G.S. Duarte, and L.A.C.P. da Mota. Pssolver: A maple implementation to solve first order ordinary differential equations with liouvillian solutions. *Computer Physics Communications*, 183(10):2313 – 2314, 2012. 53

[BC11] L. Busé and G. Chèze. On the total order of reducibility of a pencil of algebraic plane curves. *Journal of Algebra*, 341(1):256 – 278, 2011. 25

[BCCW16] A. Bostan, G. Chèze, T. Cluzeau, and J.-A. Weil. Efficient algorithms for computing rational first integrals and Darboux polynomials of planar polynomial vector fields. *Math. Comp.*, 85(299):1393–1425, 2016. 2, 5, 6, 9, 40

[BCO+07] A. Bostan, F. Chyzak, F. Ollivier, B. Salvy, É. Schost, and A. Sedoglavic. Fast computation of power series solutions of systems of differential equations. In *Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1012–1021, New York, 2007. ACM. 39, 49

[BL94] B. Beckermann and G. Labahn. A uniform approach for the fast computation of matrix-type Padé approximants. *SIAM J. Matrix Anal. Appl.*, 15(3):804–823, 1994. 50

[BLS+04] A. Bostan, G. Lecerf, E. Schost, E. Salvy, and B. Wiebelt. Complexity issues in bivariate polynomial factorization. In *Proceedings of ISSAC'04*, pages 42–49. ACM Press, 2004. 51

[Bro05] M. Bronstein. *Symbolic integration. I*, volume 1 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, second edition, 2005. Transcendental functions, With a foreword by B. F. Caviness. 21

[Cas06] G. Casale. Feuilletages singuliers de codimension un, groupoïde de galois et intégrales premières. *Annales de l'Institut Fourier*, 56(3):735–779, 2006. 17, 18, 19

[Chè11] G. Chèze. Computation of Darboux polynomials and rational first integrals with bounded degree in polynomial time. *J. Complexity*, 27(2):246–262, 2011. 2, 8, 38

[Chè15] G. Chèze. Bounding the number of remarkable values via jouanolou's theorem. *Journal of Differential Equations*, 258(10):3535 – 3545, 2015. 39

[Chr99] C. Christopher. Liouvillian first integrals of second order polynomial differential equations. *Electron. J. Differential Equations*, pages No. 49, 7 pp. (electronic), 1999. 3

[CLP07] C. Christopher, J. Llibre, and J. Pereira. Multiplicity of invariant algebraic curves in polynomial vector fields. *Pacific J. Math.*, 229(1):63–117, 2007. 25

[Dar78] G. Darboux. Mémoire sur les équations diff'érentielles du premier ordre et du premier degré. *Bull. Sci. Math.*, 32:60–96, 123–144, 151–200, 1878. 2, 8

[DDdM02a] L. G. S. Duarte, S. E. S. Duarte, and L. A. C. P. da Mota. Analysing the structure of the integrating factors for first-order ordinary differential equations with Liouvillian functions in the solution. *J. Phys. A*, 35(4):1001–1006, 2002. 8, 9

[DDdM02b] L. G. S. Duarte, S. E. S. Duarte, and L. A. C. P. da Mota. A method to tackle first-order ordinary differential equations with Liouvillian functions in the solution. *J. Phys. A*, 35(17):3899–3910, 2002. 3, 8, 9

[DLA06] F. Dumortier, J. Llibre, and J. Artés. *Qualitative theory of planar differential systems*. Universitext. Springer-Verlag, Berlin, 2006. 8

[FG10] A. Ferragut and H. Giacomini. A new algorithm for finding rational first integrals of polynomial vector fields. *Qual. Theory Dyn. Syst.*, 9(1-2):89–99, 2010. 9

[Gab95] A. Gabrielov. Multiplicities of zeroes of polynomials on trajectories of polynomial vector fields and bounds on degree of nonholonomy. *Math. Res. Lett.*, 2(4):437–451, 1995. 20

[gGG99] J. von zur Gathen and J. Gerhard. *Modern computer algebra*. Cambridge University Press, New York, 1999. 43, 49, 51

[Gor01] A. Goriely. *Integrability and nonintegrability of dynamical systems*, volume 19 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001. 8

[JNSV17] C.-P. Jeannerod, V. Neiger, E. Schost, and G. Villard. Computing minimal interpolation bases. *Journal of Symbolic Computation*, 83(Supplement C):272 – 314, 2017. Special issue on the conference ISSAC 2015: Symbolic computation and computer algebra. 50

[Lec06] G. Lecerf. Sharp precision in Hensel lifting for bivariate polynomial factorization. *Math. Comp.*, 75(254):921–933, 2006. 51

[Man93] Y.-K. Man. Computing closed form solutions of first order odes using the prelle-singer procedure. *Journal of Symbolic Computation*, 16(5):423 – 443, 1993. 2, 8

[MM97] Y.-K. Man and M. MacCallum. A rational approach to the Prelle-Singer algorithm. *J. Symbolic Comput.*, 24(1):31–43, 1997. 2, 8

[Per01] J. Pereira. Vector fields, invariant varieties and linear systems. *Ann. Inst. Fourier (Grenoble)*, 51(5):1385–1405, 2001. 8, 20, 22, 25

[Pic02] É. Picard. Sur les intégrales doubles de fonctions rationnelles dont tous les résidus sont nuls. *Bulletin des sciences mathématiques, série 2*, 26, 1902. 3

[PS83] M. Prelle and M. Singer. Elementary first integrals of differential equations. *Trans. Amer. Math. Soc.*, 279(1):215–229, 1983. 2, 3, 8

[Ris96] J.-J. Risler. A bound for the degree of nonholonomy in the plane. *Theoret. Comput. Sci.*, 157(1):129–136, 1996. Algorithmic complexity of algebraic and geometric models (Creteil, 1994). 20

[Rup86] W. Ruppert. Reduzibilität Ebener Kurven. *J. Reine Angew. Math.*, 369:167–191, 1986. 3

[Sch93] D. Schlomiuk. Elementary first integrals and algebraic invariant curves of differential equations. *Exposition. Math.*, 11(5):433–454, 1993. 8

[Sch00] A. Schinzel. *Polynomials with special regard to reducibility*, volume 77 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2000. With an appendix by Umberto Zannier. 3

[Sin92] M. Singer. Liouvillian first integrals of differential equations. *Trans. Amer. Math. Soc.*, 333(2):673–688, 1992. 2, 3, 8

[Zha17] X. Zhang. *Integrability of Dynamical Systems: Algebra and Analysis*. Developments in Mathematics. Springer Singapore, 2017. 8

[ZL12] W. Zhou and G. Labahn. Efficient algorithms for order basis computation. *J. Symbolic Comput.*, 47(7):793–819, 2012. 50

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