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A common approach to the problem of the infinitude of twin primes, primes of the form \( n! + 1 \), and primes of the form \( n! - 1 \)

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Abstract

For a positive integer \( x \), let \( \Gamma(x) \) denote \( (x - 1)! \). Let \( \text{fact}^{-1}: \{1, 2, 6, 24, \ldots\} \to \mathbb{N} \setminus \{0\} \) denote the inverse function to the factorial function. For positive integers \( x \) and \( y \), let \( \text{rem}(x, y) \) denote the remainder from dividing \( x \) by \( y \). For a positive integer \( n \), by a computation of length \( n \) we understand any sequence of terms \( x_1, \ldots, x_n \) such that \( x_1 \) is defined as the variable \( x \), and for every integer \( i \in \{2, \ldots, n\} \), \( x_i \) is defined as \( \Gamma(x_{i-1}) \), or \( \text{fact}^{-1}(x_{i-1}) \), or \( \text{rem}(x_{i-1}, x_{i-2}) \) (only if \( i \geq 3 \) and \( x_{i-1} \) is defined as \( \Gamma(x_{i-2}) \)). Let \( f(4) = 3 \), and let \( f(n + 1) = f(n)! \) for every integer \( n \geq 4 \). For an integer \( n \geq 4 \), let \( \Psi_n \) denote the following statement: if a computation of length \( n \) returns positive integers \( x_1, \ldots, x_n \) for at most finitely many positive integers \( x \), then every such \( x \) does not exceed \( f(n) \). We prove:

1. the statement \( \Psi_4 \) equivalently expresses that there are infinitely many primes of the form \( n! + 1 \);
2. the statement \( \Psi_6 \) implies that for infinitely many primes \( p \) the number \( p! + 1 \) is prime;
3. the statement \( \Psi_6 \) implies that there are infinitely many primes of the form \( n! - 1 \);
4. the statement \( \Psi_7 \) implies that there are infinitely many twin primes.

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For a positive integer \( x \), let \( \Gamma(x) \) denote \( (x - 1)! \). Let \( \text{fact}^{-1}: \{1, 2, 6, 24, \ldots\} \to \mathbb{N} \setminus \{0\} \) denote the inverse function to the factorial function. For positive integers \( x \) and \( y \), let \( \text{rem}(x, y) \) denote the remainder from dividing \( x \) by \( y \).

Definition. For a positive integer \( n \), by a computation of length \( n \) we understand any sequence of terms \( x_1, \ldots, x_n \) such that \( x_1 \) is defined as the variable \( x \), and for every integer \( i \in \{2, \ldots, n\} \), \( x_i \) is defined as \( \Gamma(x_{i-1}) \), or \( \text{fact}^{-1}(x_{i-1}) \), or \( \text{rem}(x_{i-1}, x_{i-2}) \) (only if \( i \geq 3 \) and \( x_{i-1} \) is defined as \( \Gamma(x_{i-2}) \)).

Let \( f(4) = 3 \), and let \( f(n + 1) = f(n)! \) for every integer \( n \geq 4 \). For an integer \( n \geq 4 \), let \( \Psi_n \) denote the following statement: if a computation of length \( n \) returns positive integers \( x_1, \ldots, x_n \) for at most finitely many positive integers \( x \), then every such \( x \) does not exceed \( f(n) \).

Lemma 1. For every positive integer \( n \), there are only finitely many computations of length \( n \).

Theorem 1. For every integer \( n \geq 4 \), the statement \( \Psi_n \) is true with an unknown integer bound that depends on \( n \).
Proof. It follows from Lemma 1.

Let \( \mathcal{P} \) denote the set of prime numbers.

**Lemma 2.** ([4] pp. 214–215). For every positive integer \( x \), \( \text{rem}(\Gamma(x), x) \in \mathbb{N} \setminus \{0\} \) if and only if \( x \in \{4\} \cup \mathcal{P} \).

**Theorem 2.** For every integer \( n \geq 4 \) and for every positive integer \( x \), the following computation \( \mathcal{H}_n \)

\[
\begin{align*}
    x_1 &:= x \\
    \forall i \in \{2, \ldots, n-3\} \quad x_i &:= \text{fact}^{-1}(x_{i-1}) \\
    x_{n-2} &:= \Gamma(x_{n-3}) \\
    x_{n-1} &:= \Gamma(x_{n-2}) \\
    x_n &:= \text{rem}(x_{n-1}, x_{n-2})
\end{align*}
\]

returns positive integers \( x_1, \ldots, x_n \) if and only if \( x = f(n) \).

**Proof.** We make three observations.

**Observation 1.** If \( x_{n-3} = 3 \), then \( x_1, \ldots, x_{n-3} \in \mathbb{N} \setminus \{0\} \) and \( x = x_1 = f(n) \).

If \( x = f(n) \), then \( x_1, \ldots, x_{n-3} \in \mathbb{N} \setminus \{0\} \) and \( x_{n-3} = 3 \).

Hence, \( x_{n-2} = \Gamma(x_{n-3}) = 2 \) and \( x_{n-1} = \Gamma(x_{n-2}) = 1 \). Therefore, \( x_n = \text{rem}(x_{n-1}, x_{n-2}) = 1 \).

**Observation 2.** If \( x_{n-3} = 2 \), then \( x = x_1 = \ldots = x_{n-3} = 2 \).

If \( x = 2 \), then \( x_1 = \ldots = x_{n-3} = 2 \). Hence, \( x_{n-2} = \Gamma(x_{n-3}) = 1 \) and \( x_{n-1} = \Gamma(x_{n-2}) = 1 \). Therefore, \( x_n = \text{rem}(x_{n-1}, x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\} \).

**Observation 3.** If \( x_{n-3} = 1 \), then \( x_{n-2} = \Gamma(x_{n-3}) = 1 \). Hence, \( x_{n-1} = \Gamma(x_{n-2}) = 1 \). Therefore, \( x_n = \text{rem}(x_{n-1}, x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\} \).

Observations 1–3 cover the case when \( x_{n-3} \in \{1, 2, 3\} \). If \( x_{n-3} \geq 4 \), then \( x_{n-2} = \Gamma(x_{n-3}) \) is greater than 4 and composite. By Lemma 2, \( x_n = \text{rem}(x_{n-1}, x_{n-2}) = \text{rem}(\Gamma(x_{n-2}), x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\} \).

**Corollary 1.** For every integer \( n \geq 4 \), the bound \( f(n) \) in the statement \( \Psi_n \) cannot be decreased.

**Lemma 3.** (Wilson’s theorem, [2] p. 89). For every positive integer \( x \), \( x \) divides \( \Gamma(x) + 1 \) if and only if \( x \in \{1\} \cup \mathcal{P} \).

**Corollary 2.** If \( x \in \mathcal{P} \), then \( \text{rem}(\Gamma(x), x) = x - 1 \).

**Lemma 4.** For every positive integer \( x \), the following computation \( \mathcal{A} \)

\[
\begin{align*}
    x_1 &:= x \\
    x_2 &:= \Gamma(x_1) \\
    x_3 &:= \text{rem}(x_2, x_1) \\
    x_4 &:= \text{fact}^{-1}(x_3)
\end{align*}
\]

returns positive integers \( x_1, \ldots, x_4 \) if and only if \( x = 4 \) or \( x \) is a prime number of the form \( n! + 1 \).

**Proof.** For an integer \( i \in \{1, \ldots, 4\} \), let \( A_i \) denote the set of positive integers \( x \) such that the first \( i \) instructions of the computation \( \mathcal{A} \) returns positive integers \( x_1, \ldots, x_i \). We show that

\[
A_4 = \{4\} \cup \{n! + 1 : n \in \mathbb{N} \setminus \{0\} \} \cap \mathcal{P}
\]
For every positive integer \(x\), the terms \(x_1\) and \(x_2\) belong to \(\mathbb{N} \setminus \{0\}\). By Lemma 2, the term \(x_3\) (which equals \(\text{rem}(\Gamma(x), x)\)) belongs to \(\mathbb{N} \setminus \{0\}\) if and only if \(x \in \{4\} \cup \mathcal{P}\). Hence, \(A_1 = \{4\} \cup \mathcal{P}\). If \(x = 4\), then \(x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\}\). Hence, \(4 \in A_4\). If \(x \in \mathcal{P}\), then Corollary 2 implies that \(x_3 = \text{rem}(\Gamma(x), x) = x - 1 \in \mathbb{N} \setminus \{0\}\). Therefore, for every \(x \in \mathcal{P}\), the term \(x_4 = \text{fact}^{-1}(x_3)\) belongs to \(\mathbb{N} \setminus \{0\}\) if and only if \(x \in \{n! + 1 : n \in \mathbb{N} \setminus \{0\}\}\). This proves equality 1. 

It is conjectured that there are infinitely many primes of the form \(n! + 1\), see [1] p. 443 and [5].

**Theorem 3.** The statement \(\Psi_4\) implies that the set of primes of the form \(n! + 1\) is infinite.

**Proof.** The number \(3! + 1 = 7\) is prime. By Lemma 4, for \(x = 7\) the computation \(\mathcal{A}\) returns positive integers \(x_1, \ldots, x_4\). Since \(x = 7 > 3 = f(4)\), the statement \(\Psi_4\) guarantees that the computation \(\mathcal{A}\) returns positive integers \(x_1, \ldots, x_4\) for infinitely many positive integers \(x\). By Lemma 5 there are infinitely many primes of the form \(n! + 1\). 

**Lemma 5.** If \(x \in \mathbb{N} \setminus \{0, 1\}\), then \(\text{fact}^{-1}(\Gamma(x)) = x - 1\).

**Theorem 4.** If the set of primes of the form \(n! + 1\) is infinite, then the statement \(\Psi_4\) is true.

**Proof.** There exist exactly 10 computations of length 4 that differ from \(\mathcal{H}_4\) and \(\mathcal{A}\), see Table 1. For every such computation \(\mathcal{F}_i\), we determine the set \(S_i\) of all positive integers \(x\) such that the computation \(\mathcal{F}_i\) outputs positive integers \(x_1, \ldots, x_4\) on input \(x\). We omit 10 easy proofs which use Lemmas 2 and 5. The sets \(S_i\) are infinite, see Table 1. This completes the proof.

**Table 1:** 12 computations of length 4, \(x_1 := x, \ x \in \mathbb{N} \setminus \{0\}\)

**Hypothesis.** The statements \(\Psi_4, \ldots, \Psi_7\) are true.
**Lemma 6.** For every positive integer \( x \), the following computation \( \mathcal{B} \)
\[
\begin{align*}
  x_1 & := x \\
  x_2 & := \Gamma(x_1) \\
  x_3 & := \text{rem}(x_2, x_1) \\
  x_4 & := \text{fact}^{-1}(x_3) \\
  x_5 & := \Gamma(x_4) \\
  x_6 & := \text{rem}(x_5, x_4)
\end{align*}
\]
returns positive integers \( x_1, \ldots, x_6 \) if and only if \( x \in \{4\} \cup \{p! + 1 : p \in \mathcal{P}\} \cap \mathcal{P} \).

**Proof.** For an integer \( i \in \{1, \ldots, 6\} \), let \( B_i \) denote the set of positive integers \( x \) such that the first \( i \) instructions of the computation \( \mathcal{B} \) returns positive integers \( x_1, \ldots, x_i \). Since the computations \( \mathcal{A} \) and \( \mathcal{B} \) have the same first four instructions, the equality \( B_i = A_i \) holds for every \( i \in \{1, \ldots, 4\} \). In particular,
\[
  B_4 = \{4\} \cup (\{n! + 1 : n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P})
\]
We show that
\[
  B_6 = \{4\} \cup (\{p! + 1 : p \in \mathcal{P}\} \cap \mathcal{P}) \quad (2)
\]
If \( x = 4 \), then \( x_1, \ldots, x_6 \in \mathbb{N} \setminus \{0\} \). Hence, \( 4 \in B_6 \). Let \( x \in \mathcal{P} \), and let \( x = n! + 1 \), where \( n \in \mathbb{N} \setminus \{0\} \). Hence, \( n \neq 4 \). Corollary\[2\] implies that \( x_3 = \text{rem}(\Gamma(x), x) = x - 1 = n! \). Hence, \( x_4 = \text{fact}^{-1}(x_3) = n \) and \( x_5 = \Gamma(x_4) = \Gamma(n) \in \mathbb{N} \setminus \{0\} \). By Lemma\[2\] the term \( x_6 \) (which equals \( \text{rem}(\Gamma(n), n) \)) belongs to \( \mathbb{N} \setminus \{0\} \) if and only if \( n \in \{4\} \cup \mathcal{P} \). This proves equality (2) as \( n \neq 4 \).

**Theorem 5.** The statement \( \Psi_6 \) implies that for infinitely many primes \( p \) the number \( p! + 1 \) is prime.

**Proof.** The numbers 11 and 11! + 1 are prime, see \[11\], p. 441 and \[7\]. By Lemma\[6\] for \( x = 11! + 1 \) the computation \( \mathcal{B} \) returns positive integers \( x_1, \ldots, x_6 \). Since \( x = 11! + 1 > 6! = f(6) \), the statement \( \Psi_6 \) guarantees that the computation \( \mathcal{B} \) returns positive integers \( x_1, \ldots, x_6 \) for infinitely many positive integers \( x \). By Lemma\[6\] for infinitely many primes \( p \) the number \( p! + 1 \) is prime.

**Lemma 7.** For every positive integer \( x \), the following computation \( \mathcal{C} \)
\[
\begin{align*}
  x_1 & := x \\
  x_2 & := \Gamma(x_1) \\
  x_3 & := \Gamma(x_2) \\
  x_4 & := \text{fact}^{-1}(x_3) \\
  x_5 & := \Gamma(x_4) \\
  x_6 & := \text{rem}(x_5, x_4)
\end{align*}
\]
returns positive integers \( x_1, \ldots, x_6 \) if and only if \( (x - 1)! - 1 \) is prime.

**Proof.** For an integer \( i \in \{1, \ldots, 6\} \), let \( C_i \) denote the set of positive integers \( x \) such that the first \( i \) instructions of the computation \( \mathcal{C} \) returns positive integers \( x_1, \ldots, x_i \). If \( x \in \{1, 2, 3\} \), then \( x_6 = 0 \). Therefore, \( C_6 \subseteq \mathbb{N} \setminus \{0, 1, 2, 3\} \). By Lemma\[5\] for every integer \( x \geq 4 \), \( x_4 = (x - 1)! - 1 \), \( x_5 = \Gamma((x - 1)! - 1) \), and \( x_1, \ldots, x_5 \in \mathbb{N} \setminus \{0\} \). By Lemma\[2\] for every integer \( x \geq 4 \),
\[
x_6 = \text{rem}(\Gamma((x - 1)! - 1), (x - 1)! - 1)
\]
belongs to \( \mathbb{N} \setminus \{0\} \) if and only if \( (x - 1)! - 1 \in \{4\} \cup \mathcal{P} \). The last condition equivalently expresses that \( (x - 1)! - 1 \) is prime as \( (x - 1)! - 1 \geq 5 \) for every integer \( x \geq 4 \). Hence,
\[
  C_6 = (\mathbb{N} \setminus \{0, 1, 2, 3\}) \cap \{x \in \mathbb{N} \setminus \{0, 1, 2, 3\} : (x - 1)! - 1 \in \mathcal{P}\} = \{x \in \mathbb{N} \setminus \{0\} : (x - 1)! - 1 \in \mathcal{P}\}
\]

\[\square\]
It is conjectured that there are infinitely many primes of the form \( n! - 1 \), see [1] p. 443 and [6].

**Theorem 6.** The statement \( \Psi_6 \) implies that there are infinitely many primes of the form \( x! - 1 \).

**Proof.** The number \((975 - 1)! - 1\) is prime, see [1] p. 441 and [6]. By Lemma [7], for \( x = 975 \) the computation \( \mathcal{C} \) returns positive integers \( x_1, \ldots, x_6 \). Since \( x = 975 > 720 = f(6) \), the statement \( \Psi_6 \) guarantees that the computation \( \mathcal{C} \) returns positive integers \( x_1, \ldots, x_6 \) for infinitely many positive integers \( x \). By Lemma [7] the set \( \{ x \in \mathbb{N} \setminus \{0\} : (x - 1)! - 1 \in \mathcal{P} \} \) is infinite. \( \square \)

**Lemma 8.** For every positive integer \( x \), the following computation \( \mathcal{D} \)

\[
\begin{align*}
x_1 & := x \\
x_2 & := \Gamma(x_1) \\
x_3 & := \text{rem}(x_2, x_1) \\
x_4 & := \Gamma(x_3) \\
x_5 & := \text{fact}^{-1}(x_4) \\
x_6 & := \Gamma(x_5) \\
x_7 & := \text{rem}(x_6, x_5)
\end{align*}
\]

returns positive integers \( x_1, \ldots, x_7 \) if and only if both \( x \) and \( x - 2 \) are prime.

**Proof.** For an integer \( i \in \{1, \ldots, 7\} \), let \( D_i \) denote the set of positive integers \( x \) such that the first \( i \) instructions of the computation \( \mathcal{D} \) returns positive integers \( x_1, \ldots, x_i \). If \( x = 1 \), then \( x_3 = 0 \). Hence, \( D_7 \subseteq D_3 \subseteq \mathbb{N} \setminus \{0, 1\} \). If \( x \in \{2, 3, 4\} \), then \( x_7 = 0 \). Therefore,

\[
D_7 \subseteq (\mathbb{N} \setminus \{0, 1\}) \cap (\mathbb{N} \setminus \{0, 2, 3, 4\}) = \mathbb{N} \setminus \{0, 1, 2, 3, 4\}
\]

By Lemma [2], for every integer \( x \geq 5 \), the term \( x_3 \) (which equals \( \text{rem}(\Gamma(x), x) \)) belongs to \( \mathbb{N} \setminus \{0\} \) if and only if \( x \in \mathcal{P} \setminus \{2, 3\} \). By Corollary [2] for every \( x \in \mathcal{P} \setminus \{2, 3\} \), \( x_3 = x - 1 \in \mathbb{N} \setminus \{0, 1, 2, 3\} \). By Lemma [5] for every \( x \in \mathcal{P} \setminus \{2, 3\} \), the terms \( x_4 \) and \( x_5 \) belong to \( \mathbb{N} \setminus \{0\} \) and \( x_5 = x_3 - 1 = x - 2 \). By Lemma [2] for every \( x \in \mathcal{P} \setminus \{2, 3\} \), the term \( x_7 \) (which equals \( \text{rem}(\Gamma(x_5), x_5) \)) belongs to \( \mathbb{N} \setminus \{0\} \) if and only if \( x_5 = x - 2 \in \{4\} \cup \mathcal{P} \). From these facts, we obtain that

\[
D_7 = (\mathbb{N} \setminus \{0, 1, 2, 3, 4\}) \cap (\mathcal{P} \setminus \{2, 3\}) \cap (\{6\} \cup \{p + 2 : p \in \mathcal{P}\}) = \{ p \in \mathcal{P} : p - 2 \in \mathcal{P} \}
\]

\( \square \)

A twin prime is a prime number that is either 2 less or 2 more than another prime number. The twin prime conjecture states that there are infinitely many twin primes, see [3] p. 39.

**Theorem 7.** The statement \( \Psi_7 \) implies that there are infinitely many twin primes.

**Proof.** Harvey Dubner proved that the numbers \( 459 \cdot 2^{8529} - 1 \) and \( 459 \cdot 2^{8529} + 1 \) are prime, see [8] p. 87. By Lemma [8] for \( x = 459 \cdot 2^{8529} + 1 \) the computation \( \mathcal{D} \) returns positive integers \( x_1, \ldots, x_7 \). Since \( x > 720! = f(7) \), the statement \( \Psi_7 \) guarantees that the computation \( \mathcal{D} \) returns positive integers \( x_1, \ldots, x_7 \) for infinitely many positive integers \( x \). By Lemma [8] there are infinitely many twin primes. \( \square \)
References


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