A Proximal Approach for Solving Matrix Optimization Problems Involving a Bregman Divergence
Alessandro Benfenati, Emilie Chouzenoux, Jean-Christophe Pesquet

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Abstract—In recent years, there has been a growing interest in problems such as shape classification, gene expression inference, inverse covariance estimation. Problems of this kind have a common underlying mathematical model, which involves the minimization in a matrix space of a Bregman divergence function coupled with a linear term and a regularization term. We present an application of the Douglas–Rachford algorithm which allows to easily solve the optimization problem.

In recent years, some applications such as shape classification models [1], gene expression [2], or inverse covariance estimation [3] have led to matrix variational formulations of the form:

\[
\minimize_{C \in S_n} D_f(C, S) + g(C)
\] (1)

where \(S_n\) is the cone of symmetric semidefinite positive matrices of size \(n \times n\), \(S\) is a given matrix in \(S_n\), \(f\) and \(g\) are proper lower-semicontinuous (lsc) convex functions defined on the space of \(n \times n\) matrices, and \(D_f\) is the Bregman divergence associated with \(f\). Recall that

\[
D_f(C, S) = f(C) - f(S) - \langle T(C - S) \rangle
\] (2)

where \(T \in \partial f(S) \neq \emptyset\). Note also that solving (1) amounts to computing the proximity operator of \(g + \iota_{S_n}\) at \(C\), with respect to the divergence \(D_f\), which has also been found to be useful in a number of recent works [4], [5].

Very often, due to the nature of the problems, the regularization functional \(g\) has to promote the sparsity of \(C\). A generic class of regularization is obtained by asserting that \(g = g_0 + g_1\) where

\[
g_0(C) = \begin{cases} 
\psi(d) & \text{if } C \in S_n \\
+\infty & \text{otherwise}
\end{cases}
\] (3)

where \(\psi : \mathbb{R}^n \rightarrow [0, +\infty)\) is a proper lsc function and \(d\) is the vector of eigenvalues of \(C\), whereas \(g_1\) is a function which cannot be expressed under this form. Typical examples are the nuclear norm \(\|C\|_n\) (or any Schatten norm) for \(g_0\) and the \(\ell_1\) norm \(\|\cdot\|_1\) (of the matrix elements) for \(g_1\) [6].

In this paper, we will assume that function \(f\) can be expressed similarly to \(g_0\) as \(f(C) = \varphi(d)\) if \(C \in S_n\), \(f(C) \rightarrow +\infty\) otherwise, where \(\varphi : \mathbb{R}^n \rightarrow [0, +\infty)\) is a proper lsc convex function. In particular, this assumption is satisfied when

\[
f(C) = \begin{cases} 
-\log \det(C) & \text{if } C > 0 \\
+\infty & \text{otherwise}
\end{cases}
\] (4)

Various algorithm have been proposed to solve Problem (1) when \(f\) is the above function and some specific choices of the function \(g\) are made: the popular GLASSO algorithm [3], a Gradient Projection method [1], and a splitting technique on the regularization term [6]. Here we propose to employ the Douglas–Rachford algorithm [7], which enables us to solve (1) in a fast manner, as soon as an efficient procedure for the eigenvalue decomposition is provided. The Douglas–Rachford approach alternates proximity steps on \(D_f(\cdot, S) + g_0 + \iota_{S_n}\) and on \(g_1\). For many functions \(g_1\) of practical interest, the proximity operator of \(g_1\) (e.g., \(g_1 = \|\cdot\|_1\)) has a closed form solution [7]. Let us define \(F(C) = f(C) + g_0(C)\). Let \(\gamma \in [0, +\infty]\). It can be noted that computing the proximity operator of \(\gamma(D_f(\cdot, S) + g_0 + \iota_{S_n})\) w.r.t. the Frobenius metric \(\|\cdot\|_F\), at some symmetric matrix \(\hat{C}\), is equivalent to find

\[
\hat{C} = \arg\min_{C \in \mathbb{R}^{n \times n}} \left( F(C) - \langle TC \rangle + \frac{1}{2\gamma} \|C - \hat{C}\|_F^2 \right)
\]

Classical properties of the proximity operator [7] state that

\[
\hat{C} = \text{prox}_{\gamma F_{\mathbb{R}^n \setminus \{C\}}}(\hat{C}) = \text{prox}_{\gamma F_{\mathbb{R}^n \setminus \{C\}}}(\hat{C} + \gamma T).
\]

Moreover, if \(C^\ast + \gamma T = U \text{Diag}(\sigma) U^T\) where \(U\) is an orthogonal matrix and \(\sigma \in \mathbb{R}^n\), then \(C^\ast = U D U^T\) with \(D = \text{Diag}(\text{prox}_{\gamma F_{\mathbb{R}^n \setminus \{C\}}} (\sigma))\). For example, if \(f\) is the log-det function (4) and \(g_0 = \mu \|\cdot\|_1\), then \(\gamma \in [0, +\infty)\), according to [8], the diagonal matrix of eigenvalues of \(\hat{C}\) is given by

\[
D = \frac{1}{2} \left( \Sigma - \mu I_n + \sqrt{\left( \Sigma - \mu I_n \right)^2 + 4\gamma I_n} \right)
\]

where \(\Sigma = \text{Diag}(\sigma)\). The operations to compute \(\text{prox}_{\gamma F_{\mathbb{R}^n \setminus \{C\}}} (\sigma)\) are thus component-wise.

The proposed Douglas–Rachford approach is easy to implement: if an efficient procedure for the eigenvalue decomposition is available, according to our numerical experiments, it is also very fast.

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References:


