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STÉPHANE SEURET* AND ADRIÁN UBIS

ABSTRACT. We are interested in the convergence and the local regularity of the lacunary Fourier series $F_s(x) = \sum_{n=1}^{+\infty} \frac{e^{2i\pi n^2 x}}{n^s}$. In the 1850's, Riemann introduced the series F_2 as a possible example of nowhere differentiable function, and the study of this function has drawn the interest of many mathematicians since then. We focus on the case when $1/2 < s \leq 1$, and we prove that $F_s(x)$ converges when x satisfies a Diophantine condition. We also study the L^2- local regularity of F_s, proving that the local L^2-norm of F_s around a point x behave differently around different x, according again to Diophantine conditions on x.

1. INTRODUCTION

Riemann introduced in 1857 the Fourier series

$$R(x) = \sum_{n=1}^{+\infty} \frac{\sin(2\pi n^2 x)}{n^2}$$

as a possible example of continuous but nowhere differentiable function. Though it is not the case (R is differentiable at rationals p/q where p and q are both odd [5]), the study of this function has, mainly because of its connections with several domains: complex analysis, harmonic analysis, Diophantine approximation, and dynamical systems [6, 7, 5, 4, 8, 10] and more recently [2, 3, 12].

In this article, we study the local regularity of the series

$$F_s(x) = \sum_{n=1}^{+\infty} \frac{e^{2i\pi n^2 x}}{n^s} \tag{1.1}$$

when $s \in (1/2, 1)$. In this case, several questions arise before considering its local behavior. First it does not converge everywhere, hence one needs to characterize its set of convergence points; this question was studied in [12], and we will first find a slightly more precise characterization. Then, if one wants to characterize the local regularity of a (real) function, one classically studies the pointwise Hölder exponent defined for a locally bounded function $f : \mathbb{R} \to \mathbb{R}$ at a point x by using the functional spaces $C^\alpha(x)$: $f \in C^\alpha(x)$ when there exist a constant C and a

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polynomial P with degree less than $\lfloor \alpha \rfloor$ such that, locally around x (i.e. for small H), one has

$$\|(f(\cdot) - P(\cdot - x))1\!\!1_{B(x,H)}\|_\infty := \sup(|f(y) - P(y-x)| : y \in B(x,H)) \leq CH^\alpha,$$

where $B(x,H) = \{y \in \mathbb{R} : |y-x| \leq H\}$. Unfortunately these spaces are not appropriate for our context since F_s is nowhere locally bounded (for instance, it diverges at every irreducible rational p/q such that $q \neq 2\times$odd).

Following Calderon and Zygmund in their study of local behaviors of solutions of elliptic PDE's [1], it is natural to introduce in this case the pointwise L^2-exponent defined as follows.

Definition 1.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function belonging to $L^2(\mathbb{R})$, $\alpha \geq 0$ and $x \in \mathbb{R}$. The function f is said to belong to $C_2^\alpha(x)$ if there exist a constant C and a polynomial P with degree less than $\lfloor \alpha \rfloor$ such that, locally around x (i.e. for small $H > 0$), one has

$$\left(\frac{1}{H} \int_{B(x,H)} |f(h) - P(h-x)|^2 dh \right)^{1/2} \leq CH^\alpha.$$

Then, the pointwise L^2-exponent of f at x is

$$\alpha_f(x) = \sup \left\{ \alpha \in \mathbb{R} : f \in C_2^\alpha(x) \right\}.$$

This definition makes sense for the series F_s when $s \in (1/2, 1)$, and are based on a natural generalizations of the spaces $C^\alpha(x)$ be replacing the L^∞ norm by the L^2 norm. The pointwise L^2-exponent has been studied for instance in [11], and is always greater than -1/2 as soon as $f \in L^2$.

Our goal is to perform the multifractal analysis of the series F_s. In other words, we aim at computing the Hausdorff dimension, denoted by dim in the following, of the level sets of the pointwise L^2-exponents.

Definition 1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function belonging to $L^2(\mathbb{R})$. The L^2-multifractal spectrum $d_f : \mathbb{R}^+ \cup \{+\infty\} \to \mathbb{R}^+ \cup \{-\infty\}$ of f is the mapping

$$d_f(\alpha) := \dim E_f(\alpha),$$

where the iso-Hölder set $E_f(\alpha)$ is

$$E_f(\alpha) := \{x \in \mathbb{R} : \alpha_f(x) = \alpha\}.$$

By convention one sets $d_f(\alpha) = -\infty$ if $E_f(\alpha) = \emptyset$.

Performing the multifractal analysis consists in computing its L^2-multifractal spectrum. This provides us with a very precise description of the distribution of the local L^2-singularities of f. In order to state our result, we need to introduce some notations.

Definition 1.3. Let x be an irrational number, with convergents $(p_j/q_j)_{j\geq 1}$. Let us define

$$x - \frac{p_j}{q_j} = h_j, \qquad |h_j| = q_j^{-r_j} \tag{1.2}$$

with $2 \leq r_j < \infty$. Then the approximation rate of x is defined by

$$r_{\text{odd}}(x) = \overline{\lim}\{r_j : q_j \neq 2 * \text{odd}\}.$$

This definition always makes sense because if q_j is even, then q_{j+1} and q_{j-1} must be odd (so cannot be equal to $2 * \text{odd}$). Thus, we always have $2 \leq r_{\text{odd}}(x) \leq +\infty$. It is classical that one can compute the Hausdorff dimension of the set of points with the Hausdorff dimension of the points x with a given approximation rate $r \geq 2$:

$$\text{for all } r \geq 2, \quad \dim\{x \in \mathbb{R} : r_{\text{odd}}(x) = r\} = \frac{2}{r}. \tag{1.3}$$

When $s > 1$, the series F_s converges, and the multifractal spectrum of F_s was computed by S. Jaffard in [10]. For instance, for the classical Riemann's series F_2, one has

$$d_{F_2}(\alpha) = \begin{cases} 4\alpha - 2 & \text{if } \alpha \in [1/2, 3/4], \\ 0 & \text{if } \alpha = 3/2, \\ -\infty & \text{otherwise.} \end{cases}$$

Here our aim is to somehow extend this result to the range $1/2 < s \leq 1$. The convergence of the series F_s is described by our first theorem.

Theorem 1.4. *Let $s \in (1/2, 1]$, and let $x \in (0,1)$ with convergents $(p_j/q_j)_{j\geq 1}$. We set for every $j \geq 1$*

$$\delta_j = \begin{cases} 1 & \text{if } s \in (1/2, 1) \\ \log(q_{j+1}/q_j) & \text{if } s = 1, \end{cases}$$

and

$$\Sigma_s(x) = \sum_{j:\, q_j \neq 2*\text{odd}} \delta_j \sqrt{\frac{q_{j+1}}{(q_j q_{j+1})^s}}. \tag{1.4}$$

(i) *$F_s(x)$ converges whenever $\frac{s-1+1/r_{\text{odd}}(x)}{2} > 0$. In fact, it converges whenever $\Sigma_s(x) < +\infty$.*

(ii) *$F_s(x)$ does not converge if $\frac{s-1+1/r_{\text{odd}}(x)}{2} < 0$. In fact, it does not converge whenever*

$$\overline{\lim}_{j:\, q_j \neq 2*\text{odd}}\ \delta_j \sqrt{\frac{q_{j+1}}{(q_j q_{j+1})^s}} > 0,$$

In the same way we could extend this results to rational points $x = p/q$, by proving that $F_s(x)$ converges for $q \neq 2*\text{odd}$ and does not for $q \neq 2*\text{odd}$. Observe that the convergence of $\Sigma_s(x)$ implies that $r_{odd}(x) \leq \frac{1}{1-s}$. Our result asserts that $F_s(x)$ converges as soon as $r_{\text{odd}}(x) < \frac{1}{1-s}$, and also when $r_{\text{odd}}(x) = \frac{1}{1-s}$ when $\Sigma_s(x) < +\infty$.

Jaffard's result is then extended in the following sense:

Theorem 1.5. *Let $s \in (1/2, 1]$.*

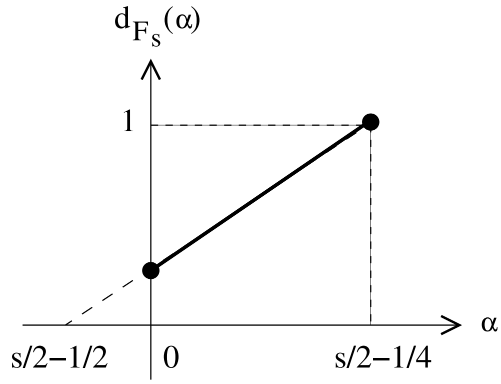

FIGURE 1. L^2-multifractal spectrum of F_s

(i) *For every x such that $\Sigma_s(x) < +\infty$, one has* $\alpha_{F_s}(x) = \dfrac{s-1+1/r_{\text{odd}}(x)}{2}$.

(ii) *For every* $\alpha \in [0, s/2 - 1/4]$

$$d_{F_s}(\alpha) = 4\alpha + 2 - 2s.$$

The second part of Theorem 1.5 follows directly from the first one. Indeed, using part (i) of Theorem 1.5 and (1.3), one gets

$$\begin{aligned} d_{F_s}(\alpha) &= \dim\left\{x : \frac{s-1+1/r_{\text{odd}}(x)}{2} = \alpha\right\} = \dim\{x : r_{\text{odd}}(x) = (2\alpha+1-s)^{-1}\} \\ &= \frac{2}{(2\alpha+1-s)^{-1}} = 4\alpha + 2 - 2s. \end{aligned}$$

The paper is organized as follows. Section 2 contains some notations and preliminary results. In Section 3, we obtain other formulations for F_s based on Gauss sums, and we get first estimates on the increments of the partial sums of the series F_s. Using these results, we prove Theorem 1.4 in Section 4. Finally, in Section 5, we use the previous estimates to obtain upper and lower bounds for the local L^2-means of the series F_s, and compute in Section 6 the local L^2-regularity exponent of F_s at real numbers x whose Diophantine properties are controlled, namely we prove Theorem 1.5.

Finally, let us mention that theoretically the L^2-exponents of a function $f \in L^2(\mathbb{R})$ take values in the range $[-1/2, +\infty]$, so they may have negative values. We believe that this is the case at points x such that $r_{\text{odd}}(x) > \frac{1}{1-s}$, so that in the end the entire L^2-multifractal spectrum of F_s would be $d_{F_s}(\alpha) = 4\alpha + 2 - 2s$ for all $\alpha \in [s/2 - 1/2, s/2 - 1/4]$.

Another remark is that for a given $s \in (1/2, 1)$, there is an optimal $p > 2$ such that F_s belongs locally to L^p, so that the p-exponents (instead of the 2-exponents) may carry some interesting information about the local behavior of F_s.

2. Notations and first properties

In all the proofs, C will denote a constant that does not depend on the variables involved in the equations.

For two real numbers $A, B \geq 0$, the notation $A \ll B$ means that $A \leq CB$ for some constant $C > 0$ independent of the variables in the problem.

In Section 2 of [3] (also in [2]), the key point to study the local behavior of the Fourier series F_s was to obtain an explicit formula for $F_s(p/q+h) - F_s(p/q)$ in the range $1 < s < 2$; this formula was just a twisted version of the one known for the Jacobi theta function. In our range $1/2 < s \leq 1$, such a formula cannot exist because of the convergence problems, but we will get some truncated versions of it in order to prove Theorems 1.4 and 1.5.

Let us introduce the partial sum

$$F_{s,N}(x) = \sum_{n=1}^{N} \frac{e^{2i\pi n^2 x}}{n^s}.$$

For any $H \neq 0$ let $\widetilde{\mu}_H$ be the probability measure defined by

$$\widetilde{\mu}_H(g) = \int_{\mathcal{C}(H)} g(h)\, \frac{dh}{2H}, \tag{2.1}$$

where $\mathcal{C}(H)$ is the annulus $\mathcal{C}(H) = [-2H, -H] \cup [H, 2H]$.

Lemma 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function in $L^2(\mathbb{R})$, and $x \in \mathbb{R}$. If $\alpha_f(x) < 1$, then*

$$\alpha_f(x) = \sup\left\{\beta \in [0,1) : \exists\, C > 0, \exists\, f_x \in \mathbb{R}\ \|f(x+\cdot) - f_x\|_{L^2(\widetilde{\mu}_H)} \leq C|H|^\beta\right\}. \tag{2.2}$$

Proof. Assume that $\alpha := \alpha_f(x) < 1$. Then, for every $\varepsilon > 0$, there exist $C > 0$ and a real number $f_x \in \mathbb{R}$ such that for every $H > 0$ small enough

$$\left(\frac{1}{H}\int_{B(x,H)} \left|f(h) - f_x\right|^2 dh\right)^{1/2} \leq CH^{\alpha-\varepsilon}.$$

Since $\mathcal{C}(H) \subset B(x, 2H)$, one has

$$\left\|f(x+\cdot) - f_x\right\|_{L^2(\widetilde{\mu}_H)} = \left(\int_{\mathcal{C}(H)} \left|f(x+h) - f_x\right|^2 \frac{dh}{2H}\right)^{1/2} \leq C|2H|^{\alpha-\varepsilon} \ll |H|^{\alpha-\varepsilon}. \tag{2.3}$$

Conversely, if (2.3) holds for every $H > 0$, then the result follows from the fact that $B(x,H) = x + \bigcup_{k\geq 1} \mathcal{C}(H/2^k)$. $\square$

So, we will use equation (2.2) as definition of the local L^2 regularity.

It is important to notice that the frequencies in different ranges are going to behave differently. Hence, it is better to look at N within dyadic intervals. Moreover, it will be easier to deal with smooth pieces. This motivates the following definition.

Definition 2.2. Let $N \geq 1$ and let $\psi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with support included in $[1/2, 2]$. One introduces the series

$$F^{\psi}_{s,N}(x) = \sum_{n=1}^{+\infty} \frac{e^{2i\pi n^2 x}}{n^s} \psi\left(\frac{n}{N}\right), \tag{2.4}$$

and for $R > 0$ one also sets

$$w^{\psi}_R(t) = e^{2i\pi R t^2} \psi(t)$$

and

$$E^{\psi}_N(x) = \frac{1}{N} \sum_{n=1}^{+\infty} w^{\psi}_{N^2 x}\left(\frac{n}{N}\right),$$

For the function $\psi_s(t) = t^{-s}\psi(t)$ (which is still C^∞ with support in $(1/2, 2)$) it is immediate to check that

$$F^{\psi}_{s,N}(x) = N^{1-s} E^{\psi_s}_N(x). \tag{2.5}$$

3. Summation Formula for $F_{s,N}$ and $F^{\psi}_{s,N}$

3.1. **Poisson Summation.** Let p, q be coprime integers, with $q > 0$. In this section we obtain some formulas for $F(p/q + h) - F(p/q)$ with $h > 0$. This is not a restriction, since

$$F_s\left(\frac{p}{q} - h\right) = \overline{F_s\left(\frac{-p}{q} + h\right)}. \tag{3.1}$$

We are going to write a summation formula for $E^{\psi}_N(p/q + h)$, with $h > 0$.

Proposition 3.2. *We have*

$$E^{\psi}_N\left(\frac{p}{q} + h\right) = \frac{1}{\sqrt{q}} \sum_{m \in \mathbb{Z}} \theta_m \cdot \widehat{w^{\psi}_{N^2 h}}\left(\frac{Nm}{q}\right). \tag{3.2}$$

where $\widehat{f}(\xi) = \displaystyle\int_{\mathbb{R}} f(t) e^{-2i\pi t\xi} dt$ *stands for the Fourier transform of* f *and* $(\theta_m)_{m \in \mathbb{Z}}$ *are some complex numbers whose modulus is bounded by* $\sqrt{2}$.

Proof. We begin by splitting the series into arithmetic progressions

$$\begin{aligned} E^{\psi}_N\left(\frac{p}{q} + h\right) &= \frac{1}{N} \sum_{n=1}^{+\infty} e^{2i\pi n^2 \frac{p}{q}} \cdot w^{\psi}_{N^2 h}\left(\frac{n}{N}\right) \\ &= \sum_{b=0}^{q-1} e^{2i\pi b^2 \frac{p}{q}} \sum_{\substack{n=1: \\ n \equiv b \mod q}}^{+\infty} \frac{1}{N} w^{\psi}_{N^2 h}\left(\frac{n}{N}\right). \end{aligned}$$

Now we apply Poisson Summation to the inner sum to get

$$\begin{aligned}\sum_{\substack{n=1:\\ n\equiv b \mod q}}^{+\infty} \frac{1}{N} w^{\psi}_{N^2h}\left(\frac{n}{N}\right) &= \sum_{n=1}^{+\infty} \frac{1}{N} w^{\psi}_{N^2h}\left(\frac{b+nq}{N}\right)\\ &= \sum_{m\in\mathbb{Z}} \frac{1}{q} e^{2i\pi\frac{bm}{q}} \cdot \widehat{w^{\psi}_{N^2h}}\left(\frac{Nm}{q}\right)\end{aligned}$$

This yields

$$E^{\psi}_N\left(\frac{p}{q}+h\right) = \frac{1}{q}\sum_{m\in\mathbb{Z}} \tau_m \cdot \widehat{w^{\psi}_{N^2h}}\left(\frac{Nm}{q}\right)$$

with

$$\tau_m = \sum_{b=0}^{q-1} e^{2i\pi\frac{pb^2+mb}{q}}.$$

This term τ_m is a Gauss sum. One has the following bounds:

- for every $m \in \mathbb{Z}$, $\tau_m = \theta_m\sqrt{q}$ with $\theta_m := \theta_m(p/q)$ satisfying
$$0 \le |\theta_m| \le \sqrt{2}.$$
- if $q = 2 * \text{odd}$, then $\theta_0 = 0$.
- if $q \ne 2 * \text{odd}$, then $1 \le \theta_0 \le \sqrt{2}$.

Finally, we get the summation formula (3.2). $\square$

3.3. **Behavior of the Fourier transform of w^{ψ}_R.** To use formula (3.2), one needs to understand the behavior of the Fourier transform of w^{ψ}_R

$$\widehat{w^{\psi}_R}(\xi) = \int_{\mathbb{R}} \psi(t)e^{2i\pi(Rt^2-\xi t)}\,dt.$$

On one hand, we have the trivial bound $|\widehat{w^{\psi}_R}(\xi)| \ll 1$ since ψ is C^∞, bounded by 1 and compactly supported. On the other hand, one has

Lemma 3.4. *Let $R > 0$ and $\xi \in \mathbb{R}$. Let ψ be a C^∞ function compactly supported inside $[1/2, 2]$. Let us introduce the mapping $g^{\psi}_R : \mathbb{R} \to \mathbb{C}$*

$$g^{\psi}_R(\xi) = e^{i\pi/4}\frac{e^{-i\pi\xi^2/(2R)}}{\sqrt{2R}}\psi\left(\frac{\xi}{2R}\right).$$

Then one has

$$\widehat{w^{\psi}_R}(\xi) = g^{\psi}_R(\xi) + \mathcal{O}_\psi\left(\frac{\rho_{R,\xi}}{\sqrt{R}} + \frac{1}{(1+R+|\xi|)^{3/2}}\right), \tag{3.3}$$

with $\rho_{R,\xi} = \begin{cases} 1 & \text{if } \xi/2R \in [1/2,2] \text{ and } R<1,\\ 0 & \text{otherwise.}\end{cases}$ *Moreover, one has*

$$\widehat{w^{\psi}_{2R}}(\xi) - \widehat{w^{\psi}_R}(\xi) = g^{\psi}_{2R}(\xi) - g^{\psi}_R(\xi) + \mathcal{O}_\psi\left(\rho_{R,\xi}\sqrt{R} + \frac{R}{(1+R+|\xi|)^{5/2}}\right), \tag{3.4}$$

Moreover, the constant implicit in $\mathcal{O}_\psi$ depends just on the L^∞-norm of a finite number of derivatives of ψ.

Proof. For $\xi/2R \notin [1/2, 2]$ the upper bound (3.3) comes just from integrating by parts several times; for $\xi, R \ll 1$ the bound (3.3) is trivial. The same properties hold true for the upper bound in (3.4).

Let us assume $\xi/2R \in [1/2, 2]$, and $R > 1$. The lemma is just a consequence of the stationary phase theorem. Precisely, Proposition 3 in Chapter VIII of [13] (and the remarks thereafter) implies that for some suitable functions $f, g : \mathbb{R} \to \mathbb{R}$, and if g is such that $g'(t_0) = 0$ at a unique point t_0, if one sets

$$S(\lambda) = \int_{\mathbb{R}} f(t) e^{i\lambda g(t)}\, dt - \sqrt{\frac{2\pi}{-i\lambda g''(t_0)}} f(t_0) e^{i\lambda g(t_0)},$$

then $|S(\lambda)| \ll \lambda^{-3/2}$ and also $|S'(\lambda)| \ll \lambda^{-5/2}$, where the implicit constants depend just on upper bounds for some derivatives of f and g, and also on a lower bound for g''.

In our case, we can apply it with $f = \psi$, $\lambda = R$, and $g(t) = 2\pi(t^2 - \xi/\lambda)$ to get precisely (3.3).

With the same choices for f and g, by applying the Mean Value Theorem and our bound for $S'(\lambda)$, we finally obtain formula (3.4). □

3.5. **Summation formula for the partial series $F_{s,N}$.** This important formula will be useful to study the convergence of $F_s(x)$.

Proposition 3.6. *Let p, q be two coprime integers. For $N \geq q$ and $0 \leq h \leq q^{-1}$, we have*

$$\begin{aligned} F_{s,2N}\left(\frac{p}{q} + h\right) - F_{s,N}\left(\frac{p}{q} + h\right) &= \frac{\theta_0}{\sqrt{q}} \int_N^{2N} \frac{e^{2i\pi h t^2}}{t^s} dt \\ &\quad + G_{s,2N}(h) - G_{s,N}(h) + \mathcal{O}(N^{\frac{1}{2}-s} \log q), \end{aligned}$$

where

$$G_{s,N}(h) = (2hq)^{s-\frac{1}{2}} e^{i\pi/4} \sum_{m=1}^{\lfloor 2Nhq \rfloor} \frac{\theta_m}{m^s} e^{-i\pi \frac{m^2}{q^2 h}}. \tag{3.5}$$

Pay attention to the fact that $G_{s,N}$ depends on p and q. We omit this dependence in the notation for clarity.

Proof. We can write

$$F_{s,2N}(x) - F_{s,N}(x) = F_{s,N}^{\mathbf{1}_{[1,2]}}(x).$$

Hence, we would like to use the formulas proved in the preceding section, but those formulas apply only to compactly supported C^∞ functions. We thus decompose the indicator function $\mathbf{1}_{[1,2]}$ into a countable sum of C^∞ functions, as follows. Let us consider η, a C^∞ function with support $[1/2, 2]$ such that

$$\eta(t) = 1 - \eta(t/2) \qquad 1 \leq t \leq 2.$$

Then, the function

$$\psi(t) = \sum_{k\geq 2} \eta\left(\frac{t}{2^{-k}}\right)$$

has support in $[0,1/2]$, equals 1 in $[0,1/4]$ and is C^∞ in $[1/4,1/2]$. Therefore, we have

$$\mathbf{1}_{[1,2]}(t) = \psi(t-1) + \psi(2-t) + \widetilde{\psi}(t) \tag{3.6}$$

with $\widetilde{\psi}$ some C^∞ function with support included in $[1,2]$.

In order to get a formula for $F_{s,N}^{\mathbf{1}_{[1,2]}}(x)$, we are going to use (3.6) and the linearity in ψ of the formula (2.4).

We will first get a formula for $F_{s,N}^{\phi}$, where ϕ is any C^∞ function supported in $[1/2,2]$. In particular, this will work with $\phi = \widetilde{\psi}$.

Lemma 3.7. *Let ϕ be a C^∞ function supported in $[1/2,2]$. Then,*

$$F_{s,N}^{\phi}\left(\frac{p}{q}+h\right) = \frac{N^{1-s}\theta_0}{\sqrt{q}} \int_{\mathbb{R}} \frac{e^{2i\pi N^2 h t^2}}{t^s}\phi(t)dt + G_{s,N}^{\phi}(h) + \mathcal{O}_\phi\left(\frac{q}{N^{1/2+s}}\right), \tag{3.7}$$

where

$$G_{s,N}^{\phi}(h) = (2hq)^{s-\frac{1}{2}} e^{i\pi/4} \sum_{m\neq 0} \frac{\theta_m}{m^s} e^{-i\pi\frac{m^2}{2q^2h}} \phi\left(\frac{m}{2Nhq}\right). \tag{3.8}$$

Proof. First, by (2.5) one has $F_{s,N}^{\phi}(x) = N^{1-s}E_N^{\phi_s}(x)$. Further, by (3.2) one has

$$E_N^{\phi_s}\left(\frac{p}{q}+h\right) = \frac{1}{\sqrt{q}} \sum_{m\in\mathbb{Z}} \theta_m \cdot \widehat{w_{N^2h}^{\phi_s}}\left(\frac{Nm}{q}\right),$$

and then, applying Lemma 3.4 with $\xi = \frac{Nm}{q}$ and $R = N^2 h$, one gets

$$\begin{aligned} E_N^{\phi_s}\left(\frac{p}{q}+h\right) &= \theta_0 \frac{\hat{w}_{N^2h}^{\phi_s}(0)}{\sqrt{q}} \\ &\quad + \frac{e^{i\pi/4}}{\sqrt{q}} \sum_{m\neq 0} \Bigg(\theta_m \phi_s\left(\frac{m}{2Nqh}\right) \frac{e^{-i\pi\frac{m^2}{2q^2h}}}{\sqrt{2N^2h}} \\ &\qquad + \mathcal{O}\left((Nm/q)^{-3/2}\right) \Bigg). \end{aligned}$$

When $\frac{\xi}{2R} = \frac{m}{2Nqh} > 2$, $\phi_s\left(\frac{m}{2Nqh}\right) = 0$. Recalling that $N \geq q$, since $\phi_s(t) = \phi(t)t^{-s}$, the above equation can be rewritten

$$\begin{aligned} F_{s,N}^{\phi}\left(\frac{p}{q}+h\right) &= \frac{N^{1-s}\theta_0}{\sqrt{q}} \int_{\mathbb{R}} \frac{e^{2i\pi N^2ht^2}}{t^s}\phi(t)dt + G_N^{\phi}(h) \\ &\quad + \frac{N^{1-s}}{\sqrt{q}} \sum_{m\geq 1} \mathcal{O}_\phi\left((Nm/q)^{-3/2}\right). \end{aligned} \tag{3.9}$$

The last term is controlled by

$$\frac{N^{1-s}}{\sqrt{q}}\left(\frac{1}{(N/q)^{3/2}}\right)=\frac{q}{N^{1/2+s}},$$

which yields (3.7). □

Now, one wants to obtain a comparable formula for $\eta^k(t) := \eta((t-1)/2^{-k})$ for all $k \geq 1$. We begin with a bound which is good just for large k.

Lemma 3.8. *For any $k \geq 1$ and $0 < h \leq 1/q$, one has*

$$F^{\eta^k}_{s,N}\left(\frac{p}{q}+h\right)=\frac{\theta_0 N^{1-s}}{\sqrt{q}}\int_{\mathbb{R}}\frac{e^{2i\pi N^2ht^2}}{t^s}\eta^k(t)dt+G^{\eta^k}_{s,N}(h)+\mathcal{O}_\eta\left(N^{1-s}2^{-k}\right).$$

Proof. First, when k becomes large, since η has support in $[1/2,2]$, one has directly:
- by (2.4):

$$|F^{\eta^k}_{s,N}(x)| \leq \sum_{n=1}^{+\infty}\frac{1}{n^s}\left|\eta\left(\frac{\frac{n}{N}-1}{2^{-k}}\right)\right| \leq \sum_{n=N+N2^{-k-1}}^{N+N2^{-k+1}}\frac{1}{n^s} \ll N^{1-s}2^{-k}$$

- by (3.8):

$$|G^{\eta^k}_{s,N}(h)| \ll (qh)^{s-\frac12}\sum_{m:\,\eta^k(\frac{m}{2Nhq})\neq 0}\frac{1}{m^s} \ll (qh)^{s-\frac12}\frac{2Nhq2^{-k}}{(2Nhq)^s} \ll \sqrt{qh}2^{-k}N^{1-s}$$

- and

$$\left|\frac{\theta_0 N^{1-s}}{\sqrt{q}}\int_{\mathbb{R}}\frac{e^{2i\pi N^2ht^2}}{t^s}\eta^k(t)dt\right| \ll N^{1-s}q^{-1/2}\int_{1+2^{-k-1}}^{1+2^{-k+1}}\frac{dt}{t^s} \ll 2^{-k}N^{1-s}q^{-1/2},$$

hence the result by (3.9), where we used that $qh \leq 1$. □

One can obtain another bound that is good for any k.

Lemma 3.9. *For every $k \geq 2$ and $0 < h \leq 1$, one has*

$$F^{\eta^k}_{s,N}\left(\frac{p}{q}+h\right)=\frac{\theta_0 N^{1-s}}{\sqrt{q}}\int_{\mathbb{R}}\frac{e^{2i\pi N^2ht^2}}{t^s}\eta^k(t)dt+G^{\eta^k}_{s,N}(h)+\mathcal{O}_\eta\left(\frac{\sqrt{q}}{N^s}+N^{1/2-s}\gamma_k\right),$$

where the sequence $(\gamma_k)_{k\geq1}$ is positive and satisfies $\sum_{k\geq1}\gamma_k \ll 1$.

Proof. The proof starts as the one of Lemma 3.7. Using the fact that for any $(\eta^k)_s(t)=\frac{\eta((t-1)/2^{-k})}{t^s}$ and that

$$\begin{aligned}
\widehat{w_R^{(\eta^k)_s}}(\xi) &= \int_{\mathbb{R}}(\eta^k)_s(t)e^{2i\pi(Rt^2-t\xi)}dt=\int_{\mathbb{R}}\frac{\eta(\frac{t-1}{2^{-k}})}{t^s}e^{2i\pi(Rt^2-t\xi)}dt\\
&= 2^{-k}e^{2i\pi(R-\xi)}\int_{\mathbb{R}}\frac{\eta(u)}{(1+u2^{-k})^s}e^{2i\pi(R2^{-2k}u^2-2^{-k}(\xi-2R)u)}du\\
&= 2^{-k}e^{2i\pi(R-\xi)}\widehat{w^{\widetilde{\eta^k}}_{R2^{-2k}}}(2^{-k}(\xi-2R)),
\end{aligned}$$

where $\widetilde{\eta^k}(u) = \frac{\eta(u)}{(1+u2^{-k})^s}$. Hence,

$$E_N^{(\eta^k)_s}\left(\frac{p}{q}+h\right) = \frac{1}{\sqrt{q}}\sum_{m\in\mathbb{Z}}\theta_m \cdot \widehat{w_{N^2h}^{(\eta^k)_s}}\left(\frac{Nm}{q}\right)$$

$$= \frac{2^{-k}}{\sqrt{q}}\sum_{m\in\mathbb{Z}}\theta_m \cdot e^{2i\pi(N^2h-\frac{Nm}{q})}\widehat{w_{N^2h2^{-2k}}^{\widetilde{\eta^k}}}\left(2^{-k}\left(\frac{Nm}{q}-2N^2h\right)\right) \tag{3.10}$$

Here we apply again Lemma 3.4 and we obtain

$$\begin{aligned} E_N^{(\eta^k)_s}\left(\frac{p}{q}+h\right) &= \frac{2^{-k}}{\sqrt{q}}\sum_{m\neq 0}\theta_m e^{2i\pi(N^2h-\frac{Nm}{q})} \\ &\quad\times\left(e^{i\pi/4}\widetilde{\eta^k}\left(\frac{2^{-k}\left(\frac{Nm}{q}-2N^2h\right)}{2N^2h2^{-2k}}\right)\frac{e^{-i\pi\frac{2^{-2k}\left(\frac{Nm}{q}-2N^2h\right)^2}{2N^2h2^{-2k}}}}{\sqrt{2N^2h2^{-2k}}}\right. \\ &\quad\left.+\mathcal{O}_{\widetilde{\eta^k}}\left(\frac{\rho_{R_k,\xi_k}}{\sqrt{2^{-2k}N^2h}}+\left(1+2^{-k}\left|\frac{Nm}{q}-2N^2h\right|+N^2h2^{-2k}\right)^{-3/2}\right)\right). \end{aligned}$$

with $R_k = 2^{-2k}N^2h$ and $\xi_k = 2^{-k}|Nm/q-2N^2h|$. Finally, after simplification, one gets

$$F_{s,N}^{\eta^k}\left(\frac{p}{q}+h\right) = N^{1-s}E_N^{(\eta^k)_s}\left(\frac{p}{q}+h\right)$$

$$= \frac{(2qh)^{s-\frac{1}{2}}}{e^{-i\pi/4}}\sum_{m\in\mathbb{Z}}\frac{\theta_m}{m^s}e^{-2i\pi\frac{m^2}{q^2h}}\eta^k\left(\frac{m}{2Nhq}\right) \tag{3.11}$$

$$= \frac{\theta_0 N^{1-s}}{\sqrt{q}}\int_{\mathbb{R}}\frac{e^{2i\pi N^2ht^2}}{t^s}\eta^k(t)dt + G_{s,N}^{\eta^k}(h) + L_N^k, \tag{3.12}$$

where by Lemmas 3.7 and 3.8 one has

$$L_N^k = \mathcal{O}_{\widetilde{\eta^k}}\left(\sum_{m\in J_N^k\cap\mathbb{Z}^*}\frac{2^{-k}N^{1-s}/\sqrt{q}}{\sqrt{2^{-2k}N^2h}}+\sum_{m\in\mathbb{Z}^*}\frac{2^{-k}N^{1-s}/\sqrt{q}}{(1+2^{-k}|\frac{Nm}{q}-2N^2h|+N^2h2^{-2k})^{3/2}}\right),$$

with $J_N^k = \left[(2+2^{-k-1})Nqh, (2+2^{-k+1})Nqh\right]$.

First, as specified in Lemma 3.4, the constants involved in the $\mathcal{O}_{\widetilde{\eta^k}}$ depend on upper bounds for some derivatives of $\widetilde{\eta^k}$, and then by the definition of $\widetilde{\eta^k}$ we can assume they are fixed and independent on both k and s.

Let $\{x\}$ stand for the distance from the real number x to the nearest integer. The first sum in L_N^k is bounded above by:

- $\sqrt{q}N^{-s} + N^{1/2-s}$ when $2^{-k} \in \left[\{2Nhq\}/4Nhq, \{2Nhq\}/Nhq\right]$,
- $\sqrt{q}N^{-s}$ otherwise.

In particular, x being fixed, the term $N^{1/2-s}$ may appear only a finite number of times when k ranges in $\mathbb{N}$.

In the second sum, there is at most one integer m for which $|Nm/q-2N^2h| < N/2q$, and the corresponding term is bounded above by

$$\begin{aligned} & 2^{-k}N^{1-s}q^{-1/2}(1+N/q2^{-k}+|N^2h|^{-2k})^{-3/2} \\ \leq\ & 2^{-k}N^{1-s}q^{-1/2}(1+N/q2^{-2k})^{-3/2} \\ =\ & N^{1/2-s}\frac{N^{1/2}q^{-1/2}2^{-k}}{(1+N/q2^{-2k})^{-3/2}} \\ \leq\ & N^{1/2-s}\gamma_k, \end{aligned}$$

where $\gamma_k = \sqrt{\frac{u_k}{(1+u_k)^3}}$ and $u_k = 2^{-2k}N/q$. The sum over k of this upper bound is finite, and this sum can be bounded above independently on N and q.

The rest of the sum is bounded, up to a multiplicative constant, by

$$\int_{u=0}^{+\infty} \frac{\sqrt{q}N^{-s}\,(2^{-k}N/q)du}{(1+N^2h2^{-2k}+2^{-k}|\frac{Nu}{q}-2N^2h|)^{3/2}} \ll \frac{\sqrt{q}N^{-s}}{(1+N^2h2^{-2k})^{1/2}} \ll \sqrt{q}N^{-s},$$

hence the result. □

Now we are ready to prove Proposition 3.6.

Recall that $N \geq q$ and $0 \leq h \leq q^{-1}$. Let K be the unique integer such that $2^{-K} \leq \frac{\sqrt{q}}{N} < 2^{-(K+1)}$. We need to bound by above the sum of the errors L_N^k:

- when $k \geq K$: we use Lemma 3.8 to get
$$\sum_{k\geq K} |L_N^k| \ll N^{1-s}2^K \ll N^{1-s}\frac{\sqrt{q}}{N} = \frac{\sqrt{q}}{N^s} \leq \frac{1}{N^{s-1/2}}.$$
- the remaining terms are simply bounded using Lemma 3.9 by
$$\sum_{k=2}^{K} |L_N^k| \ll K\frac{\sqrt{q}}{N^s}+N^{1/2-s} \ll \log N\frac{\sqrt{q}}{N^s}+N^{1/2-s} \ll \log q\frac{\sqrt{N}}{N^s} = \frac{\log q}{N^{s-1/2}},$$
where we use that the mapping $x \mapsto \frac{\sqrt{x}}{\log x}$ is increasing for large x.

Gathering all the informations, and recalling that $\sum_{k=2}^{+\infty}\eta^k(t) = \psi(t-1)$, we have that

$$F_{s,N}^{\psi(\cdot-1)}\left(\frac{p}{q}+h\right) = \frac{N^{1-s}\theta_0}{\sqrt{q}}\int_{\mathbb{R}} \frac{e^{2i\pi N^2ht^2}}{t^s}\psi(t-1)dt + G_{s,N}^{\psi(\cdot-1)}(h) + \mathcal{O}_\eta\left(\frac{\log q}{N^{s-1/2}}\right).$$

The same inequalities remain true if we use the functions $\widetilde{\eta^k} = \eta((2-t)/2^{-k})$, so the last inequality also holds for $\psi(2-\cdot)$

Finally, recalling the decomposition (3.6) expressing $\mathbf{1}_{[1,2]}$ in terms of smooth functions, we get

$$F_{s,N}^{\mathbf{1}_{[1,2]}}\left(\frac{p}{q}+h\right) = \frac{N^{1-s}\theta_0}{\sqrt{q}}\int_1^2 \frac{e^{2i\pi N^2ht^2}}{t^s}dt + G_{s,N}^{\mathbf{1}_{[1,2]}}(h) + \mathcal{O}_\eta\left(\frac{\log q}{N^{s-1/2}}\right), \tag{3.13}$$

and the result follows. □

4. Proof of the convergence theorem 1.4

4.1. Convergence part: item (i). Let x be such that (1.4) holds true.

Recall the definition (1.2) of the convergents of x. We begin by bounding $F_{s,M}(x) - F_{s,N}(x)$ for any

$$q_j/4 \leq N < M < q_{j+1}/4.$$

We apply Proposition 3.6 with $p/q = p_j/q_j$ and $h = h_j$, so that $x = p/q + h$. Due to (3.1), we can assume that $h_j > 0$. It is known that for $\frac{1}{2q_j q_{j+1}} \leq h_j = |x - p_j/q_j| < \frac{1}{q_j q_{j+1}}$.

First, since $4Nh_j q_j < 4N/q_{j+1} < 1$, the sums (3.5) appearing in $G_{s,2N}(h_j)$ and $G_{s,N}(h_j)$ have no terms, hence are equal to zero. This yields

$$F_{s,2N}(x) - F_{s,N}(x) = \frac{\theta_0}{\sqrt{q_j}} \int_N^{2N} \frac{e^{2i\pi h_j t^2}}{t^s} dt + O(N^{\frac{1}{2}-s} \log q_j).$$

It is immediate to check that $\int_a^{2a} t^{-s} e^{2i\pi t^2}\, dt \ll \min(a^{-s-1}, a^{-s+1})$, thus

$$\begin{aligned} \left| \int_N^{2N} \frac{e^{2i\pi h_j t^2}}{t^s} dt \right| &\ll |h_j|^{s/2-1/2} \left| \int_{N\sqrt{h_j}}^{2N\sqrt{h_j}} \frac{e^{2i\pi u^2}}{u^s} du \right| \\ &\ll |h_j|^{-1/2} N^{-s} \min(|N\sqrt{h_j}|^{-1}, |N\sqrt{h_j}|). \end{aligned}$$

One deduces (using that $q_j h_j$ is equivalent to q_{j+1}^{-1}) that

$$|F_{s,2N}(x) - F_{s,N}(x)| \ll |\theta_0| \frac{\sqrt{q_{j+1}}}{N^s} \min\left(\frac{N}{\sqrt{q_j q_{j+1}}}, \frac{\sqrt{q_j q_{j+1}}}{N}\right) + N^{\frac{1}{2}-s} \log q_j.$$

Thus, by writing $F_{s,M}(x) - F_{s,N}(x)$ as a dyadic sum we have

$$|F_{s,M}(x) - F_{s,N}(x)| \ll |\theta_0| \delta_j \frac{\sqrt{q_{j+1}}}{(\sqrt{q_j q_{j+1}})^s} + \frac{\log q_j}{q_j^{s-1/2}}.$$

Recalling that θ_0 is equal to zero when $q_j \neq 2\times$odd, fixing an integer $j_0 \geq 1$, for any $M > N > q_{j_0}$, one has

$$|F_{s,M}(x) - F_{s,N}(x)| \ll \sum_{j \geq j_0, q_j \neq 2*\text{odd}} \delta_j \frac{\sqrt{q_{j+1}}}{(\sqrt{q_j q_{j+1}})^s} + \sum_{j \geq j_0} \frac{\log q_j}{q_j^{s-1/2}} + \sum_{j \geq j_0} \frac{1}{q_{j+1}^{s-1/2}}.$$

The second and third series always converge when $j_0 \to \infty$, and the first does when $\Sigma_s(x) < \infty$.

4.2. **Divergence part: item (ii).** Let $0 < \varepsilon < 1/2$ a small constant. Let $N_j = \varepsilon q_j$ and $M_j = 2\varepsilon\sqrt{q_j q_{j+1}}$. Proceeding exactly as in the previous proof we get

$$F_{s,M_j}(x) - F_{s,N_j}(x) = \frac{\theta_0}{\sqrt{q_j}} \int_{N_j}^{M_j} \frac{e^{2i\pi h_j t^2}}{t^s} dt + \mathcal{O}\left(q_j^{\frac{1}{2}-s} \log q_j \right).$$

Since $e^{2i\pi h_j t^2} = 1 + \mathcal{O}(\varepsilon)$ inside the integral, as soon as $q_j \neq 2 * \text{odd}$, one has

$$|F_{s,M_j}(x) - F_{s,N_j}(x)| \geq \frac{|\theta_0|}{\sqrt{q_j}} \frac{M_j - N_j}{2 \cdot M_j^s} \geq |\theta_0| \varepsilon \frac{2\sqrt{q_{j+1}} - \sqrt{q_j}}{2^{1+s} \cdot \varepsilon^s \cdot (q_j q_{j+1})^{s/2}} \gg \sqrt{\frac{q_{j+1}}{(q_j q_{j+1})^s}},$$

which is infinitely often large by our assumption. Hence the divergence of the series.

5. Local L^2 bounds for the function F_s

Further intermediary results are needed to study the local regularity of F_s.

Proposition 5.1. *Let $h > 0$, $1/2 < s < 3/2$ and $q^2 h \ll 1$. We have*

$$\begin{aligned} F_{s,N}\left(\frac{p}{q} + 2h\right) - F_{s,N}\left(\frac{p}{q} + h\right) \quad = \quad & \frac{\theta_0}{\sqrt{q}} \int_0^N \frac{e^{2i\pi 2ht^2} - e^{2i\pi ht^2}}{t^s} dt \\ & + G_{s,N}(2h) - G_{s,N}(h) \\ & + \mathcal{O}\left(|qh|^{s-1/2} \right). \end{aligned} \tag{5.1}$$

Proof. First, one writes

$$F_{s,N}(x) = F_{s,N}^{\mathbf{1}_{[0,1]}}(x) = \sum_{m \geq 1} F_{s,N/2^m}^{\mathbf{1}_{[1,2]}}(x). \tag{5.2}$$

Observe that when N is divisible by 2, there may be some terms appearing twice in the preceding sum, so there is not exactly equality. Nevertheless, in this case, only a few terms are added and they do not change our estimates. This is left to the reader.

We are going to estimate (5.1) but with $F_{s,N}^{\mathbf{1}_{[1,2]}}$ and $G_{s,N}^{\mathbf{1}_{[1,2]}}$ instead of $F_{s,N}$ and $G_{s,N}$, with an error term suitably bounded by above. Then, using this result with N substituted by $N/2^m$, and then summing over $m = 1, ..., \lfloor \log_2 N \rfloor$ will give the result (for $m > \lfloor \log_2 N \rfloor$, the sum $F_{s,N/2^m}^{\mathbf{1}_{[1,2]}}$ is empty).

We start from equation (3.10) applied with h and $2h$, and then we apply Lemma 3.4, but this time equation (3.4) instead of (3.3). Let us introduce for all integers k the quantity

$$\begin{aligned} E_N^k \quad := \quad & F_{s,N}^{\eta^k}\left(\frac{p}{q} + 2h\right) - F_{s,N}^{\eta^k}\left(\frac{p}{q} + h\right) \\ & + \frac{\theta_0 N^{1-s}}{\sqrt{q}} \int_{\mathbb{R}} \frac{e^{2i\pi N^2 2ht^2} - e^{2i\pi N^2 ht^2}}{t^s} \eta^k(t) dt \\ & + G_{s,N}^{\eta^k}(2h) - G_{s,N}^{\eta^k}(h), \end{aligned} \tag{5.3}$$

with η^k defined as in Proposition 3.6. By the exact same computations as in Lemma 3.9, one obtains the upper bound

$$|E_N^k| \ll \beta_N^k \sum_{m\in J_k\cap\mathbb{Z}^*} \frac{2^{-k}N^{1-s}/\sqrt{q}}{(2^{-2k}N^2h)^{-1/2}} + \sum_{m\in\mathbb{Z}^*} \frac{(2^{-k}N^{1-s}/\sqrt{q})N^2h2^{-2k}}{(1+N^2h2^{-2k}+2^{-k}|\frac{Nm}{q}-2N^2h|)^{5/2}},$$

with $J_N^k = \big[(2+2^{-k-1})Nqh, (2+2^{-k+1})Nqh\big]$ and

$$\beta_N^k = \begin{cases} 1 & \text{if } 2^{-2k}N^2h \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, as at the end of the proof of Lemma 3.9, since $h \ll q^{-2}$, we can bound the sums by

$$|E_N^k| \ll \beta_N^k 2^{-2k} N^{2-s}\frac{\sqrt{h}}{\sqrt{q}} + \frac{(N^{1/2-s})\sqrt{(N/q)2^{-2k}}N^2h2^{-2k}}{(1+N^2h2^{-2k}+(N/q)2^{-2k})^{5/2}} + \frac{(\sqrt{q}N^{-s})N^2h2^{-2k}}{(1+N^2h2^{-2k})^{3/2}}$$

and then adding up in $k\ge 1$ we get

$$\sum_{k=1}^{\infty}|E_N^k| \ll \widetilde{E}_N = \frac{\sqrt{1/hq}}{N^s}\min(1,Nqh) + \frac{\sqrt{N}}{N^s}\min(1,Nqh) + \frac{\sqrt{q}}{N^s}\min(1,N^2h).$$

The same holds true for the functions $\widetilde{\eta^k} = \eta((2-t)/2^{-k})$, and for $\widetilde{\psi}$ since it is similar to η^1, so by (3.6) we finally obtain that

$$F_{s,N}^{\mathbf{1}_{[1,2]}}\left(\frac{p}{q}+2h\right) - F_{s,N}^{\mathbf{1}_{[1,2]}}\left(\frac{p}{q}+h\right) = \frac{\theta_0}{\sqrt{q}}\int_0^N \frac{e^{2i\pi 2ht^2}-e^{2i\pi ht^2}}{t^s}dt + G_{s,N}^{\mathbf{1}_{[1,2]}}(2h) - G_{s,N}^{\mathbf{1}_{[1,2]}}(h) + \mathcal{O}\left(\widetilde{E}_N\right). \tag{5.4}$$

The same holds true with $N/2^m$ instead of N. To get the result, using (5.2), it is now enough to sum the last inequality over $m = 1, ..., \lfloor \log_2 N\rfloor$. Let us treat the first term. One has

$$\begin{aligned}\sum_{m=1}^{\lfloor\log_2 N\rfloor}\frac{\sqrt{1/hq}}{N^s}\min(1,Nqh) &= \sum_{m=1}^{\lfloor\log_2 N\rfloor}\frac{\sqrt{1/hq}}{(N2^m)^s}\min(1,N2^mqh)\\ &= \frac{\sqrt{1/hq}}{N^s}\sum_{m=1}^{\lfloor\log_2 N\rfloor}\min(2^{ms},N2^{m(1-s)}qh)\\ &\le \frac{\sqrt{1/hq}}{N^s}\sum_{m=1}^{+\infty}\min(2^{ms},N2^{m(1-s)}qh)\\ &\ll \frac{\sqrt{1/hq}}{N^s}2^{Ms},\end{aligned}$$

where M is the integer part of the solution of the equation $2^{Ms} = N2^{m(1-s)}qh$, i.e. $2^M \approx Nqh$. Hence the first sum is bounded above by $\frac{\sqrt{1/qh}}{(1/qh)^s}$. The other terms are treated similarly, and finally (5.1) is true with an error term bounded by above by

$$\mathcal{O}\left(\frac{\sqrt{1/qh}}{(1/qh)^s} + \frac{\sqrt{q}}{(h^{-1/2})^{-s}}\right)$$

which is $\mathcal{O}((qh)^{s-1/2})$ on $h \ll q^{-2}$. □

We also need to control the L^2 norm of the main term.

Lemma 5.2. *Let $0 < s \leq 1$ and fix $0 < H < 1$. Let*

$$f_{s,N}(\cdot) = \int_0^N \frac{e^{2i\pi t^2(\delta+2\cdot)} - e^{2i\pi t^2(\delta+\cdot)}}{t^s} dt.$$

Then for any $N > 0$, $\|f_{s,N}(\cdot)\|_{L^2(\widetilde{\mu}_H)} \ll \min\left(H^{(s-1)/2}, H|\delta|^{(s-3)/2}\right)$.

Proof. • Let us treat first the case $|\delta| < H/4$. Using a change of variable, one has

$$f_{s,N}(h) = H^{(s-1)/2} \int_0^{N\sqrt{H}} \frac{e^{2i\pi t^2 \frac{\delta+2h}{H}} - e^{2i\pi t^2 \frac{\delta+h}{H}}}{t^s} dt.$$

We are interested in the range $H < h < 2H$, and in this case the ratios $\frac{\delta+2h}{H}$, $\frac{\delta+h}{H}$ are bounded, so that the integral is bounded by a constant independent of N. One deduces that $\|f_{s,N}(\cdot)\|_{L^2(\widetilde{\mu}_H)} \ll H^{(s-1)/2}$.

• Assume then that $|\delta| > 4H$. Assume that $\delta > 0$ (the same holds true with negative δ's). Using a change of variable, one has

$$f_{s,N}(h) = |\delta|^{(s-1)/2} \int_0^{N\sqrt{|\delta|}} \frac{e^{2i\pi t^2(1+\frac{2h}{\delta})} - e^{2i\pi t^2(1+\frac{h}{\delta})}}{t^s} dt.$$

The integral between 0 and 1 is clearly $\mathcal{O}(h/|\delta|)$. For the other part, one has (after integration by parts)

$$\int_1^{N\sqrt{\delta}} \frac{e^{2i\pi t^2(1+\frac{2h}{\delta})} - e^{2i\pi t^2(1+\frac{h}{\delta})}}{t^s} dt \quad = \quad \mathcal{O}\left(h/|\delta|\right),$$

so that $|f_{s,N}(h)| \ll H|\delta|^{(s-3)/2}$ for any $H < h < 2H$. Hence $\|f_{s,N}(\cdot)\|_{L^2(\widetilde{\mu}_H)} \ll H|\delta|^{(s-3)/2}$.

• It remains us to deal with the case $H/4 < \delta \leq 4H$. One observes that

$$f_{s,N}(h) = D(\delta+2h) - D(\delta+h) + \mathcal{O}(H), \quad \text{where } D(v) = \int_1^N t^{-s} e^{2i\pi v t^2}\, dt.$$

It is enough to get the bound

$$\int_0^H |D(v)|^2\, dv \ll H^s,$$

which follows from the fact that $|D(v)| \ll |v|^{(s-1)/2}$ when $s < 1$ and $|D(v)| \ll 1 + \log(1/|v|)$ when $s = 1$.

□

Finally, the oscillating behavior of $G_{s,N}(h)$ gives us the following.

Proposition 5.3. *Let $0 < H \le q^{-2}$ and $|\delta| \le \sqrt{H}/q$. Let*

$$g_{s,N}(\cdot) = F_{s,N}\left(\frac{p}{q} + \delta + 2\,\cdot\right) - F_{s,N}\left(\frac{p}{q} + \delta + \cdot\right) - \frac{\theta_0}{\sqrt{q}} f_{s,N}(\cdot).$$

One has $\|g_{s,N}\|_{L^2(\widetilde{\mu}_H)} \ll H^{\frac{s-1/2}{2}}$.

Proof. We consider $\mu_H = (\widetilde{\mu}_H)_{|\mathbb{R}^+}$. By (3.1), it is enough to treat the case $\delta + h > 0$ and $\delta + 2h > 0$. Proposition 5.1, applied successively with $h_n := 2^{-n}(\delta + h)$ and $\widetilde{h}_n := 2^{-n}(\delta + h/2)$, and summing over $n \ge 0$, we get that

$$g_{s,N}(h) = G_{s,N}(\delta + 2h) - G_{s,N}(\delta + h) + O\left((q(|\delta| + |h|))^{s-1/2}\right). \tag{5.5}$$

Thus, since $q(|\delta| + |h|) \ll \sqrt{H}$, it is enough to show that

$$\|G_{s,N}(\delta + \cdot)\|_{L^2(\mu_H)} \ll H^{\frac{s-1/2}{2}}. \tag{5.6}$$

Assume first that $|\delta| \ge 3H$. By expanding the square and changing the order of summation, and using that $\delta + 2H \le 2|\delta|$, we have for some $c_{n,m} \ge 0$

$$\begin{aligned}\|G_{s,N}(\delta + \cdot)\|^2_{L^2(\mu_H)} &\ll (q|\delta|)^{2s-1} \sum_{n,m=1}^{2\lfloor 2N|\delta|q\rfloor} \frac{|\theta_m|}{m^s}\frac{|\theta_n|}{n^s}\left|\int_{\delta+H+c_{n,m}}^{\delta+2H} e^{2i\pi\frac{n^2-m^2}{q^2h}}\,\frac{dh}{H}\right| \\ &\ll (q|\delta|)^{2s-1} \sum_{n,m=1}^{2\lfloor 2N|\delta|q\rfloor} \frac{|\theta_m|}{m^s}\frac{|\theta_n|}{n^s}\left|\int_{\delta+H}^{\delta+2H} e^{2i\pi\frac{n^2-m^2}{q^2h}}\,\frac{dh}{H}\right|.\end{aligned}$$

Since for $|M| \ge 1$ and $0 < \varepsilon \ll 1$

$$\int_1^{1+\varepsilon} e^{2i\pi\frac{M}{t}}\,dt \ll \frac{1}{|M|},$$

the previous sum is bounded above by

$$(q|\delta|)^{2s-1}\left[\sum_{m\ge1}\frac{1}{m^{2s}} + \frac{|\delta|}{H}q^2|\delta|\sum_{m\ge1}\frac{1}{m^{1+s}}\sum_{j\ge1}\frac{1}{j^{1+s}}\right],$$

with $j = |n - m|$. The term between brackets is bounded by a universal constant (since $q^2\delta^2/H \le 1$), hence (5.6) holds true. It is immediate that the same holds true with $(\widetilde{\mu}_H)_{|\mathbb{R}^-}$.

Further, assume that $|\delta| < 3H$. Setting $H_k = 2^{-k}H$, one has

$$\|G_{s,N}(\delta + \cdot)\|^2_{L^2(\widetilde{\mu}_H)} \le \int_0^{5H} |G_{s,N}(h)|^2\frac{dh}{H} \le \sum_{k\ge-2} 2^{-k}\|G_{s,N}(\cdot)\|^2_{L^2(\mu_{H_k})}.$$

Now, observing that $[H_k, H_{k-1}] \subset 3H_k + ([-2H_k, -H_k] \cup [H_k, 2H_k])$, one can apply (5.6) with $H = H_k$ and $\delta = 3H_k$ to get

$$\|G_{s,N}(\cdot)\|^2_{L^2(\mu_{H_k})} \le \|G_{s,N}(\delta_k + \cdot)\|_{L^2(\mu_{H_k})} \le H_k^{\frac{s-1/2}{2}} = H^{\frac{s-1/2}{2}}2^{-k\frac{s-1/2}{2}}.$$

Summing over k yields the result. $\square$

6. Proof of Theorem 1.5

6.1. **Lower bound for the local L^2-exponent α_{F_s}.** Assume that $\Sigma_s(x) < \infty$ (see equation (1.4)), so that the series $F_{s,N}(x)$ converges to $F_s(x)$. Recall that p_j/q_j stands for the partial quotients of x.

Pick N such that $0 \leq |F_s(x) - F_{s,N}(x)| < H$ and $N^{\frac{1}{2}-s} \leq H^2$. Since

$$\begin{aligned} \|F_s(x+\cdot) - F_{s,N}(x+\cdot)\|_{L^2(\widetilde{\mu}_H)} &\leq \frac{\|F_s(x+\cdot) - F_{s,N}(x+\cdot)\|_{L^2([0,1])}}{H/2} \\ &\ll \frac{N^{\frac{1}{2}-s}}{H} \leq H, \end{aligned}$$

and since one has

$$F_s(x+\cdot)-F_s(x) = F_s(x+\cdot)-F_{s,N}(x+\cdot)+F_{s,N}(x+\cdot)-F_{s,N}(x)+F_{s,N}(x)-F_s(x),$$

one deduces that

$$\|F_s(x+\cdot) - F_s(x)\|_{L^2(\widetilde{\mu}_H)} = \|F_{s,N}(x+\cdot) - F_{s,N}(x)\|_{L^2(\widetilde{\mu}_H)} + \mathcal{O}(H).$$

Thus, it is enough to take care of the local L^2-norm of $F_{s,N}(x+h) - F_{s,N}(x)$. One has

$$\begin{aligned} \|F_{s,N}(x+h) - F_{s,N}(x)\|_{L^2(\widetilde{\mu}_H)} &\leq \sum_{k\geq 1} \left\| F_{s,N}(x+2\frac{\cdot}{2^k}) - F_{s,N}(x+\frac{\cdot}{2^k})\right\|_{L^2(\widetilde{\mu}_H)} \\ &\leq \sum_{k\geq 1} \|F_{s,N}(x+2\,\cdot) - F_{s,N}(x+\cdot)\|_{L^2(\widetilde{\mu}_{H_k})}, \end{aligned} \tag{6.1}$$

where $H_k = H2^{-k}$. Let us introduce the function $f(h) = F_{s,N}(x+2\,h) - F_{s,N}(x+h)$.

Let j_H be the smallest integer such that $q_j^{-2} \leq H$. For every $k \geq 1$, and let j be the unique integer such that $q_{j+1}^{-2} \leq H_k < q_j^{-2}$ (necessarily $j \geq j_H - 1$). Using that $|x - p_j/q_j| = |h_j| \leq q_j^{-2}$, one sees that

$$\|f\|_{L^2(\widetilde{\mu}_{H_k})} = \left\| F_{s,N}\left(\frac{p_j}{q_j} + h_j + 2\,\cdot\right) - F_{s,N}\left(\frac{p_j}{q_j} + h_j + \cdot\right)\right\|_{L^2(\widetilde{\mu}_{H_k})}.$$

Since $|h_j| < 1/q_jq_{j+1} \leq \sqrt{H_k}/q_j$, we can apply Proposition 5.3 and Lemma 5.2 with H_k and $\delta = h_j$ to get

$$\begin{aligned} \|f\|_{L^2(\widetilde{\mu}_{H_k})} &\ll H_k^{\frac{s-1/2}{2}} + \frac{|\theta_0|}{\sqrt{q_j}} \min\left(H_k^{(s-1)/2}, H_k|h_j|^{(s-3)/2}\right) \\ &\ll H_k^{\frac{s-1/2}{2}} + \frac{|\theta_0|}{\sqrt{q_j}} H_k^{(s-1)/2} \min\left(1, \left|\frac{h_j}{H_k}\right|^{(s-3)/2}\right). \end{aligned}$$

In order to finish the proof we are going to consider three different cases:

(1) $s-1+1/2r_{\text{odd}}(x)>0$**:** Since $h_j = q_j^{-r_j}$ we have

$$\|f\|_{L^2(\widetilde{\mu}_{H_k})} \ll H_k^{\frac{s-1/2}{2}} + |\theta_0| H_k^{(s-1)/2} \min\left(|h_j|^{\frac{1}{2r_j}}, \frac{H_k{}^{(3-s)/2}}{|h_j|^{\frac{3-s}{2}-\frac{1}{2r_j}}}\right),$$

and optimizing in $|h_j|$ we get

$$\|f\|_{L^2(\widetilde{\mu}_{H_k})} \ll H_k^{\frac{s-1/2}{2}} + |\theta_0| H_k^{\frac{s-1+1/2r_j}{2}} \ll H_k^{(s-1+1/2r_{\text{odd}}(x)+o(H_k))/2}$$

by the definition of $r_{\text{odd}}(x)$. Adding up in k finishes the proof in this case.

(2) $s-1+1/2r_{\text{odd}}(x)=0$ **and** $s=1$**:** In this case it is enough to show that $\sum_{k\geq 1}\|f\|_{L^2(\widetilde{\mu}_{H_k})}<\infty$. We have

$$\|f\|_{L^2(\widetilde{\mu}_{H_k})} \ll H_k^{\frac{s-1/2}{2}} + \frac{|\theta_0|}{\sqrt{q_j}}$$

which implies

$$\sum_{q_{j+1}^{-2}\leq H_k\leq q_j^{-2}} \|f\|_{L^2(\widetilde{\mu}_{H_k})} \ll \sum_{q_{j+1}^{-2}\leq H_k\leq q_j^{-2}} H_k^{\frac{s-1/2}{2}} + \frac{|\theta_0|}{\sqrt{q_j}} \log(q_{j+1}/q_j).$$

This yields

$$\sum_{k\geq 1} \|f\|_{L^2(\widetilde{\mu}_{H_k})} \ll H^{\frac{s-1/2}{2}} + \sum_{j:\, q_j\neq 2*\text{odd}} \frac{1}{\sqrt{q_j}} \log\frac{q_{j+1}}{q_j} \ll 1+\Sigma_s(x) < +\infty.$$

(3) $s-1+1/2r_{\text{odd}(x)}=0$ **and** $s<1$**:** Since $h_j \asymp 1/q_jq_{j+1}$, we have

$$\|f\|_{L^2(\widetilde{\mu}_{H_k})} \ll H_k^{\frac{s-1/2}{2}} + \frac{|\theta_0|}{\sqrt{q_j}} \min\left(H_k^{(s-1)/2}, \frac{H_k}{(q_jq_{j+1})^{(s-3)/2}}\right),$$

so

$$\sum_{q_{j+1}^{-2}\leq H_k\leq q_j^{-2}} \|f\|_{L^2(\widetilde{\mu}_{H_k})} \ll (\sum_{q_{j+1}^{-2}\leq H_k\leq q_j^{-2}} H_k^{\frac{s-1/2}{2}}) + \frac{|\theta_0|}{\sqrt{q_j}}(\frac{1}{q_jq_{j+1}})^{(s-1)/2}.$$

Finally,

$$\sum_{k\geq 1} \|f\|_{L^2(\widetilde{\mu}_{H_k})} \ll H^{\frac{s-1/2}{2}} + \sum_{j, q_j\neq 2*\text{odd}} \sqrt{\frac{q_{j+1}}{(q_jq_{j+1})^s}} \ll 1+\Sigma_s(x) < \infty.$$

6.2. **Upper bound for the local L^2-exponent.** Assume first that $s<1$.

Let K be a large constant. Let $0<H\leq (1/K)q^{-2}$, with $q\neq 2*\text{odd}$ and $N>H^{-2}$. We apply Propositions 5.1 and 5.3 to get

$$\left\|F_{s,N}\left(\frac{p}{q}+2\cdot\right) - F_{s,N}\left(\frac{p}{q}+\cdot\right)\right\|_{L^2(\widetilde{\mu}_H)} = \frac{|\theta_0|}{\sqrt{q}}\left\|\widetilde{F}_s(\cdot)\right\|_{L^2(\widetilde{\mu}_H)} + \mathcal{O}(H^{\frac{s-1/2}{2}})$$

with

$$\widetilde{F_s}(h) = \int_0^N \frac{e^{4i\pi ht^2}-e^{2i\pi ht^2}}{t^s}\,dt.$$

Using a change of variable, and then after integrating by parts, one obtains

$$\begin{aligned}
\widetilde{F_s}(h) &= h^{\frac{s-1}{2}} \int_0^{N\sqrt{h}} \frac{e^{4i\pi t^2} - e^{2i\pi t^2}}{t^s}\, dt \\
&= h^{\frac{s-1}{2}} \left((2^s - 1) \int_0^{+\infty} \frac{e^{2i\pi t^2}}{t^s} dt + \mathcal{O}\left((N\sqrt{|h|})^{-s-1} \right) \right).
\end{aligned}$$

It is easily checked that $\int_0^{+\infty} \frac{e^{2i\pi t^2}}{t^s} dt$ is not zero. This leads us to the estimate

$$\left\| \widetilde{F_s}(\cdot) \right\|_{L^2(\widetilde{\mu}_H)} = C_s H^{\frac{s-1}{2}} \, (1 + \mathcal{O}(H))$$

for some non-zero constant C_s. Since $0 < H \leq q^{-2}/K$, we deduce that

$$\left\| F_{s,N}\left(\frac{p}{q} + 2\cdot\right) - F_{s,N}\left(\frac{p}{q} + \cdot\right) \right\|_{L^2(\widetilde{\mu}_H)} \geq \frac{H^{\frac{s-1}{2}}}{\sqrt{q}} \tag{6.2}$$

when H becomes small enough.

Now, pick a convergent p_j/q_j of x with $q_j \neq 2 * \text{odd}$, and take $H_j = (1/K)|h_j|$. One can check that

$$H_j \leq (1/K) \frac{1}{q_j q_{j+1}} \leq (1/K) \frac{1}{q_j^2}.$$

Then, we apply (6.2) to obtain that for every $N \geq H_j^{-2}$, one has

$$\left\| F_{s,N}\left(\frac{p_j}{q_j} + 2\cdot\right) - F_{s,N}\left(\frac{p_j}{q_j} + \cdot\right) \right\|_{L^2(\widetilde{\mu}_{H_j})} \geq \frac{H_j^{\frac{s-1}{2}}}{\sqrt{q_j}} = H_j^{\frac{s-1}{2}} h_j^{1/(2r_j)} \gg H_j^{\frac{s-1+1/r_j}{2}}.$$

On the other hand, by the triangular inequality,

$$\begin{aligned}
\left| F_{s,N}\left(\frac{p_j}{q_j} + 2h\right) - F_{s,N}\left(\frac{p_j}{q_j} + h\right) \right| &\leq \left| F_{s,N}\left(\frac{p_j}{q_j} + 2h\right) - F_{s,N}(x) \right| \\
&+ \left| F_{s,N}\left(\frac{p_j}{q_j} + h\right) - F_{s,N}(x) \right|,
\end{aligned}$$

which implies that for $\widetilde{H}_j = H_j$ or $\widetilde{H}_j = 2H_j$, one has

$$\begin{aligned}
\| F_{s,N}(x + \cdot) - F_{s,N}(x) \|_{L^2(\widetilde{\mu}_{\widetilde{H}_j})} &\geq \frac{1}{2} \left\| F_{s,N}\left(\frac{p_j}{q_j} + 2\cdot\right) - F_{s,N}\left(\frac{p_j}{q_j} + \cdot\right) \right\|_{L^2(\widetilde{\mu}_{H_j})} \\
&\gg H_j^{\frac{s-1+1/r_j}{2}}
\end{aligned}$$

Now, we can choose N so large that

$$\| F_{s,N}(x + \cdot) - F_{s,N}(x) \|_{L^2(\widetilde{\mu}_{\widetilde{H}_j})} = \| F_s(x + \cdot) - F_s(x) \|_{L^2(\widetilde{\mu}_{\widetilde{H}_j})} + \mathcal{O}(\widetilde{H}_j),$$

and we finally obtain

$$\| F_s(x + \cdot) - F_s(x) \|_{L^2(\widetilde{\mu}_{\widetilde{H}_j})} \gg \widetilde{H}_j^{\frac{s-1+1/r_j}{2}}.$$

Since this occurs for an infinite number of j, i.e. for an infinite number of small real numbers $\widetilde{H}_j$ converging to zero, one concludes that

$$\alpha_{F_s}(x) \leq \liminf_{j\to+\infty} \frac{s-1+1/r_j}{2} = \frac{s-1+1/r_{odd}(x)}{2}.$$

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