Exit-time of mean-field particles system

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Abstract

The current article is devoted to the study of a mean-field system of particles. The question that we are interested in is the behaviour of the exit-time of the first particle (and the one of any particle) from a domain $\mathcal{D}$ on $\mathbb{R}^d$ as the diffusion coefficient goes to 0. We establish a Kramers'type law. In other words, we show that the exit-time is exponentially equivalent to $\exp\left\{\frac{2}{\sigma^2}H^N\right\}$, H^N being the exit-cost. We also show that this exit-cost converges to some quantity H. To obtain this result, we proceed by two different approaches.

Key words and phrases: Exit-problem ; Large deviations ; Interacting particle systems ; Mean-field systems

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1 Introduction

The paper is devoted to the resolution of the exit-time of some mean-field interacting particles system. Let us briefly present the model. For any $i \in \mathbb{N}^*$, $\{B_t^i \ : \ t \in \mathbb{R}_+\}$ is a Brownian motion on $\mathbb{R}^d$. The Brownian motions are assumed to be independent. Each particle evolves in a non-convex landscape V, that we denote as the confining potential. Moreover, each particle interacts with any other one. We assume that the interaction does only depend on the distance between the two particles. This interacting force is odd.

In fine, the system of equations that we are interested in is the following:

$$X_t^{i,N} = x_0 + \sigma B_t^i - \int_0^t \nabla V\left(X_s^{i,N}\right) ds - \int_0^t \frac{1}{N}\sum_{j=1}^N \nabla F\left(X_s^{i,N} - X_s^{j,N}\right) ds\,, \quad \text{(I)}$$

N being arbitrarily large and σ being an arbitrarily small positive constant.

We can see the N particles in $\mathbb{R}^d$ as one diffusion in $\mathbb{R}^{dN}$. Indeed, let us write $\mathcal{X}_t^N := \left(X_t^{1,N}, \cdots, X_t^{N,N}\right)$ and $\mathcal{B}_t^N := \left(B_t^1, \cdots, B_t^N\right)$. The process $\mathcal{B}^N$ thus is a dN-dimensional Wiener process. Equation (I) can be rewritten like so:

$$\mathcal{X}_t^N = \mathcal{X}_0^N + \sigma\mathcal{B}_t^N - N\int_0^t \nabla\Upsilon^N\left(\mathcal{X}_s^N\right)ds\,. \tag{II}$$

Here, the potential on $\mathbb{R}^{dN}$ is defined by $\Upsilon^N\left(X_1,\cdots,X_N\right) := \frac{1}{N}\sum_{i=1}^N V\left(X_i\right) + \frac{1}{2N^2}\sum_{i=1}^N\sum_{j=1}^N F\left(X_i - X_j\right)$ for any $\left(X_1,\cdots,X_N\right) \in \left(\mathbb{R}^d\right)^N$.

Consequently, the whole system of particles, $\{\mathcal{X}_t^N \ : \ t \in \mathbb{R}_+\}$, is just an homogeneous and reversible diffusion in $\mathbb{R}^{dN}$ since it evolves only trough the gradient of the potential $N\Upsilon^N$.

It is well-known (see [Mél96, Szn91, BRTV98, CGM08, HT16]) that the understanding of the behaviour of the diffusion $X^{1,N}$ when N is large is linked to its hydrodynamical limit diffusion that is to say

$$\begin{cases} X_t^{1,\infty} = x_0 + \sigma B_t^1 - \int_0^t \nabla V\left(X_s^{1,\infty}\right)ds - \int_0^t \nabla F * \mu_t^\infty\left(X_s^{1,\infty}\right)ds \\ \mu_t^\infty = \mathcal{L}\left(X_t^{1,\infty}\right) \end{cases}. \tag{III}$$

We consider a domain $\mathcal{D} \subset \mathbb{R}^d$ and the associated exit-times:

$$\tau_{\mathcal{D}}^i(\sigma,N) := \inf\left\{t \geq 0 \ : \ X_t^{i,N} \notin \mathcal{D}\right\}$$

which corresponds to the first exit-time of the particle number i and

$$\tau_{\mathcal{D}}(\sigma,N) := \inf\left\{\tau_{\mathcal{D}}^i(\sigma,N) \ : \ i \in [\![1;N]\!]\right\}.$$

Let us point out that we can not directly tract the Kramers'law for $\tau_{\mathcal{D}}(\sigma,N)$ from the Kramers'law satisfied by the $\tau_{\mathcal{D}}^i(\sigma,N)$. Indeed, there is no independence since there is some interaction between the particles.

We study these exit-times in the small-noise limit with N large (but we do not take the limit as N goes to infinity).

Freidlin and Wentzell theory solves this question for time-homogeneous diffusion in finite dimension. See [DZ98, FW98] for a complete review. The main result is the following:

Theorem 1.1. *We consider a domain $\mathcal{G} \subset \mathbb{R}^k$, a potential U on $\mathbb{R}^k$ and a diffusion*

$$x_t^\sigma = x_0 + \sigma B_t - \int_0^t \nabla U\left(x_s^\sigma\right)ds\,.$$

We assume that $\mathcal{G}$ satisfies the following properties.

1. *The unique critical point of the potential U in the domain $\mathcal{G}$ is a_0. Furthermore, for any $y_0 \in \mathcal{G}$, for any $t \in \mathbb{R}_+$, we have $y_t \in \mathcal{G}$ and moreover $\lim_{t\to+\infty} y_t = a_0$ with*

$$y_t = y_0 - \int_0^t \nabla U(y_s)ds\,.$$

2. For any $y_0 \in \partial\mathcal{G}$, y_t converges toward a_0.

3. The quantity $H := \inf_{z\in\partial\mathcal{G}} (U(z) - U(a_0))$ is finite.

By $\tau_{\mathcal{G}}(\sigma)$, we denote the first exit-time of the diffusion x^σ from the domain $\mathcal{G}$. Then, for any $\delta > 0$, we have

$$\lim_{\sigma\to 0} \mathbb{P}\left\{e^{\frac{2}{\sigma^2}(H-\delta)} < \tau_{\mathcal{G}}(\sigma) < e^{\frac{2}{\sigma^2}(H+\delta)}\right\} = 1\,.$$

Furthermore, if $\mathcal{N} \subset \partial\mathcal{G}$ is such that $\inf_{z\in\mathcal{N}} U(z) > \inf_{z\in\partial\mathcal{G}} U(z)$, we know that the diffusion x^σ does not exit $\mathcal{G}$ by $\mathcal{N}$ with high probability:

$$\lim_{\sigma\to 0} \mathbb{P}\left\{x^\sigma_{\tau_{\mathcal{G}}(\sigma)} \in \mathcal{N}\right\} = 0\,.$$

We do not provide the proof which can be found in [DZ98].

In [Tug12b], we have obtained a similar result (already obtained in [HIP08]) for the self-stabilizing diffusion (III). To do so, we establish a Kramers'type law for the first particle of the mean-field system of particles. In this previous work, both the confining potential and the interacting potential are assumed to be convex.

In the current paper, we remove the hypothesis of global convexity for the confining potential.

We proceed similarly than in [Tug12b].

We now give a definition which is of crucial interest in large deviations for stochastic processes.

Definition 1.2. *Let $\mathcal{D}$ be an open domain of $\mathbb{R}^k$ and U be a potential of $\mathbb{R}^k$. In the following, we say that $\mathcal{D}$ is stable by the potential U if for any $\xi_0 \in \mathcal{D}$, for any $t \geq 0$, we have $\xi_t \in \mathcal{D}$ with*

$$\xi_t = \xi_0 - \int_0^t \nabla U(\xi_s)\, ds\,.$$

We now give the assumptions of the paper. First, we give the hypotheses on the confining potential V.

Assumption (A-1): *V is a $\mathcal{C}^2$-continuous function.*

Assumption (A-2): *For all $\lambda > 0$, there exists $R_\lambda > 0$ such that $\nabla^2 V(x) > \lambda$, for any $||x|| \geq R_\lambda$.*

Assumption (A-3) *The gradient ∇V is slowly increasing: there exist $m \in \mathbb{N}^*$ and $C > 0$ such that $||\nabla V(x)|| \leq C\left(1 + ||x||^{2m-1}\right)$, for all $x \in \mathbb{R}$.*

Let us present now the assumptions on the interaction potential F:

Assumption (A-4): *There exists a function G from $\mathbb{R}_+$ to $\mathbb{R}$ such that $F(x) = G(||x||)$.*

Assumption (A-5): *G is an even polynomial and convex function such that $\deg(G) =: 2n \geq 2$ and $G(0) = 0$.*

We finish by giving the assumptions on the open domain $\mathcal{D}$:

Assumption (A-6): *The domain $\mathcal{D}$ contains only one critical point of V: a.*

Assumption (A-7): *The domain $\mathcal{D}$ is stable by the potential $V+F*\delta_a$.*

Indeed, heuristically, the potential $V+F*\left(\sum_{j=1}^N \delta_{X_t^{j,N}}\right)$ is close to the potential $V+F*\delta_a$ in the small-noise limit. So, we can link the study of the first particle (and of any particle) with the study of a classical diffusion with potential $V+F*\delta_a$. In order to apply Freidlin-Wentzell theory, we thus assume this hypothesis.

Assumption (A-8): *There exists $\rho>0$ such that: $\langle\nabla V(x);x-a\rangle\geq\rho||x-a||^2$ for any $x\in\mathcal{D}$.*

This assumption allows us to prove that the domains that we will consider on $\mathbb{R}^{dN}$ are stable by the potential Υ^N.

Assumption (A-9): *There exists $\delta>0$ such that for any $x\in\mathcal{D}$: $V(x)-V(a)\geq\frac{\delta}{2}||x-a||^2$*

This simple hypothesis yields that the exit-cost of a ball of center $(a,\cdots,a)$ with any radius $\kappa>0$ goes to infinity as N goes to infinity.

Assumption (A-10): *By putting $\varphi_t:=x_0-\int_0^t\nabla V(\varphi_s)\,ds$, then for any $t\geq0$, $\varphi_t\in\mathcal{D}$.*

If Assumption (A-10) was not satisfied, we could easily prove that the exit-time is sub-exponential.

Assumption (A-11): *There exists an open domain $\mathcal{D}'$ which contains $\overline{\mathcal{D}}$ and which satisfies assumptions (A-6)–(A-10).*

This last assumption allows us to isolate the first particle.

Example 1.3. *We now give an example of potentials and domain satisfying Assumptions (A-1)–(A-11) in the one-dimensional case. We take $V(x):=\frac{x^4}{4}-\frac{x^2}{2}$ and $F(x):=\frac{\alpha}{2}x^2$. Then, any domain of the form $]\xi;+\infty[$ with $\xi\in]0;1[$ satisfies the assumptions with $\rho:=\xi^2+\xi>0$ and $\delta:=\left(\frac{1+\xi}{2}\right)^2$.*

The paper is organized as follows. We finish the introduction by introducing the norms that we will use. In a second section, we provide the first approach to the problem. First, we give the material then in Subsection 2.1 we obtain the stability of the studied domains (on $\mathbb{R}^{dN}$) by Υ^N. In Subsection 2.2, we compute the exit-costs of the domains. Finally, main results are stated in Subsection 2.3. Then, we provide the second approach in Section 3.

We precise the norms that we use in this work. On $\mathbb{R}^d$, we use the Euclidean norm.

Definition 1.4. *Let N be a positive integer. On $\mathbb{R}^{dN}$, we use the norm $||\,.\,||_N$ defined by*

$$||\mathcal{X}||_N^{2n}:=\frac{1}{N}\sum_{i=1}^N||X_i||\,|^{2n}\,,$$

with $\mathcal{X}:=(X_1,\cdots,X_N)\in\mathbb{R}^{dN}$. We remind the reader that $2n=\deg(G)$.

Let us observe that this norm has sense as N is large. Indeed, let $(X_i)_{i\geq 1}$ (resp. $(Y_i)_{i\geq 1}$) be a sequence of independent and identically distributed random variables with common law μ_0 (resp. ν_0). By $\mathcal{Y}_\tau$, we denote the vector $\left(Y_{\tau(1)},\cdots,Y_{\tau(n)}\right)$ for any permutation τ. Thus, the quantity

$$\inf_{\tau\in\mathcal{S}_N}||\mathcal{X}-\mathcal{Y}_\tau||_N$$

converges almost surely toward $\mathbb{W}_{2n}\left(\mu_0;\nu_0\right):=\inf\left\{\mathbb{E}\left[||X-Y||^{2n}\right]^{\frac{1}{2n}}\right\}$, the infimum being taken over X which follows μ_0 and Y which follows ν_0.

2 First approach

We now add a last technical assumption on the domain and we will discuss how we can remove it.

Definition 2.1. 1. $\mathbb{B}_\kappa^\infty(\overline{a})$ *denotes the set of all the probability measures* μ *on* $\mathbb{R}^d$ *satisfying* $\int_{\mathbb{R}^d}||x-a||^{2n}\mu(dx)\leq\kappa^{2n}$.
2. *For all the measures* μ, W_μ *is equal to* $V+F*\mu$.
3. *For all* $\nu\in\left(\mathbb{B}_\kappa^\infty(\overline{a})\right)^{\mathbb{R}_+}$ *and for all* $x\in\mathbb{R}^d$, *we also introduce the dynamical system:*

$$\psi_t^\nu(x)=x-\int_0^t\nabla W_{\nu_s}\left(\psi_s^\nu(x)\right)ds\,.$$

Assumption 2.2. *If* κ *is sufficiently small, for any* $\nu\in\left(\mathbb{B}_\kappa^\infty\right)^{\mathbb{R}_+}$, *for any* $x\in\mathcal{D}$, $\psi_t^\nu(x)\in\mathcal{D}$.

Thanks to [Tug12b], we know that if $\mathcal{D}$ satisfies Assumption (A-7) then, there exist two families of open domains $\left(\mathcal{D}_\xi^e\right)_{\xi>0}$ and $\left(\mathcal{D}_\xi^i\right)_{\xi>0}$ satisfying Assumption 2.2 and such that $\mathcal{D}_\xi^i\subset\mathcal{D}\subset\mathcal{D}_\xi^e$ and

$$\lim_{\xi\to0}\sup_{z\in\partial\mathcal{D}_\xi^i}\mathrm{d}\left(z\,;\,\mathcal{D}^c\right)=\lim_{\kappa\to0}\sup_{z\in\partial\mathcal{D}_\xi^e}\mathrm{d}\left(z\,;\,\mathcal{D}\right)=0\,.$$

Consequently, proving the Kramers'type law for a domain satisfying Assumptions (A-6)–(A-11) and Assumption 2.2 is sufficient to obtain it for a domain satisfying Assumptions (A-6)–(A-11).

Now, we give the two domains that we will study on $\mathbb{R}^{dN}$.

$$\mathcal{G}_N^1:=\left(\mathcal{D}\times\left(\mathcal{D}'\right)^{N-1}\right)\bigcap\mathbb{B}_\kappa^N\left(\overline{a}\right)\tag{1}$$

and

$$\mathcal{G}_N:=\mathcal{D}^N\bigcap\mathbb{B}_\kappa^N\left(\overline{a}\right),\tag{2}$$

where

$$\mathbb{B}_\kappa^N\left(\overline{a}\right):=\left\{\mathcal{X}\in\left(\mathbb{R}^d\right)^N\;:\;\frac{1}{N}\sum_{k=1}^N||X_k-a||^{2n}\leq\kappa^{2n}\right\}.$$

Here, $\overline{a}=(a,\cdots,a)$.

2.1 Stability of the domains by $N\Upsilon^N$

In the current work, we deal with the time-homogeneous diffusion $\mathcal{X}^N$,

$$\mathcal{X}_t^N = \overline{x_0} + \sigma\mathcal{B}_t^N - N\int_0^t \nabla\Upsilon^N\left(\mathcal{X}_s^N\right)ds\,.$$

with $\overline{x_0} := (x_0, \cdots, x_0)$ and Υ^N is a potential of $\mathbb{R}^N$.

In [Tug12b], we have used the fact that $\mathbb{B}_\kappa^N(\overline{a})$ is stable by $N\Upsilon^N$. Here, we are not able to prove this but we can circumvent the difficulty.

Proposition 2.3. *We assume Hypotheses (A-1)–(A-11) and Assumption 2.2. Then, the domains $\mathcal{G}_N^1$ and $\mathcal{G}_N$ are stable by the potential $N\Upsilon^N$ providing that κ is sufficiently small.*

Proof. We will prove the proposition only for $\mathcal{G}_N := \mathcal{D}^N \bigcap \mathbb{B}_\kappa^N(\overline{a})$. Set $\mathcal{X}_0 \in \mathcal{G}_N$. We consider the dynamical system

$$\mathcal{X}_t = \mathcal{X}_0 - N\int_0^t \nabla\Upsilon^N\left(\mathcal{X}_s\right)ds\,.$$

Let us assume that there exists $t > 0$ such that $\mathcal{X}_t \notin \mathcal{G}_N$. We consider t_0 the first time that $\mathcal{X}_{t_0} \notin \mathcal{G}_N$. Then, we have

$$\mathcal{X}_{t_0} = \mathcal{X}_0 - N\int_0^{t_0} \nabla\Upsilon^N\left(\mathcal{X}_s\right)ds\,.$$

For any $t \leq t_0$, $\mathcal{X}_t \in \mathbb{B}_\kappa^N(\overline{a})$: we deduce that the empirical measure $\frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ is in $\mathbb{B}_\kappa^\infty(\overline{a})$. According to Assumption 2.2, we deduce that $\mathcal{X}_{t_0} \in \mathcal{D}^N$.

For any $t \leq t_0$, $\mathcal{X}_t \in \mathcal{D}^N$. We deduce that

$$\begin{aligned}\frac{d}{dt}\frac{1}{N}\sum_{i=1}^N\left|X_t^i - a\right|^{2n} &= -\frac{2}{N}\sum_{i=1}^N\left\langle X_t^i - a; \nabla V(X_t^i)\right\rangle\left|X_t^i - a\right|^{2n-2}\\ &\quad - \frac{2}{N}\sum_{i=1}^N\sum_{j=1}^N\left\langle X_t^i - a; \nabla F(X_t^i - X_t^j)\right\rangle\left|X_t^i - a\right|^{2n-2}\\ &\leq -\frac{2\rho}{N}\sum_{i=1}^N\left|X_t^i - a\right|^{2n}\,.\end{aligned}$$

Consequently, $\mathcal{X}_{t_0} \in \mathbb{B}_\kappa^N(\overline{a})$. This is absurd. We deduce that $\mathcal{G}_N$ is stable by the potential $N\Upsilon^N$. □

2.2 Exit-cost of the domains

We now compute the exit-costs of the two domains:

$$H_N^1(\kappa) := \inf_{Z\in\partial\mathcal{G}_n^1}\left(N\Upsilon^N(Z) - N\Upsilon^N(\overline{a})\right)$$

and

$$H_N(\kappa) := \inf_{Z \in \partial \mathcal{G}_n} \left(N\Upsilon^N(Z) - N\Upsilon^N(\overline{a}) \right) .$$

Proposition 2.4. *We assume Hypotheses (A-1)–(A-11) and Assumption 2.2. Then, the following limits hold:*

$$\lim_{N \to \infty} H_N^1(\kappa) = H^1(\kappa) \quad \text{and} \quad \lim_{\kappa \to 0} H^1(\kappa) = H \,,$$

with $H := \inf_{z \in \mathcal{D}} V(z) + F(z-a) - V(a)$*. We also have:*

$$\lim_{N \to \infty} H_N(\kappa) = H(\kappa) \quad \text{and} \quad \lim_{\kappa \to 0} H(\kappa) = H \,.$$

Proof. We will prove the result for $H_N^1(\kappa)$. It has already been proved, see [Tug12b], that

$$\lim_{N \to \infty} \inf_{Z \in \partial \left(\mathcal{D} \times (\mathcal{D}')^{N-1} \right) \bigcap \mathbb{B}_\kappa^N(\overline{a})} \left(N\Upsilon^N(Z) - N\Upsilon^N(\overline{a}) \right) = H^1(\kappa) \,.$$

So, it is sufficient to prove

$$\lim_{N \to \infty} \inf_{Z \in \left(\mathcal{D} \times (\mathcal{D}')^{N-1} \right) \bigcap \partial \mathbb{B}_\kappa^N(\overline{a})} \left(N\Upsilon^N(Z) - N\Upsilon^N(\overline{a}) \right) = \infty \,.$$

It is immediate once we have remarked that for any $\mathcal{X} = (X_1, \cdots, X_N) \in \left(\mathcal{D} \times (\mathcal{D}')^{N-1} \right) \bigcap \partial \mathbb{B}_\kappa^N(\overline{a})$:

$$\begin{aligned}
N\Upsilon^N(X_1, \cdots, X_N) &= \sum_{i=1}^N V(X_i) + \frac{1}{2N} \sum_{i=1}^N \sum_{j=1}^N F(X_i - X_j) \\
&\geq \sum_{i=1}^N V(X_i) \\
&\geq \sum_{i=1}^N \left(V(a) + \frac{\delta}{2} \left|\left| X_i - a \right|\right|^2 \right) \\
&\geq NV(a) + \frac{\delta}{2} \sum_{i=1}^N \left|\left| X_i - a \right|\right|^2 \\
&\geq NV(a) + \frac{N^{\frac{1}{n}} \delta}{2} \left(\sum_{i=1}^N \left|\left| X_i - a \right|\right|^{2n} \right)^{\frac{1}{n}} \\
&\geq NV(a) + \frac{N^{\frac{1}{n}} \delta}{2} \kappa^2 \,.
\end{aligned}$$

This achieves the proof since $\Upsilon^N(\overline{a}) = V(a)$.

□

2.3 Main results

We now are able to obtain the Kramers'type law for the first particle or for any particle.

Theorem 2.5. *We assume Hypotheses (A-1)–(A-11) and Assumption 2.2. By $\tau_{\mathcal{D}}^{1,N}(\sigma)$, we denote the first exit-time of the diffusion $X^{1,N}$ from the domain $\mathcal{D}$. If N is large enough, for any $\delta > 0$, we have the following limit as σ goes to 0 :*

$$\mathbb{P}\left\{e^{\frac{2}{\sigma^2}\left(H_N^1-\delta\right)} \leq \tau_{\mathcal{D}}^{1,N}(\sigma) \leq e^{\frac{2}{\sigma^2}\left(H_N^1+\delta\right)}\right\} \longrightarrow 1\,,$$

where

$$\lim_{N\to\infty} H_N^1 = H\,,$$

with $H := \inf_{z\in\mathcal{D}} V(z) + F(z-a) - V(a)$.

This method relies on the second part of the result of Freidlin-Wentzell theory (the one on the exit-location).

Proof. We do not give the detailed proof since it is similar to the ones in [Tug12b]. Indeed, Proposition 2.3 and Proposition 2.4 imply that we have a Kramers'type law for the first exit-time from the domain $\mathcal{G}_n^1$. Now, we have proved that the exit-cost from the ball of center $\overline{a}$ and radius κ was larger than the one from $\mathcal{G}_n^1$ by taking N sufficiently large. In the same way, we can prove that the exit cost of the particles 2 to N from the domain $\mathcal{D}'$ is larger than the one from $\mathcal{G}_n^1$. Consequently, with a probability close to 1 as σ goes to 0, we have that $X_{\tau(N,\sigma)}^{1,N} \in \partial\mathcal{D}$, $\tau(N,\sigma)$ being the first exit-time of $\mathcal{X}^N$ from $\mathcal{G}_n^1$.

Finally, the exit-cost $H_N^1(\kappa)$ does not depend on κ if N is large enough. And, the previous paragraph applies for any $\kappa > 0$. □

We have a similar result for the exit of any particle:

Theorem 2.6. *We assume Hypotheses (A-1)–(A-11) and Assumption 2.2. By $\tau_{\mathcal{D}}^{N}(\sigma)$, we denote the first exit-time of the diffusion $\mathcal{X}$ from the domain $\mathcal{D}^N$. If N is large enough, for any $\delta > 0$, we have the following limit as σ goes to 0 :*

$$\mathbb{P}\left\{e^{\frac{2}{\sigma^2}(H_N-\delta)} \leq \tau_{\mathcal{D}}^{N}(\sigma) \leq e^{\frac{2}{\sigma^2}(H_N+\delta)}\right\} \longrightarrow 1\,,$$

where

$$\lim_{N\to\infty} H_N = H\,.$$

Remark 2.7. *Finally, we can remark that the convexity of F is not a necessary condition. Indeed, if $F(x) = F_0(x) - \frac{\alpha}{2}|x|^2$ with F_0 convex and $\alpha > 0$, it is sufficient to assume that $\min\{\rho;\delta\} > \alpha$.*

3 Second approach

To have this second approach, we add some assumptions (different from the additional assumptions in previous section).

Assumption 3.1. *We assume that* $x_0 = a$ *and that* $\deg G = 2$*, that is to say the equation that we are looking at is*

$$X_t^{i,N} = a + \sigma B_t^i - \int_0^t \nabla V\left(X_s^{i,N}\right) ds - \alpha \int_0^t \left(X_s^{i,N} - \frac{1}{N}\sum_{j=1}^N X_s^{j,N} \right) ds\,,$$

with $\alpha > 0$.

We also need the following technical hypothesis.

Assumption 3.2. *By* θ*, we denote* $\sup_{\mathbb{R}} -V'' > 0$*. Then,* $V''(a) > \frac{1}{3}\theta$.

Let us remark that an example of confining potential which satisfies this assumption is $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$. Indeed, here, $a = 1$ and $V''(a) = 2$ and $\theta = 1$.

This approach is based on the work [Tug17].

Let us remark that by taking $x_0 = a$, we immediately have $\overline{x_0} \in \mathbb{B}_\kappa^N(\overline{a})$.

We will show that the exit from the domain $\mathbb{B}_\kappa^N(\overline{a})$ does not occur (with high probability) before the time $\exp\left(\frac{2H}{\sigma^2}\right)$ if N is large then σ is small, providing that α is large enough.

Lemma 3.3. *The exit cost of the domain* $\mathbb{B}_\kappa^N$ *goes to infinity when* N *goes to infinity if* α *is large enough:*

$$\lim_{N\to+\infty} \inf_{Z\in\partial\mathbb{B}_\kappa^N} N\Upsilon^N(Z) - N\Upsilon^N(\overline{a}) = +\infty\,.$$

Proof. We have

$$\begin{aligned} N\Upsilon^N(Z) &= \sum_{i=1}^N V\left(X_i\right) + \frac{\alpha}{4N}\sum_{i=1}^N\sum_{j=1}^N \left(X_i - X_j\right)^2 \\ &= \sum_{i=1}^N \left(V\left(X_i\right) + \frac{\alpha}{2}\left(X_i - a\right)^2 - \frac{\alpha}{2}\left(X_i - a\right) m^N \right)\,, \end{aligned}$$

where $m^N := \frac{1}{N}\sum_{j=1}^N \left(X_j - a\right)$. We here assume $(X_1, \cdots, X_N) \in \partial\mathbb{B}_\kappa^N$ so that $m^N = \delta\kappa$ with $\delta \in [-1; 1]$. By taking α large enough (typically, we take $\alpha > \theta := \sup_{\mathbb{R}} -V''$), the function $x \mapsto W_\kappa(x) := V(x) + \frac{\alpha}{2}(x-a)^2 - \frac{\alpha}{2}(x-a)m^N$ is convex so it has a unique global minimum which is located in a point a_κ which satisfies

$$a_\kappa = a + \frac{\alpha}{2(V''(a)+\alpha)}\delta\kappa + o(\kappa)\,.$$

We have

$$W_\kappa(x) - W_\kappa\left(a_\kappa\right) \geq \frac{\alpha - \theta}{2}\left(x - a_\kappa\right)^2\,.$$

However, we can compute $W_\kappa\left(a_\kappa\right)$ like so:

$$W_\kappa\left(a_\kappa\right) = V(a) - \frac{\alpha^2}{8(V''(a)+\alpha)}\delta^2\kappa^2 + o(\kappa^2)\,.$$

Let $Z := (X_1, \cdots, X_N)$ be in $\partial\mathbb{B}_\kappa^N$. We have

$$\begin{aligned}
&\Upsilon^N(Z)\\
&\geq W_\kappa\left(a_\kappa\right) + \frac{\alpha-\theta}{2}\left[\kappa^2 - 2\frac{1}{N}\sum_{i=1}^N (X_i - a)(a_\kappa - a) + (a_\kappa - a)^2\right]\\
&\geq V(a) - \frac{\alpha^2}{8(V''(a)+\alpha)}\delta^2\kappa^2 + \frac{\alpha-\theta}{2}\kappa^2\left[1 - \alpha\frac{\delta^2(4V''(a)+3\alpha)}{4(V''(a)+\alpha)^2}\right] + o(\kappa^2)\\
&\geq V(a) - \frac{\alpha^2}{8(V''(a)+\alpha)}\kappa^2 + \frac{\alpha-\theta}{2}\kappa^2\left[1 - \alpha\frac{4V''(a)+3\alpha}{4(V''(a)+\alpha)^2}\right] + o(\kappa^2)\\
&\geq V(a) + \frac{(3V''(a)-\theta)\alpha^2 + 4V''(a)(V''(a)-\theta)\alpha - 4V''(a)^2\theta}{8(V''(a)+\alpha)^2}\kappa^2 + o(\kappa^2)\,.
\end{aligned}$$

As a consequence, thanks to Assumption 3.2, if α is sufficiently large then κ small enough, we have $\Upsilon^N(Z) - V(a) > 0$. Consequently, the quantity $N\Upsilon^N(Z) - N\Upsilon^N(\overline{a})$ goes to infinity as N tends to infinity. □

From this lemma, we readily obtain the following corollary.

Corollary 3.4. *Let H be an arbitrarily large positive constant. Then, for N sufficiently large, we have the limit*

$$\lim_{\sigma\to 0}\mathbb{P}\left\{\tau^N(\sigma) \leq \exp\left[\frac{2H}{\sigma^2}\right]\right\} = 0\,,$$

with $\tau^N(\sigma) := \inf\left\{t \geq 0\ :\ \left(X_t^1, \cdots, X_t^N\right) \notin \mathbb{B}_\kappa^N\right\}$.

From Corollary 3.4, we can apply the method (based on coupling result) used in [Tug17] for the McKean-Vlasov diffusion.

Definition 3.5. *We consider the diffusion Y defined by*

$$Y_t = a + \sigma W_t^1 - \int_0^t \nabla V\left(Y_s\right)ds - \alpha\int_0^t \left(Y_s - a\right)ds\,.$$

Lemma 3.6. *For any $\xi > 0$, under Assumptions 3.1–3.2 plus if α is large enough, we have:*

$$\mathbb{P}\left\{\sup_{t\in[0;\tau^N(\sigma)]}\left|\left|X_t^1 - Y_t\right|\right| \geq \xi\right\} = 0$$

if κ and σ are small enough.

Proof. Differential calculus provides

$$d\left|\left|X_t^1 - Y_t\right|\right|^2 = -2\left\langle X_t^1 - Y_t ; \nabla W_{\mu_t^N}(X_t^1) - \nabla W_{\delta_a}(Y_t)\right\rangle dt\,,$$

where $W_\mu(x) := V(x) + F * \mu(x)$ and $\mu_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$.

For any $0 \leq t \leq \tau^N(\sigma)$, we have:

$$\begin{aligned} d\left|\left|X_t^1 - Y_t\right|\right|^2 =& -2\left\langle X_t^1 - Y_t ; \nabla W_{\delta_a}(X_t^1) - \nabla W_{\delta_a}(Y_t)\right\rangle dt \\ & -2\left\langle X_t^1 - Y_t ; \nabla W_{\mu_t^N}(X_t^1) - \nabla W_{\delta_a}(X_t^1)\right\rangle dt \end{aligned}$$

The first term can be bounded like so:

$$-2\left\langle X_t^1 - Y_t ; \nabla W_{\delta_a}(X_t^1) - \nabla W_{\delta_a}(Y_t)\right\rangle \leq -2\left(\alpha - \theta\right)\left|\left|X_t^1 - Y_t\right|\right|^2\,,$$

if $\alpha > \theta$. We now bound the second term in the following way.

$$\begin{aligned} & -2\left\langle X_t^1 - Y_t ; \nabla W_{\mu_t^N}(X_t^1) - \nabla W_{\delta_a}(X_t^1)\right\rangle \\ & \leq 2\left|\left|X_t^1 - Y_t\right|\right| \times \left|\left|\nabla W_{\mu_t^N}(X_t^1) - \nabla W_{\delta_a}(X_t^1)\right|\right| \\ & \leq 2\left|\left|X_t^1 - Y_t\right|\right| \times \left|\left|\left(\nabla V(X_t^1) + \alpha\left(X_t^1 - \frac{1}{N}\sum_{j=1}^N X_t^j\right)\right) - \left(\nabla V(X_t^1) + \alpha(X_t^1 - a)\right)\right|\right| \\ & \leq 2\alpha\left|\left|X_t^1 - Y_t\right|\right| \times \left|\left|\frac{1}{N}\sum_{j=1}^N X_t^j - a\right|\right| \\ & \leq 2\alpha\left|\left|X_t^1 - Y_t\right|\right| \times \sqrt{\frac{1}{N}\sum_{j=1}^N \left|\left|X_t^j - a\right|\right|^2}\,. \end{aligned}$$

However, for any $t \leq \tau^N(\sigma)$, $\frac{1}{N}\sum_{j=1}^N \left|\left|X_t^j - a\right|\right|^2 \leq \kappa^2$. Thus, we deduce the inequality

$$\frac{d}{dt}\left|\left|X_t^1 - Y_t\right|\right|^2 \leq -2(\alpha - \theta)\left|\left|X_t^1 - Y_t\right|\right|^2 + 2\alpha\kappa\left|\left|X_t^1 - Y_t\right|\right|\,.$$

However, $X_0^1 = Y_0$. Thus, for any $t \in \left[0; \tau^N(\sigma)\right]$, we have:

$$\left|\left|X_t^1 - Y_t\right|\right| \leq \frac{\alpha}{\alpha - \theta}\kappa\,.$$

Taking $\kappa < \frac{\alpha - \theta}{\alpha}\xi$ yields the result. □

By $\tau'(\sigma)$, we denote the first exit-time of diffusion Y from a domain $\mathcal{D}'$ such that: $\mathcal{D}'$ is stable by the vector field $x \mapsto -\nabla V(x) - \nabla F(x - a)$, $\mathcal{D}' \subset \mathcal{D}$, dist $(\mathcal{D}'; \mathcal{D}^c) =: \xi > 0$ and the exit cost of diffusion Y from $\mathcal{D}'$ is larger than $H - \frac{\delta}{2}$.

The existence of such a domain is a straightforward exercice so it is left to the reader. We have:

$$\begin{aligned}\mathbb{P}\left(\tau_{\mathcal{D}}(\sigma)\leq\exp\left[\frac{2(H-\delta)}{\sigma^2}\right]\right)\leq&\mathbb{P}\left(\tau'(\sigma)\leq\exp\left[\frac{2(H-\delta)}{\sigma^2}\right]\right)\\&+\mathbb{P}\left(\tau^N(\sigma)\leq\exp\left[\frac{2(H-\delta)}{\sigma^2}\right]\right)\\&+\mathbb{P}\left(\sup_{[0;\tau^N(\sigma)]}\left|\left|X_t^1-Y_t\right|\right|\geq\xi\right)\,.\end{aligned}$$

By taking κ and σ small enough, the third term is equal to 0. Then, we observe that the first term goes to 0 since the exit cost of Y from domain $\mathcal{D}'$ is larger than $H-\frac{\delta}{2}$. Finally, by taking N large enough, the second term goes to 0 as σ goes to 0.

We deduce

$$\lim_{\sigma\to 0}\mathbb{P}\left(\tau_{\mathcal{D}}(\sigma)\leq\exp\left[\frac{2(H-\delta)}{\sigma^2}\right]\right)=0\,.$$

The upper-bound is obtained similarly so it is left to the reader.

We thus have the following theorem.

Theorem 3.7. *Under the assumptions, if α is sufficiently large then if N is large enough, we have the limit*

$$\lim_{\sigma\to 0}\mathbb{P}\left(\exp\left[\frac{2(H-\delta)}{\sigma^2}\right]\leq\tau_{\mathcal{D}}(\sigma)\leq\exp\left[\frac{2(H+\delta)}{\sigma^2}\right]\right)=1\,.$$

References

[BRTV98] S. Benachour, B. Roynette, D. Talay, and P. Vallois. Nonlinear self-stabilizing processes. I. Existence, invariant probability, propagation of chaos. *Stochastic Process. Appl.*, 75(2):173–201, 1998.

[CGM08] P. Cattiaux, A. Guillin, and F. Malrieu. Probabilistic approach for granular media equations in the non-uniformly convex case. *Probab. Theory Related Fields*, 140(1-2):19–40, 2008.

[DZ98] A. Dembo and O. Zeitouni: *Large deviations techniques and applications*, volume 38 of *Stochastic Modelling and Applied Probability.* Springer-Verlag, Berlin, 2010. Corrected reprint of the second (1998) edition.

[FW98] M. I. Freidlin and A. D. Wentzell: *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, second edition, 1998. Translated from the 1979 Russian original by Joseph Szücs.

[Mél96] Sylvie Méléard. Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models. In *Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995)*, volume 1627 of *Lecture Notes in Math.*, pages 42–95. Springer, Berlin, 1996.

[HIP08] Samuel Herrmann, Peter Imkeller, and Dierk Peithmann. Large deviations and a Kramers' type law for self-stabilizing diffusions. *Ann. Appl. Probab.*, 18(4):1379–1423, 2008.

[HT16] S. Herrmann and J. Tugaut. Mean-field limit versus small-noise limit for some interacting particle systems. To appear in Communications on Stochastic Analysis

[Szn91] Alain-Sol Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.

[Tug12b] J. Tugaut. Exit problem of McKean-Vlasov diffusions in convex landscapes. *Electronic Journal of Probability*, Vol. 17 (2012), no. 76, 1–26.

[Tug17] J. Tugaut Exit problem of McKean-Vlasov diffusion in double-wells landscape To appear in *Journal of Theoretical Probability*