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Do Markets Prove Pessimists Right?[1]

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Abstract

We study how ambiguity and ambiguity attitudes affect asset prices when consumers form their expectations based on past observations. In an OLG economy with risk-neutral yet ambiguity sensitive consumers, we describe limiting asset prices depending on the proportion of investor types. We then study the evolution of consumer type shares. With long memory, the market does not select for ambiguity-neutrality. Whenever perceived ambiguity is sufficiently small, but positive, only pessimists survive and determine prices in the limit. With one-period memory, equilibrium prices are determined by Bayesians. Yet, the average price of the risky asset is lower than its fundamental value.

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1 Introduction

There is a wide-spread view that optimism and pessimism may cause excessive pro-cyclical buying or selling in financial markets. In consequence asset prices may substantially and persistently deviate from their fundamental values. In a recent article on the financial crisis, Shefrin and Statman (2012) quote Keynes on the psychology of financial booms and crises: “The later stages of the boom are characterized by optimistic expectations as to the future yield of capital goods... of speculators who are more concerned with forecasting the next shift of market sentiment than with a reasonable estimate of the future yield of capital assets, that when disillusion falls upon an over-optimistic and over-bought market, it should fall with sudden and even catastrophic force. Moreover, the dismay and uncertainty as to the future which accompanies a collapse in the marginal efficiency of capital naturally precipitates a sharp increase in liquidity preference... it is not so easy to revive the marginal efficiency of capital, determined as it is by the uncontrollable and disobedient psychology of the business world. It is the return of confidence, to speak in ordinary language, which is so insusceptible to control in an economy of individualistic capitalism.” Keynes (1936, pp. 315-317).

In this view, investors’ trades in asset markets rely on forecasts not only about the unknown real returns, but also about prices which are endogenously generated by demand and supply. Investors’ preferences and beliefs determine the equilibrium prices which in turn feed back into the updates of these beliefs. Depending on whether investors hold identical or heterogeneous subjective beliefs, this feedback process may lead to up- or downward biased price predictions and, in consequence, to converging or cyclical price processes. Most of the literature (see Section 1.2) on the dynamics of financial markets assumes that investors are subjective expected utility (SEU) maximizers who update their beliefs according to Bayes’ law. Combined with the assumption of rational expectations, embodied in an equilibrium price functional, pro-cyclical price movements enter these models via trading frictions and constraints. The "uncontrollable and disobedient psychology of the business world" is controlled by a rational expectations equilibrium price function.

In Eichberger and Guerdjikova (2013), we provide an α-Max-Min Expected Utility (α-MEU) representation of preferences and beliefs, when information is in the form of data sets as in the case-based decision theory initiated by Gilboa and Schmeidler (2001). The beliefs are given by a set of probability distributions. They depend on the frequency of observed cases, but also on the degree of perceived ambiguity which itself is a function of the type and number of observations. Attitudes towards uncertainty make investors bias their decision-relevant beliefs, upwards for optimists and downwards for pessimists, whenever beliefs are ambiguous. If an investor feels no ambiguity, either as an SEU-maximizer or because ambiguity vanishes, possibly with large amounts of consistent data, then the individual's attitude becomes irrelevant. Thus, over and under weighting of predictions will depend both on attitudes and on the endogenously generated price data. We feel that this type of preferences and beliefs can capture some of the "uncontrollable and disobedient psychology of the business world" as mentioned by Keynes.

The dynamics of asset prices in a model with case-based investors facing ambiguity will depend on the distribution of ambiguity attitudes in the population. In turn, as the quote from Keynes suggests, the general level of optimism and pessimism in the population might itself be driven by the market. Results in social psychology show that optimism and pessimism may be intergenerationally transmitted. Zuckerman (2001, p. 184) finds that these attitudes may be adaptive traits "selected in evolution for our species" characterized by "a low [...] but significant heritability. [...] Optimism is also influenced by shared familial factors and non-shared life events, but pessimism seems to be primarily learned by events outside of the shared family environment."

Evidence also suggests that macroeconomic events may have an impact on investors' ambiguity attitudes and, thus, on their market behavior. Malmendier and Nagel (2011) find that individuals who have experienced low stock market returns during the Great depression show lower willingness to take financial risk, are less likely to participate in the stock market, invest a lower fraction of their liquid assets in stocks, and are more pessimistic about future stock returns. Such experience need not be "personal" – individual preferences may adjust in response to the observed performance of others.

In this paper, we propose a model of financial markets where (i) investors form ambiguous beliefs about future asset prices and dividends based on endogenously generated financial data, (ii) the population of investors consists of optimists, pessimists and Bayesian SEU maximizers, and (iii) the shares of these three types of investors evolve depending on their market performance. Although data about prices are generated endogenously, we treat the amount of data, i.e., the length of memory, on which investors base their predictions as an exogenous variable. We focus on two cases: short memory, containing the μ most recent observations, and long memory, containing all past observations. We also treat as exogenous the function relating the degree of ambiguity to the amount of available data. We discuss the case where ambiguity vanishes, i.e. the set of probability distributions shrinks to a singleton, as the amount of data grows and the case where ambiguity is persistent, i.e., the set of probability distributions does not converge to a singleton, no matter how much evidence is available.

1.1 Framework and results

We consider an OLG model with a risky asset and a riskless bond. Investors' preferences and beliefs are described by the case-based α-MEU representation developed in Eichberger and Guerdjikova (2013). Investors live for two periods. They observe data from an exogenous i.i.d. dividend process and endogenously determined past asset prices. Although each generation plans for just two periods, investors have access to data sets containing observations of previous generations. Based on these observations they form expectations about asset returns. Ambiguity arises because investors perceive uncertainty about the precision of their predictions depending on quantity and quality of data. We distinguish three types of investors as to their attitude towards ambiguity: optimists, pessimists, and ambiguity-neutral Bayesians.

With no ambiguity and infinitely many observations, predicted asset prices converge towards their fundamental values and in the limit investors hold rational expectations. In general, however, predictions of future returns depend on the investors' (limited) memory, on their perceived ambiguity, and on their attitude towards this ambiguity. Hence, asset prices reflect these characteristics, as well as the shares of investor types in the market. For fixed shares of types in the population, if memory includes all past cases, consumers learn the dividend process, and prices converge. However, if ambiguity is persistent, the limit price exceeds (falls below) the fundamental value for a high share of optimists (pessimists). Market prices are thus biased by the ambiguity attitudes in the population.

When the memory is short, investors cannot learn the dividend process and the asset price does not converge. The price dynamics can be described by an irreducible recurrent Markov process. The support of the invariant distribution of this process depends on the shares of types in the economy: it is shifted up (down) if optimists (pessimists) dominate the market. In general, however, short memory produces cycles of optimism, Bayesianism and pessimism, each of the regimes occurring with strictly positive probability.

To capture the idea that attitudes towards ambiguity in the population may evolve, as suggested by Zuckerman (2001) and Malmendier and Nagel (2011), we next assume (in the spirit of the indirect evolutionary approach[4] initiated by Güth and Yaari (1992)) that the proportions of investor types adjust to imitate the more successful types in previous generations. We capture this process by a replicator dynamics[5]. Equilibrium prices and population shares are now determined by the learning dynamics for a given memory length and by the replicator dynamics. When the price of the asset is constant and equals its fundamental value, the replicator dynamics favors the more cautious pessimistic investors. Hence, the state in which the asset prices are set by Bayesian investors with rational expectations and coincide with the fundamental values is not stable. With infinite memory, only pessimists survive in the unique stable steady state of the economy. Furthermore, with small but persistent ambiguity, the economy converges to this pessimistic steady state a.s. and in expectations.

In an economy with one-period memory and without Bayesian investors, cycles emerge: a sequence of high (low) dividend realizations leads to an "optimistic (pessimistic) market", in which the share of optimists (pessimists) is relatively high and the equilibrium price equals the optimists' (pessimists') reservation price.

With all three types of consumers and one-period memory no cycles occur. Almost surely, after a finite number of periods, the economy reaches a state, in which the equilibrium price in each period is set by the Bayesian investors. However, the average price lies below the fundamental value. Moreover, although the equilibrium price is determined by a "Bayesian" regime, optimists or pessimists need not disappear.

1.2 Related literature

The model presented in this paper combines elements of (i) temporary equilibrium theory with overlapping generations (OLG), (ii) case-based decision theory for agents learning from data with ambiguous beliefs, and (iii) the theory of market selection. The discussion of the literature focuses on these three aspects.

1. Rational expectations in stochastic dynamic general equilibrium and temporary equilibrium

The literature on dynamic economies with asset markets comprises two approaches which model price expectations in different ways. The temporary equilibrium approach (with or without OLG) assumes that expectations

[4] In contrast to a large literature, which will be discussed in more detail in Section 1.2, in our model evolutionary pressure does not select for "investment strategies", or beliefs, but for a feature of the preferences, specifically the attitude towards ambiguity.

[5] Hofbauer and Schlag (2000) show that the replicator dynamics can be interpreted as an imitation process.

about future prices and endowments are based on information about past prices which are generated sequentially by demand and supply. Rational expectations thus do not hold in every temporary equilibrium, but may be learned in the long run, if the environment is sufficiently stationary[6], see Bray (1982), Bray and Kreps (1987), Blume and Easley (1982), Grandmont (1998).

The approach taken by the stochastic intertemporal general equilibrium theory studies investors who plan for the full horizon of the economy. It assumes a stochastic process of endowments and investors who learn about the true parameters from observations of realizations of the endowment process and from endogenous rationally predicted asset prices which are formed according to an equilibrium pricing function. Rational expectations regarding asset prices are guaranteed by the endogenous equilibrium pricing function. Learning concerns only the stochastic process of endowments. In the tradition of Radner (1982), these models are at the core of the computable general equilibrium literature, see e.g., Marcet and Sargent (1988), Marcet and Sargent (1989), Branger, Schlag, and Wu (2015), and Chien, Cole, and Lustig (2015).

For economies with ambiguity-aversion, Condie and Ganguli (2011) show that equilibrium prices are only partially revealing for a generic set of economies. This result raises questions about the equilibrium pricing function approach in a model where decision makers face ambiguity.

In this paper, we use a Lucas-tree model common in macroeconomics and finance, see Ljungqvist and Sargent (2004). Belief formation and learning in this model have been studied both with OLG and with infinitely-lived consumers, Marcet and Sargent (1988), Marcet and Sargent (1989), Branger, Schlag, and Wu (2015).

We chose the OLG model for several reasons. First, it is not clear how to define an equilibrium with adaptive expectations in a model with infinitely-lived case-based decision makers facing ambiguity, cf. Adam and Marcet (2011). Second, the notion of ambiguity, which is usually associated with deficient data and bounded rationality, seems to be at odds with rational predictions at any point in time. Finally, in combination with the replicator dynamics over types, the OLG-model seems also better suited to incorporate the results from social psychology on the intergenerational transmission of optimism and pessimism — it captures both the hereditary mechanism and the social learning aspect of this dynamics. Hence, we follow the temporary equilibrium

[6] In general, convergence towards rational expectations fails, even in representative agent economies, see Evans and Honkapohja (2001).

approach in which the process of learning may converge to an equilibrium under rational expectations when data is abundant and the environment is sufficiently stationary.

2. Ambiguity and learning

A number of studies have shown that ambiguity aversion can explain stylized facts such as the home bias (Uppal and Wang (2003)), the equity premium puzzle (Epstein and Schneider (2007), Collard, Mukerji, Sheppard, and Tallon (2011), Zimper (2012)), or negative correlation between asset prices and returns (Ju and Miao (2012)). These models typically consider a representative investor with rational expectations.

We deviate from this literature by using the case-based decision approach of Gilboa and Schmeidler (2001) to focus on belief formation and learning. The representation developed in Eichberger and Guerdjikova (2010) and Eichberger and Guerdjikova (2013) allows us to study agents with heterogeneous attitudes towards ambiguity arising from insufficient data[7]. While for a Bayesian investor a signal which increases the probability of an event, by necessity decreases the probability of its complement, for an investor with non-additive beliefs, information has an additional value: it reduces ambiguity. For an optimist (pessimist), the additional value of information is positive (negative). Hence, the beliefs and the optimal portfolio choices of optimists / pessimists in our model cannot be mimicked by a Bayesian, Eichberger and Guerdjikova (2013, p. 1449).

A small literature studies learning under ambiguity asking whether with sufficient data ambiguity will vanish and whether the limit beliefs form a rational prediction. Contrary to intuition, such learning processes need not always converge, even when the draws are i.i.d. For an i.i.d. process with a finite set of parameter values, Marinacci (2002) identifies conditions for full Bayesian updating on a set of priors to converge to the truth. In contrast, Marinacci (1999) considers a class of capacities, which he associates with an i.i.d. process, and shows that the limit capacity corresponds to a non-singleton set of priors and ambiguity is persistent.

Epstein and Schneider (2007) examine Bayesian updating on a set of priors in combination with an α-expected maximum likelihood rule and show that this process converges to the true probability when the process is i.i.d.. Zimper and Ma (forthcoming) reexamine the results of Epstein and Schneider (2007), but using an α-log-maximum-likelihood updating rule and show that in this case convergence towards a single prior fails. Ambiguity, thus need not vanish even for an i.i.d. process. For non-i.i.d. processes, Epstein and Schneider

[7] This approach has been applied to technological adaptation in Eichberger and Guerdjikova (2012).

(2007), p. 1276, note that "ambiguity need not vanish in the long run".

Experimental evidence by Nicholls, Romm, and Zimper (2015) shows that statistical information about i.i.d. draws from an urn does not reduce the number of violations of the Sure-Thing Principle, although subjects' probability estimates converge to the truth. Convergence of beliefs and persistent ambiguity can thus coexist.

In Eichberger and Guerdjikova (2013), we address these issues in an axiomatic context identifying persistent and vanishing ambiguity from the decision maker's preferences. The case of persistent ambiguity is of particular interest in this model, since a correct prediction of prices requires learning not only the dividend and the price process but also the evolution of types. As in the changing urn example studied in Epstein and Schneider (2007), the evolution of investor types cannot be learned from price and dividend data. Investors thus might perceive ambiguity even in the limit, after having learned the dividend and price distribution.

3. Market selection

There is a large literature dealing with selection in markets, which may operate on investment strategies, beliefs (Bayesian vs. biased non-Bayesian beliefs) or preferences (discount factors, risk aversion, ambiguity attitudes). Results in this literature concern the limiting distributions of strategies or types.

Two major strands have developed in this literature: the first one, inititated by Blume and Easley (1992) and Hens and Shenk-Hoppé (2001) considers the evolution of exogenously given portfolio rules, see also Evstigneev, Hens, and Schenk-Hoppe (2002), Evstigneev, Hens, and Schenk-Hoppé (2006), Evstigneev, Hens, and Schenk-Hoppé (2008), Amir, Evstigneev, and Schenk-Hoppe (2011), Bottazi, Dindo, and Giachini (2015). A main result of this literature is the discovery of the Kelly rule, a globally stable investment rule which maximizes the log-expected utility function with correct beliefs. The second strand of the literature initiated by Blume and Easley (2006) and Sandroni (2000) studies the evolution of long-lived optimizing investors with different beliefs and preferences. In bounded economies with complete markets populated by SEU investors, only investors with correct beliefs survive. Risk preferences are irrelevant for survival.

A characteristic feature of both strands of the literature is that uncertainty concerns only the exogenous endowment process and is represented by a filtration. Investors' portfolio choices have to be measurable with respect to this filtration. Hence, this framework does not allow for adaptive learning about prices as in our paper. Bottazzi and Dindo (2014) and Brock et al. (2001) in contrast model selection in markets with adaptive learning.

They study the "deterministic skeleton" of the system, replacing the dividend process with the expectation of the dividends and show that market outcomes converge to the equilibrium under rational expectations with correct beliefs. In contrast, we study the stochastic process per se and show that such a deterministic approximation is not appropriate, when aggregate wealth is stochastic and the evolutionary dynamics is non-linear.

The assumptions of Blume and Easley (2006) have been relaxed to show that risk preferences matter for survival in unbounded economies, (Rader (1981), Kogan, Ross, Wang, and Westerfield (2006), Kogan, Ross, Wang, and Westerfield (2011) and Yan (2008)) and that with incomplete markets[8], correct beliefs are neither necessary, nor sufficient for survival, (Coury and Sciubba (2012) and Beker and Chattopadhyay (2010)). In our model, endowments are bounded, but markets are incomplete. However, our main result on survival, Proposition 15, will also hold true for a dividend process with only two realizations (rather than a continuum), i.e., for a complete market.

A small literature studies survival of investors with incorrect beliefs in OLG-models, e.g., Long, Shleifer, Summers, and Waldmann (1990), Long, Shleifer, Summers, and Waldmann (1991), Palomino (1996), and Wang (2001). In these models traders choosing riskier strategies may dominate the market.

While most of the literature works directly with the endogenous wealth dynamics, Brock, Hommes, and Wagener (2001), Alós-Ferrer and Ania (2005) and Wang (2001) introduce an exogenous dynamics over investor types. Our approach is most similar to Wang (2001) in that the replicator dynamics parallels the evolution of wealth in an economy with infinitely-lived agents with identical, but exogenous saving rates, see Remark 12.

Few papers investigate preferences other than SEU. Often, asset demand of investors with non-SEU preferences mimicks that of SEU-investors with wrong beliefs and such investors disappear. E.g., Condie (2008) and Silva (2011) show that agents with time-separable ambiguity-averse preferences can survive only if their behavior mimicks that of SEU-maximizers with correct beliefs. Without time-separability, smooth ambiguity-averse agents may engage in precautionary savings and survive with effectively wrong beliefs while driving SEU with correct beliefs out of the market, Guerdjikova and Sciubba (2015). In our OLG-model, the saving motive is absent. Nevertheless, the replicator dynamic need not select for ambiguity-neutral preferences.

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[8] Even when markets are complete, differential access to assets, (Chien, Cole, and Lustig (2015) and Guerdjikova and Quiggin (2017)) may allow agents with incorrect beliefs to survive.

Borovicka (2014) and Dindo (2016) examine survival in the context of Epstein and Zin (1989) preferences and show that time-nonseparability allows agents with wrong beliefs to survive. However, they do not discuss whether Epstein-Zin preferences can survive in the presence of SEU investors with correct beliefs. Finally, Easley and Yang (2014) analyze the survival of loss-averse decision makers and show that these agents disappear in the presence of investors with Epstein-Zin preferences who do not exhibit loss-aversion. Hence, loss-aversion cannot have a long-lasting effect on prices.

1.3 Organization of the paper

In section 2, we describe the economy, explain the process of belief formation in the light of the available data and define a temporary equilibrium for this economy. Section 3 analyzes the economy with fixed type shares for the two cases of finite and infinite memory, describes the limit distribution of prices and provide some comparative statics results. Section 4 studies the evolution of investor types with infinite and with one-period memory. Section 5 concludes. All proofs are collected in the Appendix.

2 The Model

2.1 The Economy

Consider an OLG model where in each period a mass 1 of young consumers is born. Consumers live for two periods and receive y units of the consumption good in the first period of their life and no endowment in the second period. Young consumers can use their endowment y to buy a portfolio of assets. The assets' payoffs consist of dividends and capital gains. Since consumers' endowments are 0 in the second period of their life, the returns on their portfolio determine their old-age consumption.

There are two assets (Lucas trees) in the economy: a bond in exogenous supply B which pays in each period t a riskless dividend r per unit and has a price of p_t and a risky asset in supply A which pays a random dividend $\delta \sim^{i.i.d.} \rho$ and has a price q_t. The distribution ρ is continuous with support $\left[\underline{\delta}; \bar{\delta}\right]$. Both assets can be traded in every period. Their returns can be written as $\frac{p_{t+1}+r}{p_t}$ for the bond and $\frac{q_{t+1}+\delta_{t+1}}{q_t}$ for the risky asset.

In order to focus on the portfolio choice, we will assume that consumers are interested exclusively in second-period consumption c_{t+1}. Hence, young consumers at time t will spend all their income on a portfolio $(a_t; b_t)$, whereas old consumers will consume the asset returns and capital gains of the portfolios bought when young.

Short-sales are prohibited[9], hence, $(a_t; b_t) \in \mathbb{R}_0^{2+}$. With these assumptions on preferences, Walras' law implies

$$q_t A + p_t B = y \tag{1}$$

for all asset price systems (q_t, p_t) in this economy. In consequence, there is only one free asset price, q_t, while the bond price p_t is given by $p_t = h(q_t) := \frac{1}{B}(y - q_t A)$.

2.2 Consumers' beliefs, data and learning

The bond rate of return r is assumed to be certain and publicly known. However, investors know neither the probability distribution ρ of the exogenous dividend process nor the future asset prices $(p_{t+1}; q_{t+1})$.

Consumers form beliefs about future asset prices and dividends based on past observations. In particular, in period t, a case consisting of the dividend realization δ_t of the risky asset and the equilibrium asset prices $(p_t^*; q_t^*)$ is observed. The publicly available (hard) information at time t is given by a data set P_t, which contains the cases observed in the last μ periods: $P_t = \left\{ \left(p_{t-1}^*; q_{t-1}^*; \delta_{t-1}\right) ... \left(p_{t-\mu}^*; q_{t-\mu}^*; \delta_{t-\mu}\right) \right\}$. The parameter μ describes the amount of hard information available in the economy.

Since the information contained in the data set P_t might not be sufficient to make an informed prediction about future prices and dividends, the consumers also form (subjective) perceptions about the lower and upper bounds of prices and dividends. These are given by $\underline{p}$ and $\overline{p}$ for the bond, $\underline{q}$ and $\overline{q}$ for the risky asset and $\underline{\delta}$ and $\bar{\delta}$ for the dividends[10]. Denote by $\mathcal{P} := \left\{\underline{\delta}; \overline{\delta}\right\} \times \left\{\underline{p}; \overline{p}\right\} \times \left\{\underline{q}; \overline{q}\right\}$ the set of all extreme cases.

Let $M\left(P_t \cup \mathcal{P}\right)$ be the set of distinct cases in $P_t \cup \mathcal{P}$ and m denote their number, $m = \left|M\left(P_t \cup \mathcal{P}\right)\right|$. Given the available hard information and his subjective perceptions, an investor i forms beliefs about future prices and dividends given by a set of priors

$$\Pi^i\left(P_t \cup \mathcal{P}\right) := \left(1 - \gamma_\mu^i\right) f_{P_t} + \gamma_\mu^i \Delta^m, \tag{2}$$

where f_{P_t} is the frequency of cases observed in the data set P_t, $\left(1 - \gamma_\mu^i\right)$ is the degree of confidence consumer

[9] As Remark 4 explains, introducing short-sales, but restricting consumption to be non-negative in all states of the world will leave the major part of our results unchanged.

[10] For simplicity, we assume that the perceptions of the upper and lower bounds of dividends are correct. While the lower and upper bounds are assumed to be identical across agents, we will see that pessimists will only use the lower bounds, optimists – only the upper bounds and Bayesians will not rely on these bounds at all when choosing their portfolios.

i puts in the hard information. Since the number of observations in P_t is limited, consumers might feel ambiguous about a prediction based entirely on the frequency f_{P_t}. γ^i_μ is the degree of ambiguity regarding this prediction – the weight assigned to the Δ^m, the simplex (the set of all possible probability distributions) over all possible cases in $P_t \cup \mathcal{P}$.

The degree of ambiguity γ^i_μ will depend on the length of memory μ, i.e., the number of cases in P_t. If the length of memory increases, i.e., as more information becomes available, ambiguity may shrink and might even disappear. We will assume that γ^i_μ is decreasing in μ. However, given the agents' uncertainty about the price formation process, we will not assume that ambiguity necessarily vanishes in the limit. Hence, $\lim_{\mu\to\infty} \gamma^i_\mu = \gamma^i > 0$ might obtain even when agents observe the entire history of the economy. We will refer to an agent, for whom $\gamma^i_\mu = 0$ for all $\mu \in \mathbb{N}$ as a Bayesian with a frequency-based prior or Bayesian, for short.

Following Eichberger and Guerdjikova (2013), we will assume that consumers i's preferences on the set of portfolios consisting of a^i_t units of the risky asset and b^i_t units of bonds, $\left(a^i_t; b^i_t\right)$ can be represented by the following functional[11] with a linear von Neumann-Morgenstern utility function[12]:

$$
\begin{aligned}
(3) \qquad V^i_t(a^i_t, b^i_t; P_t) &= \alpha^i \max_{\pi\in\Pi^i(P_t\cup\mathcal{P})} \sum_{(p;q;\delta)\in M(P_t\cup\mathcal{P})} ((r+p)\, b^i_t + (\delta+q)\, a^i_t)\pi\,(p;q;\delta) \\
&+ \left(1-\alpha^i\right) \min_{\pi\in\Pi^i(P_t\cup\mathcal{P})} \sum_{(p;q;\delta)\in M(P_t\cup\mathcal{P})} \left[((r+p)\, b^i_t + (\delta+q)\, a^i_t)\right] \pi\,(p;q;\delta)
\end{aligned}
$$

where $\Pi^i\,(P_t \cup \mathcal{P})$ is the set of priors defined in (2), whereas α^i is the consumers' degree of optimism.

We will distinguish three types of consumers: (i) *optimists* with $\alpha^o = 1$ respond to their ambiguity by focussing on the "best probability distribution" in $\Pi\,(P_t \cup \mathcal{P})$, (ii) *pessimists* with $\alpha^p = 0$ decide on the basis of the worst probability distribution, and (iii) *Bayesians* ($\gamma^b_\mu \equiv 0$, $\Pi^i\,(P_t \cup \mathcal{P}) = \{f_{P_t}\}$) focus on the frequency of cases f_{P_t}. We will assume that for optimists and pessimists the perceived ambiguity only depends on μ and write $\gamma^o_\mu = \gamma^b_\mu = \gamma_\mu$. The population of consumers consists of a fraction $\omega \in [0;1]$ of optimists, a fraction $\beta \in [0;1]$ of Bayesians, and a fraction $\sigma := 1-\omega-\beta \in [0;1]$ of pessimists.

The upper and lower limits of prices and dividends in $\mathcal{P}$ determine the extreme values of the expected asset

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11 Such a preference functional can be derived axiomatically from preferences over acts and data, see Eichberger and Guerdjikova (2013).

12 Risk-neutrality, i.e., a linear von Neumann-Morgenstern utility function $u(x) = x$, is a common assumption in asset demand models, e.g., Adam and Marcet (2011).

returns. In order to rule out that consumers expect returns which allow unlimited arbitrage, we will assume that beliefs about these upper and lower bounds are arbitrage-free. In particular, we will require that consumers consider the lowest possible return of the risky asset, $\frac{\underline{\delta}}{\bar{q}}$ to be lower than the minimal perceived return of the bond, $\frac{r}{\bar{p}}$ and the maximal return of the risky asset $\frac{\bar{\delta}}{\underline{q}}$ to exceed the maximal return of the bond $\frac{r}{\underline{p}}$:

$$\frac{\underline{\delta}}{\bar{q}} < \frac{r}{\bar{p}} < \frac{r}{\underline{p}} < \frac{\bar{\delta}}{\underline{q}} \tag{4}$$

Assumption A1 (Strongly Arbitrage-Free Beliefs) We call the consumers' beliefs about prices and dividends *strongly arbitrage-free* if the upper and lower bounds of prices and dividends $\bar{q}$, $\underline{q}$, $\bar{p}$, $\underline{p}$, $\bar{\delta}$ and $\underline{\delta}$ satisfy

$$\frac{\underline{\delta}}{\bar{q}} < \frac{\bar{\delta}}{\bar{q}} < \frac{r}{\bar{p}} < \frac{r}{\underline{p}} < \frac{\underline{\delta}}{\underline{q}} < \frac{\bar{\delta}}{\underline{q}} \tag{5}$$

Under Strongly Arbitrage-Free Beliefs, the ranges of returns are nested. The returns of the safe asset will always lie between the extreme returns of the risky asset, i.e., the entire range of bond returns $\left(\frac{r}{\bar{p}}; \frac{r}{\underline{p}}\right)$ lies between the highest range $\left(\frac{\underline{\delta}}{\underline{q}}; \frac{\bar{\delta}}{\underline{q}}\right)$ and the lowest range $\left(\frac{\underline{\delta}}{\bar{q}}; \frac{\bar{\delta}}{\bar{q}}\right)$ of stock returns. Hence, the expected returns of the two assets can never dominate each other. This precludes $\bar{q}$ and $\underline{q}$ from being equilibrium prices, since there would be arbitrage opportunities at these prices.

For most of our results we will rely on Assumption A1. For Proposition 17, however, a weaker notion of arbitrage-free beliefs, which allows for some overlap of the expected return ranges, is necessary.

Assumption A2 (Weakly Arbitrage-Free Beliefs) We call the consumers' beliefs about prices and dividends *weakly arbitrage-free* if the upper and lower bounds of prices and dividends $\bar{q}$, $\underline{q}$, $\bar{p}$, $\underline{p}$, $\bar{\delta}$ and $\underline{\delta}$ satisfy

(i) Larger Price Uncertainty for the Risky Asset, $\frac{\bar{q}}{\underline{q}} > \frac{\bar{p}}{\underline{p}}$, and

(ii) Overlapping Expected Returns of the Assets

$$\frac{\underline{\delta}}{\bar{q}} < \frac{r}{\bar{p}} < \frac{\bar{\delta}}{\bar{q}} < \frac{\underline{\delta}}{\underline{q}} < \frac{r}{\underline{p}} < \frac{\bar{\delta}}{\underline{q}}.$$

Notice that Strongly Arbitrage-Free Beliefs (Assumption A1) implies Larger Price Uncertainty for the Risky Asset (Assumption A2 i). Under Weakly Arbitrage-Free Beliefs the ranges of the expected returns of the assets may intersect. In this case, the extreme values of the risky asset price may obtain in equilibrium. Clearly, both Assumptions 1 and 2 are stronger than (4). Apart from being easy to interpret, the stronger constraints have the advantage of allowing us to rank the reservation prices of investors for any temporary equilibrium[13] and, thus, allow us to provide a tractable analysis of the price dynamics.

2.3 Temporary equilibrium

In each period t, Walras' law (1) relates bond prices p_t in a unique way to stock prices q_t. Hence, each case is fully described by the price q_t and the dividend δ_t. Simplifying the data set $Q_t = \{(q_t, \delta_t) \mid (h(q_t), q_t, \delta_t) \in P_t\}$, one obtains the following optimization problems.

A young consumer of type $i \in \{o; b; p\}$, chooses a portfolio $\left(a_t^i; b_t^i\right)$ so as to maximize $V_t^i(a_t^i, b_t^i; Q_t)$ given in (3) subject to the budget constraint $q_t a_t^i + p_t b_t^i \leq y$, where

- $V_t^o(a_t^o, b_t^o; Q_t) = \max\limits_{\pi \in \Pi(Q_t)} \sum_{(p;q;\delta) \in M(Q_t)} ((r+p)\, b_t^o + (\delta + q)\, a_t^o) \pi\,(p; q; \delta)$ for optimists,
- $V_t^p(a_t^p, b_t^p; Q_t) = \min\limits_{\pi \in \Pi(Q_t)} \sum_{(p;q;\delta) \in M(Q_t)} ((r+p)\, b_t^p + (\delta + q)\, a_t^p) \pi\,(p; q; \delta)$ for pessimists,
- $V_t^b(a_t^b, b_t^b; Q_t) = \sum_{(p;q;\delta) \in M(Q_t)} ((r+p)\, b_t^b + (\delta + q)\, a_t^b) f_{Q_t}\,(p; q; \delta)$ for Bayesians

Denoting by $E_{Q_t}[q] = \frac{1}{\mu} \sum_{\tau=t-1}^{t-\mu} q_\tau$ and $E_{Q_t}[\delta] = \frac{1}{\mu} \sum_{\tau=t-1}^{t-\mu} \delta_\tau$ the average dividend and the average price realized in Q_t, we obtain the following demand correspondences for the three types of consumers.

Proposition 1 *For each type of consumers $i \in \{o; b; p\}$, the demand correspondence for the risky asset is given by*

(6)
$$a_t^i\,(q_t; Q_t) = \begin{cases} \frac{y}{q_t} & \text{if } q_t < q^i\left(Q_t; \gamma_\mu\right) \\ \left[0; \frac{y}{q^i\left(Q_t; \gamma_\mu\right)}\right] & \text{if } q_t = q^i\left(Q_t; \gamma_\mu\right) \\ 0 & \text{if } q_t > q^i\left(Q_t; \gamma_\mu\right) \end{cases}$$

with reservation prices

(7)

- $q^o\left(Q_t; \gamma_\mu\right) := \frac{y\left[\left(1-\gamma_\mu\right)(E_{Q_t}[q]+E_{Q_t}[\delta])+\gamma_\mu\left(\overline{q}+\overline{\delta}\right)\right]}{\left(1-\gamma_\mu\right)y+\left[\left(1-\gamma_\mu\right)E_{Q_t}[\delta]+\gamma_\mu\left(\overline{q}+\overline{\delta}\right)\right]A+\left(\gamma_\mu\overline{p}+r\right)B}$ *for optimists,*
- $q^b\left(Q_t; \gamma_\mu\right) := \frac{y(E_{Q_t}[q]+E_{Q_t}[\delta])}{y+AE_{Q_t}[\delta]+rB}$ *for Bayesians,*
- $q^p\left(Q_t; \gamma_\mu\right) := \frac{y\left[\left(1-\gamma_\mu\right)(E_{Q_t}[q]+E_{Q_t}[\delta])+\gamma_\mu\left(\underline{q}+\underline{\delta}\right)\right]}{\left(1-\gamma_\mu\right)y+A\left[\left(1-\gamma_\mu\right)E_{Q_t}[\delta]+\gamma_\mu\left(\underline{q}+\underline{\delta}\right)\right]+\left(\gamma_\mu\underline{p}+r\right)B}$ *for pessimists.*

[13] Assumption A1 implies that the optimists' reservation price will exceed that of the Bayesians, which in turn will be higher than that of the pessimists (see Lemma 3). Assumption A2 implies that the optimists' reservation price will exceed that of the pessimists and is only used for the result in Proposition 17.

Note that for each type $i \in \{o; b; p\}$, the critical value $q^i\left(Q_t; \gamma_\mu\right)$ can also be written as $q^i\left(E_{Q_t}[q]; E_{Q_t}[\delta]; \gamma_\mu\right)$ and is increasing in the realized average dividend $E_{Q_t}[\delta]$.

Aggregating over all consumers yields the aggregate demand function for the risky asset:

$$a_t\left(q; Q_t; \gamma_\mu\right) := \omega a_t^o\left(q_t; Q_t; \gamma_\mu\right) + \beta a_t^b\left(q_t; Q_t; \gamma_\mu\right) + (1 - \omega - \beta) a_t^p\left(q_t; Q_t; \gamma_\mu\right)$$

Definition 2 *A temporary equilibrium in period t for a given data set Q_t and a given degree of ambiguity γ_μ is a stock market price system q_t^* such that $a_t\left(q_t^*; Q_t; \gamma_\mu\right) = A$.*

To simplify notation, from now on and w.l.o.g., we will normalize the exogenous income to $y = 1$ and the asset supplies to $A = B = 1$. The following Lemma shows that, as long as ambiguity matters in the limit, for sufficiently large t, the reservation prices of optimists, Bayesians and pessimists on the equilibrium path can be ranked: optimists always overvalue the risky asset relative to Bayesians, whereas pessimists undervalue it. This property can be used to compute the temporary equilibrium given $\left(Q_t; \gamma_\mu\right)$:

Lemma 3 *Suppose that memory length is either fixed at μ and $\gamma_\mu > 0$, or memory is infinite and $\lim_{\mu\to\infty} \gamma_\mu = \gamma > 0$. If beliefs are strongly arbitrage-free, there is a.s. a finite period $\bar{t}$ such that for all $t \geq \bar{t}$, $q^o\left(Q_t; \gamma_\mu\right) > q^b\left(Q_t\right) > q^p\left(Q_t; \gamma_\mu\right)$. Hence, for given shares of optimists ω, Bayesians β and pessimists σ, there is a.s. a finite $\bar{t}$ such that for all $t \geq \bar{t}$ the temporary equilibrium at time t for memory Q_t is given by:*

$$\begin{array}{llll}
\textit{(i)} & q_t^* = q^o\left(Q_t; \gamma_\mu\right), & a_t^o = \frac{1}{\omega},\ a_t^b = a_t^p = 0 & \textit{for} \quad \omega \geq q^o\left(Q_t; \gamma_\mu\right), \\
\textit{(ii)} & q_t^* = \omega, & a_t^o = \frac{1}{\omega},\ a_t^b = a_t^p = 0 & \textit{for} \quad q^o\left(Q_t; \gamma_\mu\right) > \omega \geq q^b\left(Q_t\right), \\
\textit{(iii)} & q_t^* = q^b\left(Q_t\right), & a_t^o = q^b\left(Q_t\right)^{-1}, & \\
& & a_t^b = \frac{1-\omega q^b\left(Q_t\right)^{-1}}{\beta},\ a_t^p = 0 & \textit{for} \quad \omega + \beta \geq q^b\left(Q_t\right) > \omega, \\
\textit{(iv)} & q_t^* = \omega + \beta, & a_t^o = a_t^b = (\omega+\beta)^{-1},\ a_t^p = 0 & \textit{for} \quad q^b\left(Q_t\right) > \omega + \beta \geq q^p\left(Q_t; \gamma_\mu\right), \\
\textit{(v)} & q_t^* = q^p\left(Q_t; \gamma_\mu\right), & a_t^o = a_t^b = q^p\left(Q_t; \gamma_\mu\right)^{-1}, & \\
& & a_t^p = \frac{1-(\omega+\beta) q^p\left(Q_t; \gamma_\mu\right)^{-1}}{1-\omega-\beta} & \textit{for} \quad q^p\left(Q_t; \gamma_\mu\right) > \omega + \beta.
\end{array}$$

Remark 4 *In this model, short-selling is prohibited. In fact, allowing for short-sales will not significantly alter our analysis. Short-sales in the model would still be restricted by the non-negativity constraints on consumption. As long as we require that consumption of all agents in equilibrium remains non-negative for all possible realizations of prices and dividends, we obtain that for each investor, the share of wealth invested in the risky asset should belong to a convex and compact set. Combining this result with the reservation prices in (7), we obtain the same 5 cases for the temporary equilibrium constellations, which differ from the table* *in Lemma 3 only w.r.t. the cut-off values of the investor shares ω and β. This leaves the main results of our analysis, notably the results in Section 3, as well as Proposition 14 and 15 unchanged.*

3 Equilibrium Dynamics with Optimists and Pessimists

We now study the learning dynamics in the economy driven by the exogenous dividend process and the endogenous asset price process for fixed shares of types ω, β and $\sigma = 1 - \omega - \beta$. We focus on the effects of memory length and perceived ambiguity on the limit equilibrium price process and we compare economies with infinite memory to economies with finite memory length μ. With finite memory, we assume that ambiguity γ_μ remains positive and constant. For infinite memory, we study both the case in which ambiguity vanishes, $\gamma_\mu \to 0$, as the number of observations grows and the case of persistent ambiguity where $\gamma_\mu \to \gamma > 0$.

3.1 Economies with Infinite Memory

Suppose first that investors have access to all previous observations of prices and dividends. Then, consumers "learn" both the process of dividend realizations ρ and the equilibrium price:

Proposition 5 *Suppose that consumers have access to all past cases until time t, $\mu = t$, and that perceived ambiguity γ_t converges to γ as memory length increases with time, $\lim_{t\to\infty} \gamma_t \to \gamma$. If beliefs are strongly arbitrage-free, then*

$$\lim_{t\to\infty} q_t^* \left(Q_t;\gamma_t\right) = \begin{cases} q_\infty^o =: \frac{(1-\gamma)E_\rho[\delta]+\gamma\left(\bar{q}+\bar{\delta}\right)}{(\gamma\bar{p}+r)+\left[(1-\gamma)E_\rho[\delta]+\gamma\left(\bar{q}+\bar{\delta}\right)\right]} & \text{if } \omega > q_\infty^o \\ \omega & \text{if } q_\infty^o \geq \omega > q_\infty^b \\ q_\infty^b =: \frac{E_\rho[\delta]}{E_\rho[\delta]+r} & \text{if } \omega+\beta > q_\infty^b \geq \omega \\ \omega+\beta & \text{if } q_\infty^b \geq \omega+\beta > q_\infty^p \\ q_\infty^p =: \frac{(1-\gamma)(E_\rho[\delta])+\gamma\left(\underline{q}+\underline{\delta}\right)}{\left(\gamma\underline{p}+r\right)+\left((1-\gamma)E_\rho[\delta]+\gamma\left(\underline{q}+\underline{\delta}\right)\right)} & \text{if } q_\infty^p \geq \omega+\beta \end{cases}$$

obtains a.s. Furthermore, for $\gamma > 0$, we have $q_\infty^o > q_\infty^b > q_\infty^p$ and, for $\gamma = 0$, $q_\infty^o = q_\infty^b = q_\infty^p$.

Several special cases of this proposition are of interest: first consider a representative agent economy ($\omega = 1$ or $\beta = 1$ or $\omega + \beta = 0$). If the representative agent is Bayesian ($\beta = 1$), he perceives no ambiguity and the limit price equals the fundamental value of the asset. If the representative agent is optimistic ($\omega = 1$) or pessimistic ($\omega + \beta = 0$), the price converges a.s. to a rational expectations price where the actual expected dividend of the asset is distorted upwards or downwards with a weight γ. More generally, if ambiguity is persistent, the limit price depends positively on the ambiguity attitude: a market with a higher share of optimists (pessimists) systematically over(under)values the risky asset compared to a Bayesian market. Finally, if ambiguity vanishes, the rational expectation price obtains regardless of the distribution of investor types.

3.2 Economies with Finite Memory

We now consider the case of finite memory. Since the memory length μ is fixed, we use γ to denote the perceived constant ambiguity γ_μ. We will consider two types of economies: "pure" economies, for which temporary equilibrium prices are determined by a single type of investor (even though the fractions of all three types might be positive) and "economies with different regimes" for which all three types of investors may become decisive for prices. In the latter case, different price regimes (optimistic, pessimistic or Bayesian) will obtain depending on the observed prices and dividends.

3.2.1 Optimists', Pessimists', and Bayesian Markets

Lemma 3 identifies the different equilibrium constellations which can occur depending on the shares of optimists, pessimists and Bayesians. In a first step, we consider economies in which a single type of investors determines equilibrium prices.

Definition 6 *If there is a finite period $\bar{t}$ such that in each period $t \geq \bar{t}$, the equilibrium price is given a.s. by $q_t^* = q^i(Q_t;\gamma)$ for some $i \in \{o;b;p\}$, we call the economy an*

- $\text{"optimists' market"}$ *if* $q_t^* = q^o(Q_t;\gamma)$,
- "Bayesian market" *if* $q_t^* = q^b(Q_t)$,
- $\text{"pessimists' market"}$ *if* $q_t^* = q^p(Q_t;\gamma)$.

Our next proposition identifies the sets of values of ω and β, for which each of the three cases occurs.

Proposition 7 *Suppose beliefs are strongly arbitrage-free. For $i \in \{o;b;p\}$, define $q^{\max i}$ and $q^{\max i}$ are implicitly as:*

$$q^{\max i} = q^i\left(E_{Q_t}[q] = q^{\max i}; E_{Q_t}[\delta] = \bar{\delta};\gamma\right)$$
$$q^{\max i} = q^i\left(E_{Q_t}[q] = q^{\min i}; E_{Q_t}[\delta] = \underline{\delta};\gamma\right).$$

Then,

- *for* $\omega > q^{\max o}$ *the economy is an optimists' market;*
- *for* $\omega < q^{\min b} < q^{\max b} < \omega + \beta$ *the economy is a Bayesian market;*
- *for* $\omega + \beta < q^{\min p}$ *the economy is a pessimists' market.*

Intuitively, $q^{\min i}$ is the price predicted by an investor of type i when the data set contains only observations of that same price $q^{\min i}$ and the lowest possible dividend. It is the lowest price that can be sustained if prices are set by type i investors. The maximal sustainable price $p^{\max i}$ is defined analogously.

For given values of ω and β, the dynamics of the economy is fully determined by the memory Q_t, since the temporary equilibrium in period t depends only on the price and dividend data in Q_t. This Markov property is a key feature of our analysis. For the case of a one-period memory, $\mu = 1$, the equilibrium price q_t^* is itself a Markov process, i.e., the distribution of q_t^* is fully described by the exogenous distribution of δ_{t-1}, $\rho(\delta)$ and the asset price in the previous period q_{t-1}^*. When the memory is longer, $\mu > 1$, the asset price q_t^* is no longer a Markov process. However, $(q_{t-\mu+1}^* ... q_t^*; \delta_{t-\mu+1} ... \delta_{t-1})$ is a Markov process: given the last μ price realizations $\left(q_{t-\mu}^* ... q_{t-1}^*\right)$ and the last $\mu - 1$ dividend realizations, $(\delta_{t-\mu} ... \delta_{t-2})$, the distribution of q_t^*, is fully determined by the distribution of δ_{t-1}, ρ.

We now make use of this Markov property in order to characterize the long-run equilibrium price distribution.

Proposition 8 *Assume that beliefs are strongly arbitrage-free. Let $(\omega; \beta)$ be such that the economy is either an optimists', a pessimists', or a Bayesian market and let $i \in \{o; p; b\}$ stand for the type of agent who determines market prices. Then the μ-tuples $\left(q_{t-\mu+1}^* ... q_t^*; \delta_{t-\mu+1} ... \delta_{t-1}\right)$ of equilibrium prices and dividends form a Markov process which is ψ-irreducible, positive recurrent and has an invariant probability distribution $\tilde{\pi}_\mu^i$ with a marginal distribution π_μ^i on q_t^* satisfying $\pi_\mu^i\left(\left[q^{\min i}; q^{\max i}\right]\right) = 1$.*

The proposition demonstrates the existence of an invariant distribution for the Markov process of price and dividend tuples $\left(q_{t-\mu+1}^* ... q_t^*; \delta_{t-\mu+1} ... \delta_{t-1}\right)$, $\tilde{\pi}_\mu^i$, which will describe the behavior of the economy in the long-run. For a given set $\mathbf{Q}$ of such realizations, $\tilde{\pi}_\mu^i(\mathbf{Q})$ can be interpreted as the fraction of time that the economy will spend in this set in the long-run. Of particular interest is the marginal of this distribution[14] on q_t^*, π_μ^i, i.e., the invariant distribution of equilibrium prices in the economy. Proposition 8 shows that, for parameter values $(\omega; \beta)$ which allow for markets dominated by a single type of agent $i \in \{o; b; p\}$, the support of this distribution will be given by $\left[q^{\min i}; q^{\max i}\right]$ – the range between the lowest and the highest price sustainable by this type of investor. The Proposition demonstrates that only prices in this interval will be observed in the long-run. Furthermore, any non-zero interval of prices in the respective price range will be reached in a recurrent

[14] In the Appendix, we show that the marginals of $\tilde{\pi}_\mu^i$ on $q_1 ... q_\mu$ are identical, thus we can meaningfully talk about an invariant price distribution.

manner with strictly positive probability. Similarly to the interpretation of $\tilde{\pi}^i_\mu$, for a given set of equilibrium prices $S \subset \left[q^{\min i}; q^{\max i}\right]$, in the long-run, $\pi^i_\mu(S)$ will coincide with the fraction of time, during which the equilibrium asset price falls into this range, $q^*_t \in S$.

Since $q^{\max o} > q^{\max b} > q^{\max p}$ and $q^{\min o} > q^{\min b} > q^{\min p}$ holds, see Lemma 18 in the Appendix, the recurrent prices in a market dominated by optimists will exceed those in a Bayesian market, which in turn will exceed those in a pessimistic market.

The case of a one-period memory, $\mu = 1$, is special in that the equilibrium price process q^*_t itself is a Markov process. In this case, one can explicitly compute the expected price for the invariant distribution π.

Proposition 9 *Given shares of optimists and Bayesians $(\omega; \beta)$ for which the economy is either an optimists', a pessimists', or a Bayesian market, for $\mu = 1$, the expected equilibrium price according to the long-run equilibrium distribution π^i is*

$$E_{\pi^b_1}[q] = \frac{\int \frac{\delta}{\delta+r+1}\rho'(\delta)d\delta}{\int \frac{(\delta+r)}{\delta+r+1}\rho'(\delta)d\delta} \qquad \textit{for} \quad \textit{a Bayesian market,}$$

$$E_{\pi^o_1}[q] = \frac{\int \frac{(1-\gamma)\delta+\gamma\left(\bar{q}+\bar{\delta}\right)}{\left[(1-\gamma)\delta+\gamma\left(\bar{q}+\bar{\delta}\right)\right]+(\gamma\bar{p}+r)+1-\gamma}\rho'(\delta)d\delta}{\int \frac{\left[(1-\gamma)\delta+\gamma\left(\bar{q}+\bar{\delta}\right)\right]+(\gamma\bar{p}+r)}{\left[(1-\gamma)\delta+\gamma\left(\bar{q}+\bar{\delta}\right)\right]+(\gamma\bar{p}+r)+1-\gamma}\rho'(\delta)d\delta} \qquad \textit{for} \quad \textit{an optimists' market, and}$$

$$E_{\pi^p_1}[q] = \frac{\int \frac{(1-\gamma)\delta+\gamma\left(\underline{q}+\underline{\delta}\right)}{\left[(1-\gamma)\delta+\gamma\left(\underline{q}+\underline{\delta}\right)\right]+\left(\gamma\underline{p}+r\right)+1-\gamma}\rho'(\delta)d\delta}{\int \frac{\left[(1-\gamma)\delta+\gamma\left(\underline{q}+\underline{\delta}\right)\right]+\left(\gamma\underline{p}+r\right)}{\left[(1-\gamma)\delta+\gamma\left(\underline{q}+\underline{\delta}\right)\right]+\left(\gamma\underline{p}+r\right)+1-\gamma}\rho'(\delta)d\delta} \qquad \textit{for} \quad \textit{a pessimists' market.}$$

Remark 10 *Notice that for each $i \in \{o, b, p\}$ the expected value $E_{\pi^i_1}[q]$ in Proposition 9 is lower than the limit price q^i_∞ in an economy with a representative agent of type i in Proposition 5, $E_{\pi^i_1}[q] < q^i_\infty$.*

3.2.2 Economies with Different Regimes

The class of pure economies defined in 6 is quite special. For a large range of parameter values $(\omega; \beta)$, no one type of consumers determines the long-run equilibrium price distribution. We thus extend our results to economies in which the equilibrium price is determined by the reservation price of the optimistic, pessimistic or the Bayesian consumers depending on the observed data.

Our next Proposition first identifies parameter ranges of ω and β for which the economy can exhibit all three regimes listed in Lemma 3 (i.e., prices can be determined by optimists, pessimists or Bayesians) and then establishes that in the limit, the economy will oscillate between these three different regimes spending a strictly positive fraction of time in each of them[15].

Proposition 11 *Let beliefs be strongly arbitrage-free and let γ be sufficiently small so that $q^{\min o} < q^{\max p}$. If $(\omega;\beta)$ are such that $q^{\min o} < \omega < \omega+\beta < q^{\max p}$, then, for each type of consumer $i \in \{o;b;p\}$, there are data sets $Q_t \in \left[q^{\min o};q^{\max p}\right]^{\mu} \times \left[\underline{\delta};\bar{\delta}\right]^{\mu}$ for which type i determines the equilibrium price, i.e., $q_t^* = q^i\left(Q_t;\gamma\right)$. Furthermore, the μ-tuples $(q_{t-\mu+1}^*...q_t^*;\delta_{t-\mu+1}...\delta_{t-1})$ of equilibrium prices and dividends form a Markov process which is ψ-irreducible on $\left[q^{\min o};q^{\max p}\right]^{\mu} \times \left[\underline{\delta};\bar{\delta}\right]^{\mu-1}$, positive recurrent and has an invariant probability distribution $\tilde{\pi}_\mu$ with marginal distribution on q_t^* satisfying $\pi_\mu\left(\left[q^{\min o};q^{\max p}\right]\right) = 1$.*

In order to understand the belief dynamics of the different regimes, consider a data set Q_t with observations of prices and dividends sufficiently low so that the equilibrium price is set by optimists, $q_t^* = q^o\left(Q_t;\gamma\right) < \omega$. The risky asset is then overvalued relative to its observed performance ($E_{Q_t}[q]$ and $E_{Q_t}[\delta]$) since the dominating optimists believe in higher dividend payments and capital gains $\left(\bar{q} \text{ and } \bar{\delta}\right)$. In contrast, when the observed realizations are sufficiently high, the economy is in a pessimistic regime, $q_t^* = q^p\left(Q_t;\gamma\right) > \omega+\beta$. Asset prices are then set by pessimists and the risky asset is undervalued relative to its observed performance. Finally, in the intermediate case, when Bayesians set prices, $q_t^* = q^b\left(Q_t\right) \in (\omega, \omega+\beta)$, the assets appear to be correctly priced relative to their observed returns in Q_t.

The Proposition also shows that in economies with switching regimes, the presence of optimists and pessimists leads to a narrower range of observable prices than in a purely Bayesian economy (since $q^{\min o} > q^{\min b}$ and $q^{\max p} < q^{\max b}$). This is quite different from the case of pure economies where the presence of ambiguity-sensitive consumers leads to an unambiguous upward or a downward shift in the range of observable prices.

4 Evolutionary Dynamics

So far we have treated the population shares of optimists, Bayesians and pessimists as fixed parameters. Insights from social psychology, see Zuckerman (2001), as much as experimental evidence (e.g., Apesteguia, Huck, and Oechssler (2007)) suggest, however, that these shares may change in the face of success or failure of the respective group. In phases of exuberance, when optimists tend to be successful, other consumers might

[15] While our main result is stated for an economy with all three regimes it can be easily extended to cases in which only two types of consumers determine the equilibrium prices.

imitate their behavior, thus, increasing the total share of optimists. During a depression, investors might be attracted to conservative strategies, increasing the level of pessimism. This adjustment in the aggregate attitude towards ambiguity adds a slower procyclical component to the equilibrium price dynamics.

In this section we will investigate the effect of a population share dynamics on equilibrium prices and on the long-term distribution of investor types. We assume that the population shares will adjust according to the replicator dynamics as a function of the realized portfolio returns.

Starting in $t = 0$ from population shares of optimists and Bayesians $(\omega_0; \beta_0)$, we assume that population shares evolve according to the following replicator dynamics: $\omega_1 = \omega_0$, $\beta_1 = \beta_0$ and, for $t \geq 2$,

(8)
$$\omega_{t+1} = \omega_{t-1} \frac{\frac{q_t^*+\delta_t}{q_{t-1}^*}\lambda_{t-1}^o + \frac{p_t^*+r}{p_{t-1}^*}\left(1-\lambda_{t-1}^o\right)}{1+\delta_t+r} = \omega_{t-1} \frac{\left(q_t^*+\delta_t\right)a_{t-1}^o + \left(p_t^*+r\right)b_{t-1}^o}{1+\delta_t+r}$$
$$\beta_{t+1} = \beta_{t-1} \frac{\frac{q_t^*+\delta_t}{q_{t-1}^*}\lambda_{t-1}^b + \frac{p_t^*+r}{p_{t-1}^*}\left(1-\lambda_{t-1}^b\right)}{1+\delta_t+r} = \beta_{t-1} \frac{\left(q_t^*+\delta_t\right)a_{t-1}^b + \left(p_t^*+r\right)b_{t-1}^b}{1+\delta_t+r}$$

where $\lambda_t^i := q_t^* a_t^i$ denotes the share of wealth invested into the risky asset by consumer $i \in \{o; b\}$ at time t. Obviously, the share of pessimists is given by $\sigma_t = 1 - \omega_t - \beta_t$ at each t. The evolutionary dynamics operates with a one-period lag so as to avoid that the equilibrium price at time t, q_t^* affect the population shares $(\omega_t; \beta_t)$ in the same period.

According to the first equality in (8), the share of optimists ω_{t+1} equals their initial share ω_{t-1} multiplied by the return of their portfolio relative to the return of the market portfolio. If the optimists' portfolio performs better than the market portfolio, their share in the population increases. This allows us to interpret the replicator dynamics as a process of imitation[16] that operates on preferences. The replicator dynamics (8) adjusts the proportions of different attitudes towards ambiguity according to their relative success in the past[17].

Remark 12 *An alternative way of interpreting the replicator dynamics is as evolution of wealth shares, see Wang (2001). The numerator in the second equality in (*8*) expresses the wealth of the old optimists in period* t*, while the denominator is the total wealth of the old consumers in this period. Hence, the share of young consumers of a given type in period* $t+1$ *equals to the share of wealth earned by the old consumers of the*

[16] Schlag (1998) provides a formal argument for the optimality of such an imitation rule for exogenously given stochastic payoffs and Alós-Ferrer and Schlag (2007) extend this analysis to games. Alós-Ferrer and Ania (2005) use a different imitation dynamics to model strategy selection in financial markets with market power.

[17] This approach is inspired by the findings of Malmendier and Nagel (2011) as discussed in the Introduction.

same type in period t. This dynamics thus parallels the evolution of wealth in an economy with infinitely-lived consumers provided that saving rates are identical across consumers. It follows that saving effects will not impact the process of selection, contrary, e.g. to the model of Guerdjikova and Sciubba (2015).

We first examine some properties of the evolutionary dynamics which will later allow us to analyze the joint evolution of asset prices and type shares in the economy. Note that the three extreme states $\beta = 1$, $\omega = 1$ and $\sigma = 1$, in which a single type is present, are "stationary points" of the evolutionary dynamics (8): since a representative investor must hold the market portfolio and obtain its returns, his share remains constant at 1. We would like to identify another, non-trivial "stationary" state, characterized by a price $\tilde{q}$ such that as long as $q_t^* = q_{t-1}^* = \tilde{q}$, type shares remain constant in expectation (even though they might fluctuate with the stochastic dividend realizations). The next Lemma computes the price $\tilde{q}$, but shows that it differs from the fundamental value due to the non-linearity[18] of the evolutionary dynamics in the stochastic dividend δ_t.

Lemma 13 *Given memory Q_t and consumer shares $\left(\beta_{t-1};\omega_{t-1}\right)$ and $(\beta_t;\omega_t)$:*

- *the expected share of optimists satisfies $E_\rho\left[\omega_{t+1} \mid \omega_{t-1};\omega_t;\beta_{t-1};\beta_t;Q_t\right] \geq \omega_{t-1}$ iff*

$$E_\rho\left[\frac{q_t^*+\delta_t}{1+\delta_t+r} \mid \omega_{t-1};\omega_t;\beta_{t-1};\beta_t;Q_t\right] \geq q_{t-1}^*;$$

- *the expected share of pessimists satisfies $E_\rho\left[\sigma_{t+1} \mid \omega_{t-1};\omega_t;\beta_{t-1};\beta_t;Q_t\right] \geq \sigma_{t-1}$ iff*

$$E_\rho\left[\frac{q_t^*+\delta_t}{1+\delta_t+r} \mid \omega_{t-1};\omega_t;\beta_{t-1};\beta_t;Q_t\right] \leq q_{t-1}^*;$$

- *the expected share of Bayesians satisfies $E_\rho\left[\beta_{t+1} \mid \omega_{t-1};\omega_t;\beta_{t-1};\beta_t;Q_t\right] \geq \beta_{t-1}$ iff*

$$\left(E_\rho\left[\frac{q_t^*+\delta_t}{1+\delta_t+r} \mid \omega_{t-1};\omega_t;\beta_{t-1};\beta_t;Q_t\right] - q_{t-1}^*\right)\left(a_{t-1}^b-1\right) \geq 0.$$

In particular, for $q_t^* = q_{t-1}^* = \frac{E_\rho[\delta]}{E_\rho[\delta]+r}$, the expected share of a given type in $t+1$ will exceed (lie below) its share at $t-1$ iff consumers of this type hold less (more) than one unit of the risky asset in their portfolio, whereas for $q_t^* = q_{t-1}^* = \tilde{q}$ with

$$\tilde{q} := \frac{\int \frac{\delta}{1+\delta+r}\rho'(\delta)\,d\delta}{\int \frac{\delta+r}{1+\delta+r}\rho'(\delta)\,d\delta}. \tag{9}$$

the shares of all types of consumers remain constant in expectation.

[18] The standard replicator dynamics used in game theoretical models is deterministic, see Weibull (1995), but can also be adjusted by introducing a linear noise term.

Lemma 13 shows that the evolutionary dynamics in equation (8) favors the more cautious pessimistic consumers. Even when the price of the risky asset is constant at its fundamental value and thus, expected returns of both assets are equal, the proportion of consumers who invest more in the riskless asset than its share in the market portfolio will grow in expectations. According to Lemma 3, optimists (pessimists) consistently hold a portfolio which contains at least (at most) one unit of the risky asset. Hence, for equal expected returns, the evolutionary dynamics will select against optimists and in favor of pessimists in expectation. More generally, the share of optimists (pessimists) in the population will grow in expectations whenever the initial price of the risky asset q^*_{t-1}, is relatively low (high). The fraction of the risky asset held by Bayesians can vary with the memory Q_t. Their expected share will grow when they hold a larger (smaller) share of the risky asset than in the market portfolio and the price of the risky asset is relatively low (high). Finally, if the equilibrium price is constant at $\tilde{q}$, the shares of all types of investors remain constant in expectations. Note that $\tilde{q}$ is lower than the fundamental value of the risky asset.

4.1 Evolution in Economies with Long Memory

We will first study the evolution of consumer types in an economy with infinite memory and check whether a Bayesian market is "stable" against the introduction of small proportions of optimists and pessimists[19].

From the analysis of markets with infinite memory and constant consumer shares in the previous section, we know that the equilibrium prices converge. We first consider the three extreme steady states, $(\omega = 1; q^* = q^o_\infty)$, $\left(\beta = 1; q^* = q^b_\infty\right)$ and $(\sigma = 1; q^* = q^p_\infty)$, where q^o_∞, q^b_∞ and q^p_∞ are the equilibrium prices for representative-agent economies with long memory derived in Proposition 5. Note that both q^b_∞ and q^o_∞ exceed the critical price $\tilde{q}$ determined in Lemma 13 at which consumer shares remain constant.

Consider a steady state with $\omega = 1$ or $\beta = 1$ in which we introduce an ϵ share of pessimists. For small ϵ, the equilibrium price will remain unchanged. However, since the equilibrium price exceeds the critical value $\tilde{q}$, according to Lemma 13, the pessimists' share will increase in expectation making this steady-state unstable. Similarly, the steady state with $\sigma = 1$ will be unstable if $q^p_\infty < \tilde{q}$ holds, since at this price the

[19] This analysis is similar to the notion of evolutionary stable state (ESS) used in evolutionary game theory, see Weibull (1995). A "strategy" in our context corresponds to a consumer type. It is evolutionary stable if an economy populated only by this type of consumers cannot be invaded by "mutants" using a different "strategy".

share of optimists and Bayesians would be growing in expectations. For $q^p_\infty \geq \tilde{q}$, however, the steady state $(\beta^* = 0;\ \omega^* = 0;\ q^* = q^p_\infty)$ with only pessimists in the market is stable. At this price the expected shares of both Bayesians and optimists will be decreasing. The following proposition summarizes these findings.

Proposition 14 *For $q^p_\infty < \tilde{q}$, no extreme steady-state is stable, i.e., for each extreme steady-state there is a type $i \in \{o; b; p\}$ consumers whose share is 0 in the steady-state but will grow in expectations if a small fraction ϵ of type i is introduced into the market.*
For $q^p_\infty \geq \tilde{q}$, there is one stable extreme steady-state: $(\omega^* = 0; \beta^* = 0; q^* = q^p_\infty)$.

Interestingly, a situation in which Bayesians have learned the distribution of dividends and the equilibrium price of the risky asset equals its fundamental value is not "stable" against the introduction of pessimists. This result relates to the property of the replicator dynamics to favor investors with less risky portfolios whenever the price is constant and equals the fundamental value of the risky asset. Hence, a pure Bayesian market cannot be stable, and the only potentially stable steady state is the pessimistic one.
Next, we will examine the dynamics of an economy where the pessimistic steady state is stable. We will show that the system converges a.s. to this steady state if perceived ambiguity is not too large.

Proposition 15 *Suppose that $\omega_0 + \beta_0 = \beta_1 + \omega_1 < 1$. For $q^p_\infty > \tilde{q}$, there is $\bar{\gamma} \in (0;1)$ such that for all $\gamma < \bar{\gamma}$, $\lim_{t\to\infty} \sigma_t = 1$ and $\lim_{t\to\infty} q^*_t = q^p_\infty$ a.s. in expectations.*
If $\beta_0 + \omega_0 = \beta_1 + \omega_1 = 1$, then there is a $\bar{\gamma} \in (0;1)$ such that for all $\gamma < \bar{\gamma}$, $\lim_{t\to\infty} \beta_t = 1$ and $\lim_{t\to\infty} q^*_t = q^b_\infty$ a.s. in expectations.

The analysis of economies with long memory shows that in general, evolution does not select for ambiguity-neutrality nor does it push prices towards fundamental values. Provided perceived ambiguity is small, but positive[20], the economy a.s. converges to a steady state in which only pessimists remain in the market and where the risky asset is undervalued relative to its price under rational expectations. Furthermore, Proposition 15 shows that under the replicator dynamics optimists are driven out of the market, whenever Bayesians or pessimists are present. In particular, if only Bayesians and optimists populate the economy then limit prices

[20] If ambiguity is too large, prices might fluctuate too much to ensure that the share of pessimists behaves as a submartingale and thus converges to 1. If ambiguity vanishes in the limit, all three types of investors behave as Bayesians with correct beliefs.

are determined by Bayesian consumers and converge to those under rational expectations.

4.2 Evolution in Economies with One-Period Memory

We now proceed to the analysis of economies populated by consumers with one-period memory. The first proposition shows that a.s. after a finite number of periods, an economy with one-period memory and a positive share of Bayesians will be indistinguishable from a pure Bayesian market with one-period memory.

Proposition 16 *Consider an economy with one-period memory, $\mu = 1$, strictly positive initial shares of optimists, Bayesians and pessimists, and assume beliefs are strongly arbitrage-free. Then, a.s., after a finite number of periods, the equilibrium price in each period will be determined by the reservation price of Bayesian consumers, $q_t^* = q_t^b$.*

It may surprise that the equilibrium prices do not reflect the presence of optimistic and pessimistic consumers. We know from Proposition 9, however, that the average price of the risky asset in a Bayesian market with one-period memory exactly equals the critical price $\tilde{q}$ derived in Lemma 13 at which expected consumer shares remain constant. Hence, optimists and pessimists need not disappear from the market, even though they will have no impact on equilibrium prices after a finite number of periods.

The fact that the evolutionary dynamics selects for different attitudes towards ambiguity depending on the length of the memory is interesting per se. This last result relies, however, on the fact Bayesian learning with one-period memory uses the same criterion to evaluate portfolios as does the replicator dynamics (last-period returns). This makes the case of one-period memory somewhat special and we do not expect this result to generalize to memories of arbitrary, but finite length.

Our result, however, points out a common feature between economies with infinite and those with one-period memory: in both cases, the risky asset is undervalued relative to its fundamental value. In the former case, the price converges to the pessimists' limit reservation value p_∞^p, which (in order for the result to hold) has to be larger than $\tilde{q}$, whereas in the latter case, the average price converges to $\tilde{q}$.

Since Bayesians determine prices in each period, no cycles of the type described in Proposition 11 occur in this economy with one-period memory. We next consider an economy populated only by optimists and pessimists in which beliefs are weakly arbitrage-free and show that such an economy will oscillate between periods of optimism, in which the share of optimists is high and the equilibrium price coincides with the optimists' reservation price and periods of pessimism, in which the share of pessimists is high and the equilibrium price coincides with the pessimists' reservation price.

Proposition 17 *Consider an economy with one-period memory, no Bayesians ($\beta_0 = \beta_1 = 0$) and positive initial shares of optimists and pessimists ($\omega_0 = \omega_1 \in (0;1)$). Suppose that beliefs are weakly arbitrage-free. Then there exists an open set of parameters of the model such that $q_t^o > q_t^p$ holds a.s. after a finite number of periods. For these parameters, the economy can be described by a ψ-irreducible, positive recurrent Markov process with an invariant probability distribution π. The probability distribution π assigns a positive probability to states, in which optimists determine the market price, i.e., $q_t^* = q_t^o$ and $\omega_t > q_t^*$, to states, in which pessimists determine the market price, i.e., $q_t^* = q_t^p$ and $\omega_t < q_t^*$ and to states in which the share of optimists determines the market price, $q_t^* = \omega_t$.*

Our last result shows that in economies with one-period memory and no Bayesian consumer, the evolutionary dynamics does not select for a specific attitude towards ambiguity and the economy cycles[21] between periods of optimism and pessimism, in which the corresponding type of investors determines the equilibrium price.

5 Concluding comments

Our paper contributes to a large body of literature on market selection which asks the question whether market prices correctly aggregate the available information. The two seminal papers by Blume and Easley (2006) and Sandroni (2000) showed that (absent market imperfections), heterogeneity w.r.t. to risk-aversion does not affect the ability of the market to select for agents with correct beliefs. In contrast, we show that in the presence of heterogeneity w.r.t. ambiguity-attitudes, markets might select for agents with effectively wrong beliefs. Prices might thus deviate from fundamentals not just in the short- but also in the long-run. In this sense, pessimism

[21] Note, however that cyclical behaviour only emerges if we relax the assumption of strongly arbitrage-free beliefs: two processes drive the dynamics of economy – the learning dynamics, which determines the reservation prices q_t^o and q_t^p and the evolutionary dynamics, which determines the share of optimists ω_t. To enter an "optimistic phase" with $q_t^* = q_t^o$, ω_t has to exceed q_t^o. Since both ω_t and q_t^o increase with the observed dividend realizations going from a pessimistic phase, in which $\omega_{t_0} < q_{t_0}^o$ to an optimistic one with $\omega_t > q_t^o$ requires the two processes to cross. With a one-period memory, this can occur only for dividend realizations satisfying $\delta > \frac{\bar{q}+\bar{\delta}}{r+\bar{p}} r$. Such dividend values are feasible under weakly, but not under strongly arbitrage-free beliefs. A symmetric argument applies to the case of a "pessimistic phase", where the relevant condition for the dividend is $\delta < \frac{\underline{q}+\underline{\delta}}{r+\underline{p}} r$.

might be a self-fulfilling prophecy.

There are differing views on whether pessimism is a rational response to the lack of probabilistic information or an expression of boundedly-rational behavior, cf. Gilboa, Postlewaite, and Schmeidler (2009). However, both experimental studies and market data seem to suggest that it is a prevalent characteristic that can explain certain stylized facts. Our paper shows that market forces need not select against this feature and hence, even if we consider pessimism irrational, we should not necessarily expect its effects to disappear with time.

In this model, we restrict attention to extremely optimistic and extremely pessimistic agents ($\alpha \in \{0;1\}$). Allowing for moderate attitudes towards ambiguity would make the model more realistic, but also complicate the analysis. Nevertheless, we believe that the main result of the paper will still hold in this more general setting: the evolutionary dynamics with infinite memory will not select the Bayesian investors, but rather for more pessimistic behavior which results in a lower limit price than q^{RE}, as illustrated by Lemma 13. Furthermore, for small values of γ, the only stable steady state (as in Proposition 14) will be that with $\alpha = 0$.

6 Appendix: Proofs

Proof of Proposition 1:

$$V_t^i\left(a_t^i; b_t^i; P_t\right)$$

$$= \begin{cases} (1-\gamma_\mu)\sum_{(\delta,p,q)\in M(P_t)}\left[(r+p)\,b_t+(\delta+q)\,a_t\right] f_{P_t}(\delta,p,q)+\gamma_\mu\left[(r+\overline{p})\,b_t+\left(\overline{\delta}+\overline{q}\right)a_t\right] & \text{for } i=\omega \\ \sum_{(\delta,p,q)\in M(P_t)}\left[(r+p)\,b_t+(\delta+q)\,a_t\right] f_{P_t}(\delta,p,q) & \text{for } i=\beta \\ (1-\gamma_\mu)\sum_{(\delta,p,q)\in M(P_t))}\left[(r+p)\,b_t+(\delta+q)\,a_t)\right] f_{P_t}(\delta,p,q)+\gamma_\mu\left[\left(r+\underline{p}\right)b_t+\left(\underline{\delta}+\underline{q}\right)a_t\right] & \text{for } i=p \end{cases}$$

Using Walras' Law, $p_\tau = 1-q_\tau$, $p = 1-q$ and the definitions of $E_{Q_t}[q]$ and $E_{Q_t}[\delta]$ yields the desired result.■

Proof of Lemma 3: For a given Q_t and γ_μ denote by $q^{REi}\left(Q_t;\gamma_\mu\right)$, $i \in \{o;b;p\}$ the solution to the equation:

$$q^{REi}\left(Q_t;\gamma_\mu\right) =: \begin{cases} \frac{\left[\left(1-\gamma_\mu\right)\left(q^{RE,i}\left(Q_t;\gamma_\mu\right)+E_{Q_t}[\delta]\right)+\gamma_\mu\left(\bar{q}+\bar{\delta}\right)\right]}{\left[\left(1-\gamma_\mu\right)+\left(\gamma_\mu\bar{p}+r\right)+\left[\left(1-\gamma_\mu\right)E_{Q_t}[\delta]+\gamma_\mu\left(\bar{q}+\bar{\delta}\right)\right]\right]} & \text{if } i=o \\ \frac{\left(\left(1-\gamma_\mu\right)\left(q^{RE,i}\left(Q_t;\gamma_\mu\right)+E_{Q_t}[\delta]\right)+\gamma_\mu\left(\underline{q}+\underline{\delta}\right)\right)}{\left[\left(1-\gamma_\mu\right)+\left(\gamma_\mu\underline{p}+r\right)+\left(\left(1-\gamma_\mu\right)E_{Q_t}[\delta]+\gamma_\mu\left(\underline{q}+\underline{\delta}\right)\right)\right]} & \text{if } i=p \\ \frac{\left(q^{RE,i}(Q_t)+E_{Q_t}[\delta]\right)}{1+r+E_{Q_t}[\delta]} & \text{if } i=b \end{cases},$$

or

$$q^{REi}\left(Q_t;\gamma_\mu\right)=\begin{cases}\frac{(1-\gamma_\mu)E_{Q_t}[\delta]+\gamma_\mu(\bar{q}+\bar{\delta})}{(\gamma_\mu\bar{p}+r)+\left[(1-\gamma_\mu)E_{Q_t}[\delta]+\gamma_\mu(\bar{q}+\bar{\delta})\right]} & \text{if } i=o\\ \frac{(1-\gamma_\mu)E_{Q_t}[\delta]+\gamma_\mu(\underline{q}+\underline{\delta})}{(\gamma_\mu\underline{p}+r)+\left((1-\gamma_\mu)E_{Q_t}[\delta]+\gamma_\mu(\underline{q}+\underline{\delta})\right)} & \text{if } i=p\\ \frac{E_{Q_t}[\delta]}{r+E_{Q_t}[\delta]} & \text{if } i=b\end{cases}.$$

Intuitively, q^{REi} is the price under "rational expectations" for a market with a representative agent $i\in\{o;b;p\}$ given the information contained in Q_t. Since $q^{REi}\left(Q_t;\gamma_\mu\right)$ only depends on $E_{Q_t}[\delta]$ and γ_μ, we write $q^{REi}\left(E_{Q_t}[\delta];\gamma_\mu\right)$. Simple algebra shows that

$$\text{(10)}\qquad q^{REi}\left(Q_t;\gamma_\mu\right)\underset{(<)}{>}E_{Q_t}[q]\Longleftrightarrow q^{REi}\left(Q_t;\gamma_\mu\right)\underset{(<)}{>}q^i\left(Q_t;\gamma_\mu\right)\underset{(<)}{>}E_{Q_t}[q]$$

for all $i\in\{o;b;p\}$. I.e., upon observing a data set Q_t, i's reservation price will lie strictly between the average observed price in the data and the investor i's price under "rational expectations". In a market with a representative agent i, $q_t^*=q^i\left(Q_t;\gamma_\mu\right)$ and we thus obtain that in such a market

$$\text{(11)}\qquad q^{REi}\left(Q_t;\gamma_\mu\right)\underset{(<)}{>}E_{Q_t}[q]\Longleftrightarrow q^{REi}\left(Q_t;\gamma_\mu\right)\underset{(<)}{>}q_t^*\left(Q_t;\gamma_\mu\right)\underset{(<)}{>}E_{Q_t}[q].$$

Using this observation, we can now show:

Lemma 18 *Consider a market with a representative agent $i\in\{o;b;p\}$ and finite memory length μ. In such a market, there is a.s. a finite period $\bar{t}$ such that for all $t\geq\bar{t}$, the observable prices lie in the interval $\left[q^{\min i};q^{\max i}\right]$, where $q^{\min i}$ and $q^{\max i}$ are given by:*

- *for a market with a representative optimist:* $q^{\max o}=\frac{\gamma_\mu\bar{q}+\bar{\delta}}{(\gamma_\mu\bar{p}+r)+\left[\gamma_\mu\bar{q}+\bar{\delta}\right]}$, $q^{\min o}=\frac{(1-\gamma_\mu)\underline{\delta}+\gamma_\mu(\bar{q}+\bar{\delta})}{(\gamma_\mu\bar{p}+r)+\left[(1-\gamma_\mu)\underline{\delta}+\gamma_\mu(\bar{q}+\bar{\delta})\right]}$;
- *for a market with a representative Bayesian:* $q^{\max b}=\frac{\bar{\delta}}{\bar{\delta}+r}$, $q^{\min b}=\frac{\underline{\delta}}{\underline{\delta}+r}$;
- *for a market with a representative pessimist:* $q^{\max p}=\frac{\left((1-\gamma_\mu)\bar{\delta}+\gamma_\mu(\underline{q}+\underline{\delta})\right)}{(\gamma_\mu\underline{p}+r)+\left((1-\gamma_\mu)\bar{\delta}+\gamma_\mu(\underline{q}+\underline{\delta})\right)}$, $q^{\min p}=\frac{\gamma_\mu\underline{q}+\underline{\delta}}{(\gamma_\mu\underline{p}+r)+(\gamma_\mu\underline{q}+\underline{\delta})}$.

Furthermore, if (5) holds, then $q^{\max o}>q^{\max b}>q^{\max p}$ and $q^{\min o}>q^{\min b}>q^{\min p}$.

Proof of Lemma 18: Note that $q^{\min i}=q^{REi}\left(\underline{\delta};\gamma_\mu\right)$ and $q^{\max i}=q^{REi}\left(\underline{\delta};\gamma_\mu\right)$ for $i\in\{o;b;p\}$. Suppose that in an economy with a representative agent i, $E_{Q_t}[q]<q^{\min i}$. Then, the fact that $\delta>\underline{\delta}$ w.pr. 1 combined with (10) and (11), implies that $q^i\left(Q_{\bar{t}};\gamma_\mu\right)=q_{\bar{t}}^*>q^{\min i}$ will obtain a.s. in finite time $\bar{t}$ and $q^i\left(Q_t;\gamma_\mu\right)=q_t^*>q^{\min i}$ for all $t>\bar{t}$. Similarly, if $E_{Q_t}[q]>q^{\max i}$, then, the fact that $\delta<\bar{\delta}$ w.pr. 1 combined with (10) and (11), implies that $q^i\left(Q_{\bar{t}};\gamma_\mu\right)=q_{\bar{t}}^*<q^{\max i}$ will obtain a.s. in finite time $\bar{t}$ and $q^i\left(Q_t;\gamma_\mu\right)=q_t^*<q^{\max i}$ for all $t>\bar{t}$. We conclude that $q_t^*\in\left(q^{\min i};q^{\max i}\right)$ will a.s. obtain after a finite number of periods in a market with a representative agent i.

It remains to check the ranking of the prices. Note that $q^{\max o} > q^{\max b}$ is equivalent to $\frac{\gamma_\mu \bar{q}+\bar{\delta}}{(\gamma_\mu \bar{p}+r)+[\gamma_\mu \bar{q}+\bar{\delta}]} > \frac{\bar{\delta}}{\bar{\delta}+r}$, or to $r > \frac{\bar{\delta}}{\bar{q}}\bar{p}$. $q^{\max b} > q^{\max p}$ is equivalent to $\frac{\bar{\delta}}{\bar{\delta}+r} > \frac{((1-\gamma_\mu)\bar{\delta}+\gamma_\mu(\underline{q}+\underline{\delta}))}{(\gamma_\mu \underline{p}+r)+((1-\gamma_\mu)\bar{\delta}+\gamma_\mu(\underline{q}+\underline{\delta}))}$, or to $\bar{\delta}\frac{\underline{p}}{\underline{q}} + \frac{(\bar{\delta}-\underline{\delta})}{\underline{q}}r > r$, which is satisfied whenever $r < \frac{\underline{\delta}}{\underline{q}}\underline{p}$. $q^{\min o} > q^{\min b}$ is equivalent to $\frac{(1-\gamma_\mu)\underline{\delta}+\gamma_\mu(\bar{q}+\bar{\delta})}{(\gamma_\mu \bar{p}+r)+[(1-\gamma_\mu)\underline{\delta}+\gamma_\mu(\bar{q}+\bar{\delta})]} > \frac{\underline{\delta}}{\underline{\delta}+r}$, or to $r > \frac{\underline{\delta}}{\bar{q}}\bar{p} + \frac{r(\underline{\delta}-\bar{\delta})}{\bar{q}}$, which is satisfied whenever $\frac{\underline{\delta}}{\bar{q}}\bar{p} < r$, whereas $q^{\min b} > q^{\min p}$ is equivalent to $\frac{\underline{\delta}}{\underline{\delta}+r} > \frac{\gamma_\mu \underline{q}+\underline{\delta}}{(\gamma_\mu \underline{p}+r)+(\gamma_\mu \underline{q}+\underline{\delta})}$ or to $\frac{\underline{\delta}\underline{p}}{\underline{q}} > r$. Since (5) implies $\frac{\underline{\delta}}{\underline{q}}\underline{p} > r > \frac{\bar{\delta}}{\bar{q}}\bar{p}$, we obtain the desired ranking.■

For an economy, in which all three types of agents are present, Lemma 18 then implies:

Corollary 19 *Consider an economy with finite memory of length μ, in which all three types of agents are present and let (5) hold. There is a.s. a finite period $\bar{t}$ such that for all $t \geq \bar{t}$ the observable prices lie in the interval $[q^{\min p}; q^{\max o}]$.*

According to Corollary 19, we can choose $\bar{t}$ so that for all $t \geq \bar{t}$, $q_t^* \in [q^{\min p}; q^{\max o}]$. We start by showing that for all Q_t which only contain observations of q in this range, $q^o(Q_t) > q^b(Q_t)$. Indeed, this is equivalent to:

(12) $$(\bar{q}+\bar{\delta})[1+r] > E_{Q_t}[q]\left[(\bar{p}+r)+(\bar{q}+\bar{\delta})\right] + E_{Q_t}[\delta](\bar{p}+r)$$

The inequality will thus be satisfied for all observable data sets if it is satisfied for $E_{Q_t}[q] = q^{\max o}$ and $E_{Q_t}[\delta] = \bar{\delta}$ and it is therefore enough to show that $q^{\max o} > q^b(E_{Q_t}[q] = q^{\max o}; E_{Q_t}[\delta] = \bar{\delta})$. This is equivalent to $\frac{\gamma_\mu \bar{q}+\bar{\delta}}{(\gamma_\mu \bar{p}+r)+[\gamma_\mu \bar{q}+\bar{\delta}]} > \frac{\bar{\delta}}{\bar{\delta}+r}$, which, as shown in the proof of Lemma 18 is implied by (5).

Next, we check that $q^b(Q_t) > q^p(Q_t)$, which is equivalent to

(13) $$(\underline{q}+\underline{\delta})(1+r) < E_{Q_t}[q]\left[(\underline{p}+r)+(\underline{q}+\underline{\delta})\right] + E_{Q_t}[\delta](\underline{p}+r)$$

The inequality will thus be satisfied for all observable data sets if it is satisfied for $E_{Q_t}[q] = q^{\min p}$ and $E_{Q_t}[\delta] = \underline{\delta}$ and it is therefore enough to show that $q^{\min p} < q^b(E_{Q_t}[q] = q^{\min p}; E_{Q_t}[\delta] = \underline{\delta})$. This is equivalent to $\frac{\gamma_\mu \underline{q}+\underline{\delta}}{(\gamma_\mu \underline{p}+r)+(\gamma_\mu \underline{q}+\underline{\delta})} < \frac{\underline{\delta}}{\underline{\delta}+r}$, which, as shown in the proof of Lemma 18 is implied by (5).

Next, consider a market with infinite memory, in which $\lim_{\mu\to\infty}\gamma_\mu = \lim_{t\to\infty}\gamma_t = \gamma > 0$. Since $\lim_{t\to\infty} E_{Q_t}[\delta] =$

$E_\rho[\delta]$ a.s., and since $\gamma_t \to \gamma$, we have a.s.,

$$(14) \qquad \lim_{t\to\infty} q^{REi}(Q_t;\gamma_t) = \begin{cases} q^o_\infty =: \frac{(1-\gamma)E_\rho[\delta]+\gamma\left(\bar{q}+\bar{\delta}\right)}{(\gamma\bar{p}+r)+(1-\gamma)E_\rho[\delta]+\gamma\left(\bar{q}+\bar{\delta}\right)} & \text{if } i=o \\ q^p_\infty =: \frac{(1-\gamma)E_\rho[\delta]+\gamma\left(\underline{q}+\underline{\delta}\right)}{\left(\gamma\underline{p}+r\right)+(1-\gamma)E_\rho[\delta]+\gamma\left(\underline{q}+\underline{\delta}\right)} & \text{if } i=p \\ q^b_\infty =: \frac{E_\rho[\delta]}{r+E_\rho[\delta]} & \text{if } i=b \end{cases}.$$

For $\gamma > 0$, $q^o_\infty > q^b_\infty$ is equivalent to $E_\rho[\delta] < \frac{\left(\bar{q}+\bar{\delta}\right)r}{\bar{p}+r}$, which is satisfied, since (5) implies $E_\rho[\delta] \le \bar{\delta} < \frac{\left(\bar{q}+\bar{\delta}\right)r}{\bar{p}+r}$. Similarly, $q^b_\infty > q^p_\infty$ is equivalent to $E_\rho[\delta] > \frac{r\left(\underline{q}+\underline{\delta}\right)}{\underline{p}+r}$, which is satisfied, since (5) implies $E_\rho[\delta] \ge \underline{\delta} > \frac{r\left(\underline{q}+\underline{\delta}\right)}{\underline{p}+r}$. Choose an $\xi > 0$ such that $q^o_\infty - \xi > q^b_\infty + \xi > q^b_\infty - \xi > q^p_\infty + \xi$ and note that by (10) and (11), there is a.s. a finite period $\bar{t}$ such that $q^o(Q_t;\gamma_t) < \omega$, $q^b(Q_t) \in (\omega;\omega+\beta)$ and $q^p(Q_t;\gamma_t) > \omega+\beta$ for all $t \ge \bar{t}$. For a given γ_t, $q^o(Q_t) > q^b(Q_t)$ is equivalent to (12) and $q^b(Q_t) > q^p(Q_t)$ is equivalent to (13). Now note that since $q^o_\infty > q^b_\infty > q^p_\infty$, we have by (10), $q^o_\infty > q^b\left(E_{Q_t}[q] = q^o_\infty; E_{Q_t}[\delta] = E_\rho[\delta]\right) > q^b\left(E_{Q_t}[q] = q^p_\infty; E_{Q_t}[\delta] = E_\rho[\delta]\right) > q^p_\infty$. Hence, we can choose ξ to be sufficiently small so that

$$q^o_t\left(E_{Q_t}[q] = q^o_\infty + \xi; E_{Q_t}[\delta] = E_\rho[\delta] + \xi; \gamma_t\right) > q^b\left(E_{Q_t}[q] = q^o_\infty + \xi; E_{Q_t}[\delta] = E_\rho[\delta] + \xi\right)$$

$$q^b\left(E_{Q_t}[q] = q^p_\infty - \xi; E_{Q_t}[\delta] = E_\rho[\delta] - \xi\right) > q^p_\infty\left(E_{Q_t}[q] = q^p_\infty - \xi; E_{Q_t}[\delta] = E_\rho[\delta] - \xi; \gamma_t\right)$$

hold for all values of $\gamma_t < \xi$. Hence, for all $t \ge \bar{t}$, reservation prices $q^i_t(Q_t;\gamma_t)$ are ranked as required.

Once the order of the reservation prices has been established, obtaining the temporary equilibrium is straightforward.■

Proof of Proposition 5: Consider first an economy with a representative agent of type i. Since $\lim_{t\to\infty} E_{Q_t}[\delta] = E_\rho[\delta]$ a.s., and since $\gamma_t \to \gamma$, we obtain that (14) a.s. holds. By (11), it follows that a.s., $\lim_{t\to\infty} q^*_t(Q_t;\gamma_t) = q^i_\infty$. As shown in the proof of Lemma 3, $q^o_\infty > q^b_\infty > q^p_\infty$ obtains for $\gamma > 0$ (for $\gamma = 0$, $q^o_\infty = q^b_\infty = q^p_\infty$).

Now suppose that the shares of investor types ω, β and σ are arbitrary. Suppose that ω and β satisfy one of the 5 conditions listed in the statement of the Proposition with strict inequalities. Note that for $\epsilon > 0$, on almost every path, Q, there is a period $\bar{t}(Q)$ such that $E_{Q_t}[\delta] \in [E_\rho[\delta]-\epsilon; E_\rho[\delta]+\epsilon]$, $E_{Q_t}[q] \in [q^p_\infty - \epsilon; q^o_\infty + \epsilon]$ and $\gamma_t < \gamma + \epsilon$ for all $t \ge \bar{t}(Q)$. Choose ϵ to be sufficiently small so that for all $t \ge \bar{t}(Q)$, $q^o(Q_t;\gamma_t) > q^b(Q_t) > q^p(Q_t;\gamma_t)$, $q^{REi}(Q_t;\gamma_t) \in \left(q^i_\infty - \epsilon; q^i_\infty + \epsilon\right)$ holds and so that

(i)if $\omega > q^o_\infty$, for $t \ge \bar{t}(Q)$, $\omega > q^{REo}(Q_t;\gamma_t)$

(ii)if $q^o_\infty > \omega > q^b_\infty$, for $t \ge \bar{t}(Q)$, $q^{REo}(Q_t;\gamma_t) > \omega > q^{REb}(Q_t)$;

(iii)if $q^b_\infty \in (\omega;\omega+\beta)$, for $t \ge \bar{t}(Q)$, $q^{REb}(Q_t) \in (\omega;\omega+\beta)$;

(iv)if $q^b_\infty > \omega + \beta > q^p_\infty$, for $t \geq \bar{t}(Q)$, $q^{REb}(Q_t) > \omega + \beta > q^{REp}(Q_t; \gamma_t)$;

(v)if $q^p_\infty > \omega + \beta$, for $t \geq \bar{t}(Q)$, $q^{REp}(Q_t; \gamma_t) > \omega + \beta$.

In cases (i), (iii) and (v), respectively, if for some $t \geq \bar{t}(Q)$, $q^*_t = q^i(Q_t; \gamma_t)$ for $i = o$, b, p, respectively, then

(15) $$q^o(Q_t; \gamma_t) < \omega, \text{ or } q^b(Q_t) \in (\omega; \omega + \beta), \text{ or } q^p(Q_t; \gamma_t) > \omega + \beta,$$

respectively. Assume first that $E_{Q_t}[q] \notin (q^i_\infty - \epsilon; q^i_\infty + \epsilon)$. By (14), for a given $q^{REi}(Q_t; \gamma_t)$, the equilibrium price $q^*_t = q^i(Q_t; \gamma_t)$ will lie between $E_{Q_t}[q]$ and $q^{REi}(Q_t; \gamma_t)$, and hence, $E_{Q_{t+1}}[q]$ will lie between $E_{Q_t}[q]$ and the interval $(q^i_\infty - \epsilon; q^i_\infty + \epsilon)$, or $E_{Q_{t+1}}[q] \in (q^i_\infty - \epsilon; q^i_\infty + \epsilon)$. Hence, $q^i(Q_{t+1}; \gamma_{t+1})$ will lie between $q^i(Q_t; \gamma_t)$ and the interval $(q^i_\infty - \epsilon; q^i_\infty + \epsilon)$ or in the interval $(q^i_\infty - \epsilon; q^i_\infty + \epsilon)$. Now since $q^i(Q_t; \gamma_t)$ satisfies the corresponding condition in (15), which ensures that $q^*_t = q^i(Q_t; \gamma_t)$, then by the choice of ϵ above, so does $q^i(Q_{t+1}; \gamma_{t+1})$ and hence, $q^*_{t+1} = q^i(Q_{t+1}; \gamma_{t+1})$. Suppose now that $E_{Q_t}[q] \in (q^i_\infty - \epsilon; q^i_\infty + \epsilon)$, then $q^*_t = q^i(Q_t; \gamma_t)$ will lie between $E_{Q_t}[q]$ and $q^{REi}(Q_t; \gamma_t)$, or also in the interval $(q^i_\infty - \epsilon; q^i_\infty + \epsilon)$ and so will $q^i(Q_t; \gamma_t)$. Hence, if at a $t' \geq \bar{t}(Q)$, $q^*_{t'} = q^i(Q_{t'}; \gamma_{t'})$, then $q^*_t = q^i(Q_t; \gamma_t)$ for all $t \geq t'$. It remains to show that such a period will eventually be reached. Suppose not, and let q^i_∞ not belong to the interval $\left(q^j_\infty; q^k_\infty\right)$, i.e., condition (i) or condition (v) holds. If j and k are the only ones who set prices in the economy, there will be a.s. a time $\tilde{t}(Q)$ such that for all $t \geq \tilde{t}(Q)$, $q^*_t \in \left(q^j_\infty - \epsilon; q^k_\infty + \epsilon\right)$. Since however the reservation prices of the three types of investors are strictly ordered and since condition (i) or (v) is satisfied, we know that the economy cannot have an equilibrium in this price range. Hence, there is a.s. a finite period $t' \geq \bar{t}(Q)$, such that $q^*_t = q^i(Q_t; \gamma_t)$ for all $t \geq t'$. Then, by (11), $\lim_{t\to\infty} q^*_t = \lim_{t\to\infty} q^i(Q_t; \gamma_t) = q^i_\infty$.

If condition (iii) holds, and hence, $\beta > 0$, the same argument as above can be used to show that optimists or pessimists alone cannot set prices in each period. Consider a period in which the economy switches from an optimistic state, $q^p(Q_t; \gamma_t) < q^*_t = q^o(Q_t; \gamma_t) < \omega$ to a pessimistic one, $q^*_{t+1} = q^p(Q_{t+1}; \gamma_{t+1}) > \omega + \beta$. Note however, that since the price and dividend realizations are bounded, if t is sufficiently large, $\left|q^p(Q_t; \gamma_t) - q^p(Q_{t+1}; \gamma_{t+1})\right|$ will not exceed β, a contradiction.

Next consider cases (ii) and (iv). We will show the argument for case (iv) — the one for (ii) is analogous. Note that whenever $q^b(Q_t; \gamma_t) \geq \omega + \beta$, $q^*_t = \omega + \beta$ and $q^b(Q_{t+1}; \gamma_{t+1}) \geq \omega + \beta$. If $q^*_t = q^b(Q_t; \gamma_t) < \omega + \beta$, the argument above shows that $q^*_{t+1} = q^b(Q_{t+1}; \gamma_{t+1})$ and $q^b(Q_t; \gamma_t)$ eventually converges to $\omega + \beta$. We can also replicate the arguments used above to show that optimists and pessimists cannot determine prices indefinitely, and hence, a period t with $q_t^* = q^b(Q_t; \gamma_t)$ will eventually be reached.

Finally, note that for $\omega + \beta = q_\infty^b$, the same argument as the one made for case (iv) applies. Thus, the cases $\omega = q_\infty^o$, $\omega = q_\infty^b$ and $\omega + \beta = q_\infty^p$ can be treated analogously.■

Proof of Proposition 7: Follows from combining the results of Corollary 19 and Lemma 3.

Proof of Proposition 8: Consider first the case of $\mu = 1$.

Claim 18: For $\mu = 1$, q_t^* is a scalar nonlinear state space model and, therefore, also a T-chain.

Proof of Claim 18: For $i \in \{o; b; p\}$ as in the statement of the Proposition, define $F^i\left(q_{t-1}^*; \delta_{t-1}\right) = q_t^i\left(q_{t-1}^*; \delta_{t-1}; \gamma\right) = q_t^*\left(q_{t-1}^*; \delta_{t-1}; \gamma\right)$. Since q_t^i is smooth, and the variable $(\delta_t)_t$ is i.i.d. with a density supported on an open set, F^i satisfies the conditions of a scalar nonlinear state space model (SNSS1) and (SNSS2) in Meyn and Tweedy (1993, p. 30). We now check that the Rank Condition for the Scalar CM(F) Model, (CM2), Meyn and Tweedy (1993, p. 155) is satisfied. As noted in the text, all F^i are strictly increasing in δ for any q_{t-1}^*. Finally, the condition that the distribution of δ_t is absolutely continuous, while the density of δ_t is lower-semicontinuous (SNSS3), Meyn and Tweedy (1993, p. 156) is satisfied as well. We can thus conclude, by Proposition 7.1.2., Meyn and Tweedy (1993, p. 156), that F^i is a T-chain.■

Claim 19: For $\mu = 1$, the Markov process q_t^* has an invariant set given by $\left[q^{\min i}; q^{\max i}\right]$.

Proof of Claim 19: Take an initial price $q_0 \notin \left[q^{\min i}; q^{\max i}\right]$. If $q_0 < q^{\min i}$, or $q_0 > q^{\max i}$, then the probability that the process reaches the set $\left[q^{\min i}; q^{\max i}\right]$ is 1. To see this, consider $F^i(q; \delta)$ and let $q^i(\delta) =: F^i\left(q^i(\delta); \delta\right)$. If $q \lesseqgtr q^i(\delta)$, $q^i(\delta) \gtreqless F^i(q; \delta) \gtreqless q$. If $q_0 < q^{\min i}$, since $\delta > \underline{\delta}$ with probability 1, it follows that with probability 1, the process eventually exceeds $q^{\min i}$. Symmetrically, if the process starts above $q^{\max i}$, since $\delta < \bar{\delta}$ with probability 1, eventually the process crosses $q^{\max i}$ from above.

Note, however, that once the process has reached the set $\left[q^{\min i}; q^{\max i}\right]$, the probability of leaving this set is 0. Since, by definition, for $q \in \left[q^{\min i}; q^{\max i}\right]$, $q^{\max i} = F^i\left(q^{\max i}; \bar{\delta}\right) \geq F^i(q; \delta) \geq F^i\left(q^{\min i}; \underline{\delta}\right) = q^{\min i}$, it follows that all states $q_0 \notin \left[q^{\min i}; q^{\max i}\right]$ are transient and hence, that the set $\left[q^{\min i}; q^{\max i}\right]$ is invariant.■

Claim 20: For $\mu = 1$, q_t^* is ψ-irreducible, positive recurrent and has an invariant probability distribution π_1^i with support $\left[q^{\min i}; q^{\max i}\right]$.

Proof of Claim 20: As we saw above, F^i is a T-chain on this set and furthermore, for any $q_0 \in \left[q^{\min i}; q^{\max i}\right]$, there is a positive probability that the process reaches any open set $L \subset \left[q^{\min i}; q^{\max i}\right]$. To see this, let $\left(\check{\delta}; \hat{\delta}\right)$ be such that $L = (\check{q}; \hat{q})$, where $\check{q} =: F\left(\check{q}; \check{\delta}\right)$ and $\hat{q} =: F\left(\hat{q}; \hat{\delta}\right)$. For a given q_0 and a given positive ϵ, let T denote the number of periods such that if $\delta_t \in \left(\check{\delta} + \epsilon; \hat{\delta} - \epsilon\right)$ for all $t \leq T$, then $q_T \in (\check{q}; \hat{q})$. It is obvious that T is finite, and, hence, the probability of reaching L starting from q_0 is strictly positive. By Theorem 6.0.1 in Meyn and Tweedy (1993, p. 131), F^i is ψ-irreducible on $\left[q^{\min i}; q^{\max i}\right]$ and this set is petite. Hence, by Theorem 8.3.6. in Meyn and Tweedy (1993, p. 191), the process is positive recurrent. This, in turn implies the existence of an invariant measure π_1^i and since there is a petite set $\left[q^{\min i}; q^{\max i}\right]$ such that the expected time of reaching this set starting from any element in this set is finite, the measure π_1^i is finite. It assigns a probability of 1 to $\left[q^{\min i}; q^{\max i}\right]$.■

Claim 21: For $\mu > 1$, $q_t = F^i\left(q_{t-1}...q_{t-\mu}; \delta_{t-1}...\delta_{t-\mu}\right)$ is a Nonlinear Autoregressive-Moving Average Model and the associated Markov process is a T-chain.

Proof of Claim 21: For $\mu > 1$, q_t can be written as a Nonlinear Autoregressive-Moving Average Model, see Meyn and Tweedy (1993, p. 34): $q_t = F^i\left(q_{t-1}...q_{t-\mu}; \delta_{t-1}...\delta_{t-\mu}\right)$ and F^i is a smooth function, whereas $(\delta_t) \sim i.i.d.$ Defining the state variable $x_t =: \left(q_t...q_{t-\mu+1}; \delta_{t-1}...\delta_{t-\mu+1}\right)'$, we obtain the Markov process x on $\mathbb{R}^\mu$, which has the form of a nonlinear state space model with an associated control model $q_t = F\left(x_{t-1};; \delta_t...\delta_{t-\mu+1}\right)$. We can thus apply the same reasoning as above to show that the states, in which $q_{t-k} \notin \left[q^{\min i}; q^{\max i}\right]$ for some $k \in \{0...\mu-1\}$ are transient. Just as above, the process almost surely reaches the set $(q_t...q_{t-\mu+1}) \in \left[q^{\min i}; q^{\max i}\right]^\mu$ and never leaves it afterwards. Defining $(q_\mu...q_1)$ recursively as a function of q_0 and $(\delta_0...\delta_{\mu-1})$ and taking the $\mu \times \mu$ matrix of the derivatives of $\left(\frac{\partial q_k}{\partial \delta_m}\right)_{k,m}$ with respect to δ, we obtain that for all q_0, $\left(\frac{\partial q_k}{\partial \delta_k}\right)_{k,k} > 0$ for all k, $\left(\frac{\partial q_k}{\partial \delta_m}\right)_{k,m} = 0$ for all $m > k$, and hence, $Det\left(\left(\frac{\partial q_k}{\partial \delta_m}\right)_{k,m}\right) > 0$, as required by the Rank condition (CM3) in Meyn and Tweedy (1993, p. 160). Hence, we can apply Proposition 7.1.4. in Meyn and Tweedy (1993, p. 160) to obtain that the Markov process x_t is a T-chain.■

Claim 22: For $\mu > 1$, and any $q \in \left[q^{\min i}; q^{\max i}\right]$, $\delta \in \left(\underline{\delta}; \bar{\delta}\right)$ satisfying $q = F^i\left(\underbrace{(q..q)}_{\mu\text{-times}}; \underbrace{\delta...\delta}_{(\mu-1)\text{-times}}\right)$ the state $\left(\underbrace{(q..q)}_{\mu\text{-times}}; \underbrace{\delta...\delta}_{(\mu-1)\text{-times}}\right)$ is a global attractor.

Proof of Claim 22: For every initial $\left(q_1'...q_\mu'; \delta_1'...\delta_\mu'\right)$ and every $\xi > 0$ there is a finite T and $\epsilon > 0$ such that if $\delta_{\mu+1}...\delta_{\mu+T} \in (\delta - \epsilon; \delta + \epsilon)$, then $q_{\mu+T} \in (q - \xi; q + \xi)$. Since the probability of this event is possible, the state $\left(\underbrace{(q..q)}_{\mu\text{-times}}\ \underbrace{\delta...\delta}_{(\mu-1)\text{-times}}\right)$ is accessible in finite time from any initial state, and is, hence, a global attractor.■

Claim 23: For $\mu > 1$, the Markov process $(q_t...q_{t-\mu+1};\delta_{t-1}...\delta_{t-\mu+1})$ is ψ-irreducible, positive recurrent and has an invariant probability distribution $\tilde{\pi}^i_\mu$.

Proof of Claim 23: Since the Markov process has a globally attracting state, by Proposition 7.2.5 and Theorem 7.2.6 in Meyn and Tweedy (1993, p. 164), it is ψ-irreducible on the set $\bar{S}$ reachable from $\left\{\left(\underbrace{(q..q)}_{\mu\text{-times}}\ \underbrace{\delta...\delta}_{(\mu-1)\text{-times}}\right) \mid q \in \left[q^{\min i}; q^{\max i}\right]\right\}$. By Proposition 7.2.5, $\bar{S}$ is the unique minimal set of the chain, this set is compact and hence, by Theorem 6.0.1, in Meyn and Tweedy (1993, p. 131), it is petite. Just as above, it follows that the process is positive recurrent and has a finite invariant measure $\tilde{\pi}^i_\mu$ and its support is $\bar{S} \times \left[\underline{\delta};\bar{\delta}\right]^\mu$.■

Claim 24: For $\mu > 1$, the marginals of the invariant measure $\tilde{\pi}^i_\mu$ of all components of the price vector are identical and have a support $\left[q^{\min i}; q^{\max i}\right]$.

Proof of Claim 24: By the definition of an invariant measure, we have:that for a subset S of $\bar{S}$,

$$\tilde{\pi}^i_\mu(S) = \int\int \tilde{\pi}^i_\mu\left(d\left(q'_1,q'_2...q'_\mu;\delta'_1,\delta'_2...\delta'_\mu\right)\right)\Phi\left\{S \mid q'_1,q'_2...q'_\mu;\delta'_1,\delta'_2...\delta'_\mu\right\}dq'd\delta'$$

$$= \int_{(q'_1;\delta'_1)}\int_{\left(q'_2...q'_\mu;\delta'_2...\delta'_\mu\right)\in S_{\mu-1}} \tilde{\pi}^i_\mu d\left(q'_1,q'_2...q'_\mu;\delta'_1,\delta'_2...\delta'_\mu\right) \cdot\Phi\left\{S \mid q'_1,q'_2...q'_\mu;\delta'_1,\delta'_2...\delta'_\mu\right\}d\left(q'_2...q'_\mu;\delta'_2...\delta'_\mu\right)d\left(q'_1;\delta'_1\right)$$

where Φ is the transition probability of the Markov process, $S_{\mu-1}$ is the projection of S to $\left[q^{\min i}; q^{\max i}\right]^{\mu-1} \times \left[\underline{\delta};\bar{\delta}\right]^{\mu-1}$ and the second inequality follows from the fact that $\Phi\left\{S \mid q'_1,q'_2...q'_\mu;\delta'_1,\delta'_2...\delta'_\mu\right\} = 0$ if $\left(q'_2...q'_\mu;\delta'_1,\delta'_2...\delta'_\mu\right) \notin S_{\mu-1}$.

Let S_1 be the projection of S to $\left[q^{\min i}; q^{\max i}\right]$ according to the first component of the price vector. The marginal of the first component of the price vector, $\pi^i_\mu(S_1)$, is given by:

$$\pi^i_\mu(S_1) = \int_{(q_3...q_{\mu+1};\delta_2...\delta_{\mu+1})}\int_{q\in S_1}\tilde{\pi}^i_\mu d\left(q;q_3...q_{\mu+1};\delta_2...\delta_{\mu+1}\right)$$

$$= \int_{(q'_1;\delta'_1)}\int_{\left(q'_3...q'_\mu;\delta'_2...\delta'_\mu\right)}\int_{\left(q'_{\mu+1};\delta'_{\mu+1}\right)}\int_{q\in S_1}\tilde{\pi}^i_\mu d\left(q'_1,q,q'_3...q'_\mu;\delta'_1,\delta'_2...\delta'_\mu\right) \cdot\Phi\left\{S_1 \times \left[q^{\min i}; q^{\max i}\right]^{\mu-1} \times \left[\underline{\delta};\bar{\delta}\right]^\mu \mid q'_1,q,q'_3...q'_\mu;\delta'_1,\delta'_2...\delta'_\mu\right\}$$

$$= \int_{(q_1';\delta_1')} \int_{\left(q_3'...q_\mu';\delta_2'...\delta_\mu'\right)} \int_{q\in S_1} \tilde{\pi}_\mu^i d\left(q_1', q, q_3'...q_\mu'; \delta_1', \delta_2'...\delta_\mu'\right)$$
$$\int_{\left(q_{\mu+1}';\delta_{\mu+1}'\right)} \Phi\left\{S_1 \times \left[q^{\min i}; q^{\max i}\right]^{\mu-1} \times \left[\underline{\delta}; \bar{\delta}\right]^\mu \mid q_1', q, q_3'...q_\mu'; \delta_1', \delta_2'...\delta_\mu'\right\} d\left(q_{\mu+1}'; \delta_{\mu+1}'\right)$$
$$= \int_{(q_1';\delta_1')} \int_{\left(q_3'...q_\mu';\delta_2'...\delta_\mu'\right)} \int_{q\in S_1} \tilde{\pi}_\mu^i d\left(q_1', q, q_3'...q_\mu; \delta_1', \delta_2'...\delta_\mu'\right)$$

where the last line is the marginal of the second component of the price vector and the last equality follows from the fact that for $q \in S_1$,

$$\int_{\left(q_{\mu+1}';\delta_{\mu+1}'\right)} \Phi\left\{S_1 \times \left[q^{\min i}; q^{\max i}\right]^{\mu-1} \times \left[\underline{\delta}; \bar{\delta}\right]^\mu \mid q_1', q, q_3'...q_\mu'; \delta_1', \delta_2'...\delta_\mu'\right\} d\left(q_{\mu+1}'; \delta_{\mu+1}'\right) = 1$$

Hence, the marginals of the first and the second component of the price vector coincide and reproducing the same argument, so do the marginals of all μ components of the price vector and hence, the distributions of q_1, $q_2...q_\mu$ are identical and have a support $\left[q^{\min i}; q^{\max i}\right]$ ■

Proof of Proposition 9: In the case of $\mu = 1$, the process q_t^* is itself Markov. Hence, according to the definition of the invariant probability derived in Proposition 8, we have: $\pi^i(S) = \int \pi(dq)\,\Phi(S \mid q)$, or $\int q\pi^i(dq) = \int q \int \pi^i(dq')\,\Phi(dq \mid q')$, or $E_{\pi^i}[q] = \int q^i(E_{\pi^i}[q]; \delta)\,\rho'(\delta)\,d\delta$. .Substituting for $q^i(\cdot)$ for $i \in \{o; b; p\}$ and taking the integral gives the result of the Proposition.■

Proof of Proposition 11: The existence of data sets Q_t as in the statement of the Proposition is obvious given the result of Lemma 3. It remains to check that $q^{\min o} < q^{\max p}$. This is equivalent to $\gamma^2 \underline{p}\bar{q} - \gamma^2 \bar{p}\underline{q} + r\gamma\left(\bar{q} - \underline{q}\right) < \gamma\bar{p}\bar{\delta} - \gamma\underline{p}\underline{\delta} + r(1-2\gamma)\left[\bar{\delta} - \underline{\delta}\right]$, which obtains when γ is sufficiently small.

Consider the process $\left(q_t^*...q_{t-\mu+1}^*; \delta_{t-1}...\delta_{t-\mu+1}\right)$. Using the market equilibrium conditions, we can write q_t^* as continuous functions of $\left(q_{t-1}^*...q_{t-\mu}^*; \delta_{t-2}...\delta_{t-\mu}\right)$ and δ_{t-1}. Since δ_t is i.i.d., it follows that the process $\left(q_t^*...q_{t-\mu+1}^*; \delta_{t-1}...\delta_{t-\mu+1}\right)$ is Markov and it obviously describes the state of the economy at time t. Furthermore, the process satisfies the conditions of the NSS(F) model introduced in Chapter 7 of Meyn and Tweedy (1993), except for the fact that the function F defined by the equilibrium condition derived in Lemma 3, $q_t^* = F\left(\left(q_{t-1}^*...q_{t-\mu}^*; \delta_{t-2}...\delta_{t-\mu}\right); \delta_{t-1}\right)$ is not smooth. Hence, we will have to first show that the Markov process is a T-chain, which will then allow us to use the methods in Chapter 7 of Meyn and Tweedy (1993) to demonstrate that it is ψ-irreducible and that it has an invariant distribution.

Claim 25: The Markov process $\left(q_t^*...q_{t-\mu+1}^*;\delta_{t-1}...\delta_{t-\mu+1}\right)$ is a T-chain.

Proof of Claim 25: To reach q_t^* from a given initial state $\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right)$, $\delta_{t-1}\left(\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right);q_t^*\right)$ has to satisfy:

$$\delta_{t-1}\left(\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right);q_t^*\right)\in$$

$$\in\begin{cases}\left\{\mu\frac{q_t^*[(1-\gamma)+(\gamma\bar{p}+r)]-(1-q_t^*)\left[(1-\gamma)\sum_{k=2}^{\mu}\frac{\delta_{t-k}}{\mu}+\gamma\left(\bar{q}+\bar{\delta}\right)\right]-(1-\gamma)\sum_{k=1}^{\mu}\frac{q_{t-k}}{\mu}}{(1-\gamma)(1-q_t^*)}\right\} & \text{if}\quad q_t^*<\omega\\ \begin{array}{c}[\mu\frac{q_t^*(r+1)-(1-q_t^*)\sum_{k=2}^{\mu}\frac{\delta_{t-k}}{\mu}-\sum_{k=1}^{\mu}\frac{q_{t-k}}{\mu}}{1-q_t^*};\\ \mu\frac{q_t^*[1-\gamma+\gamma\bar{p}+r]-(1-q_t^*)\left[(1-\gamma)\sum_{k=2}^{\mu}\frac{\delta_{t-k}}{\mu}+\gamma\left(\bar{q}+\bar{\delta}\right)\right]-(1-\gamma)\sum_{k=1}^{\mu}\frac{q_{t-k}}{\mu}}{(1-\gamma)(1-q_t^*)}]\end{array} & \text{if}\quad q_t^*=\omega\\ \left\{\mu\frac{q_t^*(r+1)-(1-q_t^*)\sum_{k=2}^{\mu}\frac{\delta_{t-k}}{\mu}-\sum_{k=1}^{\mu}\frac{q_{t-k}}{\mu}}{1-q_t^*}\right\} & \text{if}\quad q_t^*\in(\omega;\omega+\beta)\\ \begin{array}{c}[\mu\frac{q_t^*(r+1)-(1-q_t^*)\sum_{k=2}^{\mu}\frac{\delta_{t-k}}{\mu}-\sum_{k=1}^{\mu}\frac{q_{t-k}}{\mu}}{1-q_t^*};\\ \mu\frac{q_t^*\left[(1-\gamma)+\left(\gamma\underline{p}+r\right)\right]-(1-q_t^*)\left[(1-\gamma)\sum_{k=2}^{\mu}\frac{\delta_{t-k}}{\mu}+\gamma\left(\underline{q}+\underline{\delta}\right)\right]-(1-\gamma)\sum_{k=1}^{\mu}\frac{q_{t-k}}{\mu}}{(1-\gamma)(1-q_t^*)}]\end{array} & \text{if}\quad q_t^*=\omega+\beta\\ \left\{\mu\frac{q_t^*\left[(1-\gamma)+\left(\gamma\underline{p}+r\right)\right]-(1-q_t^*)\left[(1-\gamma)\sum_{k=2}^{\mu}\frac{\delta_{t-k}}{\mu}+\gamma\left(\underline{q}+\underline{\delta}\right)\right]-(1-\gamma)\sum_{k=1}^{\mu}\frac{q_{t-k}}{\mu}}{(1-\gamma)(1-q_t^*)}\right\} & \text{if}\quad q_t^*>\omega+\beta\end{cases}$$

While $\delta_{t-1}(\cdot)$ is a correspondence of q_t^*, for a given q_t^*, $\delta_{t-1}(\cdot)$ is either a continuous function of the initial state $\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right)$, or an interval, whose endpoints are continuous functions of the initial state. Consider a set $S\subseteq\left(\underline{q};\bar{q}\right)\times\left(\underline{\delta};\bar{\delta}\right)$. Let $S^Q=(\check{q};\hat{q})$ be the projection of S on $\left(\underline{q};\bar{q}\right)$ and let S^δ be its projection on $\left(\underline{\delta};\bar{\delta}\right)$. For a given initial state of the system $\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right)$, the probability of reaching $q_t^*\in S$ is given by: $\Pr\left\{\delta_{t-1}=\delta_{t-1}\left(\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right);q_t^*\right)\text{ for some }(\delta_{t-1};q_t^*)\in S\right\}$. Note that q_t^* is an increasing function of δ_{t-1}. Let

$$\check{\delta}_{t-1}\left(S;\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right)\right)=\min\left\{\delta_{t-1}\left(\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right);\check{q}\right)\cap S^\delta\right\}$$

denote the lowest δ_{t-1} such that S is reached starting from the initial state and let $\hat{\delta}\left(S;\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right)\right)$ stand for the maximal δ_{t-1} for which S is reached:

$$\hat{\delta}_{t-1}\left(S;\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right)\right)=\max\left\{\delta_{t-1}\left(\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right);\hat{q}\right)\cap S^\delta\right\}\text{ and}$$

$$\Pr\left\{\delta_{t-1}=\delta_{t-1}\left(\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right);q_t^*\right)\text{ for some }q_t^*\in S\right\}=\int_{\check{\delta}_{t-1}\left(S;\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right)\right)}^{\hat{\delta}_{t-1}\left(S;\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right)\right)}\rho'(\delta)\,d\delta$$

Since for a given S, the endpoints of the interval of integration are continuous functions of the initial state $\left(q_{t-1}^*...q_{t-\mu}^*;\delta_{t-2}...\delta_{t-\mu}\right)$, we obtain that the probability to reach a state[22] $\left(q_t^*;q_{t-1}^*...q_{t-\mu}^*;\delta_{t-1}...\delta_{t-\mu+1}\right)$ such

[22] Note that all other states with values of $(q_{t-1}...q_{t-\mu+1};\delta_{t-2}...\delta_{t-\mu+1})$ different from those in the initial state are

that $(q_t^*; \delta_{t-1}) \in S$ is continuous in $\left(q_{t-1}^* ... q_{t-\mu}^*; \delta_{t-2} ... \delta_{t-\mu}\right)$. Hence, the process is a T-chain.■

Claim 26: The NSS(F) model defined by the process $\left(q_t^* ... q_{t-\mu+1}^*; \delta_{t-1} ... \delta_{t-\mu+1}\right)$ is forward accessible.

Proof of Claim 26: For any initial condition $\left(q_{t-1}^* ... q_{t-\mu}^*; \delta_{t-2} ... \delta_{t-\mu}\right)$ and for $\epsilon > 0$, there is a sufficiently long sequence of dividends $(\delta_t ... \delta_K ... \delta_{K+\mu-2}) \in (\underline{\delta}; \underline{\delta} + \epsilon)$ such that $q_K^* ... q_{K+\mu-1}^* \in \left(q^{\min o}; q^{\min o} + \xi\right)$ where ξ is such that $q^{\min o} + \xi < \frac{\omega}{A}$. Choose $\left(q_{K+\mu-1}^* ... q_K^* ... \delta_{K+\mu-2} ... \delta_K\right)$ as an initial condition and observe that restricting $\delta \in (\underline{\delta}; \underline{\delta} + \epsilon)$ and, thus, $q \in \left(q^{\min o}; q^{\min o} + \xi\right)$, we obtain a system with a smooth function F, which satisfies the rank condition on p. 160 of Meyn and Tweedy (1993) (see the proof of Proposition 7). Starting from such an initial condition, the set of accessible states is open and since such an initial condition can be reached from any initial point of the original system, forward accessibility obtains.■

Claim 27: The Markov process $\left(q_t^* ... q_{t-\mu+1}^*; \delta_{t-1} ... \delta_{t-\mu+1}\right)$ is ψ-irreducible.

Proof of Claim 27: The show ψ-irreducibility, we will find a globally attracting state, which by Proposition 7.2.5 in Meyn and Tweedy (1993, p. 164), implies M-irreducibility, which in turn (see Theorem 7.2.6 in Meyn and Tweedy (1993, p. 164) implies ψ-irreducibility. Choose δ so that $\frac{\delta}{\delta + r} \in (\omega; \omega + \beta)$ and consider the state $\left(q_t^* = \frac{\delta}{\delta + r} = ... = q_{t-\mu+1}^*; \delta_{t-1} = \delta = ... \delta_{t-\mu+1}\right)$. This state is reachable from any initial state through an infinite sequence $(\delta ... \delta ...)$ and is thus, globally attracting. Thus, ψ-irreducibility obtains.

Claim 28: Let $\mu = 1$. Then the set $\left[q^{\min o}; q^{\max p}\right]$ is recurrent.

Proof of Claim 28: Just as in the proof of Proposition 8, it is easy to see that any $q \notin \left[q^{\min o}; q^{\max p}\right]$ is transient. We show that the set $\left[q^{\min o}; q^{\max p}\right]$ is recurrent. Let $(\hat{q} - \epsilon; \hat{q} + \epsilon) \subset \left[q^{\min o}; q^{\max p}\right]$. Choose a $q_0 \in \left[q^{\min o}; q^{\max p}\right]$ and ϵ sufficiently small so that one of the following cases holds:

- If $\omega \geq \hat{q} + \epsilon$, let $\delta^o(\hat{q})$ be defined as $\hat{q} = q^o(\hat{q}; \delta^o(\hat{q}))$. Then, there is a $T(\epsilon)$ such that starting from q_0 upon observing $(\delta^o(\hat{q}) - \xi; \delta^o(\hat{q}) + \xi)$ for $T(\epsilon)$ periods in a row, $q_t = q^o(Q_t) \in (\hat{q} - \epsilon; \hat{q} + \epsilon)$.
- If $\omega < \hat{q} - \epsilon < \hat{q} + \epsilon \leq \omega + \beta$, let $\delta^b(\hat{q})$ be defined as $\hat{q} = q^b\left(\hat{q}; \delta^b(\hat{q})\right)$. Then, there is a $T(\epsilon)$ such that starting from q_0 upon observing $\left(\delta^b(\hat{q}) - \xi; \delta^b(\hat{q}) + \xi\right)$ for $T(\epsilon)$ periods in a row, $q_t = q^b(Q_t) \in (\hat{q} - \epsilon; \hat{q} + \epsilon)$.
- If $\hat{q} - \epsilon > \omega + \beta$, let $\delta^p(\hat{q})$ be defined as $\hat{q} = q^p(\hat{q}; \delta^p(\hat{q}))$. In this case, there is a $T(\epsilon)$ such that starting from q_0 upon observing $(\delta^p(\hat{q}) - \xi; \delta^p(\hat{q}) + \xi)$ for $T(\epsilon)$ periods in a row, $q_t = q^{p,h}(Q_t) \in (\hat{q} - \epsilon; \hat{q} + \epsilon)$.

reached with probability 0.

We can thus select ψ to be the Lebesque measure on $\left[q^{\min o}; q^{\max p}\right]$. From Proposition 8.3.1 (ii) in Meyn and Tweedy (1993, p. 188), it follows that $\left[q^{\min o}; q^{\max p}\right]$ is indeed recurrent.

Claim 29: Let $\mu = 1$. Then the Markov process q_t^* is positive recurrent and has an invariant probability distribution π with support $\left[q^{\min o}; q^{\max p}\right]$.

Proof of Claim 29: Proposition 9.1.1 in Meyn and Tweedy (1993, p. 206) implies that q_t^* is Harris recurrent and positive and hence, has an invariant measure π. By Theorem 5.2.2 in Meyn and Tweedy (1993, p. 112), we know that there is a subset $C \subset \left[q^{\min o}; q^{\max p}\right]$, which is small and, hence, petite. Using the same argument as the one used to show that $\left[q^{\min o}; q^{\max p}\right]$ is recurrent, we can show that the first return time to C starting from C is finite. Hence, by Theorem 10.4.10 (ii) in Meyn and Tweedy (1993, p. 254), the measure π is positive and has a support $\left[q^{\min o}; q^{\max p}\right]$.■

Claim 30: Let $\mu > 1$. The the Markov process $\left(q_t^* ... q_{t-\mu+1}^*; \delta_{t-1} ... \delta_{t-\mu+1}\right)$ is ψ-irreducible and positive recurrent and has an invariant probability distribution $\tilde{\pi}$ with support $\left[q^{\min o}; q^{\max p}\right]^{\mu} \times \left[\underline{\delta}; \bar{\delta}\right]^{\mu}$. All marginal distributions of the price vector are equal and their support is $\left[q^{\min o}; q^{\max p}\right]$.

Proof of Claim 30: For $\mu > 1$, we can use the same arguments as for the case of $\mu = 1$, to construct for each $\hat{q}$ a corresponding $\hat{\delta}$ by choosing the appropriate case in the proof of Claim 28. Then, there is an $\epsilon_{\hat{q}}$ such that the sets $(\hat{q} - \epsilon; \hat{q} + \epsilon)^{\mu} \times (\delta(\hat{q}) - \xi(\epsilon); \delta(\hat{q}) + \xi(\epsilon))^{\mu-1}$ are recurrent for any $\hat{q} \in \left[q^{\min o}; q^{\max p}\right]$ and any $\epsilon \in (0; \epsilon_{\hat{q}}]$, where $\xi(\epsilon)$ are chosen as in the proof of Claim 28. Hence, we can define a measure ϕ on $\mathcal{B}\left(\left[q^{\min o}; q^{\max p}\right]^{\mu}\right)$ such that $\phi(A) > 0$ only if $A \cap (\hat{q} - \epsilon; \hat{q} + \epsilon)^{\mu}$ is a non-empty open set for some $\hat{q}$ and some $\epsilon \leq \epsilon_{\hat{q}}$ and conclude that the Markov chain is ϕ-irreducible. Proposition 4.2.2 in Meyn and Tweedy (1993, p. 90), then implies the existence of a measure ψ absolutely-continuous w.r.t. ϕ such that the Markov chain is ψ-irreducible. Proposition 8.3.1 (ii) in Meyn and Tweedy (1993) (p. 188), implies that $(\hat{q} - \epsilon; \hat{q} + \epsilon)^{\mu}$ is indeed recurrent. From Proposition 9.1.1 in Meyn and Tweedy (1993, p. 206) we further conclude that the chain is also Harris recurrent and positive. Hence, there exists an invariant measure $\tilde{\pi}$. By Theorem 5.2.2 in Meyn and Tweedy (1993) (p. 112), we know that there is a subset $C \subset (\hat{q} - \epsilon; \hat{q} + \epsilon)^{\mu}$, which is small and, hence, petite. Using the same argument as above, we can show that the first return time to C starting from C is finite. Hence, by Theorem 10.4.10 (ii) in Meyn and Tweedy (1993, p. 254), the measure $\tilde{\pi}$ is positive and its support is $\left[q^{\min o}; q^{\max p}\right]^{\mu} \times \left[\underline{\delta}; \bar{\delta}\right]^{\mu}$. The proof that all marginal distributions of the price vector are equal is the same as in the proof of Proposition 8. As shown above, their support is $\left[q^{\min o}; q^{\max p}\right]$.■

Proof of Lemma 13: First note that $\left(\frac{\left[(q_t^*+\delta_t)a_{t-1}^i+(p_t^*+r)b_{t-1}^i\right]}{1+\delta_t+r}\right)''_{\delta_t} = \frac{2(1+r-q_t^*)\left(1-a_{t-1}^i\right)}{\left(1-q_{t-1}^*\right)(1+\delta_t+r)^3}$ and the replicator dynamics is concave or convex in δ depending on whether a_{t-1}^i is above or below 1. So, the function is convex if $a_{t-1}^i < 1$ and concave if $a_{t-1}^i > 1$ and we obtain:

$$E_\rho\left[\frac{\frac{q_t^*+\delta_t}{q_{t-1}^*}\lambda_{t-1}^i + \frac{p_t^*+r}{p_{t-1}^*}\left(1-\lambda_{t-1}^i\right)}{1+\delta_t+r}\right] > \left[\frac{\frac{q_t^*+E_\rho[\delta]}{q_{t-1}^*}\lambda_{t-1}^i + \frac{p_t^*+r}{p_{t-1}^*}\left(1-\lambda_{t-1}^i\right)}{1+E_\rho\left[\delta\right]+r}\right]$$

iff $a_{t-1}^i < 1$. In particular, when the prices are such that the "expected returns" of the two assets are equal, the r.h.s. equals 1 and we have that the share of type i will be increasing in expectations iff $a_{t-1}^i < 1$. Hence, a type of consumers who hold less of the risky asset than the market portfolio will see their share in the population strictly increase in expectations, and vice versa.

Furthermore, $E_\rho\left[\omega_{t+1} \mid \omega_{t-1}; \omega_t; \beta_{t-1}; \beta_t; Q_t\right] \geq \omega_{t-1}$ is equivalent to $E_\rho\left[\frac{[q_t^*+\delta_t]\left(a_{t-1}^o-1\right)}{(1+\delta_t+r)} \mid \cdot\right] \geq \left(a_{t-1}^o - 1\right) q_{t-1}^*$. Now note that optimists always hold more than 1 of the risky asset (intuitively, the fraction of their wealth invested in the risky asset is at least as large as that of the Bayesians and the pessimists, while if everyone were to invest all of their wealth in a, everyone would hold exactly 1). Hence, $a_{t-1}^o > 1$, and $E_\rho\left[\omega_{t+1} \mid \cdot\right] \geq \omega_{t-1}$ iff $E_\rho\left[\frac{q_t^*+\delta_t}{1+\delta_t+r} \mid \cdot\right] \geq q_{t-1}^*$.

Substituting σ for ω and a_{t-1}^p, b_{t-1}^p for a_{t-1}^o, b_{t-1}^o, we obtain that the share of pessimists increases in expectation iff $E_\rho\left[\frac{[q_t^*+\delta_t]\left(a_{t-1}^p-1\right)}{(1+\delta_t+r)} \mid \cdot\right] \geq \left(a_{t-1}^p - 1\right) q_{t-1}^*$. Now note that pessimists always hold less than 1 of the risky asset (intuitively, the fraction of their wealth invested in the risky asset is at most as large as that of the Bayesians and the optimists, while if everyone were to invest all of their wealth in a, everyone would hold exactly 1). Hence, $a_{t-1}^p < 1$, and $E_\rho\left[\sigma_{t+1} \mid \cdot\right] \geq \sigma_{t-1}$ iff $E_\rho\left[\frac{q_t^*+\delta_t}{1+\delta_t+r} \mid \cdot\right] \leq q_{t-1}^*$.

Finally, substituting β for ω and a_{t-1}^b, b_{t-1}^b for a_{t-1}^o, b_{t-1}^o, we obtain that the share of Bayesians increases in expectation iff $\left[E_\rho\left[\frac{q_t^*+\delta_t}{1+\delta_t+r} \mid \cdot\right] - q_{t-1}^*\right]\left(a_{t-1}^b - 1\right) \geq 0$. To determine $\tilde{q}$, note that for $q_t^* = q_{t-1}^*$, $E_\rho\left[\frac{q_t^*+\delta_t}{1+\delta_t+r} \mid \cdot\right] - -q_t^* = 0$ is satisfied iff $q_t^* = \frac{\int \frac{\delta}{1+\delta+r}\rho'(\delta)d\delta}{\int \frac{\delta+r}{1+\delta+r}\rho'(\delta)d\delta} = \tilde{q}$.■

Proof of Proposition 15:

Claim 31: For every $\epsilon > 0$ and almost every path of the system η, there is a finite time $\bar{t}(\eta)$ such that $q_t^* \in (q_\infty^p - \epsilon; q_\infty^o + \epsilon)$ for all $t \geq \bar{t}(\eta)$.

Proof of Claim 31: Since $\lim_{t\to\infty} E_{Q_t}\left[\delta\right] = E_\rho\left[\delta\right]$ a.s., for every ξ, $\xi' > 0$ and on almost every path η, there is a $\bar{t}(\eta)$ such that $E_{Q_t}[\delta] \in \left(E_\rho[\delta] - \xi'; E_\rho[\delta] + \xi\right)$ for all $t \geq \bar{t}(\eta)$. Let ξ and ξ' be such that $q^p_\infty - \epsilon = q^{REp}\left(E_\rho[\delta] - \xi'\right)$ and $q^o_\infty + \epsilon = q^{REo}\left(E_\rho[\delta] + \xi'\right)$. Since $q^p\left(Q_{t+1}; \gamma_{t+1}\right) > q^*_t$ whenever $q^*_t < q^p_\infty$ and $q^o\left(Q_{t+1}; \gamma_{t+1}\right) < q^*_t$ whenever $q^*_t > q^o_\infty$ and since $q^*_t \in [q^p(Q_t; \gamma_t); q^o_t(Q_t; \gamma_t)]$ for all t, it follows that for almost every path η, there is a $\bar{t}(\eta)$ such that $q^*_t(\eta) \in (q^p_\infty - \epsilon; q^o_\infty + \epsilon)$.

Claim 32: There is a $\kappa > 0$ such that on a path η, $E[\sigma_{t+2} \mid \sigma_t] \leq \sigma_t$ implies $q^*_{t+1} \geq q^*_t + \kappa$ for all $t \geq \bar{t}(\eta)$

<u>Proof of Claim 32:</u> A necessary (but not sufficient)[23] condition for $E[\sigma_{t+2} \mid \sigma_t] \leq \sigma_t$ is given by

(16)
$$\int \frac{1 - q^*_{t+1} + r}{(1 - q^*_t)(1 + \delta + r)} \rho'(\delta)\, d\delta \leq 1$$

Rewrite (16) as: $q^*_{t+1} - q^*_t \geq q^*_t \frac{\int \frac{\delta+r}{1+\delta+r}\rho'(\delta)d\delta}{\int \frac{1}{1+\delta+r}\rho'(\delta)d\delta} - \frac{\int \frac{\delta}{1+\delta+r}\rho'(\delta)d\delta}{\int \frac{1}{1+\delta+r}\rho'(\delta)d\delta}$. Hence, for $q^*_t \geq q^p_\infty - \epsilon > \frac{\int \frac{\delta}{1+\delta+r}\rho'(\delta)d\delta}{\int \frac{\delta+r}{1+\delta+r}\rho'(\delta)d\delta}$,

(17)
$$q^*_{t+1} - q^*_t \geq (q^p_\infty - \epsilon) \frac{\int \frac{\delta+r}{1+\delta+r}\rho'(\delta)\, d\delta}{\int \frac{1}{1+\delta+r}\rho'(\delta)\, d\delta} - \frac{\int \frac{\delta}{1+\delta+r}\rho'(\delta)\, d\delta}{\int \frac{1}{1+\delta+r}\rho'(\delta)\, d\delta} =: \kappa.$$

Claim 33: There exists a $\bar{\gamma} > 0$ such that for all $\gamma < \bar{\gamma}$ and for almost every path η, $q^*_{t+1} - q^*_t < \kappa$ for all $t \geq \bar{t}(\eta)$.

<u>Proof of Claim 33:</u> Note that for $\gamma > 0$, $q^o_\infty > q^p_\infty$ and for $\gamma = 0$, $q^p_\infty = q^o_\infty$. Note further that since $q^p_\infty > \frac{\int \frac{\delta}{1+\delta+r}\rho'(\delta)d\delta}{\int \frac{\delta+r}{1+\delta+r}\rho'(\delta)d\delta}$, $q^p_\infty \frac{1}{\int \frac{1}{1+\delta+r}\rho'(\delta)d\delta} - \frac{\int \frac{\delta}{1+\delta+r}\rho'(\delta)d\delta}{\int \frac{1}{1+\delta+r}\rho'(\delta)d\delta} > q^p_\infty$.

Let $\bar{\gamma}$ be such that $q^o_\infty(\gamma) < q^p_\infty(\gamma) \frac{1}{\int \frac{1}{1+\delta+r}\rho'(\delta)d\delta} - \frac{\int \frac{\delta}{1+\delta+r}\rho'(\delta)d\delta}{\int \frac{1}{1+\delta+r}\rho'(\delta)d\delta}$ for all $\gamma < \bar{\gamma}$. For a given γ, let ϵ be such that

$$q^o_\infty(\gamma) < (q^p_\infty(\gamma) - \epsilon) \frac{1}{\int \frac{1}{1+\delta+r}\rho'(\delta)\, d\delta} - \frac{\int \frac{\delta}{1+\delta+r}\rho'(\delta)\, d\delta}{\int \frac{1}{1+\delta+r}\rho'(\delta)\, d\delta} - 2\epsilon$$

and hence, for $t \geq \bar{t}(\eta)$,

$$q^*_{t+1} - q^*_t \leq q^o_\infty - q^p_\infty + 2\epsilon << (q^p_\infty(\gamma) - \epsilon) \frac{1}{\int \frac{1}{1+\delta+r}\rho'(\delta)\, d\delta} - \frac{\int \frac{\delta}{1+\delta+r}\rho'(\delta)\, d\delta}{\int \frac{1}{1+\delta+r}\rho'(\delta)\, d\delta} - q^p_\infty =$$

$$= q^p_\infty \frac{\int \frac{\delta+r}{1+\delta+r}\rho'(\delta)\, d\delta}{\int \frac{1}{1+\delta+r}\rho'(\delta)\, d\delta} - \frac{\int \frac{\delta}{1+\delta+r}\rho'(\delta)\, d\delta}{\int \frac{1}{1+\delta+r}\rho'(\delta)\, d\delta} = \kappa$$

Hence, $q^*_{t+1} - q^*_t < \kappa$ for all $t \geq \bar{t}(\eta)$. ■

Claim 34: For almost every path η, there is a period $\bar{t}(\eta)$ such that the process $\sigma_0; \sigma_2 ... \sigma_t; \sigma_{t+2}...$ is a sub-

[23] Note that the condition above refers to the case, in which pessimists do not hold the risky asset in period t. If they did hold the risky asset, and the price of the risky asset were to rise, the expression above provides a lower bound for $\frac{E[\sigma_{t+2}|\sigma_t]}{\sigma_t}$. Hence, this condition is indeed necessary for the pessimists' share to decrease in expectations, regardless of the pessimists' portfolio choice at time t.

martingale for all $t \geq \bar{t}(\eta)$, i.e., $E[\sigma_{t+2} \mid \sigma_t] > \sigma_t$ holds for all $t \geq \bar{t}(\eta)$, whenever $\sigma_t < 1$.

Proof of Claim 34: For a given path η, let $t > \bar{t}(\eta)$, where $\bar{t}(\eta)$ is the period determined in Claim 31. If the price of the risky asset decreases between periods t and $t+1$, we have $\frac{\int \frac{\delta}{1+\delta+r}\rho'(\delta)d\delta}{\int \frac{\delta+r}{1+\delta+r}\rho'(\delta)d\delta} < q^p_\infty - \epsilon < q^*_{t+1} \leq q^*_t$, and hence, by Lemma 13, $E[\sigma_{t+2} \mid \sigma_t] > \sigma_t$.

We also know that for $t \geq \bar{t}(\eta)$, if the price of the risky asset decreases between periods t and $t+1$, $q^*_{t+1} - q^*_t < \kappa$, and hence, $E[\sigma_{t+2} \mid \sigma_t] > \sigma_t$. ■

Claim 35: $\lim_{k\to\infty} \sigma_t = 1$ a.s. and in expectations

Proof of Claim 35: By Claim 34, σ_{2k} is a submartingale, and hence, it converges a.s. and in expectations to a random variable. Since σ is bounded between 0 and 1 and since its expectation is growing, the expectation of the limit is clearly equal to 1 and thus, σ_{2k} converges to a deterministic variable equal to 1, i.e., $\lim_{k\to\infty} \sigma_{2k} = 1$ a.s. Since we can redo the argument for the sequence σ_{2k+1}, we conclude that $\lim_{t\to\infty} \sigma_t = 1$ a.s. and in expectations.

To prove the second part of the Proposition, exchange σ for β and q^p_∞ for q^b_∞ in the proof above.■

To shorten notation, whenever the path of the economy is fixed, we will henceforth write $q^i_t =: q^i(Q_t;\gamma)$ for the reservation price of investor $i \in \{o;b;p\}$ at time t.

Proof of Proposition 16: Consider a sequence of dividend realizations $(\delta^g...\delta^g...)$ with $\delta^g \in (\underline{\delta};\bar{\delta})$. We know that there exists a period T such that $q^*_t \in \left[\frac{(1-\gamma)\delta^g+\gamma(\underline{q}+\underline{\delta})}{[(1-\gamma)\delta^g+\gamma(\underline{q}+\underline{\delta})]+(r+\gamma\underline{p})};\frac{(1-\gamma)\delta^g+\gamma(\bar{q}+\bar{\delta})}{[(1-\gamma)\delta^g+\gamma(\bar{q}+\bar{\delta})]+(r+\gamma\bar{p})}\right]$ for all $t \geq T$. In the following 7 claims, we fix δ^g and T as above and consider $t \geq T$.

Claim 36: If in a given period, $q^*_t = \min\{q^o_t;\omega_t\}$, then $q^*_{t+2} = \min\{q^o_{t+2};\omega_{t+2}\}$.

Proof of Claim 36: $q^*_t = \min\{q^o_t;\omega_t\} \leq \omega_t$ implies $\omega_{t+2} = \omega_t \frac{\frac{q^*_{t+1}+\delta^g}{q^*_t}+(p^*_{t+1}+r)b^o_t}{\delta^g+r+1} \geq \frac{(q^*_{t+1}+\delta^g)}{\delta^g+r+1} = q^b_{t+2}$. From Proposition 3, we know that for all $t \geq T$, $q^b_{t+2} > q^p_{t+2}$ for all $\delta^g \in \left[\frac{(\underline{q}+\underline{\delta})r}{r+\underline{p}};\frac{(\bar{q}+\bar{\delta})r}{r+\bar{p}}\right]$. Hence, $q^*_{t+2} = \min\{q^o_{t+2};\omega_{t+2}\}$.

Claim 37: If in a given period, $q^*_t = \max\{q^p_t;\omega_t+\beta_t\}$, then $q^*_{t+2} = \max\{q^p_{t+2};\omega_{t+2}+\beta_{t+2}\}$

Proof of Claim 37: Since $q^*_t = \max\{q^p_t;\omega_t+\beta_t\}$ implies $q^*_t \geq \omega_t$, $\omega_{t+2} = \omega_t \frac{\frac{q^*_{t+1}+\delta^g}{q^*_t}}{\delta^g+r+1} \leq \frac{(q^*_{t+1}+\delta^g)}{\delta^g+r+1} = q^b_{t+2}$. From Proposition 3, we know that for all $t \geq T$, $q^b_{t+2} < q^p_{t+2}$ for all $\delta^g \in \left[\frac{(\underline{q}+\underline{\delta})r}{r+\underline{p}};\frac{(\bar{q}+\bar{\delta})r}{r+\bar{p}}\right]$. Hence, $q^*_{t+2} =$

$\max\left\{q_{t+2}^{p};\omega_{t+2}+\beta_{t+2}\right\}$.■

Claim 38: If in a given period, the price $q_t^* \in [\omega_t;\omega_t+\beta_t]$, then $q_{t+2}^* = q_{t+2}^b$.

Proof of Claim 38: If $q_t^* \in [\omega_t;\omega_t+\beta_t]$, we obtain: $\omega_{t+2} = \frac{\omega_t}{q_t^*}\frac{q_{t+1}^*+\delta^g}{\delta^g+r+1} \leq q_{t+2}^b$,

$$\sigma_{t+2} = 1-\left(\omega_{t+2}+\beta_{t+2}\right) = \frac{1-(\omega_t+\beta_t)}{p_t^*}\frac{p_{t+1}^*+r}{\delta^g+r+1} = \frac{1-(\omega_t+\beta_t)}{1-q_t^*}\frac{1-q_{t+1}^*+r}{\delta^g+r+1} \geq 1-q_{t+2}^b$$

and hence, $\omega_{t+2}+\beta_{t+2} \geq q_{t+2}^b$. It follows that $q_{t+2}^* = q_{t+2}^b$.■

Claim 39: Let $i \in \{o;p\}$.The economy cannot indefinitely remain in a state such that $q_t^* = q_t^i$.

Proof of Claim 39: Let $i = o$. Suppose to the contrary that $q_t^* = q_t^o$ for all $t \geq T$. Then, $\lim_{t\to\infty} q_t^* = q^{REo}(\delta^g) > \frac{\delta^g}{r+\delta^g}$. However, $\frac{\frac{q^{REo}(\delta^g)+\delta^g}{q^{REo}(\delta^g)}}{\delta^g+r+1} < 1$, and hence, $\omega_{t+2} < \omega_t$, i.e., the share of optimists will decrease, eventually falling below $q^{REo}(\delta^g)$. Hence, $q_t^* < q_t^i$ will eventually obtain, a contradiction. A symmetric argument can be made for the case $i = p$. ■

Claim 40: For a constant sequence of dividends $\delta^g \in \left[\frac{(\underline{q}+\underline{\delta})r}{r+\underline{p}};\frac{(\bar{q}+\bar{\delta})r}{r+\bar{p}}\right]$, the economy cannot forever switch in every subsequent period between an optimistic state with $q_t^* = q_t^o$ and a pessimistic state with $q_t^* = q_t^p$.

Proof of Claim 40: Assume to the contrary that the economy switches from an optimistic to a pessimistic state in each subsequent period. Then, we would have a price sequence defined by: $q_t^* = q_t^o = q^o\left(q_{t-1}^*;\delta^g\right) = q^o\left(q_{t-1}^p;\delta^g\right)$ and $q_{t+1}^* = q_t^p = q^p\left(q_{t-1}^*;\delta^g\right) = q^p\left(q_{t-1}^o;\delta^g\right)$. Assume, w.l.o.g., that $q_{\bar{t}}^o \geq q_{\bar{t}+2}^o$ for some $\bar{t}$, then $q_{\bar{t}+1}^p \geq q_{\bar{t}+3}^p$ and both sequences q_t^o and q_t^p are decreasing starting from $\bar{t}$. But these are bounded sequences, which, therefore converge. The limits q^p and q^o must, therefore satisfy $q^o(q^p;\delta^g) = q^o$ and $q^p(q^o;\delta^g) = q^p$. Hence,

$$q^o = \frac{\left[(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)\right]\left[\left[(1-\gamma)\delta^g+\gamma\left(\underline{q}+\underline{\delta}\right)\right]+\left(r+\gamma\underline{p}\right)+2(1-\gamma)\right]}{\left[\left[(1-\gamma)\delta^g+\gamma\left(\underline{q}+\underline{\delta}\right)\right]+\left(r+\gamma\underline{p}\right)+1-\gamma\right]\left[\left[(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)\right]+(r+\gamma\bar{p})+1-\gamma\right]-(1-\gamma)^2}$$

and since we have $q_t^* \leq q^{REo}(\delta^g)$, $q^o \leq \frac{(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)}{\left[(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)\right]+\left(r+\gamma\underline{p}\right)}$ must hold. Simple algebraic computations show that this is equivalent to: $\left[\left[(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)\right]+(r+\gamma\bar{p})\right] \leq \left[(1-\gamma)\delta^g+\gamma\left(\underline{q}+\underline{\delta}\right)\right]+\left(r+\gamma\underline{p}\right)$, which cannot hold. A cycle between the pessimistic and the optimistic state is thus impossible.■

Claim 41: For a constant sequence of dividends $\delta^g \in \left[\frac{(\underline{q}+\underline{\delta})r}{r+\underline{p}};\frac{(\bar{q}+\bar{\delta})r}{r+\bar{p}}\right]$, the economy cannot forever switch in every subsequent period between an optimistic state, in which $q_t^* = q_t^o$ and a Bayesian states, in which $q_t^* = q_t^b$.

Proof of Claim 41: By the same argument as above, in such a cycle, the limit prices q^o and q^b would satisfy

$q^o\left(q^b;\delta^g\right)=q^o$ and $q^b\left(q^o;\delta^g\right)=q^b$, and hence,

$$q^o=\frac{(1-\gamma)\left(\delta^g+\delta^g\left(\delta^g+r+1\right)\right)+\gamma\left(\bar{q}+\bar{\delta}\right)\left(\delta^g+r+1\right)}{\left(\delta^g+r+1\right)\left[(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)+r+\gamma\bar{p}+(1-\gamma)\right]-(1-\gamma)}$$

$$q^b=\frac{(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)+\delta^g\left[(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)+r+\gamma\bar{p}+(1-\gamma)\right]}{\left[(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)+r+\gamma\bar{p}+(1-\gamma)\right]\left(\delta^g+r+1\right)-(1-\gamma)}.$$

Note that $q^b<q^o$ is equivalent to

$$\begin{aligned}&\frac{(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)+\delta^g\left[(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)+r+\gamma\bar{p}+(1-\gamma)\right]}{\left[\left[(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)+r+\gamma\bar{p}+(1-\gamma)\right]\left(\delta^g+r+1\right)-(1-\gamma)\right]}\\ \leq\ &\frac{(1-\gamma)\left(\delta^g+\delta^g\left(\delta^g+r+1\right)\right)+\gamma\left(\bar{q}+\bar{\delta}\right)\left(\delta^g+r+1\right)}{\left(\delta^g+r+1\right)\left[(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)+r+\gamma\bar{p}+(1-\gamma)\right]-(1-\gamma)}\end{aligned}$$

or $\delta^g\leq\frac{\left(\bar{q}+\bar{\delta}\right)r}{r+\bar{p}}$, which is satisfied by definition.

Second, $q^b>\frac{\delta^g}{\delta^g+r}$ is equivalent to

$$\frac{(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)+\delta^g\left[(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)+r+\gamma\bar{p}+(1-\gamma)\right]}{\left[(1-\gamma)\delta^g+\gamma\left(\bar{q}+\bar{\delta}\right)+r+\gamma\bar{p}+(1-\gamma)\right]\left(\delta^g+r+1\right)-(1-\gamma)}>\frac{\delta^g}{\delta^g+r},$$

or $\frac{\left(\bar{q}+\bar{\delta}\right)r}{r+\bar{p}}>\delta^g$, which is always satisfied.

Consider a period t with $q_t^*=q^o$. Hence, $a_t^o>1$ and since $q^o>q^b>\frac{\delta^g}{\delta^g+r}$, by Lemma 13, if $q_{t+1}^*=q_t^*=q^o$, $\omega_{t+2}<\omega_t$ would obtain for any ω_t. Since in fact $q_{t+1}^*=q^b<q_t^*=q^o$, $\omega_{t+2}<\omega_t$ for any ω_t. Hence, $q_t^*=q_t^o$ for all t implies $\omega_{t+2}<\omega_t$ for all t and hence, $\lim_{t\to\infty}\omega_t=0$. Thus, eventually, $\omega_t<q_t^o$ obtains in contradiction with $q_t^*=q^o$ for all t and such a cycle cannot occur.■

Claim 42: For a constant sequence of dividends $\delta^g\in\left[\frac{\left(\underline{q}+\underline{\delta}\right)r}{r+\underline{p}};\frac{\left(\bar{q}+\bar{\delta}\right)r}{r+\bar{p}}\right]$, the economy cannot forever switch in every subsequent period between a pessimistic state, in which $q_t^*=q_t^p$ and a Bayesian states, in which $q_t^*=q_t^b$.

<u>Proof of Claim 42:</u> The proof of this Claim uses an argument symmetric to that in the Proof of Claim 41 and is therefore omitted.■

Claims 36 and 37 demonstrate that if the economy starts in an optimistic (pessimistic) state, $q_t^*=q_t^o$ ($q_t^*=q_t^p$) at time t, it cannot transition to a pessimistic (optimistic) state at time $t+2$, $q_{t+2}^*\neq q_t^p$ ($q_{t+2}^*\neq q_t^o$) Furthermore, by Claims 39 and 40, the economy can neither indefinitely stay in an optimistic or in a pessimistic regime, nor indefinitely cycle between those. Finally, Claims 41 and 42 also exclude cycles between Bayesian and optimistic, as well as Bayesian and pessimistic regimes. We conclude that the system eventually reaches two consecutive periods $\bar{t}$ and $\bar{t}+1$ in which $q_{\bar{t}}^*\in[\omega_{\bar{t}};\omega_{\bar{t}}+\beta_{\bar{t}}]$, $q_{\bar{t}+1}^*\in\left[\omega_{\bar{t}+1};\omega_{\bar{t}+1}+\beta_{\bar{t}+1}\right]$, upon which, as shown in Claim 38, $q^*_{t+2} = q^b_{t+2}$, $q^*_{t+3} = q^b_{t+3}$.

Claim 43: Suppose that at time t, $q^*_t \in [\omega_t; \omega_t + \beta_t]$. Then, for any $\delta_{t+1} \in \left(\underline{\delta}; \bar{\delta}\right)$, $q^*_{t+2} = q^b_{t+2}$.

Proof of Claim 43: We have $\omega_{t+2} = \frac{\omega_t}{q^*_t} \frac{q^*_{t+1}+\delta_{t+1}}{\delta_{t+1}+r+1} \leq q^b_{t+2}$ and

$$\begin{aligned} \sigma_{t+2} &= 1 - \left(\omega_{t+2} + \beta_{t+2}\right) = \frac{1-(\omega_t+\beta_t)}{p^*_t} \frac{p^*_{t+1}+r}{\delta_{t+1}+r+1} \\ &= \frac{1-(\omega_t+\beta_t)}{1-q^*_t} \frac{1-q^*_{t+1}+r}{\delta_{t+1}+r+1} \geq 1 - q^b_{t+2} \end{aligned}$$

and hence, $\omega_{t+2} + \beta_{t+2} \geq q^b_{t+2}$, which implies $q^*_{t+2} = q^b_{t+2}$.■

Denote by $U =: \left\{\omega; \beta; q; \omega'; \beta'; q'; \delta' \mid q \in (\omega; \omega+\beta)\right\}$ the set of states in which the equilibrium price q equals the reservation price of the Bayesian consumers. Since for any initial state of the system, $x_0 =: \left(\omega_0; \beta_0; q_0; \omega'_0; \beta'_0; q'_0; \delta'_0\right)$, the set of all attainable states starting from this initial state, $A_+(x_0)$ has a non-empty intersection with U. By Proposition 7.2.2 in Meyn and Tweedy (1993, p. 162), the set U is thus reached with a positive probability from any initial state. Furthermore, the closure of the set U is an absorbing set of the system Hence, a.s. after a finite number of periods, $q^*_t = q^b_t$ for all t.■

Proof of Proposition 17: Weak arbitrage-free beliefs is equivalent to $\bar{\delta} > \frac{(\bar{q}+\bar{\delta})r}{[r+\bar{p}]} > \frac{(\underline{q}+\underline{\delta})r}{r+\underline{p}} > \underline{\delta}$.

The condition $q^o(q;\delta) > q^p(q;\delta)$ for all $q \in \left[q^{\min p}; q^{\max o}\right]$ is equivalent to:

$$\left(\bar{q} + \bar{\delta} - \underline{q} - \underline{\delta}\right) r + \gamma\left(\underline{p}\bar{q} - \bar{p}\underline{q}\right) + \underline{p}\bar{\delta} - \bar{p}\left(\gamma\underline{\delta} + (1-\gamma)\bar{\delta}\right) > 0$$

Using the fact that weakly arbitrage free beliefs imply $\left(\underline{p}\bar{q} - \bar{p}\underline{q}\right) > 0$; $\bar{\delta}\underline{p} > r\underline{q}$; $r\underline{q} > \underline{\delta}\underline{p}$; $\underline{\delta}\bar{q} > \bar{\delta}\underline{q}$; $\bar{\delta}\bar{p} > r\bar{q}$; $r\bar{q} > \underline{\delta}\bar{p}$, we find that the inequality would be satisfied if $\underline{p}\bar{\delta} - \underline{q}r > \bar{p}\bar{\delta} - \bar{q}r$. E.g.: $\bar{q} = 1$, $\underline{q} = 0,001$, $\underline{\delta} = 0,0005$, $\bar{p} = 0,8$, $\underline{p} = 0,1$, $\bar{\delta} = 0,45$, $r = 0,32$ satisfy all inequalities and since those are strict, we conclude that there exists an open set of parameters satisfying the conditions of the Proposition.

Claim 45: The Markov process given by $\left(q^*_{t-1}; \delta_{t-1}; \omega_{t-2}; q^*_{t-2}; \delta_{t-2}\right)$ is a T-chain.

Proof of Claim 45: Since $\beta_t \equiv 0$, we can write q^*_t and ω_t as continuous functions of $\left(q^*_{t-1}; \delta_{t-1}; \omega_{t-2}; q^*_{t-2}; \delta_{t-2}\right)$. Since δ_t is i.i.d., the process $\left(\omega_t; q^*_t; \omega_{t-1}; q^*_{t-1}; \delta_{t-1}\right)$ is Markov and it describes the economy at time t.

For a given initial condition $\left(\omega_{t-1}; q^*_{t-1}; \omega_{t-2}; q^*_{t-2}; \delta_{t-2}\right)$ a given ω_t obtains if δ_{t-1} satisfies:

$$\delta_{t-1}\left(\omega_t; \omega_{t-1}; q^*_{t-1}; \omega_{t-2}; q^*_{t-2}; \delta_{t-2}\right) = \delta_{t-1}\left(\omega_t; q^*_{t-1}; \omega_{t-2}; q^*_{t-2}\right)$$

$$= \begin{cases} -(1+r) + \frac{(1-\omega_{t-2})\left(1+r-q^*_{t-1}\right)}{\left(1-q^*_{t-2}\right)(1-\omega_t)} & \text{if} \quad \omega_{t-2} \geq q^*_{t-2} \\ -(1+r) + \omega_{t-2} \frac{1+r-q^*_{t-1}}{\omega_{t-2}-\omega_t q^*_{t-2}} & \text{if} \quad \omega_{t-2} < q^*_{t-2} \end{cases}$$

This is obviously a continuous function of $\left(\omega_t; q^*_{t-1}; \omega_{t-2}; q^*_{t-2}\right)$. Note that $\left(\omega_t; \delta_{t-1}; q^*_{t-1}; \omega_{t-2}; q^*_{t-2}\right)$ uniquely determine the equilibrium price q^*_t, which is also a continuous function of these arguments. Since δ_{t-1} can be written as a function of $\left(\omega_t; q^*_{t-1}; \omega_{t-2}; q^*_{t-2}\right)$, we can eliminate δ_{t-1} from the arguments of q^*_t and write:

$$q^*_t = q^*_t\left(\omega_t; \delta_{t-1}(\omega_t); q^*_{t-1}; \omega_{t-2}; q^*_{t-2}\right) = q^*_t\left(\omega_t; q^*_{t-1}; \omega_{t-2}; q^*_{t-2}\right) = q^*_t(\omega_t; \cdot),$$

where $\cdot$ stands for the initial state of the system.

Consider an open and convex set $S \subseteq (0;1) \times \left(\underline{q}; \bar{q}\right)$. For a given initial state of the system $\left(q^*_{t-1}; \omega_{t-1}; q^*_{t-2}; \delta_{t-2}; \omega_{t-2}\right)$, the probability of reaching S is given by:

$$\Pr\left\{\delta_{t-1} = \delta_{t-1}\left(\omega_t; q^*_{t-1}; \omega_{t-2}; q^*_{t-2}\right) \text{ for some } \omega_t \text{ s.t. } (\omega_t; q^*_t(\omega_t; \cdot)) \in S\right\}$$

So for a given set S and given parameters ω_{t-2}, q^*_{t-1}, q^*_{t-2}, let $S_\Omega\left(\omega_{t-2}, q^*_{t-1}, q^*_{t-2}\right)$ be the set:

$$S_\Omega\left(\omega_{t-2}, q^*_{t-1}, q^*_{t-2}\right) = \left\{\omega \mid \left(\omega; q\left(\omega; \omega_{t-2}, q^*_{t-1}, q^*_{t-2}\right)\right) \in S\right\}.$$

Obviously, this is the set of points in S that is potentially reachable from the given initial state for any possible realization of δ_{t-1}.

Take a $\delta > 0$, by continuity of $q^*_t\left(\omega_t; q^*_{t-1}; \omega_{t-2}; q^*_{t-2}\right)$ and since the set S is open, it is possible to find a sufficiently small $\epsilon > 0$ such that if $\left[\left(\omega_{t-2}, q^*_{t-1}, q^*_{t-2}\right) - \left(\omega'_{t-2}, q^{*\prime}_{t-1}, q^{*\prime}_{t-2}\right)\right] < \epsilon$ and if $\left(\omega_t; q^*_t\left(\omega_t; q^*_{t-1}; \omega_{t-2}; q^*_{t-2}\right)\right) \in S$, then $\left(\omega_t; q^*_t\left(\omega_t; q^{*\prime}_{t-1}; \omega'_{t-2}; q^{*\prime}_{t-2}\right)\right) \in S$. Moreover, $q^*_t\left(\omega_t; q^*_{t-1}; \omega_{t-2}; q^*_{t-2}\right)$ is a continuous and increasing function in ω_t. So take a set $S = S^\omega \times S^q$, where S^ω and S^q are open convex subsets of $(0;1)$ and $\left(\underline{q}; \bar{q}\right)$, respectively. Then, S_Ω will also be an open convex subset of $(0;1)$. The lower and upper boundaries of this set will be continuous functions of $\left(q^*_{t-1}; \omega_{t-2}; q^*_{t-2}\right)$. Hence, the probability of reaching such a set will also be continuous in $\left(q^*_{t-1}; \omega_{t-2}; q^*_{t-2}\right)$. It thus follows that the Markov process defined above is a T-chain.■

Claim 46: The Markov process $\left(q^*_{t-1}; \delta_{t-1}; \omega_{t-2}; q^*_{t-2}; \delta_{t-2}\right)$ is forward accessible.

Proof of Claim 46: For each initial condition, the set $\left\{\omega \mid \omega = \omega_t\left(q^*_{t-1}; \delta_{t-1}; q^*_{t-2}; \omega_{t-1}; \omega_{t-2}\right) \text{ for some } \delta_{t-1} \in \left(\underline{\delta}; \bar{\delta}\right)\right\}$ has a non-empty interior and hence, since the function $q^*(\omega; \cdot)$ is continuous and increasing in ω, the set of reachable q^*'s also has a non-empty interior.■

Consider a sequence of dividend realizations $(\delta^g ... \delta^g ...)$ with $\delta^g \in \left(\frac{(\underline{q}+\underline{\delta})r}{r+\underline{p}}; \frac{(\bar{q}+\bar{\delta})r}{r+\bar{p}}\right)$. We know that there exists a period T such that $q_t^* \in \left[q^{REp}(\delta^g); q^{REo}(\delta^g)\right]$ for all $t \geq T$.

Claim 47: For a constant sequence of dividends $\delta^g \in \left[\frac{(\underline{q}+\underline{\delta})r}{r+\underline{p}}; \frac{(\bar{q}+\bar{\delta})r}{r+\bar{p}}\right]$, the economy cannot forever switch in every subsequent period between a pessimistic state with $q_t^* = q_t^p$ and an "intermediate" state with $q_t^* = \omega_t$.

Proof of Claim 47: Assume to the contrary that the economy cycles between pessimistic and intermediate states. Then, the sequence of prices satisfies $q_{t+1}^* = q^p(\omega_t; \delta^g)$ and $q_{t+2}^* = \omega_{t+2} = \frac{\omega_t \frac{q_{t+1}^*+\delta^g}{\omega_t}}{\delta^g+r+1} = \frac{\omega_t \frac{q^p(\omega_t;\delta^g)+\delta^g}{\omega_t}}{\delta^g+r+1}$ for all t. As above, w.l.o.g., assume that $q_{\bar{t}+1}^p \geq q_{\bar{t}+3}^p$. It follows that $\omega_{\bar{t}+2} \geq \omega_{\bar{t}+4}$ and since both sequences are bounded, both of them have to converge. The limits of ω and q^p then have to satisfy:

$$\omega = \frac{\left(\gamma\left(\underline{q}+\underline{\delta}\right) + \delta^g\left[\left[(1-\gamma)\delta^g + \gamma\left(\underline{q}+\underline{\delta}\right)\right] + \left(r+\gamma\underline{p}\right) + 2(1-\gamma)\right]\right)}{\left[\left[\left[(1-\gamma)\delta^g + \gamma\left(\underline{q}+\underline{\delta}\right)\right] + \left(r+\gamma\underline{p}\right) + 1-\gamma\right](\delta^g+r+1) - (1-\gamma)\right]}$$

$$q^p = \frac{(1-\gamma)(\delta^g + \delta^g(\delta^g+r+1)) + \gamma\left(\underline{q}+\underline{\delta}\right)(\delta^g+r+1)}{\left[\left[(1-\gamma)\delta^g + \gamma\left(\underline{q}+\underline{\delta}\right)\right] + \left(r+\gamma\underline{p}\right)\right](\delta^g+r+1) - (1-\gamma)}$$

We then need that $\omega < \frac{(1-\gamma)(q^p+\delta^g)+\gamma\left(\underline{q}+\underline{\delta}\right)}{\left[(1-\gamma)\delta^g+\gamma\left(\underline{q}+\underline{\delta}\right)\right]+\left(r+\gamma\underline{p}\right)+1-\gamma}$, which simplifies to:

$$\left[(1-\gamma) + \left[\gamma\left(\underline{q}+\underline{\delta}\right) + (1-\gamma)\delta^g\right] + \left(\gamma\underline{p}+r\right)\right]\left[(1-\gamma)\delta^g + \gamma\left[r\left(\delta^g-\underline{\delta}\right) - \underline{q}r + \underline{p}\delta^g\right]\right] +$$
$$+ (1-\gamma)\gamma\left(\underline{q}+\underline{\delta}\right) < 0$$

We will now show that the expression $(1-\gamma)\delta^g + \gamma\left[r\left(\delta^g-\underline{\delta}\right) - \underline{q}r + \underline{p}\delta^g\right]$ is positive. Indeed,

$$\left[(1-\gamma)\delta^g + \gamma\left[r\left(\delta^g-\underline{\delta}\right) - \underline{q}r + \underline{p}\delta^g\right]\right] \geq \frac{\underline{q}+\underline{\delta}}{\underline{p}+r}r(1-\gamma) + \gamma\left[r\left(\frac{\underline{q}+\underline{\delta}}{\underline{p}+r}r - \underline{\delta}\right) - \underline{q}r + \underline{p}\frac{\underline{q}+\underline{\delta}}{\underline{p}+r}r\right] \geq 0$$

since $\frac{1}{\underline{p}+r}(1-\gamma) \geq 0$. Hence, the condition is never true and a cycle of this type is thus impossible.■

Claim 48: For a constant sequence of dividends $\delta^g \in \left[\frac{(\underline{q}+\underline{\delta})r}{r+\underline{p}}; \frac{(\bar{q}+\bar{\delta})r}{r+\bar{p}}\right]$, the economy cannot forever switch in every subsequent period between an optimistic state, with $q_t^* = q_t^o$ and an "intermediate" state, with $q_t^* = \omega_t$.

Proof of Claim 48: The proof of this Claim is symmetric to that of Claim 47 and therefore omitted.

Claim 49: For $\delta^g \in \left(\frac{(\underline{q}+\underline{\delta})r}{r+\underline{p}}; \frac{(\bar{q}+\bar{\delta})r}{r+\bar{p}}\right)$, $x^g = (\omega^g; q^g; \omega^g; q^g; \delta^g)$ with $q^g = \omega^g = \frac{\delta^g}{\delta^g+r}$ is globally attracting.

Proof of Claim 49: Using Claims 36 – 40 from the proof of Proposition 16 (substituting $\beta = 0$ and noting that $\left(\underline{\delta}; \bar{\delta}\right) \subseteq \left(\frac{(\underline{q}+\underline{\delta})r}{r+\underline{p}}; \frac{(\bar{q}+\bar{\delta})r}{r+\bar{p}}\right)$), we can show that the economy cannot indefinitely remain in a state, in which optimists (pessimists) determine prices, $q_t^* = q_t^o$ ($q_t^* = q_t^p$). Nor can it indefinitely cycle between optimistic and pessimistic states. Furthermore, if at time t, $q_t^* = \min\{q_t^o; \omega_t\}$, then $q_{t+2}^* = \min\{q_t^o; \omega_t\}$ and if $q_t^* = \max\{q_t^p; \omega_t\}$, then $q_{t+2}^* = \max\left\{q_{t+2}^p; \omega_{t+2}\right\}$. Hence, eventually, a period t with $q_t^* = \omega_t$ is reached. If $q_{t-1}^* = q_t^p$, then $q_{t+1}^* = \max\left\{q_{t+1}^p; \omega_{t+1}\right\}$, and if $q_{t-1}^* = q_t^o$, then $q_{t+1}^* = \min\left\{q_{t+1}^o; \omega_{t+1}\right\}$. By Claims 47 and 48, a cycle between optimistic and "intermediate" states, (with $q_t^* = \omega_t$), is impossible, and so is a cycle between pessimistic and intermediate states.

This implies that there is a finite time $\bar{t} \geq T$ such that $q_{\bar{t}}^* = \omega_{\bar{t}}$ and $q_{\bar{t}+1}^* = \omega_{\bar{t}+1}$ and thus, $q_t^* = \omega_t$ for all $t \geq \bar{t}$. If $\omega_t < \omega^g$, the share of optimists will increase, whereas for $\omega_t > \omega^g$, it will decrease. It follows that for an infinite sequence of δ^g realizations, the economy will converge to x^g, regardless of the initial state x_0^g, i.e., the state is reachable from any initial state and thus globally attracting. ■

Claim 50: The Markov process defined in Claim 45 is ψ-irreducible, positive recurrent and has an invariant probability distribution $\tilde{\pi}$.

Proof of Claim 50: Since by Claim 49, a globally attracting state exists, we obtain, by Proposition 7.2.5 in Meyn and Tweedy (1993, p. 164), that the process is M-irreducible. Since, furthermore, the (NSS3) condition on the density function of δ is satisfied and the model is forward accessible, we have, by Theorem 7.2.6., in Meyn and Tweedy (1993, p. 164) that the Markov process is ψ-irreducible on the set $\bar{S}$ reachable from x^g, which is compact, see Proposition 7.2.5 in Meyn and Tweedy (1993, p. 164) and hence, by Theorem 6.0.1 in Meyn and Tweedy (1993, p. 131), it is petite. Hence, just as in the proof of Proposition 11, the process is positive recurrent and has an invariant distribution $\tilde{\pi}$.■

Claim 51: The irreducible set of the Markov process contains a state $x^o = \left(\omega^o; q^o; \omega^{o\prime}; q^{o\prime}; \delta^{o\prime}\right)$ with $\omega^o > q^o = q_t^o\left(q^{o\prime}; \delta^{o\prime}\right)$ and a state $x^p = \left(\omega^p; q^p; \omega^{p\prime}; q^{p\prime}; \delta^{p\prime}\right)$ with $1 - \omega^p > q^p = q_t^p\left(q^{p\prime}; \delta^{p\prime}\right)$, which are in the support of the invariant measure $\tilde{\pi}$ defined in Claim 50.

Proof of Claim 51: Consider the state $x^g = (\omega^g; q^g; \omega^g; q^g; \delta^g)$ as defined above with $\delta^g = \frac{r\left(\bar{q}+\bar{\delta}\right)}{\bar{p}+r}$. Let $\delta^{o\prime} > \delta_g$, then $\omega_{t+1} = \omega_{t-1}\frac{\omega_t+\delta}{\omega_{t-1}(1+\delta+r)} = \frac{\frac{\delta^g}{\delta^g+r}+\delta^{o\prime}}{1+\delta^{o\prime}+r} = \frac{\frac{\bar{q}+\bar{\delta}}{\bar{q}+\bar{\delta}+\bar{p}+r}+\delta^{o\prime}}{1+\delta+r} = q_{t+1}^b$. Furthermore, for the chosen values of $q_t = q^g = \omega^g$ and $\delta^{o\prime}$, we have that $q_{t+1}^b > q_{t+1}^o = \frac{(1-\gamma)\left(\frac{\bar{q}+\bar{\delta}}{\bar{q}+\bar{\delta}+\bar{p}+r}+\delta^{o\prime}\right)+\gamma\left(\bar{q}+\bar{\delta}\right)}{\left[(1-\gamma)\delta^{o\prime}+\gamma\left(\bar{q}+\bar{\delta}\right)\right]+(r+\gamma\bar{p})+1-\gamma}$. It follows that $\omega_{t+1} > q_{t+1}^o$ and hence, $q_{t+1}^* = q_t^o$. I.e., the state

$$x^o = (\omega^o = \frac{\bar{q}+\bar{\delta}+\delta^{o\prime}\left(\bar{q}+\bar{\delta}+\bar{p}+r\right)}{\left(1+\delta^{o\prime}+r\right)\left(\bar{q}+\bar{\delta}+\bar{p}+r\right)}; q^o = \frac{(1-\gamma)\left(\frac{\bar{q}+\bar{\delta}}{\bar{q}+\bar{\delta}+\bar{p}+r}+\delta^{o\prime}\right)+\gamma\left(\bar{q}+\bar{\delta}\right)}{\left[(1-\gamma)\,\delta^{o\prime}+\gamma\left(\bar{q}+\bar{\delta}\right)\right]+(r+\gamma\bar{p})+1-\gamma};$$

$$\omega^{o\prime} = \frac{\bar{q}+\bar{\delta}}{\bar{q}+\bar{\delta}+\bar{p}+r}; q^{o\prime} = \frac{\bar{q}+\bar{\delta}}{\bar{q}+\bar{\delta}+\bar{p}+r}; \delta^{o\prime})$$

for $\delta^{o'} > \frac{r(\bar{q}+\bar{\delta})}{\bar{p}+r}$ satisfies $\omega^o > q^o = q_t^o\left(q^{o\prime};\delta^{o\prime}\right)$ and is accessible from the globally accessible state x^g with $\delta^g = \frac{r(\bar{q}+\bar{\delta})}{\bar{p}+r}$. It is therefore part of the minimal set and hence, in the support of $\tilde{\pi}$.

A symmetric argument applies for the pessimistic states: consider x^g with $\delta^g = \frac{(\underline{q}+\underline{\delta})r}{r+\underline{p}}$ upon a dividend realization $\delta^{p'} < \delta^g$. It is easy to show that the state

$$x^p = (\omega^p = \frac{\underline{q}+\underline{\delta}+\delta^{p'}\left(\underline{q}+\underline{\delta}+\underline{p}+r\right)}{\left(1+\delta^{p'}+r\right)\left(\underline{q}+\underline{\delta}+\underline{p}+r\right)}; q^p = \frac{(1-\gamma)\left(\frac{\underline{q}+\underline{\delta}}{\underline{q}+\underline{\delta}+\underline{p}+r}+\delta^{p'}\right)+\gamma\left(\underline{q}+\underline{\delta}\right)}{\left[(1-\gamma)\,\delta^{p'}+\gamma\left(\underline{q}+\underline{\delta}\right)\right]+\left(r+\gamma\underline{p}\right)+1-\gamma};$$

$$\omega^{p\prime} = \frac{\underline{q}+\underline{\delta}}{\underline{q}+\underline{\delta}+\underline{p}+r}; q^{p\prime} = \frac{\underline{q}+\underline{\delta}}{\underline{q}+\underline{\delta}+\underline{p}+r}; \delta^{p\prime})$$

satisfies $1-\omega^p > q^p = q_t^p\left(q^{p\prime};\delta^{p\prime}\right)$ and is accessible from the globally accessible state x^g with $\delta^g = \frac{(\underline{q}+\underline{\delta})r}{r+\underline{p}}$. It is therefore part of the minimal set and hence, in the support of $\tilde{\pi}$.■

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