Proper base change over henselian pairs

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Université Paul Sabatier
UMR 5219 du CNRS
Institut de Mathématiques de Toulouse
118 route de Narbonne
F-31062 TOULOUSE Cedex 9
E-mail address: Massimo.Pippi@math.univ-toulouse.fr
ORCID ID: 0000-0002-5660-3156
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Abstract. We discuss a question which appears in [*Séminaire de Géométrie Algébrique du Bous Marie- Théorie des topos et cohomologie étale des schémas, Exposé XII, Remarks 6.13*] concerning proper base change. In particular, we propose a solution in a particular non-affine case.

1 Introduction

The question we would like to answer is the following one[2]:

Question 1. *Let (X, X_0) be an henselian couple (in the sense of* Definition 1*). Is it true that (ii) and (iii) in* [2, Exposé XII, Proposition 6.5] *hold with $\mathbb{L} = \mathbb{P}$ and for every n?*

This question appears in [2, Exposé XII, Remarks 6.13].
We can restate it as follows:

Question 2. *Let (X, X_0) be an henselian couple. Is it true that*

a. the base change functor induces an equivalence between the category of étale coverings of X and the category of étale coverings of X_0?

b. for any torsion étale sheaf $\mathscr{F}$ and for any integer n, the morphism

$$H^n(X, \mathscr{F}) \longrightarrow H^n(X_0, \mathscr{F}_{|X_0})$$

is an isomorphism?

[1]the content of this note was obtained before the author started to receive the above mentioned scholarship, but it was organized in the present manuscript afterwards.

[2]for the notation and the exact definition of the objects that are involved we refer to the original source.

Remark 1. *1. When X is proper and finitely presented over an henselian ring (A,m) and $X_0 = X \times_{Spec(A)} Spec(A/m)$, we know that the answer to* Question 1 *is affirmative. This is the proper base change theorem in étale cohomology.*

2. The case $(X,X_0) = (Spec(A), Spec(A/I))^3$, for (A,I) an henselian pair was studied an solved by R. Elkik in [4] *and by O. Gabber in* [5].

We propose a solution in the following situation:

(†) *Let X be proper over a noetherian affine scheme $Spec(A)$ and $X_0 = X \times_{Spec(A)} Spec(A/I)$ for some ideal $I \subseteq A$.*

We will see that, under these assumptions, (X, X_0) is an henselian couple for which Question 1 has a positive answer. To achieve this, we will first generalize [1, Theorem 3.1] to the following form:

Theorem 1. *Let (A,I) be an henselian pair. Let $S = Spec(A)$ and let $f : X \longrightarrow S$ be a proper finitely presented morphism. Let $X_0 = X \times_S S_0$, where $S_0 = Spec(A/I)$. Then*

$$\acute{E}t_f(X) \longrightarrow \acute{E}t_f(X_0)$$
$$Z \mapsto Z \times_S S_0$$

is an equivalence of categories.

Here $\acute{E}t_f(W)$ denotes the category of finite étale schemes over W. The key tools for the proof are Artin's approximation theory and [12, Tag 0AH5], which combined with [1, Corollary 1.8] yields the following theorem

Theorem 2. *Let (A,I) be an henselian pair with A noetherian. Let $\hat{A}$ be the I-adic completion of A and assume that one of the following hypothesis is satisfied:*

1. *$A \longrightarrow \hat{A}$ is a regular ring map;*
2. *A is a G-ring;*
3. *(A,I) is the henselization[4] of a pair (B,J), where B is a noetherian G-ring.*

Let $\mathscr{F}$ be a functor which is locally of finite presentation[5]

$$A\text{-}algebras \longrightarrow Sets$$

Given any $\hat{\xi} \in \mathscr{F}(\hat{A})$ and any $N \in \mathbb{N}$, there exists an element $\xi \in \mathscr{F}(A)$ such that

$$\xi \equiv \hat{\xi} \ mod \ I^N$$

i.e. ξ and $\hat{\xi}$ have the same image in $\mathscr{F}(A/I^N) \cong \mathscr{F}(\hat{A}/\hat{I}^N)$

[3]an affine henselian couple is the same thing as an henselian pair. See Remark 7.
[4]here the henselization is the left adjoint to the inclusion functor *Henselian Pairs* $\longrightarrow$ *Pairs*
[5]see [1, Definition 1.5]

Remark 2. *In* Theorem 2 *we have that 3.* $\Rightarrow$ *2.* $\Rightarrow$ *1. See* [12, Tag 0AH5]*.*

Remark 3. Theorem 1*, joined with* [5, Corollary 1]*, gives us a positive answer to* Question 1 *when* (X, X_0) *is proper and finitely presented over an henselian pair. Moreover, we will see how we can always reduce to this case from situation* (†)*.*

2 Proof of Theorem 1

This proof is an adaption of the one given in the local case by Artin (see [1, Theorem 3.1]). This generalization is possible thanks to Popescu's characterization of regular morphisms between noetherian rings, which provides us Theorem 2 as a corollary.
First we reduce to the case where A is the henselization of a finitely presented $\mathbb{Z}$-algebra. in order to do this, we need the following two preliminary lemmas.

Lemma 1. *Let* $S = Spec(A)$ *and let* $g : X \longrightarrow S$ *be a proper morphism of finite presentation. Then the functor*

$$\mathscr{F} : A\text{-}Algebras \longrightarrow Sets$$

$$B \mapsto \{\textit{finite étale coverings of } Spec(B) \times_S X\}/isomorphism$$

is locally of finite presentation.

Proof. See the beginning of the proof of [1, Theorem 3.1]. □

Lemma 2. *Let* $S = Spec(A)$ *and let* $g : X \longrightarrow S$ *be a proper morphism of finite presentation. Let* $Z_1 \to X$ *and* $Z_2 \to X$ *be two finite étale covers of* X*. Then the functor*

$$\mathscr{G} : A\text{-}algebras \longrightarrow Sets$$

$$B \mapsto Hom_{X \times_S Spec(B)}(Z_1 \times_S Spec(B), Z_2 \times_S Spec(B))$$

is locally of finite presentation.

Proof. The lemma is a straightforward consequence of [7, Theorem 8.8.2.(i)]. □

Let (A, I) be an henselian pair and write A as a direct limit $\varinjlim A_i$, where each A_i is a subalgebra of A that is finitely generated over $\mathbb{Z}$. Let $(A_i^h, (I \cap A_i)^h)$ be the henselization of $(A_i, (I \cap A_i))$ for each i. Then by [11, Chapter XI, Proposition 2] $\varinjlim (A_i^h, (I \cap A_i)^h)$ is an henselian pair. It is easy to see that

$$(A, I) = \varinjlim (A_i^h, (I \cap A_i)^h)$$

Write $S_i = Spec(A_i^h)$ for every index i. Then

$$S = \varprojlim S_i$$

By [7, Thereom 8.8.2. (ii)] we know that X comes from a finitely presented scheme X_{i_0} for some index i_0, i.e. $X \cong X_{i_0} \times_{S_{i_0}} S$. Moreover, by [7, Theorem 8.10.5], we can assume that X_{i_0} is also proper over S_{i_0}. As the functor

$$\mathscr{F} : A_{i_0}^h - Algebras \longrightarrow Sets$$

$$B \mapsto \{\text{finite étale coverings of } Spec(B) \times_{S_{i_0}} X_{i_0}\}/\text{isomorphism}$$

is locally of finite presentation, we have that

$$\mathscr{F}(A) = \varinjlim \mathscr{F}(A_i^h)$$

Therefore, every finite étale cover of X comes from a finite étale cover of $X_i = S_i \times_{S_{i_0}} X_{i_0}$ for a suitable index i.

Remark 4. *All schemes $X_{i_0} \times_{S_{i_0}} S_i$ and $X \cong X_{i_0} \times_{S_{i_0}} S$ are quasi-compact and quasi-separated, as they are proper over affine schemes.*

Let $Z \to X$ and $W \to X$ be two finite étale covers of X. Then we can assume without loss of generality that they come from two finite étale covers $Z_{i_0} \to X_{i_0}$, $W_{i_0} \to X_{i_0}$. Then by Lemma 2 we see that

$$\varinjlim Hom_{X_i}(Z_i, W_i) = Hom_X(Z, W)$$

It is then clear that we can reduce the proof of Theorem 1 to the case where (A, I) is the henselization of a pair (B, J), where B is finitely generated over $\mathbb{Z}$. In particular, B is a G-ring and Theorem 2 holds.

Lemma 3. *The functor in Theorem 1 is essentially surjective.*

Proof. Consider a finite étale morphism $X_0' \longrightarrow X_0$. Label $\hat{A}$ the completion of A with respect to the ideal I and let $\hat{S} = Spec(\hat{A})$, $\hat{X} = X \times_S \hat{S}$. Notice that $\hat{A}$ is a complete separated ring by Krull's theorem (see [3, Theorem 10.17]). By [8, Theorem 18.3.4], we have that the functor

$$\acute{E}t_f(\hat{X}) \longrightarrow \acute{E}t_f(X_0)$$

$$Z \mapsto Z \times_S S_0$$

is an equivalence of categories. Then there exists some $[\hat{X}' \longrightarrow \hat{X}] \in \mathscr{F}(\hat{A})$ such that

$$\hat{X}' \times_{\hat{S}} S_0 \cong X_0'$$

By Theorem 2 we get that there exists some finite étale morphism $X' \longrightarrow X$ which is congruent modulo I to $\hat{X}' \longrightarrow \hat{X}$, i.e.

$$X' \times_S S_0 \cong X_0'$$

$\square$

It remains only to show that the functor in Theorem 1 is fully faithful.

Lemma 4. *The functor in* Theorem 1 *is fully faithful.*

Proof. Let X' and X'' be two finite étale schemes over X and let $\phi \in Hom_X(X', X'')$. The morphism ϕ corresponds uniquely to its graph $\Gamma_\phi : X' \longrightarrow X' \times_X X''$, which is an open immersion as both X' and X'' are of finite type over X and as X'' is étale over X (see [9, Corollaire 3.4]). Also notice that Γ_ϕ is a closed immersion (see [10, Exercise 3.3.10]). If we assume that X' is connected and nonempty, ϕ corresponds uniquely to a connected component of $X' \times_X X''$ of degree one over X'. The degree of such a component can be measured at any point of X'. We conclude therefore by applying the next lemma to a component of $X' \times_X X''$. $\square$

Lemma 5. *X is nonempty and connected if and only if the same is true for X_0.*

Proof. We are given the following cartesian square

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
S_0 \arrow[r] & S
\end{tikzcd}$$

If X is connected and nonempty, then $f(X) \subseteq S$ is a nonempty closed subset of S (as f is proper). Let J be an ideal of A that identifies $f(X)$. Let $f(x) = p \in V(J)$ be a closed point of S. As I is contained in the Jacobson radical of A, the prime ideal p lies in S_0. Then

$$\begin{tikzcd}
\{x\} \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
& X_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& S_0 \arrow[r] & S
\end{tikzcd}$$

In particuar, X_0 is nonempty. Furthermore, as this argument can be used for any connected component of X, if X is disconnected then also X_0 is disconnected. Conversely, assume that X_0 is disconnected. Label C_0 a nonempty connected component of X_0. As the scheme X_0 is quasi-compact, C_0 is open and closed in X_0. Therefore, $C_0 \longrightarrow X_0$ is a finite étale morphism. By Lemma 3, there exists a finite étale morphism $C \longrightarrow X$ which induces $C_0 \longrightarrow X_0$. As C_0 is connected and nonempty, the same is true for C. The morphism $C \longrightarrow X$ is therefore of degree 1 at every point of C. As it is also finite and étale, it is both an open and a closed immersion, i.e. C is a connected component of X. If $C = X$, we would get $C_0 = X_0$, a contradiction. Then X is disconnected. Finally, it is clear that if X_0 is nonempty, X is nonempty too. $\square$

Theorem 1 follows immediately from Lemma 3 and Lemma 4.

3 Henselian couples

Recall that an henselian pair (A, I) is a ring A together with an ideal $I \subseteq$ such that

1. I is contained in the Jacobson ideal of A;
2. for every finite A algebra B, there is a bijection between the set of idempotent elements of B and the set of idempotent elements of $B \otimes_A A/I$.

For more details, see [11].
Let (A, I) be an henselian pair. Then for every finite morphism $Spec(B) = X \longrightarrow Spec(A)$, we have a bijection

$$Id(B) = Of(X) = Of(X_0) = Id(B/IB) \quad \text{where } X_0 = X \times_{Spec(A)} Spec(A/I)$$

Here $Of(Z)$ denotes the set of subsets of Z which are both open and closed.
This fact suggests the following definition (see [8, Définition 18.5.5]), which is meant to generalize the notion of henselian pair to the non-affine setting.

Definition 1. *Let X be a scheme and let X_0 be a closed subscheme. We say that (X, X_0) is an henselian couple if for every finite morphism $Y \longrightarrow X$ we have a bijection*

$$Of(Y) = Of(Y_0)$$

where $Y_0 = Y \times_X X_0$.

Remark 5. *If X is locally noetherian, it is a consequence of* [6, Proposition 6.1.4] *and* [6, Corollaire 6.1.9] *that connected sets in $Of(X)$ (resp. $Of(X_0)$) are in bijection with $\Pi_0(X)$ (resp. $\Pi_0(X_0)$), the set of connected components of X (resp. X_0).*

Remark 6. *It is a consequence of* [6, Corollary 5.1.8] *that (X, X_0) is an henselian couple if and only if $(X_{red}, (X_0)_{red})$ is an henselian couple as well.*

Remark 7. *It is immediate to observe that if (A, I) is a pair and $(Spec(A), Spec(A/I))$ is an henselian couple, then I is contained in the Jacobson radical of A. In fact, if $m \subseteq A$ is a maximal ideal, then we have a bijection*

$$Of(Spec(A/m)) = Of(Spec(A/m \otimes_A A/I))$$

In particular, $Spec(A/m \otimes_A A/I)$ can not be the empty scheme. Therefore, as it is a closed subscheme of $Spec(A/m)$, we must have an equality $Spec(A/m) = Spec(A/m \otimes_A A/I)$, whence $I \subseteq m$. Moreover, if $Z \longrightarrow Spec(A)$ is a finite morphism, then $Z = Spec(B)$ is affine and the corresponding morphism $A \longrightarrow B$ is finite. Then we have bijections

$$Id(B) = Of(Spec(B)) = Of(Spec(B/IB)) = Id(B/IB)$$

We have just showed that an affine henselian couple is an henselian pair. The converse was observed at the beginning of this section.

Lemma 6. *Let (A, I) be an henselian pair with A noetherian and let X be a proper A-scheme. Set $S = Spec(A)$, $S_0 = Spec(A/I)$ and let $X_0 = X \times_S S_0$. Then (X, X_0) is an henselian couple.*

Proof. This is a trivial consequence of Theorem 1 and [2, Exposé XII, Proposition 6.5 (i)]. $\square$

Lemma 7. *Let X be a scheme and let X_0 be a closed subscheme. Let A be a noetherian ring and assume that X is proper over $Spec(A)$. Also assume that $X_0 = X \times_{Spec(A)} Spec(A/I)$ for some ideal $I \subseteq A$. Put $J = ker(B = \mathscr{O}_X(X) \longrightarrow \mathscr{O}_{X_0}(X_0))$. If (B, J) is an henselian pair, then (X, X_0) is an henselian couple.*

Proof. Let (A^h, I^h) be the henselization of the couple (A, I) given by [12, Tag 0A02]. Then we have the following diagram

$$\begin{tikzcd} & (X, X_0) \arrow[d, "f"] \\ (Spec(A^h), Spec(A^h/I^h)) \arrow[r, "\gamma"] & (Spec(A), Spec(A/I)) \end{tikzcd}$$

which induces the following diagram of pairs:

$$\begin{tikzcd} & (B, J) \\ (A^h, I^h) \arrow[ur, dashed, "\psi"] & (A, I) \arrow[l] \arrow[u] \end{tikzcd}$$

The morphism ψ is the one induced by the universal property of (A^h, I^h). As

$$Hom_{Rings}(A^h, B) = Hom_{Schemes}(X, Spec(A^h))$$

the homomorphism ψ identifies a unique morphism of schemes $\phi : X \longrightarrow Spec(A^h)$. Thus we get the following commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[d, "f"] \\ Spec(A^h) \arrow[r, "\gamma"] & Spec(A) \end{tikzcd}$$

Moreover, by [12, Tag 0AGU], we get that

$$\gamma^{-1}(Spec(A/I)) = Spec(A^h \otimes_A A/I) = Spec(A^h/I^h)$$

whence

$$X \times_{Spec(A^h)} Spec(A^h/I^h) = X_0$$

Therefore, the couple (X, X_0) lies over the henselian couple $(Spec(A^h), Spec(A^h/I^h))$. Furthermore, A^h is a noetherian ring (see [12, Tag 0AGV]). Finally, as f is a proper morphism and γ is separated, we get that ϕ is proper as well by [10, Proposition 3.3.16]. Then we can conclude that (X, X_0) is an henselian couple by the previous lemma. □

The previous lemma tells us that, under some appropriate hypothesis, if the pair

$$(\mathscr{O}_X(X), ker(\mathscr{O}_X(X) \longrightarrow \mathscr{O}_{X_0}(X_0)))$$

is henselian, then (X, X_0) is an henselian couple. It is natural to ask if the converse is true, i.e. if given an henselian couple (X, X_0) the associated pair is henselian. An answer is provided by the next lemma.

Lemma 8. *Let X be a quasi-compact and quasi-separated scheme and let $i : X_0 \longrightarrow X$ be a closed immersion such that (X, X_0) is an henselian couple. Then $(B, J) = (\mathscr{O}_X(X), ker(\mathscr{O}_X(X) \longrightarrow \mathscr{O}_{X_0}(X_0)))$ is an henselian pair.*

Proof. By [12, Tag 09XI], it is sufficient to show that for every étale ring map $B \longrightarrow C$ together with a B-morphism $\sigma : C \longrightarrow B/J$, there exists a B-morphism $C \longrightarrow B$ which lifts σ.
Consider the cartesian diagram

$$\begin{tikzcd} X_C = X \times_{Spec(B)} Spec(C) \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(C) \arrow[r] & Spec(B) \end{tikzcd}$$

As $Spec(C) \longrightarrow Spec(B)$ is étale and separated, the morphism $X_C \longrightarrow X$ is étale and separated as well. Then, by [8, Proposition 18.5.4], we have a bijection

$$\Gamma(X_C/X) \longrightarrow \Gamma(X_C \times_X X_0/X_0)$$

between the sections of $X_C \longrightarrow X$ and those of $X_C \times_X X_0 \longrightarrow X_0$.
Observation 1. The universal property of $X_C \times_X X_0$ tells us that

$$\Gamma(X_C \times_X X_0/X_0) \cong Hom_X(X_0, X_C)$$

Observation 2. Let $\mathscr{J} \subseteq \mathscr{O}_X$ be the sheaf of ideals associated to X_0. Then we have a short exact sequence of $\mathscr{O}_X$-modules

$$0 \longrightarrow \mathscr{J} \longrightarrow \mathscr{O}_X \longrightarrow i_*\mathscr{O}_{X_0} \longrightarrow 0$$

Applying the global sections functor, we get an exact sequence

$$0 \longrightarrow J = \mathscr{J}(X) \longrightarrow \mathscr{O}_X(X) = B \longrightarrow \mathscr{O}_{X_0}(X_0)$$

Hence, we have an homomorphism

$$B/J \longrightarrow \mathscr{O}_{X_0}(X_0)$$

Therefore, we get a morphism of schemes

$$X_0 \longrightarrow Spec(\mathscr{O}_{X_0}(X_0)) \longrightarrow Spec(B/J)$$

Also notice that the diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X \arrow[d] \\
Spec(B/J) \arrow[r] & Spec(B)
\end{tikzcd}$$

is commutative.

Now consider the diagram

$$\begin{tikzcd}
X_0 \arrow[dr, dashed, "\exists!\, \tilde{\alpha}"'] \arrow[drr] \arrow[ddr] \arrow[ddd] & & \\
& X_C \arrow[r] \arrow[d] & X \arrow[d] \\
& Spec(C) \arrow[r] & Spec(B) \\
Spec(B/J) \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

Label $\tilde{\alpha} : X_0 \longrightarrow X_C$ the X-morphism provided by the universal property of X_C and let $\alpha : X \longrightarrow X_C$ be the corresponding X-morphism in $\Gamma(X_C/X)$. Consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[rr, bend left, "id_X"] \arrow[dr] & X_C \arrow[r] \arrow[d] & X \arrow[d] \\
& Spec(C) \arrow[r] & Spec(B) \\
& & Spec(B/J) \arrow[ul] \arrow[u]
\end{tikzcd}$$

and the corresponding commutative diagram in *Rings*:

$$\begin{tikzcd}
B \arrow[rr, bend left, "id_B"] & \mathscr{O}_{X_C}(X_C) \arrow[l, "\alpha"'] & B \arrow[l] \\
& C \arrow[u] \arrow[ul, "\psi"] \arrow[dr, "\sigma"'] & B \arrow[l, "\phi"] \arrow[u, "id_B"'] \arrow[d, "\pi"] \\
& & B/J
\end{tikzcd}$$

It is then clear that ψ is the B-morphism we were looking for. This concludes the proof of the lemma. $\square$

Corollary 1. *Let (X, X_0) be an henselian couple. Assume that X is proper over a noetherian ring A and that $X_0 = X \times_{Spec(A)} Spec(A/I)$ for some ideal $I \subseteq A$. Then (X, X_0) is proper over an henselian pair.*

Proof. As X is proper over $Spec(A)$, it is a quasi-compact and quasi-separated scheme. Hence, by Lemma 8, $(\mathscr{O}_X(X), ker(\mathscr{O}_X(X) \longrightarrow \mathscr{O}_{X_0}(X_0)))$ is an henselian pair. Therefore, by the same construction described in Lemma 7, we get that (X, X_0) is proper over (A^h, I^h). $\square$

Corollary 2. *Let (X, X_0) be a couple and assume that X is proper over a noetherian ring A and that $X_0 = X \times_{Spec(A)} Spec(A/I)$ for some ideal $I \subseteq A$. Then (X, X_0) is an henselian couple if and only if $(\mathscr{O}_X(X), ker(\mathscr{O}_X(X) \longrightarrow \mathscr{O}_{X_0}(X_0)))$ is an henselian pair.*

By Remark 3 every henselian couple (X, X_0) which arises as in Lemma 6 satisfies conditions *2.* and *3.* in [2, Exposé XII, Proposition 6.5] with $\mathbb{L} = \mathbb{P}$ and for every n. Then, applying Corollary 1, we get the following result:

Theorem 3. *Let (X, X_0) be an henselian couple. Assume that X is proper over a noetherian ring A and that $X_0 = X \times_{Spec(A)} Spec(A/I)$ for some ideal $I \subseteq A$. Then conditions 2. and 3. in* [2, Exp. XII, Remarks 6.13] *are satisfied with $\mathbb{L} = \mathbb{P}$ and for every n.*

This gives a positive answer to Question 1 if we assume that hypothesis (†) hold.

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