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PROPER BASE CHANGE OVER HENSELIAN PAIRS

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ABSTRACT. We discuss a question which appears in [*Séminaire de Géométrie Algébrique du Bous Marie- Théorie des topos et cohomologie étale des schémas, Exposé XII, Remarks 6.13*] concerning proper base change. In particular, we propose a solution in a particular non-affine case.

1 INTRODUCTION

The question we would like to answer is the following one²:

Question 1. *Let (X, X_0) be an henselian couple (in the sense of Definition 1). Is it true that (ii) and (iii) in [2, Exposé XII, Proposition 6.5] hold with $\mathbb{L} = \mathbb{P}$ and for every n ?*

This question appears in [2, Exposé XII, Remarks 6.13].

We can restate it as follows:

Question 2. *Let (X, X_0) be an henselian couple. Is it true that*

- a. the base change functor induces an equivalence between the category of étale coverings of X and the category of étale coverings of X_0 ?*
- b. for any torsion étale sheaf \mathcal{F} and for any integer n , the morphism*

$$H^n(X, \mathcal{F}) \longrightarrow H^n(X_0, \mathcal{F}|_{X_0})$$

is an isomorphism?

¹the content of this note was obtained before the author started to receive the above mentioned scholarship, but it was organized in the present manuscript afterwards.

²for the notation and the exact definition of the objects that are involved we refer to the original source.

Remark 1. 1. When X is proper and finitely presented over an henselian ring (A, m) and $X_0 = X \times_{\text{Spec}(A)} \text{Spec}(A/m)$, we know that the answer to Question 1 is affirmative. This is the proper base change theorem in étale cohomology.

2. The case $(X, X_0) = (\text{Spec}(A), \text{Spec}(A/I))^3$, for (A, I) an henselian pair was studied and solved by R. Elkik in [4] and by O. Gabber in [5].

We propose a solution in the following situation:

(†) Let X be proper over a noetherian affine scheme $\text{Spec}(A)$ and $X_0 = X \times_{\text{Spec}(A)} \text{Spec}(A/I)$ for some ideal $I \subseteq A$.

We will see that, under these assumptions, (X, X_0) is an henselian couple for which Question 1 has a positive answer. To achieve this, we will first generalize [1, Theorem 3.1] to the following form:

Theorem 1. Let (A, I) be an henselian pair. Let $S = \text{Spec}(A)$ and let $f : X \rightarrow S$ be a proper finitely presented morphism. Let $X_0 = X \times_S S_0$, where $S_0 = \text{Spec}(A/I)$. Then

$$\begin{aligned} \mathcal{E}t_f(X) &\longrightarrow \mathcal{E}t_f(X_0) \\ Z &\mapsto Z \times_S S_0 \end{aligned}$$

is an equivalence of categories.

Here $\mathcal{E}t_f(W)$ denotes the category of finite étale schemes over W . The key tools for the proof are Artin's approximation theory and [12, Tag 0AH5], which combined with [1, Corollary 1.8] yields the following theorem

Theorem 2. Let (A, I) be an henselian pair with A noetherian. Let \hat{A} be the I -adic completion of A and assume that one of the following hypothesis is satisfied:

1. $A \rightarrow \hat{A}$ is a regular ring map;
2. A is a G -ring;
3. (A, I) is the henselization⁴ of a pair (B, J) , where B is a noetherian G -ring.

Let \mathcal{F} be a functor which is locally of finite presentation⁵

$$A\text{-algebras} \longrightarrow \text{Sets}$$

Given any $\hat{\xi} \in \mathcal{F}(\hat{A})$ and any $N \in \mathbb{N}$, there exists an element $\xi \in \mathcal{F}(A)$ such that

$$\xi \equiv \hat{\xi} \pmod{I^N}$$

i.e. ξ and $\hat{\xi}$ have the same image in $\mathcal{F}(A/I^N) \cong \mathcal{F}(\hat{A}/\hat{I}^N)$

³an affine henselian couple is the same thing as an henselian pair. See Remark 7.

⁴here the henselization is the left adjoint to the inclusion functor *Henselian Pairs* \rightarrow *Pairs*

⁵see [1, Definition 1.5]

Remark 2. In Theorem 2 we have that $3. \Rightarrow 2. \Rightarrow 1.$ See [12, Tag 0AH5].

Remark 3. Theorem 1, joined with [5, Corollary 1], gives us a positive answer to Question 1 when (X, X_0) is proper and finitely presented over an henselian pair. Moreover, we will see how we can always reduce to this case from situation (\dagger) .

2 PROOF OF THEOREM 1

This proof is an adaption of the one given in the local case by Artin (see [1, Theorem 3.1]). This generalization is possible thanks to Popescu's characterization of regular morphisms between noetherian rings, which provides us Theorem 2 as a corollary.

First we reduce to the case where A is the henselization of a finitely presented \mathbb{Z} -algebra. in order to do this, we need the following two preliminary lemmas.

Lemma 1. *Let $S = \text{Spec}(A)$ and let $g : X \rightarrow S$ be a proper morphism of finite presentation. Then the functor*

$$\mathcal{F} : A\text{-Algebras} \rightarrow \text{Sets}$$

$$B \mapsto \{\text{finite étale coverings of } \text{Spec}(B) \times_S X\} / \text{isomorphism}$$

is locally of finite presentation.

Proof. See the beginning of the proof of [1, Theorem 3.1]. □

Lemma 2. *Let $S = \text{Spec}(A)$ and let $g : X \rightarrow S$ be a proper morphism of finite presentation. Let $Z_1 \rightarrow X$ and $Z_2 \rightarrow X$ be two finite étale covers of X . Then the functor*

$$\mathcal{G} : A\text{-algebras} \rightarrow \text{Sets}$$

$$B \mapsto \text{Hom}_{X \times_S \text{Spec}(B)}(Z_1 \times_S \text{Spec}(B), Z_2 \times_S \text{Spec}(B))$$

is locally of finite presentation.

Proof. The lemma is a straightforward consequence of [7, Theorem 8.8.2.(i)]. □

Let (A, I) be an henselian pair and write A as a direct limit $\varinjlim A_i$, where each A_i is a subalgebra of A that is finitely generated over \mathbb{Z} . Let $(A_i^h, (I \cap A_i)^h)$ be the henselization of $(A_i, (I \cap A_i))$ for each i . Then by [11, Chapter XI, Proposition 2] $\varinjlim (A_i^h, (I \cap A_i)^h)$ is an henselian pair. It is easy to see that

$$(A, I) = \varinjlim (A_i^h, (I \cap A_i)^h)$$

Write $S_i = \text{Spec}(A_i^h)$ for every index i . Then

$$S = \varprojlim S_i$$

By [7, Theorem 8.8.2. (ii)] we know that X comes from a finitely presented scheme X_{i_0} for some index i_0 , i.e. $X \cong X_{i_0} \times_{S_{i_0}} S$. Moreover, by [7, Theorem 8.10.5], we can assume that X_{i_0} is also proper over S_{i_0} . As the functor

$$\mathcal{F} : A_{i_0}^h - \text{Algebras} \longrightarrow \text{Sets}$$

$$B \mapsto \{\text{finite étale coverings of } \text{Spec}(B) \times_{S_{i_0}} X_{i_0}\} / \text{isomorphism}$$

is locally of finite presentation, we have that

$$\mathcal{F}(A) = \varinjlim \mathcal{F}(A_i^h)$$

Therefore, every finite étale cover of X comes from a finite étale cover of $X_i = S_i \times_{S_{i_0}} X_{i_0}$ for a suitable index i .

Remark 4. *All schemes $X_{i_0} \times_{S_{i_0}} S_i$ and $X \cong X_{i_0} \times_{S_{i_0}} S$ are quasi-compact and quasi-separated, as they are proper over affine schemes.*

Let $Z \rightarrow X$ and $W \rightarrow X$ be two finite étale covers of X . Then we can assume without loss of generality that they come from two finite étale covers $Z_{i_0} \rightarrow X_{i_0}$, $W_{i_0} \rightarrow X_{i_0}$. Then by Lemma 2 we see that

$$\varinjlim \text{Hom}_{X_i}(Z_i, W_i) = \text{Hom}_X(Z, W)$$

It is then clear that we can reduce the proof of Theorem 1 to the case where (A, I) is the henselization of a pair (B, J) , where B is finitely generated over \mathbb{Z} . In particular, B is a G-ring and Theorem 2 holds.

Lemma 3. *The functor in Theorem 1 is essentially surjective.*

Proof. Consider a finite étale morphism $X'_0 \rightarrow X_0$. Label \hat{A} the completion of A with respect to the ideal I and let $\hat{S} = \text{Spec}(\hat{A})$, $\hat{X} = X \times_S \hat{S}$. Notice that \hat{A} is a complete separated ring by Krull's theorem (see [3, Theorem 10.17]). By [8, Theorem 18.3.4], we have that the functor

$$\mathcal{E}t_f(\hat{X}) \longrightarrow \mathcal{E}t_f(X_0)$$

$$Z \mapsto Z \times_S S_0$$

is an equivalence of categories. Then there exists some $[\hat{X}' \rightarrow \hat{X}] \in \mathcal{F}(\hat{A})$ such that

$$\hat{X}' \times_{\hat{S}} S_0 \cong X'_0$$

By Theorem 2 we get that there exists some finite étale morphism $X' \rightarrow X$ which is congruent modulo I to $\hat{X}' \rightarrow \hat{X}$, i.e.

$$X' \times_S S_0 \cong X'_0$$

□

It remains only to show that the functor in Theorem 1 is fully faithful.

Lemma 4. *The functor in Theorem 1 is fully faithful.*

Proof. Let X' and X'' be two finite étale schemes over X and let $\phi \in \text{Hom}_X(X', X'')$. The morphism ϕ corresponds uniquely to its graph $\Gamma_\phi : X' \rightarrow X' \times_X X''$, which is an open immersion as both X' and X'' are of finite type over X and as X'' is étale over X (see [9, Corollaire 3.4]). Also notice that Γ_ϕ is a closed immersion (see [10, Exercise 3.3.10]). If we assume that X' is connected and nonempty, ϕ corresponds uniquely to a connected component of $X' \times_X X''$ of degree one over X' . The degree of such a component can be measured at any point of X' . We conclude therefore by applying the next lemma to a component of $X' \times_X X''$. \square

Lemma 5. *X is nonempty and connected if and only if the same is true for X_0 .*

Proof. We are given the following cartesian square

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & & \downarrow f \\ S_0 & \longrightarrow & S \end{array}$$

If X is connected and nonempty, then $f(X) \subseteq S$ is a nonempty closed subset of S (as f is proper). Let J be an ideal of A that identifies $f(X)$. Let $f(x) = p \in V(J)$ be a closed point of S . As I is contained in the Jacobson radical of A , the prime ideal p lies in S_0 . Then

$$\begin{array}{ccccc} \{x\} & & & & \\ & \searrow & & & \\ & & X_0 & \longrightarrow & X \\ & \swarrow & \downarrow & & \downarrow f \\ & & S_0 & \longrightarrow & S \end{array}$$

In particular, X_0 is nonempty. Furthermore, as this argument can be used for any connected component of X , if X is disconnected then also X_0 is disconnected. Conversely, assume that X_0 is disconnected. Label C_0 a nonempty connected component of X_0 . As the scheme X_0 is quasi-compact, C_0 is open and closed in X_0 . Therefore, $C_0 \rightarrow X_0$ is a finite étale morphism. By Lemma 3, there exists a finite étale morphism $C \rightarrow X$ which induces $C_0 \rightarrow X_0$. As C_0 is connected and nonempty, the same is true for C . The morphism $C \rightarrow X$ is therefore of degree 1 at every point of C . As it is also finite and étale, it is both an open and a closed immersion, i.e. C is a connected component of X . If $C = X$, we would get $C_0 = X_0$, a contradiction. Then X is disconnected. Finally, it is clear that if X_0 is nonempty, X is nonempty too. \square

Theorem 1 follows immediately from Lemma 3 and Lemma 4.

3 HENSELIAN COUPLES

Recall that an henselian pair (A, I) is a ring A together with an ideal $I \subseteq A$ such that

1. I is contained in the Jacobson ideal of A ;
2. for every finite A algebra B , there is a bijection between the set of idempotent elements of B and the set of idempotent elements of $B \otimes_A A/I$.

For more details, see [11].

Let (A, I) be an henselian pair. Then for every finite morphism $\text{Spec}(B) = X \rightarrow \text{Spec}(A)$, we have a bijection

$$\text{Id}(B) = \text{Of}(X) = \text{Of}(X_0) = \text{Id}(B/IB) \quad \text{where } X_0 = X \times_{\text{Spec}(A)} \text{Spec}(A/I)$$

Here $\text{Of}(Z)$ denotes the set of subsets of Z which are both open and closed. This fact suggests the following definition (see [8, Définition 18.5.5]), which is meant to generalize the notion of henselian pair to the non-affine setting.

Definition 1. *Let X be a scheme and let X_0 be a closed subscheme. We say that (X, X_0) is an henselian couple if for every finite morphism $Y \rightarrow X$ we have a bijection*

$$\text{Of}(Y) = \text{Of}(Y_0)$$

where $Y_0 = Y \times_X X_0$.

Remark 5. *If X is locally noetherian, it is a consequence of [6, Proposition 6.1.4] and [6, Corollaire 6.1.9] that connected sets in $\text{Of}(X)$ (resp. $\text{Of}(X_0)$) are in bijection with $\Pi_0(X)$ (resp. $\Pi_0(X_0)$), the set of connected components of X (resp. X_0).*

Remark 6. *It is a consequence of [6, Corollary 5.1.8] that (X, X_0) is an henselian couple if and only if $(X_{\text{red}}, (X_0)_{\text{red}})$ is an henselian couple as well.*

Remark 7. *It is immediate to observe that if (A, I) is a pair and $(\text{Spec}(A), \text{Spec}(A/I))$ is an henselian couple, then I is contained in the Jacobson radical of A . In fact, if $m \subseteq A$ is a maximal ideal, then we have a bijection*

$$\text{Of}(\text{Spec}(A/m)) = \text{Of}(\text{Spec}(A/m \otimes_A A/I))$$

In particular, $\text{Spec}(A/m \otimes_A A/I)$ can not be the empty scheme. Therefore, as it is a closed subscheme of $\text{Spec}(A/m)$, we must have an equality $\text{Spec}(A/m) = \text{Spec}(A/m \otimes_A A/I)$, whence $I \subseteq m$. Moreover, if $Z \rightarrow \text{Spec}(A)$ is a finite morphism, then $Z = \text{Spec}(B)$ is affine and the corresponding morphism $A \rightarrow B$ is finite. Then we have bijections

$$\text{Id}(B) = \text{Of}(\text{Spec}(B)) = \text{Of}(\text{Spec}(B/IB)) = \text{Id}(B/IB)$$

We have just showed that an affine henselian couple is an henselian pair. The converse was observed at the beginning of this section.

Lemma 6. *Let (A, I) be an henselian pair with A noetherian and let X be a proper A -scheme. Set $S = \text{Spec}(A)$, $S_0 = \text{Spec}(A/I)$ and let $X_0 = X \times_S S_0$. Then (X, X_0) is an henselian couple.*

Proof. This is a trivial consequence of Theorem 1 and [2, Exposé XII, Proposition 6.5 (i)]. \square

Lemma 7. *Let X be a scheme and let X_0 be a closed subscheme. Let A be a noetherian ring and assume that X is proper over $\text{Spec}(A)$. Also assume that $X_0 = X \times_{\text{Spec}(A)} \text{Spec}(A/I)$ for some ideal $I \subseteq A$. Put $J = \ker(B = \mathcal{O}_X(X) \rightarrow \mathcal{O}_{X_0}(X_0))$. If (B, J) is an henselian pair, then (X, X_0) is an henselian couple.*

Proof. Let (A^h, I^h) be the henselization of the couple (A, I) given by [12, Tag 0A02]. Then we have the following diagram

$$\begin{array}{ccc} & & (X, X_0) \\ & & \downarrow f \\ (Spec(A^h), Spec(A^h/I^h)) & \xrightarrow{\gamma} & (Spec(A), Spec(A/I)) \end{array}$$

which induces the following diagram of pairs:

$$\begin{array}{ccc} & & (B, J) \\ & \nearrow \psi & \uparrow \\ (A^h, I^h) & \longleftarrow & (A, I) \end{array}$$

The morphism ψ is the one induced by the universal property of (A^h, I^h) . As

$$\text{Hom}_{\text{Rings}}(A^h, B) = \text{Hom}_{\text{Schemes}}(X, \text{Spec}(A^h))$$

the homomorphism ψ identifies a unique morphism of schemes $\phi : X \rightarrow \text{Spec}(A^h)$. Thus we get the following commutative diagram

$$\begin{array}{ccc} & & X \\ & \swarrow \phi & \downarrow f \\ Spec(A^h) & \xrightarrow{\gamma} & Spec(A) \end{array}$$

Moreover, by [12, Tag 0AGU], we get that

$$\gamma^{-1}(\text{Spec}(A/I)) = \text{Spec}(A^h \otimes_A A/I) = \text{Spec}(A^h/I^h)$$

whence

$$X \times_{\text{Spec}(A^h)} \text{Spec}(A^h/I^h) = X_0$$

Therefore, the couple (X, X_0) lies over the henselian couple $(\text{Spec}(A^h), \text{Spec}(A^h/I^h))$. Furthermore, A^h is a noetherian ring (see [12, Tag 0AGV]). Finally, as f is a proper morphism and γ is separated, we get that ϕ is proper as well by [10, Proposition 3.3.16]. Then we can conclude that (X, X_0) is an henselian couple by the previous lemma. \square

The previous lemma tells us that, under some appropriate hypothesis, if the pair

$$(\mathcal{O}_X(X), \ker(\mathcal{O}_X(X) \rightarrow \mathcal{O}_{X_0}(X_0)))$$

is henselian, then (X, X_0) is an henselian couple. It is natural to ask if the converse is true, i.e. if given an henselian couple (X, X_0) the associated pair is henselian. An answer is provided by the next lemma.

Lemma 8. *Let X be a quasi-compact and quasi-separated scheme and let $i : X_0 \rightarrow X$ be a closed immersion such that (X, X_0) is an henselian couple. Then $(B, J) = (\mathcal{O}_X(X), \ker(\mathcal{O}_X(X) \rightarrow \mathcal{O}_{X_0}(X_0)))$ is an henselian pair.*

Proof. By [12, Tag 09XI], it is sufficient to show that for every étale ring map $B \rightarrow C$ together with a B -morphism $\sigma : C \rightarrow B/J$, there exists a B -morphism $C \rightarrow B$ which lifts σ .

Consider the cartesian diagram

$$\begin{array}{ccc} X_C = X \times_{\text{Spec}(B)} \text{Spec}(C) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec}(C) & \longrightarrow & \text{Spec}(B) \end{array}$$

As $\text{Spec}(C) \rightarrow \text{Spec}(B)$ is étale and separated, the morphism $X_C \rightarrow X$ is étale and separated as well. Then, by [8, Proposition 18.5.4], we have a bijection

$$\Gamma(X_C/X) \rightarrow \Gamma(X_C \times_X X_0/X_0)$$

between the sections of $X_C \rightarrow X$ and those of $X_C \times_X X_0 \rightarrow X_0$.

Observation 1. The universal property of $X_C \times_X X_0$ tells us that

$$\Gamma(X_C \times_X X_0/X_0) \cong \text{Hom}_X(X_0, X_C)$$

Observation 2. Let $\mathcal{J} \subseteq \mathcal{O}_X$ be the sheaf of ideals associated to X_0 . Then we have a short exact sequence of \mathcal{O}_X -modules

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_X \rightarrow i_*\mathcal{O}_{X_0} \rightarrow 0$$

Applying the global sections functor, we get an exact sequence

$$0 \rightarrow J = \mathcal{J}(X) \rightarrow \mathcal{O}_X(X) = B \rightarrow \mathcal{O}_{X_0}(X_0)$$

Hence, we have an homomorphism

$$B/J \longrightarrow \mathcal{O}_{X_0}(X_0)$$

Therefore, we get a morphism of schemes

$$X_0 \longrightarrow \text{Spec}(\mathcal{O}_{X_0}(X_0)) \longrightarrow \text{Spec}(B/J)$$

Also notice that the diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec}(B/J) & \longrightarrow & \text{Spec}(B) \end{array}$$

is commutative.

Now consider the diagram

$$\begin{array}{ccccc} X_0 & & & & \\ \downarrow & \searrow \tilde{\alpha} & & \searrow & \\ & X_C & \longrightarrow & X & \\ & \downarrow & & \downarrow & \\ & \text{Spec}(C) & \longrightarrow & \text{Spec}(B) & \\ & \nearrow & \nearrow & & \\ \text{Spec}(B/J) & & & & \end{array}$$

Label $\tilde{\alpha} : X_0 \rightarrow X_C$ the X -morphism provided by the universal property of X_C and let $\alpha : X \rightarrow X_C$ be the corresponding X -morphism in $\Gamma(X_C/X)$. Consider the following commutative diagram

$$\begin{array}{ccccc} & & \text{id}_X & & \\ & \longleftarrow & \text{---} & \longrightarrow & \\ X & \xrightarrow{\alpha} & X_C & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & \text{Spec}(C) & \longrightarrow & \text{Spec}(B) \\ & & \nwarrow & \nearrow & \\ & & \text{Spec}(B/J) & & \end{array}$$

and the corresponding commutative diagram in *Rings*:

$$\begin{array}{ccccc}
 & & \xrightarrow{id_B} & & \\
 & & \curvearrowright & & \\
 B & \xleftarrow{\alpha} & \mathcal{O}_{X_C}(X_C) & \xleftarrow{\quad} & B \\
 & \searrow \psi & \uparrow & & \uparrow id_B \\
 & & C & \xleftarrow{\quad} & B \\
 & & & \searrow \phi & \downarrow \pi \\
 & & & & B/J \\
 & & & \searrow \sigma & \\
 & & & &
 \end{array}$$

It is then clear that ψ is the B -morphism we were looking for. This concludes the proof of the lemma. \square

Corollary 1. *Let (X, X_0) be an henselian couple. Assume that X is proper over a noetherian ring A and that $X_0 = X \times_{\text{Spec}(A)} \text{Spec}(A/I)$ for some ideal $I \subseteq A$. Then (X, X_0) is proper over an henselian pair.*

Proof. As X is proper over $\text{Spec}(A)$, it is a quasi-compact and quasi-separated scheme. Hence, by Lemma 8, $(\mathcal{O}_X(X), \ker(\mathcal{O}_X(X) \rightarrow \mathcal{O}_{X_0}(X_0)))$ is an henselian pair. Therefore, by the same construction described in Lemma 7, we get that (X, X_0) is proper over (A^h, I^h) . \square

Corollary 2. *Let (X, X_0) be a couple and assume that X is proper over a noetherian ring A and that $X_0 = X \times_{\text{Spec}(A)} \text{Spec}(A/I)$ for some ideal $I \subseteq A$. Then (X, X_0) is an henselian couple if and only if $(\mathcal{O}_X(X), \ker(\mathcal{O}_X(X) \rightarrow \mathcal{O}_{X_0}(X_0)))$ is an henselian pair.*

By Remark 3 every henselian couple (X, X_0) which arises as in Lemma 6 satisfies conditions 2. and 3. in [2, Exposé XII, Proposition 6.5] with $\mathbb{L} = \mathbb{P}$ and for every n . Then, applying Corollary 1, we get the following result:

Theorem 3. *Let (X, X_0) be an henselian couple. Assume that X is proper over a noetherian ring A and that $X_0 = X \times_{\text{Spec}(A)} \text{Spec}(A/I)$ for some ideal $I \subseteq A$. Then conditions 2. and 3. in [2, Exp. XII, Remarks 6.13] are satisfied with $\mathbb{L} = \mathbb{P}$ and for every n .*

This gives a positive answer to Question 1 if we assume that hypothesis (†) hold.

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