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ALGEBRAIC APPROXIMATIONS OF UNIRULLED COMPACT KÄHLER THREEFOLDS

by

Hsueh-Yung Lin

Abstract. — We study deformations of non-algebraic compact Kähler threefolds admitting a conic bundle structure and prove as a corollary of the main result that every uniruled compact Kähler threefold has an arbitrarily small deformation to some projective variety.

1 Introduction

Given a compact Kähler manifold $X$, we say that $X$ admits an algebraic approximation if there exist arbitrarily small deformations of $X$ to some projective variety. Whether a compact Kähler manifold admits an algebraic approximation is known as the Kodaira problem. This problem has a positive answer for all compact Kähler manifolds of dimension $\leq 2 \ [13, 4]$. However starting from dimension 4 and on, compact Kähler manifolds which do not admit any algebraic approximation are known to exist in each dimension \([20]\) and there even exist such manifolds which are homotopically obstructed to being projective.

For compact Kähler threefolds the Kodaira problem is still open in general, but some partial results are known. For a compact Kähler threefold $X$ of non-negative Kodaira dimension $\kappa$ (i.e. non-uniruled threefolds thanks to the abundance conjecture \([8]\)), the "bimeromorphic" version of the Kodaira problem has a positive answer, namely one can always find a (smooth) bimeromorphic model of $X$ which admits an algebraic approximation \([12, 14, 9]\). Note that by a conjecture of Peternell (cf. \([12, \text{Conjecture 1.2}]\)), the same conclusion is expected to hold for all compact Kähler manifolds with $\kappa \geq 0$, whereas for uniruled manifolds, there exist examples (of dimension $\geq 8$) which answer negatively the bimeromorphic Kodaira problem \([21]\). We also know that threefolds with $\kappa = 0$ or 1 admit algebraic approximations \([15]\).

For uniruled threefolds, the non-algebraic ones are all bimeromorphic to $\mathbb{P}^1$-fibrations over a non-algebraic surface. Furthermore, these $\mathbb{P}^1$-fibrations are bimeromorphic to standard conic bundles (cf. Proposition 3.1 for the precise statement and the definition of standard conic bundles). The Kodaira problem for some conic bundles of dimension 3 had already been studied in \([19]\). In this work, we study the existence of good algebraic approximations of standard conic bundles of dimension 3. The main result is the following.

Theorem 1.1. — Let $f : X \to S$ be a standard conic bundle over a non-algebraic smooth compact Kähler surface $S$. There exists a deformation $f : \mathcal{X} \to \mathcal{S} \to \Delta$ of $f$ such that for each proper subvariety $C \subset S$ (not necessarily equidimensional), the family $\mathcal{X} \to \Delta$ is an $f^{-1}(C)$-locally trivial algebraic approximations of $X$. 
As a corollary, we will see that in spite of Voisin’s examples of uniruled manifolds in higher dimension constructed in [21] mentioned above, the Kodaira problem has a positive answer for uniruled threefolds.

**Corollary 1.2.** — Every uniruled compact Kähler threefold has an algebraic approximation.

Now we outline the structure of the proof of Theorem 1.1. The algebraic dimension $a(S)$ of $S$ is either 0 or 1. In the case where $a(S) = 0$, the standard conic bundle $f$ is in fact a $\mathbb{P}^1$-bundle (Proposition 4.1). Since $a(S) = 0$, the minimal model of $S$ is either a 2-torus or a K3 surface. The former case is easier and will be studied in Section 4. As for the latter case, regarding a $\mathbb{P}^1$-bundle as the projectivization of a twisted vector bundle $E$ of rank 2, we will study instead the algebraic approximation of the pair $(S, E)$. This will be carried out in Section 5. If $a(S) = 1$, then $S$, being an elliptic fibration $p : S \to B$, admits a strongly locally trivial algebraic approximation (cf. Section 2 for the definition). In Section 6, we show that such an algebraic approximation can always be lifted to a strongly locally trivial algebraic approximation of $X \to B$, which implies Theorem 1.1 in the case where $a(S) = 1$.

### 2 Convention, terminologies, and a projectivity criterion

In this article, a *complex variety* is a connected, Hausdorff, reduced, and irreducible complex space. When a complex variety is smooth, it is called a *complex manifold*. A *threefold* is a complex manifold of dimension 3. A *fibration* is a surjective holomorphic map $f : X \to Y$ between complex spaces with connected fibers. The fiber $f^{-1}(t)$ of $f$ over $t \in Y$ is denoted by $X_t$.

For the reader’s convenience, we recollect some terminologies from [15] that we will keep using. Let $X$ be a complex variety. A deformation of $X$ is a surjective flat holomorphic map $\mathcal{X} \to \Delta$ containing $X$ as a fiber. Let $C$ be a subvariety of a complex variety $X$. We say that a deformation $\mathcal{X} \to \Delta$ of $X$ is *$\Delta$-locally trivial* if there exist a Zariski closed subset $\mathcal{C} \subset \mathcal{X}$ and a neighborhood(1) $\mathcal{U} \subset \mathcal{X}$ of $\mathcal{C}$ such that $\mathcal{C} \cap X = C$ and that we have isomorphisms $\mathcal{C} \cong C \times \Delta$ and $\mathcal{U} \cong (\mathcal{U} \cap X) \times \Delta$ over $\Delta$. Let $f : X \to B$ be a fibration (endowed with an equivariant action by a finite group $G$). A (G-equivariant) deformation of $f$ is a deformation $\pi : \mathcal{X} \to \mathcal{B} \to \Delta$ of $X$ (preserving the $G$-action) which factorizes through a deformation $\mathcal{B} \to \Delta$ of $B$ (preserving the $G$-action) such that $q : \mathcal{X}_0 \to \mathcal{B}_0$ is isomorphic to $f$. A deformation of $f$ fixing the base $B$ is a deformation of $f$ such that $\mathcal{B} \to \Delta$ is isomorphic to the second projection map $B \times \Delta \to \Delta$. Such a deformation is called *strongly locally trivial* if for every point $t \in B$, there exists a neighborhood $U$ of $t$ such that $q^{-1}(U \times \Delta)$ is isomorphic to $f^{-1}(U) \times \Delta$ over $\Delta$. In particular, a strongly locally trivial deformation of $f : X \to B$ is $X_t$-locally trivial for every $t \in B$ (the converse is also true if $f$ is proper). Finally an *algebraic approximation* of $X$ is a deformation $\pi : \mathcal{X} \to \Delta$ of $X$ such that there exists a sequence of points $(t_n)_{n \in \mathbb{N}}$ in $\Delta$ converging to $\pi(X)$ and parameterizing algebraic members of $\pi$.

Recall that a complex manifold $X$ is *algebraically connected* if for any general pair of points $x, y \in X$, there exists a proper, connected (not necessarily irreducible) curve in $X$ containing $x$ and $y$. The following projectivity criterion, due to Campana and Moishezon, will be used many times in this article.

**Theorem 2.1 (Moishezon’s projectivity criterion + Campana [7, Corollaire in p.212])**

A compact complex manifold is projective if and only if it is Kähler and algebraically connected.

A $\mathbb{P}^1$-fibration is a fibration $f : X \to B$ whose general fiber is isomorphic to $\mathbb{P}^1$. An immediate consequence of the above criterion is the following projectivity criterion for $\mathbb{P}^1$-fibrations.

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(1)Unless otherwise indicated, neighborhoods and open subsets in this article are defined with respect to the Euclidean topology.
Corollary 2.2. — Let \( f : X \to B \) be a \( \mathbb{P}^1 \)-fibration. Assume that \( X \) is a compact Kähler manifold and \( B \) is projective, then \( X \) is projective.

Proof. — If \( B \) is algebraic connected, since \( f^{-1}(C) \) is a uniruled surface, hence algebraic for every curve \( C \subseteq B \), the threefold \( X \) is also algebraically connected. Therefore \( X \) is projective by Theorem 2.1. □

3 \( \mathbb{P}^1 \)-fibrations and standard conic bundles

In this article, a \( \mathbb{P}^1 \)-bundle is a smooth \( \mathbb{P}^1 \)-fibration. A conic bundle is a \( \mathbb{P}^1 \)-fibration \( f : X \to B \) such that \( f = \pi_X \) where \( X \) is identified with the zero locus of a non-trivial section \( \sigma \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \pi^*\mathcal{L}) \) for some locally free sheaves \( \mathcal{E} \) and \( \mathcal{L} \) over \( B \) of rank 3 and 1 respectively and \( \pi : \mathbb{P}(\mathcal{E}) \to B \) is the standard projection. The section \( \sigma \) defines a map \( \sigma : \mathcal{E} \to \mathcal{E}^\vee \otimes \mathcal{L} \), and hence a map \( \det \sigma : \det(\mathcal{E}) \to \det(\mathcal{E})^\vee \otimes \mathcal{L}^\otimes 3 \). The divisor \( D \) defined by \( \det \sigma \in H^0(\det(\mathcal{E})^\otimes 2 \otimes \mathcal{L}^\otimes 3) \) is called the discriminant locus of the conic bundle \( f \). As a set, this is the locus where the quadratic form defined by the restriction of \( \sigma \) to the fibers of \( \mathbb{P}^2(\mathcal{E}^\vee) \otimes \mathcal{L} \) does not have maximal rank.

A flat \( \mathbb{P}^1 \)-fibration \( f : X \to B \) with \( X \) and \( B \) assumed to be smooth is an example of conic bundles: By flatness of \( f \) which implies the vanishing of \( R^1f_*\mathcal{O}_X \) [3, Corollary III.11.2], the sheaf \( \mathcal{E}' := (f_*\alpha_X^\vee)^\vee \) is locally free of rank 3. Since the line bundle \( \alpha_X^\vee \) is relatively very ample, \( \alpha_X^\vee \) defines an embedding \( X \hookrightarrow \mathbb{P}(\mathcal{E}) \) and for each fiber \( \mathbb{P}(\delta_b) \) of \( \mathbb{P}(\mathcal{E}) \to B \), by construction \( X \cap \mathbb{P}(\delta_b) \) is a curve of degree 2 in \( \mathbb{P}(\delta_b) \). So \( X \) is the zero locus of a section \( \sigma \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \pi^*\mathcal{L}) \) for some line bundle \( \mathcal{L} \) over \( B \).

Given an algebraic \( \mathbb{P}^1 \)-fibration \( f : X \to B \), Sarkisov proved that it is always bimeromorphic to a standard conic bundle [18, Proposition 1.13]. In the following we extend Sarkisov’s theorem to non-algebraic \( \mathbb{P}^1 \)-fibrations, under the assumption that \( \dim B = 2 \).

Proposition 3.1 (cf. Miyanishi [17, Theorem in page 89] and Sarkisov [18, Proposition 1.13])

Let \( f : X \to B \) be a \( \mathbb{P}^1 \)-fibration over a compact complex surface \( B \). There exists a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\sim} & X' \\
\downarrow f & & \downarrow f' \\
B & \xrightarrow{\sim} & B'
\end{array}
\]

where the horizontal arrows are bimeromorphic maps and \( f' : X' \to B' \) is a standard conic bundle: Namely, \( f' \) is a flat conic bundle whose discriminant locus is a simple normal crossing divisor and \( X' \) and \( B' \) are compact complex manifolds with \( \text{Pic}(X') = f'^*\text{Pic}(B') \oplus \mathbb{Z}\omega_X \).

Remark 3.2. — When \( B \) is a projective surface, Proposition 3.1 was already proven in [17, Theorem in page 89] by Miyanishi and Zagorskih. In [18], Sarkisov proved a similar statement for any complete algebraic variety \( B \). In the complex analytic setting, it would be important to know whether Proposition 3.1 still holds without the hypothesis \( \dim B = 2 \) in order to study uniruled compact Kähler manifolds of higher dimension (e.g. the Kodaira problem). As we will see below, the assumption \( \dim B = 2 \) allows us first to find a conic bundle (over a compact base) which is bimeromorphic to the original one. Once we obtain a conic bundle bimeromorphic to \( f : X \to B \), the rest of the argument follows mutatis mutandis from the proof of Sarkisov’s theorem mentioned above.
Proof of Proposition 3.1. — Up to base-changing \( f \) with a desingularization of \( B \) and resolving the singularities, we can assume that both \( X \) and \( B \) are smooth. Let \( \mathcal{E} := (f, K_X^2)^{\vee} \). Since \( B \) is a smooth surface, the reflexive sheaf \( \mathcal{E} \) is locally free. Let \( B' \subseteq B \) be a nonempty Zariski open over which the fibration \( f \) is flat, then \( (f, K_X^2)_{|B'} \) is already locally free so \( X' := f^{-1}(B') \) is the zero locus of a section \( \sigma \in H^0(\mathcal{P}(\mathcal{E}), \mathcal{O}_{\mathcal{P}(\mathcal{E})}(2) \otimes \mathcal{N}^{\vee}) \) for some line bundle \( \mathcal{L}' \) over \( B' \). Therefore the closure \( \overline{X'} \) of \( X' \) in \( \mathcal{P}(\mathcal{E}) \) is the zero locus of some section \( \sigma \in H^0(\mathcal{P}(\mathcal{E}), \mathcal{O}_{\mathcal{P}(\mathcal{E})}(2) \otimes \mathcal{N}^{\vee}) \) for some line bundle \( \mathcal{L}' \) over \( B' \). As \( \overline{X'} \) is bimeromorphic to \( X \) over \( B' \), up to replacing \( X \) by \( \overline{X'} \) we can assume that \( f : X \to B \) is a conic bundle embedded into \( \mathcal{P}(\mathcal{E}) \).

Starting from the conic bundle \( f : X \to B \), the rest of the proof is almost the same as the proof of [18, Proposition 1.13] or [17, Theorem in page 89]. We shall only provide an outline and refer to loc. cit.\(^2\) for details.

For each point \( o \in B \), we can find a neighborhood \( U \subseteq B \) of \( o \) and local coordinates \( (x \in U; [X_0 : X_1 : X_2] \in \mathcal{P}(\mathcal{E})) \) of \( \mathcal{P}(\mathcal{E}) \) such that the restriction of \( \sigma \) to \( U \) viewed as a family of quadratic forms is of the form \( a_0(x)X_0^2 + a_1(x)X_1^2 + a_2(x)X_2^2 \). Let \( Z_i \subseteq B \) denote the locus where \( \sigma \) is of rank \( i \). Since \( X \) is irreducible, if \( Z_0 \neq \emptyset \) then \( \dim Z_0 = 0 \). Up to replacing \( X \) with the strict transformation of \( X \) in \( \mathcal{P}(\mathcal{E}) \) for some vector bundle \( \mathcal{E}' \) over the blow-up \( B' \) of \( B \) along \( Z_0 \) such that \( \mathcal{E}'_{|\overline{B'} \setminus \sigma(Z_0)} \cong \mathcal{E}_{\overline{B} \setminus \sigma \circ \rho(Z_0)} \) where \( \nu : B' \to B \) is the blow-up map, we can assume that \( Z_0 = \emptyset \) (cf. [17, page 90]).

Let \( C \subseteq B \) be the discriminant divisor of \( f \). Up to base-changing \( f : X \to B \) with a log-resolution of the pair \((B, C_{\mathrm{red}})\), we can assume that \( C_{\mathrm{red}} \) is a simple normal crossing divisor. By performing elementary transformations of \( f : X \to B \) along the non-reduced irreducible components of \( C \), we can further assume that \( C \) is reduced (cf. [18, Lemma 1.14] or [17, page 90 and 91]). Therefore locally around a point \( o \in C \setminus \Sing(C) \) (resp. \( o \in \Sing(C) \)), there exist local coordinates \((u, v)\) in which \( \sigma \) is of the form \( X_0^2 + uX_1^2 + X_2^2 \) (resp. \( X_0^2 + uX_1^2 + vX_2^2 \)). Accordingly \( Z_1 = \Sing(C) \) and \( Z_2 = C \setminus \Sing(C) \), so we can conclude by [18, Corollary 1.11] that \( f \) is flat and \( X \) is smooth.

It remains to show that \( f : X \to B \) can be contracted to a standard conic bundle. Suppose that \( C' \subseteq C \) is an irreducible component of \( C \) such that \( X_{C'} := f^{-1}(C') \) is reducible, then \( C' \cap \Sing(C) = \emptyset \). Indeed, if \( o \in C' \cap \Sing(C) \), then as we can see from the local expression of \( \sigma \) above, the monodromy action around \( o \in C' \) on the double cover of \( C' \setminus \Sing(C) \) parameterizing lines in the fibers of \( f \) exchanges the two lines in \( f^{-1}(p) \) for every \( p \in C' \), so the total space \( X_{C'} \) would be irreducible.

The divisor \( X_{C'} \) has two irreducible components \( E_1 \) and \( E_2 \): both \( E_1 \) and \( E_2 \) are ruled surfaces over \( C' \), and \( E_1 \cap E_2 \) is a section of both \( E_1 \to C' \) and \( E_2 \to C' \). It follows that \( F \cdot E_1 = -1 \) where \( F \) is a fiber of \( E_1 \to C' \). So one can blow down the divisor \( E_1 \) in \( X \) to a curve isomorphic to \( C' \) [10, Theorem 2] and obtain a new conic bundle which is smooth along the smooth curve \( C' \subseteq B \) as \( C' \cap \Sing(C) = \emptyset \). After contracting all such ruled surfaces in the same way, we finally obtain a conic bundle \( f : X \to B \) satisfying \( \rho(X) = \rho(B) + 1 \). \( \square \)

4 Case where the MRC-quotient is a 2-torus of algebraic dimension 0

In Section 4 and 5, we study deformations of standard conic bundles over a compact Kähler surface of algebraic dimension 0. First we note that such conic bundles are in fact smooth.

Proposition 4.1. — If \( X \to \tilde{S} \) is a standard conic bundle over a surface of algebraic dimension \( a(\tilde{S}) = 0 \), then it is a \( \mathbb{P}^1 \)-bundle.

\(^2\)N.B. The terminology used in [18] is different from ours. In [18], \( \mathbb{P}^1 \)-fibrations are called conic bundles and conic bundles are called embedded conic bundles.
Proof. — The minimal model $S$ of $	ilde{S}$ is either a K3 surface or a 2-torus. As $a(S) = 0$, there exists only finitely many curves in $S$ and the union of curves in $S$ has normal crossings and is a disjoint union of trees of (−2)-curves. Since $\tilde{S}$ is a sequence of blow-ups of $S$, the union of curves in $S$ is also a normal crossing divisor and the dual graph of which is still a disjoint union of trees. Accordingly connected components of the discriminant locus $C$ of $X \to \tilde{S}$ are also trees. So if $C$ is not empty, $C$ would contain an irreducible component $C'$ meeting the closure of $\setminus C'$ in $C$ in exactly one point, which is in contradiction with [19, Proposition 3.1]. □

The minimal model $S$ of a compact Kähler surface $\tilde{S}$ of algebraic dimension 0 is either a K3 surface or a 2-torus. In this section we study the easier case where $S$ is a 2-torus.

**Proposition 4.2.** — Let $f : X \to \tilde{S}$ be a $\mathbb{P}^1$-bundle over a (smooth) compact Kähler surface $\tilde{S}$ bimeromorphic to a torus $S$ of algebraic dimension 0. For every proper Zariski closed subset $C \subset \tilde{S}$, there exists an $f^{-1}(C)$-locally trivial algebraic approximation of $f$.

**Proof.** — Since $f$ is a $\mathbb{P}^1$-bundle, for every finite subset $Z \subset \tilde{S}$, every small deformation of $f$ is an $f^{-1}(Z)$-locally trivial deformation. Therefore we can assume that $C$ is a union of curves. Since $a(\tilde{S}) = 0$, there are only finitely many curves in $\tilde{S}$. So we can further assume that $C$ is the union of all curves in $\tilde{S}$.

First we treat the case where $\tilde{S} = S$ (so $C = \emptyset$). Let $\rho : \tilde{S} \to S$ be the universal cover of $S$ and let $f : \tilde{X} \to \tilde{S}$ be the pullback of $f$ under $\rho$. The fundamental group $\Lambda = \pi_1(S)$ of $S$ acts on $\tilde{S}$ by deck transformations and the pullback $\tilde{f} : \tilde{X} \to \tilde{S}$ induces a $\Lambda$-action on $\tilde{X}$ which is $f$-equivariant. Since $\tilde{S} \simeq \mathbb{C}^2$, the $\mathbb{P}^1$-fibration $f$ is trivial by the Oka principle; let us fix an identification $\tilde{X} = \mathbb{C}^2 \times \mathbb{P}^1$.

For each $g \in \Lambda$, the map

$$
\tau_g : \mathbb{C}^2 \to \text{Aut}(\mathbb{P}^1) \simeq \text{PGL}_2(\mathbb{C}) \quad x \mapsto (y \mapsto p_2(g \cdot (x, y))).
$$

is $\Lambda$-invariant (here $p_2 : \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the second projection). So $\tau_g$ descends to a map $\tau_g : S \to \text{PGL}_2(\mathbb{C})$. As $S$ is of algebraic dimension 0, the composition $S \twoheadrightarrow \tau_g \twoheadrightarrow \text{PGL}_2(\mathbb{C}) \to \mathbb{P}^1$ is constant, namely the image of $\tau_g$ is contained in a fiber of $\text{PGL}_2(\mathbb{C}) \to \mathbb{P}^1$, which is isomorphic to $\mathbb{C} \times \mathbb{C}^\times$. Therefore since $S$ is compact, by the the maximum principle the map $\tau_g$ is constant.

By abuse of notation, let $\tau_g \in \text{Aut}(\mathbb{P}^1)$ denote also the constant image of $\tau_g$. Then the map

$$
\Lambda \to \text{Aut}(\mathbb{P}^1)
$$

$$
g \mapsto \tau_g. \quad (4.1)
$$

is a group homomorphism. We conclude that under the identification $\tilde{X} = \mathbb{C}^2 \times \mathbb{P}^1$, the $\Lambda$-action on $\mathbb{C}^2 \times \mathbb{P}^1$ is diagonal.

Now let $\pi : \mathcal{U} \to \Lambda$ be an algebraic approximation of $S$ over a one-dimensional disc and let $\overline{\mathcal{U}} \to \mathcal{U}$ be the universal cover of $\mathcal{U}$. As $\Lambda$ is simply connected, for each $t \in \Lambda$, the fiber over $t \in \Lambda$ of the composition $\pi : \overline{\mathcal{U}} \to \mathcal{U} \to \Lambda$ is the universal cover of the torus $\mathcal{U}_t := \pi^{-1}(t)$. Also the restriction of the group action $\Lambda \simeq \pi_1(\mathcal{U})$ on $\overline{\mathcal{U}}$ by deck transformations to each fiber $\overline{\mathcal{U}}_t := \pi^{-1}(t)$ is again the $\Lambda$-action by deck transformations.

Let $\overline{\mathcal{X}} := \overline{\mathcal{U}} \times \mathbb{P}^1$ be endowed with the diagonal $\Lambda$-action where the $\Lambda$-action on $\mathbb{P}^1$ is given by (4.1) and let $\mathcal{X} := \overline{\mathcal{X}} / \Lambda$ be the quotient. Since fibers of $\overline{\mathcal{X}} \to \Lambda$ are $\Lambda$-invariant and since $\overline{\mathcal{X}} \to \overline{\mathcal{U}}$ is $\Lambda$-equivariant, the composition $\overline{\mathcal{X}} \to \overline{\mathcal{U}} \to \Lambda$ descends to the quotient.
The later is a family of $\mathbb{P}^1$-fibrations containing $f : X \to S$ as a member. For every $t \in \Delta$ such that $\mathcal{X}_t$ is algebraic, since $\mathcal{X}_t := \Pi^{-1}(t)$ is a $\mathbb{P}^1$-fibration over $\mathcal{X}_t$, the total space $\mathcal{X}_t$ is also algebraic by Corollary 2.2. As $\Delta$ contains a dense subset of points parameterizing algebraic members of $\mathcal{X} \to \Delta$, we conclude that (4.2) is an algebraic approximation of $f : X \to S$.

Now we deal with the general case. Let $\eta : \tilde{S} \to S$ be the contraction of the $(-1)$-curves of $\tilde{S}$ and let $P \subset S$ be the image of the exceptional divisor of $\eta$. For each $p \in P$, let $U_p \subset S$ be a sufficiently small Stein neighborhood of $p$ and let $U = \cup_{p \in P} U_p$. Let $\tilde{U} := \eta^{-1}(U)$. We claim that $H^2(\tilde{U}, \mathcal{O}_{\tilde{U}}) = 0$. Indeed, it suffices to show that $H^2(\tilde{U}, \mathcal{Z}) = 0$ and $H^2(\tilde{U}, \mathcal{O}_{\tilde{U}}) = 0$. Since $\tilde{U}$ retracts to trees of curves $D$, we have $H^2(U, \mathcal{Z}) = H^2(D, \mathcal{Z}) = 0$. As $U$ is Stein and fibers of $\eta$ are of dimension $\leq 1$, it follows that $H^2(\tilde{U}, \mathcal{O}_{\tilde{U}}) = H^0(U, R^1\eta_*\mathcal{O}_U) = 0$. So every $\mathbb{P}^1$-bundle over $\tilde{U}$ is the projectivization of some vector bundle $\mathcal{E}_U$, which is necessarily trivial when restricted to $\tilde{U}^\times := \eta^{-1}(U^\times) \approx U^\times := U\setminus P$, since $U$ is Stein and codim$U(U) = 2$.

Therefore we can replace $f^{-1}(\tilde{U}) \to \tilde{U}$ with $\mathbb{P}^1 \times U \to U$ in $f : X \to \tilde{S}$ and obtain a $\mathbb{P}^1$-bundle $g : Y \to S$ which is bimeromorphic to $f : X \to \tilde{S}$ and the restriction of the bimeromorphic map to $Y_{1^r} := g^{-1}(U^\times)$ is an isomorphism onto $X_{1^r} := f^{-1}(\tilde{U}^\times)$ over $U^\times \approx \tilde{U}^\times$. According to the above, there exists a strongly locally trivial algebraic approximation $\mathcal{Y} \xrightarrow{\approx} S \times \Delta \to \Delta$ of $g : Y \to S$. Up to shrinking $U$, we can use the isomorphisms $g^{-1}(U^\times \times \Delta) \approx Y_{1^r} \times \Delta \approx X_{1^r} \times \Delta$ to replace $g^{-1}(U \times \Delta)$ with $X_{1^r} \times \Delta$ in $\mathcal{Y}$ and obtain an algebraic approximation of $f : X \to S$, which is $f^{-1}(C)$-locally trivial as $f^{-1}(C) \subset X_{1^r}$. 

5 Case where the MRC-quotient is a K3 surface of algebraic dimension 0

Since $\mathbb{P}^1$-bundles are the projectivization of twisted vector bundles of rank 2, given Proposition 4.1, the deformation problem of standard conic bundles over a compact Kähler surface of algebraic dimension 0 is directly related to the deformation-obstruction theory of twisted vector bundles that we shall study now.

5.1 Semi-regularity maps and deformations of twisted vector bundles

The reader is referred to [6, Chapter 1] for basic knowledge concerning twisted sheaves.

Let $X$ be a complex space. Given a 2-cocycle $\alpha = \{a_{ijk}\}$ representing some element $[\alpha] \in H^2(X, \mathcal{O}_X^*)$ with respect to a Čech cover $\{U_i\}_{i \in I}$ of $X$, an $\alpha$-twisted vector bundle $(E, \alpha)$ of rank $r$ is, up to passing to some refinement of $\{U_i\}_{i \in I}$, a collection of holomorphic maps $g_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})$ for all $i \neq j \in I$ such that $g_{ji} \cdot g_{ij} = \Id$ and that $g_{ki} \cdot g_{kj} \cdot g_{ij} = a_{ijk} \cdot \Id_{U_i \cap U_j \cap U_k}$. Here $\Id_{U_i \cap U_j \cap U_k} : U_i \cap U_j \cap U_k \to GL(r, \mathbb{C})$ is the constant map to the identity $\Id \in GL(r, \mathbb{C})$. Similarly and more generally we can define $\alpha$-twisted coherent sheaves, which form an abelian category $\text{Coh}(X, \alpha)$. Up to equivalence of categories, $\text{Coh}(X, \alpha)$ does not depend on the representative $\alpha$ of $[\alpha] \in H^2(X, \mathcal{O}_X^*)$.

Let $g_{ij}$ be given as above and let $g_{ij}$ denote the composition of $g_{ij}$ with the map $GL(r, \mathbb{C}) \to PGL(r, \mathbb{C})$. Since the image of $g_{ij} \cdot g_{kj} \cdot g_{ij}$ consists of scalar matrices, the collection $\{g_{ij}\}$ defines a closed 1-cocycle and thus a $\mathbb{P}^{r-1}$-bundle over $X$ denoted by $\mathbb{P}(E, \alpha) \to X$ or simply $\mathbb{P}(E)$. Reciprocally, every $\mathbb{P}^{r-1}$-bundle over

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This category depends tacitly on the Čech cover $\{U_i\}_{i \in I}$. It is only up to equivalence of categories that $\text{Coh}(X, \alpha)$ is independent of $\{U_i\}_{i \in I}$. 

---
X is isomorphic over X to some $\mathbf{P}(E, \alpha) \to X$. Moreover, $(E, \alpha)$ can be chosen such that $[\alpha] \in H^2(X, \mathcal{O}_X^\times)$ is torsion [6, p.9 and Example 1.2.2]. Also if $\alpha'$ is another representative of $[\alpha]$ and if $(E, \alpha) \simeq (E', \alpha')$, then $\mathbf{P}(E, \alpha)$ is isomorphic to $\mathbf{P}(E', \alpha')$ over X.

Let $\mathcal{X} \to B$ be a holomorphic map and $\mathcal{E}$ an $\alpha$-twisted vector bundle over $\mathcal{X}$ which is flat over B. As in the untwisted case, we define the Atiyah class $\operatorname{At}(\mathcal{E})$ as follows. Let $\Delta_{\mathcal{X}} \subset \mathcal{X} \times_B \mathcal{X}$ denote the relative diagonal of $\mathcal{X}$ and let $I_{\Delta_{\mathcal{X}}}$ be its ideal sheaf. Let $p_i : \mathcal{X} \times_B \mathcal{X} \to \mathcal{X}$ be the $i$-th projection. Regarding $\mathcal{O}_{\mathcal{X} \times_B \mathcal{X}}/I_{\Delta_{\mathcal{X}}}^2$ as an $(\alpha^{-1} \otimes \alpha)$-twisted sheaf, the first jet bundle of $\mathcal{E}$ is the $\alpha$-twisted sheaf $J^1(\mathcal{E})$ defined by

$$J^1(\mathcal{E}) := p_1^*(\mathcal{E} \otimes \mathcal{O}_{\mathcal{X} \times_B \mathcal{X}}/I_{\Delta_{\mathcal{X}}}^2).$$

This sheaf sits in the middle of the short exact sequence

$$0 \longrightarrow \mathcal{E} \otimes \Omega_{\mathcal{X}/B}^1 \longrightarrow J^1(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0$$

and the corresponding element $\operatorname{At}(\mathcal{E}) \in \operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_{\mathcal{X}/B}^1)$ is called the Atiyah class of $\mathcal{E}$.

The following result is classical in the untwisted case (see for example [5, Theorem 5.1] and its proof) with a similar proof.

**Proposition 5.1.** — Let $X$ be a compact Kähler manifold and $E$ an $\alpha$-twisted vector bundle over $X$. Assume that the trace map $\operatorname{tr} : \operatorname{Ext}^2(E, E) \to H^2(X, \mathcal{O}_X)$ is injective. Then for every smooth deformation $\pi : \mathcal{X} \to S$ of $X$ over a smooth variety $S$, whenever the class $\operatorname{tr}(\operatorname{At}(E)) \in H^1(X, \mathcal{O}_X^\times) \subset H^2(X, \mathbb{C})$ remains of type $(1, 1)$ along $S$ under the parallel transport with respect to the Gauss-Manin connection, up to shrinking $S$ there exists a twisted vector bundle $\mathcal{E}$ on $\mathcal{X}$ such that $\mathcal{E}$ is flat over $S$ and $\mathcal{E}_X \simeq E$.

**Proof.** — Let $o \in S$ be the point parameterizing the central fiber $X$. By Artin’s approximation theorem [1], it suffices to show that $E$ extends to a twisted vector bundle over the formal neighborhood of $o \in S$. To this end, we need to prove by induction on $n$ starting from $\mathcal{E}_0 := E$, that if $\mathcal{X}_n \to \Delta_n := \operatorname{Spec}(\mathbb{C}[[\epsilon_1, \ldots, \epsilon_{\dim S}]/(\epsilon_1, \ldots, \epsilon_{\dim S})^{n+1})$ denotes the base change of $\pi : \mathcal{X} \to S$ with the inclusion $\Delta_n \hookrightarrow S$ of the $n$-th infinitesimal neighborhood of $o$, then $\mathcal{E}_{n-1}$ extends to a twisted vector bundle $\mathcal{E}_n$ on $\mathcal{X}_n$ which is flat over $\Delta_n$.

**Lemma 5.2.** — Let $\mathcal{X}_n \to \Delta_n$ be a smooth morphism whose special fiber is isomorphic to a compact Kähler manifold $X$. Let $\mathcal{X}_{n+1} \to \Delta_{n+1}$ be a flat extension of $\mathcal{X}_n \to \Delta_n$ and let $\theta \in \operatorname{Ext}^1(\Omega_{\mathcal{X}_n/\Delta_n}, I_{\mathcal{X}_n})$ be the corresponding element. Let $\mathcal{E}_n$ be an extension on $\mathcal{X}_n$ of the twisted sheaf $E$ defined on $X$. Define

$$\mathbf{ob}_{\mathcal{E}_n} : \operatorname{Ext}^1(\Omega_{\mathcal{X}_n/\Delta_n}, I_{\mathcal{X}_n}) \xrightarrow{\operatorname{id}_{\mathcal{E}_n} \otimes} \operatorname{Ext}^1(\mathcal{E}_n \otimes \Omega_{\mathcal{X}_n/\Delta_n}, \mathcal{E}_n \otimes I_{\mathcal{X}_n}) \xrightarrow{\mathbf{y}_{\mathcal{E}}(\theta)} \mathbf{Ext}^2(\mathcal{E}_n, \mathcal{E}_n \otimes I_{\mathcal{X}_n})$$

where the last arrow is the Yoneda product with $\operatorname{At}(\mathcal{E}_n)$. If $\mathbf{ob}_{\mathcal{E}_n}(\theta) = 0$, then $\mathcal{E}_n$ can be further extended to a twisted sheaf $\mathcal{E}_{n+1}$ on $\mathcal{X}_{n+1}$.

**Proof.** — Let $I : \mathcal{X}_n \hookrightarrow \mathcal{X}_{n+1}$ be the inclusion of $\mathcal{X}_n$ and $I := I_{\mathcal{X}_n}$ the associated ideal sheaf. As $I^2 = 0$, the restriction map $\mathcal{O}_{\mathcal{X}_{n+1}} \to \mathcal{O}_{\mathcal{X}_n}$ induces an $\mathcal{O}_{\mathcal{X}_n}$-module structure on $I$. Since $\mathbf{ob}_{\mathcal{E}_n}(\theta) = 0$, by [11, Exercise III.5.3] there exists $\mathcal{F}$ sitting in the middle of the following commutative diagram with exact rows and
columns.

\[
\begin{array}{ccccccccc}
 & 0 & 0 \\
\downarrow & & & \downarrow & & & \downarrow & & & \downarrow \\
\mathcal{E}_n \otimes I & \mathcal{E}_n \otimes I & \mathcal{E}_n \otimes I \\
\downarrow & & & \downarrow & & & \downarrow & & & \downarrow \\
0 & \mathcal{E}_n \otimes \Omega_{X_{n+1}/\Delta_{n+1},X_n} & \mathcal{F} & \mathcal{E}_n & 0 \\
\downarrow & & & \downarrow & & & \downarrow & & & \downarrow \\
0 & \mathcal{E}_n \otimes \Omega_{X_n/\Delta_n} & j^!(\mathcal{E}_n) & \mathcal{E}_n & 0 \\
\downarrow & & & \downarrow & & & \downarrow & & & \downarrow \\
0 & 0 & 0 & 0 & 0
\end{array}
\]  
\quad (5.1)

Let \( j : \mathcal{E}_n \to j^!(\mathcal{E}_n) \) be a map sending each local section of \( \mathcal{E}_n \) to its jet of first order. The map \( j \), instead of being \( \mathcal{O}_{X_n} \)-linear, satisfies the “Leibniz rule”: given local sections \( \sigma \) and \( f \) of \( \mathcal{E}_n \) and \( \mathcal{O}_{X_n} \) respectively, we have \( j(f \sigma) = f \cdot j(\sigma) + df \cdot \sigma \). We define

\[
\mathcal{E}_{n+1} := \ker \left( r \circ pr_1 - j \circ pr_2 : \mathcal{F} \oplus \mathcal{E}_n \to j^!(\mathcal{E}_n) \right),
\]

which can be endowed with a twisted sheaf structure over \( X_{n+1} \) as follows. Regarding \( \mathcal{O}_{X_n} \), as the kernel

\[
\mathcal{O}_{X_n} := \ker \left( r' \circ pr_1 - d \circ pr_2 : \Omega_{X_{n+1}/\Delta_{n+1},X_n} \oplus \mathcal{O}_{X_n} \to \Omega_{X_n/\Delta_n} \right)
\]

where \( r' : \Omega_{X_{n+1}/\Delta_{n+1},X_n} \to \Omega_{X_n/\Delta_n} \) is the restriction map, given local sections \((a,f)\) of \( \mathcal{O}_{X_{n+1}} \) and \((\beta,\sigma)\) of \( \mathcal{E}_{n+1} \) where \( a, f, \beta, \sigma \) are local sections of \( \Omega_{X_{n+1}/\Delta_{n+1},X_n} \), \( \mathcal{O}_{X_n} \), \( \mathcal{F} \), and \( \mathcal{E}_n \) respectively, as in the untwisted case we define

\[
(a,f) \cdot (\beta,\sigma) := (f \beta + \sigma \alpha, f \sigma).
\]  
\quad (5.2)

One verifies that (5.2) defines locally an \( \mathcal{O}_{X_{n+1}} \)-module structure on \( \mathcal{E}_{n+1} \) which is locally free and flat over \( \Delta_{n+1} \) satisfying \( \mathcal{E}_{n+1}\vert_{\Delta_{n+1}} = \mathcal{E}_n \).

Let \( \{U_i^{n+1}\}_{i \in I} \) be an open cover of \( X_{n+1} \) such that \( U_i^{n+1} \) is isomorphic to \( U_i \times \Delta_{n+1} \) over \( \Delta_{n+1} \) for some sufficiently fine open cover \( \{U_i\}_{i \in I} \) of \( X \) so that \( \mathcal{E}_{n+1}\vert_{U_i^{n+1}} \) and \( (\mathcal{F} \oplus \mathcal{E}_n)\vert_{U_i} \) are free where \( U_i^n := U_i^{n+1} \cap X_n \). We fix an isomorphism \( (\mathcal{F} \oplus \mathcal{E}_n)\vert_{U_i} = \mathcal{O}_{U_i}^N \) for each \( i \) and let \( \theta_{ij} \in \text{GL}_X(U_i \cap U_j) \) be the transition functions of \( \mathcal{F} \oplus \mathcal{E}_n \) from \( U_i^n \) to \( U_j^n \). With the \( \mathcal{O}_{X_{n+1}} \)-module structure on \( \mathcal{E}_{n+1} \) defined above, the restriction \( \theta_{ij} \) of \( \theta_{ij} \) to \( \mathcal{E}_{n+1} \) is \( \mathcal{O}_{U_i^{n+1}\cap U_j^{n+1}} \)-linear. Since \( \theta_{ij} \theta_{jk} \) \( \theta_{ki} \) \( \mathcal{E}_{n+1} \) \( \mathcal{E}_{n+1} \) \( \mathcal{E}_{n+1} \) is central, the product \( \theta_{ij} \theta_{jk} \) \( \mathcal{E}_{n+1} \) \( \mathcal{E}_{n+1} \) \( \mathcal{E}_{n+1} \) is central. Let \( \alpha' := \{\alpha_{ij}^{jk}\} \) be the Čech 2-cocycle of the sheaf \( \mathcal{O}_{X_{n+1}} \), with respect to the open cover \( \{U_i^{n+1}\} \) such that \( \theta_{ij} \theta_{jk} \) \( \mathcal{E}_{n+1} \) \( \mathcal{E}_{n+1} \) \( \mathcal{E}_{n+1} \) \( \mathcal{E}_{n+1} \). It follows that \( \mathcal{E}_{n+1} \) is an \( \alpha' \)-twisted locally free sheaf which is flat over \( \Delta_{n+1} \) and whose restriction to \( X_n \) is isomorphic to \( \mathcal{E}_n \).

We can compose \( \text{ob}_E \) with the trace map \( \text{tr} : \text{Ext}^2(\mathcal{E}_n, \mathcal{E}_n \otimes I_{X_n}) \to H^2(X_n, I_{X_n}) \). As the trace map commutes with the Yoneda product, we have

\[
\text{tr} \circ \text{ob}_{\mathcal{E}_n}(\theta) = \theta \sim \text{tr}(\text{At}(\mathcal{E}_n)) \in H^2(X_n, I_{X_n}).
\]

As \( \text{tr}(\text{At}(E)) \) remains of type \((1,1)\), by [5, Lemma 5.8] we have \( \text{tr} \circ \text{ob}_{\mathcal{E}_n}(\theta) = 0 \). Since \( \text{tr} : \text{Ext}^2(E, E) \to H^2(X, \mathcal{O}_X) \) is injective, by [5, Lemma 5.10], whose proof works for twisted sheaves as well, the trace map \( \text{tr} : \text{Ext}^2(\mathcal{E}_n, \mathcal{E}_n \otimes I_{X_n}) \to H^2(X_n, I_{X_n}) \) is also injective. Thus \( \text{ob}_{\mathcal{E}_n}(\theta) = 0 \), so \( \mathcal{E}_n \) extends to a twisted sheaf \( \mathcal{E}_{n+1} \) over \( X_{n+1} \) by Lemma 5.2.

5.2 Deformations of \( \mathbb{P}^1 \)-bundles
Let $\tilde{S}$ be a compact Kähler surface with $a(\tilde{S}) = 0$ whose minimal model is a K3 surface. Since $a(\tilde{S}) = 0$, there exist only finitely many irreducible curves in $\tilde{S}$ and in $\tilde{S}$. Let $C$ (resp. $\tilde{C}$) be the union of all curves in $S$ (resp. $\tilde{S}$).

**Proposition 5.3.** — Given a $\mathbb{P}^1$-bundle $f : X \to \tilde{S}$ over $\tilde{S}$ and a deformation $\pi : \mathcal{F} \to \Delta$ of $S$ such that the classes in $\text{NS}(S) \subset H^2(X, \mathbb{Q})$ remains of type $(1, 1)$ under the parallel transport along $\Delta$ with respect to the Gauss-Mannin connection. Up to shrinking $\Delta$, the deformation $\pi : \mathcal{F} \to \Delta$ is $C$-locally trivial and it lifts to a $\tilde{C}$-locally trivial deformation $\tilde{\mathcal{F}} \to \Delta$ of $\tilde{S}$, which further lifts to a deformation of the $\mathbb{P}^1$-bundle $f$.

**Remark 5.4.** — In particular if $\text{NS}(S) = 0$, there is no obstruction to deforming a $\mathbb{P}^1$-bundle over $S$ along any small deformation of $S$.

**Proof.** — That $\mathcal{F} \to \Delta$ is $C$-locally trivial follows from the second statement of [15, Lemma 4.7].

Let $\eta : \tilde{S} = S_0 \to \cdots \to S_0 = S$ be a sequence of contractions of $(-1)$-curve to the minimal model $S$ of $\tilde{S}$. Let $\mathcal{C} = C \times \Delta \subset \mathcal{F}$ denote the $C$-locally trivial deformation of the pair $(S, C)$ over $\Delta$. Let $p \in S$ be the blow-up center of $S_1 \to S$ and let $\mathcal{P} \subset \mathcal{C}$ be a deformation of $p$ over $\Delta$ such that the incidence relations between $\mathcal{P}_t$ and different components of $\mathcal{C}_t$ do not depend on $t \in \Delta$. It follows that the blow-up $\mathcal{F}_t \to \Delta$ of $\mathcal{F}$ along $\mathcal{P}$ is a $C_1$-locally trivial deformation of $S_1$ where $C_1$ is the pre-image of $C$ in $S_1$. By iterating the same construction for each blow-up $S_{i+1} \to S_i$, we obtain a $\tilde{C}$-locally trivial deformation $\tilde{\mathcal{C}} = (\tilde{C} \times \Delta) \subset \tilde{\mathcal{F}} \to \Delta$ of $\tilde{C} \subset \tilde{S}$.

Let $E$ be an $\alpha$-twisted locally free sheaf of rank 2 over $\tilde{S}$ whose projectivization $\mathbb{P}(E) \to \tilde{S}$ is isomorphic to the $\mathbb{P}^1$-bundle $f$. We can assume that $[\alpha] \in H^2(\tilde{S}, \Theta^*_\tilde{S})$ is torsion. It suffices to show that up to shrinking $\Delta$, the twisted sheaf $E$ extends to some twisted locally free sheaf $\mathcal{E}$ over $\tilde{\mathcal{F}}$ which is flat over $\Delta$.

First we assume that $E$ is the extension of two torsion free $\alpha$-twisted sheaves of rank 1. As torsion free twisted sheaves of rank 1 are isomorphic to untwisted ones, the $\alpha$-twisted sheaf $E$ is also isomorphic to an untwisted sheaf (still denoted by $E$) and can be considered as an extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow L' \otimes I_Y \longrightarrow 0$$

for some line bundles $L$ and $L'$ over $\tilde{S}$ and some $0$-dimensional subscheme $Y \subset \tilde{S}$.

Let $L$ and $L'$ be line bundles over $S$ such that $L = \eta^*L(D)$ and $L' = \eta^*L'(D')$ for some divisors $D$ and $D'$ supported on the exceptional locus of $\eta : \tilde{S} \to S$. Since the Noether-Lefschetz loci of $C_1(L)$ and $C_1(L')$ in the universal deformation space of $S$ contains $\Delta$ by assumption, there exist line bundles $\mathcal{L}$ and $\mathcal{L}'$ over $\mathcal{F}$ which are deformations of $L$ and $L'$ respectively. Also by construction, there exist divisors $\mathcal{D}$ and $\mathcal{D}'$ on $\tilde{\mathcal{F}}$ supported on the exceptional locus of the blow-up $\tilde{\mathcal{F}} \to \mathcal{F}$ and flat over $\Delta$ such that $\mathcal{D} \cap S = D$ and $\mathcal{D}' \cap S = D'$.

Let $Y \subset \tilde{\mathcal{F}}$ be a deformation of the subscheme $Y \subset \tilde{S}$ such that the incidence relations between $\mathcal{D}_t$ and each irreducible component of $\mathcal{C}_t$ do not depend on $t$. By abuse of notation, let $\eta : \tilde{\mathcal{F}} \to \mathcal{F}$ denote the contraction constructed above and let $\tilde{\mathcal{L}} := \eta^*L(\mathcal{D})$ and $\tilde{\mathcal{L}}' := \eta^*L'(\mathcal{D}')$. To show that $E$ deforms with $\tilde{S}$ over $\Delta$, we only need to show that $\dim \text{Ext}^1(\tilde{\mathcal{L}}' \otimes I_{\mathcal{D}}, \tilde{\mathcal{L}})$ does not depend on $t \in \Delta$ up to shrinking $\Delta$. By Serre duality, it suffices to show that $h^1(\tilde{\mathcal{L}}, (\tilde{\mathcal{L}}' \otimes I_{\mathcal{D}})_{|\tilde{\mathcal{L}}})$ is constant in $t$ for any line bundles $\tilde{\mathcal{L}}_0$ over $\tilde{\mathcal{F}}$ of the form $\tilde{\mathcal{L}}_0 = \eta^*L_0(\mathcal{D}_0)$ where $L_0$ is a line bundle over $\mathcal{F}$ and $\mathcal{D}_0$ is a divisor on $\tilde{\mathcal{F}}$ supported on the exceptional locus of $\tilde{\mathcal{F}} \to \mathcal{F}$, both flat over $\Delta$. To this end, it suffices to show that $h^0(\tilde{\mathcal{L}}, (\tilde{\mathcal{L}}_0 \otimes I_{\mathcal{D}})_{|\tilde{\mathcal{L}}})$ and
\( h^i \left( \mathcal{F}, \mathcal{Z}_{0, t} \right) \) are constant in \( t \) then conclude by the exact sequence

\[
0 \rightarrow H^0 \left( \mathcal{F}, (\mathcal{Z}_0 \otimes I_{\mathcal{Y}})_{t} \right) \rightarrow H^0 \left( \mathcal{F}, \mathcal{Z}_{0, t} \right) \rightarrow H^0 \left( \mathcal{F}, \mathcal{Z}_{0, 0} \right) \\
\rightarrow H^1 \left( \mathcal{F}, (\mathcal{Z}_0 \otimes I_{\mathcal{Y}})_{t} \right) \rightarrow H^1 \left( \mathcal{F}, \mathcal{Z}_{0, t} \right) \rightarrow H^1 \left( \mathcal{F}, \mathcal{Z}_{0, 0} \right) = 0.
\]

By assumption \( \eta_t(\mathcal{Z}_0 \otimes I_{\mathcal{Y}}) = \mathcal{L}_0 \otimes \eta_t(\mathcal{E}_t(\mathcal{Z}_0) \otimes I_{\mathcal{Y}}) \approx \mathcal{L}_0 \otimes I_{\mathcal{Y}} \) where \( \mathcal{Z} \subset \mathcal{Y} \) is a family of 0-dimensional subschemes over \( \Delta \). Since \( \mathcal{F} \to \Delta \) is a \( (\text{supp} \mathcal{Z}_0) \cap \tilde{\mathcal{S}} \)-locally trivial deformation and since the incidence relations between \( \mathcal{Y}_t \) and irreducible components of \( \mathcal{Z}_{t_i} \) do not depend on \( t \), the incidence relations between \( \mathcal{Y}_t \) and irreducible components of \( \mathcal{Z}_{t_i} \) do not depend on \( t \) either.

If \( h^0 \left( \mathcal{S}, \mathcal{L}_{0, \mathcal{S}} \otimes I_{\mathcal{Y}} \right) = 0 \), then \( t \mapsto h^0 \left( \mathcal{F}_t, \mathcal{Z}_{0, \mathcal{S}} \right) \) is constant by upper semi-continuity. If \( h^0 \left( \mathcal{S}, \mathcal{L}_{0, \mathcal{S}} \otimes I_{\mathcal{Y}} \right) > 0 \), then \( h^0 \left( \mathcal{S}, \mathcal{L}_{0, \mathcal{S}} \otimes I_{\mathcal{Y}} \right) = h^0 \left( \mathcal{S}, \mathcal{L}_{0, \mathcal{S}} \right) = 1 \) since \( a(S) = 0 \). By [19, Proposition 4.3], \( \pi, \mathcal{L}_0 \) is a line bundle over \( \Delta \). Let \( \sigma \in H^0(\Delta, \pi, \mathcal{L}_0) \) be a section which does not vanish. On the one hand since \( \mathcal{F} \to \Delta \) is a \( C \)-locally trivial deformation and the vanishing locus \( V(\sigma_{\mathcal{S}}) \) of \( \sigma_{\mathcal{S}} \in H^0(\mathcal{S}, \mathcal{L}_{0, \mathcal{S}}) \) is contained in \( C \) and on the other hand since Pic(\( \mathcal{S} \)) is discrete and \( h^0 \left( \mathcal{F}_t, \mathcal{Z}_{0, t} \right) = 1 \) for every \( t \in \Delta \), the divisor \( \sigma \) defined by \( \sigma \in H^0(\mathcal{F}, \mathcal{L}_0) \) is necessarily the trivial deformation of \( V(\sigma_{\mathcal{S}}) \). We deduce that the incidence relations between the irreducible components of \( \mathcal{Z}_t \) and that of \( V(\sigma_{\mathcal{S}}) \) do not depend on \( t \). Therefore \( \mathcal{Z}_t \subset V(\sigma_{\mathcal{S}}) \), so \( 1 \leq h^0 \left( \mathcal{F}_t, \mathcal{Z}_{0, \mathcal{S}} \right) \leq h^0 \left( \mathcal{F}_t, \mathcal{Z}_{0, \mathcal{S}} \right) = 1 \). In either case we deduce that \( t \mapsto h^0 \left( \mathcal{S}, \mathcal{Z}_0 \otimes I_{\mathcal{Y}} \right) \) is constant.

Similarly, \( h^0 \left( \mathcal{F}, \mathcal{Z}_{0, t} \right) \) does not depend on \( t \). By Serre duality, \( h^0 \left( \mathcal{F}, \mathcal{Z}_{0, t} \right) = h^0 \left( \mathcal{F}, \mathcal{Z}_{0, t} \otimes \omega_{\mathcal{F}_t} \right) \), so \( t \mapsto h^0 \left( \mathcal{F}, \mathcal{Z}_{0, t} \right) \) is also constant according to the above. Since \( \mathcal{Z}_0 \) is flat over \( \Delta \), we also deduce that \( t \mapsto h^1 \left( \mathcal{F}, \mathcal{Z}_{0, t} \right) \) is constant. Therefore up to shrinking \( \Delta \), we can now conclude that there exists a locally free sheaf \( \mathcal{E} \) of rank 2 over \( \mathcal{F} \) such that \( \mathcal{E}_{t_0} = E \). So \( \mathcal{P}(\mathcal{E}) \to \Delta \) is a deformation of the \( \mathcal{P}^1 \)-bundle \( f : X \to \mathcal{S} \) lifting \( \mathcal{F} \to \Delta \).

Now we turn to the case where \( E \) is not the extension of any pair of twisted sheaves of rank 1. In particular, this assumption implies that \( E \) is simple. As \( [a_1] \in H^2(\mathcal{S}, \mathcal{O}_{\mathcal{S}}^2) \) is torsion, there exists \( n \in \mathbb{Z}_{\geq 0} \) such that \( E^{\otimes n} \) is isomorphic to an untwisted vector bundle. Since \( n \cdot \text{tr}(\text{At}(\mathcal{E})) = \text{tr}(\text{At}(\mathcal{E}^{\otimes n})) = -c_1(\mathcal{E}^{\otimes n}) \in H^2(\mathcal{S}, \mathbb{Q} \cap H^{1,1}(\mathcal{S})) \), the class \( \text{tr}(\text{At}(\mathcal{E})) \) remains of type \( (1, 1) \) under the parallel transport by the Gauss-Manin connection along \( \Delta \) by assumption. Thus by Proposition 5.1 it suffices to show that \( \text{tr} : \text{Ext}^1(E, E) \to H^2(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \) is injective.

By Serre duality, the trace map is isomorphic to \( \text{tr} : \text{Hom}(E, E \otimes \omega_{\mathcal{S}}) \to H^0(\mathcal{S}, \omega_{\mathcal{S}}) \). Since \( \mathcal{S} \) is a sequence of blow-ups of a K-trivial surface, the line bundle \( \omega_{\mathcal{S}} \) is effective. A non-trivial map \( \mathcal{O}_{\mathcal{S}} \to \omega_{\mathcal{S}} \) gives rise to a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(E, E) & \longrightarrow & \text{Hom}(E, E \otimes \omega_{\mathcal{S}}) \\
\downarrow \text{tr} & & \downarrow \text{tr} \\
H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) & \longrightarrow & H^0(\mathcal{S}, \omega_{\mathcal{S}}).
\end{array}
\]

Since \( E \) is simple, the trace map on the left is an isomorphism. As \( \kappa(\mathcal{S}) = 0 \), the inclusion map below is in fact an isomorphism. Hence \( \text{tr} : \text{Hom}(E, E \otimes \omega_{\mathcal{S}}) \to H^0(\mathcal{S}, \omega_{\mathcal{S}}) \) is surjective. To show that \( \text{tr} : \text{Hom}(E, E \otimes \omega_{\mathcal{S}}) \to H^0(\mathcal{S}, \omega_{\mathcal{S}}) \) is injective, we only need to show that \( \dim \text{Hom}(E, E \otimes \omega_{\mathcal{S}}) \leq 1 \). To this end, consider the determinant \( \det : \text{Hom}(E, E \otimes \omega_{\mathcal{S}}) \to H^0(\mathcal{S}, \omega_{\mathcal{S}}^2) \). Since \( E \) does not have any quotient of rank 1, the determinant is injective. Again as \( \kappa(\mathcal{S}) = 0 \), we have \( 1 = h^0(\mathcal{S}, \omega_{\mathcal{S}}^2) \geq \dim \text{Hom}(E, E \otimes \omega_{\mathcal{S}}) \), which finishes the proof of Proposition 5.3.
Corollary 5.5. — Let \( f : X \to \hat{S} \) be a \( \mathbb{P}^1 \)-bundle over a smooth compact Kähler surface \( \hat{S} \) of algebraic dimension 0 bimeromorphic to a K3 surface. For every proper Zariski closed subset \( C \subset \hat{S} \), there exists an \( f^{-1}(C) \)-locally trivial algebraic approximation of \( f \).

Proof. — As in the beginning of the proof of Proposition 4.2 we can assume that \( C \) is the union of all curves in \( \hat{S} \). Let \( \mathcal{F} \to \Delta \) be the universal deformation of \( S \) such that each element in \( \text{NS}(S) \) remains of type \((1,1)\) under the parallel transport along \( \Delta \) with respect to the Gauss-Manin connection. By Proposition 5.3, up to shrinking \( \Delta \) the deformation \( \mathcal{F} \to \Delta \) lifts to a deformation \( \mathcal{X} \to \Delta \) of the \( \mathbb{P}^1 \)-bundle \( f : X \to \hat{S} \).

Let \( S \to S_{\text{can}} \) be the contraction of all \((-2)\)-curves in \( S \) and let \( \tilde{\eta} : \hat{S} \to S \to S_{\text{can}} \) be its composition with \( \eta \). As \( \tilde{\eta}(C) \) is a finite set of points, there exists a Stein neighborhood \( U \subset S_{\text{can}} \) of \( \tilde{\eta}(C) \); let \( \tilde{U} := \tilde{\eta}^{-1}(U) \). By \( C \)-local triviality of the deformation \( \mathcal{F} \to \Delta \), up to shrinking \( \tilde{U} \) we can assume the existence of a neighborhood \( \tilde{U} \subset \mathcal{F} \) of the trivial deformation \( \mathcal{U} \subset \mathcal{F} \to \Delta \) with the first projection \( \tilde{U} \to \Delta \) is Stein and fibers of \( \phi := (\tilde{\eta} \times \text{Id}_\Delta) : \tilde{U} \to \Delta \) are of dimension \( \leq 1 \), it follows that \( H^2(\tilde{U}, \mathcal{O}_{\tilde{U}}) = 0 \).

As \( H^2(\tilde{U}, \mathcal{O}_{\tilde{U}}) = 0 \), there exists a locally free sheaf \( \mathcal{F} \) of rank 2 over \( \tilde{U} \) such that \( f^{-1}(\tilde{U}) = \mathbb{P}(\mathcal{F}) \) over \( \tilde{U} \) where \( f \) is the map \( \mathcal{F} \to \mathcal{F} \). Regarding \( (\phi, \mathcal{F})^{\text{VV}} \) as a family of reflexive sheaves \( \mathcal{G}_t \) on \( U \) parameterized by \( t \in \Delta \), since \( U \) has at worst rational double points as singularity, Artin-Verdier’s theorem shows that \( \mathcal{G}_0 \) is isomorphic to a constant family \([2, \text{Theorem } 1.11] \)

So \( \mathcal{F}^{-1}(\tilde{U} - C) \) is isomorphic to \( \mathcal{F}(\tilde{U} - C) \) where \( \mathcal{F}(\tilde{U} - C) = \text{pr}_1(\mathcal{F}^{-1}(\tilde{U} - C)) \)

Thus we can glue \( \mathcal{F}^{-1}(\tilde{U}) \) along \( f^{-1}(\tilde{U}) \subset \mathcal{F}^{-1}(\tilde{U} - C) \) and \( f^{-1}(\tilde{U} - C) \) using the isomorphisms

\[
\mathcal{F}^{-1}(\tilde{U} - C) = \mathbb{P}(\mathcal{F})(\tilde{U} - C) \cong \mathbb{P}(\text{pr}_1(\mathcal{F}^{-1}(\tilde{U} - C))) = f^{-1}(\tilde{U} - C) \\
\]

to obtain a new deformation \( \mathcal{X}' \to \mathcal{F} \to \Delta \) of the \( \mathbb{P}^1 \)-fibration \( f : X \to \hat{S} \), which is \( f^{-1}(C) \)-locally trivial. Since \( \mathcal{F} \to \Delta \) is an algebraic approximation of \( S \) by \([15, \text{Lemma } 4.7] \), the family \( \mathcal{F} \to \Delta \) is an algebraic approximation of \( \hat{S} \). Therefore \( \mathcal{X}' \to \mathcal{F} \to \Delta \) is an algebraic approximation of \( f \) by Corollary 2.2.

Remark 5.6. — We could have proven Proposition 5.3 and Corollary 5.5 for \( \mathbb{P}^1 \)-bundles over a surface bimeromorphic to a 2-torus of algebraic dimension 0 as well. However, since the proof of Proposition 4.2 is simpler and does not rely on results of deformations of twisted sheaves, we decide to give separate proofs according to whether the minimal model of the base of the \( \mathbb{P}^1 \)-bundle is a 2-torus or a K3 surface.

6 Case where the MRC-quotient is of algebraic dimension 1

Now we turn to study deformations of a \( \mathbb{P}^1 \)-fibration \( X \to S \) over a compact Kähler surface \( S \) of algebraic dimension \( \text{algebraic dim}(S) = 1 \). Such a surface \( S \) is a non-algebraic elliptic surface (and vice versa), so let \( p : S \to B \) be a surjective map whose general fiber is a smooth curve of genus 1. By the work of Kodaira [13], the elliptic surface \( S \to B \) has a strongly locally trivial algebraic approximation. In this section, we show that this algebraic approximation can always be lifted to a strongly locally trivial algebraic approximation of \( X \to B \).
Proposition 6.1. — Let $X$ be a compact Kähler threefold which is a $\mathbb{P}^1$-fibration $f : X \to S$ over a non-algebraic smooth elliptic surface $p : S \to B$. Then there exists a strongly locally trivial algebraic approximation of $\pi := p \circ f : X \to B$.

The schema of proof of Proposition 6.1 is simple. Since strongly locally trivial deformations of $p$ and $f$ (say parameterized by $\Delta$) are given by families over $\Delta$ of (closed) 1-cocycles $\eta$ and $\bar{\eta}$ of local sections of $\text{Aut}^0(S/B) \to B$ and $\text{Aut}^0(X/B) \to B'$ respectively, Proposition 6.1 will be proven if we know that every such family of 1-cocycles $\eta$ can be lifted to some $\bar{\eta}$. A general fiber $F$ of $\pi : X \to B$ is a ruled surface over an elliptic curve $C$. Let $E$ be a vector bundle of rank 2 over $C$ such that $P(E) \cong F$ and let $g : P(E) \to C$ be the projection. Since $g(C) > 0$, the image of a fiber of $g$ under an automorphism of $F$ is still a fiber of $g$. This property allows us to define a group homomorphism $g : \text{Aut}(F) \to \text{Aut}(C)$. Here is a sufficient condition for which elements in $\text{Aut}^0(C)$ lifts to elements in $\text{Aut}^0(F)$.

Proposition 6.2 (Maruyama [16, Lemma 8]). — If the rank-two vector bundle $E$ is indecomposable or $E \cong L \oplus L'$ for some line bundles $L$ and $L'$ with $\deg(L) = \deg(L')$, then $g_*\text{Aut}^0(F) = \text{Aut}^0(C)$.

We will show that in the setting of Proposition 6.1, the ruled surface $X_b := \pi^{-1}(b)$ for a general point $b \in B$ satisfies the sufficient condition in Proposition 6.2. This will allow us to lift the 1-cocycles in question.

Proof of Proposition 6.1. — Since $S$ is not algebraic, every subvariety of $S$ is contained in a finite union of fibers of $p : S \to B$. In particular by generic smoothness of $f$, the image of the non-smooth locus of $f$ in $S$ is supported on a finite union of fibers of $p$. Let $\Sigma \subset B$ be a finite subset of points such that the restriction $f^\circ : X^\circ \to S^\circ$ (resp. $p^\circ : S^\circ \to B^\circ$) of $f$ (resp. $p$) to $\pi^{-1}(B^\circ)$ (resp. $p^{-1}(B^\circ)$) is smooth where $B^\circ := B \setminus \Sigma$, $S^\circ := S \setminus (\Sigma)$, and $X^\circ := X \setminus \pi^{-1}(\Sigma)$. It follows that $\pi^\circ := p^\circ \circ f^\circ$ is a smooth family of ruled surfaces.

We show in this paragraph that for every $b \in B^\circ$, the ruled surface $X_b$ satisfies the sufficient condition in Proposition 6.2. Assume that it is not the case, then there exists $b \in B^\circ$ and a line bundle $L_b$ over $S_b$ with $\deg(L_b) < 0$ such that $X_b := \pi^{-1}(b)$ is isomorphic to $P(\mathcal{O}_{S_b} \oplus L_b)$ over $S_b$. Since $f$ is proper, the direct image sheaf $f_*\mathcal{K}_{X/B}^\vee$ is coherent. As $S$ is a surface, the dual $(f_*\mathcal{K}_{X/B}^\vee)^\vee$ is locally free. Consider the line bundle $\mathcal{L} := \text{det}(f_*\mathcal{K}_{X/S}^\vee)^\vee$. Since $X_b \cong P(\mathcal{O}_{S_b} \oplus L_b)$, we have

\[ \mathcal{K}_{X_b/S_b} \cong \mathcal{O}(-2) \otimes f_b^* \text{det}(\mathcal{O}_{S_b} \oplus L_b^\vee). \]

By the base change theorem, there exists a non-empty Zariski open $U \subset S$ such that for every subvariety $Z \subset U$, we have $f_*\mathcal{K}_{X/Z}^\vee \cong f_*\mathcal{K}_{X_b/Z_b}^\vee$. Again as $S$ is non-algebraic, $S \setminus U$ is contained in a finite union of fibers of $p$, so there exists a fiber $S_0$ of $p$ contained in $U \cap S^\circ$. Therefore,

\[ \mathcal{L}|_{S_0} \cong \text{det}(f_*\mathcal{K}_{X/S_0}^\vee) \cong \text{det}(f_*\mathcal{K}_{X_b/S_b}^\vee) \cong \left(\text{det} \text{Sym}^2(\mathcal{O}_{S_0} \oplus L_0^\vee)\right) \otimes L_0 = L_0^{\otimes 2}, \]

which is of positive degree as $\deg(L_0) < 0$. Accordingly $c_1(\mathcal{L}^\vee \otimes \mathcal{O}(mS_0)) > 0$ whenever $m \gg 0$, which is in contradiction with the assumption that $S$ is non-algebraic [3, Proposition 6.2].

There exists a Galois cover $u : \tilde{B} \to B$ of $B$ such that the elliptic fibration $\tilde{p} : \tilde{S} := S \times_B \tilde{B} \to \tilde{B}$ has a $G$-invariant collection of local sections where $G := \text{Gal}(\tilde{B}/B)$. To prove Proposition 6.1, it suffices to prove that $\pi : X := X \times_S \tilde{S} \to B$ has a $G$-equivariant strongly locally trivial algebraic approximation. Note that the truth of the latter statement is invariant under any $G$-equivariant bimeromorphic modification of $X$ because such a modification is necessarily supported on a finite union of fibers of $\pi$, otherwise the support would

\(^{(4)}\text{Aut}^0(X/B)\text{ denotes the identity component of the Lie group of automorphisms of }S\text{ preserving fibers of }p\text{ and inducing the identity map on }B. \text{ Idem for }\text{Aut}^0(S/B).\)
contain a curve dominating $B$ and as fibers of $X \to B$ are algebraic, this would imply that $X$ is algebraically connected, which contradicts the assumption that $X$ is non-algebraic.

By [14, Theorem 1.6] (which in the case of elliptic surfaces, is already a consequence of Kodaira’s work [13]), there exists a strongly locally trivial $G$-equivariant algebraic approximation

$$\begin{align*}
(\hat{S} \subset \hat{\mathcal{F}}) & \xrightarrow{q} \Delta \times B \\
\pi & \downarrow \quad (o \in \Delta)
\end{align*} \tag{6.1}$$

of $\hat{p} : \hat{S} \to \hat{B}$ fixing the base $\hat{B}$. Let $\mathcal{U} := \{\Delta_i\}_{i \in \mathcal{I}}$ be a $G$-invariant finite collection of open discs covering $\hat{B}$ such that

- The restriction of $\pi$ to $q^{-1}(\Delta \times \Delta_i)$ is a trivial deformation of $\hat{p}^{-1}(\Delta_i)$ for every $i \in \mathcal{I}$.
- For all $i, j, k \in \mathcal{I}$, the intersection $\Delta_i \cap \Delta_j$ is connected and simply connected and $\Delta_i \cap \Delta_j \cap \Delta_k = \emptyset$;
- If $\Sigma := q^{-1}(\Sigma)$, then $\Delta_i \cap \Sigma$ is the origin of $\Delta_i$ or empty for every $i \in \mathcal{I}$;
- For each pair of indices $i, j \in \mathcal{I}$ such that $i \neq j$, we have $\Sigma \cap \Delta_i \cap \Delta_j = \emptyset$.

Let $\hat{B}^o := q^{-1}(B^o)$, $\hat{S}^o := \hat{p}^{-1}(B^o)$, and let $\mathcal{F}$ be the sheaf of sections of the Jacobian fibration associated to $\hat{S}^o \to \hat{B}^o$. The elliptic fibration $q : \hat{\mathcal{F}} \to \Delta \times B$ can be represented by a $G$-invariant 1-cocycle $(\eta_{ij})_{i,j \in \mathcal{I}}$ of the sheaf $\mathcal{P} \mathcal{F}$ with respect to the open cover $\{\Delta \times \Delta_i\}_{i \in \mathcal{I}}$ of $\Delta \times B$ where $j : \hat{B}^o \hookrightarrow \Delta \times B$ is the inclusion and $\mathcal{P} \mathcal{F} : \Delta \times B \to B$ denotes the second projection (cf. [14, Section 5.4] for instance). Each $\eta_{ij}$ defines a map $\eta_{ij} : \Delta_i \times \Delta_j \to \text{Aut}(\hat{p}^{-1}(\Delta_i))/\Delta_{ij}$ where $\Delta_{ij} := \Delta_i \cap \Delta_j$.

Let $\mu_{ij} := \eta_{ij}(o)^{-1} \circ \eta_{ij} : \Delta_i \to \text{Aut}(\hat{p}^{-1}(\Delta_i))/\Delta_{ij}$, and define a $G$-equivariant collection of isomorphisms over $\{\Delta_i\}_{i \in \mathcal{I}}$ which defines the fibration $\hat{\pi} : \hat{X} \to \hat{B}$ by gluing $U_i := \pi^{-1}(\Delta_i)$ and $U_j$ along $U_{ij}$ with the identification $g_{ij} : U_{ij} \to U_j$. If we define

$$h_{ij} : \Delta_i \to \text{Aut}(\pi^{-1}(\Delta_i))/\Delta_{ij}$$

$$t \mapsto g_{ij}(t) \circ \mu_{ij}(t),$$

then the $G$-equivariant collection of maps $[h_{ij}]_{i,j \in \mathcal{I}}$ defines a strongly locally trivial deformation

$$\hat{\mathcal{F}} \to \hat{\mathcal{F}} \to \Delta \times B \xrightarrow{\text{pr}_2} \Delta$$

of $\hat{X} \to \hat{B}$ lifting $\hat{S} \to \hat{B}$ and preserving the equivariant $G$-action. The quotient

$$\mathcal{F} := \hat{\mathcal{F}}/G \to \mathcal{F} := \hat{\mathcal{F}}/G \to \Delta \xrightarrow{\text{pr}_2} \Delta$$

is thus a strongly locally trivial deformation of $X \to B$ lifting (6.1).

The fiber $\mathcal{X}_i$ of $\mathcal{F} \to \Delta$ over $t$ is a $\mathbb{P}^1$-fibration over the elliptic surface $\mathcal{X}_i$, so $\mathcal{X}_i$ is algebraic if $\mathcal{X}_i$ is algebraic by Corollary 2.2. Since the subset of $\Delta$ parameterizing algebraic members of the family $\mathcal{X} \to \Delta$ is dense, $\mathcal{F} \to B \times \Delta \to \Delta$ is thus a strongly locally trivial algebraic approximation of $\pi : X \to B$, which concludes the proof of Proposition 6.1. \qed
Corollary 6.3. — Let $X$ be a Kähler threefold which is a $\mathbb{P}^1$-fibration $f : X \to S$ over a smooth compact Kähler surface $S$ with of algebraic dimension 1. Then for every subvariety $C \subset S$, the fibration $f$ has a $f^{-1}(C)$-locally trivial algebraic approximation.

Proof. — As $a(S) = 1$, $S$ is an elliptic surface $p : S \to B$. Since $S$ is non-algebraic, every irreducible component of a subvariety $C$ of $S$ is contained in a fiber of $p$. Therefore Proposition 6.1 implies in particular Corollary 6.3. □

7 Conclusion

Proof of Theorem 1.1. — The algebraic dimension $a(S)$ of $S$ is either 0 or 1. If $a(S) = 0$, then the minimal model of $S$ is either a 2-torus or a K3 surface. In this case the conjunction of Proposition 4.1, Proposition 4.2 and Corollary 5.5 allows to conclude. In the case where $a(S) = 1$, Theorem 1.1 follows from Corollary 6.3. □

Proof of Corollary 1.2. — Let $X$ be a uniruled compact Kähler threefold and $X \to B$ an MRC fibration of $X$ over a compact Kähler manifold $B$. Since $X$ is uniruled, $\dim B \leq 2$. If $\dim B = 0$, then $X$ is already projective as $X$ is rationally connected. If $\dim B = 1$, then since a general fiber of $X \to B$ does not have any global holomorphic 1-form and 2-form, we have $H^0(X, \Omega^2_X) = 0$. So $X$ is also projective in this case.

Suppose that $\dim B = 2$, then $X$ is bimeromorphic to a $\mathbb{P}^1$-fibration $X' \to S$ over a non-uniruled compact Kähler surface. Furthermore by Proposition 3.1, we can assume that $f : X' \to S$ is a standard conic bundle. We can also assume that $S$ is non-algebraic, because otherwise $X$ will already be projective by Corollary 2.2. Theorem 1.1 implies that for every subvariety $C \subset X'$ of dimension $\leq 1$ (not necessarily equidimensional), the pair $(X', C)$ has a $C$-locally trivial algebraic approximation. We conclude by [15, Proposition 1.4] that $X$ admits an algebraic approximation. □

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