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Density of the set of probability measures with the martingale representation property

Dmitry Kramkov* Sergio Pulido†

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Abstract

Let ψ be a multi-dimensional random variable. We show that the set of probability measures \mathbb{Q} such that the \mathbb{Q} -martingale $S_t^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t]$ has the Martingale Representation Property (MRP) is either empty or dense in \mathcal{L}_{∞} -norm. The proof is based on a related result involving analytic fields of terminal conditions $(\psi(x))_{x \in U}$ and probability measures $(\mathbb{Q}(x))_{x \in U}$ over an open set U . Namely, we show that the set of points $x \in U$ such that $S_t(x) = \mathbb{E}^{\mathbb{Q}(x)}[\psi(x) | \mathcal{F}_t]$ does not have the MRP, either coincides with U or has Lebesgue measure zero. Our study is motivated by the problem of endogenous completeness in financial economics.

Keywords: martingale representation property, martingales, stochastic integrals, analytic fields, endogenous completeness, complete market, equilibrium.

AMS Subject Classification (2010): 60G44, 60H05, 91B51, 91G99.

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1 Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, \mathbb{Q} be an equivalent probability measure, and $S = (S_t^i)$ be a multi-dimensional martingale under \mathbb{Q} . It is often important to know whether S has the Martingale Representation Property (MRP), that is, whether every local martingale under \mathbb{Q} is a stochastic integral with respect to S . For instance, in mathematical finance such MRP corresponds to the *completeness* of the market with stock prices S .

In many applications, S is defined in a *forward form*, as a solution of an SDE, and the verification of the MRP is quite straightforward. Suppose, for example, that S is an Itô process with drift vector-process $b = (b_t)$ and volatility matrix-process $\sigma = (\sigma_t)$. Then the MRP holds if and only if σ has full rank $d\mathbb{P} \times dt$ almost surely. The density process Z of the martingale measure \mathbb{Q} is computed by Girsanov's theorem.

We are interested in the situation where both S and Z are described in a *backward form* through their terminal values:

$$\begin{aligned} Z_\infty &= \frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\zeta}{\mathbb{E}[\zeta]}, \\ S_t &= \mathbb{E}^\mathbb{Q}[\psi | \mathcal{F}_t], \quad t \geq 0, \end{aligned} \tag{1}$$

where $\zeta > 0$ and $\psi = (\psi^i)$ are *given* random variables. Such setup naturally arises in the problem of *endogeneous completeness* of financial economics, where the random variable ψ represents the terminal values of the traded securities and \mathbb{Q} defines an equilibrium pricing measure. The term “endogenous” indicates that the stock prices $S = (S^i)$ are *computed* by (1) as part of the solution. The examples include the construction of Radner equilibrium [1, 4, 10, 6] and the verification of the completeness property for a market with options [2, 11].

The main focus of the existing literature has been on the case when the random variables ζ and ψ are defined in terms of a Markov diffusion in a form consistent with Feynman-Kac formula. The proofs have relied on PDE methods and, in particular, on the theory of analytic semigroups [7]. A key role has been played by the assumption that time-dependencies are analytic.

In this paper we do not impose any conditions on the form of the random variables ζ and ψ . Our main results are stated as Theorems 2.3 and 3.1. In

Theorem 2.3 we show that the set

$$\mathcal{Q}(\psi) \triangleq \left\{ \mathbb{Q} \sim \mathbb{P} : S_t^{\mathbb{Q}} \triangleq \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t] \text{ has the MRP} \right\}$$

is either empty or \mathcal{L}_{∞} -dense in the set of all equivalent probability measures. In Theorem 3.1 we consider analytic fields of probability measures $(\mathbb{Q}(x))_{x \in U}$ and terminal conditions $(\psi(x))_{x \in U}$ over an open set U . We prove that the exception set

$$I \triangleq \left\{ x \in U : S_t(x) \triangleq \mathbb{E}^{\mathbb{Q}(x)}[\psi(x) | \mathcal{F}_t] \text{ does not have the MRP} \right\}$$

either coincides with U or has Lebesgue measure zero.

We expect the results of this paper to be useful in problems of financial economics involving the endogeneous completeness property. In particular, they play a key role in our work, in progress, on the problem of optimal investment in a “backward” model of price impact [3, 8], where stock prices and wealth processes solve a coupled system of quadratic BSDEs. Theorem 2.3 allows us to relax this apparently complex stochastic control problem into a simple *static* framework, where maximization is performed over a set of random variables. We show that the solution ξ of the static problem yields the optimal investment strategy provided that the stock prices S given by (1) have the MRP. Using Theorem 3.1 we prove such MRP for almost all values of the model parameters.

2 Density of the set of probability measures with the MRP

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions of completeness and right-continuity; the initial σ -algebra \mathcal{F}_0 is trivial and $\mathcal{F} = \mathcal{F}_{\infty}$. We denote by $\mathcal{L}_1 = \mathcal{L}_1(\mathbf{R}^d)$ and $\mathcal{L}_{\infty} = \mathcal{L}_{\infty}(\mathbf{R}^d)$ the Banach spaces of (equivalence classes of) d -dimensional random variables ξ with the norms $\|\xi\|_{\mathcal{L}_1} \triangleq \mathbb{E}[|\xi|]$ and $\|\xi\|_{\mathcal{L}_{\infty}} \triangleq \inf \{c > 0 : \mathbb{P}[|\xi| \leq c] = 1\}$. We use same notation \mathcal{L}_1 for the isometric Banach space of uniformly integrable martingales M with the norm $\|M\|_{\mathcal{L}_1} \triangleq \|M_{\infty}\|_{\mathcal{L}_1}$.

For a matrix $A = (A^{ij})$ we denote its transpose by A^* and define its norm as

$$|A| \triangleq \sqrt{\text{tr } AA^*} = \sqrt{\sum_{i,j} |A^{ij}|^2}.$$

If X is a m -dimensional semimartingale and γ is a $m \times n$ -dimensional X -integrable predictable process, then $\gamma \cdot X = \int \gamma^* dX$ denotes the n -dimensional stochastic integral of γ with respect to X . We recall that a $n \times k$ -dimensional predictable process ζ is $(\gamma \cdot X)$ -integrable if and only if $\gamma\zeta$ is X -integrable. In this case, $\zeta \cdot (\gamma \cdot X) = (\gamma\zeta) \cdot X$ is a k -dimensional semimartingale.

Definition 2.1. Let \mathbb{Q} be an equivalent probability measure ($\mathbb{Q} \sim \mathbb{P}$) and S be a d -dimensional local martingale under \mathbb{Q} . We say that S has the *Martingale Representation Property (MRP)* if every local martingale M under \mathbb{Q} is a stochastic integral with respect to S , that is, there is a predictable S -integrable process γ with values in \mathbf{R}^d such that

$$M = M_0 + \gamma \cdot S.$$

Remark 2.2. Jacod's theorem in [5, Section XI.1(a)] states that S has the MRP if and only if there is only one $\mathbb{Q} \sim \mathbb{P}$ such that S is a local martingale under \mathbb{Q} . Thus, there is no need to mention \mathbb{Q} in the definition of the MRP.

Let $\psi = (\psi^i)_{i=1,\dots,d}$ be a d -dimensional random variable. We denote by $\mathcal{Q}(\psi)$ the family of probability measures $\mathbb{Q} \sim \mathbb{P}$ such that $\mathbb{E}^{\mathbb{Q}}[|\psi|] < \infty$ and the \mathbb{Q} -martingale

$$S_t^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t], \quad t \geq 0,$$

has the MRP.

This is our first main result.

Theorem 2.3. Suppose that $\psi \in \mathcal{L}_1(\mathbf{R}^d)$ and $\mathcal{Q}(\psi) \neq \emptyset$. Then for every $\epsilon > 0$ there is $\mathbb{Q} \in \mathcal{Q}(\psi)$ such that

$$\left\| \frac{d\mathbb{Q}}{d\mathbb{P}} - 1 \right\|_{\mathcal{L}_{\infty}} \leq \epsilon.$$

The proof is based on Theorem 3.1 from Section 3 and on the following elementary lemma. We recall the definition of an analytic function with values in a Banach space at the beginning of Section 3.

Lemma 2.4. Let ζ be a nonnegative random variable. Then the map $x \mapsto e^{-x\zeta}$ from $(0, \infty)$ to \mathcal{L}_{∞} is analytic.

Proof. Fix $y > 0$. For every $\omega \in \Omega$ the function $x \mapsto e^{-x\zeta(\omega)}$ has a Taylor's expansion

$$e^{-x\zeta(\omega)} = \sum_{n=0}^{\infty} A_n(y)(\omega)(x-y)^n, \quad x \in \mathbf{R}, \quad (2)$$

where

$$A_n(y) = \frac{1}{n!} \frac{d^n}{dx^n} (e^{-x\zeta})|_{x=y} = \frac{1}{n!} (-1)^n \zeta^n e^{-y\zeta}.$$

We deduce that

$$\|A_n(y)\|_{\mathcal{L}_\infty} \leq \frac{1}{n!} \max_{t \geq 0} (t^n e^{-yt}) = \frac{1}{n!} \left(\frac{n}{ey} \right)^n \leq K \frac{1}{\sqrt{n}} \left(\frac{1}{y} \right)^n,$$

where the existence of a constant $K > 0$ follows from Sterling's formula:

$$\lim_{n \rightarrow \infty} \frac{\sqrt{2\pi n}}{n!} \left(\frac{n}{e} \right)^n = 1.$$

It follows that the series in (2) converges in \mathcal{L}_∞ provided that $|x - y| < y$. \square

Proof of Theorem 2.3. We take $\mathbb{R} \in \mathcal{Q}(\psi)$, denote $\zeta \triangleq \frac{d\mathbb{R}}{d\mathbb{P}}$, and for $x > 0$ define the random variables

$$\begin{aligned} \zeta(x) &\triangleq \frac{1 - e^{-x\zeta}}{x} + \frac{x}{1+x}, \\ \xi(x) &\triangleq \zeta(x)\psi, \end{aligned}$$

and a probability measure $\mathbb{Q}(x)$ such that

$$\frac{d\mathbb{Q}(x)}{d\mathbb{P}} = \frac{\zeta(x)}{\mathbb{E}[\zeta(x)]}.$$

We set $\zeta(0) \triangleq \zeta$, $\xi(0) \triangleq \zeta\psi$, and $\mathbb{Q}(0) \triangleq \mathbb{R}$ and observe that for every $\omega \in \Omega$ the functions $x \mapsto \zeta(x)(\omega)$ and $x \mapsto \xi(x)(\omega)$ on $[0, \infty)$ are continuous. Since

$$|\zeta(x)| \leq \zeta \sup_{t \geq 0} \frac{1 - e^{-t}}{t} + \frac{x}{1+x} \leq \zeta + 1,$$

the dominated convergence theorem yields that $x \mapsto \zeta(x)$ and $x \mapsto \xi(x)$ are continuous maps from $[0, \infty)$ to \mathcal{L}_1 . By Lemma 2.4, $x \mapsto \zeta(x)$ is an analytic map from $(0, \infty)$ to \mathcal{L}_∞ and thus $x \mapsto \zeta(x)$ and $x \mapsto \xi(x)$ are analytic maps from $(0, \infty)$ to \mathcal{L}_1 . Theorem 3.1 then implies that the exception set

$$I \triangleq \{x > 0 : \mathbb{Q}(x) \notin \mathcal{Q}(\psi)\}$$

is at most countable.

Choose now any $\epsilon > 0$. Since

$$-\frac{1}{1+x} \leq \zeta(x) - 1 \leq \frac{1}{x} - \frac{1}{1+x},$$

there is $x_0 = x_0(\epsilon)$ such that the assertion of the theorem holds for every $\mathbb{Q}(x)$ with $x \geq x_0$ and $x \notin I$. \square

3 The MRP for analytic fields of martingales

Let \mathbf{X} be a Banach space and U be an open connected set in \mathbf{R}^d . We recall that a map $x \mapsto X(x)$ from U to \mathbf{X} is *analytic* if for every $y \in U$ there exist a number $\epsilon = \epsilon(y) > 0$ and elements $(Y_\alpha(y))$ in \mathbf{X} such that the ϵ -neighborhood of y belongs to U and

$$X(x) = \sum_{\alpha} Y_{\alpha}(y)(x - y)^{\alpha}, \quad |y - x| < \epsilon.$$

Here the series converges in the norm $\|\cdot\|_{\mathbf{X}}$ of \mathbf{X} , the summation is taken with respect to multi-indices $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{Z}_+^d$ of non-negative integers, and if $x = (x_1, \dots, x_d) \in \mathbf{R}^d$, then $x^{\alpha} \triangleq \prod_{i=1}^d x_i^{\alpha_i}$.

This is our second main result.

Theorem 3.1. *Let U be an open connected set in \mathbf{R}^l and suppose that the point $x_0 \in \mathbf{R}^l$ belongs to the closure of U . Let $x \mapsto \zeta(x)$ and $x \mapsto \xi(x)$ be continuous maps from $U \cup \{x_0\}$ to $\mathcal{L}_1(\mathbf{R})$ and $\mathcal{L}_1(\mathbf{R}^d)$, respectively, whose restrictions to U are analytic. For every $x \in U \cup \{x_0\}$, assume that $\zeta(x) > 0$ and define a probability measure $\mathbb{Q}(x)$ and a $\mathbb{Q}(x)$ -martingale $S(x)$ by*

$$\frac{d\mathbb{Q}(x)}{d\mathbb{P}} = \frac{\zeta(x)}{\mathbb{E}[\zeta(x)]}, \quad S_t(x) = \mathbb{E}^{\mathbb{Q}(x)} \left[\frac{\xi(x)}{\zeta(x)} \middle| \mathcal{F}_t \right].$$

If $S(x_0)$ has the MRP, then the exception set

$$I \triangleq \{x \in U : S(x) \text{ does not have the MRP}\}$$

has Lebesgue measure zero. If, in addition, U is an interval in \mathbf{R} , then the set I is at most countable.

The following example shows that *any* countable set I in \mathbf{R} can play the role of the exception set of Theorem 3.1. In this example we choose $\zeta(x) = 1$ (so that $\mathbb{Q}(x) = \mathbb{P}$) and take $x \mapsto \xi(x)$ to be a *linear* map from \mathbf{R} to $\mathcal{L}_{\infty}(\mathbf{R})$.

Example 3.2. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$ be a filtered probability space, where the filtration is generated by independent Bernoulli random variables (ϵ_n) with

$$\mathbb{P}[\epsilon_n = 1] = \mathbb{P}[\epsilon_n = -1] = \frac{1}{2}.$$

It is well-known that every martingale (N_n) admits the unique “integral” representation:

$$N_n = N_0 + \sum_{k=1}^n h_k(\epsilon_1, \dots, \epsilon_{k-1}) \epsilon_k, \quad (3)$$

for some functions $h_k = h_k(x_1, \dots, x_{k-1})$, $k \geq 1$, where h_1 is just a constant.

Let $I = (x_n)$ be an arbitrary sequence in \mathbf{R} . We define a linear map $x \mapsto \xi(x)$ from \mathbf{R} to $\mathcal{L}_\infty(\mathbf{R})$ by

$$\xi(x) = \sum_{n=1}^{\infty} \frac{(x - x_n)}{2^n(1 + |x_n|)} \epsilon_n = \psi_0 + \psi_1 x,$$

where ψ_0 and ψ_1 are bounded random variables:

$$\psi_0 = - \sum_{n=1}^{\infty} \frac{x_n}{2^n(1 + |x_n|)} \epsilon_n, \quad \psi_1 = \sum_{n=1}^{\infty} \frac{1}{2^n(1 + |x_n|)} \epsilon_n.$$

We have that

$$M_n(x) = \mathbb{E}[\xi(x) | \mathcal{F}_n] = \mathbb{E}[\xi(x) | \epsilon_1, \dots, \epsilon_n] = \sum_{k=1}^n \frac{(x - x_k)}{2^k(1 + |x_k|)} \epsilon_k$$

and thus

$$\Delta M_n(x) = M_n(x) - M_{n-1}(x) = \frac{(x - x_n)}{2^n(1 + |x_n|)} \epsilon_n.$$

If $x \notin I$, then the martingale (N_n) from (3) is a stochastic integral with respect to $M(x)$:

$$N_n = N_0 + \sum_{k=1}^n h_k(\epsilon_1, \dots, \epsilon_{k-1}) \frac{2^k(1 + |x_k|)}{(x - x_k)} \Delta M_k(x).$$

However, if $x_m \in I$, then the martingales $M(x_m)$ and

$$L_n^{(m)} = \sum_{k=1}^n 1_{\{k=m\}} \epsilon_k = 1_{\{n \geq m\}} \epsilon_m, \quad n \geq 0,$$

are orthogonal. Hence, $L^{(m)}$ does not admit an integral representation with respect to $M(x_m)$.

The rest of the section is devoted to the proof of Theorem 3.1. It relies on Theorems A.1 and B.1 from the appendices and on the lemmas below.

Let X be a (uniformly) square integrable martingale taking values in \mathbf{R}^m . We denote by $\langle X \rangle = (\langle X^i, X^j \rangle)$ its predictable process of quadratic variation, which takes values in the cone \mathcal{S}_+^m of symmetric nonnegative $m \times m$ -matrices, and define the predictable increasing process

$$A^X \triangleq \text{tr} \langle X \rangle = \sum_{i=1}^m \langle X^i, X^i \rangle.$$

Standard arguments show that there is a predictable process κ^X with values in \mathcal{S}_+^m such that

$$\langle X \rangle = (\kappa^X)^2 \cdot A^X.$$

On the predictable σ -algebra \mathcal{P} of $[0, \infty) \times \Omega$ we introduce a measure

$$\mu^X(dt, d\omega) \triangleq dA_t^X(\omega) \mathbb{P}[d\omega].$$

For a nonnegative predictable process γ the expectation under μ^X is given by

$$\mathbb{E}^{\mu^X}[\gamma] = \mathbb{E} \left[\int_0^\infty \gamma_t dA_t^X \right].$$

We observe that this measure is finite:

$$\mu^X([0, \infty) \times \Omega) = \mathbb{E} [A_\infty^X] = \mathbb{E} [|X_\infty - X_0|^2] < \infty.$$

For predictable processes (γ^n) and γ the notation $\gamma^n \xrightarrow{\mu^X} \gamma$ stands for the convergence in measure μ^X :

$$\forall \epsilon > 0 : \quad \mu^X[|\gamma^n - \gamma| > \epsilon] \rightarrow 0, \quad n \rightarrow \infty.$$

Lemma 3.3. *Let X be a square integrable martingale with values in \mathbf{R}^m and γ be a predictable m -dimensional process. Then γ is X -integrable and $\gamma \cdot X = 0$ if and only if $\kappa^X \gamma = 0$, μ^X -a.s..*

Proof. Since $\gamma 1_{\{|\gamma| \leq n\}} \cdot X \rightarrow \gamma \cdot X$ as $n \rightarrow \infty$ in the semimartingale topology, we can assume without a loss in generality that γ is bounded. Then $\gamma \cdot X$ is a square integrable martingale with predictable quadratic variation

$$\langle \gamma \cdot X \rangle_t = \int_0^t |\kappa^X \gamma|^2 dA^X$$

and the result follows from the identity:

$$\mathbb{E} [(\gamma \cdot X)_\infty^2] = \mathbb{E} [\langle \gamma \cdot X \rangle_\infty] = \mathbb{E} \left[\int_0^\infty |\kappa^X \gamma|^2 dA^X \right] = \mathbb{E}^{\mu^X} [|\kappa^X \gamma|^2].$$

□

For every predictable process ζ taking values in \mathcal{S}_+^m we can naturally define a \mathcal{S}_+^m -valued predictable process ζ^\oplus such that for all (ω, t) the matrix $\zeta_t^\oplus(\omega)$ is the pseudo-inverse to the matrix $\zeta_t(\omega)$.

From Lemma 3.3 we deduce that if α is an integrand for X then the predictable process

$$\beta \triangleq \kappa^{X^\oplus} \kappa^X \alpha$$

is also X -integrable and $\alpha \cdot X = \beta \cdot X$. Moreover, $|\beta| \leq |\alpha|$, by the minimal norm property of the pseudo-inverse matrices. In view of this property, we call a predictable m -dimensional process γ a *minimal integrand* for X if γ is X -integrable and

$$\gamma = \kappa^{X^\oplus} \kappa^X \gamma.$$

From the definition of a minimal integrand we immediately deduce that

$$|\kappa^X \gamma| \leq |\kappa^X| |\gamma|, \quad |\gamma| \leq |\kappa^{X^\oplus}| |\kappa^X \gamma|. \quad (4)$$

We denote by $\mathcal{H}_1 = \mathcal{H}_1(\mathbf{R}^d)$ the Banach space of uniformly integrable d -dimensional martingales M with the norm:

$$\|M\|_{\mathcal{H}_1} \triangleq \mathbb{E} \left[\sup_{t \geq 0} |M_t| \right].$$

We say that a sequence (N^n) of local martingales converges to a local martingale N in $\mathcal{H}_{1,loc}$ if there are stopping times (τ^m) such that $\tau^m \uparrow \infty$ and $N^{n,\tau^m} \rightarrow N^{\tau^m}$ in \mathcal{H}_1 . Here as usual, we write $Y^\tau \triangleq (Y_{\min(t,\tau)})$ for a semi-martingale Y stopped at a stopping time τ .

Lemma 3.4. *Let X be a square integrable martingale with values in \mathbf{R}^m and (γ^n) be a sequence of predictable m -dimensional X -integrable processes such that the stochastic integrals $(\gamma^n \cdot X)$ converge to 0 in $\mathcal{H}_{1,loc}$. Then $\kappa^X \gamma^n \xrightarrow{\mu^X} 0$. If, in addition, (γ^n) are minimal integrands then $\gamma^n \xrightarrow{\mu^X} 0$.*

Proof. It is sufficient to consider the case of minimal integrands. By localization, we can suppose that $\gamma^n \cdot X \rightarrow 0$ in \mathcal{H}_1 , which by Davis' inequality is equivalent to the convergence of $([\gamma^n \cdot X]_\infty^{1/2})$ to 0 in \mathcal{L}_1 .

Assume for a moment that $|\gamma^n| \leq 1$. Then $[\gamma^n \cdot X] \leq [X]$ and the theorem on dominated convergence yields that $[\gamma^n \cdot X]_\infty \rightarrow 0$ in \mathcal{L}_1 . It follows that

$$\lim_{n \rightarrow \infty} \mathbb{E} [[\gamma^n \cdot X]_\infty] = \lim_{n \rightarrow \infty} \mathbb{E}^{\mu^X} [\kappa^X \gamma^n]^2 = 0.$$

Hence, $\kappa^X \gamma^n \xrightarrow{\mu^X} 0$, which in view of (4), also implies that $\gamma^n \xrightarrow{\mu^X} 0$.

In the general case, we observe that

$$\beta^n \triangleq \frac{1}{1 + |\gamma^n|} \gamma^n$$

are minimal integrands for X such that $|\beta^n| \leq 1$ and $[\beta^n \cdot X] \leq [\gamma^n \cdot X]$. Hence, by what we have already proved, $\beta^n \xrightarrow{\mu^X} 0$, which clearly yields that $\gamma^n \xrightarrow{\mu^X} 0$ and then that $\kappa^X \gamma^n \xrightarrow{\mu^X} 0$. \square

Lemma 3.5. *Let X be a square integrable m -dimensional martingale and $\gamma = (\gamma^{ij})$ be a predictable X -integrable process with values in $\mathbf{R}^{m \times d}$. Then X is a stochastic integral with respect to $Y \triangleq \gamma \cdot X$, that is $X = X_0 + \zeta \cdot Y$ for some predictable Y -integrable $d \times m$ -dimensional process ζ , if and only if*

$$\text{rank } \kappa^X \gamma = \text{rank } \kappa^X, \quad \mu^X - a.s.. \quad (5)$$

Proof. We recall that a predictable process ζ is $Y = \gamma \cdot X$ -integrable if and only if $\gamma \zeta$ is X -integrable. From Lemma 3.3 we deduce that ζ is Y -integrable and satisfies

$$X = X_0 + \zeta \cdot Y = X_0 + \zeta \cdot (\gamma \cdot X) = (\gamma \zeta) \cdot X$$

if and only if

$$\kappa^X \gamma \zeta = \kappa^X, \quad \mu^X - a.s..$$

However, the solvability of this linear equation with respect to ζ is equivalent to (5) by an elementary argument from linear algebra. \square

Lemma 3.6. *Let U be an open connected set in \mathbf{R}^d and $x \mapsto \sigma(x)$ be an analytic map with values in $k \times l$ -matrices. Then there is a nonzero real-analytic function f on U such that*

$$E \triangleq \left\{ x \in U : \text{rank } \sigma(x) < \sup_{y \in U} \text{rank } \sigma(y) \right\} = \{x \in U : f(x) = 0\}.$$

In particular, the set E has Lebesgue measure zero and if $d = 1$, then it consists of isolated points.

Proof. Let $m \triangleq \sup_{y \in U} \text{rank } \sigma(y)$. If $m = 0$, then the set E is empty and we can take $f = 1$. If $m > 0$, then the result holds for

$$f(x) = \sum_{\alpha} \det \sigma_{\alpha}(x) \sigma_{\alpha}^*(x),$$

where (σ_{α}) is the family of all $m \times m$ sub-matrices of σ . The remaining assertions follow from the well-known properties of zero-sets of real-analytic functions. \square

Proof of Theorem 3.1. Without restricting generality we can assume that $\zeta(x_0) = 1$ and, hence, $\mathbb{Q}(x_0) = \mathbb{P}$. Proposition 2 in [9] shows that if some multi-dimensional local martingale has the MRP, then there is a bounded, hence square integrable, m -dimensional martingale X that has the MRP. We fix such X and use for it the \mathcal{S}_+^m -valued predictable process κ^X and the finite measure μ^X on the predictable σ -algebra \mathcal{P} introduced just before Lemma 3.3.

We define the martingales

$$Y_t(x) \triangleq \mathbb{E} [\zeta(x) | \mathcal{F}_t], \quad R_t(x) \triangleq \mathbb{E} [\xi(x) | \mathcal{F}_t],$$

and observe that $R(x) = S(x)Y(x)$. Let $\alpha(x)$ and $\beta(x)$ be integrands for X with values in \mathbf{R}^m and $\mathbf{R}^{m \times d}$, respectively, such that

$$\begin{aligned} Y(x) &= Y_0(x) + Y_-(x)\alpha(x) \cdot X, \\ R(x) &= R_0(x) + Y_-(x)\beta(x) \cdot X. \end{aligned}$$

Integration by parts yields that

$$dR(x) - S_-(x)dY(x) = Y_-(x)d(S(x) + [S(x), \alpha(x) \cdot X]).$$

It follows that

$$S(x) + [S(x), \alpha(x) \cdot X] = S_0(x) + \sigma(x) \cdot X,$$

where

$$\sigma(x) = \beta(x) - \alpha(x)S_-^*(x).$$

From Theorem B.1 we deduce that $S(x)$ has the MRP (under $\mathbb{Q}(x)$) if and only if the stochastic integral $\sigma(x) \cdot X$ has the MRP. By Lemma 3.5 the latter property is equivalent to

$$\text{rank } \kappa^X \sigma(x) = \text{rank } \kappa^X, \quad \mu^X - a.s.,$$

and therefore, the exception set I admits the description:

$$I = \{x \in U : \mu^X [D(x)] > 0\},$$

where for $x \in U \cup \{x_0\}$ the predictable set $D(x)$ is given by

$$D(x) = \{(\omega, t) : \text{rank } \kappa_t^X(\omega) \sigma_t(x)(\omega) < \text{rank } \kappa_t^X(\omega)\}.$$

From Theorem A.1 we deduce the existence of the integrands $\alpha(x)$ and $\beta(x)$ and of the modifications of the martingales $Y(x)$ and $R(x)$ such that for every $(\omega, t) \in \Omega \times [0, \infty)$ the function

$$x \mapsto \sigma_t(x)(\omega) = \beta_t(x)(\omega) - \alpha_t(x)(\omega) \frac{R_{t-}^*(x)(\omega)}{Y_{t-}(x)(\omega)},$$

taking values in the space of $m \times d$ -matrices, is analytic on U . Hereafter, we shall use these versions.

Let λ be the Lebesgue measure on \mathbf{R}^l and $\mathcal{B} = \mathcal{B}(U)$ be the Borel σ -algebra on U . Since for every (ω, t) the function $x \mapsto \sigma_t(x)(\omega)$ is continuous on U , the function $(\omega, t, x) \mapsto \sigma_t(x)(\omega)$ is $\mathcal{P} \times \mathcal{B}$ -measurable. It follows that

$$E \triangleq \{(\omega, t, x) : \text{rank } \kappa_t^X(\omega) \sigma_t(x)(\omega) < \text{rank } \kappa_t^X(\omega)\} \in \mathcal{P} \times \mathcal{B}.$$

From Fubini's theorem we deduce the equivalences:

$$(\mu^X \times \lambda) [E] = 0 \quad \Leftrightarrow \quad \mu^X [F] = 0 \quad \Leftrightarrow \quad \lambda [I] = 0,$$

where

$$F \triangleq \{(\omega, t) : \lambda [\{x \in U : \text{rank } \kappa_t^X(\omega) \sigma_t(x)(\omega) < \text{rank } \kappa_t^X(\omega)\}] > 0\}.$$

Hence to obtain the multi-dimensional version of the theorem we need to show that $\mu^X(F) = 0$.

From Lemma 3.6 and the analyticity of the function $x \mapsto \sigma_t(x)(\omega)$ we deduce that

$$F = \{(\omega, t) : \text{rank } \kappa_t^X(\omega) \sigma_t(x)(\omega) < \text{rank } \kappa_t^X(\omega), \forall x \in U\}. \quad (6)$$

We recall now that if (x_n) is a sequence in U that converges to x_0 , then the martingales $(R(x_n), Y(x_n))$ converge to the martingale $(R(x_0), Y(x_0)) = (S(x_0), 1)$ in \mathcal{L}_1 . By Lemma A.3, passing to a subsequence, we can assume that $(R(x_n), Y(x_n)) \rightarrow (R(x_0), Y(x_0))$ in $\mathcal{H}_{1,loc}$. From Lemma 3.4 we deduce that

$$\begin{aligned}\kappa^X \alpha(x_n) &\xrightarrow{\mu^X} 0, \\ \kappa^X \beta(x_n) &\xrightarrow{\mu^X} \kappa^X \beta(x_0) = \kappa^X \sigma(x_0).\end{aligned}$$

It follows that

$$\kappa^X \sigma(x_n) = \kappa^X (\beta(x_n) - \alpha(x_n) S_-^*(x_n)) \xrightarrow{\mu^X} \kappa^X \beta(x_0) = \kappa^X \sigma(x_0).$$

Passing to a subsequence we can choose the sequence (x_n) so that

$$\kappa^X \sigma(x_n) \rightarrow \kappa^X \sigma(x_0), \quad \mu^X - a.s..$$

As $a \mapsto \text{rank } a$ is a lower-semicontinuous function on matrices, it follows that

$$\liminf_n \text{rank } \kappa^X \sigma(x_n) \geq \text{rank } \kappa^X \sigma(x_0), \quad \mu^X - a.s..$$

Accounting for (6) we obtain that

$$F \subset D(x_0), \quad \mu^X - a.s..$$

However, as $S(x_0)$ has the MRP, Lemma 3.5 yields that $\mu^X [D(x_0)] = 0$ and the multi-dimensional version of the theorem follows.

Assume now that U is an open interval in \mathbf{R} and that contrary to the assertion of the theorem the exception set I is uncountable. Then there are $\epsilon > 0$, a closed interval $[a, b] \subset U$, and a sequence $(x_n) \subset [a, b]$ such that

$$\mu^X [D(x_n)] \geq \epsilon, \quad n \geq 1.$$

Since for every (ω, t) the function $x \mapsto \sigma_t(x)(\omega)$ is analytic, we deduce from Lemma 3.6 that on every closed interval the integer-valued function $x \mapsto \text{rank}(\sigma_t(x)(\omega))$ has constant value except for a *finite* number of points, where its values are smaller. It follows that

$$\limsup_n D(x_n) \triangleq \bigcap_n \bigcup_{m \geq n} D(x_m) = F$$

and thus

$$\mu^X [F] \geq \limsup_n \mu^X [D(x_n)] \geq \epsilon.$$

However, as we have already shown, $\mu^X [F] = 0$ and we arrive to a contradiction. \square

A Analytic fields of martingales and stochastic integrals

We denote by $\mathbf{D}^\infty([0, \infty), \mathbf{R}^d)$ the Banach space of RCLL (right-continuous with left limits) functions $f : [0, \infty) \rightarrow \mathbf{R}^d$ equipped with the uniform norm: $\|f\|_\infty \triangleq \sup_{t \geq 0} |f(t)|$.

Theorem A.1. *Let U be an open connected set in \mathbf{R}^l and $x \mapsto \xi(x)$ be an analytic map from U to $\mathcal{L}_1(\mathbf{R}^d)$. Then there are modifications of the accompanying d -dimensional martingales*

$$M_t(x) \triangleq \mathbb{E} [\xi(x) | \mathcal{F}_t],$$

such that for every $\omega \in \Omega$ the maps $x \mapsto M_t(x)(\omega)$ taking values in $\mathbf{D}^\infty([0, \infty), \mathbf{R}^d)$ are analytic on U .

If in addition, the MRP holds for a local martingale X with values in \mathbf{R}^m , then there is a stochastic field $x \mapsto \sigma(x)$ of integrands for X such that

$$M(x) = M_0(x) + \sigma(x) \cdot X,$$

and for every $(\omega, t) \in \Omega \times [0, \infty)$ the function $x \mapsto \sigma_t(x)(\omega)$ taking values in $m \times d$ -matrices is analytic on U .

The proof of the theorem is divided into a series of lemmas. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbf{Z}_+^l$ we denote

$$|\alpha| \triangleq \alpha_1 + \dots + \alpha_l.$$

The space \mathcal{H}_1 has been introduced just before Lemma 3.4.

Lemma A.2. *Let $(M^\alpha)_{\alpha \in \mathbf{Z}_+^l}$ be uniformly integrable martingales with values in \mathbf{R}^d such that*

$$\sum_{\alpha} 2^{|\alpha|} \|M^\alpha\|_{\mathcal{L}_1} < \infty.$$

Then there is an increasing sequence (τ_m) of stopping times such that $\{\tau_m = \infty\} \uparrow \Omega$ and

$$\sum_{\alpha} \|M^{\alpha, \tau_m}\|_{\mathcal{H}_1} < \infty, \quad m \geq 1.$$

Proof. We define the martingale

$$L_t \triangleq \mathbb{E} \left[\sum_{\alpha} 2^{|\alpha|} |M_{\infty}^{\alpha}| \middle| \mathcal{F}_t \right], \quad t \geq 0,$$

and stopping times

$$\tau_m \triangleq \inf \{t \geq 0 : L_t \geq m\}, \quad m \geq 1.$$

Clearly, $\{\tau_m = \infty\} \uparrow \Omega$ and $|M^{\alpha}| \leq 2^{-|\alpha|} L$. Moreover,

$$\|L^{\tau_m}\|_{\mathcal{H}_1} = \mathbb{E} \left[\sup_{0 \leq t \leq \tau_m} L_t \right] \leq m + \mathbb{E} [L_{\tau_m}] = m + L_0 < \infty.$$

It follows that

$$\sum_{\alpha} \|M^{\alpha, \tau_m}\|_{\mathcal{H}_1} \leq \|L^{\tau_m}\|_{\mathcal{H}_1} \sum_{\alpha} 2^{-|\alpha|} < \infty.$$

□

Lemma A.3. *Let (M^n) and M be uniformly integrable martingales such that $M^n \rightarrow M$ in \mathcal{L}_1 . Then there exists a subsequence of (M^n) that converges to M in $\mathcal{H}_{1,loc}$.*

Proof. Since $M^n \rightarrow M$ in \mathcal{L}_1 there exists a subsequence (M^{n_k}) such that

$$\sum_{k=1}^{\infty} \|M^{n_{k+1}} - M^{n_k}\|_{\mathcal{L}_1} 2^k < \infty.$$

Lemma A.2 implies that $M^{n_k} \rightarrow M$ in $\mathcal{H}_{1,loc}$.

□

Let X be a square integrable martingale taking values in \mathbf{R}^m . As in Section 3 we associate with X the increasing predictable process $A^X \triangleq \text{tr} \langle X \rangle$, the \mathcal{S}_+^m -valued predictable process κ^X such that $\langle X \rangle = (\kappa^X)^2 \cdot A^X$, and a finite measure $\mu^X(dt, d\omega) \triangleq dA_t^X(\omega) \mathbb{P}[d\omega]$ on the predictable σ -algebra \mathcal{P} of $\Omega \times [0, \infty)$. We recall that an integrand γ for X is *minimal* if

$$\gamma = \kappa^{X^{\oplus}} \kappa^X \gamma. \tag{7}$$

Lemma A.4. *Let X be a bounded martingale with values in \mathbf{R}^m and $(\gamma^\alpha)_{\alpha \in \mathbf{Z}_+^l}$ be minimal integrands for X such that*

$$\sum_{\alpha} \|\gamma^\alpha \cdot X\|_{\mathcal{H}^1} < \infty. \quad (8)$$

Then

$$\sum_{\alpha} |\gamma^\alpha|^2 < \infty, \quad \mu^X - a.s.. \quad (9)$$

Proof. By Davis' inequality, (8) is equivalent to

$$\sum_{\alpha} \mathbb{E} \left[[\gamma^\alpha \cdot X]_{\infty}^{1/2} \right] < \infty.$$

By replacing if necessary γ^α with $\frac{1}{1+|\gamma^\alpha|} \gamma^\alpha$, we can assume without a loss of generality that $|\gamma^\alpha| \leq 1$. Let us show that in this case the increasing optional process

$$B_t \triangleq \sum_{\alpha} [\gamma^\alpha \cdot X]_t, \quad t \geq 0,$$

is locally integrable. Since

$$B_{\infty} = \sum_{\alpha} [\gamma^\alpha \cdot X]_{\infty} \leq \left(\sum_{\alpha} [\gamma^\alpha \cdot X]_{\infty}^{1/2} \right)^2 < \infty,$$

we only need to check that the positive jump process ΔB is locally integrable. Actually, we shall show that $\sup_{t \geq 0} \Delta B_t$ is integrable. Indeed, as X is bounded, there is a constant $c > 0$ such that $|(\gamma^\alpha)^* \Delta X| \leq c$. Hence,

$$\sup_{t \geq 0} \Delta B_t \leq \sum_{\alpha} ((\gamma^\alpha)^* \Delta X)^2 \leq c \sum_{\alpha} |(\gamma^\alpha)^* \Delta X| \leq c \sum_{\alpha} [\gamma^\alpha \cdot X]_{\infty}^{1/2},$$

where the right-hand side has finite expected value.

Since for every stopping time τ

$$\mathbb{E} [B_{\tau}] = \sum_{\alpha} \mathbb{E} [[\gamma^\alpha \cdot X]_{\tau}] = \sum_{\alpha} \mathbb{E} \left[\int_0^{\tau} |\kappa^X \gamma^\alpha|^2 dA^X \right],$$

the local integrability of B yields the existence of stopping times (τ^m) such that $\tau_m \uparrow \infty$ and

$$\sum_{\alpha} \mathbb{E} \left[\int_0^{\tau_m} |\kappa^X \gamma^\alpha|^2 dA^X \right] = \sum_{\alpha} \mathbb{E}^{\mu^X} \left[|\kappa^X \gamma^\alpha|^2 1_{[0, \tau_m]} \right] < \infty.$$

It follows that

$$\sum_{\alpha} |\kappa^X \gamma^{\alpha}|^2 < \infty, \quad \mu^X - a.s..$$

This convergence implies (9) in view of inequalities (4) for minimal integrands. \square

Lemma A.5. *Let X be a square integrable martingale taking values in \mathbf{R}^m and (γ^n) be minimal integrands for X such that $(M^n \triangleq \gamma^n \cdot X)$ are uniformly integrable martingales. Suppose that there are a uniformly integrable martingale M and a predictable process γ such that $M^n \rightarrow M$ in \mathcal{L}_1 and $\gamma_t^n(\omega) \rightarrow \gamma_t(\omega)$ for every (ω, t) . Then γ is a minimal integrand for X and $M = \gamma \cdot X$.*

Proof. In view of characterization (7) for minimal integrands, the minimality of every element of (γ^n) implies the minimality of γ provided that the latter is X -integrable. Thus we only need to show that γ is X -integrable and $M = \gamma \cdot X$.

By Lemma A.3, passing to subsequences, we can assume that $M^n = \gamma^n \cdot X \rightarrow M$ in $\mathcal{H}_{1,loc}$. Since the space of stochastic integrals is closed under the convergence in $\mathcal{H}_{1,loc}$, there is a X -integrable predictable process $\tilde{\gamma}$ such that $M = \tilde{\gamma} \cdot X$. From Lemma 3.4 we deduce that

$$\kappa^X(\gamma^n - \tilde{\gamma}) \xrightarrow{\mu^X} 0.$$

It follows that

$$\kappa^X(\tilde{\gamma} - \gamma) = 0, \quad \mu^X - a.s.,$$

and Lemma 3.3 yields the result. \square

Proof of Theorem A.1. It is sufficient to prove the existence of the required analytic versions only locally, in a neighborhood of every $y \in U$. Hereafter, we fix $y \in U$. There are $\epsilon = \epsilon(y) \in (0, 1)$ and a family $(\zeta_{\alpha} = \zeta_{\alpha}(y))_{\alpha \in \mathbf{Z}_+^l}$ in \mathcal{L}_1 such that

$$\begin{aligned} \xi(x) &= \xi(y) + \sum_{\alpha} \zeta_{\alpha}(x - y)^{\alpha}, \quad \max_i |x_i - y_i| < 2\epsilon, \\ \sum_{\alpha} \mathbb{E}[|\zeta_{\alpha}|] (2\epsilon)^{|\alpha|} &< \infty, \end{aligned}$$

where the first series converges in \mathcal{L}_1 .

By taking conditional expectations with respect to \mathcal{F}_t we obtain that

$$M_t(x) = M_t(y) + \sum_{\alpha} L_t^{\alpha} (x - y)^{\alpha}, \quad \max_i |x_i - y_i| < 2\epsilon, \quad (10)$$

where $L_t^{\alpha} \triangleq \mathbb{E}[\zeta_{\alpha} | \mathcal{F}_t]$ and the series converges in \mathcal{L}_1 . Lemma A.2 yields an increasing sequence (τ_m) of stopping times such that $\{\tau_m = \infty\} \uparrow \Omega$ and

$$\sum_{\alpha} \|L^{\alpha, \tau_m}\|_{\mathcal{H}_1} \epsilon^{|\alpha|} < \infty, \quad m \geq 1.$$

It follows that

$$\sum_{\alpha} \sup_{t \geq 0} |L_t^{\alpha}(\omega)| \epsilon^{|\alpha|} < \infty, \quad \mathbb{P} - a.s.$$

and we can modify the martingales (L^{α}) so that the above convergence holds true for every $\omega \in \Omega$. Then the series in (10) converges uniformly in t for every $\omega \in \Omega$ and every x such that $\max_i |x_i - y_i| < \epsilon$. Thus, it defines the modifications of $M(x)$ for such x with the required analytic properties.

For the second part of the theorem we observe that the statement is invariant with respect to the choice of the local martingale X that has the MRP. Proposition 2 in [9] shows that we can choose X to be a bounded m -dimensional martingale.

As X has the MRP, there are minimal integrands $\sigma(y)$ and (γ^{α}) such that

$$\begin{aligned} M(y) &= M_0(y) + \sigma(y) \cdot X, \\ L^{\alpha} &= L_0^{\alpha} + \gamma^{\alpha} \cdot X, \quad \alpha \in \mathbf{Z}_+^l. \end{aligned}$$

From Lemma A.4 we deduce that

$$\sum_{\alpha} |\gamma_t^{\alpha}(\omega)|^2 \epsilon^{2|\alpha|} < \infty$$

for all (ω, t) except a predictable set of μ^X -measure 0. By Lemma 3.3 we can set $\gamma^{\alpha} = 0$ on this set without changing $\gamma^{\alpha} \cdot X$. Then the series converges for every (ω, t) . As $\epsilon \in (0, 1)$, we deduce that

$$\sum_{\alpha} |\gamma_t^{\alpha}(\omega)| \epsilon^{2|\alpha|} < \infty$$

and thus for $x = (x_1, \dots, x_l)$ such that $\max_i |x_i - y_i| < \epsilon^2$ and every (ω, t) we can define

$$\sigma_t(x)(\omega) \triangleq \sigma_t(y)(\omega) + \sum_{\alpha} \gamma_t^{\alpha}(\omega) (x - y)^{\alpha}.$$

By construction, the function $x \rightarrow \sigma_t(x)(\omega)$ is analytic in a neighborhood of y . By Lemma A.5, for every x such that $\max_i |x_i - y_i| < \epsilon^2$ the predictable process $\sigma(x)$ is an integrand for X and

$$\begin{aligned} M(x) &= M(y) + \sum_{\alpha} L^{\alpha}(x - y)^{\alpha} \\ &= M_0(x) + \sigma(y) \cdot X + \sum_{\alpha} (\gamma^{\alpha} \cdot X)(x - y)^{\alpha} \\ &= M_0(x) + \sigma(x) \cdot X. \end{aligned}$$

□

B The MRP under the change of measure

Let X be a d -dimensional local martingale and $Z > 0$ be the density process of $\tilde{\mathbb{P}} \sim \mathbb{P}$. We denote by $\tilde{Z} \triangleq 1/Z$ the density process of \mathbb{P} under $\tilde{\mathbb{P}}$ and set $L \triangleq \tilde{Z}_- \cdot Z$ and $\tilde{L} \triangleq Z_- \cdot \tilde{Z}$. Using integration by parts we deduce that

$$d(\tilde{Z}X) = X_- d\tilde{Z} + \tilde{Z}_- d\tilde{X},$$

where

$$\tilde{X} = X + [X, \tilde{L}].$$

It follows that \tilde{X} is a d -dimensional local martingale under $\tilde{\mathbb{P}}$. Of course, this is just a version of Girsanov's theorem.

We observe that the relations between X and \tilde{X} are symmetric in the sense that

$$X = \tilde{X} + [\tilde{X}, L].$$

Indeed, as we have already shown, $Y \triangleq \tilde{X} + [\tilde{X}, L]$ is a d -dimensional local martingale. Clearly, the local martingales \tilde{X} and Y have the same initial values and the same continuous martingale parts. Finally, they have identical jumps:

$$\begin{aligned} \Delta(Y - X) &= \Delta([X, \tilde{L}] + [\tilde{X}, L]) = \Delta X(\Delta \tilde{L} + \Delta L + \Delta \tilde{L} \Delta L) \\ &= \Delta X \Delta(Z \tilde{Z}) = 0. \end{aligned}$$

Theorem B.1. *The MRP holds under X if and only if it holds under \tilde{X} .*

Proof. By symmetry, it is sufficient to prove only one of the implications. We assume that X has the MRP. Let \tilde{M} be a local martingale under $\tilde{\mathbb{P}}$. The arguments before the statement of the theorem yield the unique local martingale M such that

$$\tilde{M} = M + [M, \tilde{L}].$$

If now H is an integrand for X such that $M = M_0 + H \cdot X$, then

$$\tilde{M} = \tilde{M}_0 + H \cdot (X + [X, \tilde{L}]) = \tilde{M}_0 + H \cdot \tilde{X}.$$

□

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